《计算机辅助几何设计》作业

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1. Given the following cubic polynomial curve:

$$P(u) = -\begin{pmatrix} 7/8 \\ 5/8 \end{pmatrix} u^3 + \begin{pmatrix} 9 \\ 15/4 \end{pmatrix} u^2 - \begin{pmatrix} 57/2 \\ 9/2 \end{pmatrix} u + \begin{pmatrix} 30 \\ -1 \end{pmatrix}$$

1) Calculate its polar form and the vertices of its Bézier control polygon P_0, P_1, P_2, P_3 within the interval[2,4], and roughly sketch this control polygon;

2) Use the de Casteljau algorithm to calculate the polynomial curve at sample $u = \frac{5}{2}$, 3, $\frac{7}{2}$, and draw it in the figure in 1);

3) Using the results from 2) to subdivide the curve at $u = 3$, then subdivide the right portion at its midpoint $u = \frac{7}{2}$. Draw the control polygon in the figure in 1), and draw the curve $P(u)$

解：

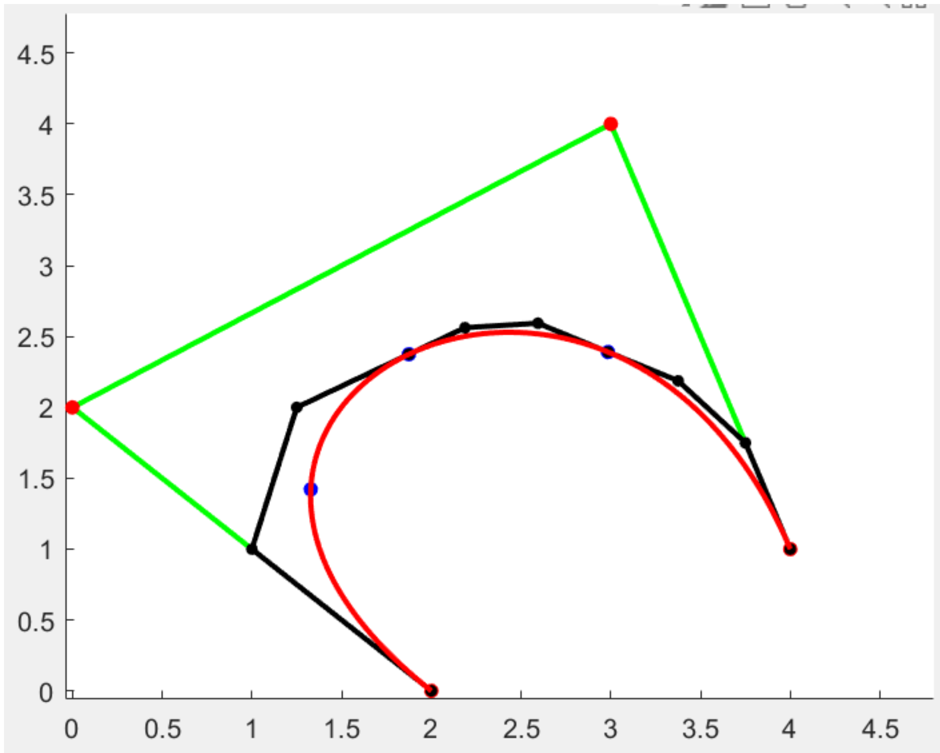

图 1:

1)

$$f(u_1,u_2,u_3) = -\begin{pmatrix} 7/8 \\ 5/8 \end{pmatrix} u_1u_2u_3 + \begin{pmatrix} 9 \\ 15/4 \end{pmatrix} \frac{u_1u_2+u_2u_3+u_1u_3}{3} - \begin{pmatrix} 57/2 \\ 9/2 \end{pmatrix} \frac{u_1+u_2+u_3}{3} + \begin{pmatrix} 30 \\ -1 \end{pmatrix}$$

$P_0 = f(2,2,2) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $P_1 = f(2,2,4) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$P_2 = f(2,4,4) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $P_3 = f(4,4,4) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

2)

