16-720 Computer Vision: 3D Reconstruction

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Q1.1

Since $\mathbf{p}_l = \mathbf{p}_r = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ and the relationship between $\mathbf{p}_l$ and $\mathbf{p}_r$ is $\mathbf{p}_r^T \mathbf{F} \mathbf{p}_l = 0$,

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow F_{33} = 0$$

Q1.2

Suppose the world coordinate frame is fixed to the first camera, so for the first camera,

$$\mathbf{R} = \mathbf{I} \qquad \mathbf{t}_l = \mathbf{0}$$

For the second camera, it only differs from the first camera by a pure translation, which are

$$\mathbf{R} = \mathbf{I} \qquad \mathbf{t}_r = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

Suppose the 2D point on the left image is $\mathbf{p}_l = \begin{pmatrix} x_l & y_l & 1 \end{pmatrix}^T$, the 2D point on the right image is $\mathbf{p}_r = \begin{pmatrix} x_r & y_r & 1 \end{pmatrix}^T$. Then for $\mathbf{p}_l$,

$$\mathbf{p}_l \equiv [\mathbf{I} \quad \mathbf{0}]\,\mathbf{P} \equiv \mathbf{P} \qquad \mathbf{p}_r \equiv [\mathbf{I} \quad \mathbf{t}_r]\,\mathbf{P}$$

$$\Rightarrow \mathbf{p}_r^T(\mathbf{t}_r \times \mathbf{I}\mathbf{p}_l) = 0 \qquad \Rightarrow \mathbf{p}_r^T(\mathbf{E}\mathbf{p}_l) = 0$$

$$\mathbf{E} = \mathbf{t}_r \times \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{pmatrix}$$

The epipolar line where $\mathbf{p}_l$ lies is

$$l_l = \mathbf{E}\mathbf{p}_l = \begin{pmatrix} 0 \\ -t \\ ty_l \end{pmatrix}$$

$$l_l : -ty + ty_l = 0$$

Therefore, l_l is parallel to the x-axis.

For $\mathbf{p}_r$,

$$\mathbf{p}_l^T(\mathbf{E}^T\mathbf{p}_r) = 0$$

$$\mathbf{E}^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & -t & 0 \end{pmatrix}$$

The epipolar line where $\mathbf{p}_r$ lies is

$$l_r = \mathbf{E}^T\mathbf{p}_r = \begin{pmatrix} 0 \\ t \\ -ty_r \end{pmatrix}$$

$$l_r : ty - ty_l = 0$$

Therefore, l_r is parallel to the x-axis.

Q1.3

Suppose the 3D point is $\mathbf{P}$, the 2D point at frame 1 is $\mathbf{p}_1$, the 2D point at frame 2 is $\mathbf{p}_2$. Then the projection from 3D point to 2D points are

$$\begin{aligned}\mathbf{p}_1 &\equiv \mathbf{K}[\mathbf{R}_1 \quad \mathbf{t}_1]\mathbf{P} \equiv \mathbf{K}(\mathbf{R}_1\mathbf{P} + \mathbf{t}_1) \\ &\Rightarrow \mathbf{P} = \mathbf{R}_1^{-1}(\mathbf{K}^{-1}\mathbf{p}_1 - \mathbf{t}_1) \\ \mathbf{p}_2 &\equiv \mathbf{K}[\mathbf{R}_2 \quad \mathbf{t}_2]\mathbf{P} \equiv \mathbf{K}(\mathbf{R}_2\mathbf{P} + \mathbf{t}_2)\end{aligned}$$

Therefore, the relationship between $\mathbf{p}_1$ and $\mathbf{p}_2$ is

$$\begin{aligned}\mathbf{p}_2 &\equiv \mathbf{K}(\mathbf{R}_2[\mathbf{R}_1^{-1}(\mathbf{K}^{-1}\mathbf{p}_1 - \mathbf{t}_1)] + \mathbf{t}_2) \\ &\equiv \mathbf{K}(\mathbf{R}_2\mathbf{R}_1^{-1}\mathbf{K}^{-1}\mathbf{p}_1 - \mathbf{R}_2\mathbf{R}_1^{-1}\mathbf{t}_1 + \mathbf{t}_2) \\ &\equiv \mathbf{K}(\mathbf{R}_{rel}\mathbf{p}_1 + \mathbf{t}_{rel}) \\ \Rightarrow \mathbf{R}_{rel} &= \mathbf{R}_2\mathbf{R}_1^{-1}\mathbf{K}^{-1} \qquad \mathbf{t}_{rel} = -\mathbf{R}_2\mathbf{R}_1^{-1}\mathbf{t}_1 + \mathbf{t}_2\end{aligned}$$

