一、函数逼近

设不同的基函数为 $\{1, g_1(x), g_2(x), \cdots, g_{n-1}(x)\}$

拟合的函数为 $f(x) = \sum_{i=0}^{n-1} \lambda_i g_i(x)$

已知 $(x_0, y_0), \cdots (x_{n-1}, y_{n-1})$ 共n个数据点,

则可建立方程组

$$\begin{pmatrix} 1 & g_1(x_0) & g_2(x_0) & \cdots & g_{n-1}(x_0) \\ & \cdots & \cdots & \cdots & \\ 1 & g_1(x_{n-1}) & g_2(x_{n-1}) & \cdots & g_{n-1}(x_{n-1}) \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

例: $B = \{1, x, x^2\}$ 已知 $[(x,y)] = \{(0,2), (1,0), (2,3)\}$

则 $$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \Rightarrow \begin{cases} \lambda_0 = 2 \\ \lambda_2 = -\frac{9}{2} \\ \lambda_3 = \frac{5}{2} \end{cases}$$

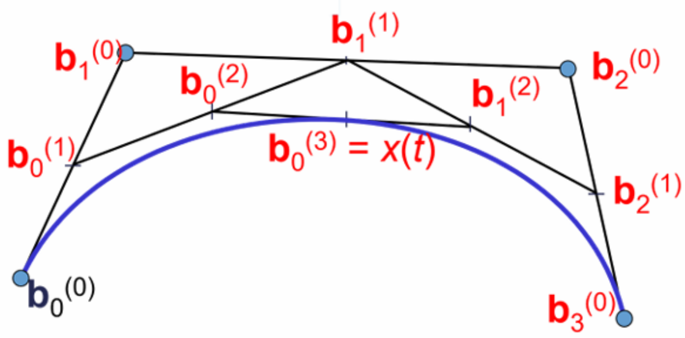
$$f(x) = 2 - \frac{9}{2}x + \frac{5}{2}x^2$$

二、Bezier 曲线

1、De Casteljau algorithm

有0到n共(n+1)个控制点

对n阶 Bezier 曲线



Algorithm:

for r=1..n

　for i=0..n-r

$$b_i^{(r)} = (1-t)\, b_i^{(r-1)} + t\, b_{i+1}^{(r-1)}$$

　end

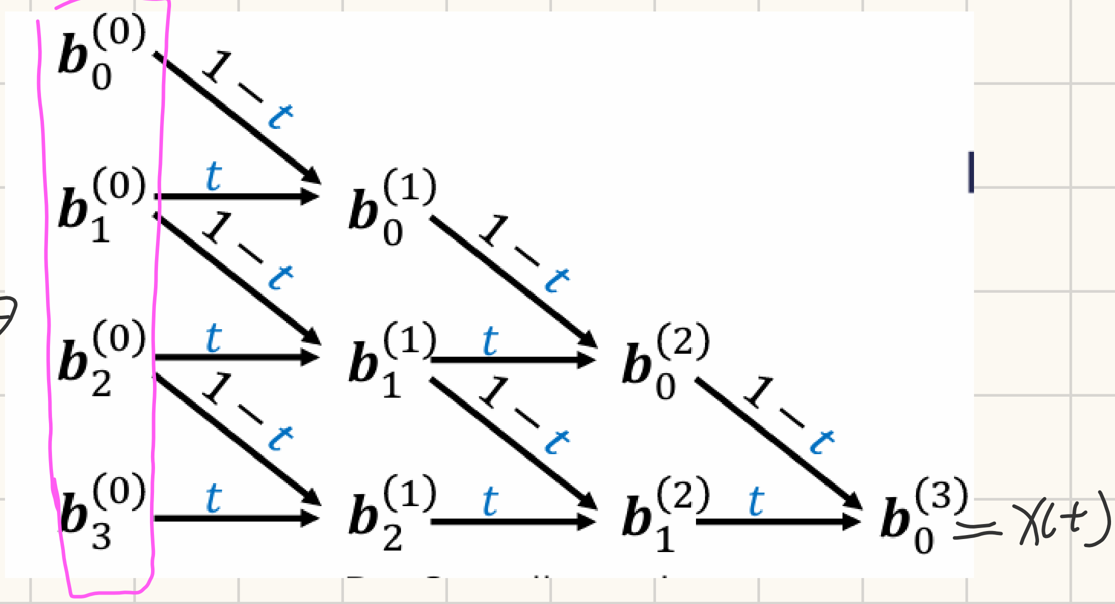
end

return $b_0^{(n)}$

The whole algorithm consists only of repeated linear interpolations.

