Multilevel Variance Estimation in the Log-Euclidean Geometry*

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Abstract. Recent research has led to the definition of a new covariance estimator: the Multilevel Monte-Carlo (MLMC) estimator in log-Euclidean geometry. This estimator would make it possible to avoid having the loss of positivity in MLMC covariance estimation. In this paper, we will study the efficiency of this estimator, notably through its Mean Squared Error. Proving that the first order of the MSE of the log-Euclidean estimator is proportional to the MSE of the Euclidean estimator allows us to compare precisely both estimators. What's more, our numerical results show asymptotic unbiasedness as a function of the budget. Furthermore, the MSE of the log-Euclidean MLMC Variance estimator converges asymptotically and its MSE is lower than the Euclidean MLMC Variance estimator. We observed that the log-Euclidean MLMC Variance Estimator is well suited in applications with low budget application. However, it is important to note that this estimator is biased at lower budget levels.

Key words. Monte Carlo, Multilevel Monte Carlo, Log-Euclidean Geometry, Covariance, Variance, LaTeX.

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1. Introduction.

In the realm of uncertainty quantification, the Multilevel Monte Carlo (MLMC) method has emerged as a pivotal approach for efficiently estimating statistical parameters in complex systems. Traditional MLMC approaches the estimations of covariance matrices in Euclidean geometry, such as those defined by Giles [4] and utilised by Mycek & de Lozzo [7], face challenges due to the loss of positivity during subtraction operations, which can introduce significant errors as matrix size increase. This issue prompted the development of the Multilevel Monte Carlo Covariance Estimator in log-Euclidean geometry by Maurais et al. (2023), aimed at addressing these inaccuracies.

Previous studies, such as [7], have extensively utilised the standard MLMC method in Euclidean settings to compute Sobol' indices, demonstrating its effectiveness in reducing computational costs while maintaining precision.

Despite these advancements, a significant gap persists in the comparative analysis of these methodologies in log-Euclidean versus Euclidean geometries. Current literature lacks a comprehensive, side-by-side evaluation of the MLMC approach across these two geometric frameworks, particularly in terms of mean squared error (MSE) and overall computational efficiency at equivalent costs. This study aims to address this research gap by critically analysing and comparing the performance of the log-Euclidean Multilevel Monte Carlo Variance Estimator against the traditional MLMC estimator. Through rigorous mathematical modelling and numerical computation, we seek to establish whether the log-Euclidean metric provides a measurable improvement in accuracy and cost-effectiveness for covariance estimation in uncertainty quantification.

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2. Context and Importance of Covariance Estimation in Various Fields.

Covariance matrices are crucial tools used across various scientific and engineering disciplines to understand relationships between variables and manage uncertainties in complex systems. The significance of these matrices is highlighted by [3] in the field of uncertainty quantification in risk analysis. This research introduces novel computational techniques that simplify the inversion process of these matrices from cubic to quadratic complexity, significantly improving efficiency for large datasets. This advancement highlights the critical role of covariance matrices in effectively managing complexities in risk analysis and other related fields .

In climate modelling, for instance, [1] demonstrates the importance of efficient covariance computation for large-scale simulations, enabling advanced weather prediction and climate analysis on high-performance computing architectures.

Additionally, [8] highlights the importance of covariance matrices in defining Gaussian Processes (GPs) for uncertainty quantification. The use of covariance functions in the GP framework is essential for accurate approximation of posterior statistics, facilitating efficient and precise uncertainty management in high-dimensional stochastic inputs and complex simulations.

2.1. Advantages of Multilevel Monte Carlo (MLMC) Methods.

The Multilevel Monte Carlo (MLMC) methods, as described by Michael B. Giles in both his 2008 foundational paper [4] and subsequent 2015 overview [5], offer significant computational advantages over traditional Monte Carlo approaches. MLMC techniques reduce the variance of estimators and the computational cost by combining simulations at multiple levels of accuracy, efficiently converging to a solution with fewer high-fidelity model evaluations. This approach not only saves computational resources but also accelerates the convergence to statistical estimates, making it especially beneficial for applications involving expensive model evaluations.

2.2. Challenges with MLMC Covariance Estimators.

However, when applying MLMC methods to covariance estimation, certain challenges arise. Notably, the MLMC covariance estimator can produce matrices that are not positive definite due to the subtraction operations involved in the multilevel estimator calculations. This issue leads to potential negative eigenvalues in the aggregated result.

2.3. Common Remedies and Their Drawbacks.

A common technique to address the issue of non-positive matrices involves diagonalising these matrices and truncating negative eigenvalues. While this method rectifies the problem of non-positivity, it comes with significant drawbacks. First, the diagonalisation of large covariance matrices, as often encountered in fields like climate modelling or financial econometrics, is computationally expensive and scales poorly with matrix size. Secondly, truncating negative eigenvalues can lead to a loss of information, which is particularly detrimental in uncertainty quantification where preserving the integrity of data is crucial. This loss can lead to inaccurate estimates and predictions, undermining the efficacy of the low-cost advantage purported by MLMC methods.

2.4. Concluding Comments.

This section has highlighted the essential role and challenges of covariance matrices in various disciplines. As we move forward, the "Proposed Solutions and Methodology" section will introduce innovative approaches such as A. Maurais' Log-Euclidean MLMC estimator [6]. The aim of this project is to study the behaviour of this estimator, in the simplified case of a scalar value, i.e. a variance instead of a covariance matrix.

3. Proposed Solution and Methodology.

The Euclidean geometry, the canonical MLMC approach to estimating the variance $\mathbb{V}[Y]$ of a random variable $Y : \Omega \to \mathbb{R}$ is to take $2L+1 \in \mathbb{N}$ samples of $(Y_\ell)_{\ell\in\{0,\dots,L\}}$ and for each level ℓ, generating M_ℓ random vectors Y_ℓ and $Y_{\ell-1}$. Then, the MLMC estimator $\hat{V}_L^{ML}[Y]$ of the variance $\mathbb{V}[Y]$ defined in [7] is:

$$\hat{V}_L^{ML}[Y] = \sum_{\ell=0}^{L} \hat{V}_{M_\ell}^{(\ell)}[Y_\ell] - \hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}] \qquad \text{with } Y_{-1} = 0_{\mathbb{R}^d} \tag{3.1}$$

where $\hat{V}_{M_\ell}^{(\ell)}[Y_\ell]$ and $\hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}]$ are single-level Multi-Carlo variance estimator (commonly known as the corrected empirical variance):

$$\begin{aligned} \hat{V}_{M_\ell}^{(\ell)}[Y_\ell] &= \frac{M_\ell}{M_\ell-1}\hat{E}_{M_\ell}^{(\ell)}\left[\left(Y_\ell - \hat{E}_{M_\ell}^{(\ell)}[Y_\ell]\right)^2\right] \\ &= \frac{1}{M_\ell-1}\sum_{i=1}^{M_\ell}\left[\left(Y_\ell^{(\ell,i)} - \frac{1}{M_\ell}\sum_{i=1}^{M_\ell}\left[Y_\ell^{(\ell,i)}\right]\right)^2\right] \end{aligned} \tag{3.2}$$

$$\begin{aligned} \hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}] &= \frac{M_\ell}{M_\ell-1}\hat{E}_{M_\ell}^{(\ell)}\left[\left(Y_{\ell-1} - \hat{E}_{M_\ell}^{(\ell)}[Y_{\ell-1}]\right)^2\right] \\ &= \frac{1}{M_\ell-1}\sum_{i=1}^{M_\ell}\left[\left(Y_{\ell-1}^{(\ell,i)} - \frac{1}{M_\ell}\sum_{i=1}^{M_\ell}\left[Y_{\ell-1}^{(\ell,i)}\right]\right)^2\right] \end{aligned} \tag{3.3}$$

Remark 3.1. These random variables $\hat{V}_{M_\ell}^{(\ell)}[Y_\ell]$ and $\hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}]$ takes values in the space $\mathbb{R}_+^*$ of strictly positive real numbers but calculate their difference can occasionally lead to a negative MLMC variance estimations, especially when $\mathbb{V}[Y]$ is small. In the previous subsection 2.3, we discussed several common remedies for this issue. However, these solutions come with notable limitations. Instead, the following subsection introduces an alternative estimator that operates within a different geometric framework.

3.1. MLMC Variance Estimator in Log-Euclidean Geometry.

Use of the Log-Euclidean geometry for multilevel estimations was presented and reviewed in [6]. Here, we present a connected, yet different MLMC estimator of a scalar variance in the log-Euclidean geometry. The estimator $\hat{V}_L^{LML}[Y]$ of the variance $\mathbb{V}[Y]$ is:

$$\hat{V}_L^{LML}[Y] = \exp\left(\log \hat{V}_{M_0}^{(0)}[Y_0] + \sum_{\ell=1}^{L}\log \hat{V}_{M_\ell}^{(\ell)}[Y_\ell] - \log \hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}]\right) \tag{3.4}$$

where $\hat{V}_{M_\ell}^{(\ell)}[Y_\ell]$ and $\hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}]$ are single-level MC estimator for variances defined in (3.2) and (3.3), based on the same input sample $(Y_\ell)_{\ell\in\{0,\dots,L\}}$.

Proposition 3.2. *Let $a \in]0,+\infty[\setminus\{1\}$. For any $\ell \in \{1,\dots,L\}$, using the rescaling $\tilde{Y}_\ell = a^{-1/2}Y_\ell$, the Log-MLMC estimator can be written in base a as:*

$$\hat{V}_L^{LML}[Y] = a\exp_a\left(\log_a \hat{V}_{M_0}^{(0)}[Y_0] + \sum_{\ell=1}^{L}\log_a \hat{V}_{M_\ell}^{(\ell)}[\tilde{Y}_\ell] - \log_a \hat{V}_{M_\ell}^{(\ell)}[\tilde{Y}_{\ell-1}]\right) = a\hat{V}_L^{LML,a}[\tilde{Y}] \tag{3.5}$$

where $\log_a$ represents the logarithm in base a and $\log$ represents the natural logarithm and $\exp_a\colon x \mapsto a^x$.