Suppose $\hat{\mathbf{p}}_l$ and $\hat{\mathbf{p}}_r$ are in normalized coordinate,

$$\mathbf{p}_l \equiv \mathbf{K}\hat{\mathbf{p}}_l \quad \mathbf{p}_r \equiv \mathbf{K}\hat{\mathbf{p}}_r$$

$$\begin{aligned}\hat{\mathbf{p}}_r^T(\mathbf{t}_{rel} \times \mathbf{R}_{rel}\hat{\mathbf{p}}_l) = \hat{\mathbf{p}}_r^T\mathbf{E}\hat{\mathbf{p}}_l = 0 \quad &\Rightarrow \quad \mathbf{E} = \mathbf{t}_{rel} \times \mathbf{R}_{rel} \\ \mathbf{p}_r^T\mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1}\mathbf{p}_l = \mathbf{p}_r^T\mathbf{F}\mathbf{p}_l = 0 \quad &\Rightarrow \quad \mathbf{F} = \mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1} = \mathbf{K}^{-T}\mathbf{t}_{rel} \times \mathbf{R}_{rel}\mathbf{K}^{-1}\end{aligned}$$

Q1.4

Since all points on the object $\mathbf{X}$ are of equal distance to the mirror, suppose the distance from the object points to the mirror is d, then its reflection $\mathbf{X}'$ has all of its points distance d from the mirror and is also flat. Therefore, $\mathbf{X}$ can move perpendicular to the mirror by d to get $\mathbf{X}'$. $\mathbf{X}'$ only differs from $\mathbf{X}$ by a pure translation $\mathbf{t} = \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix}^T$. The relationship between two objects is

$$\mathbf{X}' = [\mathbf{I} \quad \mathbf{t}]\mathbf{X}$$

Then the projection from 3D points to 2D points are

$$\mathbf{p} \equiv \mathbf{K}[\mathbf{I} \quad \mathbf{0}]\mathbf{X}$$
$$\mathbf{p}' \equiv \mathbf{K}\mathbf{X}' \equiv \mathbf{K}[\mathbf{I} \quad \mathbf{t}]\mathbf{X}$$

Suppose $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}'$ are in normalized coordinate,

$$\mathbf{p} \equiv \mathbf{K}\hat{\mathbf{p}} \quad \mathbf{p}' \equiv \mathbf{K}\hat{\mathbf{p}}'$$

$$\hat{\mathbf{p}}'^T(\mathbf{t} \times \mathbf{I}\hat{\mathbf{p}}) = \hat{\mathbf{p}}'^T\mathbf{E}\hat{\mathbf{p}} = 0 \quad \Rightarrow \quad \mathbf{E} = \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix} \quad \mathbf{E}^T = -\mathbf{E}$$
$$\mathbf{p}_r^T\mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1}\mathbf{p}_l = \mathbf{p}_r^T\mathbf{F}\mathbf{p}_l = 0 \quad \Rightarrow \quad \mathbf{F} = \mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1}$$

Since $\mathbf{F}^T = (\mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1})^T = \mathbf{K}^{-T}\mathbf{E}^T\mathbf{K}^{-1} = -\mathbf{K}^{-T}\mathbf{E}\mathbf{K}^{-1} = -\mathbf{F}$, the fundamental matrix is a skew-symmetric matrix. Therefore, the two images of the object are related by a skew-symmetric fundamental matrix.

