CS101 Algorithms and Data Structures

Stack

Textbook Ch 10.1

Outline

- Stack ADT
- Implementation
- Example applications

Reverse-Polish Notation

Normally, mathematics is written using what we call *in-fix* notation:

$$(3 + 4) \times 5 - 6$$

The operator is placed between two operands

One weakness: parentheses are required

$$(3 + 4) \times 5 - 6 = 29$$

$$3 + 4 \times 5 - 6 = 17$$

$$3 + 4 \times (5 - 6) = -1$$

$$(3 + 4) \times (5 - 6) = -7$$

Reverse-Polish Notation

Alternatively, we can place the operands first, followed by the operator:

$$(3 + 4) \times 5 - 6$$

$$3\ 4\ +\ 5\ \times\ 6\ -$$

Parsing reads left-to-right and performs any operation on the last two operands:

$$3\ 4\ +\ 5\ \times\ 6\ -$$

$$7\quad 5\ \times\ 6\ -$$

$$35\quad 6\ -$$

$$29$$

Reverse-Polish Notation

Other examples:

3 4 5 × + 6 –

3 20 + 6 –

23 6 –

17

$3 + 4 \times 5 - 6 = 17$

3 4 5 6 – × +

3 4 –1 × +

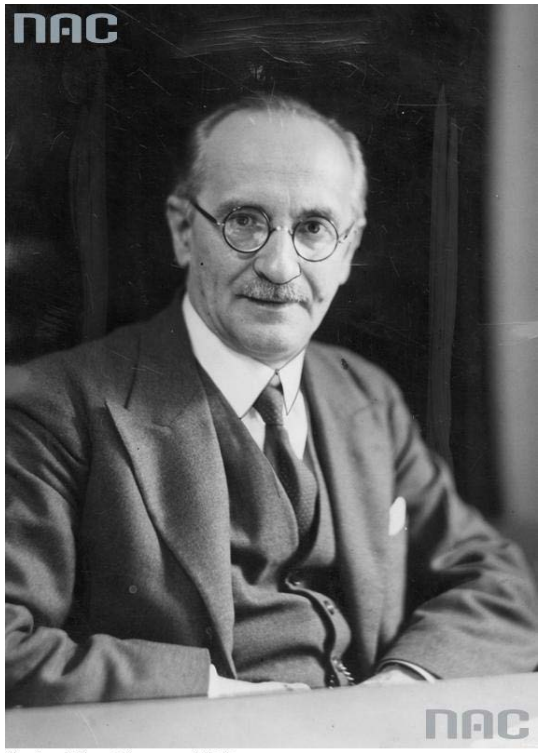
3 –4 +

–1

$3 + 4 \times (5 - 6) = -1$

Reverse-Polish Notation

This is called *reverse-Polish* notation after the mathematician Jan Łukasiewicz



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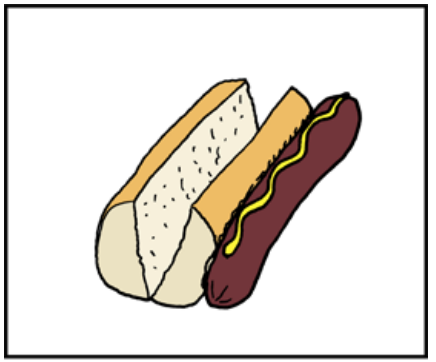


http://xkcd.com/645/

Reverse-Polish Notation

Benefits:

- No ambiguity and no brackets are required
- It is the same process used by a computer to perform computations:
  - operands must be loaded into registers before operations can be performed on them

Reverse-Polish Notation

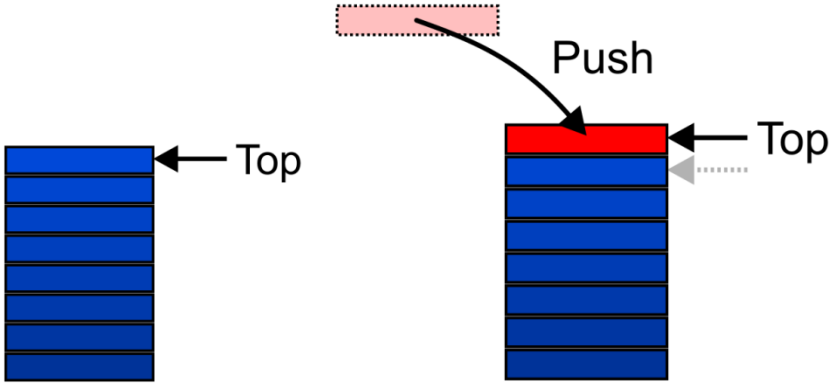
The easiest way to parse reverse-Polish notation is to use an operand stack:

- operands are processed by pushing them onto the stack
- when processing an operator:
  - pop the last two items off the operand stack,
  - perform the operation, and
  - push the result back onto the stack

Stack ADT

Also called a *last-in–first-out* (LIFO) behaviour

– Graphically, we may view these operations as follows:



Applications

Numerous applications:

- Parsing code:
  - Matching parenthesis
  - XML (e.g., XHTML)
- Tracking function calls
- Dealing with undo/redo operations
- Reverse-Polish calculators
- Assembly language