Proof. Let $a > 0$ and $a \neq 1$. For all $\ell \in \{1,\dots,L\}$:

$$\begin{aligned} \tilde{Y}_\ell = a^{-1/2}Y_\ell \implies \log_a \mathbb{V}(\tilde{Y}_\ell) &= \log_a \mathbb{V}\left(\frac{Y_\ell}{\sqrt{a}}\right) \\ &= \log_a\left(\frac{1}{a}\mathbb{V}[Y_\ell]\right) \\ &= \log_a \mathbb{V}[Y_\ell] - \log_a(a) \end{aligned} \tag{3.6}$$

Hence:

$$\begin{aligned} \hat{V}_L^{LML,a}[\tilde{Y}] &= \exp_a\left(\log_a(\hat{V}_{M_0}^{(0)})[\tilde{Y}_0] + \sum_{\ell=1}^{L}\log_a \hat{V}_{M_\ell}^{(\ell)}[\tilde{Y}_\ell] - \log_a \hat{V}_{M_\ell}^{(\ell)}[\tilde{Y}_{\ell-1}]\right) \\ &= \exp\left[\left(\log_a(\hat{V}^{(0)})[\tilde{Y}_0] + \sum_{\ell=1}^{L}\log_a \hat{V}_{M_\ell}^{(\ell)}[\tilde{Y}_\ell] - \log_a \hat{V}_{M_\ell}^{(\ell)}[\tilde{Y}_{\ell-1}]\right)\log(a)\right] \\ &= \exp\left[\left(\log_a(\hat{V}^{(0)})[Y_0] - \log_a(a) + \sum_{\ell=1}^{L}\log_a \hat{V}_{M_\ell}^{(\ell)}[Y_\ell] - \log_a(a) - \log_a \hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}]\right.\right. \\ &\qquad\qquad \left.\left. + \log_a(a)\right)\log(a)\right] \\ &= \exp\left[\left(\log_a(\hat{V}^{(0)})[Y_0] + \sum_{\ell=1}^{L}\left(\log_a \hat{V}_{M_\ell}^{(\ell)}[Y_\ell] - \log_a \hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}]\right) - \log_a(a)\right)\log(a)\right] \\ &= \exp\left[\left(\frac{\log \hat{V}_{M_0}^{(0)}[Y_0]}{\log(a)} + \sum_{\ell=1}^{L}\left(\frac{\log \hat{V}_{M_\ell}^{(\ell)}[Y_\ell]}{\log(a)} - \frac{\log \hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}]}{\log(a)}\right) - 1\right)\log(a)\right] \\ &= \exp\left(\log \hat{V}_{M_0}^{(0)}[Y_0] + \sum_{\ell=1}^{L}\left(\log \hat{V}_{M_\ell}^{(\ell)}[Y_\ell] - \log \hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}]\right) - \log(a)\right) \\ &= \frac{1}{a}\hat{V}_L^{LML}[Y] \end{aligned} \tag{3.7}$$

∎

Remark 3.3. The Log-Euclidean Geometry defined in [2] is reviewed in [6]. Maurais then constructs the Log-Euclidean MLMC covariance estimator and proves its existence and its positive definiteness almost surely whenever $M_0, M_1, \dots, M_L > d$. In our case, it translates that the variance estimator (3.4) exists and is posisive almost surely, whenever $M_0, M_1, \dots, M_L > d$. Using the estimator in (3.4) presents a promising approach for applying MLMC methods to compute variances while ensuring positivity. Prior to concluding, it is essential to evaluate the mean squared error (MSE), bias, and variance of this estimator, and subsequently compare these metrics with those of the estimator in (3.1).

3.2. Main results. Let x, y be in $\mathbb{R}_+^*$. We define the log-Euclidean distance as:

$$d_{LE}(x, y) = |\log(x) - \log(y)|^2$$

Proposition 3.4 (MSE in the log-Euclidean metric of the Log-MLMC estimator (3.4) [6]). *Assume that $|V - V_\ell|< h/4$ with $h < 1$ for $\ell = 0, \dots, L$. Then, for sufficiently large samples $M_\ell \in \mathbb{N}^{L+1}$, the MSE in the log-Euclidean metric of the Log-MLMC estimator defined in (3.4) is*

$$\begin{aligned}\mathbb{E}\left[d_{LE}\left(\hat{V}_L^{LML}, V\right)^2\right] &= \mathbb{E}\left[\left|\log \hat{V}_L^{LML} - \log V\right|^2\right] \\ &= \frac{1}{a^2}\mathbb{E}\left[\left|\hat{V}_L^{ML} - V\right|^2\right] + \mathcal{O}\left(h^3\right)\end{aligned}$$

Proof. To simplify, we introduce the following notations:

- $V = \mathbb{V}[Y]$,
- $\hat{V}_L^{ML} = \hat{V}_L^{ML}[Y]$,
- $\hat{V}_L^{LML} = \hat{V}_L^{LML}[Y]$
- $\hat{V}_\ell = \hat{V}_{M_\ell}^{(\ell)}[Y_\ell]$,
- $\hat{V}_{\ell-1} = \hat{V}_{M_\ell}^{(\ell)}[Y_{\ell-1}]$,
- $\hat{V}_{-1} = 0$ in the euclidean metric,
- $\log(\hat{V}_{-1}) = 0$ in the log-euclidean metric.

Show that:

$$\mathbb{E}\left[\left|\log(\hat{\tilde{V}}_L^{LML}) - \log(\tilde{V})\right|^2\right] - \mathbb{E}\left[\left|\hat{\tilde{V}}_L^{ML} - \tilde{V}\right|^2\right] = \mathcal{O}\left(h^3\right)$$

Let $a \in]0; +\infty[\setminus\{1\}$ such that $|\frac{1}{a}V - 1|= |\tilde{V} - 1|< \frac{h}{4}$ and $\frac{1}{a}|V - V_\ell|= |\tilde{V}_\ell - \tilde{V}|< \frac{h}{4}$ $\forall \ell \in \{0, \dots, L\}$ .Thus we are going to do the proof on the rescaled samples $\tilde{Y}_\ell = a^{-1/2} Y_\ell$ $\forall \ell \in \{0, \dots, L\}$. Introducing this rescaling gives:

$$
\begin{aligned}
\hat{\hat{V}}_L^{ML} &= \sum_{\ell=0}^{L} \left(\hat{\hat{V}}_\ell - \hat{\hat{V}}_{\ell-1} \right) \\
&= \frac{1}{a} \sum_{\ell=0}^{L} \left(\hat{V}_\ell - \hat{V}_{\ell-1} \right) \\
&= \frac{1}{a} \hat{V}_L^{ML}
\end{aligned}
\tag{3.8}
$$

According to (3.4) we can write our estimators as:

$$
\begin{aligned}
\left|\log(\hat{V}_L^{LML}) - \log(V)\right| &= \left|\log(a\hat{\hat{V}}_L^{LML}) - \log(a\tilde{V})\right| \\
&= \left|\log(\hat{\hat{V}}_L^{LML}) + log(a) - \log(\tilde{V}) - log(a)\right| \\
&= \left|\log(\hat{\hat{V}}_L^{LML}) - \log(\tilde{V})\right| \\
&= \left|\left(\log_a(\hat{\hat{V}}_0^{(0)}) + \sum_{l=1}^{L} \left(\log_a(\hat{\hat{V}}_\ell^{(\ell)}) - \log_a(\hat{\hat{V}})_{\ell-1}^{(\ell)})\right)\right) \log(a) - \log(\tilde{V})\right|
\end{aligned}
\tag{3.9}
$$

Taylor expansion in logarithmic in base a series can be written for x in the vicinity of b as follows:

$$
\log_a(x) = \log_a(b) + \sum_{k=1}^{\infty} \frac{(x-b)^k(-1)^{k+1}}{kb^k \log(a)}
\tag{3.10}
$$

With $x = \tilde{V}_\ell$ and $b = 1$:

$$
\log_a(\tilde{V}_\ell) = \sum_{k=1}^{\infty} \frac{(\tilde{V}_\ell - 1)^k(-1)^{k+1}}{k \log(a)}
\tag{3.11}
$$

And the Taylor expansion of the natural logarithm for x in the vicinity of b is :

$$
\log(x) = \log(b) + \sum_{k=1}^{\infty} \frac{(x-b)^k(-1)^{k-1}}{kb^k}
\tag{3.12}
$$

With $x = \tilde{V}$ and $b = 1$:

$$
\log(\tilde{V}) = \sum_{k=1}^{\infty} \frac{(\tilde{V} - 1)^k(-1)^{k-1}}{k}
\tag{3.13}
$$

To prove that this series converges, we will employ the alternating series criterion which states that an alternate series $\sum_{k=1}^{\infty}(-1)^{k+1}u_k$, converges if the following conditions are satisfied:

1. $|u_k|$ decreases monotonically, i.e. $|u_{k+1}|< |u_k|$ and,
2. $\lim_{k\to\infty} u_k = 0$.

Here, $u_k = \frac{(\tilde{V}-1)^k}{k}$. Since $|\tilde{V}-1|^k< h/4$, then $\frac{1}{k}|\tilde{V}-1|^k < \frac{h}{4k} < 1$ because $k > 1$. Thus:

1. $\frac{1}{k}|\tilde{V}-1|^k$ decreases monotonically and,
2. $\lim_{k\to\infty} \frac{1}{k}|\tilde{V}-1|^k = 0$.

So the series defined in (3.11) converges.

Moreover, by the triangular inequality:

$$|\tilde{V}_\ell - 1|= |\tilde{V}_\ell - \tilde{V} + \tilde{V} - 1| \leq |\tilde{V}_\ell - \tilde{V}|+|\tilde{V} - 1| < \frac{h}{4} + \frac{h}{4} < \frac{h}{2} < 1 \tag{3.14}$$

By definition of our estimators $\hat{\tilde{V}}_\ell^{(\ell)}$ and $\hat{\tilde{V}}_{\ell-1}^{(\ell)}$ for all $\ell \in \{0, \dots, L\}$, the strong law of large numbers (Kolmogorov's law) implies almost surely that for any $\delta_\ell > 0$ and for a sufficiently large sample M_ℓ guarantees:

$$\max\left(\left|\hat{\tilde{V}}_\ell^{(\ell)} - \tilde{V}_\ell\right|, \left|\hat{\tilde{V}}_{\ell-1}^{(\ell)} - \tilde{V}_{\ell-1}\right|\right) < \delta_\ell \tag{3.15}$$

$$\lim_{M_\ell\to\infty} \left|\hat{\tilde{V}}_{\ell-1}^{(\ell)} - \tilde{V}_\ell\right| \leq \left|\tilde{V}_{\ell-1} - \tilde{V}\right| + \left|\tilde{V} - \tilde{V}_\ell\right| < \frac{h}{4} + \frac{h}{4} < \frac{h}{2} < 1 \tag{3.16}$$

