Quasisymmetric invariant for families of posets

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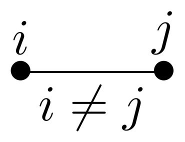
Many conjectures

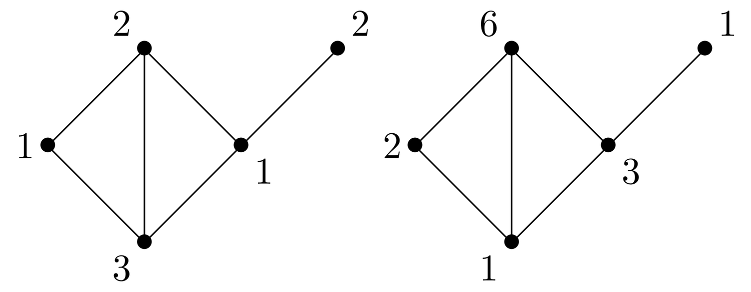
Invariant: map $\phi : \mathscr{C} \to H$ s.t. $A \sim B \Rightarrow \phi(A) = \phi(B)$.

Graphs



$i \neq j$

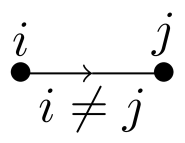
Chromatic symmetric functions $X_G \in \mathrm{Sym}$



$x_1^2x_2^3x_3^1 \quad + \quad x_1^2x_2^1x_3^1x_6^1 \ + \dots$

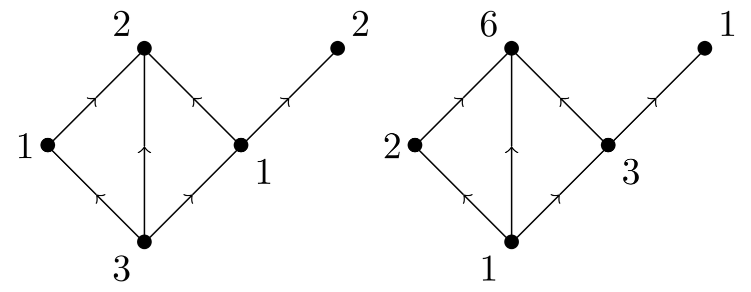
CONJ. [Sta95]: X distinguishes trees

(Acyclic) digraphs



$i \neq j$

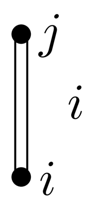
Chromatic quasisymmetric functions $\vec{X}_G \in \mathrm{QSym}[t]$ (t counts ascents)



$x_1^2x_2^3x_3^1 \cdot t^3 + \ x_1^2x_2^1x_3^1x_6^1 \cdot t^5 + \dots$

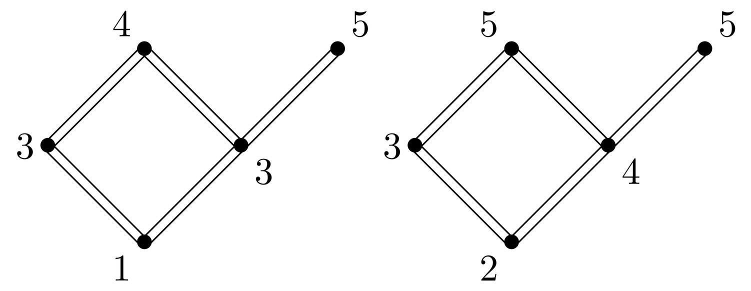
CONJ. [AS21]: $\vec{X}$ distinguishes oriented trees

Posets



$i < j$

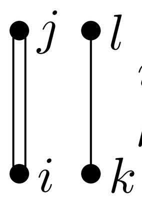
Strict partition enumerator $\bar{K}_P \in \mathrm{QSym}$ (lead. coeff. of $\vec{X}_G$)