Reverse-Polish Notation

Evaluate the following reverse-Polish expression using a stack:

$$1\ \ 2\ \ 3\ +\ 4\ \ 5\ \ 6\ \times\ -\ 7\ \times\ +\ -\ 8\ \ 9\ \times\ +$$

Reverse-Polish Notation

Push 1 onto the stack

1 2 3 + 4 5 6 × – 7 × + – 8 9 × +

Reverse-Polish Notation

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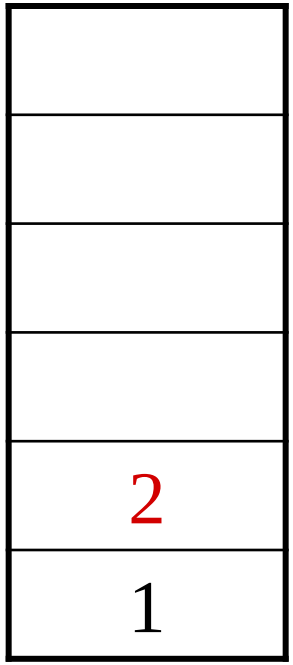
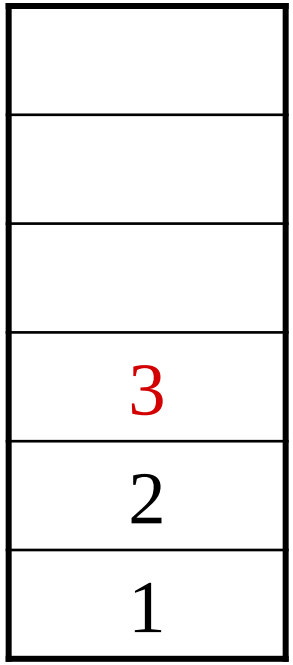


Reverse-Polish Notation

Push 3 onto the stack

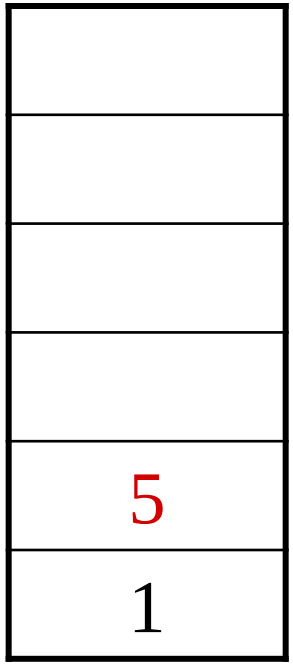
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Pop 3 and 2 and push 2 + 3 = 5

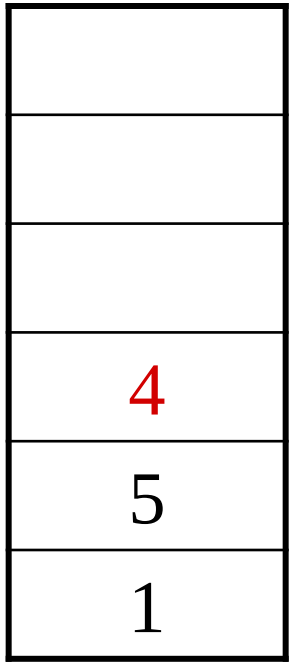
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Push 4 onto the stack

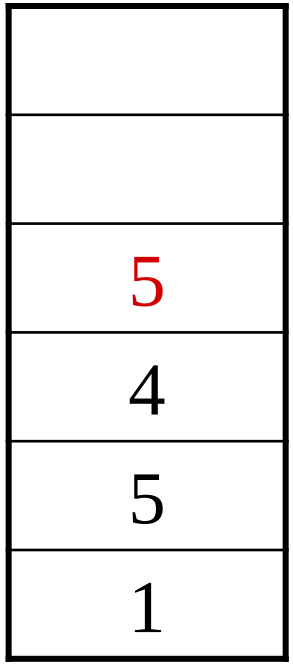
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Push 5 onto the stack

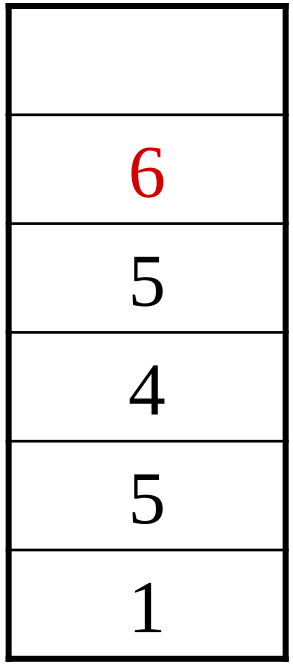
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Push 6 onto the stack

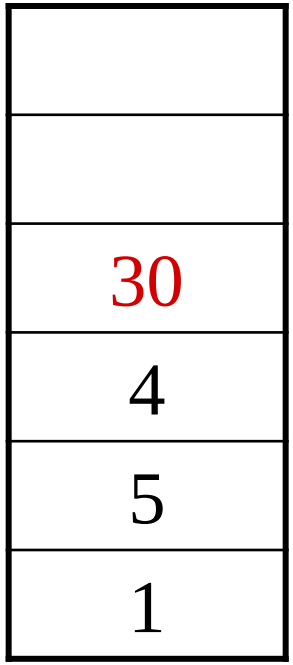
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Pop 6 and 5 and push 5 × 6 = 30

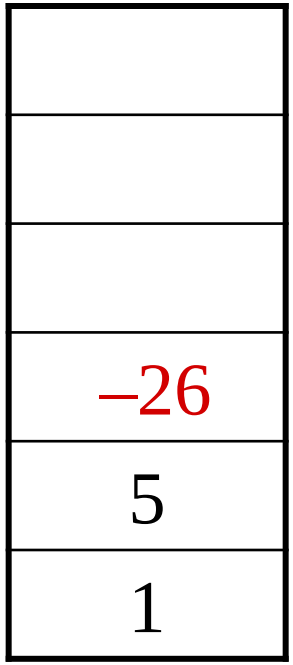
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Pop 30 and 4 and push 4 – 30 = –26