具体而言

起始控制点



2、Bernstein基

$X(t)=\sum_{i=0}^{n} B_i^n(t)\cdot b_i$ 其中 $B_i^n(t)=\binom{n}{i}t^i(1-t)^{n-i}$

实际上为 $(1-t+t)^n$ 的二项式展开

$$
\begin{array}{llll}
B_0^{(0)} := 1 & & & \\
B_0^{(1)} := 1-t & B_1^{(1)} := t & & \\
B_0^{(2)} := (1-t)^2 & B_1^{(2)} := 2t(1-t) & B_2^{(2)} := t^2 & \\
B_0^{(3)} := (1-t)^3 & B_1^{(3)} := 3t(1-t)^2 & B_2^{(3)} := 3t^2(1-t) & B_3^{(3)} := t^3
\end{array}
$$

仿射不变性：

$$\sum_{i=0}^{n} B_i^n(t)\cdot(Ab_i+a) = A\left(\sum_{i=0}^{n} B_i^n(t)\cdot b_i\right) + \sum_{i=0}^{n} B_i^n(t)\cdot a$$

$$= A\,X(t) + a\sum_{i=0}^{n} B_i^n(t)$$

Convex Hull:

besier曲线始终位于其控制点所围成的凸包之中

凸包 $\Omega=\{x\in\mathbb{R}^d \mid x=\sum_{i=1}^{n}\lambda_i p_i,\ 满足\ \lambda_i\ge 0\ 且\ \sum_{i=1}^{n}\lambda_i=1,\ p_i为控制点\}$

直接理解就是最外层控制点连成的凸多边形

Variation Reduction:

一条直线穿过凸包，与贝塞尔曲线交点数不超过与控制多边形交点数，说明贝塞尔曲线比控制多边形更简单

Degree elevation

增加控制点：从 $b_0,\dots,b_n \to x(t)$ 变为 $\bar{b}_0,\dots,\bar{b}_n,\bar{b}_{n+1} \to \bar{x}(t)$

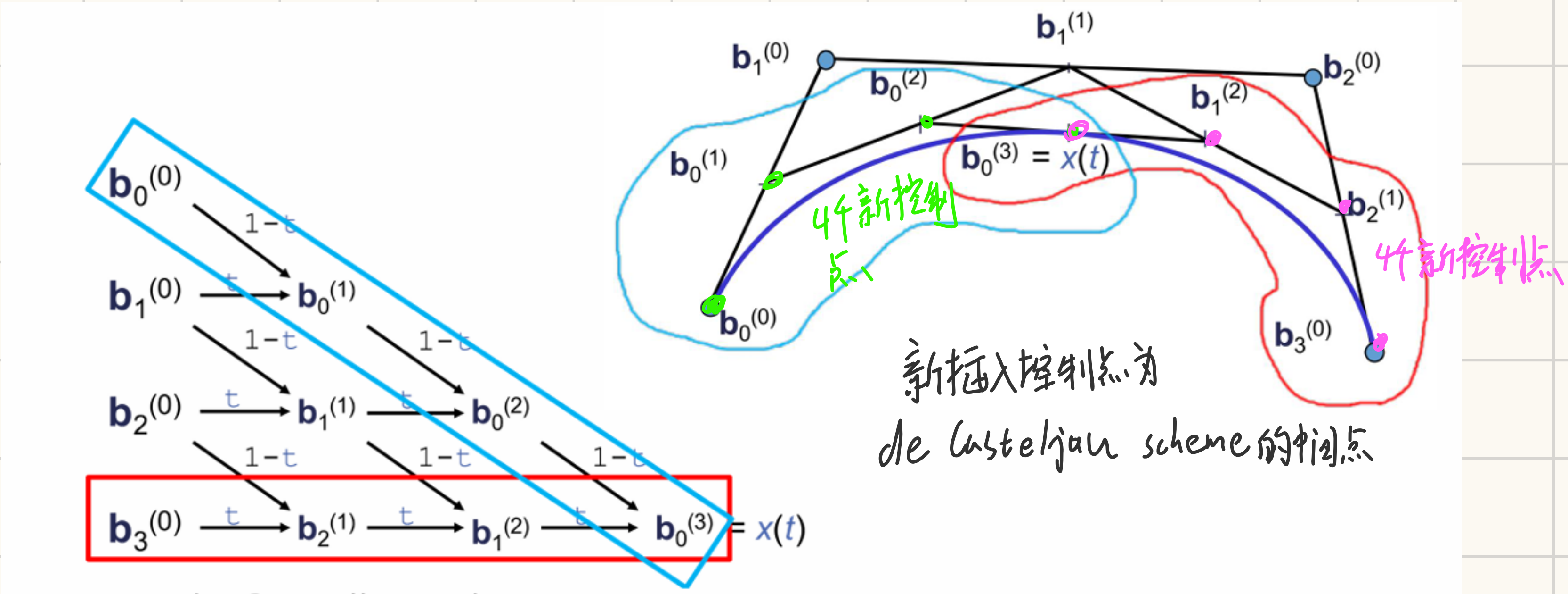
方法 $\bar{b}_0=b_0,\ \bar{b}_{n+1}=b_n$

$$\bar{b}_j=\frac{j}{n+1}b_{j-1}+\left(1-\frac{j}{n+1}\right)b_j$$

证明见PPT chapter 3

Subdivision

将一条 $b_0,\dots,b_n \to x(t)$ 分为 $b_0^{(1)},\dots,b_n^{(1)} \to x^{(1)}(t)$ 与 $b_0^{(2)},\dots,b_n^{(2)} \to x^{(2)}(t)$ 两条曲线