Q2.1

Recovered fundamental matrix **F**:

```
[[ 9.78833283e-10 -1.32135929e-07  1.12585666e-03]
 [-5.73843315e-08  2.96800276e-09 -1.17611996e-05]
 [-1.08269003e-03  3.04846703e-05 -4.47032655e-03]]
```

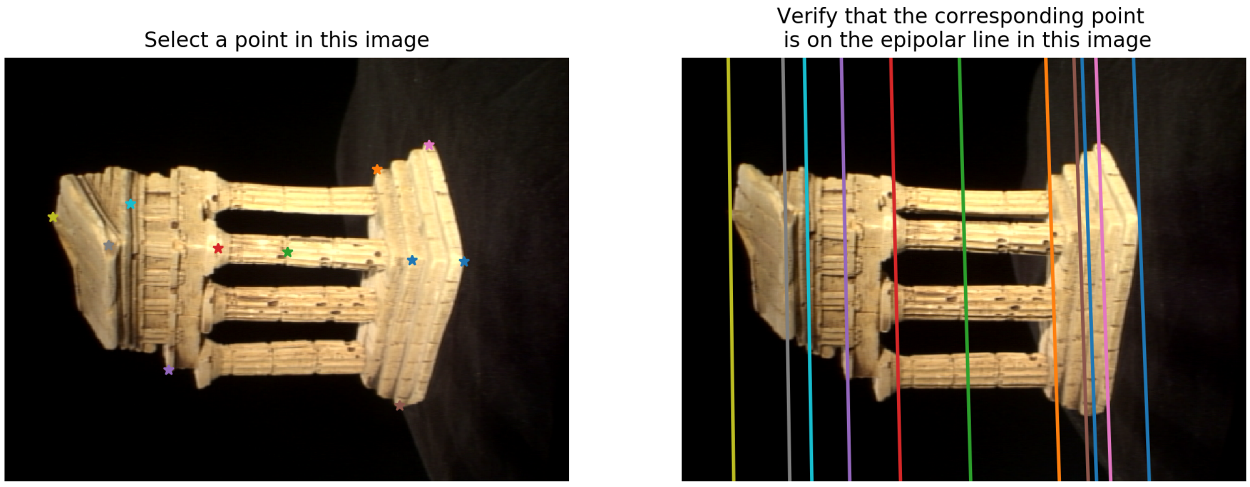


Figure 1: Epipolar lines visualization example.

Q3.1

Since $\hat{\mathbf{p}}_2^T \mathbf{E} \hat{\mathbf{p}}_1 = 0$, $\mathbf{p}_2 \equiv \mathbf{K}_2 \hat{\mathbf{p}}_2$ and $\mathbf{p}_1 \equiv \mathbf{K}_1 \hat{\mathbf{p}}_1$,

$$\mathbf{p}_2^T \mathbf{K}_2^{-T} \mathbf{E} \mathbf{K}_1^{-1} \mathbf{p}_1 = \mathbf{p}_2^T \mathbf{F} \mathbf{p}_1 = 0$$

$$\Rightarrow \mathbf{F} = \mathbf{K}_2^{-T} \mathbf{E} \mathbf{K}_1^{-1}$$
$$\Rightarrow \mathbf{E} = \mathbf{K}_2^{T} \mathbf{F} \mathbf{K}_1$$

Estimated essential matrix $\mathbf{E}$:

```
[[ 2.26268684e-03 -3.06552495e-01  1.66260633e+00]
 [-1.33130407e-01  6.91061098e-03 -4.33003420e-02]
 [-1.66721070e+00 -1.33210351e-02 -6.72186431e-04]]
```

Q3.2

Suppose that camera matrices `C1` and `C2` are

$$C1 = \begin{pmatrix} C1_{11} & C1_{12} & C1_{13} & C1_{14} \\ C1_{21} & C1_{22} & C1_{23} & C1_{24} \\ C1_{31} & C1_{32} & C1_{33} & C1_{34} \end{pmatrix} \qquad C2 = \begin{pmatrix} C2_{11} & C2_{12} & C2_{13} & C2_{14} \\ C2_{21} & C2_{22} & C2_{23} & C2_{24} \\ C2_{31} & C2_{32} & C2_{33} & C2_{34} \end{pmatrix}$$

Suppose that $\mathbf{w}_i$, $\mathbf{x}_{1i}$ and $\mathbf{x}_{2i}$ are

$$\mathbf{w}_i = \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \qquad \mathbf{x}_{1i} = \begin{pmatrix} u_{1i} \\ v_{1i} \\ 1 \end{pmatrix} \qquad \mathbf{x}_{2i} = \begin{pmatrix} u_{2i} \\ v_{2i} \\ 1 \end{pmatrix}$$