Setting $\delta_\ell = h/4$ and applying the inequality triangular and the result gives:

$$\begin{aligned} \max\left\{\left|\hat{\tilde{V}}_\ell^{(\ell)} - 1\right|, \left|\hat{\tilde{V}}_{\ell-1}^{(\ell)} - 1\right|\right\} &= \max\left\{\left|\hat{\tilde{V}}_\ell^{(\ell)} - \tilde{V}_\ell + \tilde{V}_\ell - 1\right|, \left|\hat{\tilde{V}}_{\ell-1}^{(\ell)} - \tilde{V}_{\ell-1} + \tilde{V}_{\ell-1} - 1\right|\right\} \\ &\leq \max\left\{\left|\hat{\tilde{V}}_\ell^{(\ell)} - \tilde{V}_\ell\right| + \left|\tilde{V}_\ell - 1\right|, \left|\hat{\tilde{V}}_{\ell-1}^{(\ell)} - \tilde{V}_{\ell-1}\right| + \left|\tilde{V}_{\ell-1} - 1\right|\right\} \\ &\leq \delta_\ell + \frac{h}{2} \leq \frac{h}{4} + \frac{h}{2} = \frac{3h}{4} < 1 \end{aligned} \tag{3.17}$$

Therefore, the series defined in (3.13) converges for the same alternating series criterion used to demonstrate the convergence of (3.11). Expanding the Taylor series for each logarithm to the first order in the vicinity of 1 (i.e. $\log_a(\tilde{V}) \sim (\tilde{V}-1)/\log(a)$ and $\log(\tilde{V}) \sim \tilde{V} - 1$) gives:

$$
\begin{aligned}
\log(\hat{\hat{V}}_L^{LML}) - \log(\tilde{V}) &= \left(\log_a(\hat{\hat{V}}_0^{(0)}) + \sum_{\ell=1}^{L}\left(\log_a(\hat{\hat{V}}_\ell^{(\ell)}) - \log_a(\hat{\hat{V}}_{\ell-1}^{(\ell)})\right)\right)\log(a) - \log(\tilde{V}) \\
&= \left(\frac{\hat{\hat{V}}_0^{(0)} - 1}{\log(a)} + \sum_{\ell=1}^{L}\left[\frac{\hat{\hat{V}}_\ell^{(\ell)} - 1}{\log(a)} - \frac{\hat{\hat{V}}_{\ell-1}^{(\ell)} - 1}{\log(a)}\right]\right)\log(a) - (\tilde{V} - 1) + R \\
&= \hat{\hat{V}}_0^{(0)} + \sum_{\ell=1}^{L}\left(\hat{\hat{V}}_\ell^{(\ell)} - \hat{\hat{V}}_{\ell-1}^{(\ell)}\right) - \tilde{V} + R \\
&= \hat{\hat{V}}_L^{ML} - \tilde{V} + R
\end{aligned} \tag{3.18}
$$

Where R is the remainder of the first-order Taylor expansion:

$$
\begin{aligned}
R = &\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}\left(\hat{\hat{V}}_0^{(0)} - 1\right)^k \\
&+ \sum_{\ell=1}^{L}\left(\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}\left(\hat{\hat{V}}_\ell^{(\ell)} - 1\right)^k\right) \\
&- \sum_{\ell=1}^{L}\left(\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}\left(\hat{\hat{V}}_{\ell-1}^{(\ell)} - 1\right)^k\right) \\
&- \sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}\left(\tilde{V} - 1\right)^k.
\end{aligned} \tag{3.19}
$$

By definition of the MLMC variance estimator (3.8):

$$
|\hat{\hat{V}}_L^{ML} - \tilde{V}| = \left|\hat{\hat{V}}_0^{(0)} + \sum_{\ell=1}^{L}\left(\hat{\hat{V}}_\ell^{(\ell)} - \hat{\hat{V}}_{\ell-1}^{(\ell)}\right) - \tilde{V}\right|
$$

So by the triangular inequality and the reverse triangle inequality ($||a| - |b|| \leq |a-b|$), and by (3.18) :

$$
\begin{aligned}
\left||\log(\hat{\hat{V}}_L^{LML}) - \log(\tilde{V})| - |\hat{\hat{V}}_L^{ML} - \tilde{V}|\right| &\leq \left|\log(\hat{\hat{V}}_L^{LML}) - \log(\tilde{V}) - \hat{\hat{V}}_L^{ML} + \tilde{V}\right| \\
&\leq |R|
\end{aligned} \tag{3.20}
$$

From (3.18) and triangle inequality we also have :

$$
\begin{aligned}
|\log\hat{\hat{V}}_L^{LML} - \log\tilde{V}| + |\hat{\hat{V}}_L^{ML} - \tilde{V}| &= |\hat{\hat{V}}_L^{ML} - \tilde{V}| + |\hat{\hat{V}}_L^{ML} - \tilde{V} + R| \\
&\leq 2|\hat{\hat{V}}_L^{ML} - \tilde{V}| + |R|
\end{aligned} \tag{3.21}
$$

$\forall \alpha \in \mathbb{R}, |\alpha^k| = |\alpha|^k, k \in \mathbb{N}$. Then by the triangle inequality:

$$
\begin{aligned}
|R| \leq & \sum_{k=2}^{\infty} \frac{1}{k} \left| \hat{\tilde{V}}_0^{(0)} - 1 \right|^k \\
& + \sum_{\ell=1}^{L} \left(\sum_{k=2}^{\infty} \frac{1}{k} \left| \hat{\tilde{V}}_\ell^{(\ell)} - 1 \right|^k \right) \\
& + \sum_{\ell=1}^{L} \left(\sum_{k=2}^{\infty} \frac{1}{k} \left| \hat{\tilde{V}}_{\ell-1}^{(\ell)} - 1 \right|^k \right) \\
& + \sum_{k=2}^{\infty} \frac{1}{k} \left| \tilde{V} - 1 \right|^k .
\end{aligned} \tag{3.22}
$$

Then, since $|\tilde{V} - 1| < h/4 < 1$ due to the rescaling and since (3.17), each term in the four series of (3.22) decreases monotonically and tends to 0 as k tends to infinity. Hence, each series in (3.22) converges absolutely. Bounding $1/k$ with 1, factoring a squared term into each sum and re-indexing the sums in (3.22) gives:

$$
\begin{aligned}
|R| \leq & \left| \hat{\tilde{V}}_0^{(0)} - 1 \right|^2 \sum_{k=0}^{\infty} \left| \hat{\tilde{V}}_0^{(0)} - 1 \right|^k \\
& + \sum_{\ell=1}^{L} \left| \hat{\tilde{V}}_\ell^{(\ell)} - 1 \right|^2 \left(\sum_{k=0}^{\infty} \left| \hat{\tilde{V}}_\ell^{(\ell)} - 1 \right|^k \right) \\
& + \sum_{\ell=1}^{L} \left| \hat{\tilde{V}}_{\ell-1}^{(\ell)} - 1 \right|^2 \left(\sum_{k=0}^{\infty} \left| \hat{\tilde{V}}_{\ell-1}^{(\ell)} - 1 \right|^k \right) \\
& + \left| \tilde{V} - 1 \right|^2 \sum_{k=0}^{\infty} \left| \tilde{V} - 1 \right|^k .
\end{aligned} \tag{3.23}
$$

In the inequality, we replace the geometric series with their values, i.e. $\sum_{k=0}^{+\infty} r^k = \frac{1}{1-r}$ with $r < 1$:

$$|R| \leq \frac{\left|\hat{\tilde{V}}_0^{(0)} - 1\right|^2}{1 - \left|\hat{\tilde{V}}_0^{(0)} - 1\right|^k} + \sum_{\ell=1}^{L} \frac{\left|\hat{\tilde{V}}_\ell^{(\ell)} - 1\right|^2}{1 - \left|\hat{\tilde{V}}_\ell^{(\ell)} - 1\right|^k} + \sum_{\ell=1}^{L} \frac{\left|\hat{\tilde{V}}_{\ell-1}^{(\ell)} - 1\right|^2}{1 - \left|\hat{\tilde{V}}_{\ell-1}^{(\ell)} - 1\right|^k} + \frac{\left|\tilde{V} - 1\right|^2}{1 - \left|\tilde{V} - 1\right|^k} \tag{3.24}$$

Now we have the following two inequalities.
By growth of $x \mapsto \frac{1}{1-x}$ on $]-\infty, 1[$ and (3.17):

$$\max\left\{ \frac{1}{1 - \left|\hat{\tilde{V}}_\ell^{(\ell)} - 1\right|}, \frac{1}{1 - \left|\hat{\tilde{V}}_{\ell-1}^{(\ell)} - 1\right|} \right\} < \frac{1}{1 - \frac{3h}{4}} < 4 \tag{3.25}$$

and similarly

$$\frac{1}{1 - \left|\tilde{V} - 1\right|} < \frac{1}{1 - \frac{h}{4}} = \frac{4}{3} \tag{3.26}$$

By replacing in (3.24):

$$|R| \leq 4\left|\hat{\tilde{V}}_0^{(0)} - 1\right|^2 + 4\sum_{\ell=1}^{L} \left|\hat{\tilde{V}}_\ell^{(\ell)} - 1\right|^2 + 4\sum_{\ell=1}^{L} \left|\hat{\tilde{V}}_{\ell-1}^{(\ell)} - 1\right|^2 + \frac{4}{3}\left|\tilde{V} - 1\right|^2. \tag{3.27}$$

The final step is to apply the inequality form (3.17)

$$
\begin{aligned}
|R| &\leq 4\left(\frac{3h}{4}\right)^2 + 4\sum_{\ell=1}^{L}\left(\frac{3h}{4}\right)^2 + 4\sum_{\ell=1}^{L}\left(\frac{3h}{4}\right)^2 + \frac{4}{3}\left(\frac{h}{4}\right)^2 \\
&\leq \left(\frac{7}{3} + \frac{9}{2}\sum_{\ell=1}^{L}\right) h^2 \\
&= Kh^2.
\end{aligned} \tag{3.28}
$$

Combining (3.28) with (3.20) gives

$$
\Big| |\log(\hat{\tilde{V}}_L^{LML}) - \log(\tilde{V})| - |\hat{\tilde{V}}_L^{ML} - \tilde{V}| \Big| \leq Kh^2 \tag{3.29}
$$

By definition of our estimators, since $\forall \ell, \ell' \in \{0, \ldots, L\}, \tilde{V}_{\ell'} < \infty$, the strong low of large numbers (Kolmogorov's law) implies almost surely that :

$$
\lim_{M_\ell \to +\infty} \hat{\tilde{V}}_{\ell'}^{(\ell)} = \tilde{V}_{\ell'} \tag{3.30}
$$