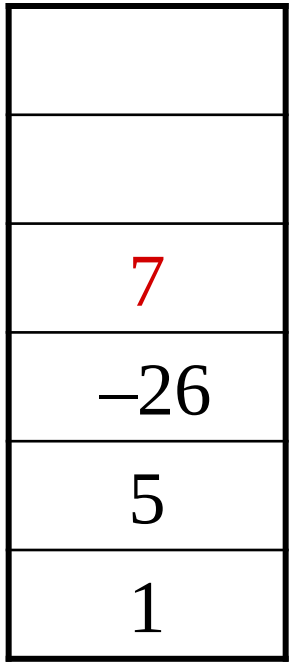
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Push 7 onto the stack

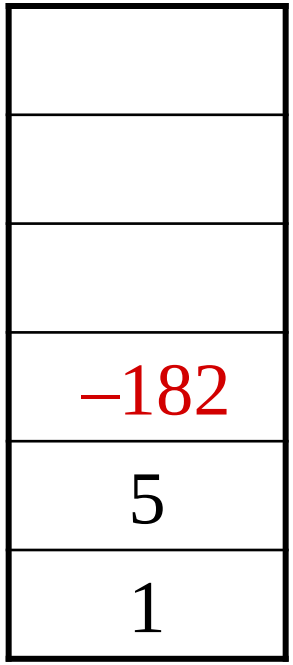
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Pop 7 and –26 and push –26 × 7 = –182

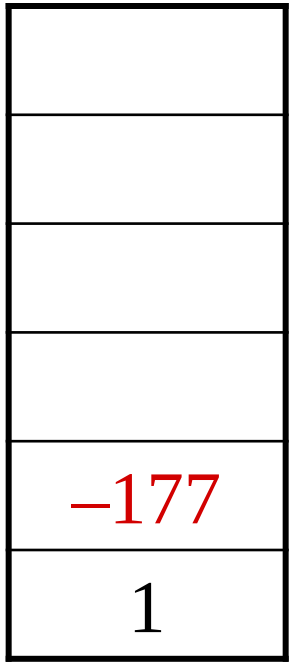
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Pop –182 and 5 and push –182 + 5 = –177

1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

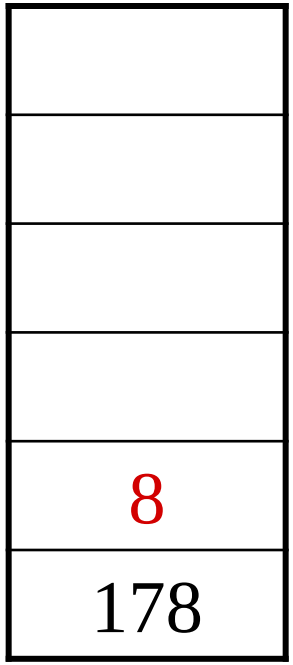
Pop –177 and 1 and push 1 – (–177) = 178

1 2 3 + 4 5 6 × – 7 × + – 8 9 × +

Reverse-Polish Notation

Push 8 onto the stack

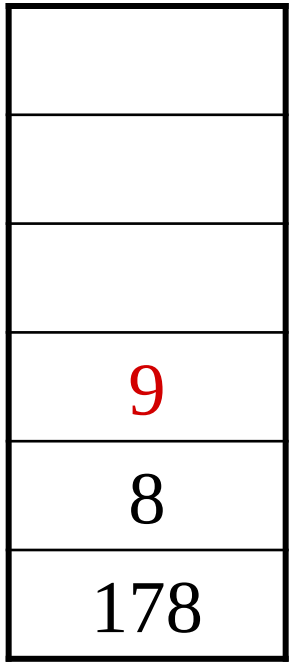
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Push 1 onto the stack

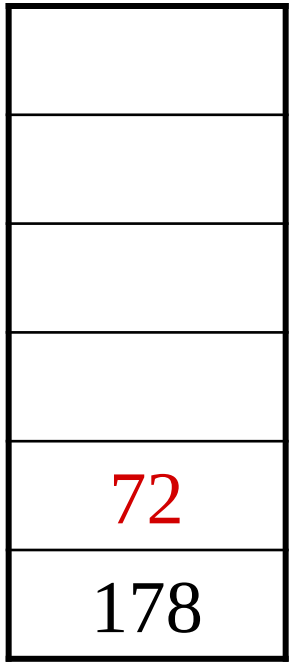
1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Pop 9 and 8 and push 8 × 9 = 72

1 2 3 + 4 5 6 × – 7 × + – 8 9 × +



Reverse-Polish Notation

Pop 72 and 178 and push 178 + 72 = 250

1 2 3 + 4 5 6 × – 7 × + – 8 9 × +

Reverse-Polish Notation

Thus

$$1\ \ 2\ \ 3\ +\ \ 4\ \ 5\ \ 6\ \times\ –\ \ 7\ \times\ +\ –\ \ 8\ \ 9\ \times\ +$$

evaluates to the value on the top: 250

The equivalent in-fix notation is

$$((1 – ((2 + 3) + ((4 – (5 \times 6)) \times 7))) + (8 \times 9))$$

We reduce the parentheses using order-of-operations:

$$1 – (2 + 3 + (4 – 5 \times 6) \times 7) + 8 \times 9$$

Stack ADT

- Uses an explicit linear ordering
- Two principal operations
  - *Push*: insert an object onto the top of the stack
  - *Pop*: erase the object on the top of the stack
  - *CreateStack:* generate an empty stack
  - *IsEmpty*: determine if stack is empty
  - *IsFull*: determine if stack is full

Outline

- Stack ADT
- Implementation
- Example applications

Implementations

We will look at two implementations of stacks:

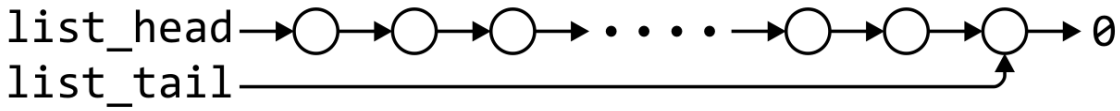
- Singly linked lists
- One-ended arrays