Therefore,

$$\mathbf{x}_{1i} \equiv C1\mathbf{P}_i \qquad \mathbf{x}_{2i} \equiv C2\mathbf{P}_i$$

$$\begin{pmatrix} u_{1i} \\ v_{1i} \end{pmatrix} = \begin{pmatrix} C1_{11} & C1_{12} & C1_{13} & C1_{14} \\ C1_{21} & C1_{22} & C1_{23} & C1_{24} \\ C1_{31} & C1_{32} & C1_{33} & C1_{34} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \qquad \begin{pmatrix} u_{2i} \\ v_{2i} \end{pmatrix} = \begin{pmatrix} C2_{11} & C2_{12} & C2_{13} & C2_{14} \\ C2_{21} & C2_{22} & C2_{23} & C2_{24} \\ C2_{31} & C2_{32} & C2_{33} & C2_{34} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow u_{1i} &= \frac{C1_{11}X + C1_{12}Y + C1_{13}Z + C1_{14}}{C1_{31}X + C1_{32}Y + C1_{33}Z + C1_{34}} \\ v_{1i} &= \frac{C1_{21}X + C1_{22}Y + C1_{23}Z + C1_{24}}{C1_{31}X + C1_{32}Y + C1_{33}Z + C1_{34}} \\ u_{2i} &= \frac{C2_{11}X + C2_{12}Y + C2_{13}Z + C2_{14}}{C2_{31}X + C2_{32}Y + C2_{33}Z + C2_{34}} \\ v_{2i} &= \frac{C2_{21}X + C2_{22}Y + C2_{23}Z + C2_{24}}{C2_{31}X + C2_{32}Y + C2_{33}Z + C2_{34}} \end{aligned}$$

$$\Rightarrow \begin{pmatrix} u_{1i}C1_{31} - C1_{11} & u_{1i}C1_{32} - C1_{12} & u_{1i}C1_{33} - C1_{13} & u_{1i}C1_{34} - C1_{14} \\ v_{1i}C1_{31} - C1_{21} & v_{1i}C1_{32} - C1_{22} & v_{1i}C1_{33} - C1_{23} & v_{1i}C1_{34} - C1_{24} \\ u_{2i}C2_{31} - C2_{11} & u_{2i}C2_{32} - C2_{12} & u_{2i}C2_{33} - C2_{13} & u_{2i}C2_{34} - C2_{14} \\ v_{2i}C2_{31} - C2_{21} & v_{2i}C2_{32} - C2_{22} & v_{2i}C2_{33} - C2_{23} & v_{2i}C2_{34} - C2_{24} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = 0 \quad \mathbf{A}_i\mathbf{w}_i = 0$$

$$\Rightarrow \mathbf{A}_i = \begin{pmatrix} u_{1i}C1_{31} - C1_{11} & u_{1i}C1_{32} - C1_{12} & u_{1i}C1_{33} - C1_{13} & u_{1i}C1_{34} - C1_{14} \\ v_{1i}C1_{31} - C1_{21} & v_{1i}C1_{32} - C1_{22} & v_{1i}C1_{33} - C1_{23} & v_{1i}C1_{34} - C1_{24} \\ u_{2i}C2_{31} - C2_{11} & u_{2i}C2_{32} - C2_{12} & u_{2i}C2_{33} - C2_{13} & u_{2i}C2_{34} - C2_{14} \\ v_{2i}C2_{31} - C2_{21} & v_{2i}C2_{32} - C2_{22} & v_{2i}C2_{33} - C2_{23} & v_{2i}C2_{34} - C2_{24} \end{pmatrix}$$