Applying (3.30) in (3.8) gives almost surely:

$$
\begin{aligned}
\lim_{\substack{M_\ell \to +\infty \\ \ell \in \{0,\ldots,L\}}} \hat{\tilde{V}}_L^{ML} &= \sum_{\ell=0}^{L} \tilde{V}_\ell - \tilde{V}_{\ell-1} \\
&= \tilde{V}_0 - 0 + \tilde{V}_1 - \tilde{V}_0 + \cdots + \tilde{V}_L - \tilde{V}_{L-1} \\
&= \tilde{V}_L
\end{aligned} \tag{3.31}
$$

In the same way, applying (3.30) in (3.5) gives almost surely:

$$
\begin{aligned}
\lim_{\substack{M_\ell \to +\infty \\ \ell \in \{0,\ldots,L\}}} \log(\hat{\tilde{V}}_L^{LML}) &= \left(\log_a(\tilde{V}_0) + \sum_{\ell=1}^{L}\Big(\log_a(\tilde{V}_\ell) - \log_a(\tilde{V}_{\ell-1})\Big)\right)\log(a) \\
&= \Big(\log_a(\tilde{V}_0) + \log_a(\tilde{V}_1) - \log_a(\tilde{V}_0) + \cdots + \log_a(\tilde{V}_L) - \log_a(\tilde{V}_{L-1})\Big)\log(a) \\
&= \log(\tilde{V}_L)
\end{aligned} \tag{3.32}
$$

Hence, result of (3.31) is equivalent to:

$$
\forall \varepsilon > 0, \exists \eta_0 \in \mathbb{N}, \forall \ell \in \{0, \ldots, L\}, \forall M_\ell \geq \eta_0, |\hat{\tilde{V}}_L^{ML} - \tilde{V}_L| < \varepsilon \quad \text{almost surely} \tag{3.33}
$$

And result of (3.32) is equivalent to:

$$
\forall \varepsilon > 0, \exists \eta_0 \in \mathbb{N}, \forall \ell \in \{0, \ldots, L\}, \forall M_\ell \geq \eta_0, |\log(\hat{\tilde{V}}_L^{LML}) - \log(\tilde{V}_L)| < \varepsilon \quad \text{almost surely} \tag{3.34}
$$

In this section, we show that:

$$\mathbb{E}\left[\left|\log(\hat{V}_L^{LML}) - \log(V)\right|^2\right] \le \mathbb{E}\left[\left|\hat{V}_L^{ML} - V\right|^2\right] + \mathcal{O}\left(h^2\right) \tag{3.35}$$

Setting $\varepsilon = h/4$ in (3.33) and using the assumption $|\tilde{V} - \tilde{V}_L|< h/4$ results in:

$$\begin{aligned} |\hat{\tilde{V}}_L^{ML} - \tilde{V}| &= |\hat{\tilde{V}}_L^{ML} - \tilde{V}_L + \tilde{V}_L - \tilde{V}| \\ &\le |\hat{\tilde{V}}_L^{ML} - \tilde{V}_L|+|\tilde{V} - \tilde{V}_L| \\ &< \frac{h}{4} + \frac{h}{4} = \frac{h}{2} \end{aligned} \tag{3.36}$$

Using this result in (3.21) provides:

$$|\log \hat{\tilde{V}}_L^{LML} - \log \tilde{V}|+|\hat{\tilde{V}}_L^{ML} - \tilde{V}| \le h + Kh^2 \tag{3.37}$$

Factoring using $a^2 - b^2 = (a-b)(a+b)$ and substituting (3.29) and (3.37) produces:

$$\begin{aligned} &\left|\left|\log(\hat{\tilde{V}}_L^{LML}) - \log(\tilde{V})\right|^2 - \left|\hat{\tilde{V}}_L^{ML} - \tilde{V}\right|^2\right| \\ &= \left|\left|\log(\hat{\tilde{V}}_L^{LML}) - \log(\tilde{V})\right| + \left|\hat{\tilde{V}}_L^{ML} - \tilde{V}\right|\right| \times \left|\left|\log(\hat{\tilde{V}}_L^{LML}) - \log(\tilde{V})\right| - \left|\hat{\tilde{V}}_L^{ML} - \tilde{V}\right|\right| \\ &\le (h + Kh^2) \times Kh^2 \\ &\le (Kh^3 + K^2h^4) \end{aligned} \tag{3.38}$$

Taking the expectation and using the linearity of the expectation ($\mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[X+Y]$) leads to:

$$\begin{aligned} &\left|\mathbb{E}\left[\left|\log(\hat{\tilde{V}}_L^{LML}) - \log(\tilde{V})\right|^2\right] - \mathbb{E}\left[\left|\hat{\tilde{V}}_L^{ML} - \tilde{V}\right|^2\right]\right| \\ &= \left|\mathbb{E}\left[\left|\log(\hat{\tilde{V}}_L^{LML}) - \log(\tilde{V})\right|^2 - \left|\hat{\tilde{V}}_L^{ML} - \tilde{V}\right|^2\right]\right| \\ &\le (Kh^3 + K^2h^4) \end{aligned} \tag{3.39}$$

Since $K > 0$, (3.39) is equivalent to:

$$\mathbb{E}\left[\left|\log(\hat{\tilde{V}}_L^{LML}) - \log(\tilde{V})\right|^2\right] - \mathbb{E}\left[\left|\hat{\tilde{V}}_L^{ML} - \tilde{V}\right|^2\right] = \mathcal{O}\left(h^3\right) \tag{3.40}$$

so that

$$\begin{aligned}\mathbb{E}\left[\left|\log(\hat{V}_L^{LML}) - \log(V)\right|^2\right] &= \mathbb{E}\left[\left|\hat{\tilde{V}}_L^{ML} - \tilde{V}\right|^2\right] + \mathcal{O}\left(h^3\right) \\ &= \frac{1}{a^2}\mathbb{E}\left[\left|\hat{V}_L^{ML} - V\right|^2\right] + \mathcal{O}\left(h^3\right)\end{aligned} \tag{3.41}$$

■

Remark 3.5. Note that Proposition 3.4 is used to determine optimal allocation sample $M^* = M_0, \ldots, M_L$. Using c_ℓ as the cost of each level, the costs of the multilevel estiamtor is defined as:

$$c(M) = \sum_{\ell=0}^{L} c_\ell M_\ell \tag{3.42}$$

Then, for a given budget $B > 0$, the optimal sample allocation M^* of the $\hat{V}_L^{LML}$ estimator is found by minimising the first-order MSE of the estimator [6]. In other words, the first-order optimal sample allocation M^* is the one that solves:

$$\begin{aligned} &\min_{M \in \mathbb{R}^{L+1}} \mathbb{E}\left[d_{LE}\left(\hat{V}_L^{LML}, V\right)^2\right] \\ &\text{such that} \qquad c(M) \leq B\end{aligned} \tag{3.43}$$

Since the first-order approximation of MSE of the log-euclidean estimator is equal up to a factor $1/a^2$, to the MSE of the euclidean estimator in their in their respective geometries (3.4), the first-order optimal sample allocation M^* of $\hat{V}_L^{LML}$ is the same as for $\hat{V}_L^{ML}$. Note that due to this $1/a^2$ factor, the minimum of (3.43) is not the same as the minimum of $\mathbb{E}[|\hat{V}_L^{ML} - V|^2]$. But M^* that minimise the first one are the same that minimise the second one. We will introduce the optimal allocation sample in the subsection 4.3.

Remark 3.6. By looking closely to (3.38), we found a different way to write and prove the results of Proposition 3.4 established in [6].

Proposition 3.7 (MSE in the log-Euclidean metric of the Log-MLMC estimator). *Assume that $|\hat{V}_L - V| < h$ and $|\log \hat{V}_L - \log V| < h$ with $h \in \mathbb{R}^*$. Let η_0 be such that $\forall M_0 > \eta_0$*

$$\begin{cases} |\hat{V}_0^{(0)} - V_0| < h \\ |\log \hat{V}_0^{(0)} - \log V_0| < h \end{cases} \qquad \textit{almost surely}$$

and, $\forall \ell \in \{1, \ldots, L\}$, let η_ℓ be such that $\forall M_\ell > \eta_\ell$

$$\begin{cases} |\hat{V}_\ell^{(\ell)} - V_\ell| < h \\ |\log \hat{V}_\ell^{(\ell)} - \log V_\ell| < h \\ |\hat{V}_{\ell-1}^{(\ell)} - V_{\ell-1}| < h \\ |\log \hat{V}_{\ell-1}^{(\ell)} - \log V_{\ell-1}| < h \end{cases} \qquad \textit{almost surely}$$

Then, for sufficiently large samples $M_\ell > \eta_\ell$ with $\ell \in \{0, \dots, L\}$, the MSE in the log-Euclidean metric of the Log-MLMC estimator defined in (3.4) is

$$\mathbb{E}\left[d_{LE}\left(\hat{V}_L^{LML}, V\right)^2\right] = \mathbb{E}\left[\left|\log \hat{V}_L^{LML} - \log V\right|^2\right]$$
$$= \mathbb{E}\left[\left|\hat{V}_L^{ML} - V\right|^2\right] + \mathcal{O}\left(h^2\right)$$

Proof. **Existence of $\boldsymbol{\eta} = \boldsymbol{\eta_0}, \dots, \boldsymbol{\eta_L}$**
Let $\ell \in \{0, \dots, L\}$. By definition of our estimators, since $V_\ell < \infty$, the strong low of large numbers (Kolmogorov's law) implies almost surely that $\forall \varepsilon = \varepsilon_0, \dots, \varepsilon_L > 0$, $\exists \eta = \eta_0, \dots, \eta_L \in \mathbb{N}^{N+1}$ such as for all $M_\ell > \eta_\ell$:

$$\begin{cases} |\hat{V}_0^{(0)} - V_0| < \varepsilon_0 \\ |\log \hat{V}_0^{(0)} - \log V_0| < \varepsilon_\ell \\ |\hat{V}_\ell^{(\ell)} - V_\ell| < \varepsilon_\ell \\ |\log \hat{V}_\ell^{(\ell)} - \log V_\ell| < \varepsilon_\ell \\ |\hat{V}_{\ell-1}^{(\ell)} - V_{\ell-1}| < \varepsilon_{\ell-1} \\ |\log \hat{V}_{\ell-1}^{(\ell)} - \log V_{\ell-1}| < \varepsilon_{\ell-1} \end{cases}$$

By replacing $\varepsilon_\ell = h$, we have the assumptions and the existence of η.