The optimal asymptotic run time of any algorithm is $\Theta(1)$

- The run time of the algorithm is independent of the number of objects being stored in the container

Linked-List Implementation

Operations at the front of a singly linked list are all $\Theta(1)$



	Front/1^{st}	Back/n^{th}
Find	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(1)$	$\Theta(1)$
Erase	$\Theta(1)$	$\Theta(n)$

The desired behavior of an Abstract Stack may be reproduced by performing all operations at the front

void push_front(int)

We could, however, note that when the list is empty, `list_head == 0`, thus we could shorten this to:

```
void List::push_front( int n ) {
    list_head = new Node( n, list_head );
}
```

If it is empty, we start with:

and, if we try to add 81, we should end up with:

void push_front(int)

We could, however, note that when the list is empty, `list_head == 0`, thus we could shorten this to:

```
void List::push_front( int n ) {
    list_head = new Node( n, list_head );
}
```

If it is not empty, we start with:

and, if we try to add 70, we should end up with:

int pop_front()

The correct implementation assigns a temporary pointer to point to the node being deleted:

```
int List::pop_front() {
    if ( empty() ) {
        throw underflow();
    }

    int e = front();
    Node *ptr = list_head;
    list_head = list_head->next();
    delete ptr;
    return e;
}
```

int front() const

```
int List::front() const {
    if ( empty() ) {
        throw underflow();
    }

    return head()->retrieve();
}
```

int pop_front()

The correct implementation assigns a temporary pointer to point to the node being deleted:

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    list_head = list_head->next();
    delete ptr;
    return e;
}
```

e = 70

int front() const

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int List::front() const {
    if ( empty() ) {
        throw underflow();
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    return head()->retrieve();
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    return e;
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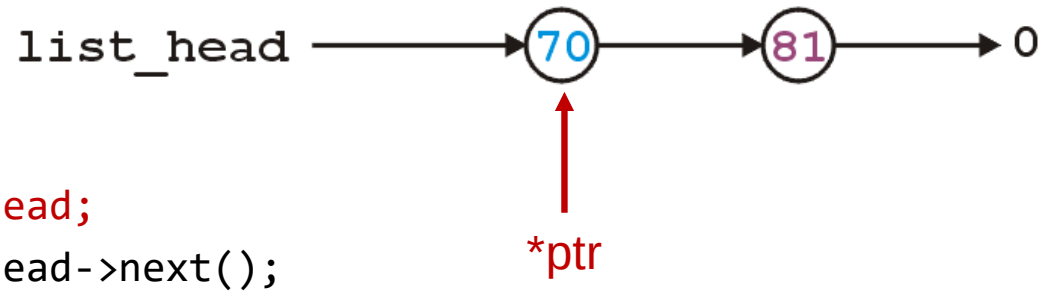

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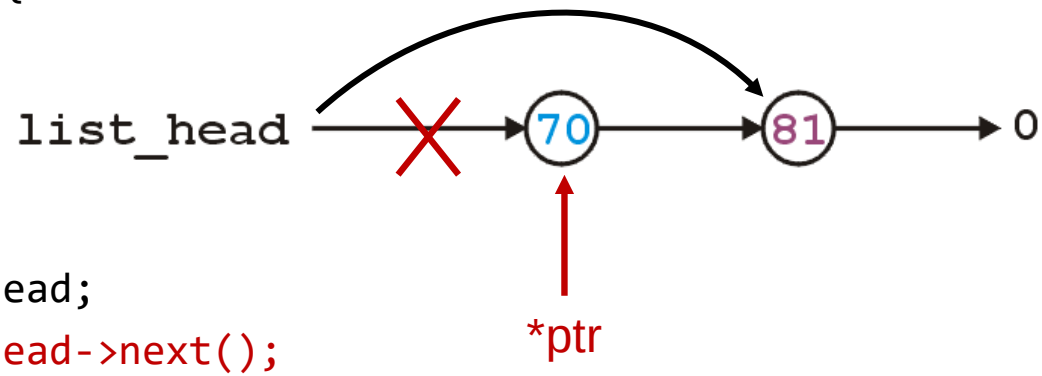
    delete ptr;
    return e;
}
```



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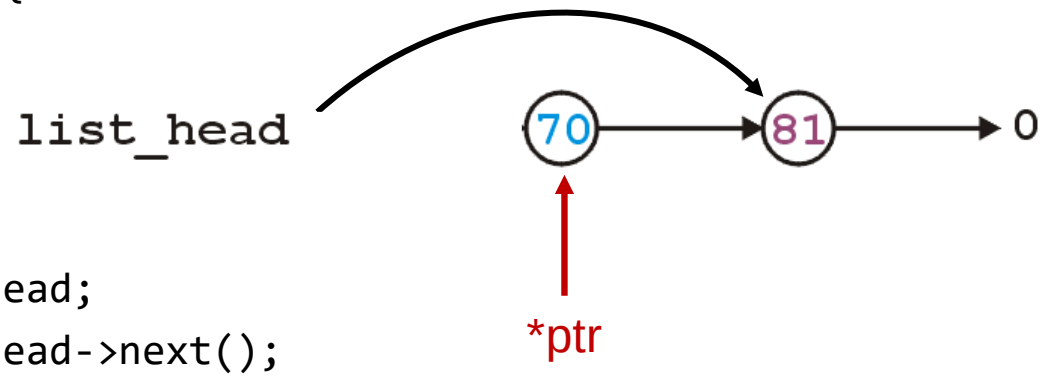

