Discrete to Continuous Optimization

September 18, 2023

1 Definitions and notations

Let $V = \{1, \dots, n\}$ be a finite and non-empty set, and denote $\mathcal{P}(V)$ its power set. Define also $\mathcal{P}(V)^k = \{(S_1, \dots, S_k) : S_i \subset V\}$ and $\mathcal{P}_k(V) = \{(S_1, \dots, S_k)\} \in \mathcal{P}(V)^k : S_i \cap S_j = \emptyset\}$.

We say that two vectors $x, y \in \mathbb{R}^n$ are comonotonic if $(x_i - x_j)(y_i - y_j) \geq 0$ for all i, j i.e. they have the same "variations".

A discrete function $F : \mathcal{A} \to \mathbb{R}$ is submodular if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \in \mathcal{A}$. A continuous function $F : \mathbb{R}^n \to \mathbb{R}$ is submodular if $F(x) + F(y) \geq F(\min(x, y)) + F(\max(x, y))$ where the min and max are taken elementwise.

2 Lovasz extensions

2.1 Definitions

Given a function $f : \mathcal{P}(V) \to \mathbb{R}$, its Lovasz extension extends the domain of f to $\mathbb{R}^n$:

> Denote $x = (x_1, \dots, x_n) \in \mathbb{R}$, and let $\sigma : V \cup \{0\} \to V \cup \{0\}$ be a permutation such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \dots \leq x_{\sigma(n)}$ and $\sigma(0) = 0$ where we add $x_0 = 0$ to x. The Lovasz extension is defined as
>
> $$f^L(x) = \sum_{i=0}^{n-1} (x_{\sigma(i+1)} - x_{\sigma(i)}) f(V^{\sigma(i)}(x)) \tag{1}$$
>
> where $V^0 = V$ and $V^{\sigma(i)}(x) = \{j \in V : x_j > x_{\sigma(i)}\}$.

The above formula can be rewritten as an integral :

$$f^L(x) = \int_{\min x_i}^{\max x_i} f(V^t(x))dt + f(V) \min x_i \tag{2}$$

where $V^t(x) = \{i \in V, x_i > t\}$ ($V^t(x)$ is constant by parts and $x_0 = 0$) and with the assumption $f(\emptyset) = 0$,

$$f^L(x) = \int_{-\infty}^{0} (f(V^t(x)) - f(V))dt + \int_{0}^{+\infty} f(V^t(x))dt \tag{3}$$

Indeed, when $t < \min x_i$, $V^t(x) = V$ so in this case the left term is equal to 0, while when $t > \max x_i$, $V^t(x) = \emptyset$ so by assumption in this case the right term is equal to 0.

2.2 Key properties

Proposition 2.1. *For a set S, denoting its characteristic vector $\mathbf{1}_S \in \{0, 1\}^n$, $f^L(\mathbf{1}_S) = f(S)$.*

Proof. All the term in the above sum are equal to 0 except for the first appearing 1 in the sorted x (when $x_{\sigma(i+1)} = 1$ and $x_{\sigma(i)} = 0$) because in these cases $x_{\sigma(i+1)} = x_{\sigma(i)} = 0$. For the only non-null term, $x_{\sigma(i)} = 0$ so $V^{\sigma(i)} = S$. □

Proposition 2.2. *f^L is the unique function that is affine on each polyhedral cone $\mathbb{R}^n_\sigma = \{x \in \mathbb{R}^n, x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}\}$ and for which $f^L(\mathbf{1}_S) = f(S)$ for every set S.*

Proposition 2.3. *f^L is positively one-homogeneous, piecewise-linear and Lipschitzian continuous.*

Proposition 2.4. *$f^L(x + t\mathbf{1}_V) = f^L(x) + tf(V)$.*

Proof. We have

$$f^L(x + t\mathbf{1}_V) = \sum_{i=0}^{n-1}((x + t\mathbf{1}_V)_{\sigma(i+1)} - (x + t\mathbf{1}_V)_{\sigma(i)})f(V^{\sigma(i)}(x + t\mathbf{1}_V)) \tag{4}$$

In the above formula, $V^{\sigma(i)}(x + t\mathbf{1}_V) = V^{\sigma(i)}(x)$ because translating all indices by t doesn't change their sorted order, and when $i \neq 0$

$$(x + t\mathbf{1}_V)_{\sigma(i+1)} - (x + t\mathbf{1}_V)_{\sigma(i)} = x_{\sigma(i+1)} - x_{\sigma(i)} \tag{5}$$

while when $i = 0$ this is equal to $x_{\sigma(1)} + t$. □

Proposition 2.5. *A continuous function $F : \mathbb{R}^V \to \mathbb{R}$ is a Lovasz extension of some $f : \mathcal{P}(V) \to \mathbb{R}$ if and only if $F(x + y) = F(x) + F(y)$ for x, y comonotonic.*

Proposition 2.6. *f is submodular $\iff$ f^L is convex $\iff$ f^L is submodular.*

Theorem 2.7. *A one homogeneous function $F : \mathbb{R}^V \to \mathbb{R}$ is a Lovasz extension of some submodular function if and only if $F(x + t\mathbf{1}_V) = F(x) + tF(\mathbf{1}_V)$ and $F(x) + F(y) \geq F(\max(x, y)) + F(\min(x, y))$ where the min and max are taken elementwise.*

2.3 Connection to Morse theory

Will be done later. Apparently this allows to establish a discrete Morse theory on hypergraphs.

3 Multi-way extensions

3.1 Definitions

The k-way Lovasz extension of a function $f : \mathcal{P}(V_1) \times \cdots \times \mathcal{P}(V_k) \to \mathbb{R}$ is defined as

$$f^L(x^1, \ldots, x^k) = \int_{\min x}^{\max x} f(V_1^t(x^1), \ldots, V_k^t(x^k))dt + f(V_1, \ldots, V_k) \min x \tag{6}$$

$$= \int_{-\infty}^{0} (f(V_1^t(x^1), \ldots, V_k^t(x^k)) - f(V_1, \ldots, V_k))dt + \int_{0}^{+\infty} f(V_1^t(x^1), \ldots, V_k^t(x^k))dt \tag{7}$$

where the V_i^t are defined in the same way as for the classical case. Beware that the x^k are all vectors of $\mathbb{R}^n$.

In this case, we take $\mathcal{D}_{\mathcal{A}} = \{x \in \mathbb{R}_+^{kn} | ((V^t(x^1), \ldots, V^t(x^k)) \in \mathcal{A} \text{for all } t \in \mathbb{R}\}$.