Show that

$$\mathbb{E}\left[\left|\log(\hat{V}_L^{LML}) - \log(V)\right|^2\right] - \mathbb{E}\left[\left|\hat{V}_L^{ML} - V\right|^2\right] = \mathcal{O}\left(h^2\right)$$

By the triangular inequality:

$$\left|\left|\log(\hat{V}_L^{LML}) - \log(V)\right| - \left|\hat{V}_L^{ML} - V\right|\right| \leq \left|\log(\hat{V}_L^{LML}) - \log(V)\right| + \left|\hat{V}_L^{ML} - V\right| \tag{3.44}$$

Factoring using $a^2 - b^2 = (a-b)(a+b)$ and substituting (3.44) produces:

$$\begin{aligned} &\left|\left|\log(\hat{V}_L^{LML}) - \log(V)\right|^2 - \left|\hat{V}_L^{ML} - V\right|^2\right| \\ &= \left|\left|\log(\hat{V}_L^{LML}) - \log(V)\right| + \left|\hat{V}_L^{ML} - V\right|\right| \times \left|\left|\log(\hat{V}_L^{LML}) - \log(V)\right| - \left|\hat{V}_L^{ML} - V\right|\right| \\ &\leq \left(\left|\log(\hat{V}_L^{LML}) - \log(V)\right| + \left|\hat{V}_L^{ML} - V\right|\right)^2 \end{aligned} \tag{3.45}$$

Let's bound $|\hat{V}_L^{ML} - V|$:

$$\left|\hat{V}_L^{ML} - V\right| = \left|\hat{V}_L^{ML} - V_L + V_L - V\right| \leq |\hat{V}_L^{ML} - V_L| + |V_L - V| \tag{3.46}$$

Using the assumptions leads to:

$$\begin{aligned}
|\hat{V}_L^{ML} - V_L| &= \left|\hat{V}_0^{(0)} + \sum_{\ell=1}^{L}\left(\hat{V}_\ell^{(\ell)} - \hat{V}_{\ell-1}^{(\ell)}\right) - \left[V_0 + \sum_{\ell=1}^{L}(V_\ell - V_{\ell-1})\right]\right| \\
&= \left|\left(\hat{V}_0^{(0)} - V_0\right) + \sum_{\ell=1}^{L}\left[\left(\hat{V}_\ell^{(\ell)} - V_\ell\right) - \left(\hat{V}_{\ell-1}^{(\ell)} - V_{\ell-1}\right)\right]\right| \\
&\leq \left|\left|\hat{V}_0^{(0)} - V_0\right| + \sum_{\ell=1}^{L}\left[\left|\hat{V}_\ell^{(\ell)} - V_\ell\right| + \left|\hat{V}_{\ell-1}^{(\ell)} - V_{\ell-1}\right|\right]\right| \\
&< \left|h + \sum_{\ell=1}^{L}(h+h)\right| = (2L+1)h \qquad \text{almost surely}
\end{aligned} \tag{3.47}$$

Hence, (3.46) becomes:

$$\begin{aligned}
\left|\hat{V}_L^{ML} - V\right| &\leq |\hat{V}_L^{ML} - V_L| + |V_L - V| \\
&< (2L+1)h + h \quad \text{almost surely}
\end{aligned} \tag{3.48}$$

In the same way:

$$\left|\log \hat{V}_L^{LML} - \log V\right| \leq |\log \hat{V}_L^{LML} - \log V_L| + |\log V_L - \log V| \tag{3.49}$$

$$\begin{aligned}
\left|\log \hat{V}_L^{LML} - \log V_L\right| &= \left|\log \hat{V}_0^{(0)} + \sum_{\ell=1}^{L}\left(\log \hat{V}_\ell^{(\ell)} - \log \hat{V}_{\ell-1}^{(\ell)}\right) - \left[\log V_0 + \sum_{\ell=1}^{L}(\log V_\ell - \log V_{\ell-1})\right]\right| \\
&\leq \left|\left|\log \hat{V}_0^{(0)} - \log V_0\right| + \sum_{\ell=1}^{L}\left[\left|\log \hat{V}_\ell^{(\ell)} - \log V_\ell\right| + \left|\log \hat{V}_{\ell-1}^{(\ell)} - \log V_{\ell-1}\right|\right]\right| \\
&< \left|h + \sum_{\ell=1}^{L}(h+h)\right| \\
&= (2L+1)h
\end{aligned} \tag{3.50}$$

Hence, (3.49) becomes:

$$\begin{aligned}
\left|\log \hat{V}_L^{LML} - \log V\right| &\leq |\log \hat{V}_L^{LML} - \log V_L| + |\log V_L - \log V| \\
&< (2L+1)h + h \quad \text{almost surely}
\end{aligned} \tag{3.51}$$

By applying (3.48) and (3.51) in (3.45), we derive:

$$\left|\left|\log(\hat{V}_L^{LML}) - \log(V)\right|^2 - \left|\hat{V}_L^{ML} - V\right|^2\right| \leq \left(\left|\log(\hat{V}_L^{LML}) - \log(V)\right| + \left|\hat{V}_L^{ML} - V\right|\right)^2 < Ch^2 \tag{3.52}$$

with $C = (4L+4)^2 \in \mathbb{R}_+$.

Taking the expectation and using the linearity of the expectation ($\mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[X+Y]$) leads to:

$$\begin{aligned}&\left|\mathbb{E}\left[\left|\log(\hat{V}_L^{LML}) - \log(V)\right|^2\right] - \mathbb{E}\left[\left|\hat{V}_L^{ML} - V\right|^2\right]\right| \\ &= \left|\mathbb{E}\left[\left|\log(\hat{V}_L^{LML}) - \log(V)\right|^2 - \left|\hat{V}_L^{ML} - V\right|^2\right]\right| \\ &\leq Ch^2\end{aligned} \tag{3.53}$$

Since $C > 0$, (3.53) is equivalent to:

$$\mathbb{E}\left[\left|\log(\hat{V}_L^{LML}) - \log(V)\right|^2\right] - \mathbb{E}\left[\left|\hat{V}_L^{ML} - V\right|^2\right] = \mathcal{O}\left(h^2\right) \tag{3.54}$$

so that

$$\mathbb{E}\left[\left|\log(\hat{V}_L^{LML}) - \log(V)\right|^2\right] = \mathbb{E}\left[\left|\hat{V}_L^{ML} - V\right|^2\right] + \mathcal{O}\left(h^2\right) \tag{3.55}$$

■

Remark 3.8. Proposition 3.4 and Proposition 3.7 present the mean squared error (MSE) of the estimator (3.4) within the context of the log-Euclidean metric. A direct comparison with the MSE of the estimator (3.1) is not feasible, as the latter is defined within the Euclidean metric framework.

In the subsequent section, we will define the mathematical constructs necessary for the numerical implementation of the two MLMC estimators discussed in section 3. This will be followed by a presentation of the experimental results comparing the mean squared error (MSE), bias, and variance of the estimators (3.1) and (3.4).

4. Mathematical constructs for MLMC Estimator Implementation. We want to implement the two MLMC Estimator of the variance as presented in (3.1) and (3.4) with an abstract numerical simulator described by the function f that returns a scalar value, defined as:

$$\begin{aligned} f : \mathbb{R}^d &\to \mathbb{R} \\ x &\mapsto v^{\mathrm{T}} x, \end{aligned} \tag{4.1}$$

with

$$x = \begin{bmatrix} x_1 & x_2 & \dots & x_d \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^d,$$

and

$$v = \begin{bmatrix} v_1 & v_2 & \cdots & v_d \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^d.$$

We are interested in the random variable $Y = f \circ X = f(X) : \Omega \to \mathbb{R}$ where $X = \begin{bmatrix} X_1 & \dots & X_d \end{bmatrix}^{\mathrm{T}} : \Omega \to \mathbb{R}^d$ is a random vector i.i.d. components.

4.1. Multilevel introduction.

In the Classic Monte Carlo method, we only use one numerical simulator that is usually high in fidelity and has a high cost. On the other hand, in the Multilevel Monte Carlo method, we use different fidelity simulators that has different costs in addition to the high fidelity one. It allows us to lower the number of high fidelity simulations and hence, to reduce the total cost or, at equivalent cost, enhance the precision of our estimator [4] [5].

Let $\ell \in \{0, \dots, L\}$ be our fidelity indices, for our numerical simulators f_ℓ. The fidelity increases with the indices, which means that f_0 is our lowest fidelity simulator and f_L is our highest (usually $f_L \equiv f$ in the classic Monte Carlo method). The more these f_ℓ are correlated, the more we gain in cost and precision. (**Add some citation**) For each level of fidelity, we decide to run M_ℓ simulations with $M_\ell \geq M_{\ell+1} \geq d$. These M_ℓ can be optimise depending on our problems parameters ([7], 2.39).

For each f_ℓ, we then have a vector $v_\ell = \begin{bmatrix} v_{\ell,1} & v_{\ell,2} & \cdots & v_{\ell,d} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^d$ with each v_ℓ highly correlated one to another.

4.2. Multilevel example.

Instead of redoing what has been done in [7], we will focus on giving a concrete example with a two-level Monte Carlo in the Euclidean geometry.

Let $v_1 \in \mathbb{R}^d$ for our high fidelity simulation function f_1 (previous f). We now have a low fidelity simulation function f_0 based on the high fidelity function f_1 with $v_0 = v_1 + \varepsilon$, $\varepsilon \sim \mathcal{N}_d(0_d, \sigma^2 I_d)$ and $\sigma^2 = 0.01$ or 0.1.

For each multilevel estimation, we generate these random vectors:

$$X^{(1,i)} = \begin{bmatrix} X_1^{(1,i)} & X_2^{(1,i)} & \dots & X_d^{(1,i)} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^d, \quad i \in \{1, \dots, M_1\}$$

$$X^{(0,i)} = \begin{bmatrix} X_1^{(0,i)} & X_2^{(0,i)} & \dots & X_d^{(0,i)} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^d, \quad i \in \{1, \dots, M_0\}$$

This two sequences of random vectors are i.i.d and follow the normal distribution $\mathcal{N}_d(0, 1)$.