Q4.1

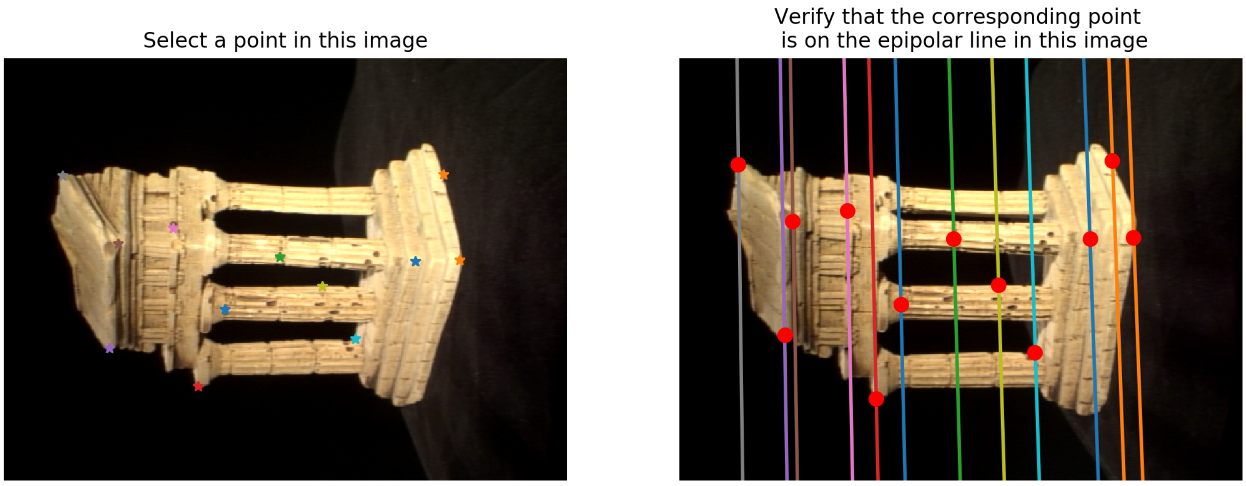


Figure 2: Example corresponding points by searching along epipolar lines.