de Casteljau scheme

三、Bezier样条

对于曲线 c，其曲率为 $k(t)=\frac{\|c'(t)\times c''(t)\|}{\|c'(t)\|^3}$

例：圆 $c=(r\cos t, r\sin t)$

$c'(t)=(-r\sin t, r\cos t)$　　$c''(t)=(-r\cos t, -r\sin t)$

$k(t)=r^2/r^3=\frac{1}{r}$ 即圆的曲率为定值

曲线 c 的长度　$len(c)=\int_a^b \|c'\|dt$　$t\in[a,b]$ 构成曲线

如圆 $c=(r\cos t, r\sin t)$ $len(c)=\int_0^{2\pi} r\,dt=2\pi r$

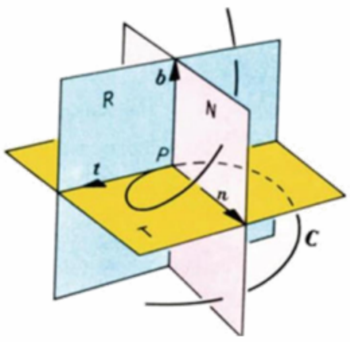
记长度为 s　　则 $s=\int_0^t \|c'\|dt$

则 c 可以表示为 s 的函数，称为 arc-length 参数化

如 $c=\begin{pmatrix}\cos t\\ \sin t\\ t\end{pmatrix}$　　$s(t)=\int_0^t \sqrt{2}\,dt=\sqrt{2}t$　　$t=\frac{s}{\sqrt{2}}$

则 arc-length 为　$c=\begin{pmatrix}\cos\frac{s}{\sqrt{2}}\\ \sin\frac{s}{\sqrt{2}}\\ \frac{s}{\sqrt{2}}\end{pmatrix}$

- The tangent（切线）$\boldsymbol{t}=\frac{c'}{\|c'\|}$, the normal plane $(p-p_0)\cdot\boldsymbol{t}=0$ 粉色N
- The binormal（副法线）$\boldsymbol{b}=\frac{c'\times c''}{\|c'\times c''\|}$, the osculating plane $(p-p_0)\cdot\boldsymbol{b}=0$ 黄色T
- The principal normal $\boldsymbol{n}=\boldsymbol{b}\times\boldsymbol{t}$ $=\frac{t'}{\|t'\|}=\frac{c''}{\|c''\|}$, the rectifying plane $(p-p_0)\cdot\boldsymbol{n}=0$ 蓝色R
- The curvature $\kappa(t)=\frac{c'\times c''}{\|c'\|^3}$ $=t'=c''$
- The torsion $\tau(t)=\frac{(c'\times c'')\cdot c'''}{\|c'\times c''\|^2}$ $=-b'\cdot n$



3维 Frenet Frame (arc-length 参数化)

$$\begin{pmatrix}e_1(s)\\ e_2(s)\\ e_3(s)\end{pmatrix}'=\begin{pmatrix}0 & k(s) & 0\\ -k(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}e_1(s)\\ e_2(s)\\ e_3(s)\end{pmatrix}$$

其中 $e_1(s)=c'(s)$　　$e_2(s)=\frac{c''(s)}{\|c''(s)\|}$　　$e_3(s)=e_1(s)\times e_2(s)$

C^0 连续：曲线连续

C^1：一阶导连续

C^2：二阶导连续

给定 $[0,1]$ 上的 Bezier 曲线 $y(u)$ 与 $[t_i, t_{i+1}]$ 上的 Bezier 曲线 $[t_i, t_{i+1}]$

令 $u(t) = \frac{t-t_i}{t_{i+1}-t_i}$ 　　则 $x(t) = y(u(t))$

$$x'(t) = y'(u(t)) \cdot u'(t) = \frac{y'(u(t))}{t_{i+1}-t_i}$$

归纳可得 $x^{[n]}(t) = \frac{y^{[n]}(u(t))}{(t_{i+1}-t_i)^n}$

例 $f(t) = \sum_{i=0}^{n} B_i^n(t) P_i$ 在 $[0,1]$ 上

$f(0) = P_0$ 　 $f(1) = P(1)$ 　 $f'(0) = n(P_1 - P_0)$ 　 $f'(1) = n(P_n - P_{n-1})$

$f''(0) = n(n-1)(P_2 - 2P_1 + P_0)$ 　 $f''(1) = n(n-1)(P_n - 2P_{n-1} + P_{n-2})$