Then for each multilevel estimation, with $M_0 \geq M_1$, we compute M_0 and M_1 low fidelity simulations

$$\begin{cases} Y_0^{(0)} = f_0\left(X^{(0)}\right) \\ Y_0^{(1)} = f_0\left(X^{(1)}\right) \end{cases}$$

and M_1 high fidelity simulations

$$Y_1^{(1)} = f_1(X^{(1)}).$$

In Euclidean multilevel Monte Carlo, we have for the expectation:

$$\begin{aligned} \hat{E}_1^{ML}[Y] &= \frac{1}{M_0}\sum_{i=1}^{M_0} Y_0^{(0,i)} + \frac{1}{M_1}\sum_{i=1}^{M_1}\left[Y_1^{(1,i)} - Y_0^{(1,i)}\right] \\ &= \frac{1}{M_0}\sum_{i=1}^{M_0} f_0\left(X^{(0,i)}\right) + \frac{1}{M_1}\sum_{i=1}^{M_1}\left[f_1\left(X^{(1,i)}\right) - f_0\left(X^{(1,i)}\right)\right] \end{aligned}$$

and for the variance (3.1):

$$\begin{aligned} \hat{V}_1^{ML}[Y] &= \hat{V}_{M_0}^{(0)}[Y_0] + \hat{V}_{M_1}^{(1)}[Y_1] - \hat{V}_{M_1}^{(1)}[Y_0] \\ &= \frac{1}{M_0-1}\sum_{i=1}^{M_0}\left[\left(Y_0^{(0,i)} - \frac{1}{M_0}\sum_{i=1}^{M_0}\left[Y_\ell^{(0,i)}\right]\right)^2\right] \\ &\quad + \frac{1}{M_1-1}\sum_{i=1}^{M_1}\left[\left(Y_1^{(1,i)} - \frac{1}{M_1}\sum_{i=1}^{M_1}\left[Y_\ell^{(1,i)}\right]\right)^2 - \left(Y_0^{(1,i)} - \frac{1}{M_1}\sum_{i=1}^{M_1}\left[Y_0^{(1,i)}\right]\right)^2\right] \end{aligned}$$

In the log-Euclidean metric, we have for the variance (3.4):

$$\begin{aligned} \hat{V}_1^{LML}[Y] &= \exp\left(\log \hat{V}_{M_0}^{(0)}[Y_0] + \log \hat{V}_{M_1}^{(1)}[Y_1] - \log \hat{V}_{M_1}^{(1)}[Y_0]\right) \\ &= \exp\left(\log\left(\frac{1}{M_0-1}\sum_{i=1}^{M_0}\left[\left(Y_0^{(0,i)} - \frac{1}{M_0}\sum_{i=1}^{M_0}\left[Y_\ell^{(0,i)}\right]\right)^2\right]\right)\right. \\ &\qquad + \log\left(\frac{1}{M_1-1}\sum_{i=1}^{M_1}\left[\left(Y_1^{(1,i)} - \frac{1}{M_1}\sum_{i=1}^{M_1}\left[Y_\ell^{(1,i)}\right]\right)^2\right]\right) \\ &\qquad \left. - \log\left(\frac{1}{M_1-1}\sum_{i=1}^{M_1}\left[\left(Y_0^{(1,i)} - \frac{1}{M_1}\sum_{i=1}^{M_1}\left[Y_0^{(1,i)}\right]\right)^2\right]\right)\right) \end{aligned}$$

4.3. Target cost strategy. In order to find optimum allocation sample M_ℓ for a given budget B, we must [7]:

- Compute $\mathcal{V}_0 = \mathbb{V}[f_0(X)^2]$ and $\mathcal{V}_\ell = \mathbb{V}[f_\ell(X)^2 - f_{\ell-1}(X)^2]$ for $\ell = 1, \ldots, L$
    - Estimated with a pilot sample of size $n = 10000$, independant of the rest
- Define the cost of each level. It is common to take $\mathcal{C}_\ell = 2\mathcal{C}_{\ell-1}$ with $\mathcal{C}_L = 1$

Then we can compute for each $\ell \in \{0, \ldots, \ell\}$

$$M_\ell = 1 + \lfloor (B/\mathcal{S}_L)\sqrt{\mathcal{V}_\ell/\mathcal{C}_\ell} \rfloor$$

with

$$\mathcal{S}_L = \sum_{\ell \leq L} \sqrt{\mathcal{V}_\ell \mathcal{C}_\ell}$$

5. Experimental Comparison of MLMC Estimators. In this section, we present the numerical results of the Multilevel Monte Carlo (MLMC) method for estimating variance in log-Euclidean geometry framework. All numerical result in this section are based on the model presented in subsection 4.2.

5.1. Bias Comparison as a Function of Buget. Figure 1 compares the bias of the variance estimates $\hat{V}_L^{ML}$ (using Euclidean geometry) and $\hat{V}_L^{LML}$ (using log-Euclidean geometry) as a function of the budget η. These box-plots show the bias in Euclidean metric for various budget levels on a logarithmic scale.

Figure 1 reveals a notable difference in the evolution of the bias of the two estimators. Indeed, the bias of $\hat{V}_L^{ML}$ fluctuates slightly around 10^{-5}, which confirms that $\hat{V}_L^{ML}$ is unbiased by construction. However, $\hat{V}_L^{LML}$ decreases as the budget B increases, which indicates that the estimator is asymptotically unbiased.

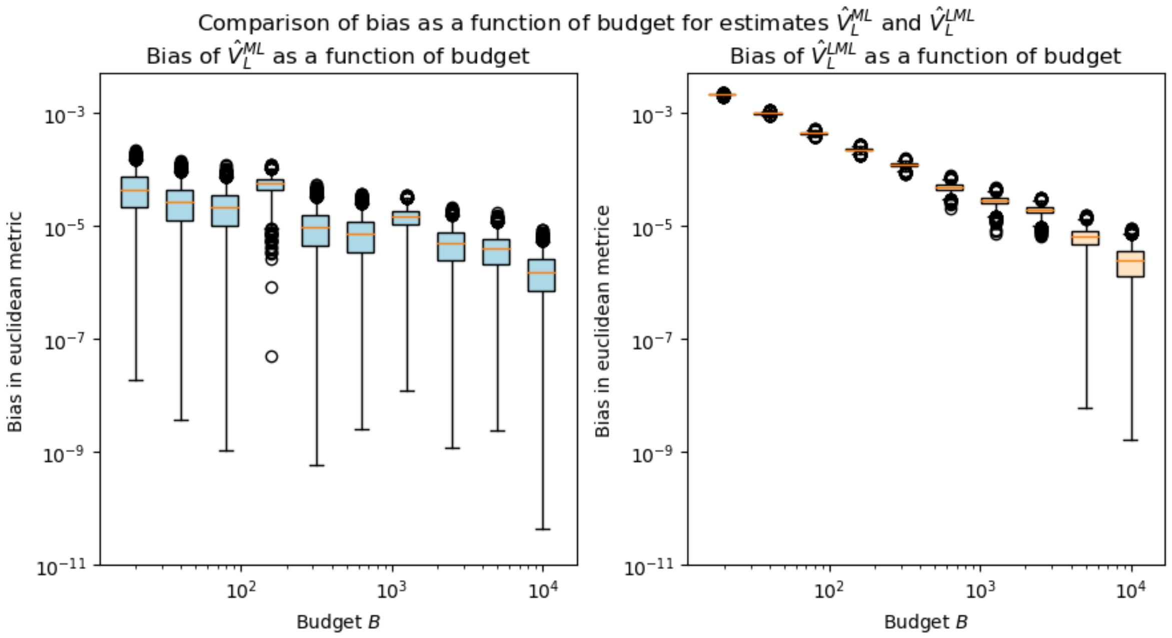

Figure 1. *Comparison of bias as a function of budget for the estimators $\hat{V}_L^{ML}$ and $\hat{V}_L^{LML}$. The bias for each estimator was computed from 1,000,000 realisations. Subsequently, the bootstrap method was employed to obtain the distribution of the bias, resulting in the boxplots presented. A total of 10,000 bootstrap samples, each comprising 1,000,000 estimates of $\hat{V}_L^{ML}$ and $\hat{V}_L^{LML}$, were generated.*

5.2. Distribution of Variance Estimates for Low Budget. Figure 2 compares the distribution of the variances for a low budget. The histogram of the distribution of $\hat{V}_L^{LML}$ confirms that this estimator ensures the positivity of all variance estimate. Nevertheless, on average, for low budgets, the difference between the estimator and the expected variance is greater for $\hat{V}_L^{LML}$ than for $\hat{V}_L^{ML}$.

Indeed for low budgets, Figure 1 illustrates that while $\hat{V}_L^{ML}$ estimator is unbiased, $\hat{V}_L^{LML}$ is biased. This issue may present challenges in certain practical applications and should be thoroughly evaluated before selecting this estimator.

Moreover, the results presented in Table 1 provide a comparative analysis of the bias and the rate of negative values for our estimators across different budget levels. For lower budget scenarios, the estimator $\hat{V}_L^{ML}$ exhibits an insignificant bias but produces approximately 2.5% negative estimates. Conversely, the estimator $\hat{V}_L^{LML}$ avoids negative values entirely, though it demonstrates a slightly higher bias compared to $\hat{V}_L^{ML}$. In higher budget contexts, neither estimator yields negative values, and the bias of $\hat{V}_L^{ML}$ remains smaller than that of $\hat{V}_L^{LML}$.

Although these findings are specific to the practical case discussed in subsection 4.2, it is important to note that there are instances where, for a given budget ($B = 500$), both estimators produce non-negative estimates, yet $\hat{V}_L^{LML}$ still exhibits a noticeable bias.