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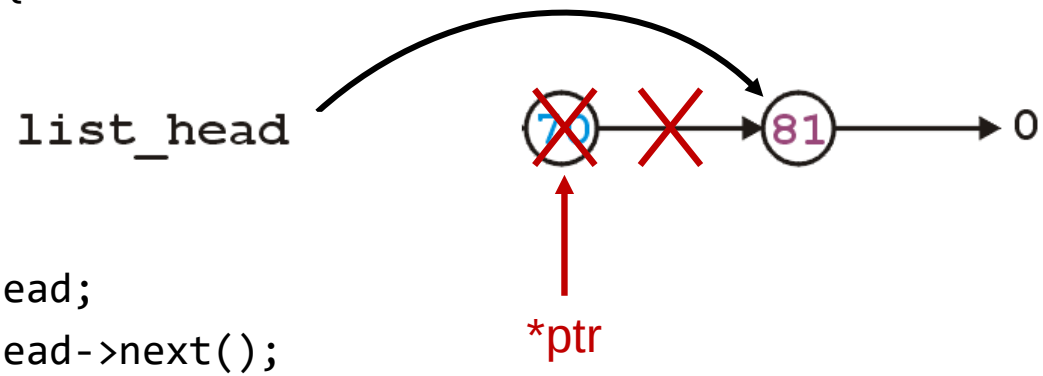

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    int e = front();
    Node *ptr = list_head;
    list_head = list_head->next();
    delete ptr;
    return e;
}
```

Array Implementation

For one-ended arrays, all operations at the back are $\Theta(1)$

	Front/1^{st}	Back/n^{th}
Find	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(n)$	$\Theta(1)$
Erase	$\Theta(n)$	$\Theta(1)$

Top

If there are n objects in the stack, the last is located at index $n - 1$

```
template <typename Type>
Type Stack<Type>::top() const {
    if ( empty() ) {
        throw underflow();
    }

    return array[stack_size - 1];
}
```

Pop

Removing an object simply involves reducing the size

- By decreasing the size, the previous top of the stack is now at the location `stack_size`

```
template <typename Type>
Type Stack<Type>::pop() {
    if ( empty() ) {
        throw underflow();
    }

    --stack_size;
    return array[stack_size];
}
```

Push

Pushing an object onto the stack can only be performed if the array is not full

```
template <typename Type>
void Stack<Type>::push( Type const &obj ) {
    if ( stack_size == array_capacity ) {
        throw overflow();
    }

