

Computer Aided Geometric Design

Fall Semester 2024

Blossoming and Polar Forms
Bézier Splines and B-Splines Revisited

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A Short Step Back

Bézier & Monomials

Matrix Form

Matrix Notation: Bézier → Monomials

$$\mathbf{f}(t) = (1 \quad t \quad t^2) \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix}$$

(quadratic case)

$$\mathbf{f}(t) = (1 \quad t \quad t^2 \quad t^3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix}$$

(cubic case)

Format Conversion

Conversion: Compute Bézier coefficients from monomial coefficients

$$\begin{pmatrix} \mathbf{c}_0^{(Bez.)} \\ \mathbf{c}_1^{(Bez.)} \\ \mathbf{c}_2^{(Bez.)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} \quad (\text{quadratic case})$$

$$\begin{pmatrix} \mathbf{c}_0^{(Bez.)} \\ \mathbf{c}_1^{(Bez.)} \\ \mathbf{c}_2^{(Bez.)} \\ \mathbf{c}_3^{(Bez.)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{pmatrix} \quad (\text{cubic case})$$

Format Conversion

Conversion: quadratic to cubic

$$\begin{pmatrix} \mathbf{c}_0^{(3)} \\ \mathbf{c}_1^{(3)} \\ \mathbf{c}_2^{(3)} \\ \mathbf{c}_3^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{c}_0^{(2)} \\ \mathbf{c}_1^{(2)} \\ \mathbf{c}_2^{(2)} \\ 0 \end{pmatrix}$$

Convert to monomials and back to Bézier coefficients. (other degrees similar)

Example application: Output of TrueType fonts in Postscript.

Polar Forms & Blossoms

Idea & Definition

Affine Combinations

Definition (reminder):

- An *affine combination* of n points $\in \mathbb{R}^d$ is given by:

$$P_\alpha = \sum_{i=1}^n \alpha_i p_i \text{ with } \sum_{i=1}^n \alpha_i = 1$$

- A function f is said to be *affine* in its parameter x_i , if:

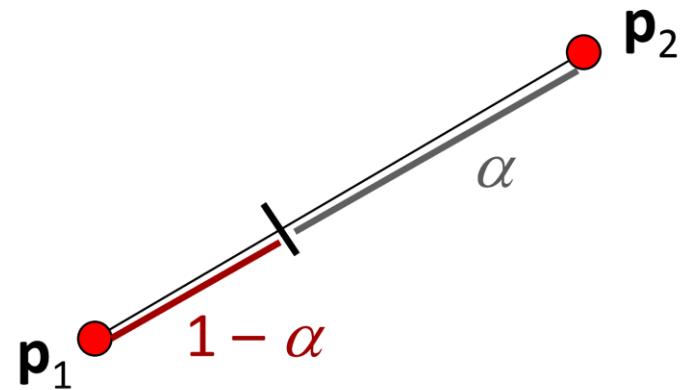
$$f\left(x_1, \dots, \sum_{i=1}^n \alpha_i x_i^{(k)}, \dots, x_m\right) = \sum_{i=1}^n \alpha_i f\left(x_1, \dots, x_i^{(k)}, \dots, x_m\right) \text{ for } \sum_{i=1}^n \alpha_i = 1$$

Affine Combinations

Examples:

- Linear (affine) interpolation of 2 points:

$$\mathbf{p}_\alpha = \alpha \mathbf{p}_1 + (1 - \alpha) \mathbf{p}_2$$



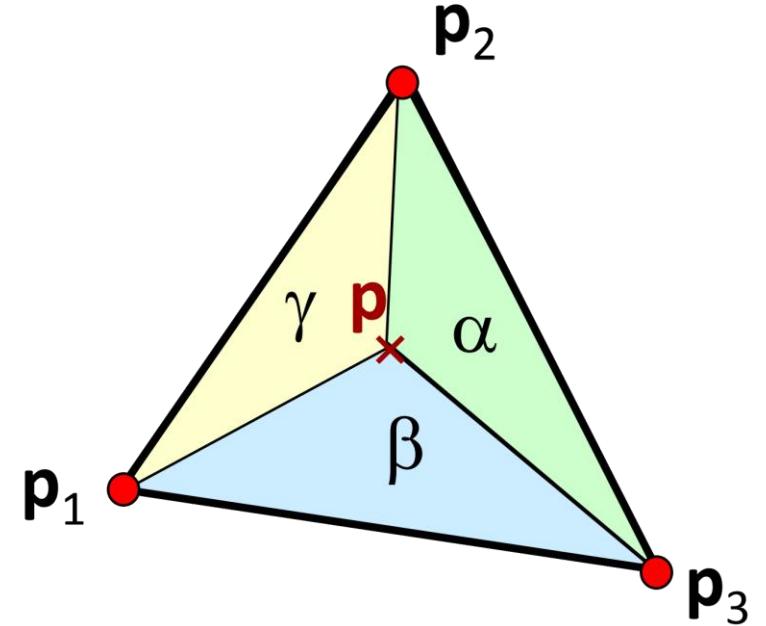
Affine Combinations

Examples:

- Barycentric combinations of 3 points

("barycentric coordinates")

$$p = \alpha p_1 + \beta p_2 + \gamma p_3, \text{ with } \alpha + \beta + \gamma = 1$$



Properties:

$$\gamma = 1 - \alpha - \beta$$

$$\alpha = \frac{\Delta(p_2, p_3, p)}{\Delta(p_1, p_2, p_3)},$$

$$\beta = \frac{\Delta(p_1, p_3, p)}{\Delta(p_1, p_2, p_3)},$$

$$\gamma = \frac{\Delta(p_1, p_2, p)}{\Delta(p_1, p_2, p_3)}$$

Transformation to barycentric coordinates is a linear map
(heights in triangles)

Formalizing the Idea

- **Idea:** Express (piecewise) polynomial curves as iterated linear (affine) interpolations
- **First steps:**
 - A polynomials: $P(t) = at^3 + bt^2 + ct + d$
 - Can be written as: $P(t) = a \cdot t \cdot t \cdot t + b \cdot t \cdot t + c \cdot t + d$
 - Interpret each variable t as a separate parameter:

new function

$$p(t_1, t_2, t_3) = a \cdot t_1 \cdot t_2 \cdot t_3 + b \cdot t_1 \cdot t_2 + c \cdot t_1 + d$$

Polar Forms

Solution: Polar Forms / Blossoms

A *polar form* or *blossom* f of a polynomial F of degree d is a function in d variables:

$$F: \mathbb{R} \rightarrow \mathbb{R}$$

$$f: \mathbb{R}^d \rightarrow \mathbb{R}$$

with the following properties:

- *Diagonality*: $f(t, t, \dots, t) = F(t)$
- *Symmetry*: $f(t_1, t_2, \dots, t_d) = f(t_{\pi(1)}, t_{\pi(2)}, \dots, t_{\pi(d)})$
for all permutations of indices π
- *Multi-affine*: $\sum \alpha_k = 1$
 $\Rightarrow f(t_1, t_2, \dots, \sum \alpha_k t_i^{(k)}, \dots, t_d)$
 $= \alpha_1 f(t_1, t_2, \dots, t_i^{(1)}, \dots, t_d) + \dots + \alpha_n f(t_1, t_2, \dots, t_i^{(n)}, \dots, t_d)$

Polar Forms

Rationale:

- Model polynomial as multi-affine function
 - (multi-affine property)
- Plug in a common parameter to obtain the original polynomial
 - (diagonal of the blossom)
- Symmetry property – makes the solution unique
 - There is exactly one polar form for each polynomial
 - This standardization makes different polars “compatible”, we can compare them with each other
 - We will see how this works in a few slides …

Properties

The mapping from polynomials to their corresponding polar forms is one-to-one

- For each polar form $f(t_1, t_2, \dots, t_n)$,
a unique polynomials $F(\mathbf{t})$ exists
- For each polynomial $F(\mathbf{t})$,
a unique polar form $f(t_1, t_2, \dots, t_n)$ exists

Properties

Polar forms of monomials:

- Degree 0: $1 \rightarrow ?$
- Degree 1:
- Degree 2:
- Degree 3:

Properties

Polar forms of monomials:

- Degree 0: $1 \rightarrow 1$
- Degree 1: $1 \rightarrow 1, t \rightarrow t$
- Degree 2: $1 \rightarrow 1, t \rightarrow \frac{t_1+t_2}{2}, t^2 \rightarrow t_1 t_2$
- Degree 3: $1 \rightarrow 1, t^2 \rightarrow \frac{t_1 t_2 + t_2 t_3 + t_1 t_3}{3}$
 $t \rightarrow \frac{t_1 + t_2 + t_3}{3}, t^3 \rightarrow t_1 t_2 t_3$

Properties

Polar forms of monomials:

- Degree 0: $f = c_0$

- Degree 1: $f(t_1) = c_0 + c_1 t_1$

- Degree 2: $f(t_1, t_2) = c_0 + c_1 \frac{t_1+t_2}{2} + c_2 t_1 t_2$

- Degree 3:
$$f(t_1, t_2, t_3) = c_0 + c_1 \frac{t_1+t_2+t_3}{3} + c_2 \frac{t_1 t_2 + t_2 t_3 + t_1 t_3}{3}$$
$$+ c_3 t_1 t_2 t_3$$

Properties

General Case:

$$f(t_1, \dots, t_n) = \sum_{i=0}^n c_i \binom{n}{i}^{-1} \sum_{\substack{S \subseteq \{1 \dots n\}, \\ |S|=i}} \prod_{j \in S} t_j$$

- The c_i are the monomial coefficients
- Idea: use all possible subsets of t_i to make it symmetric
- This solution is unique
- Without the symmetry property, there would be a large number of solutions

$|S|$ denotes the cardinality of the set S

Generalizations

Blossoms for polynomials curves (points as output):

- Polar form of a polynomial curve of degree d :

$$\begin{aligned} F: \quad \mathbb{R} &\rightarrow \mathbb{R}^n \\ f: \quad \mathbb{R}^d &\rightarrow \mathbb{R}^n \end{aligned}$$

new

- Required Properties:

- Diagonality: $f(t, t, \dots, t) = F(t)$

- Symmetry: $f(t_1, t_2, \dots, t_d) = f(t_{\pi(1)}, t_{\pi(2)}, \dots, t_{\pi(d)})$
for all permutations of indices π

- Multi-affine: $\sum \alpha_k = 1$

$$\begin{aligned} &\Rightarrow f\left(t_1, t_2, \dots, \sum \alpha_k t_i^{(k)}, \dots, t_d\right) \\ &= \alpha_1 f\left(t_1, t_2, \dots, t_i^{(1)}, \dots, t_d\right) + \dots + \alpha_n f\left(t_1, t_2, \dots, t_i^{(n)}, \dots, t_d\right) \end{aligned}$$

Generalizations

Blossoms with points as arguments:

- Polar form of degree d with points as input and output:

$$\begin{aligned} F: \quad & \mathbb{R}^m \xrightarrow{\text{new}} \mathbb{R}^n \\ f: \quad & \mathbb{R}^{d \times m} \xrightarrow{\text{new}} \mathbb{R}^n \end{aligned}$$

- Required Properties:

- Diagonality: $f(\mathbf{t}, \mathbf{t}, \dots, \mathbf{t}) = F(\mathbf{t})$

- Symmetry: $f(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_d) = f(\mathbf{t}_{\pi(1)}, \mathbf{t}_{\pi(2)}, \dots, \mathbf{t}_{\pi(d)})$

for all permutations of indices π

- Multi-affine: $\sum \alpha_k = 1$

$$\begin{aligned} & \Rightarrow f\left(\mathbf{t}_1, \mathbf{t}_2, \dots, \sum \alpha_k \mathbf{t}_i^{(k)}, \dots, \mathbf{t}_d\right) \\ & = \alpha_1 f\left(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_i^{(1)}, \dots, \mathbf{t}_d\right) + \dots + \alpha_n f\left(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_i^{(n)}, \dots, \mathbf{t}_d\right) \end{aligned}$$

Generalizations

Vector arguments

- We will have to distinguish between *points* and *vectors* (differences of points)
- Use “hat” notation $\hat{v} = p - q$ to denote vectors
- Also defined in the one dimensional case (vectors in \mathbb{R})
- One vector: $\hat{1} = 1 - 0$, $\hat{\mathbf{1}} = [1, \dots, 1]^T - \mathbf{0}$
- Define shorthand notation (recursive) to define vectors in polar form:

$$f(t_1, \dots, t_{n-k}, \hat{v}_1, \dots, \hat{v}_k) := f(t_1, \dots, t_{n-k}, p_1, \hat{v}_2, \dots, \hat{v}_k) - f(t_1, \dots, t_{n-k}, q_1, \hat{v}_2, \dots, \hat{v}_k)$$

$n-k$ k $n-k$ $k-1$ $n-k$ $k-1$

Properties

Derivatives of blossoms:

$$f(t_1, \dots, t_n) = \sum_{i=0}^n c_i \binom{n}{i}^{-1} \sum_{\substack{S \subseteq \{1 \dots n\}, j \in S \\ |S|=i}} \prod_{j \in S} t_j$$

- The c_i are related to the derivatives at $t = 0$
- Hence: $c_k = \frac{1}{k!} \frac{d^k}{dt^k} F(0) = \binom{n}{k} f\left(\underbrace{0, \dots, 0}_{n-k}, \underbrace{\hat{1}, \dots, \hat{1}}_k\right)$
- In general: $\frac{d^k}{dt^k} F(t) = \frac{n!}{(n-k)!} f\left(\underbrace{t, \dots, t}_{n-k}, \underbrace{\hat{1}, \dots, \hat{1}}_k\right)$

Example

$$f(t_1, t_2, t_3) = c_0 + c_1 \frac{t_1 + t_2 + t_3}{3} + c_2 \frac{t_1 t_2 + t_1 t_3 + t_2 t_3}{3} + c_3 t_1 t_2 t_3$$

$$\begin{aligned} f'(t) &= \frac{3!}{2!} \left[\left(c_0 + c_1 \frac{1+t+t}{3} + c_2 \frac{1t+tt+1t}{3} + c_3 1tt \right) - \left(c_0 + c_1 \frac{0+t+t}{3} + c_2 \frac{tt}{3} \right) \right] \\ &= 3 \left(c_1 \frac{1}{3} + c_2 \frac{2t}{3} + c_3 tt \right) \\ &= 3c_3 t^2 + 2c_2 t + c_1 \end{aligned}$$

Continuity Condition

Theorem: continuity condition for polynomials

The following statements are equivalent:

1. F and G are C^k -continuous at t
2. $\forall t_1, \dots, t_k: f(t, \dots, t, t_1, \dots, t_k) = g(t, \dots, t, t_1, \dots, t_k)$
3. $f(t, \dots, t, \underbrace{\hat{1}, \dots, \hat{1}}_{k\text{-times}}) = g(t, \dots, t, \underbrace{\hat{1}, \dots, \hat{1}}_{k\text{-times}})$

$$\begin{aligned} 2 \Leftrightarrow 3: \quad & f(t, \dots, t, t_1) = f(t, \dots, t, t_1 - 0) \\ &= t_1(f(t, \dots, t, 1) - f(t, \dots, t, 0)) \\ &= t_1 f(t, \dots, t, \hat{1}) \end{aligned}$$

Continuity Condition

Examples:

- $\forall \textcolor{blue}{t}_1, \textcolor{blue}{t}_2, \textcolor{blue}{t}_3: f(\textcolor{blue}{t}_1, \textcolor{blue}{t}_2, \textcolor{blue}{t}_3) = g(\textcolor{blue}{t}_1, \textcolor{blue}{t}_2, \textcolor{blue}{t}_3) \Rightarrow$ same curve
- $\forall \textcolor{blue}{t}_1, \textcolor{blue}{t}_2: f(\textcolor{blue}{t}_1, \textcolor{blue}{t}_2, \textcolor{red}{t}) = g(\textcolor{blue}{t}_1, \textcolor{blue}{t}_2, \textcolor{red}{t}) \Rightarrow C^2$ at $\textcolor{red}{t}$
- $\forall \textcolor{blue}{t}_1: f(\textcolor{blue}{t}_1, \textcolor{red}{t}, \textcolor{red}{t}) = g(\textcolor{blue}{t}_1, \textcolor{red}{t}, \textcolor{red}{t}) \Rightarrow C^1$ at $\textcolor{red}{t}$
- $f(\textcolor{red}{t}, \textcolor{red}{t}, \textcolor{red}{t}) = g(\textcolor{red}{t}, \textcolor{red}{t}, \textcolor{red}{t}) \Rightarrow C^0$ at $\textcolor{red}{t}$

Raising the Degree

Raising the degree of a blossom:

- Can we directly construct a polar form with degree elevated by one from a lower degree one, without changing the polynomial?
- [Other than transforming to monomials, adding $0 \cdot t^{d+1}$, and transforming back?]

