Exploring Extensions of Maximum Clique Problems

from 2-Graphs to k-Uniform Hypergraphs

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Introduction

- My end-of-study internship took place from the 1st pf May to the 20th of October at the University of Padua, department of applied mathematics.
- I was guided by Mr. Francesco Rinaldi, with support from Mr. Damiano Zeffiro (a former Ph.D. student).
- The objective was to generalize several techniques for solving clustering problems in hypergraphs, focusing on the Maximum Clique Problem (MCP) and related relaxations.

Definitions

- We work on graphs represented as $G = (V, E)$ where V is a list of vertices and E is a list of edges.
- An **hypergraph** generalizes traditional graphs by allowing edges to connect more than two vertices. Here we are interested in k-uniform hypergraphs where all of the hyperedges connect the same number of vertices.
- In a graph $G = (V, E)$, a clique is a subset $S \subseteq V$ such that every two distinct vertices in S are adjacent, i.e., $\{u, v\} \in E$ for all $u, v \in S$.
- The MCP is the search of the largest clique in an arbitrary graph. It is an NP-hard problem.

First Part: A Continuous Formulation for the Maximum s-Defective Clique Problem

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- The first part of my internship focused on developing a continuous formulation for the Maximum s-Defective Clique Problem (MsdCP) in hypergraphs.
- This formulation was inspired by existing formulations for the simpler case $k = 2$, and its equivalence with the combinatorial problem was proven.
- We then generalized results about variants of the Frank-Wolfe (FW) algorithm applied to this problem.

Motivation

- Since the strict requirement that every two elements have a direct relation is often not satisfied in practice (e.g., due to experimental errors in protein research), relaxations of the clique model have been proposed.
- Here we are interested in the concept of s-defective cliques, allowing up to s missing links.

A bit of history

Turán's theorem states:

$$\max_{x \in \Delta} x^T A x = 1 - \frac{1}{\omega(G)} \tag{5}$$

where $\Delta = \{x \in \mathbb{R}^n \mid x \geq 0 \text{ and } 1^T x = 1\}$, A is the adjacency matrix, and $\omega(G)$ is the size of the maximum clique.

To address spurious solutions (continuous solutions that has no interpretation in terms of set), regularization was introduced:

$$\max_{x \in \Delta} x^T A x + \alpha \|x\|_2^2 \tag{6}$$

with $\alpha \in (0, 1)$. Global and local maximizers are in one-one correspondence with characteristic vectors of maximum and maximal cliques.

A bit of history

The continuous unregularized formulation for the MCP was generalized to the MsdCP:

$$\max_{(x,y)\in\Delta\times D_s(G)} x^T(A+A(y))x \tag{7}$$

where $\Delta = \{x \in \mathbb{R}^n \mid x \geq 0 \text{ and } 1^T x = 1\}$, and $D_s(G) = \{y \in \{0,1\}^{\overline{E}|1^T y \leq s}\}$. x corresponds to the node chosen in the s-defective clique, and y corresponds to the missing edges in the s-defective clique.

The formulation can be seen as two nested problems : first, finding the best s edges to add to the graph in order to have the largest clique in the augmented graph, and then finding this clique.

A bit of history

Regularization was added to suppress spurious maxima, ensuring full equivalence with the discrete problem:

$$\max_{(x,y)\in\Delta\times D_s(G)} x^T(A + A(y))x + \alpha\|x\|_2^2 + \beta\|y\|_2^2$$

where $0 < \alpha < 2$ and $\beta > 0$.

A bit of history

The regularized maximum clique formulation was generalized to k-uniform hypergraphs:

$$\min_{x \in \Delta} \underbrace{\sum_{e \in \overline{E}} \prod_{i \in e} x_i}_{L_{\overline{G}}(x)} + \tau \|x\|_k^k$$

where $0 < \tau \leq \frac{1}{k(k-1)}$ (strict inequality when $k = 2$). Note that this formulation searches for a minimum on the complement of the graph.

Formulation for the MsdCP on hypergraphs

We came up with the following formulation :

$$\min_{\text{s.t. } (x,y)\in\mathcal{P}_s} L_{\overline{G}}(x) - \underbrace{L_{G(y)}(x)}_{\sum_{e\in\overline{E}} y_e \prod_{i\in e} x_i} + \alpha\|x\|_k^k - \beta\|y\|_2^2$$

where $0 < \alpha \leq \frac{1}{k(k-1)}$ (strict inequality for $k = 2$) and $\beta > 0$.