则对于 $[t_i, t_{i+1}]$ 时 $x'(t_i) = \frac{n(P_1 - P_0)}{t_{i+1}-t_i}$ 　 $x''(t_{i+1}) = \frac{n(P_n - P_{n-1})}{(t_{i+1}-t_i)^2}$

若 $b_0^-, b_1^-, \dots, b_n^-$ 与 $b_0^+, b_1^+ \dots b_n^+$ 两段 bezier $(b_n^- = b_0^+)$

C^1 满足 $\frac{b_n^- - b_{n-1}^-}{t_j - t_{j-1}} = \frac{b_1^+ - b_0^+}{t_{j+1} - t_j}$ （一阶导相容）

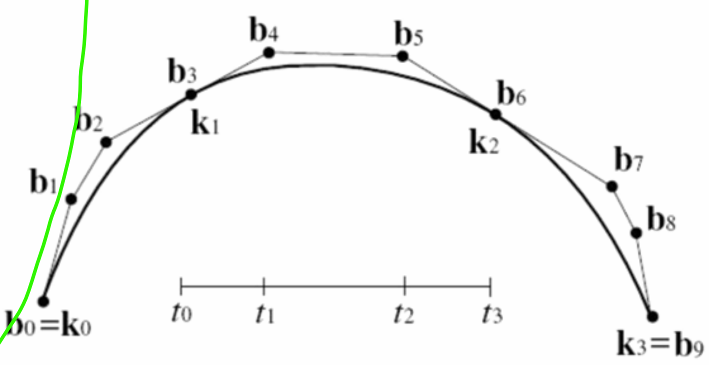
C^2 $\frac{b_n^- - 2b_{n-1}^- + b_{n-2}^-}{(t_j - t_{j-1})^2} = \frac{b_2^+ - 2b_1^+ + b_0^+}{(t_{j+1} - t_j)^2}$ （二阶导相容）

C^2 三次 Bezier 样条

Cubic Bézier Splines

- $3n+1$ unknown points
- $b_{3i} = k_i$ for $i = 0, \dots, n$ 已知点 $(n+1)$
  $n+1$ equations
- C^1 in points k_i for $i = 1, \dots, n-1$
  $n-1$ equations 一阶连续 $(n-1)$
- C^2 in points k_i for $i = 1, \dots, n-1$
  $n-1$ equations 二阶连续 $(n-1)$

$3n-1$ equations



边界条件

⇒ 2 additional conditions necessary: end conditions

如自然边界条件：$x''(t_0) = 0 \Rightarrow b_2 - 2b_1 + b_0 = 0$ 　 $b_1 = \frac{b_0 + b_2}{2}$

$x''(t_n) = 0 \Rightarrow b_{3n} - 2b_{3n-1} + b_{3n-2} = 0$ 　 $b_{3n-1} = \frac{b_{3n} + b_{3n-2}}{2}$

B样条

B样条的基函数

$$N_{i,1}(t)=\begin{cases}1, & t_i \le t < t_{i+1}\\ 0, & \text{otherwise}\end{cases}$$

$$N_{i,k}(t)=\frac{t-t_i}{t_{i+k-1}-t_i}N_{i,k-1}(t)+\frac{t_{i+k}-t}{t_{i+k}-t_{i+1}}N_{i+1,k-1}(t)$$

for $k>1$ and $i=0,\dots,n$

给定 $n+1$ 个控制点 $d_0,\dots,d_n \in R^3$，结点向量 $T=(t_0,\cdots,t_n,\cdots t_{n+k})$

对应的B样条为 $x(t)=\sum_{i=0}^{n} N_{i,k}(t)\cdot d_i$　d_i 为 de Boor 点　K表示K阶样条

多节效应：$t_0=t_1=\cdots=t_{k-1}$，$t_{n+1}=t_{n+2}=\cdots=t_{n+k}$

提 $T=(\underbrace{t_0,t_0,\cdots,t_0}_{k次},t_k,t_{k+1},\cdots,t_{n-1},t_n,\underbrace{t_{n+1},\cdots t_{n+1}}_{k次})$

端点处　$x(t_0)=d_0$　$x(t_{n+1})=d_n$　$x'(t_0)=\frac{k-1}{t_k-t_0}(d_1-d_0)$

de Boor 算法

1. Search index r with $t_r \le t < t_{r+1}$ 找下标
2. for $i=r-k+1,\dots,r$

   $d_i^0=d_i$

- for $j=1,\dots,k-1$

  for $i=r-k+1+j,\dots,r$ 迭代

  $d_i^j=\left(1-\alpha_i^j\right)\cdot d_{i-1}^{j-1}+\alpha_i^j\cdot d_i^{j-1}$

  with $\alpha_i^j=\frac{t-t_i}{t_{i+k-j}-t_i}$

  Then: $d_r^{k-1}=x(t)$

$$\begin{array}{lllll}
d_{r-k+1}=d_{r-k+1}^0 & & & & \\
d_{r-k+2}=d_{r-k+2}^0 & d_{r-k+2}^1 & & & \\
\cdots & & & & \\
d_{r-1}=d_{r-1}^0 & d_{r-1}^1 & \cdots & d_{r-1}^{k-2} & \\
d_r=d_r^0 & d_r^1 & \cdots & d_r^{k-2} & d_r^{k-1}=x(t)
\end{array}$$