The disjoint-pair Lovasz extension of a function $f : \mathcal{P}_2(V) \to \mathbb{R}$ is defined as

Set $f(\emptyset, \emptyset) = 0$, f^L can be written as

$$f^L(x) = \int_0^{+\infty} f(V_+^t(x), V_-^t(x))dt \tag{8}$$

or with a sum

$$f^L(x) = \sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) f(V_{\sigma(i)}^+(x), V_{\sigma(i)}^-(x))dt \tag{9}$$

where σ is a permutation sorting $|x|$ i.e. sorting x by the absolute value of its components and $\sigma(0) = 0$, where we add $x_0 = 0$ to x and

$$V_{\sigma(i)}^+ = \{j \in V, +x_j > |x_{\sigma(i)}|\} \tag{10}$$

with $V_{\sigma(i)}^-$ being defined the same way.

The identification $\mathbf{1}_S \iff S$ is replaced here by $\mathbf{1}_S - \mathbf{1}_T \iff (S, T)$ where S and T are disjoint. In this case, we take $\mathcal{D}_{\mathcal{A}} = \{x \in \mathbb{R}_+^V | ((V_+^t(x), V_-^t(x)) \in \mathcal{A} \text{for all } t \geq 0\}$.

It is possible to define a k-way analog for disjoint-pair Lovasz extensions :

Given $v_i = \{1, \ldots, n_i\}$ and a function $f : \mathcal{P}_2(V_1) \times \cdots \times \mathcal{P}_2(V_k)$, define $f^L : \mathbb{R}^{V_1} \times \cdots \times \mathbb{R}^{V_k} \to \mathbb{R}$ by

$$f^L(x^1, \ldots, x^k) = \int_0^{\|x\|_\infty} f(V_{1,t}^+(x^1), V_{1,t}^-(x^1), \ldots, V_{k,t}^+(x^k), V_{k,t}^-(x^k))dt \tag{11}$$

where $V_{i,t}^+(x^i) = \{j \in V_i, +x_j^i > t\}$. We can replace $\|x\|_\infty$ by $+\infty$ if we set $f(\emptyset, \emptyset) = 0$.

k-way submodularity is defined as follows :

> Given a tuple $V = (V_1, \ldots, V_k)$ of finite sets and $\mathcal{A} \subset \{(A_1, \ldots, A_k) | A_i \subset V_i\}$, a discrete function $f : \mathcal{A} \to \mathbb{R}$ is k-way submodular if $f(A_1, \ldots, A_k) + f(B_1, \ldots, B_k) \geq f(\min(A_1, B_1), \ldots, \min(A_k, B_k)) + f(\max(A_1, B_1), \ldots, \max(A_k, B_k))$.

3.2 Key properties

Proposition 3.1. *For the multi-way Lovasz extension f^L, we have*
(a) $f^L(\cdot)$ is positively one-homogeneous, piecewise linear and Lipschitz continuous
(b) $(\lambda f)^L = \lambda f^L$

Proposition 3.2. *For the disjoint-pair Lovasz extension f^L, we have*
(a) f^L is Lipschitz continuous, and $|f^L(x) - f^L(y)| \leq 2 \max_{(S,T) \in \mathcal{P}_2(V)} f(S, T) \|x - y\|_1$ for all $x, y \in \mathbb{R}^n$. Also, $|f^L(x) - f^L(y)| \leq 2 \sum_{(S,T) \in \mathcal{P}_2(V)} f(S, T) \|x - y\|_\infty$ for all $x, y \in \mathbb{R}^n$.
(b) $f^L(-x) = \pm f^L(x)$ for all $x \in \mathbb{R}^n$ if and only if $f(S, T) = \pm f(T, S)$ for all $(S, T) \in \mathcal{P}_2(V)$.
(c) $f^L(x+y) = f^L(x) + f^L(y)$ whenever $V_\pm(y) \subset V_\pm(\tilde{x})$ where $\tilde{x}$ has components $\tilde{x}_i = x_i$ if $|x_i| = \|x_i\|_\infty$ and 0 otherwise.

Proposition 3.3. *A continuous function F is a disjoint-pair Lovasz extension of some function $f : \mathcal{P}_2(V) \to \mathbb{R}$ if and only if $F(x) + F(y) = F(x + y)$ whenever $|x|$ and $|y|$ are comonotonic.*

Proof. By the definition of the disjoint-pair Lovasz extension, we know that F is a disjoint-pair Lovasz extension of some function $f : \mathcal{P}_2(V) \to \mathbb{R}$ if and only if $\lambda F(x) + (1-\lambda)F(y) = F(\lambda x + (1-\lambda)y)$ for all vectors x, y such that $|x|$ and $|y|$ are absolutely comonotonic, for all $\lambda \in [0,1]$. Therefore we only need to prove the sufficiency part.

For $x \in \mathbb{R}^n$, since sx and tx with $s, t \geq 0$ are absolutely comonotonic, $F(sx) + F(tx) = F((s+t)x)$ which yields a Cauchy equation on the halft-line. Thus the continuity assumption implies the linearity of F on the ray $\mathbb{R}^+ x$, which implies the property $F(tx) = tF(x)$ for all $t \geq 0$, and hence $\lambda F(x) + (1-\lambda)F(y) = F(\lambda x + (1-\lambda)y)$ for any vectors x and y such that $|x|$ and $|y|$ are comonotonic and for all $\lambda \in [0,1]$.

This completes the proof. $\square$

For relations between the original and the disjoint-pair Lovasz extension, we also have

Proposition 3.4. *For $h : \mathcal{P}(V) \to [0, +\infty)$ with $h(\emptyset) = 0$ and $f : \mathcal{P}_2(V) \to [0, +\infty)$ with $f(\emptyset, \emptyset) = 0$, we have :*

(a) If $f(S,T) = h(S) + h(V \setminus T) - h(V)$ for all $(S,T) \in \mathcal{P}_2(V)$ then $f^L = h^L$.

(b) If $f(S,T) = h(S) + h(T)$ and $h(S) = h(V \setminus S)$ for all $(S,T) \in \mathcal{P}_2(V)$ then $f^L = h^L$.

(c) If $f(S,T) = h(S)$ for all $(S,T) \in \mathcal{P}_2(V)$ then $f^L(x) = h^L(x)$ for all $x \in [0, +\infty)^n$.

(d) If $f(S,T) = h(S \cup T)$ for all $(S,T) \in \mathcal{P}_2(V)$ then $f^L(x) = h^L(|x|)$.

(e) If $f(S,T) = h(S) \pm h(T)$ for all $(S,T) \in \mathcal{P}_2(V)$ then $f^L(x) = h^L(x^+) \pm h^L(x^-)$ where $x^\pm = \max(0, \pm x)$.