Remark

To explain the origin of the formulation, combining the ideas of the MsdCP on 2-graphs and the MCP on k-graphs, we come to this starting point :

$$\min_{s.t.\ (x,y)\in\mathcal{P}_s} L_{\overline{G\cup G(y)}}(x) + \alpha\|x\|_k^k \pm \beta\|y\|_l^l$$

Then it seems natural to write $L_{\overline{G\cup G(y)}}(x) = L_{\overline{G}\setminus G(y)}(x) = L_{\overline{G}}(x) - L_{G(y)}(x)$.

Formal Equivalence between the Discrete and the Continuous Problems

Denoting $h(x,y) := L_{\overline{G}}(x) - L_{G(y)}(x) + \alpha\|x\|_k^k - \beta\|y\|_2^2$.

Proposition

The following are equivalent :
(i) $p = (x,y) \in \mathcal{P}_s$ is a local minimizer for $h(x,y)$
(ii) p is a strict local minimizer
(iii) $p = (x^{(C)}, y^{(p)})$, where $s \geq l = \mathbf{1}^T y^{(p)} \in \mathbb{N}$, with C an l-defective clique in G which is also a maximal clique in $G \cup G(y^{(p)})$, and $y^{(p)} \in \{0,1\}^{\overline{E}}$ such that $y_e^{(p)} = 1$ for every $e \in \binom{C}{k} \cap \overline{E}$ and with supp$(y^{(p)})$ of maximum cardinality under these constraints.

In either of these equivalent cases, we have $h(p) = \alpha|C|^{1-k} - \beta l$.

Frank-Wolfe Algorithm Overview

- The Frank-Wolfe (FW) algorithm is an optimization scheme suitable for problems with a search space represented as a polytope, as it is in our case.
- At each step, the algorithm moves the argument towards a vertex of the polytope in order to maximize a first-order local approximation of the objective function.
- The convexity of the search space ensures that the updated position x_{k+1} is feasible without the need for a costly projection step into the feasible domain.

FW algorithm

```
Initialize x^0 within the feasible polytope P = conv(Q) with |Q| < ∞.
for t = 1, 2, ... do
    Compute the gradient: ∇f(x^t).
    Find the vertex v ∈ Q that maximizes ∇f(x^t)^T v.
    Determine the step size: γ_t.
    Update the solution: x^{t+1} = x^t + γ_t(v − x^t).
end for
```

Initialize x^0 within the feasible polytope $\mathcal{P} = \text{conv}(\mathcal{Q})$ with $|\mathcal{Q}| < \infty$.

for $t = 1, 2, \ldots$ **do**

  Compute the gradient: $\nabla f(x^t)$.

  Find the vertex $v \in \mathcal{Q}$ that maximizes $\nabla f(x^t)^T v$.

  Determine the step size: γ_t.

  Update the solution: $x^{t+1} = x^t + \gamma_t(v - x^t)$.

end for

FDFW

1: **Initialize:** $w_0 \in \mathcal{Q}$, $k = 0$
2: **if** w_k is stationary **then**
3:     **STOP**
4: **end if**
5: Find $s_k \in \arg\max_{y \in \mathcal{Q}} \nabla f(w_k)^T y$ and $d_k^{\mathcal{FW}} = s_k - w_k$.
6: Find $v_k \in \arg\min_{y \in \mathcal{F}(w_k)} \nabla f(w_k)^T y$ and $d_k^{\mathcal{FD}} = w_k - v_k$.
7: **if** $\nabla f(w_k)^T d_k^{\mathcal{FW}} \geq \nabla f(w_k)^T d_k^{\mathcal{FD}}$ **then**
8:     $d_k = d_k^{\mathcal{FW}}$
9: **else**
10:     $d_k = d_k^{\mathcal{FD}}$
11: **end if**
12: Choose the step size $\alpha_k \in (0, \alpha_k^{\max}]$ using a suitable criterion.
13: Update $w_{k+1} = w_k + \alpha_k d_k$.
14: Set $k = k + 1$ and go to step 2.

Motivation for the FWdc

- The convergence of the previous optimization scheme applied to our objective can be slow and inefficient, due to the structural differences between x and y.
- Since the two variables are tied, it is difficult to efficiently adjust the regularization parameters. The algorithm may either ignore x if the penalty coefficient on the y variable is large or ignore y if this coefficient is small.
- This limitation motivated the introduction of the following FW variant, named FWdc.