Solution:

- Given: $f(t_1, \dots, t_d)$
- We obtain: $f^{(+1)}(t_1, \dots, t_{d+1}) = \frac{1}{d+1} \sum_{i=1}^{d+1} f(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{d+1})$



leave out t_i

Raising the Degree

Proof:

$$\begin{aligned}\forall t: f^{(+1)}(t, \dots, t) &= \frac{1}{d+1} \sum_{i=1}^{d+1} f(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{d+1}) \Big|_{t_1=\dots=t_{d+1}=t} \\ &= \frac{1}{d+1} \sum_{i=1}^{d+1} f(t, \dots, t) \\ &= f(t, \dots, t) \\ \Rightarrow F^{(+1)}(t) &= F(t)\end{aligned}$$

Polars and Control Points

Interpretation (Examples):

- Multi-variate function: $f(t_1, t_2, t_3)$
 - Describes a point depending on three parameters
 - Where $f(t_1, t_2, t_3)$ moves for changing (t_1, t_2, t_3)
(think of storing monomial coefficients inside)
- Polynomial value: $f(1.5, 1.5, 1.5)$
 - One value of the polynomial curve: $F(1.5)$
- Off-curve points: $f(1, 2, 3)$
 - Evaluate points not necessarily on the polynomial curve
 - Question: what meaning do various off-curve points have?
 - We will use off-curve points as control points

Polars and Control Points

Interpretation (Examples):

- Specifying: $f(t_1, t_2, t_3)$
 - Assume, f is not known yet
 - We want to determine a polar (i.e. a polynomial)
- On-curve points:
 $\{f(0,0,0) = x_0, f(1,1,1) = x_1, f(2,2,2) = x_2, f(3,3,3) = x_3\}$
 - Degree d polynomial has $d + 1$ degrees of freedom
 - We know already how to do polynomial interpolation
- Off-curve points:
 $\{f(1, 2, 3) = x_{123}, f(2, 3, 4) = x_{234}\}$
 - We can also use off-curve points to specify the polar/polynomial
 - This is the main motivation for the whole formalism

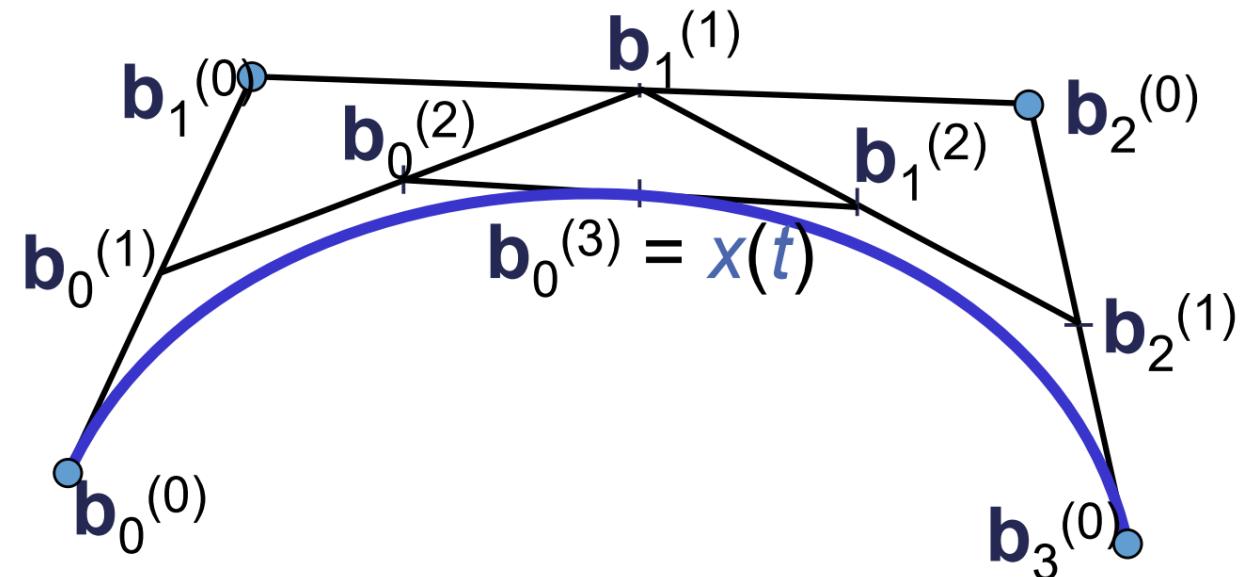
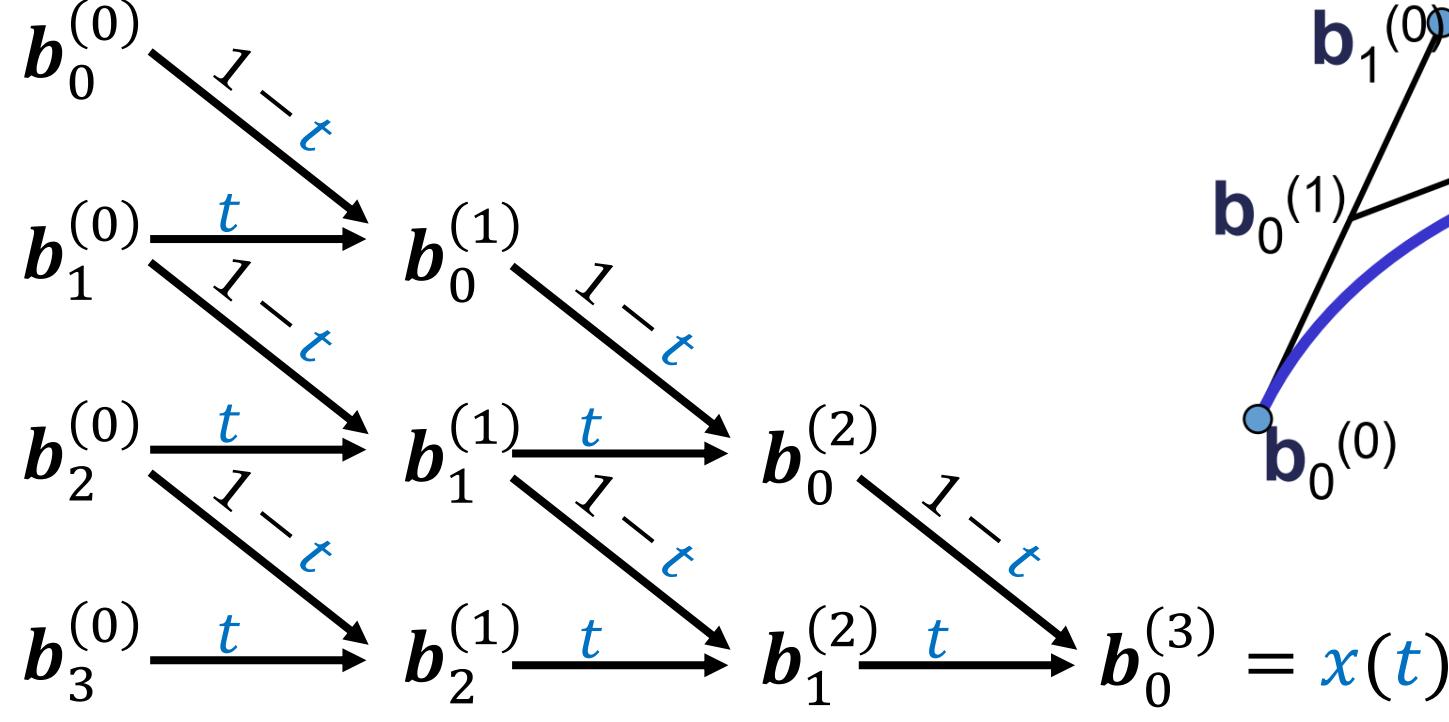
Polar Forms & Blossoms

Bézier Splines

De Casteljau algorithm (from earlier)

Repeated convex combination of control points

$$\mathbf{b}_i^{(r)} = (1 - t)\mathbf{b}_i^{(r-1)} + t\mathbf{b}_{i+1}^{(r-1)}$$



De Casteljau Algorithm

The de Casteljau algorithm is simple to state with blossoms:

- We just have to exchange the labels
- Then use the multi-affinity property in order to compute intermediate points
- With this view, we can easily show that the de Casteljau algorithm is equivalent to the formulation based on Bernstein polynomials

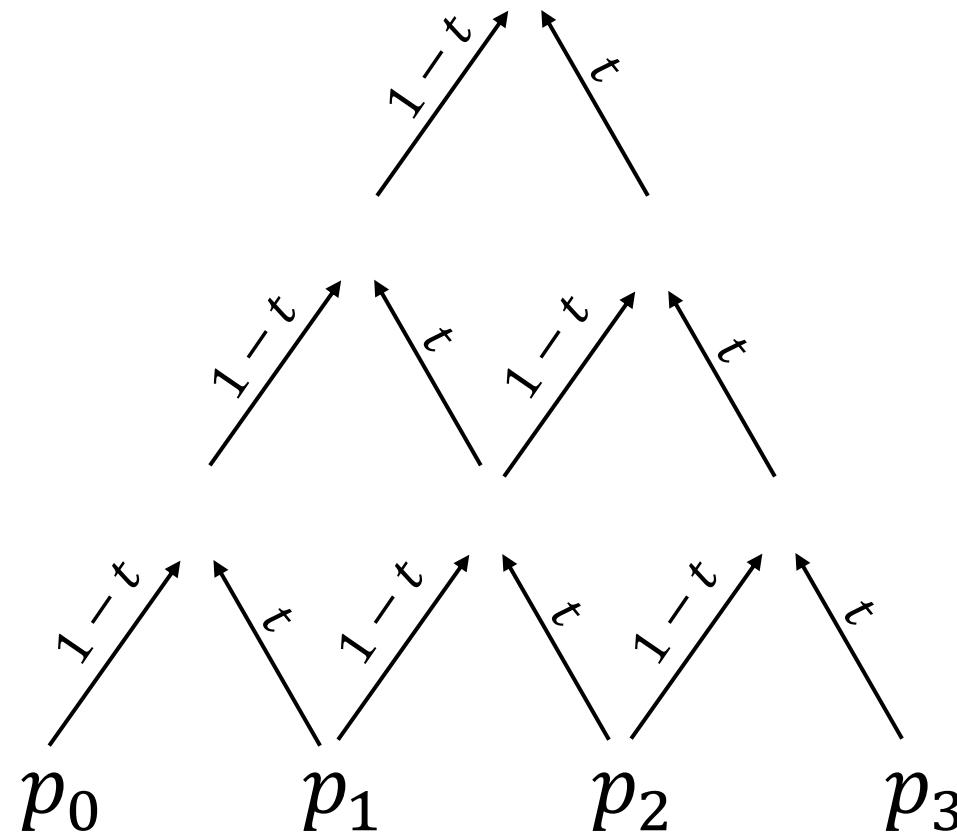
Key observation

$$b(0, 0(1 - t) + 1t, 1)$$
$$b(0, 0, 1)$$
$$b(0, 1, 1)$$

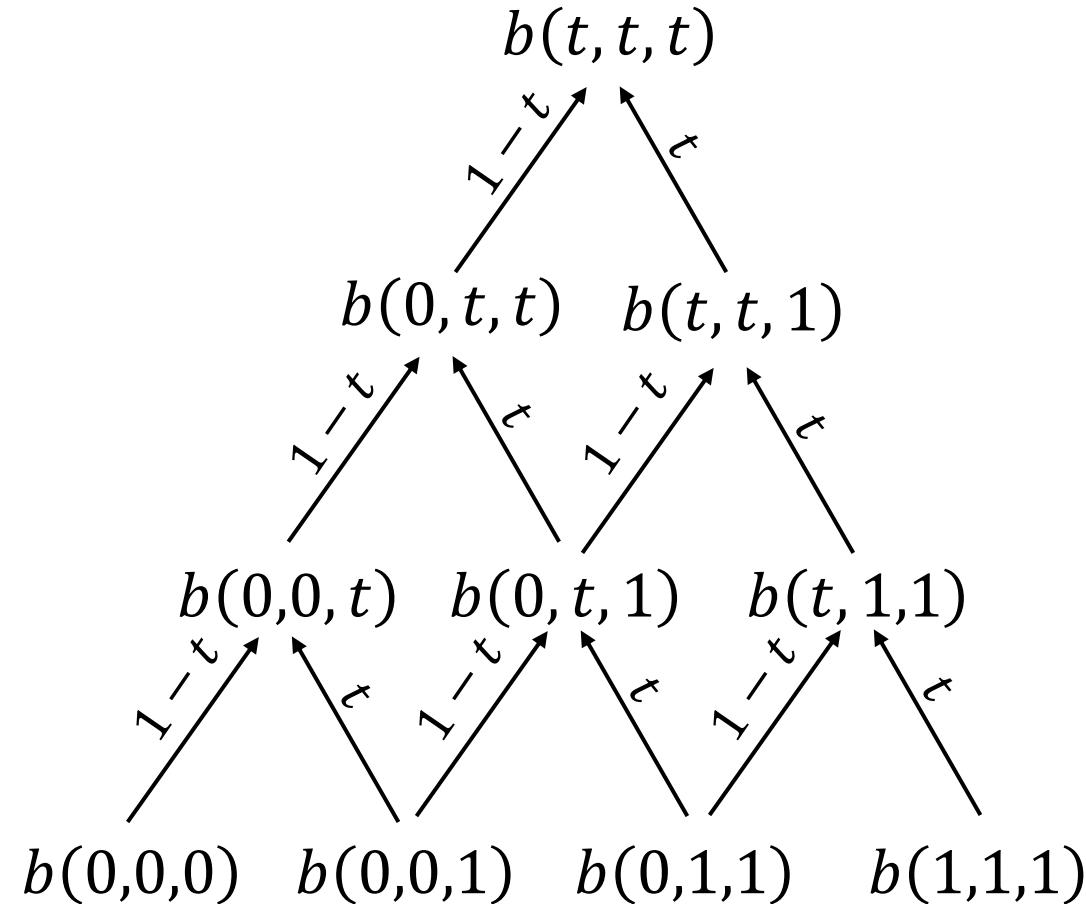
A diagram illustrating a vector operation in a 3D space. Two points are defined: $b(0, 0, 1)$ at the bottom left and $b(0, 1, 1)$ at the bottom right. A dashed arrow labeled $1-t$ originates from the origin and points towards the first point. A solid arrow labeled t originates from the first point and points towards the second point.