Proposition 3.5. *For the simple k-way Lovasz extension, if f has a separable summation form $f(S_1, \ldots, S_k) = \sum_{i=0}^k f_i(S_i)$, we have $f^L(x^1, \ldots, x^k) = \sum_{i=1}^k f_i^L(x^i)$ for all $(x^1, \ldots, x^k)$.*

Similarly, for the disjoint-pair Lovasz extension, if f have a separable summation form $f(S_1, T_1, \ldots, S_k, T_k) = \sum_{i=0}^k f_i(S_i, T_i)$ where $S_i \cap T_i = \emptyset$ for all i, we have $f^L(x^1, \ldots, x^k) = \sum_{i=1}^k f_i^L(x^i)$ for all $(x^1, \ldots, x^k)$.

Proposition 3.6. *The following statements are equivalent :*

a) f is k-way submodular on $\mathcal{A}$

b) the k-way Lovasz extension f^L is convex on each convex subset of $\mathcal{D}_\mathcal{A}$

c) the k-way Lovasz extension f^L is submodular on $\mathcal{D}_\mathcal{A}$

4 Combinatorial and continuous optimization links via Lovasz extensions

4.1 Definitions

Here, f^L denote any of the different Lovasz extensions defined above. $\mathcal{A}$ is a restricted family of $\mathcal{P}(V)^k$.

4.2 Key results

Theorem 4.1. *Given set functions $f_1, \ldots, f_n : \mathcal{A} \to \mathbb{R}^+$ and a zero-homogeneous function $H : (\mathbb{R}^+)^m \setminus \{0\} \to \mathbb{R} \cup \{+\infty\}$ with $H(a+b) \geq \min(H(a), H(b))$ for all $a, b \in (\mathbb{R}^+)^m \setminus \{0\}$, we have*

$$\min_{S \in \mathcal{A}'} H(f_1(S), \ldots, f_n(S)) = \inf_{x \in \mathcal{D}'} H(f_1^L(x), \ldots, f_n^L(x)) \tag{12}$$

where $\mathcal{A}' = \{S \in \mathcal{A} : (f_1(S), \ldots, f_n(S)) \in Dom(H)\}$, $\mathcal{D}' = \{x \in \mathcal{D}_\mathcal{A} \cap (\mathbb{R}^+)^V : (f_1^L(x), \ldots, f_n^L(x)) \in Dom(H)\}$ and $Dom(H) = \{a \in (\mathbb{R}^+)^m \setminus \{0\} : H(a) \in \mathbb{R}\}$.

Proof. By the property of H,

$$H(\sum_{i=1}^m t_i a_{i,1}, \ldots, \sum_{i=1}^m t_i a_{i,n}) = H(\sum_{i=1}^m t_i a^i) \geq \min H(t_i a^i) \tag{13}$$

$$= \min H(a^i) = \min H(a_{i,1}, \ldots, a_{i,n}) \tag{14}$$

For the original Lovasz extension,

$$H(f_1^L(x),\dots,f_n^L(x)) \tag{15}$$
$$= H(\int_{\min x}^{\max x} f_1(V^t(x))dt + f_1(V)\min x,\dots,\int_{\min x}^{\max x} f_n(V^t(x))dt + f_n(V)\min x) \tag{16}$$
$$= H(\sum_{i=1}^{m}(t_i - t_{i-1})f_1(V^{t_i-1}(x)),\dots,\sum_{i=1}^{m}(t_i - t_{i-1})f_n(V^{t_i-1}(x))) \tag{17}$$
$$\geq \min_i H(f_1(V^{t_i-1}(x)),\dots,f_n(V^{t_i-1}(x))) \tag{18}$$
$$\geq \min_{\mathcal{A}\in\mathcal{A}'} H(f_1(S),\dots,f_n(S)) \tag{19}$$
$$= \min_{\mathcal{A}\in\mathcal{A}'} H(f_1^L(\mathbf{1}_S),\dots,f_n^L(\mathbf{1}_S)) \tag{20}$$
$$\geq \inf_{x\in\mathcal{D}'} H(f_1^L(x),\dots,f_n^L(x)) \tag{21}$$

Thus taking the inf and min on the adequate inequality we obtain that $\min_{\mathcal{A}\in\mathcal{A}'} H(f_1(S),\dots,f_n(S)) = \min_{x\in\mathcal{D}'} H(f_1^L(S),\dots,f_n^L(S))$.

The proof is similar for multi-way extensions. $\square$

Proposition 4.2. *Given two set functions $f,g:\mathcal{A}\to[0,\infty)$, let $\tilde{f},\tilde{g}:\mathcal{D}_{\mathcal{A}}\to\mathbb{R}$ satisfying $\tilde{f}\geq f^L$, $\tilde{g}\leq g^L$, $\tilde{f}(\mathbf{1}_S)=f(S)$ and $\tilde{g}(\mathbf{1}_S)=g(S)$. Then*

$$\min_{S\in\mathcal{A}\cap supp(g)} \frac{f(S)}{g(S)} = \inf_{\Psi\in\mathcal{D}_{\mathcal{A}}\cap supp(\tilde{g})} \frac{\tilde{f}(\Psi)}{\tilde{g}(\Psi)} \tag{22}$$

If we replace the conditions by $\tilde{f}\leq f^L$, $\tilde{g}\geq g^L$. Then

$$\max_{S\in\mathcal{A}\cap supp(g)} \frac{f(S)}{g(S)} = \sup_{\Psi\in\mathcal{D}_{\mathcal{A}}\cap supp(\tilde{g})} \frac{\tilde{f}(\Psi)}{\tilde{g}(\Psi)} \tag{23}$$

Proof. Obviously,

$$\inf_{\Psi\in\mathcal{D}_{\mathcal{A}}\cap\mathrm{supp}(\tilde{g})} \frac{\tilde{f}(\Psi)}{\tilde{g}(\Psi)} \leq \min_{S\in\mathcal{A}\cap\mathrm{supp}(g)} \frac{\tilde{f}(\mathbf{1}_S)}{\tilde{g}(\mathbf{1}_S)} = \min_{S\in\mathcal{A}\cap\mathrm{supp}(g)} \frac{f(S)}{g(S)} \tag{24}$$