FWdc

1: **Initialize:** $z_0 = (x_0, y_0) \in \Delta \times \mathcal{D}_s(G)$, $k = 0$
2: **if** z_k is stationary **then**
3:     **STOP**
4: **end if**
5: Compute x_{k+1} by applying one step of the previous algorithm with $w_0 = x_k$ and $f(w) = \tilde{h}(w, y_k)$.
6: Find $y_{k+1} \in \arg\max_{y \in \mathcal{D}_s(G)} \nabla_y h_G(x_{k+1}, y_k)^T y$.
7: Set $k = k + 1$ and go to step 2.

Rationale

- The FWdc algorithm exploits the cross-product structure of the search space by splitting the update rules for x and y.
- This allows for a more efficient adjustment of the regularization parameters since the updates for x and y are decoupled.
- As the convergence for y is extremely slow (due to its high dimensionality), its update step is replaced by a full FW step.

FDFW Local Identification and Convergence

Theorem

Let $p := (x^{(C)}, y^{(p)})$ *be a strict minimizer, let* z_k *be a sequence generated by the FDFW. Then under certain conditions there exists a neighborhood* $U(p)$ *of* p *such that if* $K := \min\{k \in \mathbb{N} | z_k \in U(p)\}$ *we have the following properties :*
(a) if $\tilde{h}(z_k)$ *is monotonically increasing, then* $supp(z_k) = C$ *and* $y_k = y^{(p)}$ *for every* $k \geq K + dim\ \mathcal{F}(w_k)$
(b) under additional conditions, then $z_k \to p$.

FDFW Global Convergence

Corollary

Let $\{z_k\}$ *be a sequence generated by the FDFW, and assume that there are no saddle points in the limit set of* $\{z_k\}$*. Then under certain conditions on the step size we have* $z_k \to p := (x^{(C)}, y^{(p)})$ *with* p *a strict minimizer such that* $supp(x_k) \subset C$ *and* $y_k = y^{(p)}$ *for* k *large enough.*

Limited Changes in FWdc Variant

Proposition

In the FWdc variant, if $\tilde{h}(z_k)$ is increasing at each separate update of x_k and y_k, then $\{y_k\}$ can change at most $l + \frac{|\overline{E}|+\alpha\left(1-|C^*|^{1-k}\right)}{\beta}$ times, with C^* a maximum l-defective clique if we consider the MsdCP.

Explicit Bound in FWdc for Local Identification

Proposition

Let C be a clique in $G \cup G(\overline{y})$ and $\delta_{\max}$ be the maximum eigenvalue of the adjacency tensor $\mathcal{A}(\overline{G \cup G(\overline{y})})$, and define :

$$m_\alpha(C, G \cup G(\overline{y})) := \min_{v \in V \setminus C} E^C(v) - \alpha k$$

Let K be a fixed index in $\mathbb{N}$ and I^c be the components of supp(x_K) with index not in C, and $L := \frac{1}{(k-2)!}\delta_{\max} + k(k-1)\alpha$. Assume that y is stationary, and that

$$\|x_K - x^{(C)}\|_1 \leq \frac{m_\alpha(C, G \cup G(\overline{y}))}{m_\alpha(C, G \cup G(\overline{y})) + 2|C|^{k-1}\left(\frac{1}{(k-2)!}\delta_{\max} + k(k-1)\alpha\right)}$$

Then supp$(x_{K+|I^c|}) = C$.

Failure

Theorem

Let $\{z_k\}$ be a sequence generated by the FDFW, with regularization coefficient $\alpha = 1$. Under certain conditions on the step sizes, $\{z_k\}$ converges to a stationary point and identifies its support infinite time.