Q4.2

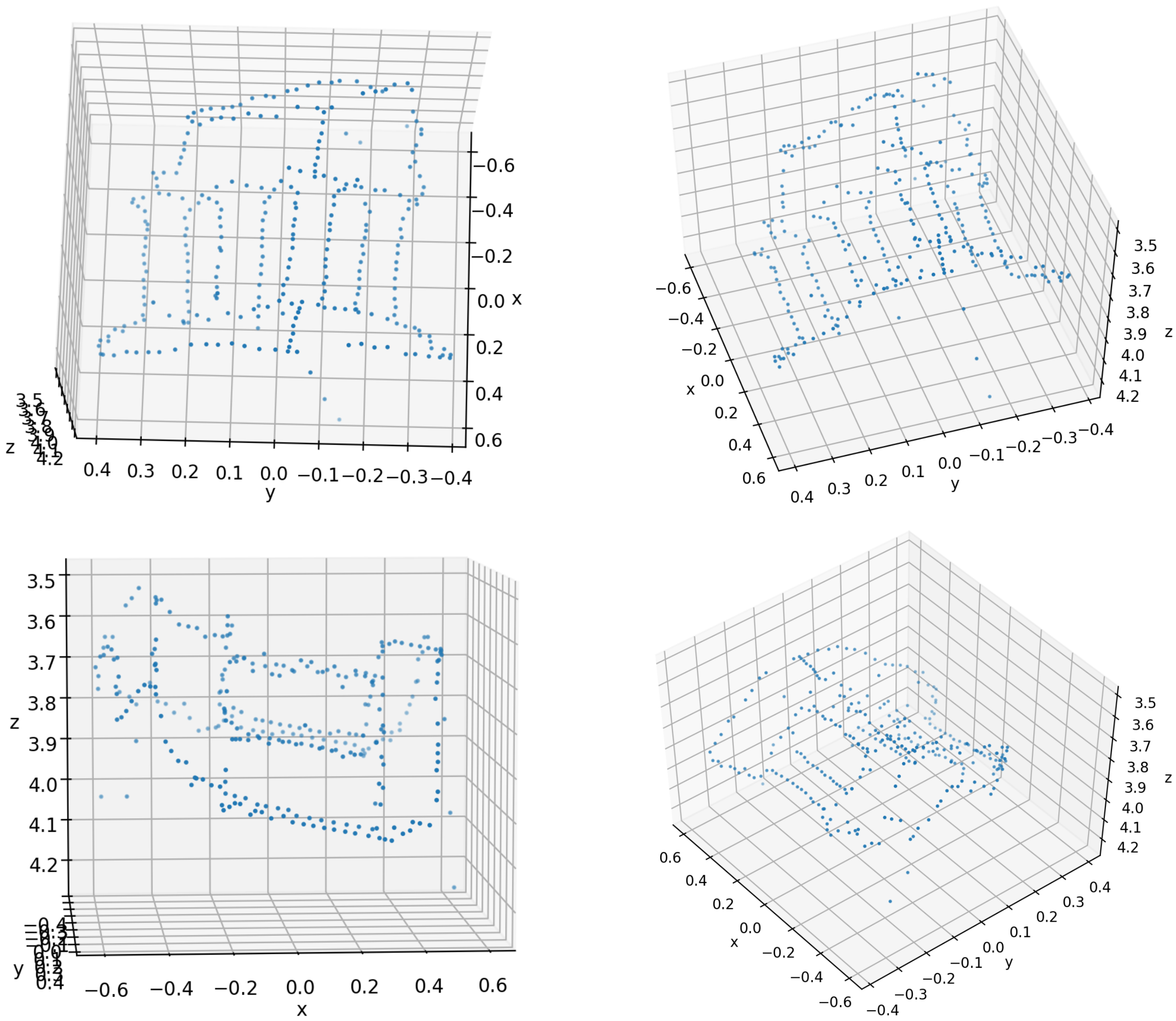


Figure 3: 3D reconstruction point clouds.

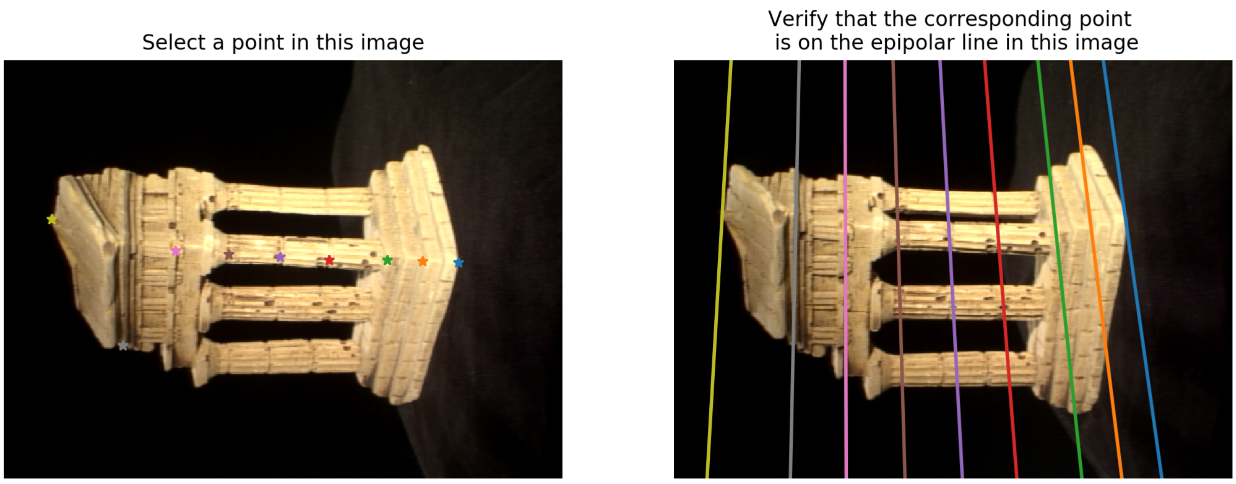
Q5.1

RANSAC optimizes the fundamental matrix such that when applying the fundamental matrix to points in image 1, the generated epipolar lines pass through corresponding points in image 2, and vice versa. Therefore, the error metrics I used here is the squared distance between each 2D point and its epipolar line in two images. The equation is:

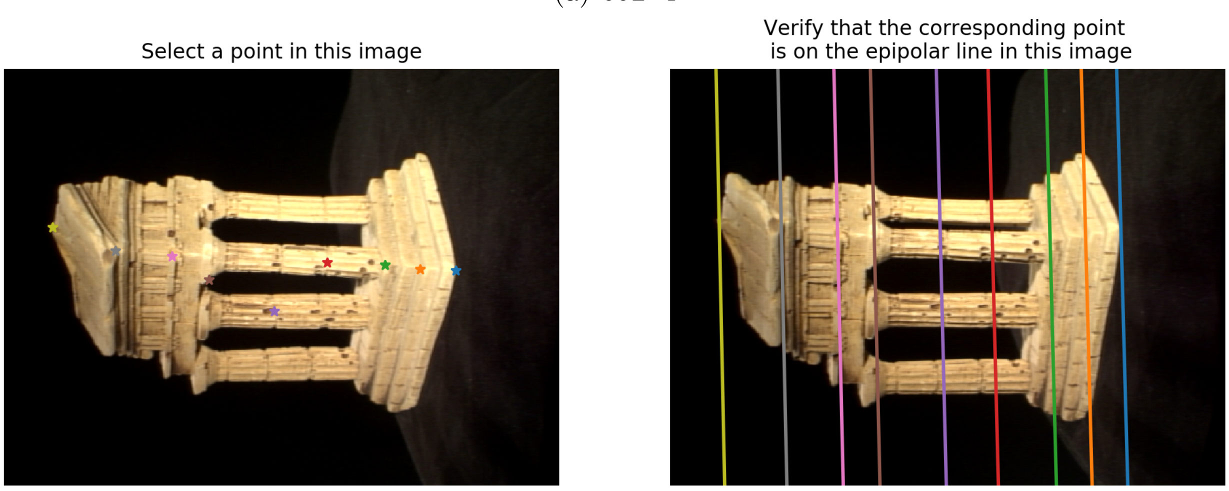
$$error(i) = d^2(p_{1i}, l_{1i}) + d^2(p_{2i}, l_{2i}) = \frac{p_{1i}^T l_{1i}}{l_{1i,1}^2 + l_{1i,2}^2} + \frac{p_{2i}^T l_{2i}}{l_{2i,1}^2 + l_{2i,2}^2}$$