On the other hand, for any $\phi\in\mathcal{D}_{\mathcal{A}}\cap\mathrm{supp}(\tilde{g})$, $g^L(\phi)\geq\tilde{g}(\phi)>0$. Hence there exists $t\in(\min\tilde{\beta}\phi-1,\max\tilde{\beta}\phi+1)$ satisfying $g(V^t(\phi))>0$. Here $\beta\phi=\phi$ (resp. $|\phi|$) if f^L represents either the original or the k-way Lovasz extension of f (resp., either the disjoint-pair or the k-way disjoint-pair Lovasz extension). So, the set $W(\phi)=\{t\in\mathbb{R}, g(V^t(\phi))>0\}$ is non-empty. Since $\{V^t(\phi), t\in W(\phi)\}$ is finite, there exists $t_0\in W(\phi)$ such that $\frac{f(V^{t_0}(\phi))}{g(V^{t_0}(\phi))}=\min_{t\in W(\phi)}\frac{f(V^t(\phi))}{g(V^t(\phi))}$. Accordingly, $f(V^t(\phi))\geq\frac{f(V^{t_0}(\phi))}{g(V^{t_0}(\phi))}g(V^t(\phi))$ for any $t\in W(\phi)$, and thus

$$f(V^t(\phi))\geq Cg(V^t(\phi)) \text{ with } C=\min_{t\in W(\phi)}\frac{f(V^t(\phi))}{g(V^t(\phi))}\geq 0 \tag{25}$$

holds for any $t\in\mathbb{R}$ (because $g(V^t(\phi))=0$ for $t\in\mathbb{R}\setminus W(\phi)$ which means that the above inequality automatically holds). Consequently,

$$\tilde{f}(\phi)\geq f^L(\phi) \tag{26}$$
$$= \int_{\min\tilde{\beta}\phi}^{\max\tilde{\beta}\phi} f(V^t(\phi))dt + f(V(\phi))\min\tilde{\beta}\phi \tag{27}$$
$$\geq C\left[\int_{\min\tilde{\beta}\phi}^{\max\tilde{\beta}\phi} g(V^t(\phi))dt + g(V(\phi))\min\tilde{\beta}\phi\right] \tag{28}$$
$$= Cg^L(\phi)\geq C\tilde{g}(\phi) \tag{29}$$

It follows that

$$\frac{\tilde{f}(\phi)}{\tilde{g}(\phi)} \geq C \geq \min_{S\in\mathcal{A}\cap\mathrm{supp}(g)} \frac{f(S)}{g(S)} \tag{30}$$

and thus the proof is completed. The dual case is similar. □

Proposition 4.3. *Let $f, g : \mathcal{A} \to [0, +\infty)$ be two set functions and $f = f_1 - f_2$ and $g = g_1 - g_2$ be decompositions of differences of submodular functions.*

Let $\tilde{f}_2, \tilde{g}_1$ be the restriction of positively one-homogeneous convex functions onto $\mathcal{D}_{\mathcal{A}}$, with $f_1(S) = \tilde{f}_1(\mathbf{1}_S)$ and $g_2(S) = \tilde{g}(\mathbf{1}_S)$. Define $\tilde{f} = f_1^L - \tilde{f}_2$ and $\tilde{g} = \tilde{g}_1 - g_2^L$. Then,

$$\min_{S\in\mathcal{A}\cap supp(g)} \frac{f(S)}{g(S)} = \min_{x\in\mathcal{D}_{\mathcal{A}}\cap supp(\tilde{g})} \frac{\tilde{f}(S)}{\tilde{g}(S)} \tag{31}$$

5 Homogeneous and piecewise multilinear extensions

5.1 Definitions

Given $V_i = \{1, \dots, n_i\}$ and the power set $\mathcal{P}(V_i)$ for $i = 1, \dots, k$, for a discrete function $f : \mathcal{P}(V_1) \times \dots \times \mathcal{P}(V_k) \to \mathbb{R}$, its piecewise multilinear extension is defined on $\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k}$ by

$$f^M(x^1, \dots, x^k) = \sum_{i_1\in V_1,\dots,i_k\in V_k} \prod_{l=1}^{k} (x^l_{\sigma_l(i_l)} - x^l_{\sigma_l(i_{l-1})}) f(V^{i_1}(x^1), \dots, V^{i_k}(x^k)) \tag{32}$$

where $V^i(x^l) = \{j \in V_l | x^l_j > x^l_{l-1}\}$ for $i \geq 2$, $V^1(x^l) = V_l$, σ_l is a permutation of indices sorting x_l by non-decreasing order, and we add $x^l_0 = 0$ to each x^l.

For such a piecewise multilinear extension, its piecewise polynomial extension is defined for all $x \in \mathbb{R}$ as

$$f^M_\Delta(x) = f^M(x, \dots, x) \tag{33}$$

Under the assumption that $f(A_1, \dots, A_k) = 0$ whenever $A_i \in \{V, \emptyset\}$ for some i, we have the following integral representation

$$f^M(x^1, \dots, x^k) = \int_{\min x^k}^{\max x^k} \cdots \int_{\min x^1}^{\max x^1} f(V^{t_1}(x^1), \dots, V^{t_k}(x^k)) dt_1 \dots dt_k \tag{34}$$

where $V^{t_l}(x^l) = \{j \in V | x^l_j > t_l\}$.

This is under the above assumption, I will write down the explicit formula when this is not equal to 0 later.

The disjoint-pair Lovasz extension can be generalized : for a function $f : \mathcal{P}_2(V_1) \times \dots \times \mathcal{P}_2(V_k) \to \mathbb{R}$, the multiple integral extension on $\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k}$ is defined as

$$f^M(x^1, \dots, x^k) = \int_0^{\|x^k\|_\infty} \cdots \int_0^{\|x^1\|_\infty} f(V^{t_1}_+(x^1), V^{t_1}_-(x^1), \dots, V^{t_k}_+(x^k), V^{t_k}_-(x^k)) dt_1 \dots dt_k \tag{35}$$

where $V^{t_l}_\pm = \{j \in V_l | \pm x^l_j > t_l\}$.

5.2 Key properties

Proposition 5.1. *Let $f : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k} \to \mathbb{R}$ and define $\tilde{f} : A \in \mathcal{P}(V_1) \to f^M(\mathbf{1}_A, x^2, \dots, x^k) \in \mathbb{R}$. Then $\tilde{f}^L(x) = f^M(x, x^2, \dots, x^k)$ for any $x \in \mathbb{R}^{n_1}$.*

Proposition 5.2. *A function $f : \mathcal{P}(V_1) \times \dots \mathcal{P}(V_k) \to \mathbb{R}$ is modular on each component if and only if f^M is multilinear. Then we have*

$$f^M(x^1, \dots, x^k) = \int_0^{\max x^k} \cdots \int_0^{\max x^1} f(V^{t_1}(x^1), \dots, V^{t_k}(x^k))dt_1 \dots dt_k \tag{36}$$

where $V^{t_l}(x^l) = \{j \in V | x^l_j > t_l\}$.

Proof. Assume f is modular on each component. Then by the definition of f^M, it must be linear on each component, thus f^M is a k-homogeneous polynomial and is linear on each variable x^l_i. Therefore the explicit expression is uniquely determined by the data on the subset $\{x^1 \in \mathbb{R}^{\ltimes\rtimes} | \min x^1 = 0\} \times \dots \{x^k \in \mathbb{R}^{\ltimes\urcorner} | \min x^k = 0\}$. On such a subset, the above formula can be derived directly form the definition of f^M.