$u = 5/2$:

$$f(2,2,5/2) = 3/4f(2,2,2) + 1/4f(2,2,4) = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$f(2,5/2,4) = 3/4f(2,2,4) + 1/4f(2,4,4) = \begin{pmatrix} 3/4 \\ 5/2 \end{pmatrix}$$

$$f(5/2,4,4) = 3/4f(2,4,4) + 1/4f(4,4,4) = \begin{pmatrix} 13/4 \\ 13/4 \end{pmatrix}$$

$$f(2,5/2,5/2) = 3/4f(2,2,5/2) + 1/4f(2,5/2,4) = \begin{pmatrix} 21/16 \\ 1 \end{pmatrix}$$

$$f(5/2,5/2,4) = 3/4f(2,5/2,4) + 1/4f(5/2,4,4) = \begin{pmatrix} 11/8 \\ 43/16 \end{pmatrix}$$

$$P(5/2) = f(5/2,5/2,5/2) = 3/4f(2,5/2,5/2) + 1/4f(5/2,5/2,4) = \begin{pmatrix} 85/64 \\ 91/64 \end{pmatrix}$$

$u = 3$:

$$f(2,2,3) = 1/2f(2,2,2) + 1/2f(2,2,4) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f(2,3,4) = 1/2f(2,2,4) + 1/2f(2,4,4) = \begin{pmatrix} 3/2 \\ 3 \end{pmatrix}$$

$$f(3,4,4) = 1/2f(2,4,4) + 1/2f(4,4,4) = \begin{pmatrix} 7/2 \\ 5/2 \end{pmatrix}$$

$$f(2,3,3) = 1/2f(2,2,3) + 1/2f(2,3,4) = \begin{pmatrix} 5/4 \\ 2 \end{pmatrix}$$

$$f(3,3,4) = 1/2f(2,3,4) + 1/2f(3,3,4) = \begin{pmatrix} 5/2 \\ 11/4 \end{pmatrix}$$

$$P(3) = f(3,3,3) = 1/2f(2,3,3) + 1/2f(3,3,4) = \begin{pmatrix} 15/8 \\ 19/8 \end{pmatrix}$$

$u = 7/2$

$$f(2,2,7/2) = 1/4f(2,2,2) + 3/4f(2,2,4) = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}$$

$$f(2,7/2,4) = 1/4f(2,2,4) + 3/4f(2,4,4) = \begin{pmatrix} 9/4 \\ 7/2 \end{pmatrix}$$

$$f(7/2,4,4)=1/4f(2,4,4)+3/4f(4,4,4)=\begin{pmatrix}15/4\\7/4\end{pmatrix}$$

$$f(2,7/2,7/2)=1/4f(2,2,7/2)+3/4f(2,7/2,4)=\begin{pmatrix}29/16\\3\end{pmatrix}$$

$$f(7/2,7/2,4)=1/4f(2,7/2,4)+3/4f(7/2,4,4)=\begin{pmatrix}27/8\\35/16\end{pmatrix}$$

$$P(7/2)=f(7/2,7/2,7/2)=1/4f(2,7/2,7/2)+3/4f(7/2,7/2,4)=\begin{pmatrix}191/64\\153/64\end{pmatrix}$$

3)

第一段控制点：

$$Q_1^{(0)}=f(2,2,2)=\begin{pmatrix}2\\0\end{pmatrix},Q_1^{(1)}=f(2,2,3)=\begin{pmatrix}1\\1\end{pmatrix}$$

$$Q_1^{(2)}=f(2,3,3)=\begin{pmatrix}5/4\\2\end{pmatrix},Q_1^{(3)}=f(3,3,3)=\begin{pmatrix}15/8\\19/8\end{pmatrix}$$

第二段控制点：

$$Q_2^{(0)}=f(3,3,3)=\begin{pmatrix}15/8\\19/8\end{pmatrix}$$

$$Q_2^{(1)}=f(3,3,7/2)=1/4f(2,3,3)+3/4f(3,3,4)==\begin{pmatrix}35/16\\41/16\end{pmatrix}$$

$$Q_2^{(2)}=f(3,7/2,7/2)=1/2f(2,7/2,7/2)+1/2f(7/2,7/2,4)=\begin{pmatrix}83/32\\83/32\end{pmatrix}$$

$$Q_2^{(3)}=f(3,7/2,7/2)=\begin{pmatrix}191/64\\153/64\end{pmatrix}$$

第三段控制点：

$$Q_3^{(0)}=f(7/2,7/2,7/2)=\begin{pmatrix}191/64\\153/64\end{pmatrix},Q_3^{(1)}=f(7/2,7/2,4)=\begin{pmatrix}27/8\\35/16\end{pmatrix}$$

$$Q_3^{(2)}=f(7/2,4,4)=\begin{pmatrix}15/4\\7/4\end{pmatrix},Q_3^{(3)}=f(4,4,4)=\begin{pmatrix}4\\1\end{pmatrix}$$