De Castljau Algorithms for Bézier Curves

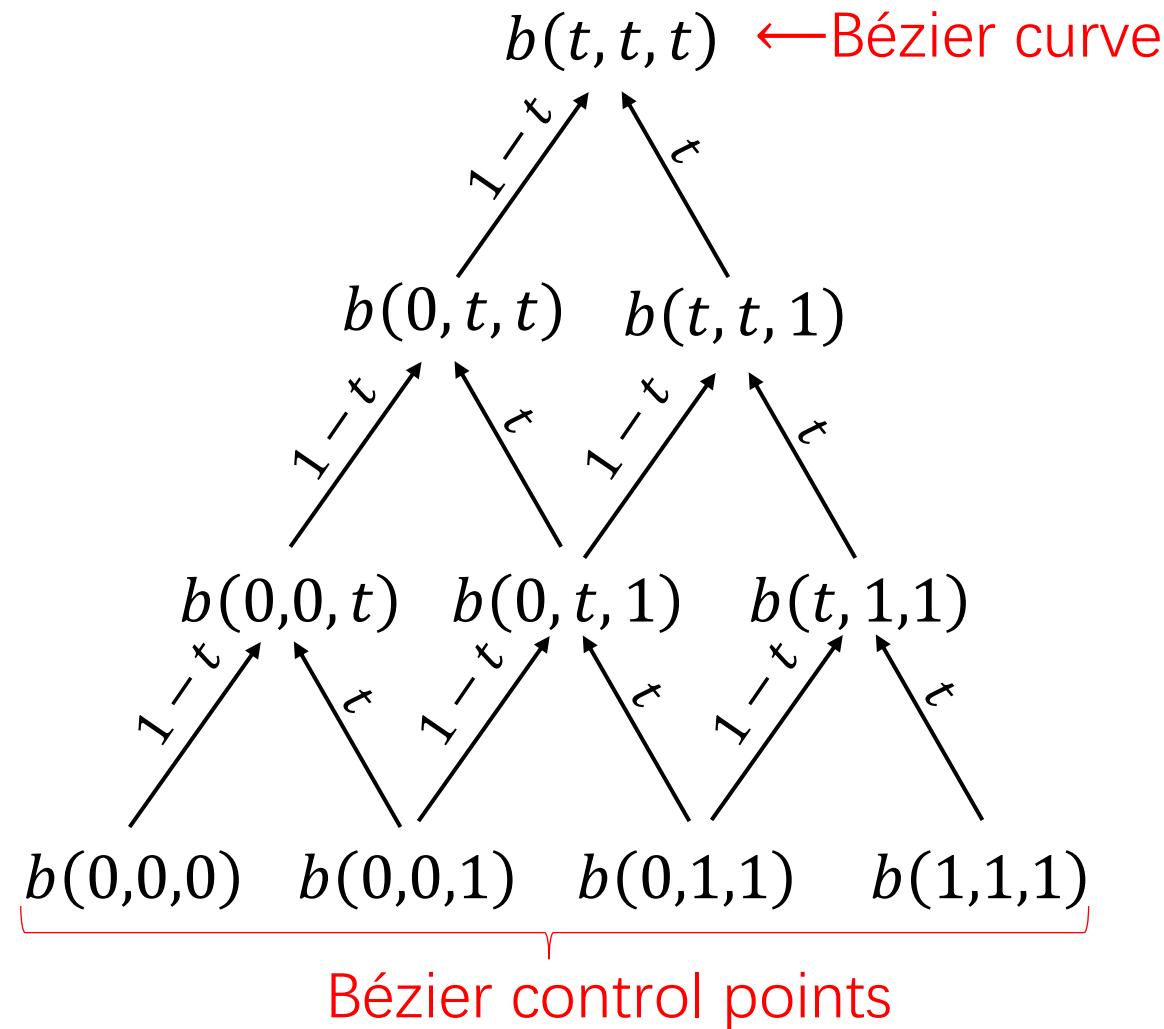
$$(1 - t)^3 p_0 + 3(1 - t)^2 t p_1 + 3(1 - t)t^2 p_2 + t^3 p_3$$



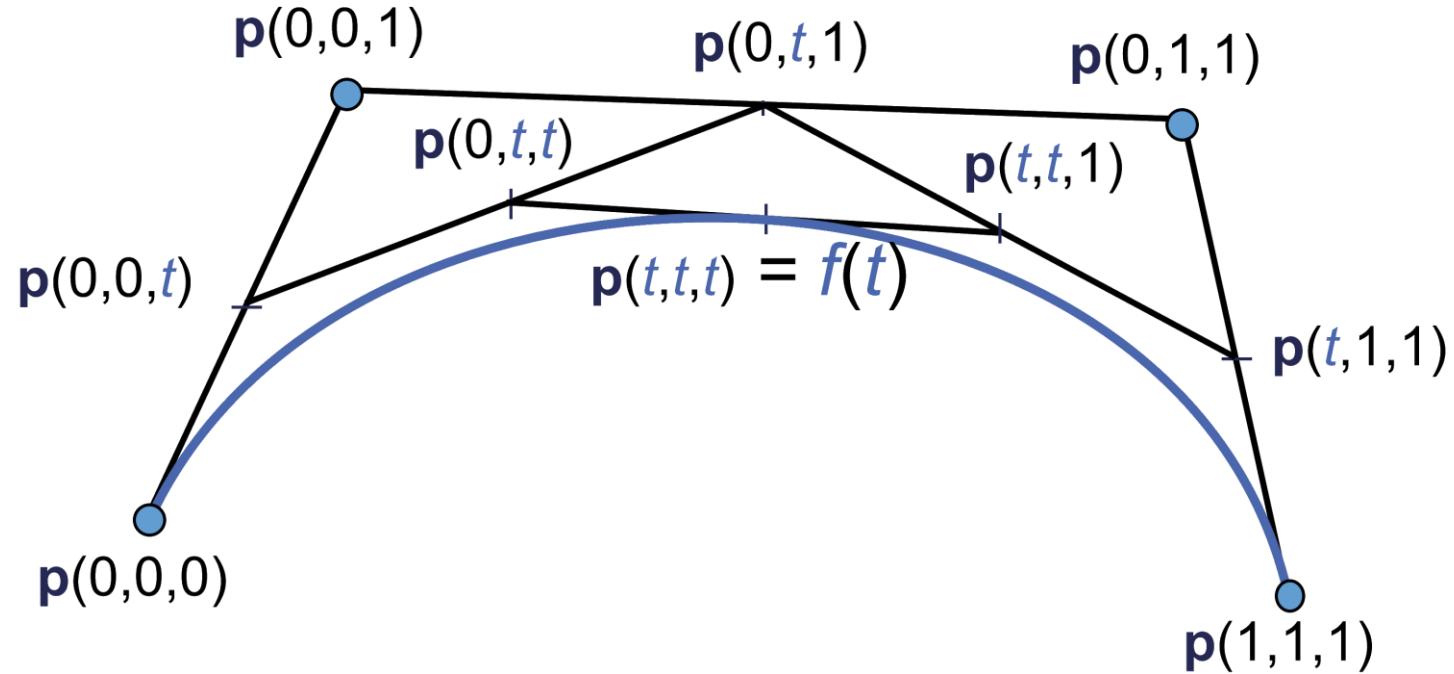
De Castljau Algorithms for Bézier Curves



De Castljau Algorithms for Bézier Curves



De Casteljau (Polar forms)



Bézier control points: $p(0,0,0)$, $p(0,0,1)$, $p(0,1,1)$, $p(1,1,1)$

Analysis

Transforming a polar to the Bernstein basis:

$$f(t, \dots, t) = (1 - t)f(t, \dots, t, 0) + tf(t, \dots, t, 1)$$

Analysis

Transforming a polar to the Bernstein basis:

$$\begin{aligned}f(t, \dots, t) &= (1 - t)f(t, \dots, t, 0) + tf(t, \dots, t, 1) \\&= (1 - t)[(1 - t)f(t, \dots, t, 0, 0) + tf(t, \dots, 0, 1)] + t[(1 - t)f(t, \dots, t, 1, 0) + tf(t, \dots, t, 1, 1)]\end{aligned}$$

Analysis

Transforming a polar to the Bernstein basis:

$$\begin{aligned}f(t, \dots, t) &= (1-t)f(t, \dots, t, 0) + tf(t, \dots, t, 1) \\&= (1-t)[(1-t)f(t, \dots, t, 0, 0) + tf(t, \dots, 0, 1)] + t[(1-t)f(t, \dots, t, 1, 0) + tf(t, \dots, t, 1, 1)] \\&= (1-t)^2 f(t, \dots, t, 0, 0) + 2t(1-t)tf(t, \dots, 0, 1) + t^2 f(t, \dots, t, 1, 1)\end{aligned}$$

Analysis

Transforming a polar to the Bernstein basis:

$$\begin{aligned} f(t, \dots, t) &= (1-t)f(t, \dots, t, 0) + tf(t, \dots, t, 1) \\ &= (1-t)[(1-t)f(t, \dots, t, 0, 0) + tf(t, \dots, 0, 1)] + t[(1-t)f(t, \dots, t, 1, 0) + tf(t, \dots, t, 1, 1)] \\ &= (1-t)^2f(t, \dots, t, 0, 0) + 2t(1-t)tf(t, \dots, 0, 1) + t^2f(t, \dots, t, 1, 1) \\ &= \dots \\ &= \sum_{i=0}^n \binom{n}{i} t^i (1-t)^{n-i} f(\underbrace{0, \dots, 0}_{n-i}, \underbrace{1, \dots, 1}_i) \end{aligned}$$

Analysis

De Castlejau Algorithm: Perform this in reverse order

- Bézier points:
- Intermediate points:
- Recursive computation:

$$p_i^{(0)}(t) = f(0, \underbrace{\dots, 0}_{d-i}, \underbrace{1, \dots, 1}_i)$$

$$p_i^{(j)}(t) = f(0, \underbrace{\dots, 0}_{d-i-j}, \underbrace{1, \dots, 1}_i, \underbrace{t, \dots, t}_j)$$

$$p_i^{(j)}(t) = f(0, \underbrace{\dots, 0}_{d-i-j}, \underbrace{t, \dots, t}_j, \underbrace{1, \dots, 1}_i)$$

$$= (1-t)f(0, \underbrace{\dots, 0}_{d-i-j+1}, \underbrace{t, \dots, t}_{j-1}, \underbrace{1, \dots, 1}_i) + tf(0, \underbrace{\dots, 0}_{d-i-j}, \underbrace{t, \dots, t}_{j-1}, \underbrace{1, \dots, 1}_{i+1})$$

$$= (1-t)p_i^{(j-1)}(t) + tp_{i+1}^{(j-1)}(t)$$

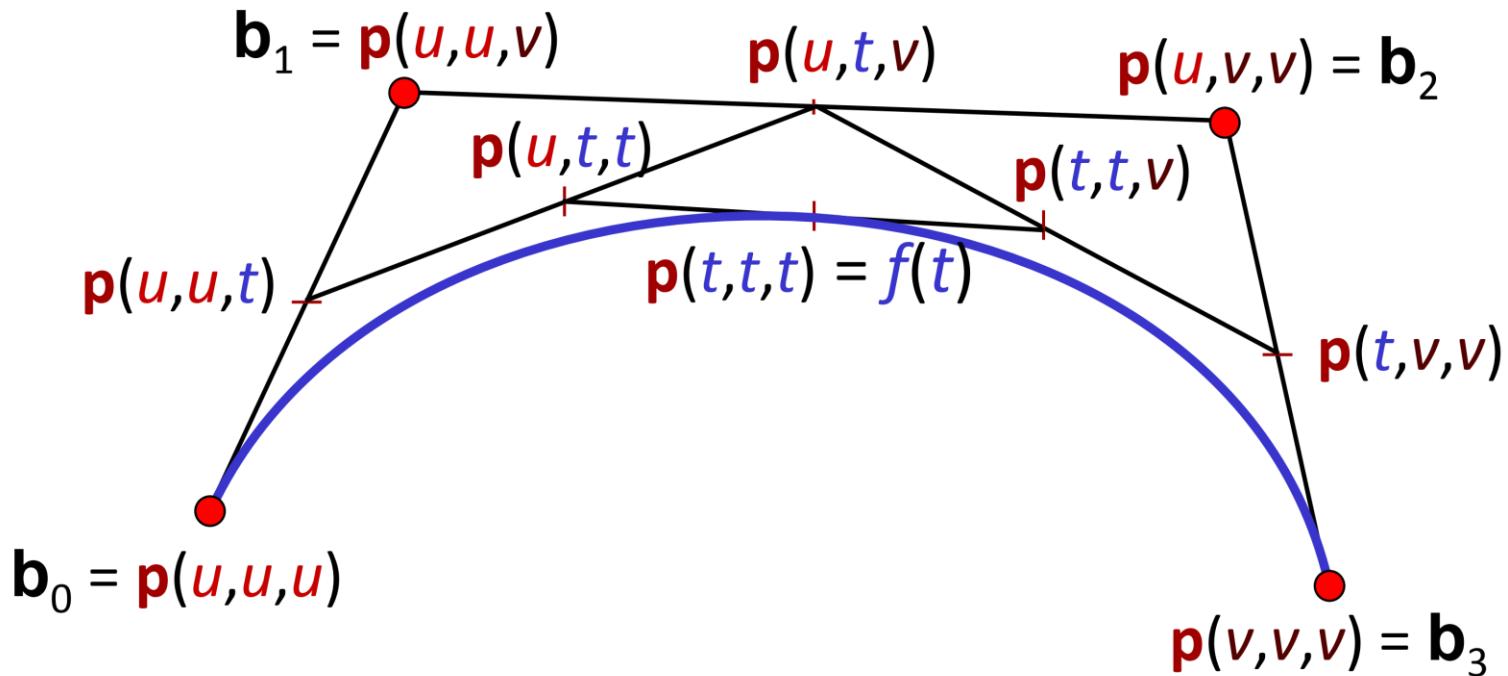
Consequence: Bernstein / de Casteljau lead to the same result