For the converse, assume that f^M is multilinear and f is not modular on its first component. Then the restriction of f^M to its first component $f^M(x, \mathbf{1}_{A_2}, \dots, \mathbf{1}_{A_k}) = \tilde{f}^L(x)$ is the Lovasz extension of a non-modular function, which implies that f^M is not linear on its first component, a contradiction. $\square$

Proposition 5.3. *If f is of the form $f(A_1, \dots, A_k) = \prod_{i=1}^k f_i(A_i)$ then $f^M(x^1, \dots, x^k) = \prod_{i=1}^k f_i^L(x^i)$.*

6 Combinatorial and continuous optimization links via piecewise multilinear extensions

6.1 Definitions

Given constraints sets $\mathcal{A} \subset (\mathcal{P}(V) \setminus \{\emptyset\})^k$ (or $\mathcal{A} \subset (\mathcal{P}_2(V) \setminus \{\emptyset, \emptyset\})^k$) and $\mathcal{D} \subset (\mathbb{R}^n_+)^k$, the feasible sets $\mathcal{A}(\mathcal{D})$ and $\mathcal{D}(\mathcal{A})$ are defined in two different ways.

For the piecewise multilinear extension, $\mathcal{D}(\mathcal{A}) = \{(x^1, \dots, x^k) \in (\mathbb{R}^n_+)^k | (V^{t_1}(x^1), \dots, V^{t_k}(x^k)) \in \mathcal{A}$, for all $t_i < \max x^i\}$ and $\mathcal{A}(\mathcal{D}) = \{(V^{t_1}(x^1), \dots, V^{t_k}(x^k)) \in (\mathcal{P}(V) \setminus \{\emptyset\})^k$, for all $(x^1, \dots, x^k) \in (\mathbb{R}^n_+)^k, t_1, \dots, t_k \in \mathbb{R}\}$.

For the multiple integral extension, $\mathcal{D}(\mathcal{A}) = \{(x^1, \dots, x^k) \in (\mathbb{R}^n_+)^k | (V^{t_1}_+(x^1), V^{t_1}_-(x^1), \dots, V^{t_k}_+(x^k), V^{t_k}_-(x^k)) \in \mathcal{A}$, for all $t_i < \|x^i\|_\infty\}$ and $\mathcal{A}(\mathcal{D}) = \{(V^{t_1}_+(x^1), V^{t_1}_-(x^1), \dots, V^{t_k}_+(x^k), V^{t_k}_-(x^k)) \in (\mathcal{P}_2(V) \setminus \{\emptyset, \emptyset\})^k$, for all $(x^1, \dots, x^k) \in (\mathbb{R}^n_+)^k, t_1, \dots, t_k \ge 0\}$.

$(\mathcal{A}, \mathcal{D})$ is called a perfect domain pair if $\mathcal{A} = \mathcal{A}(\mathcal{D})$ and $\mathcal{D} = \mathcal{D}(\mathcal{A})$.

6.2 Key results

Theorem 6.1. *Given $f : \mathcal{A} \to \mathbb{R}$ and $g : \mathcal{A} \to [0, +\infty)$, we have*

$$\sup_{A \in \mathcal{A} \cap supp(g)} \frac{f(A)}{g(A)} \le \sup_{x \in \mathcal{D} \cap supp(g^M)} \frac{f^M(x)}{g^M(x)} \le \sup_{A \in \tilde{\mathcal{A}}} \frac{f(A)}{g(A)} \tag{37}$$

whenever $\{\mathbf{1}_A : A \in \mathcal{A}\} \subset \mathcal{D}$ and $\mathcal{A}(D) \subset \tilde{\mathcal{A}}$. The above inequality still holds replacing sup *and $\le$ by* inf *and $\ge$.*

If we further assume that $(\mathcal{A}, \mathcal{D})$ is a perfect domain pair, and $supp(f) \subset supp(g)$, then

$$\max_{A \in \mathcal{A} \cap supp(g)} \frac{f(A)}{g(A)} = \max_{x \in \mathcal{D} \cap supp(g^M)} \frac{f^M(x)}{g^M(x)} \tag{38}$$

and the same holds replacing max *with* min*.*

Proof. Since $g^M(\mathbf{1}_A) = g(A)$, we have $\mathbf{1}_A \in \mathcal{D} \cap \operatorname{supp}(g^M)$ whenever $A \in \mathcal{A} \cap \operatorname{supp}(g)$. Thus, the first inequality is proved. Note that for any $x \in \mathcal{D} \cap \operatorname{supp}(g^M)$, $g^M > 0$, and every multiple upper level set

$(V^{t_1}(x^1), \ldots, V^{t_k}(x^k))$ belongs to $\mathcal{A}(\mathcal{D}) \subset \tilde{\mathcal{A}}$. Hence, an approach similar to the proof in TO CITE can derive the second inequality. In fact, we also have

$$\sup_{A \in \mathcal{A}} \frac{f(A)}{g(A)} \leq \sup_{x \in \mathcal{D}} \frac{f^M(x)}{g^M(x)} \leq \sup_{A \in \tilde{\mathcal{A}}} \frac{f(A)}{g(A)} \tag{39}$$

For a perfect domain pair $(\mathcal{A}, \mathcal{D})$, taking $\tilde{\mathcal{A}} = \mathcal{A}(\mathcal{D}) = \mathcal{A}$, we immediately get

$$\sup_{A \in \mathcal{A}} \frac{f(A)}{g(A)} = \sup_{x \in \mathcal{D}} \frac{f^M(A)}{g^M(A)} \tag{40}$$

and the equivalent equality replacing sup by inf.

The additional condition supp($f \subset$ supp(g) implies that $f(A) = 0$ whenever $g(A) = 0$. Thus, by the definition of piecewise multilinear extensions, for any $x \in \mathcal{D} \cap \mathrm{supp}(g^M)$,

$$\frac{f^M(x)}{g^M(x)} \in \mathrm{conv}\left\{\frac{f(A)}{g(A)} | A \text{ is a multiple upper level set of } x, \text{ and } g(A) > 0\right\} \tag{41}$$

The proof is completed. □

Theorem 6.2. *Let $H : \mathbb{R}_+^* \to \mathbb{R} \cup \{+\infty\}$ be a zero-homogeneous and quasi-concave function. For any function $f_1, \ldots, f_n : \mathcal{A} \to \mathbb{R}_+$, we have*

$$\min_{A \in \mathcal{A}} H(f_1(A), \ldots, f_n(A)) = \inf_{x \in \mathcal{D}} H(f_1^M(x), \ldots, f_n^M(x)) \tag{42}$$

where $(\mathcal{A}, \mathcal{D})$ forms a perfect domain pair w.r.t. the piecewise multilinear extension.