where p_{1i} is the coordinates of the i^{th} point in image 1, l_{1i} is the epipolar line for the i^{th} point in image 1. Similar for points in image 2.

The inliers are 2D points whose distance to their epipolar lines are within a threshold `tol`. The best fundamental matrix is the one that has the largest amount of inliers. Figures below are generated by using the best fundamental matrix under different thresholds.



(a) `tol=4`



(b) `tol=8`

Figure 4: Epipolar line matching results with `tol=4,8`.

Threshold	`tol=4`	`tol=8`
# Inliers	52	89

Table 1: The number of inliers under different values of `tol`.

Q5.3

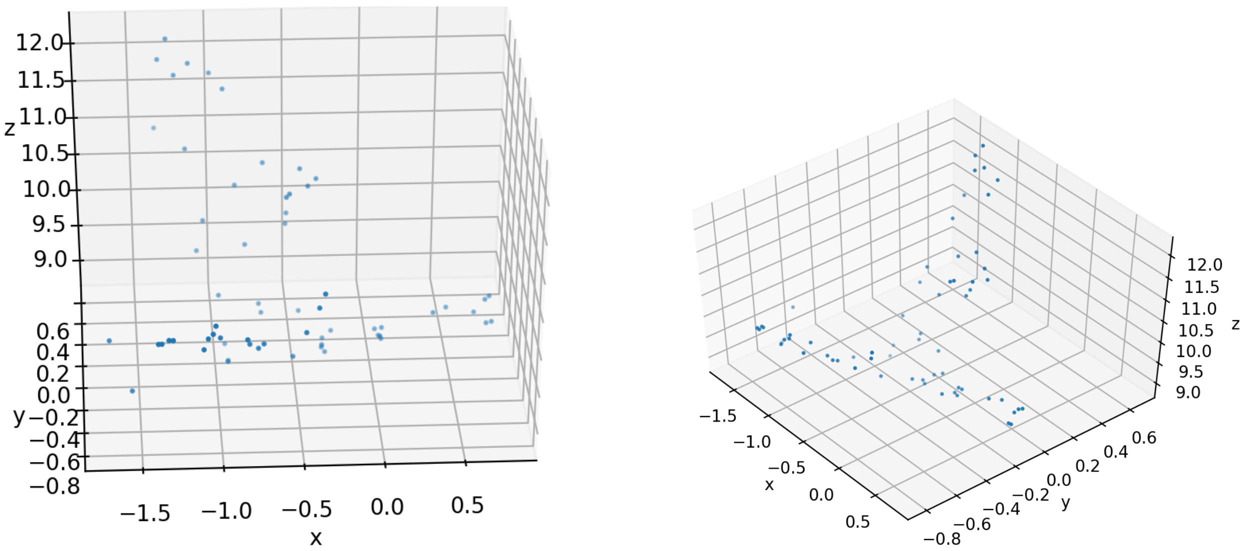


Figure 5: 3D reconstruction point clouds.

M2:

```
  [[-0.99827222  0.05177314 -0.02778693  0.04058628]
[ 0.05874897  0.88800331 -0.45606872  1.        ]
[ 0.00106277 -0.45691319 -0.88951066 -0.32232399]]
```