例题见PPT chapter 6最后

C^2 3次 B样条（分段3次，此时 $k=3+1=4$，4阶样条）

则 $T=(t_0,t_1,\cdots,t_{n+6})=(\underbrace{s_0,s_0,s_0,s_0}_{4},s_1,\cdots s_{n-1},\underbrace{s_n,s_n,s_n,s_n}_{4})$

$$\begin{cases} x(s_0)=k_0=d_0 \\ x(s_i)=k_i=N_{i,4}(s_i)d_i+N_{i+1,4}(s_i)d_{i+1}+N_{i+2,4}(s_i)d_{i+2} & i=1,\cdots,n-1 \\ x(s_n)=k_n=d_{n+2} \end{cases}$$

加2个边界条件
如自然边界

$$\begin{cases} x''(s_0)=0 \Longleftrightarrow \dfrac{d_2-d_1}{s_2-s_0}=\dfrac{d_1-d_0}{s_1-s_0} \\ x''(s_n)=0 \Longleftrightarrow \dfrac{d_{n+2}-d_{n+1}}{s_n-s_{n-1}}=\dfrac{d_{n+1}-d_n}{s_n-s_{n-2}} \end{cases}$$

方程形式

$$\begin{pmatrix} 1 & & & & & & \\ \alpha_0 & \beta_0 & \gamma_0 & & & & \\ & \alpha_1 & \beta_1 & \gamma_1 & & & \\ & & & \ddots & & & \\ & & & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} & \\ & & & & \alpha_n & \beta_n & \gamma_n \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_n \\ d_{n+1} \\ d_{n+2} \end{pmatrix} = \begin{pmatrix} k_0 \\ 0 \\ k_1 \\ \vdots \\ k_{n-1} \\ 0 \\ k_n \end{pmatrix}$$

边界条件

解方程得 de boor 点后
使用 de boor 算法

- with
$$\alpha_0=s_2-s_0$$
$$\beta_0=-(s_2-s_0)-(s_1-s_0)$$
$$\gamma_0=s_1-s_0$$

$$\alpha_n=s_n-s_{n-1}$$
$$\beta_n=-(s_n-s_{n-1})-(s_n-s_{n-2})$$
$$\gamma_n=s_n-s_{n-2}$$

Natural end conditions

$$\alpha_i=N_{i,4}(s_i)$$
$$\beta_i=N_{i+1,4}(s_i)$$
$$\gamma_i=N_{i+2,4}(s_i)$$
for $i=1,\ldots,n-1$

极坐标与开花算法

对角 $F(t) = f(t, t, \cdots, t)$

对称 对$\forall$排列π，有 $f(t_1, t_2, \cdots, t_d) = f(t_{\pi(1)}, t_{\pi(2)}, \cdots, t_{\pi(d)})$

多重仿射 $\sum a_k = 1$ $f(t_1, t_2, \cdots, \sum a_k t_i^{(k)}, \cdots t_d) = \sum a_k f(t_1, t_2, \cdots t_i^{(k)}, \cdots, t_d)$

Polar forms of monomials: 极坐标

- Degree 0: $f = c_0$
- Degree 1: $f(t_1) = c_0 + c_1 t_1$
- Degree 2: $f(t_1, t_2) = c_0 + c_1 \frac{t_1 + t_2}{2} + c_2 t_1 t_2$
- Degree 3: $f(t_1, t_2, t_3) = c_0 + c_1 \frac{t_1 + t_2 + t_3}{3} + c_2 \frac{t_1 t_2 + t_2 t_3 + t_1 t_3}{3}$ $+ c_3 t_1 t_2 t_3$

更一般情况 $f(t_1, \cdots t_n) = \sum_{i=0}^{n} \frac{c_i}{\binom{n}{i}} \sum_{\substack{S \subseteq \{1, \cdots n\} \\ |S| = i}} \prod_{j \in S} t_j$

$C_k = \frac{1}{k!} \frac{d^k}{dt^k} F(0) = \binom{n}{k} f(\underbrace{0, \cdots, 0}_{(n-k)个}, \underbrace{\hat{1}, \cdots, \hat{1}}_{k个})$ $\hat{1} = 1 - 0$

$\frac{d^k}{dt^k} F(t) = \frac{n!}{(n-k)!} f(\underbrace{t, \cdots, t}_{n-k}, \underbrace{\hat{1}, \cdots \hat{1}}_{k})$