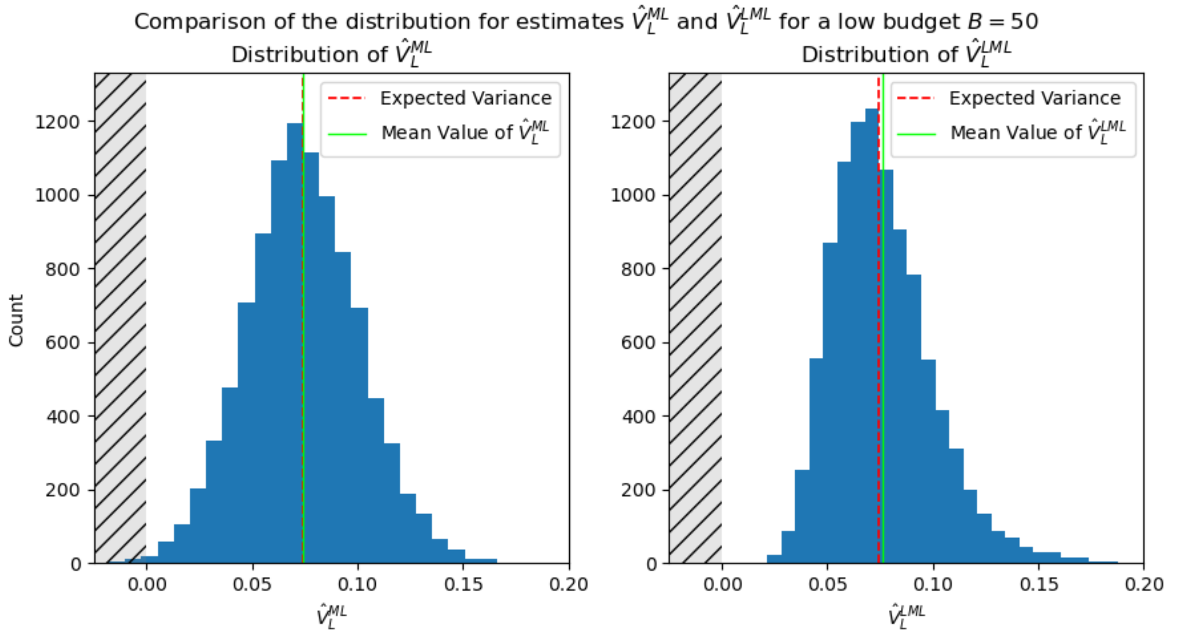

Figure 2. *Comparison of the distribution for the estimators $\hat{V}_L^{ML}$ and $\hat{V}_L^{LML}$ for a low budget $B = 50$. 10,000 realisations have been made. The red dashed line indicates the expected variance, while the green line represents the mean value of the estimates. The gray shaded highlights negative estimates.*

Budget B	50	100	500
Absolute bias of $\hat{V}_L^{ML}$	2.37e-04	1.72e-05	1.06e-05
Proportion of negative values $\hat{V}_L^{ML}$	25.55‰	0.42‰	0
Absolute bias of $\hat{V}_L^{LML}$	1.63e-03	6.49e-04	1.38e-04

Table 1

Comparison of the absolute bias and the proportion of negative values for the estimators $\hat{V}_L^{ML}$ and $\hat{V}_L^{LML}$ across different budget levels (B). Each metric was computed based on 1,000,000 estimates.

5.3. Comparison of Mean Squared Error (MSE). Figure 3 presents the Mean Squared Error (MSE) in the log-Euclidean metric for the estimator $\hat{V}_L^{LML}$ across budgets ranging from 10 to 10^7. The figure clearly illustrates that the MSE value decreases exponentially as the budget increases. This indicates that the estimator $\hat{V}_L^{LML}$ converges asymptotically to the theoretical value of the variance it aims to estimate.

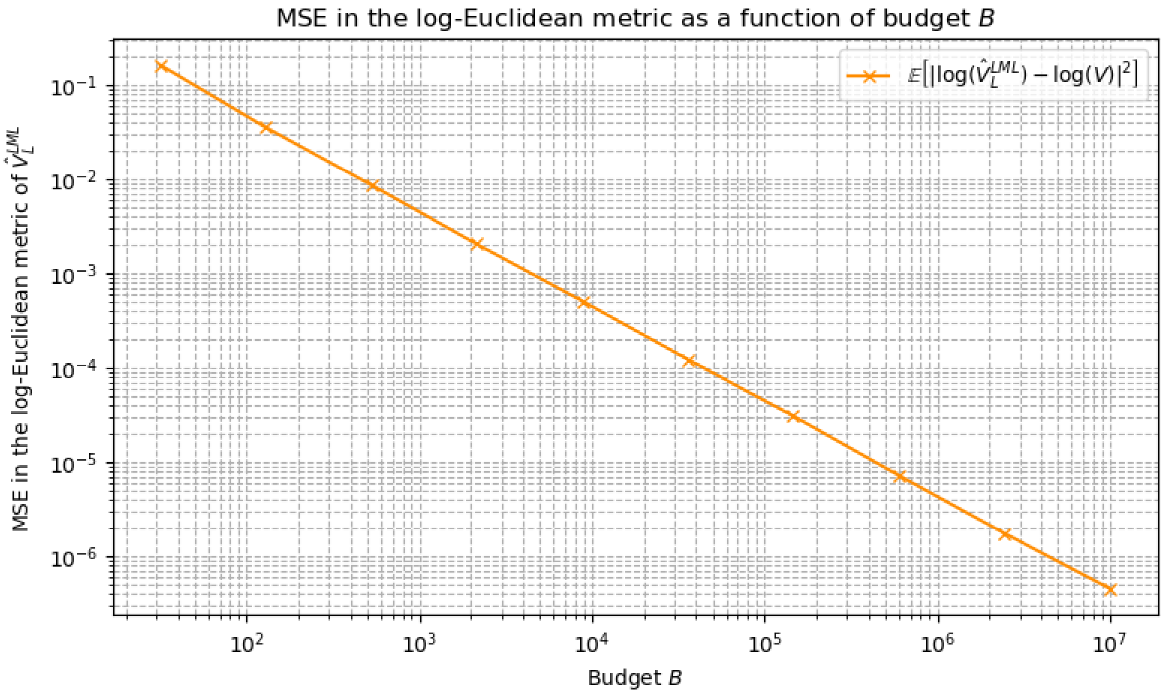

Figure 3. *Mean Squared Error (MSE) in the log-Euclidean metric for the estimator $\hat{V}_L^{LML}$ as a function of budget B. The plot demonstrates that the MSE decreases exponentially with increasing budget, indicating that the estimator $\hat{V}_L^{LML}$ converges asymptotically to the theoretical variance. Each point of the graph was computed based on 100,000 estimates.*

Figure 4 illustrates the findings presented in Proposition 3.4 and Proposition 3.7. This figure indicates that the difference between the MSE in the log-Euclidean metric for the estimator $\hat{V}_L^{LML}$ and the MSE in the Euclidean metric for the estimator $\hat{V}_L^{ML}$ decreases quadratically as the budget allocation increases. This observation is consistent with the $\mathcal{O}(h^3)$ convergence rate established in Proposition 3.4 and the $\mathcal{O}(h^2)$ rate presented in Proposition 3.7. This leads to the minimisation of the first-order MSE of $\hat{V}_L^{LML}$ and the determination of the optimal sample allocation M^*, which coincides with the sample allocation for $\hat{V}_L^{ML}$.

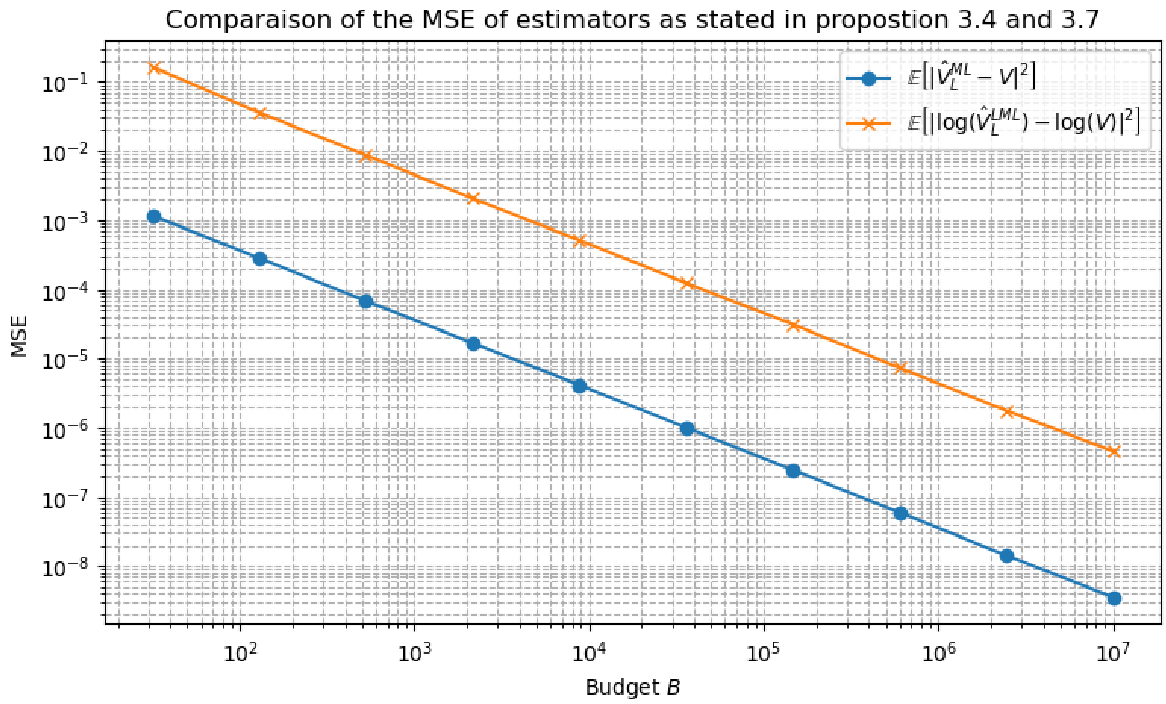

Figure 4. *Comparison of the MSE in the Euclidean metric for the estimator $\hat{V}_L^{ML}$ and the MSE in the log-Euclidean metric for the estimator $\hat{V}_L^{LML}$ as a function of budget B. Both MSE decreases exponentially with the budget and their difference decreases as well exponentially. Each point of the graph was computed based on 100,000 estimates.*

Figure 5 reveals a small difference between the MSE in Euclidean metric of our two estimators. We see that the MSE of $\hat{V}_L^{LML}$ is slightly smaller, and therefore converges faster. Separation occurs at low budget values, so once again we see that $\hat{V}_L^{LML}$ is interesting relative to $\hat{V}_L^{ML}$ at low budgets.

The analysis reveals that the MSE of $\hat{V}_L^{LML}$ is slightly smaller, indicating a faster convergence rate. This distinction becomes apparent at lower budget values, demonstrating that $\hat{V}_L^{LML}$ is particularly advantageous compared to $\hat{V}_L^{ML}$ in low budget scenarios.