2.Given the following cubic polynomial curve and parameter interval [0,1]：

$$F(u)=\begin{pmatrix}15\\-6\end{pmatrix}u^3+\begin{pmatrix}27\\10\end{pmatrix}u^2-\begin{pmatrix}9\\9\end{pmatrix}u$$

1) Calculate its first and second derivatives;

2) Calculate its polar form $f(u_1,u_2,u_3)$ and the polar forms of the derivatives F' and F'', prove that they equal to $3f(u_1,u_2,\hat{1})$ and $6f(u_1,\hat{1},\hat{1})$ respectively. Note that $f(u_1,u_2,\hat{1})=f(u_1,u_2,1)`f(u_1,u_2,0)$.

解：

1)

$$F'(u)=\begin{pmatrix}45\\-18\end{pmatrix}u^2+\begin{pmatrix}54\\20\end{pmatrix}u-\begin{pmatrix}9\\9\end{pmatrix}$$

$$F''(u)=\begin{pmatrix}90\\-36\end{pmatrix}u+\begin{pmatrix}54\\20\end{pmatrix}$$

2)

$$f(u_1,u_2,u_3)=\begin{pmatrix}15\\-6\end{pmatrix}u_1u_2u_3+\begin{pmatrix}27\\10\end{pmatrix}\frac{u_1u_2+u_2u_3+u_1u_3}{3}-\begin{pmatrix}9\\9\end{pmatrix}\frac{u_1+u_2+u_3}{3}$$

$$f'(u_1,u_2)=\begin{pmatrix}45\\-18\end{pmatrix}u_1u_2+\begin{pmatrix}54\\20\end{pmatrix}\frac{u_1+u_2}{2}-\begin{pmatrix}9\\9\end{pmatrix}$$

$$f''(u_1)=\begin{pmatrix}90\\-36\end{pmatrix}u_1+\begin{pmatrix}54\\20\end{pmatrix}$$

Proof：

$$\begin{aligned}f(u_1,u_2,\hat{1})=&f(u_1,u_2,1)-f(u_1,u_2,0)\\=&\begin{pmatrix}15\\-6\end{pmatrix}(u_1u_2-0)+\begin{pmatrix}27\\10\end{pmatrix}(\frac{u_1u_2+u_1+u_2}{3}-\frac{u_1u_2}{3})\\&-\begin{pmatrix}9\\9\end{pmatrix}(\frac{u_1+u_2+1}{3}-\frac{u_1+u_2}{3})\\=&\begin{pmatrix}45\\-18\end{pmatrix}\frac{u_1u_2}{3}+\begin{pmatrix}54\\20\end{pmatrix}\frac{u_1+u_2}{3}-\begin{pmatrix}9\\9\end{pmatrix}\frac{1}{3}\\=&\frac{1}{3}f'(u_1,u_2)\end{aligned}$$

$$
\begin{aligned}
f(u_1,\hat{1},\hat{1}) =& f(u_1,\hat{1},1)-f(u_1,\hat{1},0)\\
=&(f(u_1,1,1)-f(u_1,0,1))-(f(u_1,1,0)-f(u_1,0,0))\\
=&f(u_1,1,1)-2f(u_1,1,0)+f(u_1,0,0)\\
=&\begin{pmatrix}15\\-6\end{pmatrix}(u_1-0+0)+\begin{pmatrix}27\\10\end{pmatrix}(\frac{2u_1+1}{3}-2\cdot\frac{u_1}{3}+0)\\
&-\begin{pmatrix}9\\9\end{pmatrix}(\frac{u_1+2}{3}-2\cdot\frac{u_1+1}{3}+\frac{u_1}{3})\\
=&\begin{pmatrix}90\\-36\end{pmatrix}\frac{u_1}{6}+\begin{pmatrix}54\\20\end{pmatrix}\frac{1}{6}\\
=&\frac{1}{6}f''(u_1)
\end{aligned}
$$

3.Given a uniform B-spline defined by the following four points and knot vector [0,0,1,2,3,4,5,5]:

$$
P_0=\begin{pmatrix}-2\\-10\end{pmatrix},P_1=\begin{pmatrix}-4\\2\end{pmatrix},P_2=\begin{pmatrix}6\\5\end{pmatrix},P_3=\begin{pmatrix}4\\-7\end{pmatrix}
$$

1) Use the de Boor algorithm to calculate the curve position at $t=2.5$. Sketch the control polygon and the relevant points constructed by this algorithm.