例: $F(t) = f(t_1, t_2, t_3)$

有 $F'(t) = \frac{3!}{(3-1)!} f(t, t, \hat{1}) = 3(f(t, t, 1) - f(t, t, 0)) = 3c_3 t^2 + 2c_2 t + c_1$

$\forall t_1, t_2, t_3$ $f(t_1, t_2, t_3) = g(t_1, t_2, t_3) \Rightarrow f \equiv g$ $\forall t_1, t_2$ $f(t_1, t_2, t) = g(t_1, t_2, t) \Rightarrow C^2$ at t

提升多项式的度：

$f^{(+1)}(t_1, \cdots, t_{d+1}) = \frac{1}{d+1} \sum_{i=1}^{d+1} f(t_1, \cdots t_{i-1}, t_{i+1}, \cdots, t_{d+1})$ 没有t_i

$f(t, \cdots, t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} f(\underbrace{0, \cdots, 0}_{n-i}, \underbrace{1, \cdots 1}_{i})$

在$[u, v]$上3次Bezier曲线的4个控制点为 $p(u,u,u)$ $p(u,u,v)$ $p(u,v,v)$ $p(v,v,v)$

以3次为例 由单项式系数与Bezier系数转换

设 $p(t)=1+2t+3t^2-t^3$

① 3次Bezier曲线为 $f(t)=\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}\begin{pmatrix}1 & & & \\ -3 & 3 & & \\ 3 & -6 & 3 & \\ -1 & 3 & -3 & 1\end{pmatrix}\begin{pmatrix}P_0\\ P_1\\ P_2\\ P_3\end{pmatrix}$

则Bezier系数为 $\begin{pmatrix}1 & & & \\ -3 & 3 & & \\ 3 & -6 & 3 & \\ 1 & 3 & -3 & 1\end{pmatrix}^{-1}\begin{pmatrix}1\\ 2\\ 3\\ -1\end{pmatrix}=\begin{pmatrix}1\\ \frac{5}{3}\\ \frac{11}{3}\\ 5\end{pmatrix}$

② $b(t_0,t_1,t_2)=1+2\,\dfrac{t_0+t_1+t_2}{3}+3\,\dfrac{t_0t_1+t_1t_2+t_0t_2}{3}-t_0t_1t_2$

对应的4个点为 $b(0,0,0)=1$ $b(0,0,1)=\frac{5}{3}$ $b(0,1,1)=\frac{10}{3}$ $b(1,1,1)=5$

与Bezier等价的B样条

从 $[i,i,i]$ 到 $[i+1,i+1,i+1]$ 共4个节点

依次类推,用开花算法计算区间的所有节点即为贝塞尔曲线控制点。

有理曲线

相机参数：

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$\begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}$ 为一个正交矩阵，表示自然坐标系向相机坐标系变换

每列向量 $(d,0,0)^T$ $(0,d,0)^T$ $(0,0,d)^T$ 分别表示 x, y, z 轴方向

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 表示相机位置

有理Bezier曲线

有理 de Casteljau 算法

- Rational de Casteljau algorithm

$$\boldsymbol{b}_i^{(r)}(t) = (1-t)\frac{\omega_i^{(r-1)}(t)}{\omega_i^{(r)}(t)}\boldsymbol{b}_i^{(r-1)}(t) + t\frac{\omega_{i+1}^{(r-1)}(t)}{\omega_i^{(r)}(t)}\boldsymbol{b}_{i+1}^{(r-1)}(t)$$

with $\omega_i^{(r)}(t) = (1-t)\omega_i^{(r-1)}(t) + t\omega_{i+1}^{(r-1)}(t)$

$$f^{(eucl)}(t) = \sum_{i=0} P_i \frac{B_i^{(d)}(t)w_i}{\sum_{j=0} B_j^{(d)}(t)w_j}$$

化简得：

$$\boldsymbol{f}^{(eucl)}(t) = \frac{B_0^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\,\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t})\omega_2 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\,\omega_1 + B_2^{(2)}(\tilde{t})\omega_2}$$

$$= \frac{B_0^{(2)}(\tilde{t})\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\sqrt{\frac{1}{\omega_0\omega_2}}\,\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t}) + B_1^{(2)}(\tilde{t})\sqrt{\frac{1}{\omega_0\omega_2}}\,\omega_1 + B_2^{(2)}(\tilde{t})}$$

$$= \frac{B_0^{(2)}(\tilde{t})\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\omega\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t}) + B_1^{(2)}(\tilde{t})\omega + B_2^{(2)}(\tilde{t})} \quad \text{with } \omega := \sqrt{\frac{1}{\omega_0\omega_2}}\,\omega_1$$