$x_1^1x_3^2x_4^1x_5^1 \quad + \quad x_2^1x_3^1x_4^1x_5^2 \ + \dots$

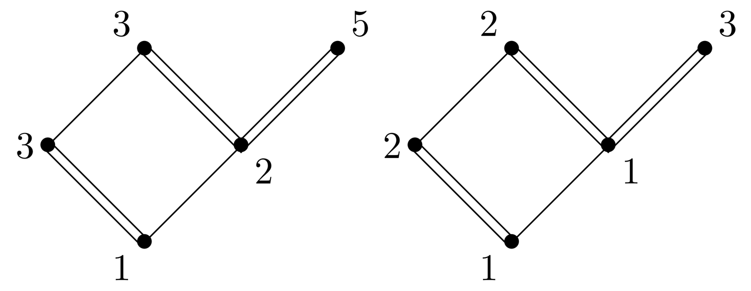
CONJ. [HT17]: $\bar{K}$ distinguishes trees

Decorated posets



$i < j$
$k \leq l$

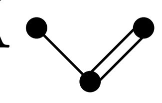
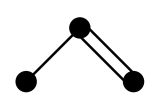
Partition enumerator $K_{P,\omega} \in \mathrm{QSym}$



$x_1^1x_2^1x_3^2x_5^1 \quad + \quad x_1^2x_2^2x_3^1 \quad + \dots$

CONJ. [ADM23+]: K distinguishes rooted trees

A hard truth

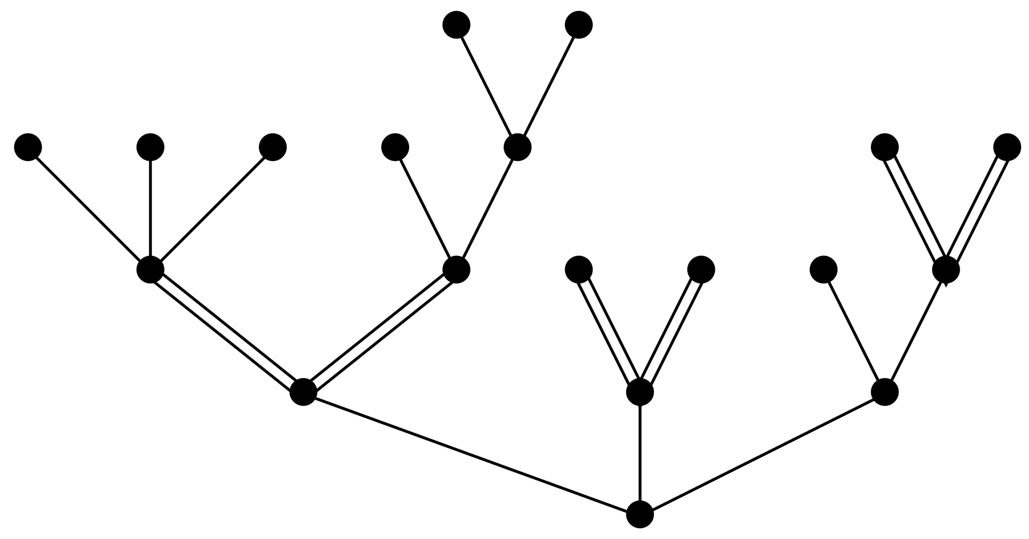
K  $=$ K 

Some results

$\bar{K}$ distinguishes:

- bowtie and N-free posets,
- width 2 posets,
- rooted trees,
- series-parallel posets,
- ...

THM. [ADM23+]:
K distinguishes fair trees: same type of edge with all children.



First result on decorated posets.

Hopf algebras

FQSym $:=$ non-commutative Hopf algebra where monomials with same standardization of indices have same coefficient.
Fundamental basis: $\mathbb{F}_{231} = a_2a_3a_1 + a_4a_7a_2 + a_{32}a_{44}a_{17} + \dots$
$\mathbb{F}_{12} \cdot \mathbb{F}_{2|1} = \mathbb{F}_{124|3} + \mathbb{F}_{14|23} + \mathbb{F}_{4|123} + \mathbb{F}_{14|32} + \mathbb{F}_{4|13|2} + \mathbb{F}_{4|3|12}$.