Generalized Parameter Intervals

- Let $f(t)$ be a Bézier curve of degree d over the domain $t \in [u, v]$
- Let p be the polar form of f
- Then the Bézier points of f are given in polar form as:
 - $b_i = p(u, \dots, u, v, \dots, v)$

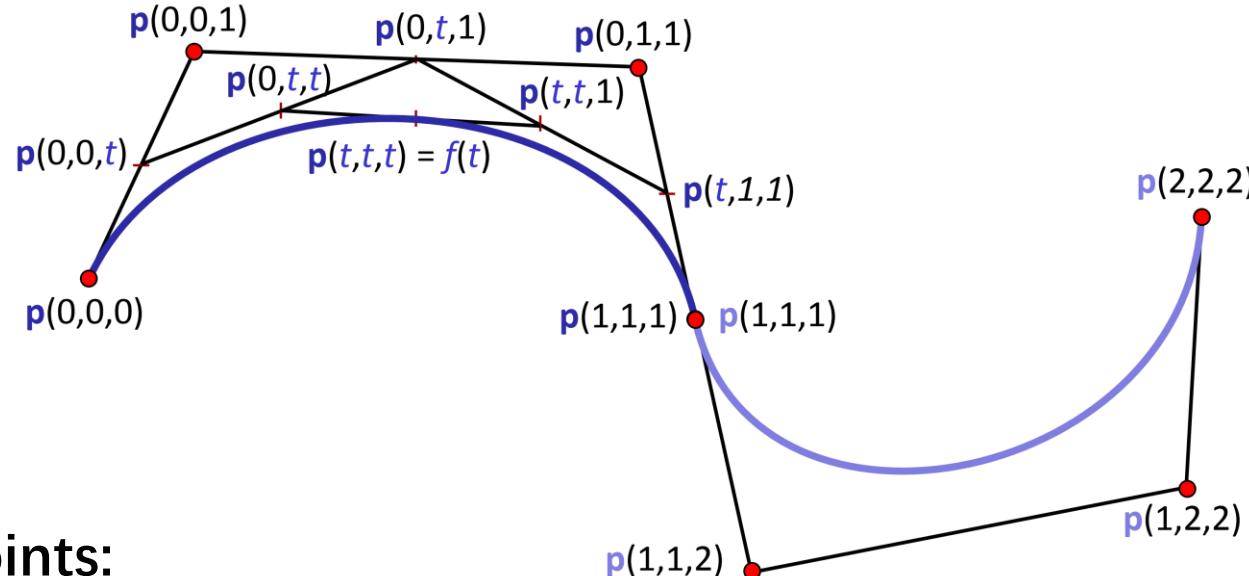
Example for a cubic Bézier curve:
 $p(u, u, u), p(u, u, v), p(u, v, v), p(v, v, v)$

Generalized Parameter Intervals



Example for a cubic Bézier curve:
 $\mathbf{p}(u, u, u), \mathbf{p}(u, u, v), \mathbf{p}(u, v, v), \mathbf{p}(v, v, v)$

Multiple Segments



Bézier Control points:

$$p(0,0,0), p(0,0,1), p(0,1,1), p(1,1,1) = p(1,1,1), p(1,1,2), p(1,2,2), p(2,2,2)$$

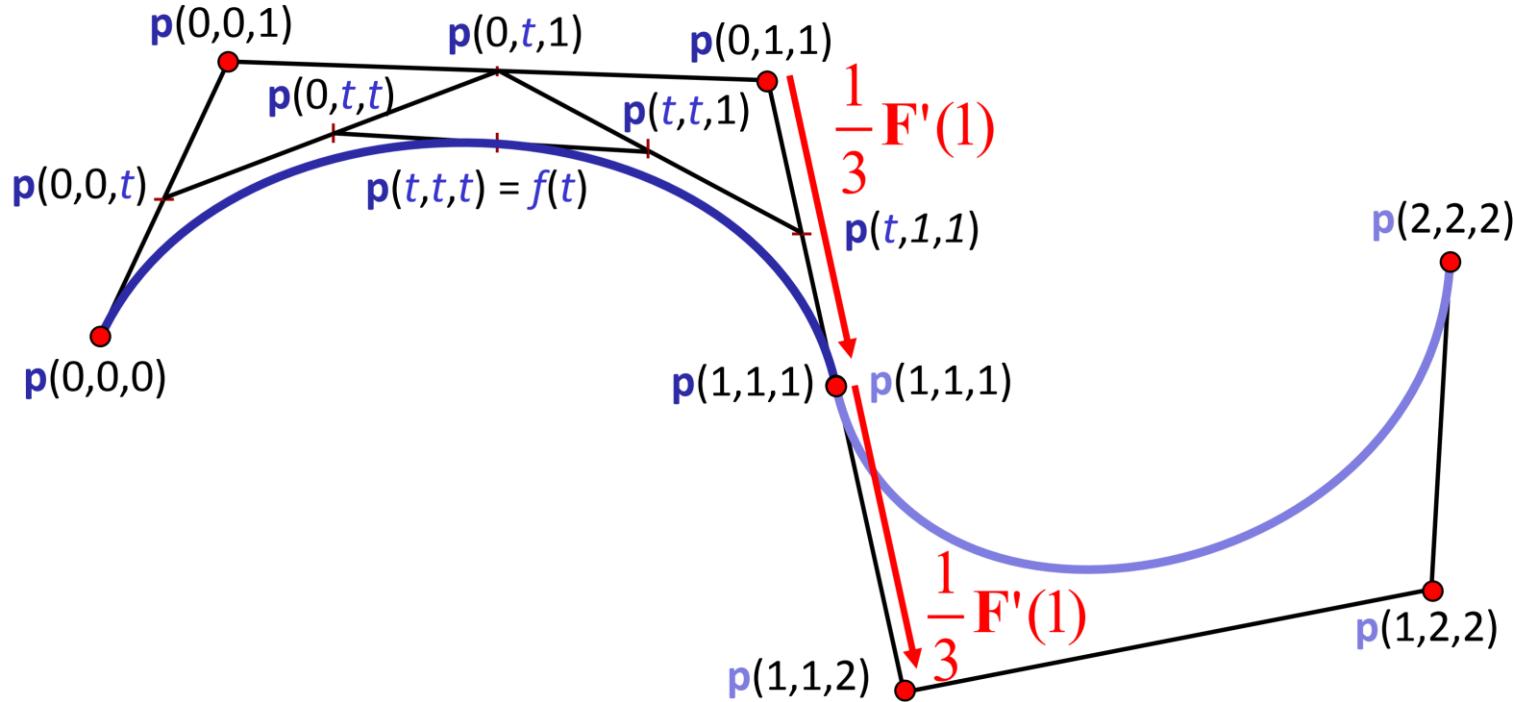
Two Curve Segments:

$$\{p(0,0,0), p(0,0,1), p(0,1,1), p(1,1,1)\}, \{p(1,1,1), p(1,1,2), p(1,2,2), p(2,2,2)\}$$

Remark: no intersection between different segments

(e.g.: combination of $p(0,1,1)$ and $p(2,1,1)$ is not defined)

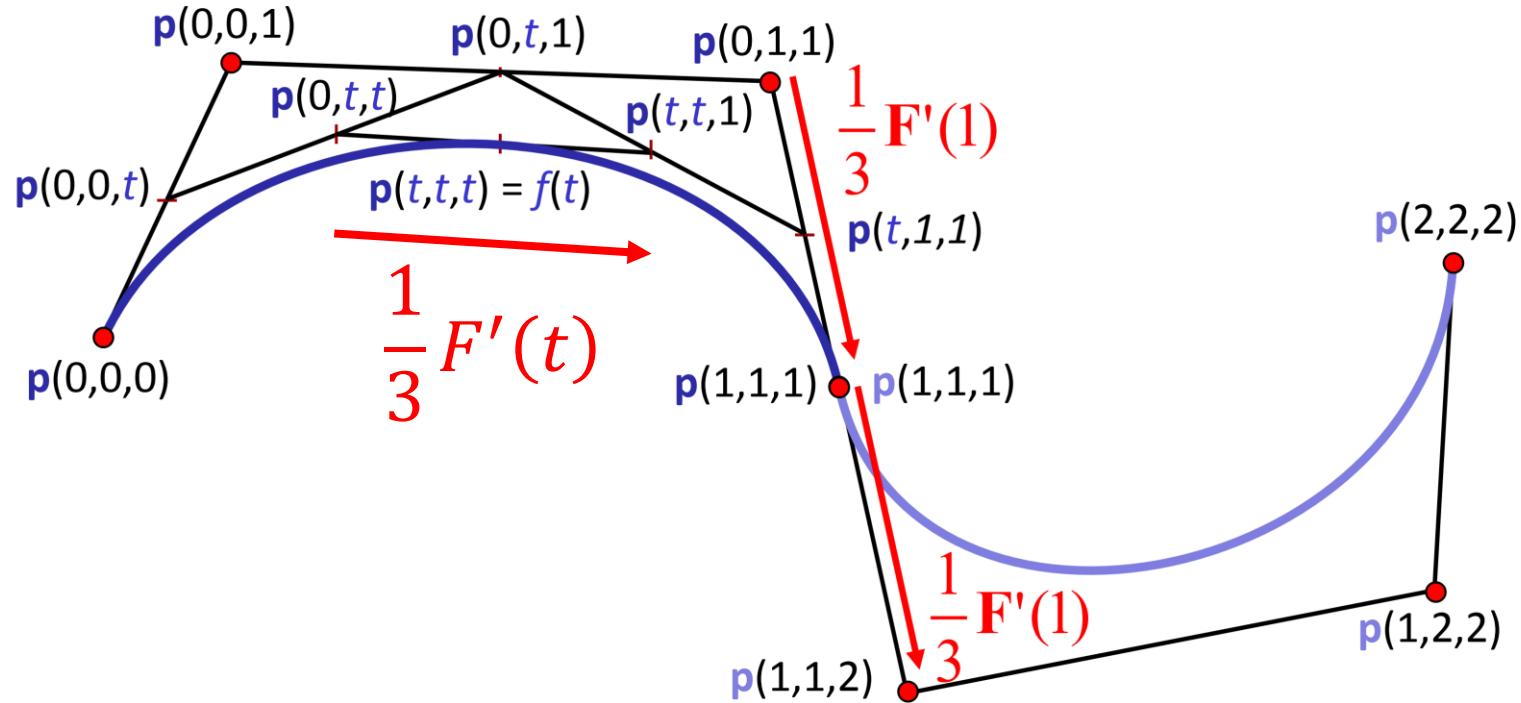
More Observations



Derivatives:

- $\frac{d}{dt} F(t) = df(t, \dots, t, \hat{1}) = d(f(t, \dots, t, 1) - f(t, \dots, t, 0))$ (degree d)
- C^1 continuity condition follows

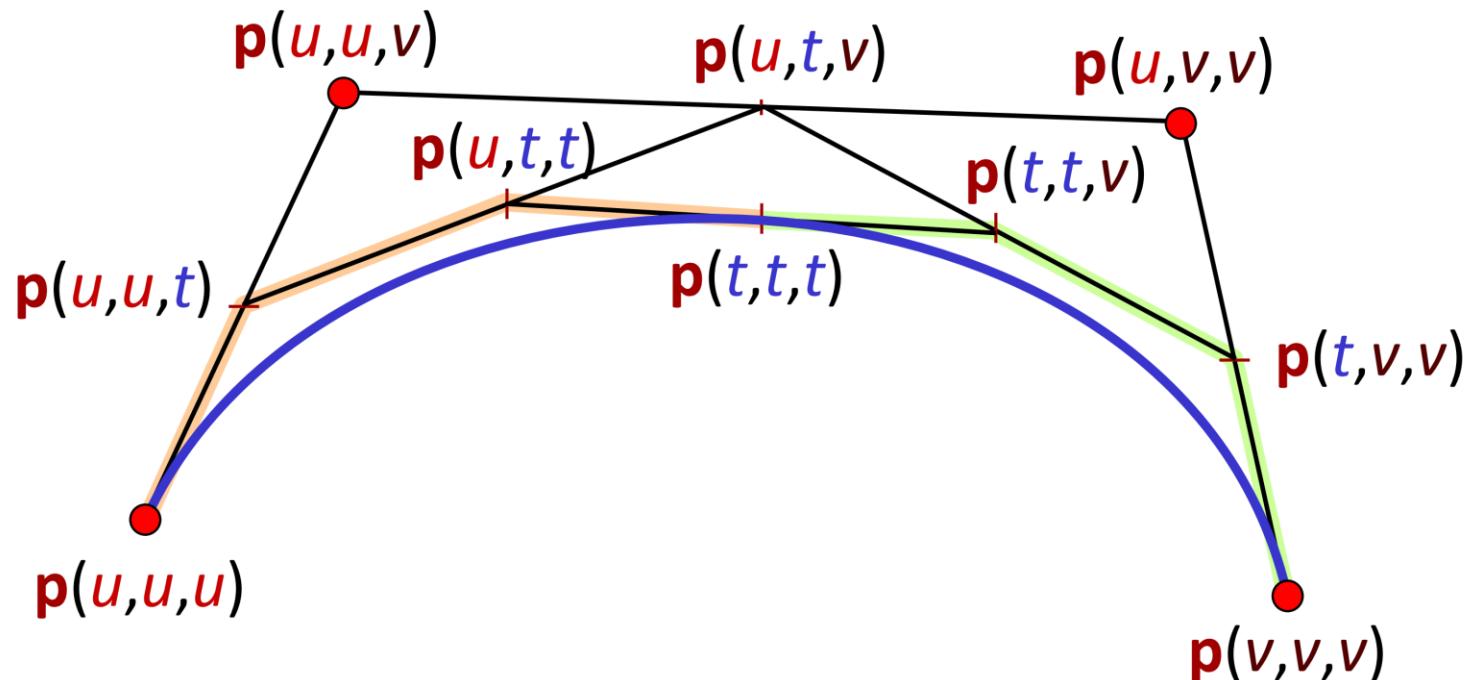
More Observations



Derivatives:

- de Casteljau algorithm computes tangent vectors at any points as a byproduct
- Proportional to last line segment that is bisected

More Observations



Subdivision:

- After each de Casteljau step, we obtain two new control polygons left and right of $f(t)$ describing the same curve.
- We can divide a segment into two
- Recursive subdivision can be used for rendering

Observations

Remark: The de Casteljau algorithm for computing

- Derivatives
 - At end points
 - At inner points t
- Subdivisions

Hold for Bézier curves of arbitrary degree $d \geq 1$

(General degree derivatives: $F'(t)/d$)

More Bézier Curve Properties...