Second Part : Study of the Regularizer in the Continuous MCP Formulation

Second Part : Study of Regularizer in the Continuous MCP Formulation

The second part of the internship focused on generalizing the study of the regularizer function. For $k = 2$, the MCP formulation was relaxed to :

$$\max_{\text{s.t. } x \in \Delta} x^T A x + \phi(x)$$

where ϕ satisfies the following conditions :

- $\nabla^2 \phi(x) > 0$ i.e. ϕ is strictly convex
- $\|\nabla^2 \phi(x)\|_2 < 2$
- $\phi(x)$ is invariant by permutation of x

Examples of Regularizers

Examples of regularizers satisfying these conditions:

- $\phi(x) = \frac{1}{2}\|x\|_2^2$
- $\alpha_1\|x + \epsilon 1\|_p^p$ with $\epsilon \geq 0$, $p \geq 2$ and $0 \leq \alpha_1 \leq \frac{2}{p(p-1)(1+\epsilon)^{p-2}}$
- $\phi(x) = \alpha_2 \sum_{i=1}^n (e^{-\beta x_i} - 1)$, with $\beta > 0$ and $0 \leq \alpha_2 \leq \frac{2}{\beta^2}$ (approximation of $-\alpha_2\|x\|_0$)

Generalized Relaxed Formulation

We generalized the relaxed formulation to :

$$\min_{\text{s.t. } x \in \Delta} L_{\overline{G}}(x) + \Phi(x) \tag{1}$$

where Φ satisfies the three following assumptions :

- $\Phi \in \mathcal{C}^2(\mathbb{R}^n)$ *and for all face* S *of* Δ, $\nabla^2 \Phi_S(x) > 0$ *for all* $x \in \Delta^{(S)}$ *i.e. the restriction of* Φ *to any face of* Δ *is strictly convex*
- $\Phi(\overline{x}) = \Phi(x)$ *for all* $\overline{x}$ *permutation of the indices of* x *i.e.* Φ *is symmetric / permutation invariant*
- $(\mathbf{e}_i - \mathbf{e}_j)^T \nabla^2 \Phi(x)(\mathbf{e}_i - \mathbf{e}_j) < x_i^{k-2} + x_j^{k-2}$ *for every* $x \in \Delta$, $i, j \in supp(x)$ *with* $i \neq j$

Sum of Real Functions

For a regularizer that is a sum of real functions (i.e. $\phi(x) = \sum_i q(x_i)$), the conditions can be equivalently expressed as:

Equivalent Conditions

- $q \in C^2(\mathbb{R})$ and $q''(y) > 0$ when $y > 0$.
- $q''(y) < y^{k-2}$ for $0 < y \leq \frac{1}{2}$ and $q''(y) \leq y^{k-2}$ for $\frac{1}{2} < y \leq 1$.
- $q''(y) < y^{k-2}$ for $0 < y \leq \frac{1}{2}$ and $q''(y) \leq y^{k-2}$ for $\frac{1}{2} < y \leq 1$, ensuring strict convexity and symmetry.

Generalization of the Regularizers

- Original regularizers :
  1. $\phi(x) = \frac{1}{2}\|x\|_2^2$
  2. $\alpha_1\|x+\varepsilon_1\|_p^p$ with $\varepsilon \geq 0$, $p \geq 2$, and $0 \leq \alpha_1 \leq \frac{2}{p(p-1)(1+\varepsilon)p-2}$
  3. $\phi(x) = \alpha_2 \sum_{i=1}^n (e^{-\beta x_i} - 1)$ with $\beta > 0$ and $0 \leq \alpha_2 \leq \frac{2}{\beta^2}$ (approximation of $-\alpha_2\|x\|_0$)
- Generalizations :
  1. $\Phi_B(x) = \alpha\|x\|_k^k$ with $0 < \alpha < \frac{1}{k(k-1)}$
  2. $\Phi_1(x) = \alpha_1\|x+\varepsilon_1\|_p^p - \frac{n}{2}\alpha_1 p(p-1)\varepsilon^{p-2}x^2$ with $\varepsilon > 0$, $p \geq k$, and $0 < \alpha_1 \leq \frac{1}{p(p-1)(1+\varepsilon)p-2}$
  3. More complicated, so no easy generalization

Third Part : Lovász Extensions

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Given a function $f : P(V) \to \mathbb{R}$, its Lovász extension extends the domain of f to $\mathbb{R}^n$:

Denote $x = (x_1, \ldots, x_n) \in \mathbb{R}$, and let $\sigma : V \cup \{0\} \to V \cup \{0\}$ be a permutation such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n)}$ and $\sigma(0) := 0$ where we add $x_0 := 0$ to x. The Lovász extension is defined as:

$$f_L(x) = \sum_{i=0}^{n-1} (x_{\sigma(i+1)} - x_{\sigma(i)}) \cdot f(V_{\sigma(i)}(x))$$

where $V^0 := V$ and $V_{\sigma(i)}(x) := \{j \in V : x_j > x_{\sigma(i)}\}$.