    array[stack_size] = obj;
    ++stack_size;
}
```

Array Capacity

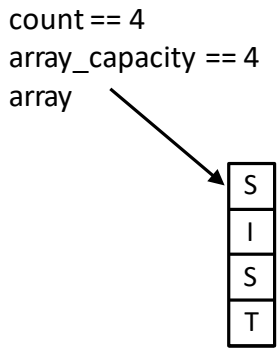
The best option is to increase the array capacity

If we increase the array capacity, the question is:

- How much?
- By a constant? `array_capacity += c;`
- By a multiple? `array_capacity *= c;`

Array Capacity

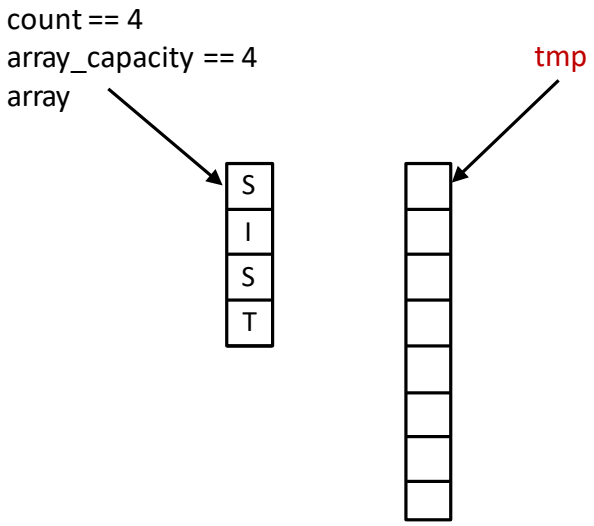
First, let us visualize what must occur to allocate new memory



Array Capacity

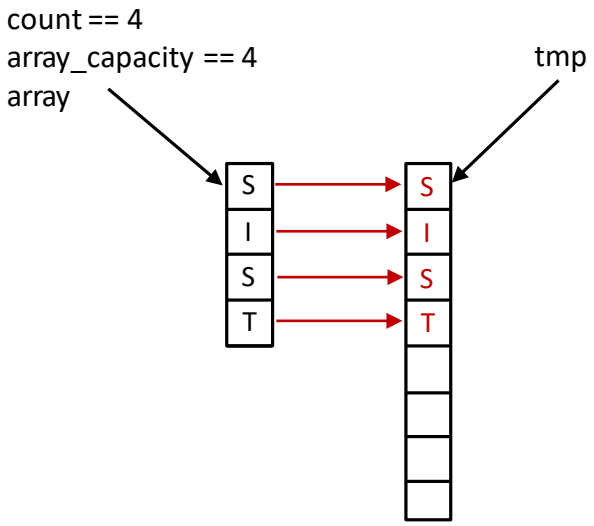
First, this requires a call to `new Type[N]` where *N* is the new capacity

- We must have access to this so we must store the address returned by new in a local variable, say `tmp`



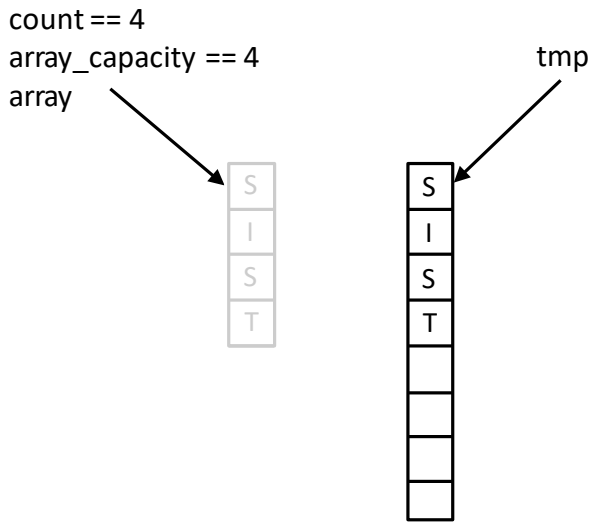
Array Capacity

Next, the values must be copied over



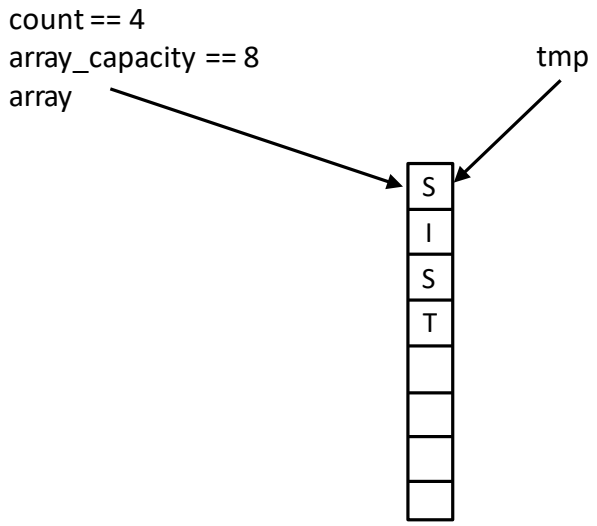
Array Capacity

The memory for the original array must be deallocated



Array Capacity

Finally, the appropriate member variables must be reassigned



Array Capacity

Back to the original question:

- How much do we change the capacity?
- Add a constant?
- Multiply by a constant?

First, we recognize that any time that we `push` onto a full stack, this requires n copies and the run time is $\Theta(n)$