General degree elevation

- Increase the degree of a Bézier curve segment by one
- What are the new control points?

Polar forms:

- Old curve: $b(t_1, \dots, t_d)$

- New curve: $b^{(+1)}(t_1, \dots, t_{d+1}) = \frac{1}{d+1} \sum_{i=1}^{d+1} b(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{d+1})$



leave out t_i

Degree Elevation

$$\mathbf{b}^{(+1)}(0, \dots, 0) = \frac{1}{d+1} \sum_{i=1}^{d+1} \mathbf{b}(0, \dots, 0) = \mathbf{b}(0, \dots, 0)$$

$$\mathbf{b}^{(+1)}(1, \dots, 0) = \frac{1}{d+1} \mathbf{b}(0, \dots, 0) + \frac{d}{d+1} \mathbf{b}(1, 0, \dots, 0)$$

$$\mathbf{b}^{(+1)}(1, 1, 0, \dots, 0) = \frac{2}{d+1} \mathbf{b}(0, \dots, 0) + \frac{d-1}{d+1} \mathbf{b}(1, 1, 0, \dots, 0)$$

$$\mathbf{b}^{(+1)}(1, 1, 1, \dots, 1, 0) = \frac{d}{d+1} \mathbf{b}(1, \dots, 1, 0) + \frac{1}{d+1} \mathbf{b}(1, \dots, 1)$$

$$\mathbf{b}^{(+1)}(1, \dots, 1) = \mathbf{b}(1, \dots, 1)$$

Degree Elevation

Result: new control points

$$\mathbf{p}_i^{(+1)} = \frac{i}{n+1} \mathbf{p}_{i-1} + \left(1 - \frac{i}{n+1}\right) \mathbf{p}_i, \quad i = 0, \dots, n+1 \text{ (zero points if out of range)}$$

Repeated degree elevation:

$$\mathbf{p}_i^{(+k)} = \sum_{j=1}^{d+1} \mathbf{p}_j \frac{\binom{d}{j} \binom{k}{i-j}}{\binom{d+k}{j}} \text{ (proof by induction)}$$

Repeating degree elevation let the control point converge to the Bézier curve in the limit

Change of basis, the easy way

- Given: Polynomial $p(t)$ of degree n in monomial form
- Wanted: coefficients of the same Bézier curve

Change of basis, the easy way

- Given: Polynomial $p(t)$ of degree n in monomial form
- Wanted: coefficients of the same Bézier curve
- Solution:

$$p(t) \rightarrow b(t_1, \dots, t_n)$$

$$\text{coefficients: } b(0, \dots, 0, \underbrace{1, \dots 1}_{k}, \dots, 1)$$

$\underbrace{ }_{n+1-k}$

- This is a direct implication of de Casteljau in polar form

Example

- Example: Bézier coefficients of $p(t) = 1 + 2t + 3t^2 - t^3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5/3 \\ 10/3 \\ 5 \end{pmatrix}$$

Example

- Using polar forms

$$b(t_0, t_1, t_2) = 1 + 2 \frac{t_0 + t_1 + t_2}{3} + 3 \frac{t_0 t_1 + t_1 t_2 + t_0 t_2}{3} - t_0 t_1 t_2$$

$$b(0,0,0) = 1, \quad b(0,0,1) = \frac{5}{3}, \quad b(0,1,1) = \frac{10}{3}, \quad b(1,1,1) = 5$$

Polar Forms & Blossoms

B-Splines

B-Spline Curves in Polar Form

An unique description in polar form exists for piecewise polynomial curves as well

- Given: B-Spline curve \mathbf{x} of order k
 - with knot vector $T = (t_0, \dots, t_{n+k})$
 - and de Boor points $\mathbf{d}_0, \dots, \mathbf{d}_n$
- Let $\underline{\mathbf{x}}$ be the polar form of \mathbf{x}
- Then the de Boor points of \mathbf{x} are given as:

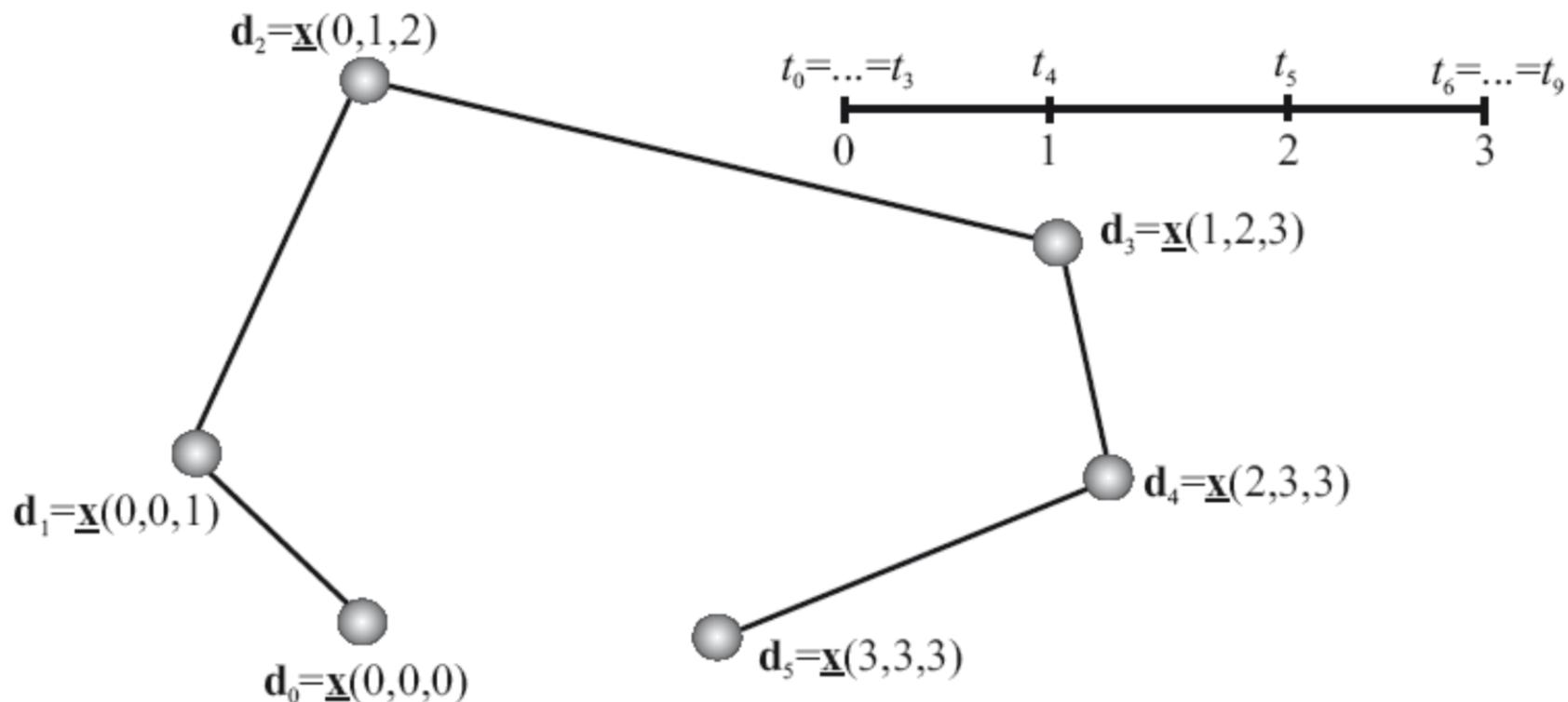
- $\mathbf{d}_i = \underline{\mathbf{x}}(t_{i+1}, \dots, t_{i+k-1})$



*we just use consecutive knot values
as blossom arguments*

B-Spline Curves in Polar form

- Example: $k = 4, n = 5$



De Boor Algorithm in Polar Form

de Boor algorithm in polar form

- To Define the curve value at t , we look for the relevant part of the de Boor polygon, i.e.,
→ the (partial) knot sequence

$$r_{k-1} \leq \dots \leq r_1 < s_1 \leq \dots \leq s_{k-1}$$

$$\text{with } r_1 \leq t < s_1$$

De Boor Algorithm in Polar Form

de Boor algorithm in polar form

- Then the intermediate points of the de Boor algorithm result are

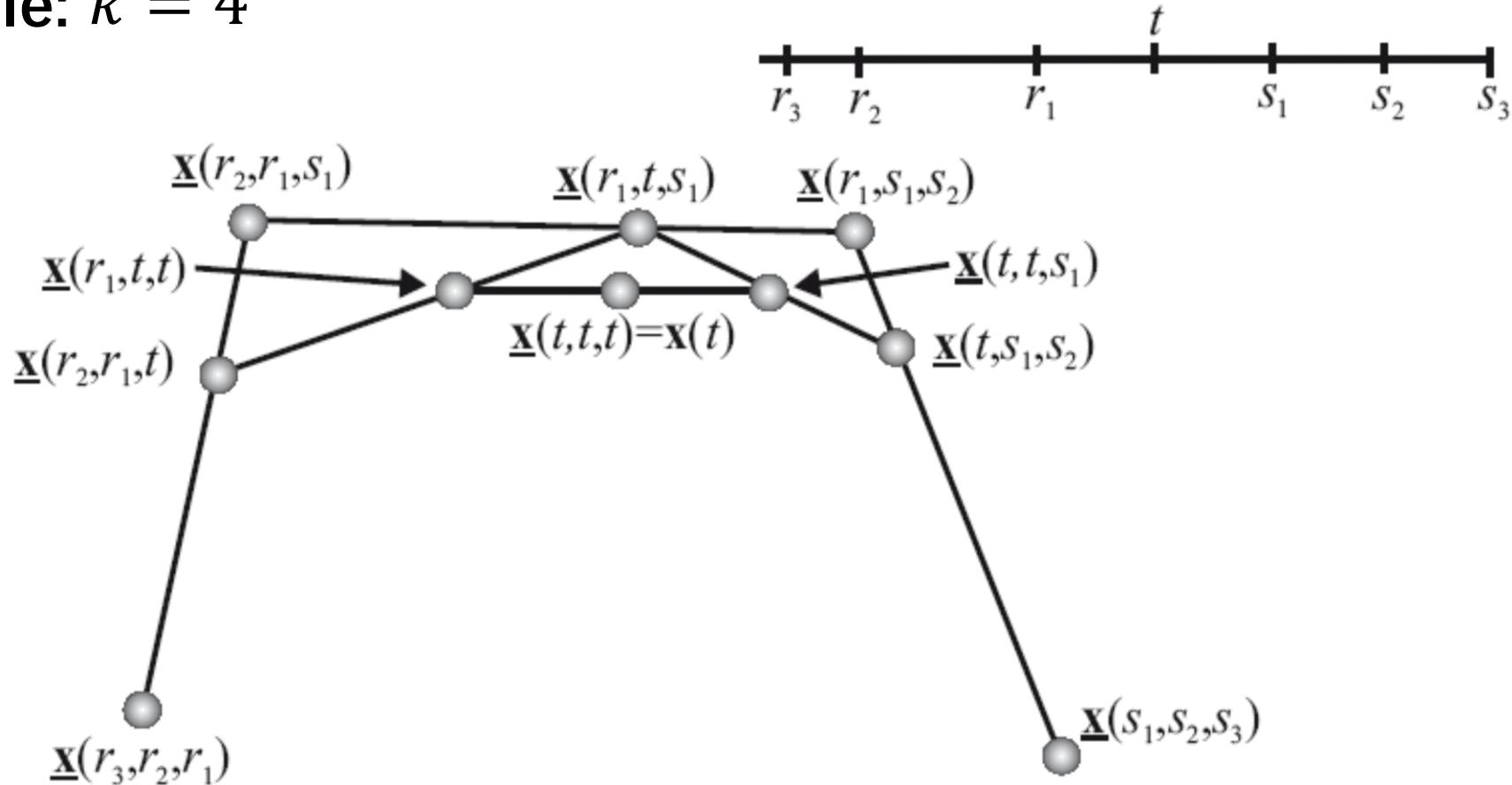
$$d_j^l(t) = \underline{x}(r_1, \dots, r_{k-1-l-j}, \underbrace{t, \dots, t}_l, s_1, \dots, s_j)$$