2) For the B-spline in 1), calculate the corresponding Bézier control points that represent the same curve. Draw the control vertices and Bézier curve in the figure in 1).

解：

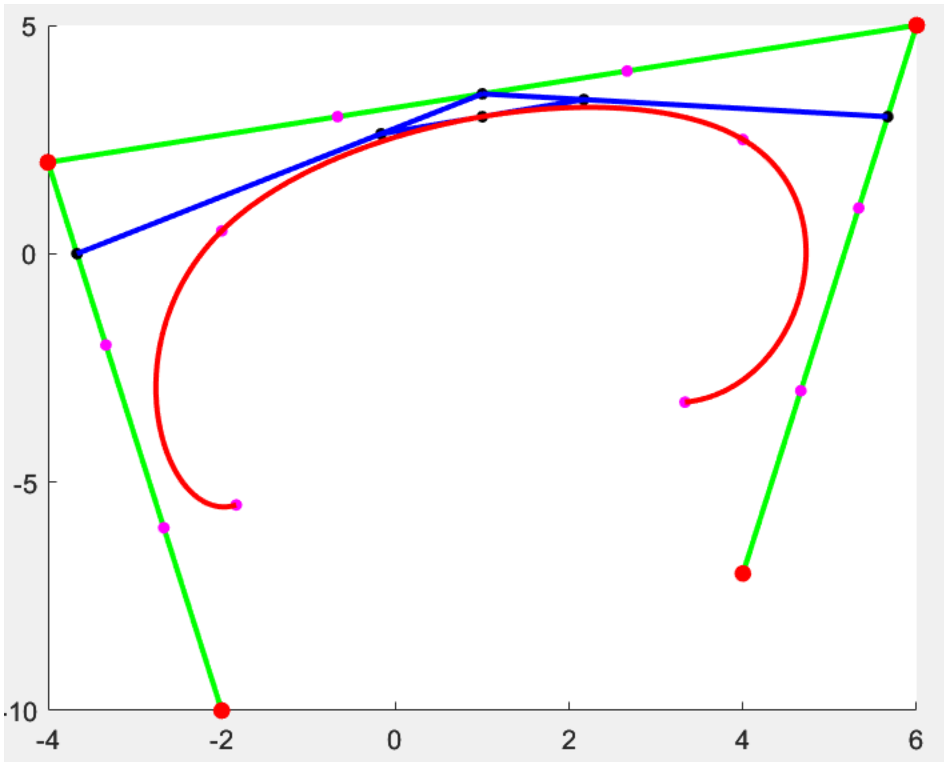

图 2:

1)

$$
\begin{aligned}
x(0,1,2) = P_0 &= \begin{pmatrix} -2 \\ -10 \end{pmatrix} \\
x(1,2,3) = P_1 &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} \\
x(2,3,4) = P_2 &= \begin{pmatrix} 6 \\ 5 \end{pmatrix} \\
x(3,4,5) = P_3 &= \begin{pmatrix} 4 \\ -7 \end{pmatrix}
\end{aligned} \tag{1}
$$

由(1):

$$
\begin{aligned}
x(1,2,2.5) &= 1/6x(0,1,2) + 5/6x(1,2,3) = \begin{pmatrix} -11/3 \\ 0 \end{pmatrix} \\
x(2,2.5,3) &= 1/2x(1,2,3) + 1/2x(2,3,4) = \begin{pmatrix} 1 \\ 7/2 \end{pmatrix} \\
x(2.5,3,4) &= 5/6x(2,3,4) + 1/6x(3,4,5) = \begin{pmatrix} 17/3 \\ 3 \end{pmatrix}
\end{aligned} \tag{2}
$$

由(2):

$$
\begin{aligned}
x(2,2.5,2.5) &= 1/4x(1,2,2.5) + 3/4x(2,2.5,3) = \begin{pmatrix} -1/6 \\ 21/8 \end{pmatrix} \\
x(2.5,2.5,3) &= 3/4x(2,2.5,3) + 1/4x(2.5,3,4) = \begin{pmatrix} 13/6 \\ 27/8 \end{pmatrix}
\end{aligned} \tag{3}
$$