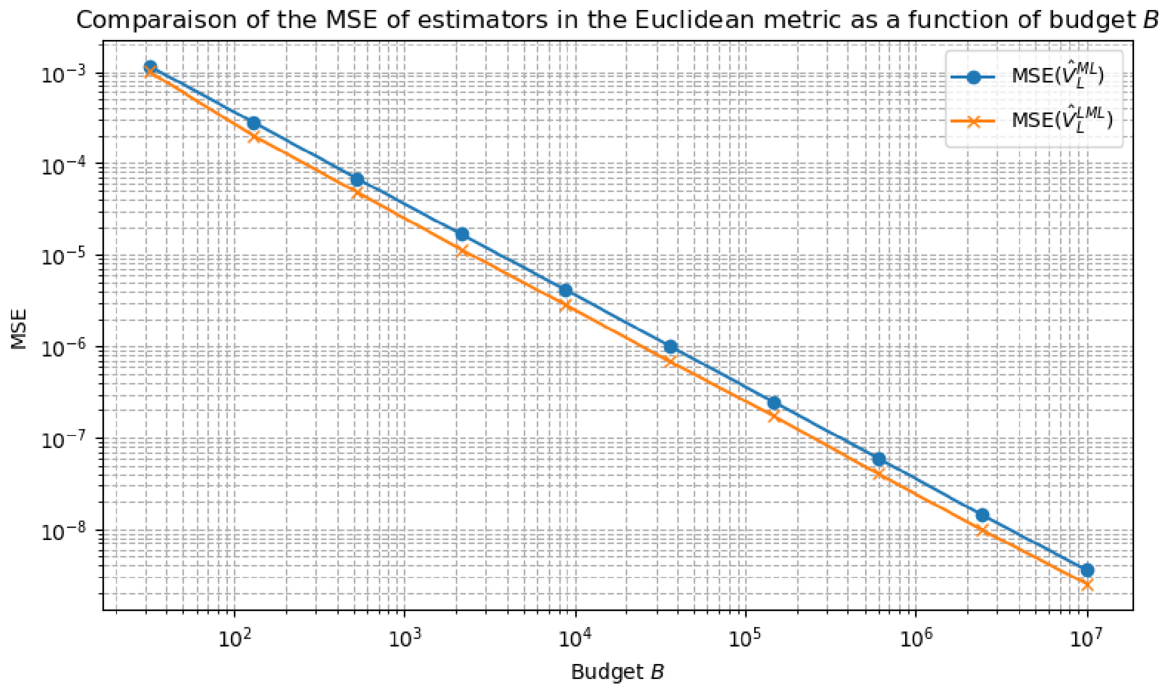

Figure 5. *Comparison of the MSE for the estimators $\hat{V}_L^{ML}$ and $\hat{V}_L^{LML}$ in the Euclidean metric as a function of budget B. Both MSE decreases exponentially with the budget but the MSE of the $\hat{V}^{LML}$ estimator is slightly better than the MSE of $\hat{V}_L^{ML}$ in the Euclidean metric. Each point of the graph was computed based on 100,000 estimates.*

5.4. Comparison of the Variances.

Figure 6 presents an insightful analysis of the variance of the two estimators. $\hat{V}_L^{LML}$'s variance is lower than $\hat{V}_L^{ML}$'s variance, even with low budget. Furthermore, Figure 5 and Figure 6 reveals a notable similarity between the comportment of the MSE and the variance of our estimators. Indeed as $\mathrm{MSE}(\hat{\theta}) = \mathbb{V}(\hat{\theta}) + \mathrm{Bias}(\hat{\theta}, \theta)^2$ for an estimator $\hat{\theta}$ of θ and since the bias is small relative to the variance, practical use must take into account the variance factor of the estimators to a considerable extent.

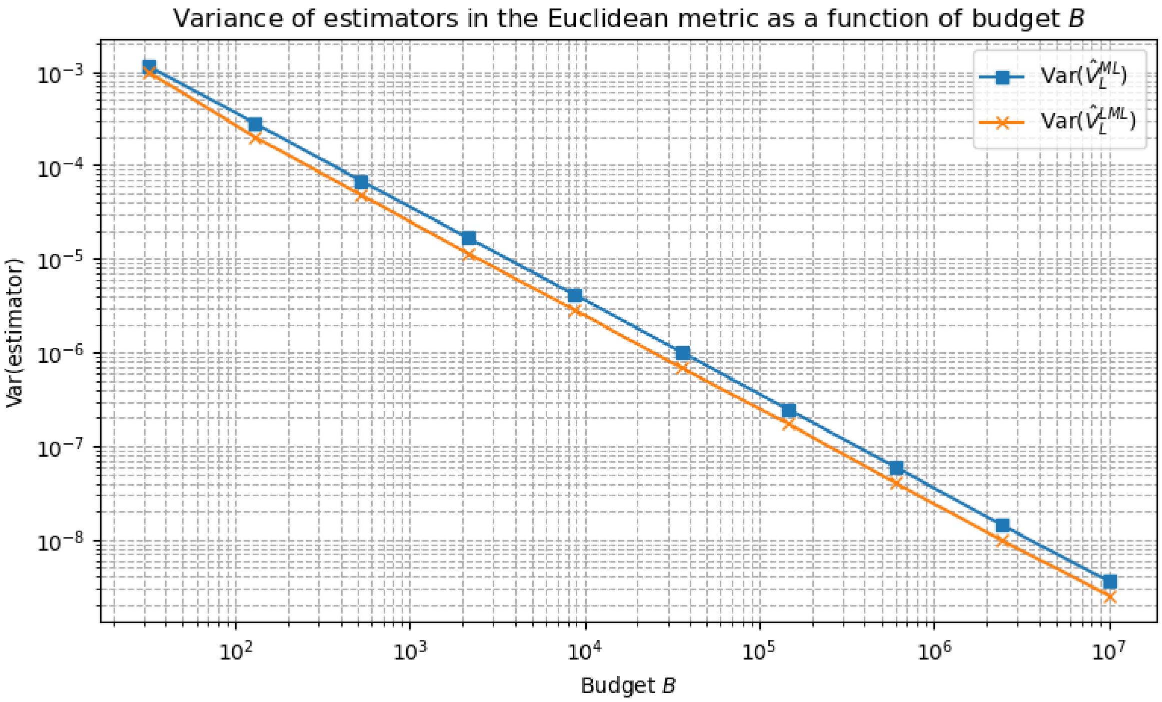

Figure 6. *Comparison of the variance for the estimators $\hat{V}_L^{ML}$ and $\hat{V}_L^{LML}$ in the Euclidean metric as a function of budget B. These two variances were estimated on 100,000 estimates at each budget point. This graph is similar to Figure 5 due to the MSE properties in Euclidean geometry.*

6. Conclusions.

This study aimed to have further information in theory and benchmarking of the estimator Multi-Level Monte Carlo (MLMC) of the variance in the log-Euclidean geometry, introduced in [6] to deal with negativity estimations using MLMC in the Euclidean metric. Our main contribution was to clear up the proof of the Proposition 3.4 (originally presented in [6]) under conditions where the theoretical variance $\mathbb{V}[Y]$ significantly deviates from 1. Maurais suggested a rescaling to meet the condition $|\mathbb{V}[\tilde{Y}] - 1| < h/4$ but did not provide any further details. In the proof of Proposition 3.4, we meticulously addressed this rescaling and developed the key passages that required more detail. Additionally, we discovered an alternative formulation for this proposition, which significantly simplifies the proof (see Proposition 3.7).

Using these propositions, we were able to apply the same target cost strategy for optimal allocation sample to the $\hat{V}_L^{LML}$ estimator as that used for the $\hat{V}_L^{ML}$ estimator. This allowed us to compute numerical estimations for direct comparison between the two estimators.

The experimental results presented in the section 5 demonstrate the comparative performance of the MLMC estimators, $\hat{V}_L^{ML}$ and $\hat{V}_L^{LML}$, for different values of the budget. The bias analysis revealed that while $\hat{V}_L^{ML}$ is unbiased by construction, it occasionally produces negative variance estimates at low budgets. In contrast, $\hat{V}_L^{LML}$ ensures positive variance estimates but at the cost of higher bias in low-budget scenarios, indicating asymptotic unbiasedness.

This trade-off should be carefully considered depending on the application requirements.

Furthermore, the Mean Squared Error (MSE) analysis confirmed that both estimators exhibit exponential decay in error with increasing budget, but $\hat{V}_L^{LML}$ displayed a slightly faster convergence rate. This is particularly advantageous in low-budget scenarios, where $\hat{V}_L^{LML}$ outperforms $\hat{V}_L^{ML}$ in terms of MSE, despite its initial bias.

Lastly, the variance comparisons aligned with the MSE findings, highlighting that the variance of $\hat{V}_L^{LML}$ is consistently lower than that of $\hat{V}_L^{ML}$, reinforcing its robustness in practical applications. These results emphasise the suitability of $\hat{V}_L^{LML}$ for scenarios where ensuring positive variance estimates is crucial, such as climate modelling, Gaussian Processes and risk analysis. These findings also underscore the importance of considering both bias and variance when evaluating estimator performance.

Future work includes exploring application of this method to datasets of different nature, and integrating the Multi-Level approach with methods of variance and covariance estimation, such as shrinkage. This could also extend to high-dimension covariance estimation.

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Appendix A. Code Availability. Python Implementation of Euclidean and log-Euclidean MLMC estimators are available at: https://github.com/Dorianhgn/Log-Euclidean-MLMC-Estimator

REFERENCES

[1] K. Akbudak, H. Ltaief, A. Mikhalev, and D. Keyes, *Tile low rank cholesky factorization for climate/weather modeling applications on manycore architectures*, in International Conference on High Performance Computing, Springer, 2017, pp. 22–40.

[2] V. Arsigny, P. Fillard, X. Pennec, and N. Ayache, *Log-euclidean metrics for fast and simple calculus on diffusion tensors*, Magnetic Resonance in Medicine: An Official Journal of the International Society for Magnetic Resonance in Medicine, 56 (2006), pp. 411–421.

[3] C. Bekas, A. Curioni, and I. Fedulova, *Low cost high performance uncertainty quantification*, in Proceedings of the 2nd Workshop on High Performance Computational Finance, 2009, pp. 1–8.

[4] M. B. Giles, *Multilevel monte carlo path simulation*, Operations Research, 56 (2008), pp. 607–617, https://doi.org/10.1287/opre.1070.0496, https://doi.org/10.1287/opre.1070.0496, https://arxiv.org/abs/https://doi.org/10.1287/opre.1070.0496.

[5] M. B. Giles, *Multilevel monte carlo methods*, Acta numerica, 24 (2015), pp. 259–328.

[6] A. Maurais, T. Alsup, B. Peherstorfer, and Y. Marzouk, *Multi-fidelity covariance estimation in the log-euclidean geometry*, 2023, https://arxiv.org/abs/2301.13749.

[7] P. Mycek and M. de Lozzo, *Multilevel Monte Carlo covariance estimation for the computation of Sobol' indices*, SIAM/ASA Journal on Uncertainty Quantification, 7 (2019), pp. pp. 1323–1348, https://doi.org/10.1137/18M1216389, https://hal.science/hal-01894503.

[8] J. Nitzler, J. Biehler, N. Fehn, P.-S. Koutsourelakis, and W. A. Wall, *A generalized probabilistic learning approach for multi-fidelity uncertainty propagation in complex physical simulations*, 2022, https://arxiv.org/abs/2001.02892.