and the desired curve point is

$$x(t) = \underline{x}(\underbrace{t, \dots, t}_{k-1})$$

De Boor Algorithm in Polar form

Example: $k = 4$

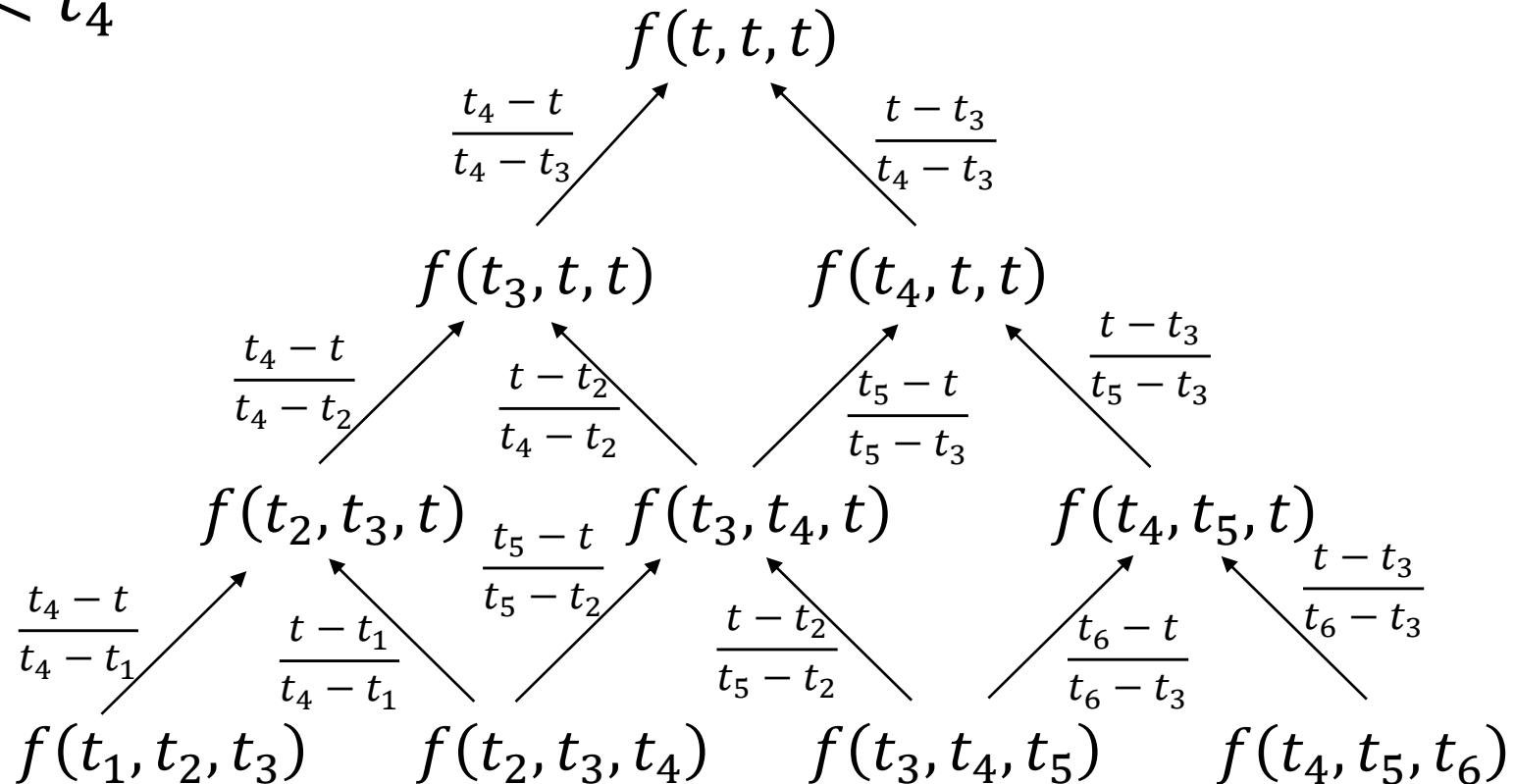


Key observation

$$\begin{array}{ccc} & f(t_2, t_3, t) & \\ \frac{t_4 - t}{t_4 - t_1} \nearrow & & \swarrow \frac{t - t_1}{t_4 - t_1} \\ f(t_1, t_2, t_3) & & f(t_2, t_3, t_4) \end{array}$$

De Boor Alg. In Polar form

- For $t_3 \leq t < t_4$



De Boor Algorithm in Polar Form

for $2 \leq t < 3$

$$\frac{3-t}{3} p(0,1,2) + \left[1 - \frac{3-t}{3}\right] p(3,1,2)$$

$$\frac{4-t}{3} p(1,2,3) + \left[1 - \frac{4-t}{3}\right] p(4,2,3)$$

$$\frac{5-t}{3} p(2,3,4) + \left[1 - \frac{5-t}{3}\right] p(5,3,4)$$

$$\frac{3-t}{2} p(t,1,2)$$

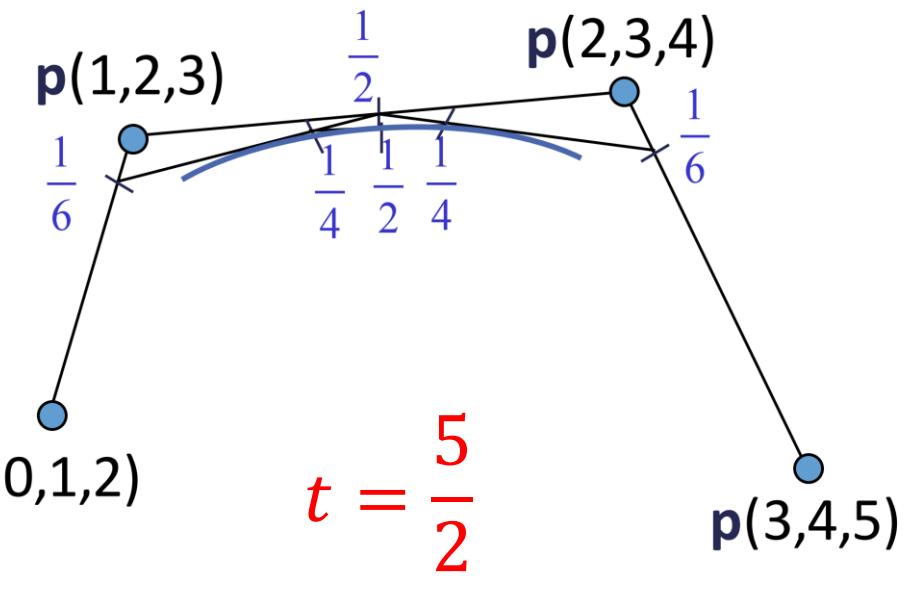
$$+ \left[1 - \frac{3-t}{2}\right] p(t,3,2)$$