Key Properties of Lovász Extension

Proposition

- *For a set* S*, denoting its characteristic vector* $1_S \in \{0,1\}^n$*,* $f_L(1_S) = f(S)$*.*
- f_L *is the unique function that is affine on each polyhedral cone* $R^n_\sigma := \{x \in R^n, x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n)}\}$ *and for which* $f_L(1_S) = f(S)$ *for every set* S*.*
- f_L *is positively one-homogeneous, piecewise-linear, and Lipschitzian continuous.*
- f *is submodular* $\Leftrightarrow$ f_L *is convex* $\Leftrightarrow$ f_L *is submodular.*

Key Results

Theorem

Given set functions $f_1, \ldots, f_n : A \to \mathbb{R}^+$ *and a zero-homogeneous function* $H : (\mathbb{R}^+)^m \setminus \{0\} \to \mathbb{R} \cup \{+\infty\}$ *with* $H(a+b) \geq \min(H(a), H(b))$ *for all* $a, b \in (\mathbb{R}^+)^m \setminus \{0\}$*, we have*

$$\min_{S \in A'} H(f_1(S), \ldots, f_n(S)) = \inf_{x \in D'} H(f_L^1(x), \ldots, f_L^n(x))$$

where $A' = \{S \in A : (f_1(S), \ldots, f_n(S)) \in Dom(H)\}$*,*
$D' = \{x \in DA \cap (\mathbb{R}^+)^V : (f_L^1(x), \ldots, f_L^n(x)) \in Dom(H)\}$*, and*
$Dom(H) = \{a \in (\mathbb{R}^+)^m \setminus \{0\} : H(a) \in \mathbb{R}\}$*.*

Key Results

Theorem

Given two set functions $f, g : A \to [0, \infty)$, *let* $\tilde{f}, \tilde{g} : DA \to \mathbb{R}$ *satisfying* $\tilde{f} \geq f_L$, $\tilde{g} \leq g_L$, $\tilde{f}(1_S) = f(S)$, *and* $\tilde{g}(1_S) = g(S)$. *Then*

$$\min_{S \in A \cap supp(g)} \frac{f(S)}{g(S)} = \inf_{\Psi \in DA \cap supp(\tilde{g})} \frac{\tilde{f}(\Psi)}{\tilde{g}(\Psi)}$$

If we replace the conditions by $\tilde{f} \leq f_L$, $\tilde{g} \geq g_L$, *then*

$$\max_{S \in A \cap supp(g)} \frac{f(S)}{g(S)} = \sup_{\Psi \in DA \cap supp(\tilde{g})} \frac{\tilde{f}(\Psi)}{\tilde{g}(\Psi)}$$

Key Results

Theorem

Let $f, g : A \to [0, +\infty)$ be two set functions with decompositions $f = f_1 - f_2$ and $g = g_1 - g_2$ as differences of submodular functions. Let $\tilde{f}_2, \tilde{g}_1$ be the restrictions of positively one-homogeneous convex functions onto DA, with $f_2(S) = \tilde{f}_2(1_S)$ and $g_1(S) = \tilde{g}_1(1_S)$. Define $\tilde{f} = f_L^1 - \tilde{f}_2$ and $\tilde{g} = \tilde{g}_1 - g_L^2$. Then,

$$\min_{S \in A \cap supp(g)} \frac{f(S)}{g(S)} = \min_{x \in DA \cap supp(\tilde{g})} \frac{\tilde{f}(S)}{\tilde{g}(S)}$$

Piecewise Multilinear Extensions

For now, Lovász extensions satisfy the property $(f+g)^L = f^L + g^L$, and homogeneous and piecewise multilinear extensions extend this property to $(gf)^L = g^L f^L$, sacrificing the equalities between the discrete and continuous problems and replacing them by inequalities, the reverse inequalities having to be proved on a case-by-case basis.

Piecewise Multilinear Extension

Given $V_i := \{1, \ldots, n_i\}$ and the power set $P(V_i)$ for $i = 1, \ldots, k$, for a discrete function $f : P(V_1) \times \ldots \times P(V_k) \to \mathbb{R}$, its piecewise multilinear extension is defined on $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k}$ by

$$f^M(x_1, \ldots, x_k) := \sum_{i_1 \in V_1, \ldots, i_k \in V_k} \prod_{l=1}^{k} (x_l^{\sigma_l(i_l)} - x_l^{\sigma_l(i_l - 1)}) f(V_{i_1}(x_1), \ldots, V_{i_k}(x_k)$$

where $V_i(x_l) := \{j \in V_l \mid x_{l,j} > x_{l,l-1}\}$ for $i \geq 2$, $V_1(x_l) := V_l$, σ_l is a permutation of indices sorting x_l by non-decreasing order, and we add $x_{l,0} := 0$ to each x_l.