由(3):

$$
x(2.5,2.5,2.5) = 1/2x(2,2.5,2.5) + 1/2x(2.5,2.5,3) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{4}
$$

2)
设控制点为$Q_i(i=0,1,\ldots,9)$

$$
\begin{aligned}
s(t) = &\sum_{i=0}^{3} P_i N_i^m(t) \\
= &x(0,0,1)N_{-1}^m(t) \\
&+ x(0,1,2)N_0^m(t) + x(1,2,3)N_1^m(t) + x(2,3,4)N_2^m(t) + x(3,4,5)N_3^m(t) \\
&+ x(4,5,5)N_4^m(t)
\end{aligned}
$$

可得

$$
x(0,0,1) = 0, x(4,5,5) = 0
$$

$$
\begin{aligned}
x(1,2,2) =& 1/3x(0,1,2) + 2/3x(1,2,3) = \begin{pmatrix} -10/3 \\ -2 \end{pmatrix} \\
x(1,1,2) =& 1/2x(0,1,2) + 1/2x(1,2,2) = \begin{pmatrix} -8/3 \\ -6 \end{pmatrix} \\
x(0,1,1) =& 1/2x(0,0,1) + 1/2x(0,1,2) = \begin{pmatrix} -1 \\ -5 \end{pmatrix} \\
x(1,1,1) =& 1/2x(0,1,1) + 1/2x(1,1,2) = \begin{pmatrix} -11/6 \\ -11/2 \end{pmatrix} \\
x(2,3,3) =& 1/3x(1,2,3) + 2/3x(2,3,4) = \begin{pmatrix} 8/3 \\ 4 \end{pmatrix} \\
x(2,2,3) =& 1/2x(1,2,3) + 1/2x(2,3,3) = \begin{pmatrix} -2/3 \\ 3 \end{pmatrix} \\
x(2,2,2) =& 1/2x(1,12,2) + 1/2x(2,2,3) = \begin{pmatrix} -2 \\ 1/2 \end{pmatrix} \\
x(3,4,4) =& 1/3x(2,3,4) + 2/3x(3,4,5) = \begin{pmatrix} 14/3 \\ -3 \end{pmatrix} \\
x(3,3,4) =& 1/2x(2,3,4) + 1/2x(3,4,4) = \begin{pmatrix} 16/3 \\ 1 \end{pmatrix} \\
x(3,3,3) =& 1/2x(2,3,3) + 1/2x(3,3,4) = \begin{pmatrix} 4 \\ 5/2 \end{pmatrix} \\
x(4,4,5) =& 1/2x(3,4,5) + 1/2x(4,5,5) = \begin{pmatrix} 2 \\ -7/2 \end{pmatrix} \\
x(4,4,4) =& 1/2x(3,4,4) + 4/2x(4,4,5) = \begin{pmatrix} 10/3 \\ -13/4 \end{pmatrix}
\end{aligned} \tag{5}
$$

由于是同一条曲线，因此开花形式一样，因此对应的Bezier控制点即为:

$$
\begin{aligned}
Q_0 =& x(1,1,1) = \begin{pmatrix} -11/6 \\ -11/2 \end{pmatrix} \\
Q_1 =& x(1,1,2) = \begin{pmatrix} -8/3 \\ -6 \end{pmatrix} \\
Q_2 =& x(1,2,2) = \begin{pmatrix} -10/3 \\ -2 \end{pmatrix} \\
Q_3 =& x(2,2,2) = \begin{pmatrix} -2 \\ 1/2 \end{pmatrix} \\
Q_4 =& x(2,2,3) = \begin{pmatrix} -2/3 \\ 3 \end{pmatrix} \\
Q_5 =& x(2,3,3) = \begin{pmatrix} 8/3 \\ 4 \end{pmatrix} \\
Q_6 =& x(3,3,3) = \begin{pmatrix} 4 \\ 5/2 \end{pmatrix} \\
Q_7 =& x(3,3,4) = \begin{pmatrix} 16/3 \\ 1 \end{pmatrix} \\
Q_8 =& x(3,4,4) = \begin{pmatrix} 14/3 \\ -3 \end{pmatrix} \\
Q_9 =& x(4,4,4) = \begin{pmatrix} 10/3 \\ -13/4 \end{pmatrix}
\end{aligned}
\tag{6}
$$