QSym $:=$ polynomial Hopf algebra where monomials with same signature have same coefficient.
Monomial basis: $M_{131} = x_1^1x_2^3x_3^1 + x_2^1x_4^3x_5^1 + x_{12}^1x_{42}^3x_{77}^1 + \dots$
Fundamental basis : $F_\alpha = \sum_{\beta \leq \alpha} M_\beta$.
$F_2 \cdot F_{11} = F_{31} + F_{22} + F_{13} + F_{22} + F_{121} + F_{112}$.

THM. [Sta71]:
Label P with $[n]$ such that labels increase (resp. decrease) along simple (resp. double) edges. Then :

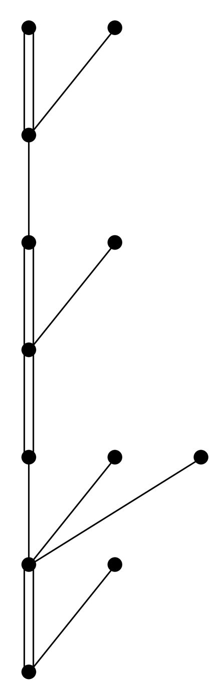
$$K_{P,\omega} = \sum_{\pi \in \mathrm{Lin}(P,\omega)} F_{\mathrm{des}(\pi)}$$

Cypress trees

$\mathrm{jump}(x)$ (resp. up-jump) $:=$ max number of double edges to get to a minimum (resp. maximum).

PROP. [LW20]:
The partition enumerator determines the joint distribution of the jumps and up-jumps.

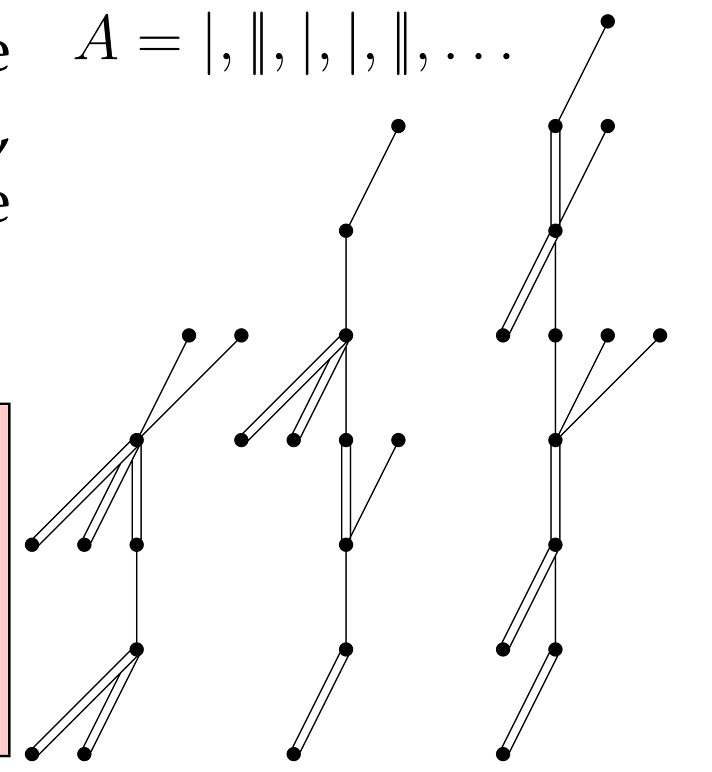
THM. [AAM23+]:
K distinguishes all cypress trees.



Centipedes

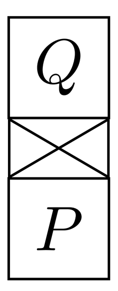
DEF.: Let $A \in \{|, \|\}^{\mathbb{N}}$. A A-centipede is a caterpillar poset with spine $A_{[n]}$, simple edges going up and double edges going down.