取 $w = -w$ 代入得到对偶曲线拼合整条曲线

例：2维双曲线的有理曲线

$$\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$$

令 $t = \tan\frac{\theta}{2}$ 得到 $\begin{pmatrix} a(1+t^2) \\ 2bt \\ 1-t^2 \end{pmatrix}$

$a(1+t^2) = a(B_0^{(2)} + B_1^{(2)} + 2B_2^{(2)})$

$2bt = b(B_1^{(2)} + 2B_2^{(2)})$

$1-t^2 = B_0^{(2)} + B_1^{(2)}$

$$f^{(hom)} = \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} B_0^{(2)} + \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} B_1^{(2)} + \begin{pmatrix} 2a \\ 2b \\ 0 \end{pmatrix} B_2^{(2)}$$

则 $$f^{(eucl)} = \frac{\begin{pmatrix} a \\ 0 \end{pmatrix} B_0^{(2)} + \begin{pmatrix} a \\ b \end{pmatrix} B_1^{(2)} + \begin{pmatrix} 2a \\ 2b \end{pmatrix} B_2^{(2)}}{B_0^{(2)} + B_1^{(2)}}$$

或使用极坐标

$\begin{pmatrix} a(1+t_1t_2) \\ b(t_1+t_2) \\ 1-t_1t_2 \end{pmatrix}$ 控制点 $(0,0)$, $(0,1)$, $(1,1)$

得到 $\bar{P}_0 = \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix}$ $\bar{P}_1 = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$ $\bar{P}_2 = \begin{pmatrix} 2a \\ 2b \\ 0 \end{pmatrix}$

$$f^{(hom)} = \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} B_0^{(2)} + \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} B_1^{(2)} + \begin{pmatrix} 2a \\ 2b \\ 0 \end{pmatrix} B_2^{(2)}$$

样条曲面

张量积形式的Bernstein基: $f(u,v)=\sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u)\, B_j^{(d)}(v)\, P_{ij}$

Rational Patch

贝塞尔

$$f^{(hom)}(u,v)=\sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u)B_j^{(d)}(v)\begin{pmatrix}\omega_{i,j}\boldsymbol{p}_{i,j}\\ \omega_{i,j}\end{pmatrix}$$

$$(B_1(u)\ \ B_2(u)\ \cdots\ B_n(u))\,[W_{ij}P_{ij}]\begin{pmatrix}B_1(v)\\ B_2(v)\\ \vdots\\ B_n(v)\end{pmatrix}$$

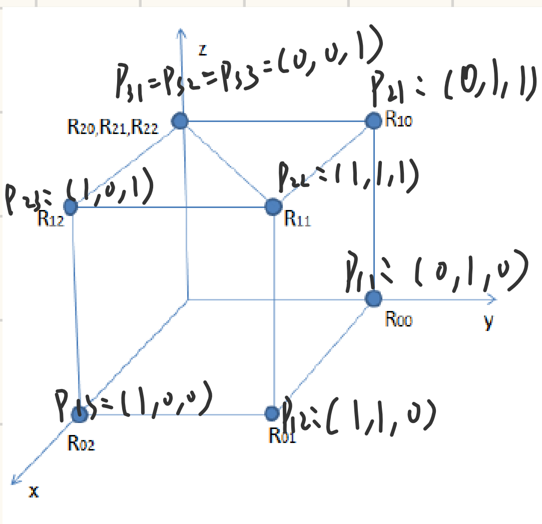
$$f^{(Eucl)}(u,v)=\frac{\sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u)B_j^{(d)}(v)\omega_{i,j}\boldsymbol{p}_{i,j}}{\sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u)B_j^{(d)}(v)\omega_{i,j}}$$

B样条

$$f^{(hom)}(u,v)=\sum_{i=0}^{d}\sum_{j=0}^{d} N_i^{(d)}(u)N_j^{(d)}(v)\begin{pmatrix}\omega_{i,j}\boldsymbol{p}_{i,j}\\ \omega_{i,j}\end{pmatrix}$$