Key Results

Theorem

Given $f : \mathcal{A} \to \mathbb{R}$ *and* $g : \mathcal{A} \to [0, +\infty)$, *we have*

$$\sup_{A \in \mathcal{A} \cap supp(g)} \frac{f(A)}{g(A)} \le \sup_{x \in D \cap supp(g^M)} \frac{f^M(x)}{g^M(x)} \le \sup_{A \in \tilde{\mathcal{A}}} \frac{f(A)}{g(A)}$$

whenever $\{1_A : A \in \mathcal{A}\} \subset D$ *and* $\mathcal{A}(D) \subset \tilde{\mathcal{A}}$. *The above inequality still holds replacing* $\sup$ *and* $\le$ *by* $\inf$ *and* $\ge$. *If we further assume that* $(\mathcal{A}, D)$ *is a perfect domain pair, and* $supp(f) \subset supp(g)$, *then*

$$\max_{A \in \mathcal{A} \cap supp(g)} \frac{f(A)}{g(A)} = \max_{x \in D \cap supp(g^M)} \frac{f^M(x)}{g^M(x)}$$

and the same holds replacing $\max$ *with* $\min$.

Key results

Theorem

Let $H : \mathbb{R}^*_+ \to \mathbb{R} \cup \{+\infty\}$ be a zero-homogeneous and quasi-concave function. For any function $f_1, \ldots, f_n : A \to \mathbb{R}^+$, we have

$$\min_{A \in A} H(f_1(A), \ldots, f_n(A)) = \inf_{x \in D} H(f_1^M(x), \ldots, f_n^M(x))$$

where (A, D) forms a perfect domain pair w.r.t. the piecewise multilinear extension.
In addition, if $H : \mathbb{R}^*_+ \to \mathbb{R} \cup \{-\infty\}$ is a zero-homogeneous and quasi-convex function, for any function $f_1, \ldots, f_n : A \to \mathbb{R}^+$, we have

$$\max_{A \in A} H(f_1(A), \ldots, f_n(A)) = \sup_{x \in D} H(f_1^M(x), \ldots, f_n^M(x))$$

Recovering the Classical MCP in 2-graphs

We have

$$\max_{S \in P(V) \setminus \{\emptyset\}} \underbrace{\frac{|E(S,S)|}{|S|^2}}_{\substack{\text{Maximized when} \\ S \text{ is a maximum clique}}} = \max_{\mathbf{x} \neq \mathbf{0}} \frac{2 \sum_{(i,j) \in E} x_i x_j}{\|\mathbf{x}\|_2^2}$$

where $P(V)$ is the power set of V, $E(S,S)$ denotes the set of edges between vertices in set S.

Recovering the MCP for Hypergraphs

We have:

$$\min_{S \in P(V) \setminus \{\emptyset\}} \underbrace{\frac{|E(S,\ldots,S)| + \tau|S|}{|S|^k}}_{\substack{\text{Maximized when} \\ S \text{ is a maximum clique}}} = \min_{\mathbf{x} \neq \mathbf{0}} \frac{k! \sum_{e \in E} \prod_{i \in e} x_i + \tau \|\mathbf{x}\|_k^k}{\|\mathbf{x}\|_k^k}$$

where $0 < \tau < \frac{1}{|V|(|V|^{k-1}-1)}$, $P(V)$ is the power set of V and $E(S,\ldots,S)$ represents the set of hyperedges with all vertices in set S.

Starting Point for the MsdcP

I suggested starting from the following function :

$$h(S) := (s+2)^{|S|}\left(s+1-\left(\frac{|S|(|S|-1)}{2}-|E(S,S)|\right)\right) \tag{2}$$

Conclusion

The internship work progressed along three main directions:

- Developed a novel continuous formulation for MsdCP in hypergraphs, studied algorithm convergence, and provided implementation.
- Extended the continuous formulation of MCP in hypergraphs to a broader class of regularizers.
- Laid the groundwork for using continuous extensions to find formulations for discrete problems within the MCP family.