In addition, if $H : \mathbb{R}_+^* \to \mathbb{R} \cup \{-\infty\}$ is a zero-homogeneous and quasi-convex function, for any function $f_1, \ldots, f_n : \mathcal{A} \to \mathbb{R}_+$, we have

$$\max_{A \in \mathcal{A}} H(f_1(A), \ldots, f_n(A)) = \sup_{x \in \mathcal{D}} H(f_1^M(x), \ldots, f_n^M(x)) \tag{43}$$

where $(\mathcal{A}, \mathcal{D})$ forms a perfect domain pair w.r.t. the piecewise multilinear extension.

Corollary 6.3. *For a log-concave polynomial P of degree d in n variables, and for $f_1, \ldots, f_, : \mathcal{A} \to [0, +\infty)$, we have*

$$\min_{A \in \mathcal{A}} \frac{P(f_1(A), \ldots, f_n(A)}{(f_1(A) + \cdots + f_n(A))^d} = \inf_{x \in \mathcal{D}} \frac{P(f_1^M(A), \ldots, f_n^M(A)}{(f_1^M(A) + \cdots + f_n^M(A))^d} \tag{44}$$

where $(\mathcal{A}, \mathcal{D})$.

7 Applications

7.1 Lemmas

Lemma 7.1. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth functions such that g is positive on $(\mathbb{R}_+^*)^n \setminus \{0\}$. For a maximizer (resp. minimizer) x of $\frac{f}{g}_{|(\mathbb{R}_+^*)^n \setminus \{0\}}$ (if it exists), let v be such that $x + v \in (\mathbb{R}_+^*)^n \setminus \{0\}$, supp($v$) $\subset$ supp(x), $t \in \mathbb{R} \to g(x + tv)$ is constant, and $\frac{\partial^2 f}{\partial y_i \partial y_j}(y) = 0$ for all $i, j \in$ supp(v), for all $y \in \mathbb{R}^n$. If we further assume that f is real analytic, then $x + v$ is also a maximizer of $\frac{f}{g}_{|(\mathbb{R}_+^*)^n \setminus \{0\}}$.*

Proof. Let x be a critical point of $\frac{f}{g}_{|(\mathbb{R}_+^*)^n\setminus\{0\}}$ and let $v \in \mathbb{R}^n$ be such that $\langle \nabla g(x), v\rangle = 0$ and $\text{supp}(v) \subset \text{supp}(x)$, then $\langle \nabla f(x, v\rangle = 0$. Indeed, for any $i \in \text{supp}(x)$, we have $\frac{\partial}{\partial x_i}\frac{f}{g}(x) = 0$ because x is a critical point, thus $\frac{\partial f}{\partial x_i}(x) = \frac{f}{g}(x)\frac{\partial g}{\partial x_i}(x)$ for any $i \in \text{supp}(x)$. Because $v_i = 0$ whenever $i \notin \text{supp}(x)$, we have

$$\langle \nabla f(x), v\rangle = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(x) = \sum_{i\in\text{supp}(x)} v_i \frac{\partial f}{\partial x_i}(x) \tag{45}$$

$$= \sum_{i\in\text{supp}(x)} v_i \frac{f}{g}(x)\frac{\partial g}{\partial x_i}(x) = \frac{f}{g}(x)\langle \nabla g(x), v\rangle = 0 \tag{46}$$

Now we prove the lemma. It follows from $g(x+tv) = g(x)$ for all $t \in \mathbb{R}$ that $\langle \nabla g(x), v\rangle = 0$, and thus by the above claim, we have $\langle \nabla f(x), v\rangle = 0$. Since f is a real analytic function, $t \to f(x+tv)$ must be real analytic. Note that $\frac{df(x+tv)}{dt}(t=0) = \langle \nabla f(x), v\rangle = 0$ and for any $k \geq 2$,

$$\frac{d^k f(x+tv)}{dt^k}(t=0) = \sum_{i_1,\ldots,i_k=1}^{n} v_{i_1}\ldots v_{i_k} \frac{\partial^k f}{\partial x_{i_1}\ldots\partial x_{i_k}}(x) \tag{47}$$

$$= \sum_{i_1,\ldots,i_k\in\text{supp}(v)} v_{i_1}\ldots v_{i_k} \frac{\partial^k f}{\partial x_{i_1}\ldots\partial x_{i_k}}(x) = 0 \tag{48}$$

where the last equality is due to the condition that $\frac{\partial f}{\partial x_i \partial x_j}(x) = 0$ for all $i, j \in \text{supp}(v)$.

Therefore, the real analytic function $t \to f(x+tv)$ is constant. This implies that $f(x+v) = f(x)$, and hence $\frac{f(x+v}{g(x+v)} = \frac{f(x)}{g(x)}$ thus $x+v$ is also a maximizer of $\frac{f}{g}_{|(\mathbb{R}_+^*)^n\setminus\{0\}}$.

The case of minimizer is similar. ☐

7.2 Application to the MCP on 2-graphs

7.2.1 Definitions

Let $G = (V, E)$ be an undirected k-graph with no edge between a node and itself. Denote A its adjacency matrix.

Denote $E(S, T)$ the number of edges $e = (i, j)$ of the graph where $i \in S$ and $j \in T$.

7.3 Result

Let $G = (V, E)$ be an undirected graph with no edge between a node and itself. Denote A its adjacency matrix.

The following problem

$$\max_{S\in\mathcal{P}(V)\setminus\{\emptyset\}} \frac{|E(S,S)|}{|S|^2} \tag{49}$$

has solutions that are maximum cliques. Indeed, for any set S,

$$\frac{|E(S,S)|}{|S|^2} \leq \frac{|S|(|S|-1)}{|S|^2} \tag{50}$$

with equality only if S is a clique, and the above is maximized with S a maximum clique.

Define $f(S,T) = |E(S,T)|$ and $g(S,T) = |S||T|$. Then the piecewise multilinear extensions of f and g defined on $\mathbb{R}_+^n$ satisfy $f^M(x,y) = x^T A y$ and $g^M(x,y) = \|x\|_1\|y\|_1$.

Then

$$\max_{S\in\mathcal{P}(V)\setminus\{\emptyset\}} \frac{|E(S,S)|}{|S|^2} = \max_{x\neq 0} \frac{2\sum_{(i,j)\in E} x_i x_j}{\|x\|_1^2} \tag{51}$$

7.3.1 Proof

$g(S,T) = \tilde{g}(S)\tilde{g}(T)$ and $\tilde{g}$ is modular so $\tilde{g}^M(x) = \langle u, x\rangle^2$ where $u = (\tilde{g}(\{1\}), \dots, \tilde{g}(\{n\})) = \mathbf{1}$.

f is modular on each component so its extension f^M must be multilinear and thus $f^M(x,y) = x^T M y$ where $M = (f(\{i\},\{j\}))_{n\times n} = A$.