$$+ \left[1 - \frac{4-t}{2}\right] p(t,3,4)$$

$$\frac{3-t}{1} p(t,t,2)$$

$$+ \left[1 - \frac{3-t}{1}\right] p(t,t,3)$$

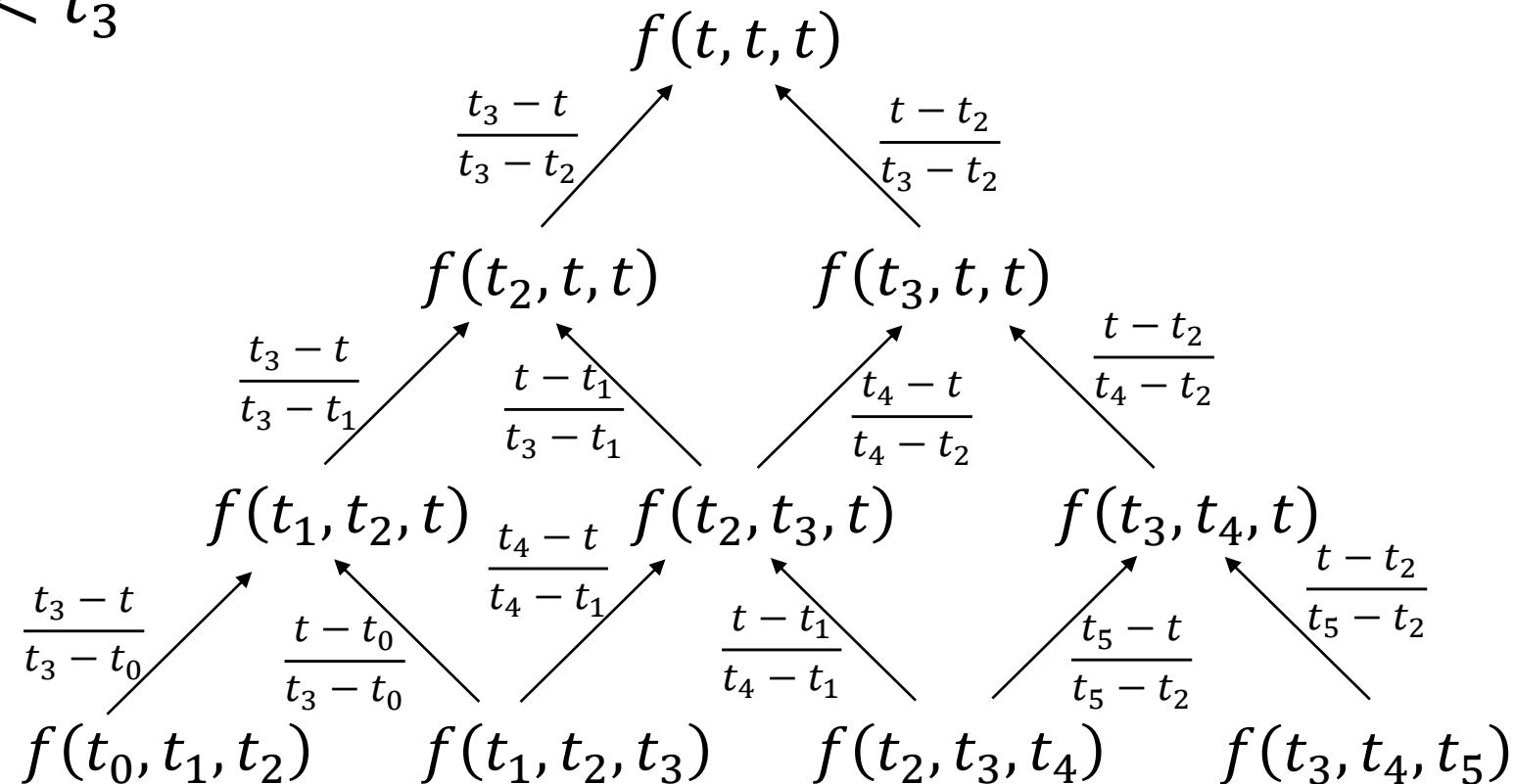
$$p(t,t,t)$$



$$t = \frac{5}{2}$$

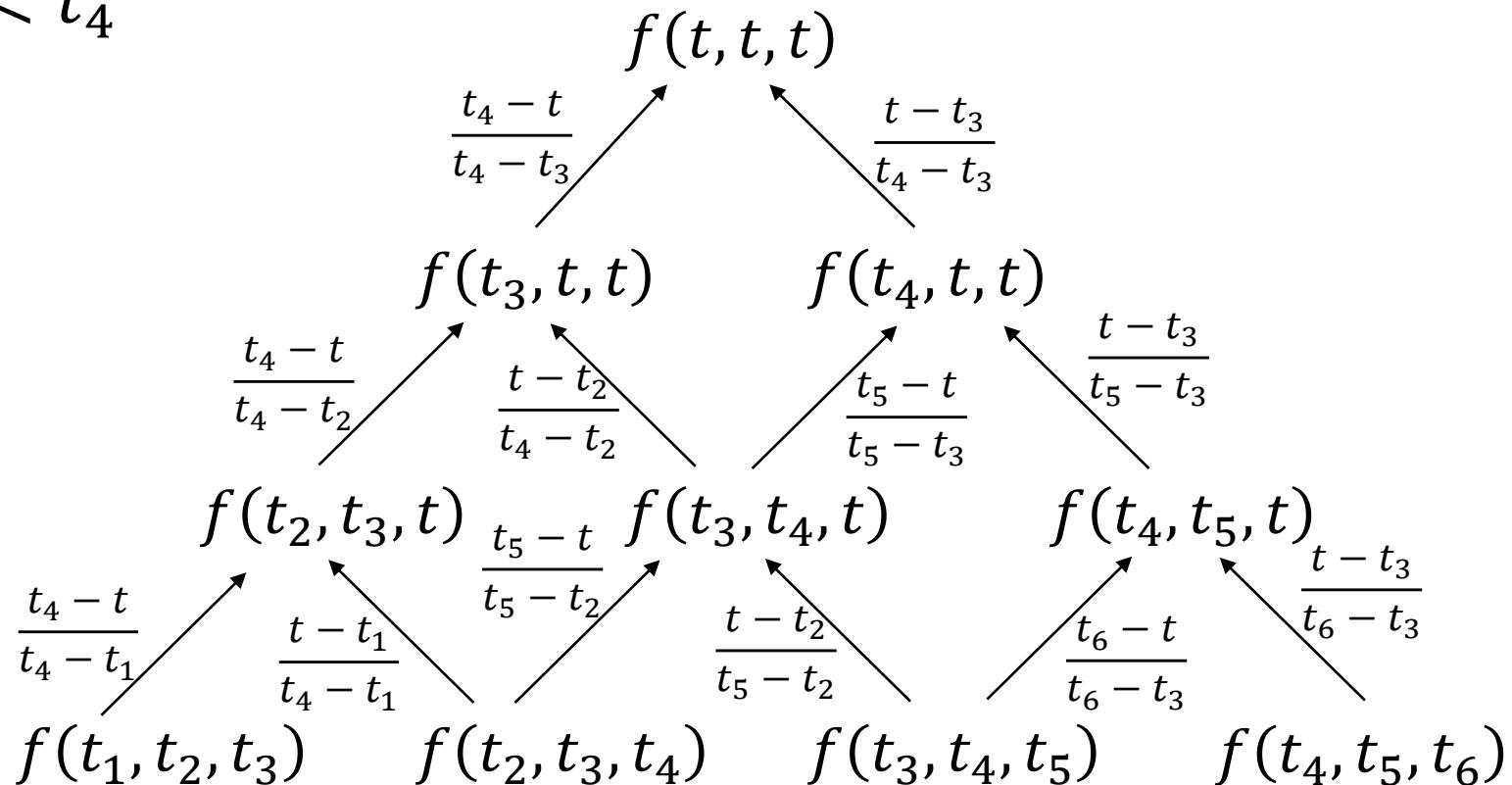
B-Splines in Polar form

- For $t_2 \leq t < t_3$

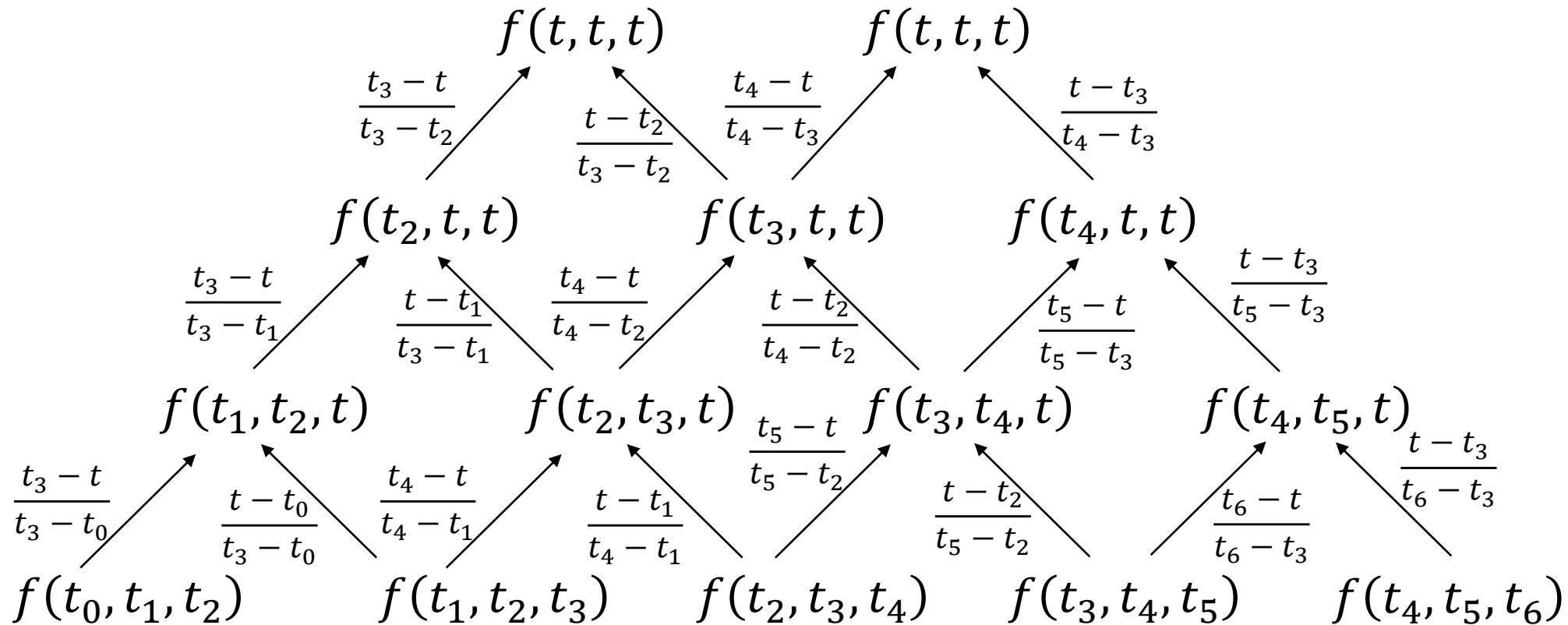


B-Splines in Polar form

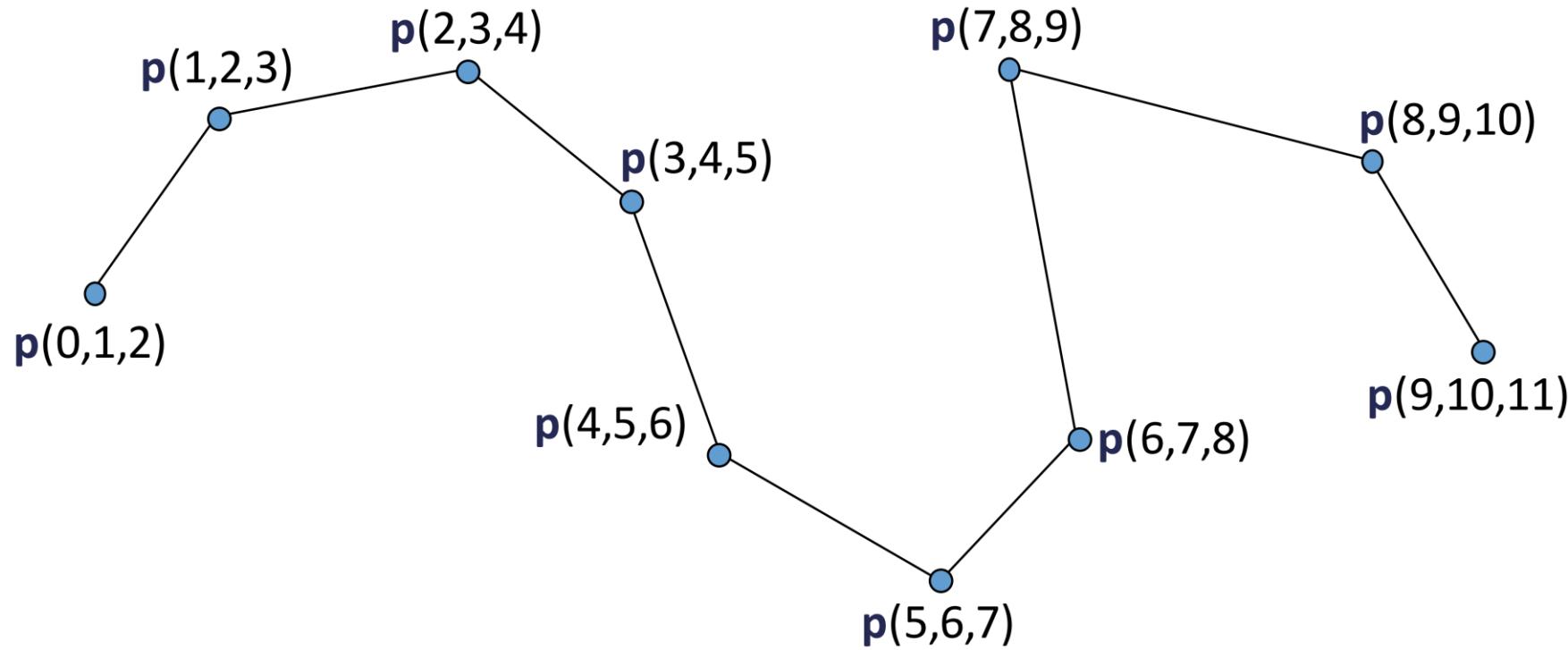
- For $t_3 \leq t < t_4$



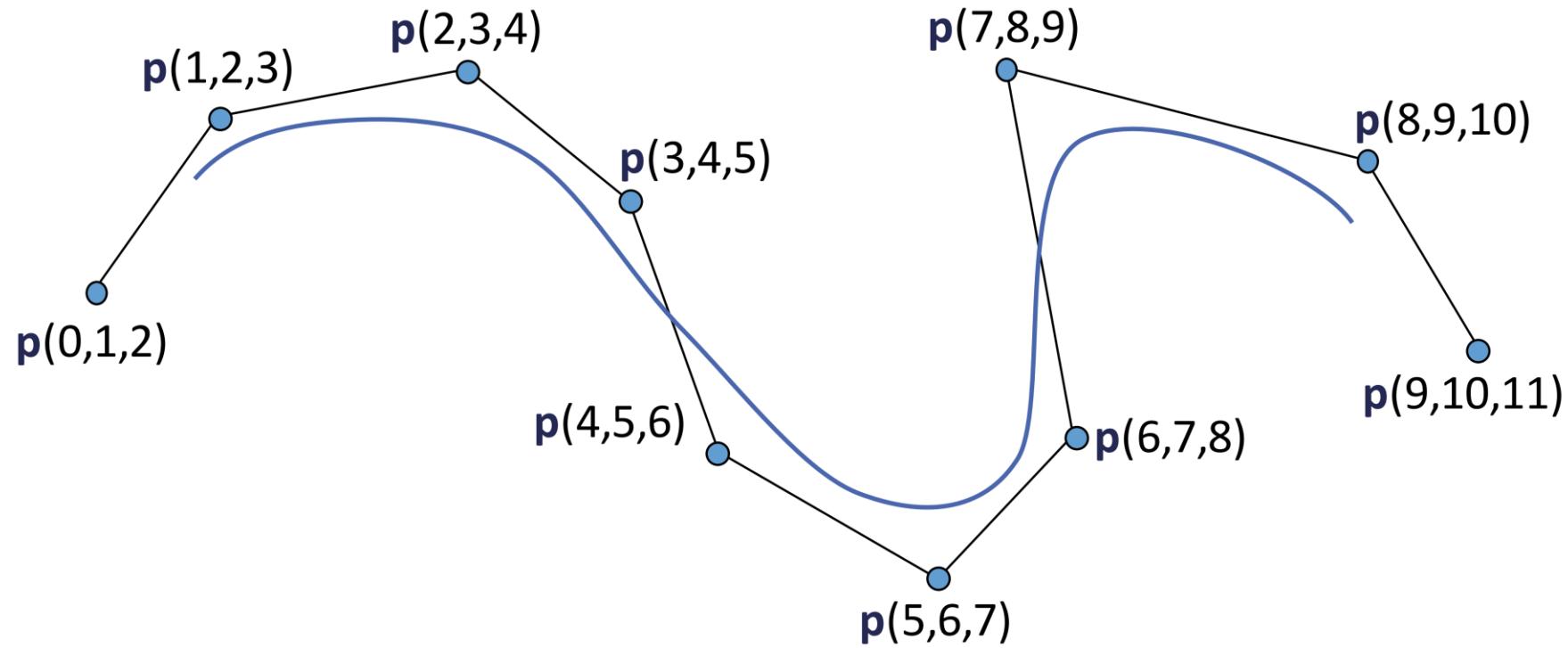
B-Splines in Polar form



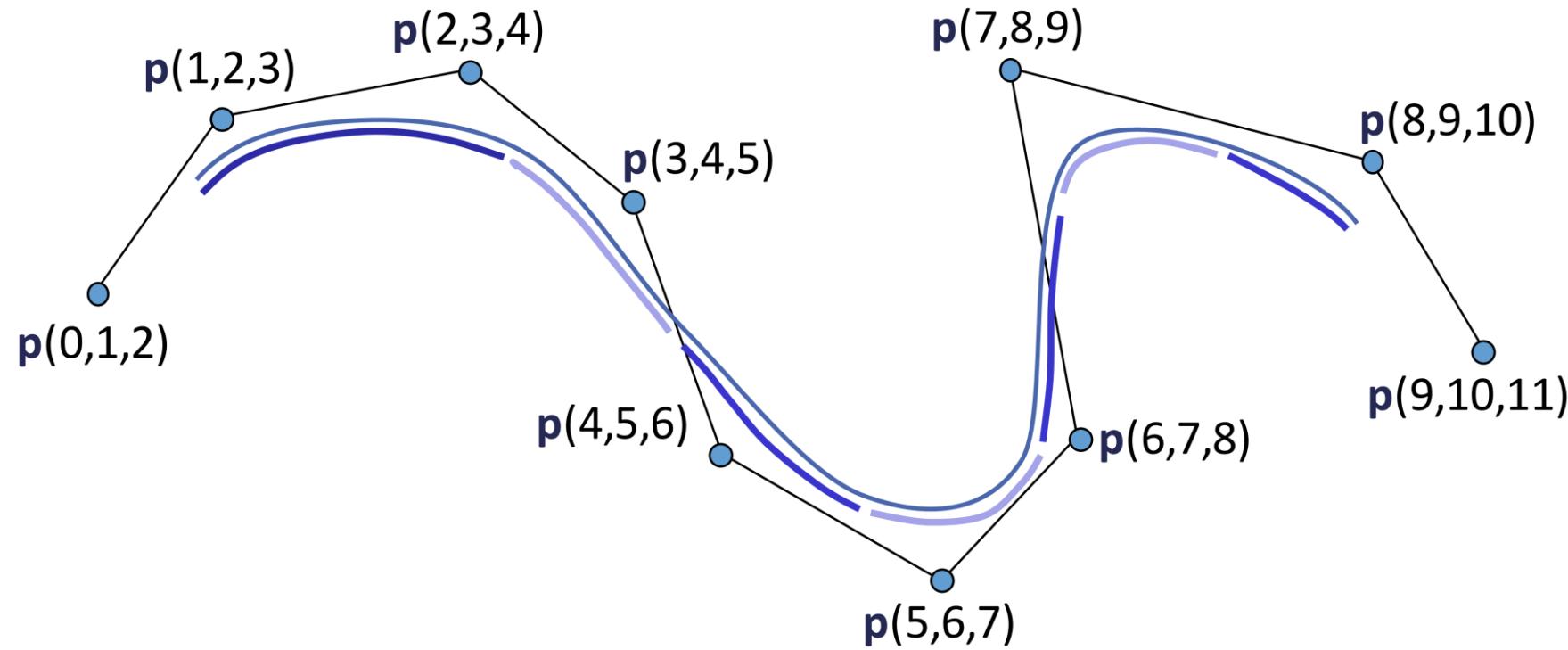
Example: General Case



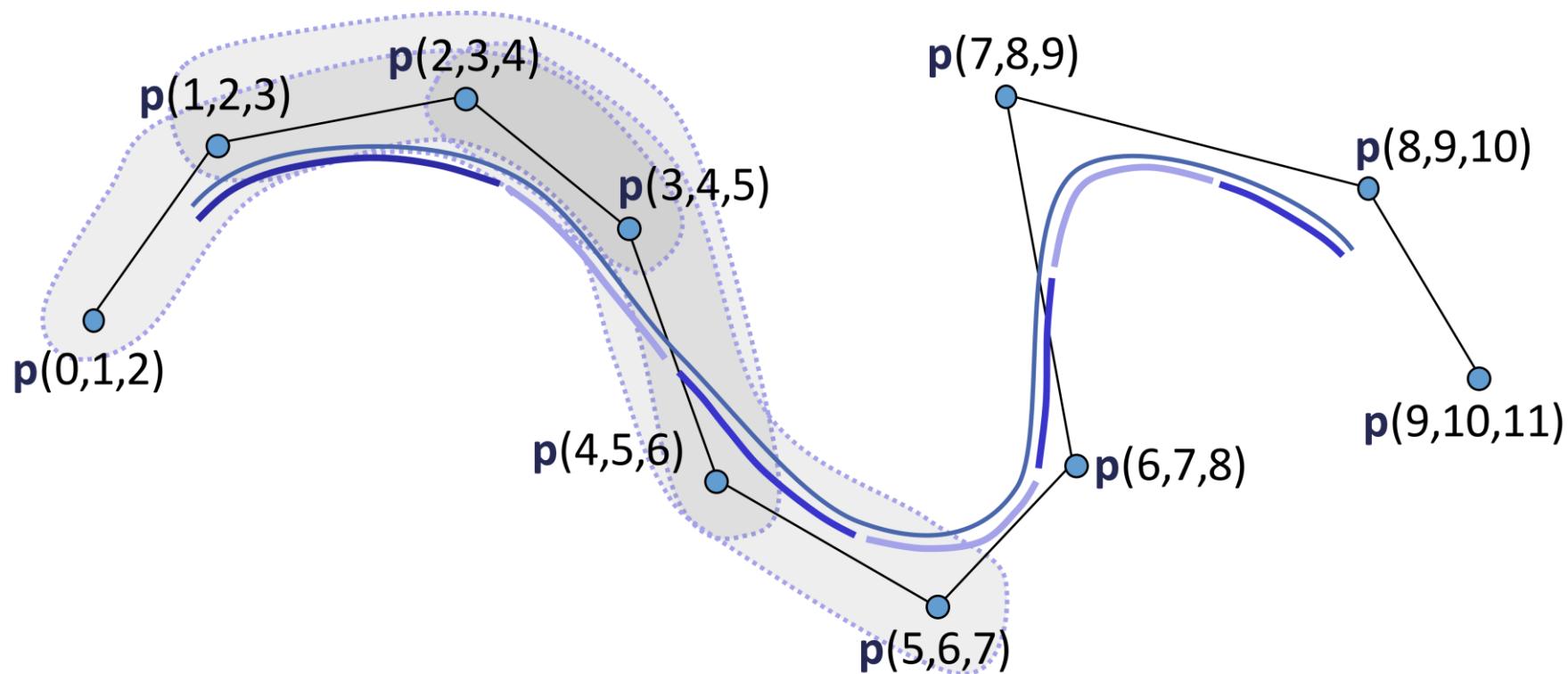
Example: General Case



Example: General Case