Therefore, `push` is usually $\Theta(1)$ except when new memory is required

Array Capacity

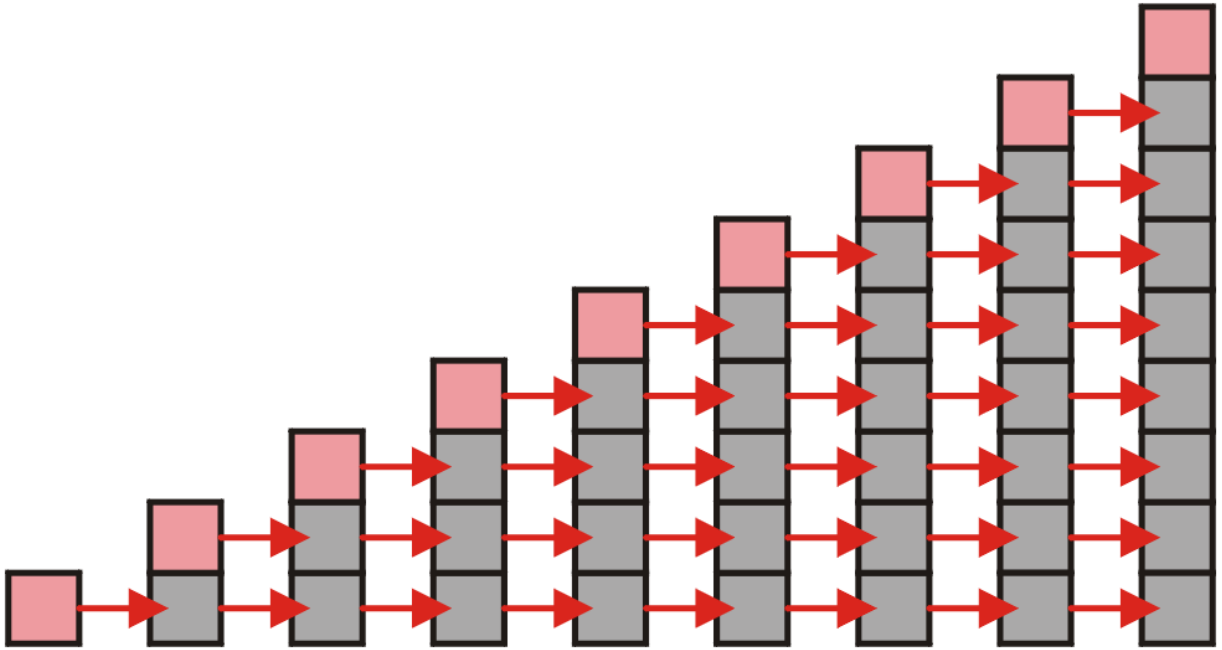
To state the average run time, we will introduce the concept of amortized time:

- If n operations requires $\Theta(f(n))$, we will say that an individual operation has an amortized run time of $\Theta(f(n)/n)$
- Therefore, if inserting n objects requires:
  - $\Theta(n^2)$ copies, the amortized time is $\Theta(n)$
  - $\Theta(n)$ copies, the amortized time is $\Theta(1)$

Array Capacity

Let us consider the case of increasing the capacity by 1 each time the array is full

- With each insertion when the array is full, this requires all entries to be copied



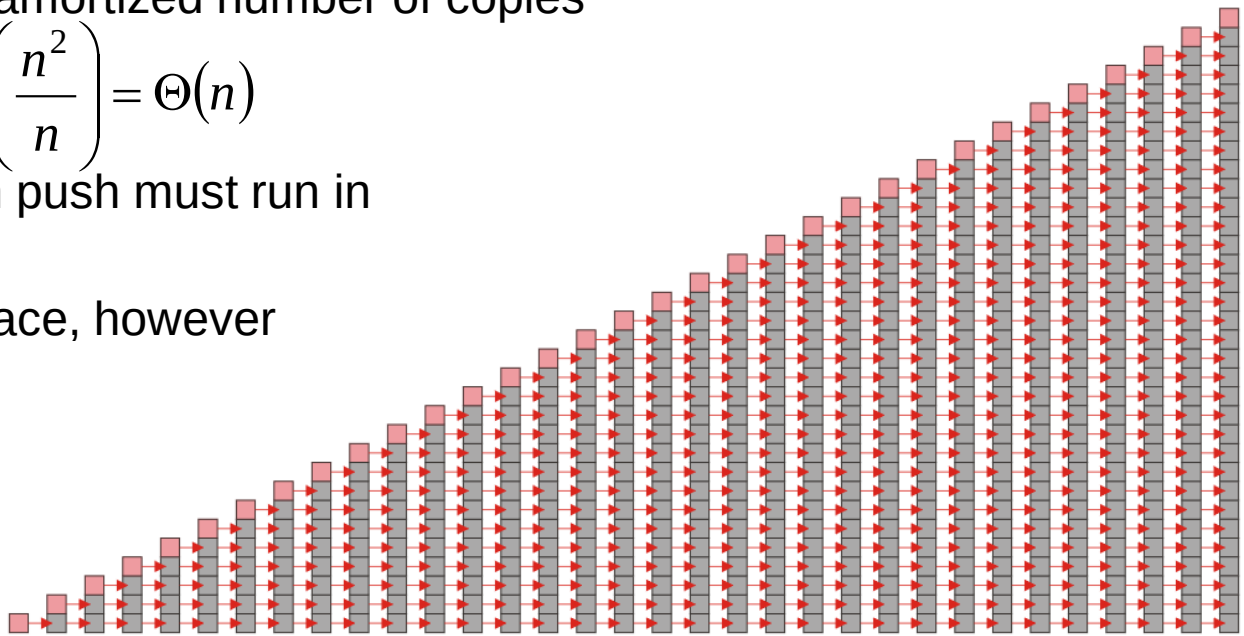
Array Capacity

Suppose we insert k objects

- The pushing of the k^{th} object on the stack requires k-1 copies
- The total number of copies is now given by:

$$\sum_{k=1}^{n}(k-1)=\left(\sum_{k=1}^{n}k\right)-n=\frac{n(n+1)}{2}-n=\frac{n(n-1)}{2}=\Theta\left(n^2\right)$$

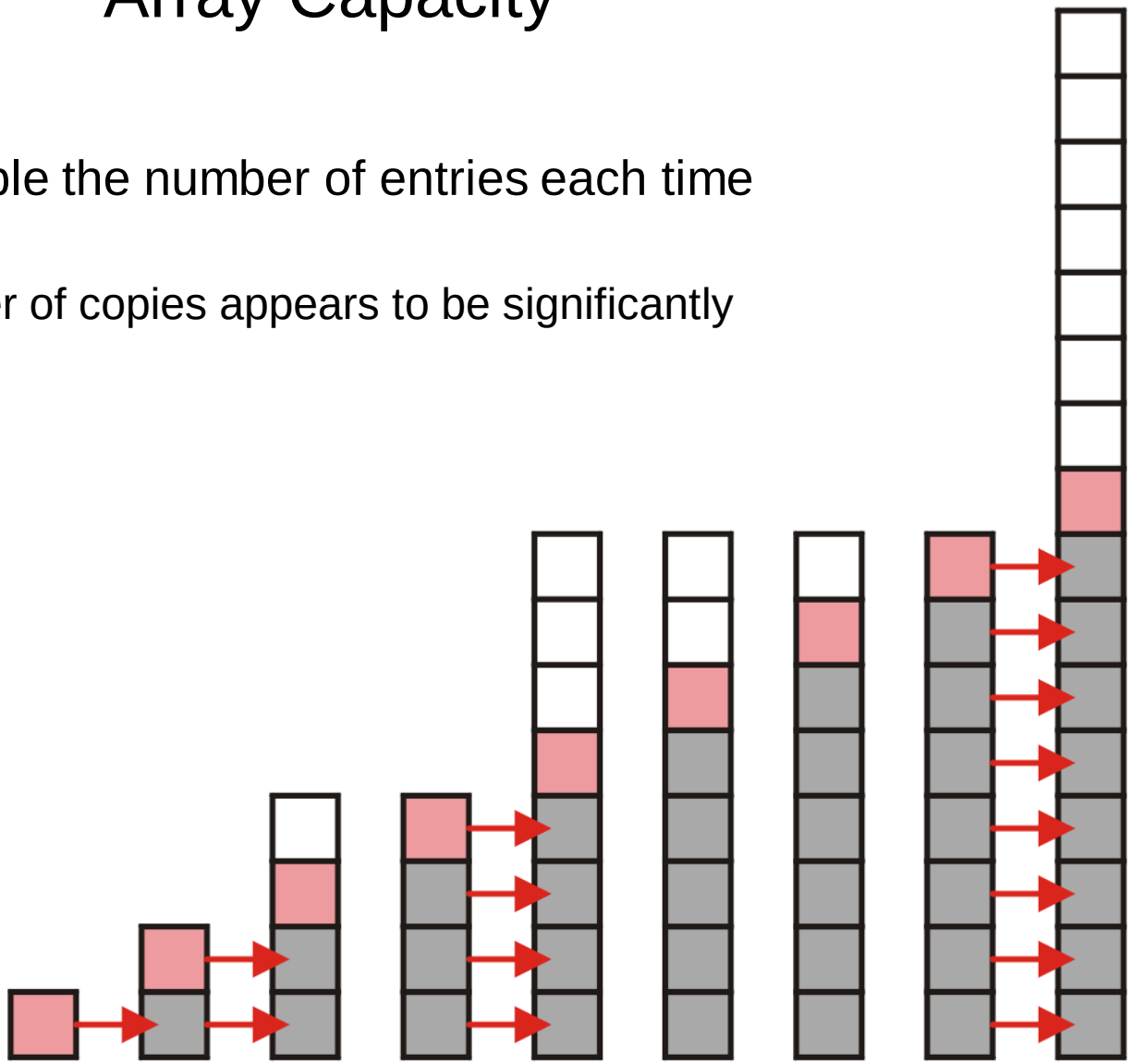
- Therefore, the amortized number of copies is given by $\Theta\left(\frac{n^2}{n}\right)=\Theta(n)$