A similar result to 6.1 holds for multilinear extensions evaluated diagonally, which gives us the inequality

$$\max_{S\in\mathcal{P}(V)\setminus\{\emptyset\}} \frac{|E(S,S)|}{|S|^2} \le \max_{x\in\mathbb{R}_+^n\setminus\{0\}} \frac{2\sum_{(i,j)\in E} x_i x_j}{\|x\|_1^2} = \max_{x\neq 0} \frac{2\sum_{(i,j)\in E} x_i x_j}{\|x\|_1^2} \tag{52}$$

For the equality, we have to show that there exists a set S such that $\mathbf{1}_S$ is a maximizer of the right side of (52). Let x be a maximizer of the right side in $\mathbb{R}_+^n \setminus \{0\}$. If $f(\{i\},\{j\}) = 0$ and $x_i x_j > 0$ for some $i \neq j$, then taking v defined as $v_i = -x_i$, $v_j = x_i$ and $v_l = 0$ for $l \neq i, j$, we have that $x + v$ is also a maximizer by 7.1, and repeating the process, we finally obtain a subset $S \subset V$ satisfying $\operatorname{supp}(x) = S$ and $f(\{i\},\{j\}) > 0$ for all $i \neq j$ in S i.e. S is a clique. Then

$$\frac{f_\Delta^M(x)}{g_\Delta^M(x)} = \frac{x^T A x}{(\mathbf{1}^T x)^2} = 1 - \frac{\sum_{i\in S} x_i^2}{(\sum_{i\in S} x_i)^2} \le 1 - \frac{1}{|S|} \tag{53}$$

and the equality holds if and only if $x_i = \text{Const}$ for $i \in S$. In consequence, $\mathbf{1}_S$ is a maximizer of the right side of (52). The proof is completed.

7.3.2 Remarks

There is no guarantee that the maximizer x^* of the right side of (56) has a meaning in terms of set (i.e. there is no guarantee that $x^* = \mathbf{1}_S$ for a particular set S). Such a solution is sometimes called a "spurious" solution. We only have that a set S maximizing the left side implies that $\mathbf{1}_S$ maximizes the right side.

The authors of [1] added a regularizer $\alpha\|x\|_2^2$ with $\alpha \in (0,2)$ which in [2] was further relaxed to a function Φ satisfying $\nabla^2\Phi \geq 0$, $\nabla^2\phi < 2$ and Φ permutation invariant. The first inequality can be replaced by a strict one to ensure that not only the global maxima but also the local ones are in relation with the characteristic vectors of sets.

7.4 Application to the MCP on k-graphs induced by k-cliques of a 2-graph

7.4.1 Definitions

Let $G^* = (V, E^*)$ be an undirected 2-graph, and let $G = (V, E)$ be the graph induced by its k-cliques, i.e. the graph mapping every k-cliques of G^* to an edge in G. Denote $\mathcal{A}$ its adjacency tensor.

Denote $E(S_1, \dots, S_k)$ the number of edges $e = (i_1, \dots, i_k)$ of the graph where $i_j \in S_j$ for all j.

7.4.2 Result

The following problem

$$\max_{S\in\mathcal{P}(V)\setminus\{\emptyset\}} \frac{|E(S,\dots,S)|}{|S|^k} \tag{54}$$

has solutions that are maximum cliques. Indeed, for any set S,

$$\frac{|E(S,\dots,S)|}{|S|^k} \le \frac{\prod_{i=0}^{k-1}(|S| - k)}{|S|^k} \tag{55}$$

with equality only if S is a clique, and the above is maximized with S a maximum clique.

Define $f(S_1, \dots, S_k) = |E(S_1, \dots, S_k)|$ and $g(S_1, \dots, S_k) = |S_1| \dots |S_k|$. Then the piecewise multilinear extensions of f and g defined on $\mathbb{R}_+^n$ satisfy $f^M(x^1, \dots, x^k) = \mathcal{A}x^1 \dots x^k$ and $g^M(x^1, \dots, x^k) = \|x^1\|_1 \dots \|x^k\|_1$.

Then

$$\max_{S\in\mathcal{P}(V)\setminus\{\emptyset\}} \frac{|E(S,\dots,S)|}{|S|^k} = \max_{x\neq 0} \frac{k!\sum_{e\in E}\prod_{i\in e} x_i}{\|x\|_1^k} \tag{56}$$

7.4.3 Proof

$g(S_1,\dots,S_k) = |S_1|\dots|S_k|$ and $\tilde{g}$ is modular so $\tilde{g}^M(x) = \langle u, x\rangle^k$ where $u = (\tilde{g}(\{1\}),\dots,\tilde{g}(\{n\})) = \mathbf{1}$.

f is modular on each component so its extension f^M must be multilinear and thus $f^M(x^1,\dots,x^k) = \mathcal{M}x^1\dots x^k$ where $\mathcal{M} = (f(\{i_1\},\dots,\{i_k\}))_{n\times\dots\times n} = \mathcal{A}$.

A similar result to 6.1 holds for multilinear extensions evaluated diagonally, which gives us the inequality

$$\max_{S\in\mathcal{P}(V)\setminus\{\emptyset\}} \frac{|E(S,\dots,S)|}{|S|^k} \leq \max_{x\in\mathbb{R}^n_+\setminus\{0\}} \frac{k!\sum_{e\in E}\prod_{i\in e} x_i}{\|x\|_1^k} = \max_{x\neq 0} \frac{k!\sum_{e\in E}\prod_{i\in e} x_i}{\|x\|_1^k} \tag{57}$$

For the equality, we have to show that there exists a set S such that $\mathbf{1}_S$ is a maximizer of the right side of (65). Let x be a maximizer of the right side in $\mathbb{R}^n_+\setminus\{0\}$. If there exist $i,j\in\mathrm{supp}(x)$ such that for all edges $e=(e_1,\dots,e_k)\in E$ such that $i,j\in e$ and $x_{e_l}>0$ for all l, we have $f(\{e_1\},\dots,\{e_k\}) = 0$ (or in other words if we have two indices such that no k-cliques in the original graph contains them both), then taking v defined as $v_i = -x_i$, $v_j = x_i$ and $v_l = 0$ for $l\neq i,j$ where $i,j\in e$, we have that $x+v$ is also a maximizer by 7.1, and repeating the process, we finally obtain a subset $S\subset V$ satisfying $\mathrm{supp}(x) = S$ and $f(\{i_1\},\dots\{i_k\})>0$ for all $i_1,\dots,i_k$ in S (remind that G is induced by k-cliques of G^*), i.e. S is a clique. Then by McLaurin inequality,

$$\frac{f^M_\Delta(x)}{g^M_\Delta(x)} = \frac{k!\sum_{e\in E}\prod_{i\in e} x_i}{(\mathbf{1}^T x)^k} = \frac{k!\sum_{e\in E(S)}\prod_{i\in e} x_i}{\|x\|_1^k} \tag{58}$$

$$\leq \frac{\binom{|S|}{k}\left(\frac{\|x\|_1}{|S|}\right)^k}{\|x\|_1^k} \tag{59}$$

and the equality holds if and only if $x_i = \mathrm{Const}$ for $i\in S$. In consequence, $\mathbf{1}_S$ is a maximizer of the right side of (65). The proof is completed.