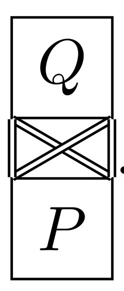
$A = |, \|, |, |, \|, \dots$



THM. [AAM23+]:
Fix $A \in \{|, \|\}^{\mathbb{N}}$. K distinguishes all A-centipedes *with some constraints at the top and bottom.*

Fair series-parallel posets

DEF.: $\bullet$ or $\boxed{P}\ \boxed{Q}$ or 

or 

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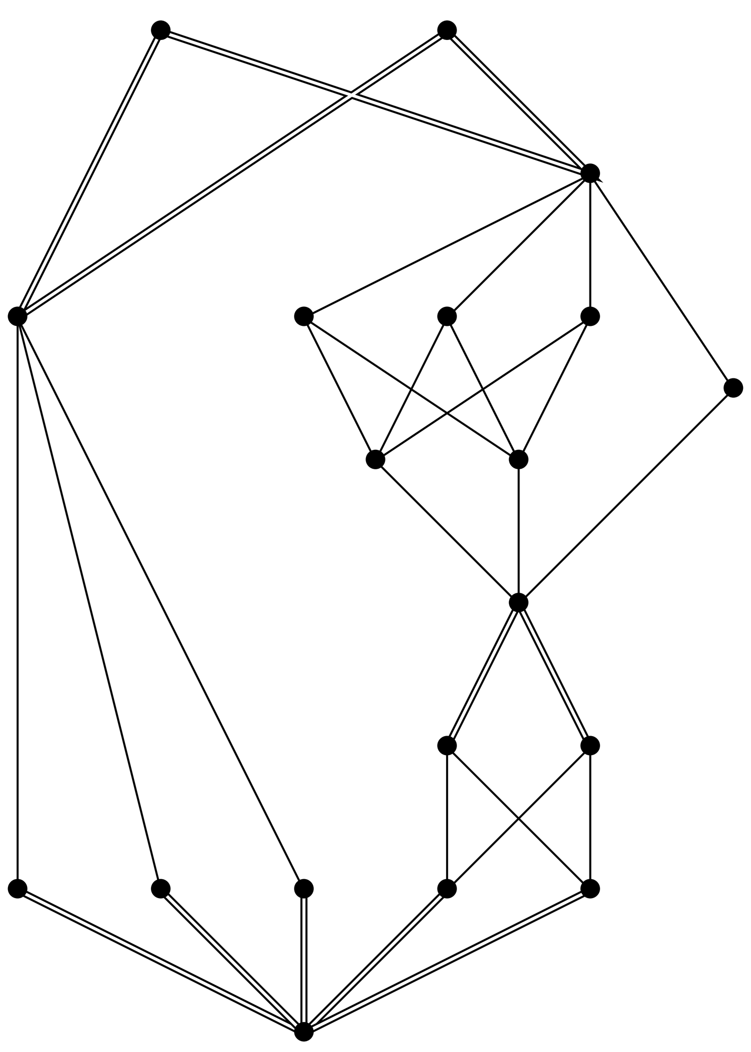
PROP. [AAM23+]:
Partition enumerators of **connected** fair series-parallel posets are **irreducible** in QSym.

THM. [AAM23+] :
K distinguishes fair series-parallel posets.



References

[AAM23+]: Albertin, Aval & Mlodecki, to be written.
[ADM23+]: Aval, Djenabou & McNamara, *Quasisymmetric functions distinguishing trees*.
[AS21]: Alexandersson & Sulzgruber, *P-partitions and p-positivity*.
[HT17]: Hasebe & Tsujie, *Order quasisymmetric functions distinguish rooted trees*.
[LW20]: Liu & Weselcouch, *P-partition generating function equivalence of naturally labeled posets*.
[Sta71]: Stanley, *Ordered structures and partitions*.
[Sta95]: Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*.