- Therefore each push must run in $\Theta(n)$ time
- The wasted space, however is $\Theta(1)$



Array Capacity

Suppose we double the number of entries each time the array is full

- Now the number of copies appears to be significantly fewer



Array Capacity

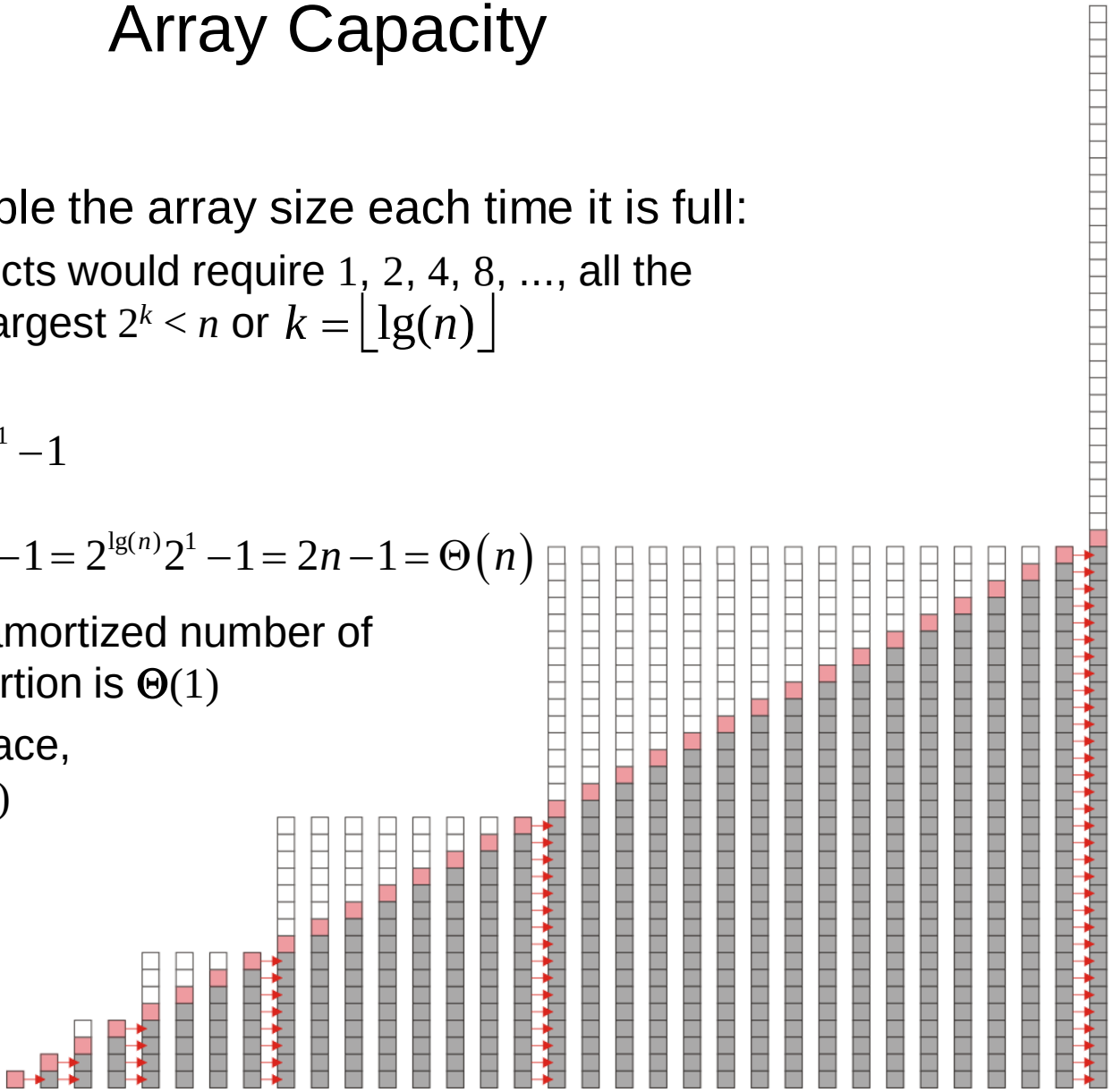
Suppose we double the array size each time it is full:

- Inserting n objects would require $1, 2, 4, 8, \ldots$, all the way up to the largest $2^k < n$ or $k = \lfloor \lg(n) \rfloor$

$$\sum_{k=0}^{\lfloor \lg(n) \rfloor} 2^k = 2^{\lfloor \lg(n) \rfloor + 1} - 1$$

$$\leq 2^{\lg(n)+1} - 1 = 2^{\lg(n)} 2^1 - 1 = 2n - 1 = \Theta(n)$$

- Therefore the amortized number of copies per insertion is $\Theta(1)$
- The wasted space, however is $\mathbf{O}(n)$



Array Capacity

Note the difference in worst-case amortized scenarios:

	Copies per Insertion	**Unused Memory**
Increase by 1	$n-1$	0
Increase by m	n/m	$m-1$
Increase by a factor of 2	1	n
Increase by a factor of $r > 1$	$1/(r-1)$	$(r-1)n$

Summary

- Stack ADT
  - Push, pop, LIFO
- Implementation
  - Linked list
  - Array
    - How to increase the array capacity
- Applications
  - Parsing XHTML
  - Function calls
  - Reverse-Polish Notation