7.4.4 Remarks

This result only holds for k-graphs induced by k-cliques of a 2-graphs. The impossibility to fully generalize this technique to k-graphs is discussed in the introduction of [3].

7.5 Application to the MCP on k-graphs, based on the problem derived in [3]

7.5.1 Definitions

Let $G=(V,E)$ be an undirected k-graph. Denote $\mathcal{A}$ the adjacency tensor of the complement of G.

Denote $\overline{E}(S_1,\dots,S_k)$ the number of edges $e=(i_1,\dots,i_k)$ of the complement of the graph where $i_j\in S_j$ for all j.

7.5.2 Result

The following problem

$$\min_{S\in\mathcal{P}(V)\setminus\{\emptyset\}} \frac{|\overline{E}(S,\dots,S)|+\tau|S|}{|S|^k} \tag{60}$$

has solutions that are maximum cliques. Indeed, denote C the maximum clique in the graph. For any set S that is not a clique,

$$|C| \geq |S| - k|\overline{E}(S,\dots,S)| \iff |\overline{E}(S,\dots,S)| \geq \frac{|S|-|C|}{k} \tag{61}$$

because by removing all the vertices of the missing edges (and there are at most $k|\overline{E}(S,\dots,S)|$), we obtain a clique. Then,

$$\frac{|\overline{E}(S,\dots,S)|+\tau|S|}{|S|^k} \geq \frac{|S|-|C|}{k|S|^k} + \frac{\tau}{|S|^{k-1}} \tag{62}$$

$$= \frac{1}{k|S|^{k-1}} + \frac{\tau}{|S|^{k-1}} - \frac{|C|}{k|S|^k} \tag{63}$$

Define $f(S_1,\dots,S_k) = |E(S_1,\dots,S_k)| + \tau|S_1 \cap \dots \cap S_k|$ and $g(S_1,\dots,S_k) = |S_1|\dots|S_k|$. Then the piecewise multilinear extensions of f and g defined on $\mathbb{R}_+^n$ satisfy $f^M(x^1,\dots,x^k) = \mathcal{A}(\overline{G})x^1\dots x^k + \tau\|x\|_k^k$ and $g^M(x^1,\dots,x^k) = \|x^1\|_1\dots\|x^k\|_1$.

Then

$$\min_{S\in\mathcal{P}(V)\setminus\{\emptyset\}} \frac{|\overline{E}(S,\dots,S)|+\tau|S|}{|S|^k} = \min_{x\neq 0} \frac{k!\sum_{e\in\overline{E}}\prod_{i\in e}x_i + \tau\|x\|_k^k}{\|x\|_1^k} \tag{64}$$

7.5.3 Proof

$g(S_1,\dots,S_k) = |S_1|\dots|S_k|$ and $\tilde{g}$ is modular so $\tilde{g}^M(x) = \langle u,x\rangle^k$ where $u = (\tilde{g}(\{1\}),\dots,\tilde{g}(\{n\})) = \mathbf{1}$.

f is modular on each component so its extension f^M must be multilinear and thus $f^M(x^1,\dots,x^k) = \mathcal{M}x^1\dots x^k$ where $\mathcal{M} = (f(\{i_1\},\dots,\{i_k\}))_{n\times\dots\times n} = \mathcal{A}(\overline{G}) + \tau\mathcal{I}$.

A similar result to 6.1 holds for multilinear extensions evaluated diagonally, which gives us the inequality

$$\min_{S\in\mathcal{P}(V)\setminus\{\emptyset\}} \frac{|E(S,\dots,S)|}{|S|^k} \geq \min_{x\in\mathbb{R}_+^n\setminus\{0\}} \frac{k!\sum_{e\in E}\prod_{i\in e}x_i + \tau\|x\|_k^k}{\|x\|_1^k} \tag{65}$$

For the equality, we have to show that there exists a set S such that $\mathbf{1}_S$ is a minimizer of the right side of (65). Let x be a minimizer of the right side in $\mathbb{R}_+^n \setminus \{0\}$.

7.5.4 Remarks

7.6 Extension for s-defective cliques

7.6.1 Proposal

$$h(S) = (s+2)^{|S|}\left(s+1-(|S|(|S|-1)-|E(S,S)|)\right) \tag{66}$$

This can also be rewritten in fractional form :

$$h(S) = \frac{(s+1-(|S|(|S|-1)-|E(S,S)|))}{\left(\frac{1}{s+2}\right)^{|S|}} \tag{67}$$

7.6.2 Motivation

The second factor can be rephrased as "$s+1$ - missing edges" and is there to ensure that there is indeed less than s missing edges (or else the factor would be equal to 0 or negative).

The first factor is to ensure that at a specific number of missing edges, the function is growing with the cardinal of the set. Its form is there to ensure that the tradeoff between the penalty of having additional missing edges and the increase in the first factor is always in favor of the latter, i.e. the biggest decrease the second factor can have is $s+1 \to 1$ when going from a clique to a set with s missing edges i.e. dividing by $s+1$, so to ensure that an increase in the cardinal is profitable (even if we add s missing edges), we need the first factor to grow at least $s+2 > s+1$.

7.6.3 Continuous extension of the problem

Denote

$$f(S) = s + 1 - (|S|(|S| - 1) - |E(S,S)|) = s + 1 - |S|^2 - |S| + |E(S,S)| \tag{68}$$

and

$$g(S) = \left(\frac{1}{s+2}\right)^{|S|} \tag{69}$$

f is a sum of simple functions so its piecewise multilinear extension is

$$f^M(x) = \tag{70}$$

g can be extended to $[0, +\infty)^n$ by

$$\tilde{g}(x) = \left(\frac{1}{s+2}\right)^{\mathbf{1}^T x} \tag{71}$$

Is it useful ?

References

[1] Immanuel M. Bomze, Francesco Rinaldi, and Damiano Zeffiro. *Fast cluster detection in networks by first-order optimization*. 2021. arXiv: 2103.15907 [math.OC].

[2] James T. Hungerford and Francesco Rinaldi. *A General Regularized Continuous Formulation for the Maximum Clique Problem*. 2017. arXiv: 1709.02486 [math.OC].

[3] Marcello Pelillo Samuel Rota Bulò. "A generalization of the Motzkin–Straus theorem to hypergraphs". In: *Optimization Letters* 3 (2009), pp. 287–295. DOI: 10.1007/s11590-008-0108-3.