Example: General Case



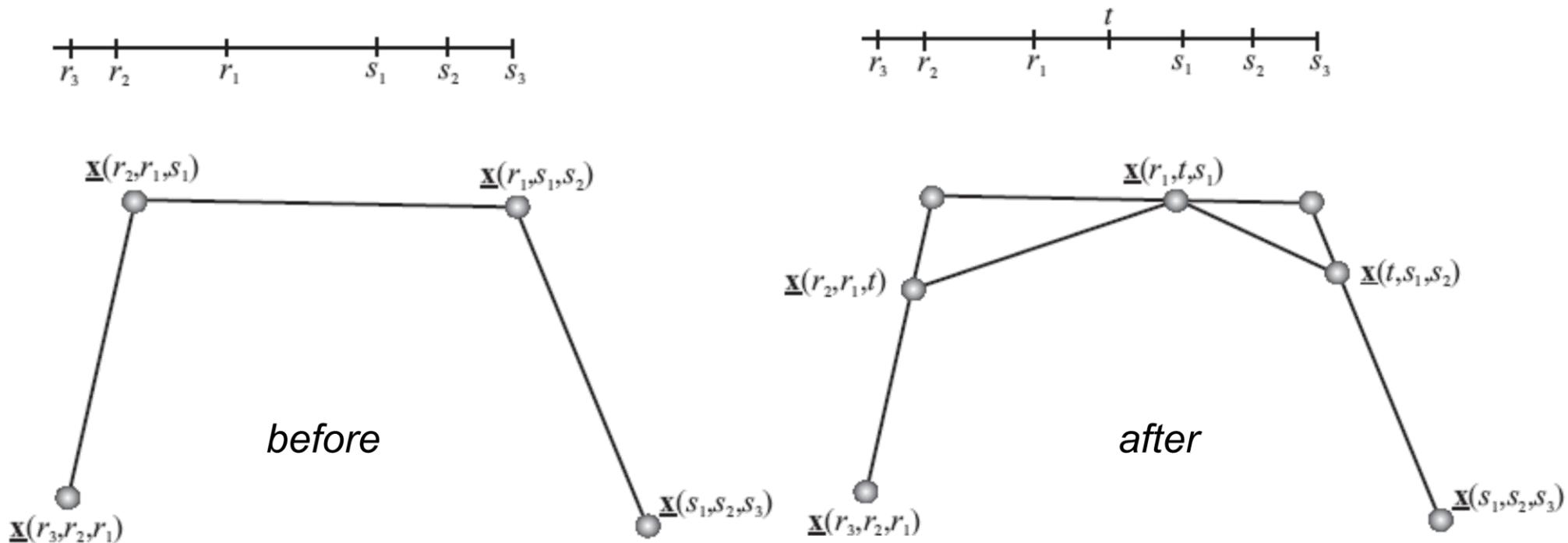
Knot Insertion

Knot insertion

- Increases the number of curve segments, but not the polynomial degree
- Insertion at t : First step of the de Boor algorithm!

Knot Insertion

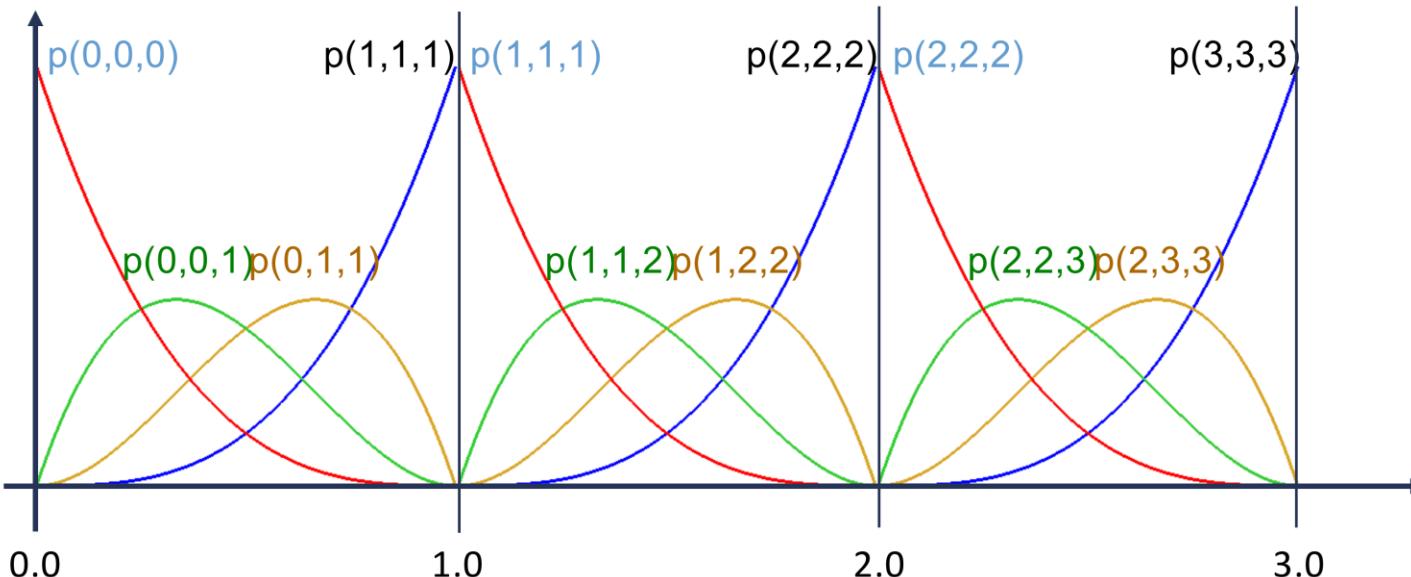
- Insertion of knots Example



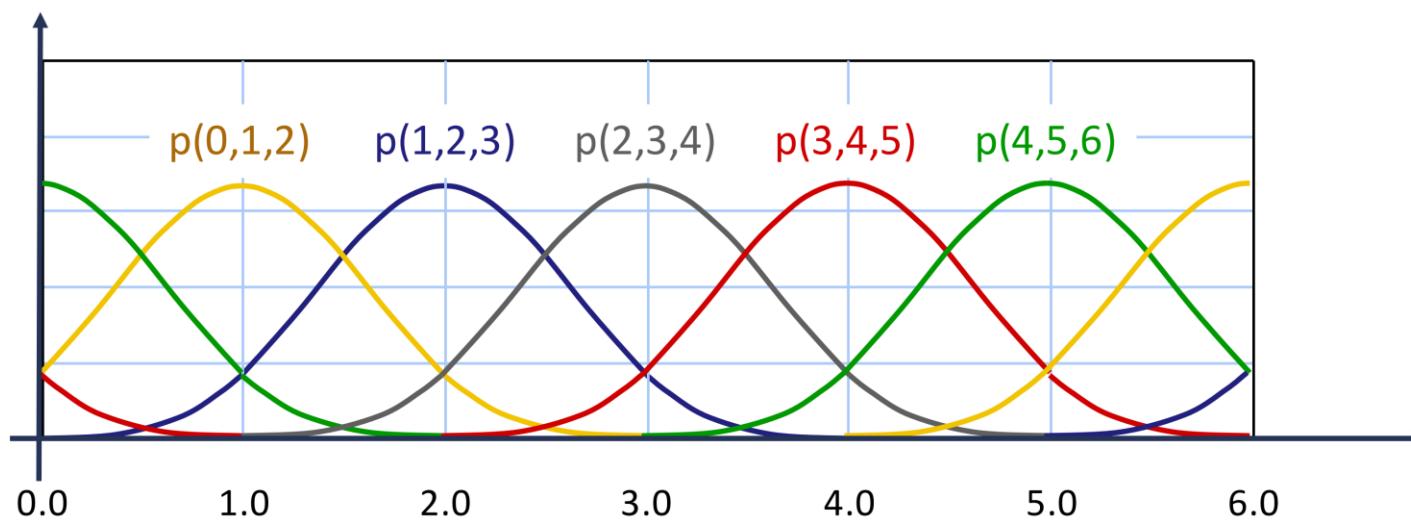
Polar Forms & Blossoms

Illustrations

Structure

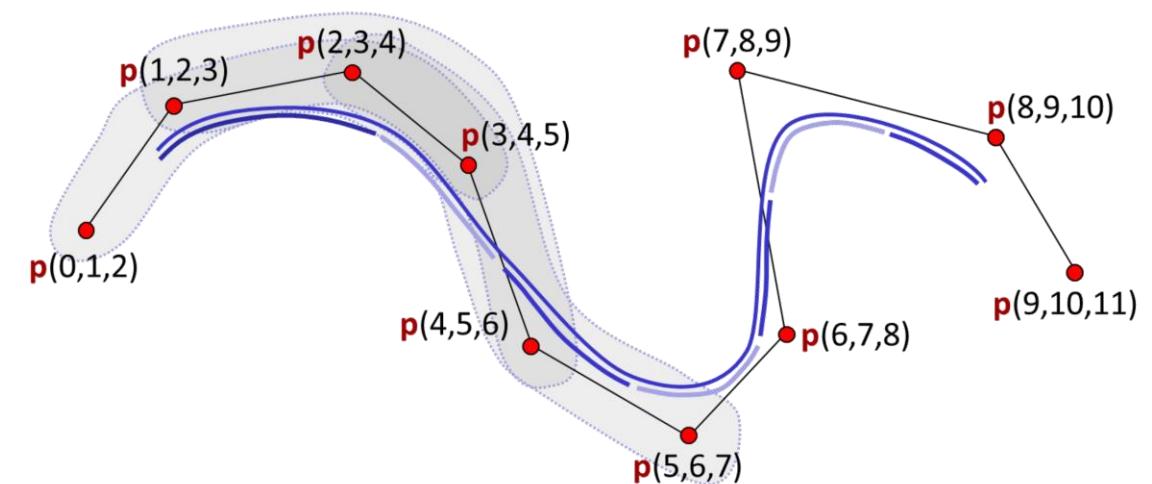
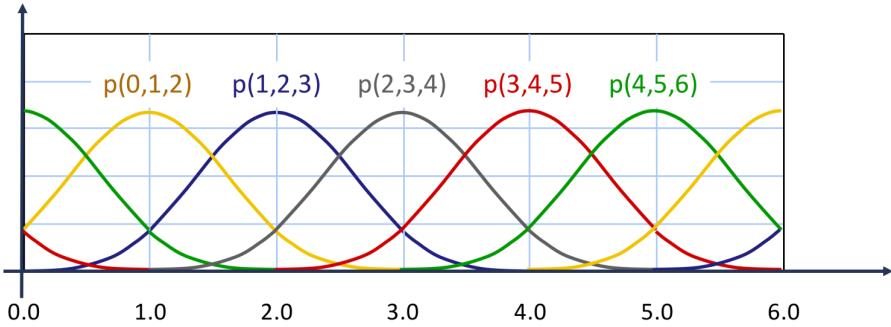
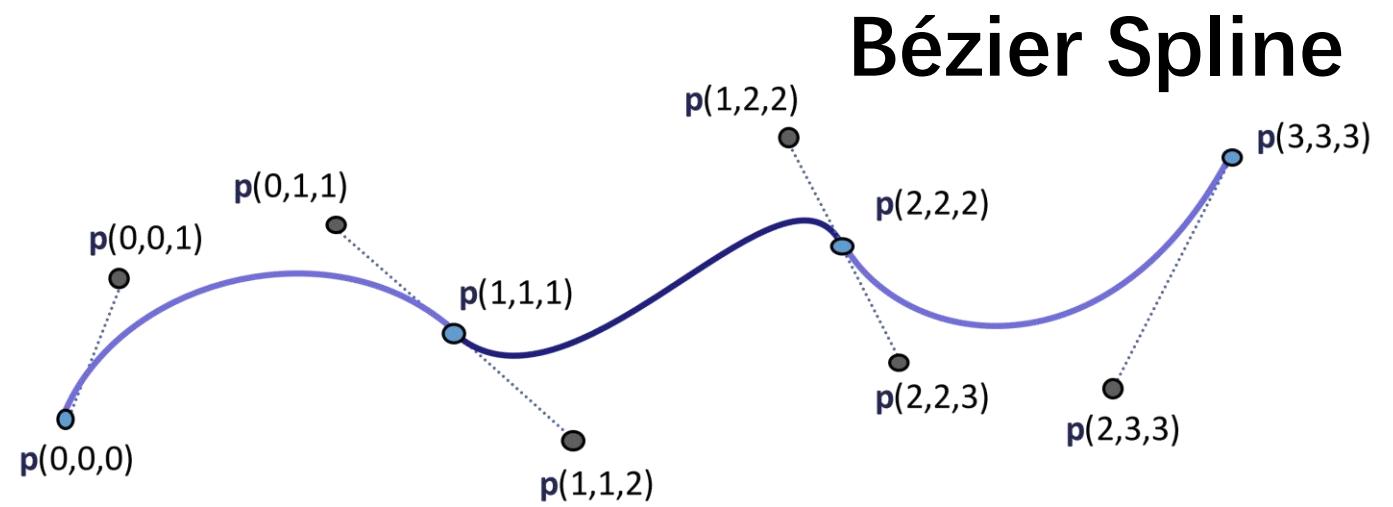
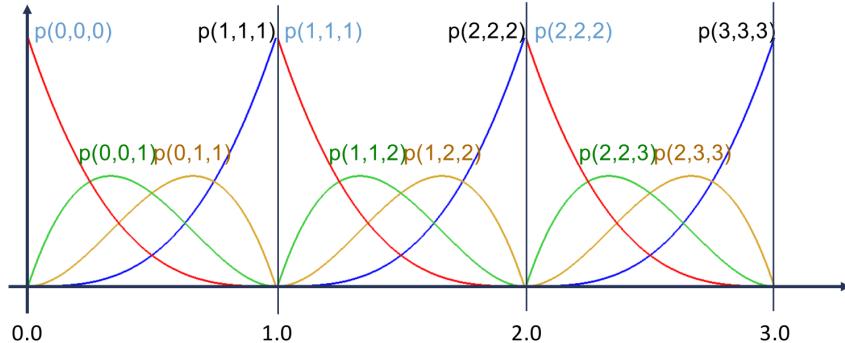


Bézier Spline



B-Spline

Structure



Bézier Spline

B-Spline