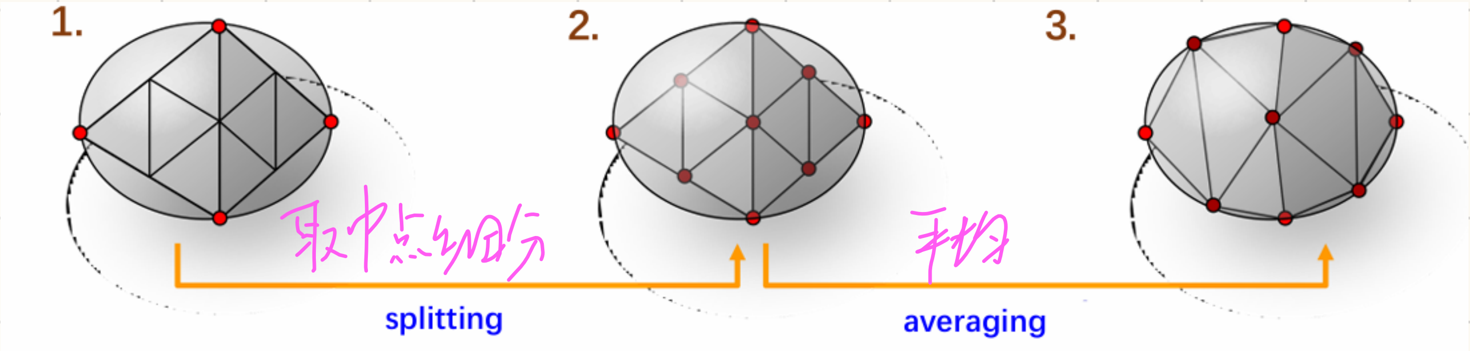
$$f^{(Eucl)}(u,v)=\frac{\sum_{i=0}^{d}\sum_{j=0}^{d} N_i^{(d)}(u)N_j^{(d)}(v)\omega_{i,j}\boldsymbol{p}_{i,j}}{\sum_{i=0}^{d}\sum_{j=0}^{d} N_i^{(d)}(u)N_j^{(d)}(v)\omega_{i,j}}$$

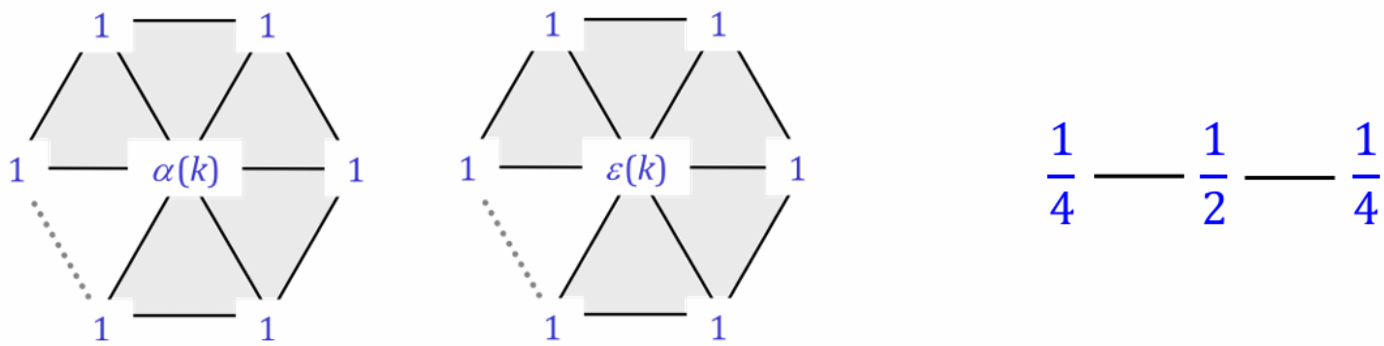
双二次曲面控制点选取(画单位球)



$$W=\begin{pmatrix}1\\ \frac{\sqrt2}{2}\\ 1\end{pmatrix}\left(1,\frac{\sqrt2}{2},1\right)=\begin{pmatrix}1 & \frac{\sqrt2}{2} & 1\\ \frac{\sqrt2}{2} & 1 & \frac{\sqrt2}{2}\\ 1 & \frac{\sqrt2}{2} & 1\end{pmatrix}$$

细化曲线、曲面





averaging mask　　evaluation (limit) mask　　boundary/sharp crease mask

$$\alpha(k) = k(1-\beta(k))/\beta(k) \qquad \varepsilon(k) = 3k/(4\beta(k))$$

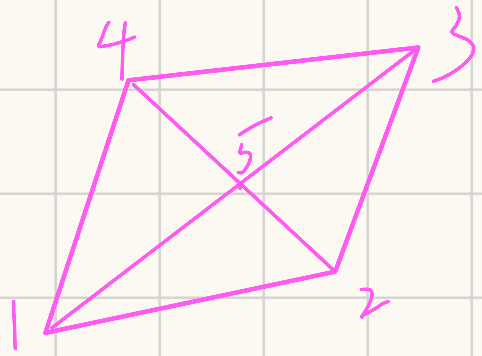
$$\beta(k) = \frac{5}{4} - \frac{(3 + 2\cos(2\pi/k))^2}{32}$$ k为邻边

创建邻接矩阵，将邻接矩阵对角元变为α(k)，并将每行归一化

参数化

Tutte：

下图为例：



1、得到边界点(1,2,3,4)

2、设置矩阵

$$A=\left.\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}\right.$$

对于边界点，该行只有该元素为1，其他为0

非边界点，该元素为1，其余除它外每个邻边元素为 $-\frac{1}{d}$

$$b=\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ 0 \end{pmatrix}$$

边界点已知被映射后的结果（如映射到单位圆）

非边界点设为0

3、解 $Ay=b$　y即为映射后的结果