FEDERAL STATE AUTONOMOUS EDUCATIONAL INSTITUTION
OF HIGHER EDUCATION
ITMO UNIVERSITY

Report
on the practical task No. 1
”Experimental time complexity analysis”

Performed by
Grigoryev Mikhail (370852) &
Semenova Valeria (370061)
Academic group J4133c
Accepted by
Dr Petr Chunaev

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Goal

Experimental study of the time complexity of different algorithms.

Formulation of the problem

For each n in range [1, 2000] the average computer execution time of programs implementing the algorithms and functions below (for five runs) is to be measured using timestamps. The obtained data is to be plotted showing the average execution time as a function of n. Theoretical time complexities are to be calculated and compared to empyrical.

1. Generate a random vector $v = [v_1, v_2, \cdots, v_n]$ with non-negative elements. For v, implement:
   - $f(v) = const$;
   - $f(v) = \sum_{k=1}^{n} v_k$;
   - $f(v) = \prod_{k=1}^{n} v_k$;
   - supposing the elements of v are the coefficients of a polynomial P of degree $n-1$, calculate the value $P(1.5)$ by a direct calculation of $P(x) = \sum_{k=1}^{n} v_k x^{k-1}$ and by Horner's method by representing the polynomial as:

     $$P(x) = v_1 + x(v_2 + x(v_3 + \cdots))$$

   - Bubble Sort of the elements of v;
   - Quick Sort of the elements of v;
   - Timsort of the elements of v.
2. Generate random matrices A and B of size $n \times n$ with non-negative elements. Find the usual matrix product for A and B.
3. Describe the data structures and design techniques used within the algorithms.

Brief theoretical part

Time complexity of an algorithm on a dataset of size n refers to the amount of time that a certain computer requires to execute the algorithm on that dataset. As computers have vastly different computing powers, time complexity is usually represented in the "big O" notation.

Definition:

$$f(n) = O(g(n)) \quad \Leftrightarrow \quad \exists n_0, c > 0 : \quad \forall n > n_0 \quad 0 \leq f(n) \leq cg(n)$$

This represents that $f(N)$ grows no faster than $g(N)$, starting with datasets of size N_0.

Algorithms and functions implemented in this task are:

- Constant function (implementation is trivial). Time complexity is $O(1)$ as there are no calculations which depend on the size of the input dataset.

- Sum function is taken from python3 numpy library. Time complexity is $O(n)$, as to add all items within a vector, we must iterate over all of them.

- Product function is also taken from numpy library. Time complexity is again $O(n)$ due to iteration over all items within the random input vector.

- Direct polynomial evaluation was implemented trivially, as we needed to calculate each term individually. For x^{k-1} numpy power function was used. Time complexity of numpy.power is $O(1)$ so total complexity of the computation is proportional to the number of terms, or $O(n)$.

- Horner method of polynomial evaluation was implemented via a recursive function. Time complexity is still $O(n)$, as the number of recursive function calls is equal to n and all of them are executed at a constant rate.

- Bubble sort algorithm was faster to write by hand than to find an implementation of as it is rarely used due to its inefficiency. Time complexity is $O(n^2)$, as in worst case scenario (reversed order) for every item in the random input vector we will have to iterate over every other item. Average time of bubble sort can vary vastly, as it works fine with inputs close to being sorted.

- Quicksort algorithm was taken from numpy library. Time complexity of quicksort depends on how the pivot point is selected. Generally, it has complexity of $O(n \log n)$.

- Timsort is python's default sorting algorithm in its standard library, so standard library was used. It finds ordered runs within the input vector and utilizes them to sort the vector faster (with a combination of merge and insertion sorting algorithms). It has time complexity of $O(n \log n)$ as it is impossible to sort a vector faster than $O(n \log n)$ (unless the vector consists only of numerical values, then sorting can be done in linear time).

- Trivial implementation of matrix multiplication without any optimization proved to calculate the matrix product of A and B very slowly. Thus, numpy library function matmul uses optimized BLAS (Basic Linear Algebra Subprograms) method. It has time complexity of $O(n^3)$ but supposedly low constants, so computations don't take years.

For convenience, random input vector (and matrices) was generated once per all 2000 measurements and smaller datasets were created by slicing the initial random vector. E.g. if $n = 420$, random vector for this computation will be *v[:420]* and random matrix will be *A[:420, :420]*.

For random input vector and random input matrices numpy arrays (dynamic arrays, essentially) were used.

Results

Measurements of average execution time of constant function with time complexity $O(n)$ showed that constant time is needed in order to return a constant.

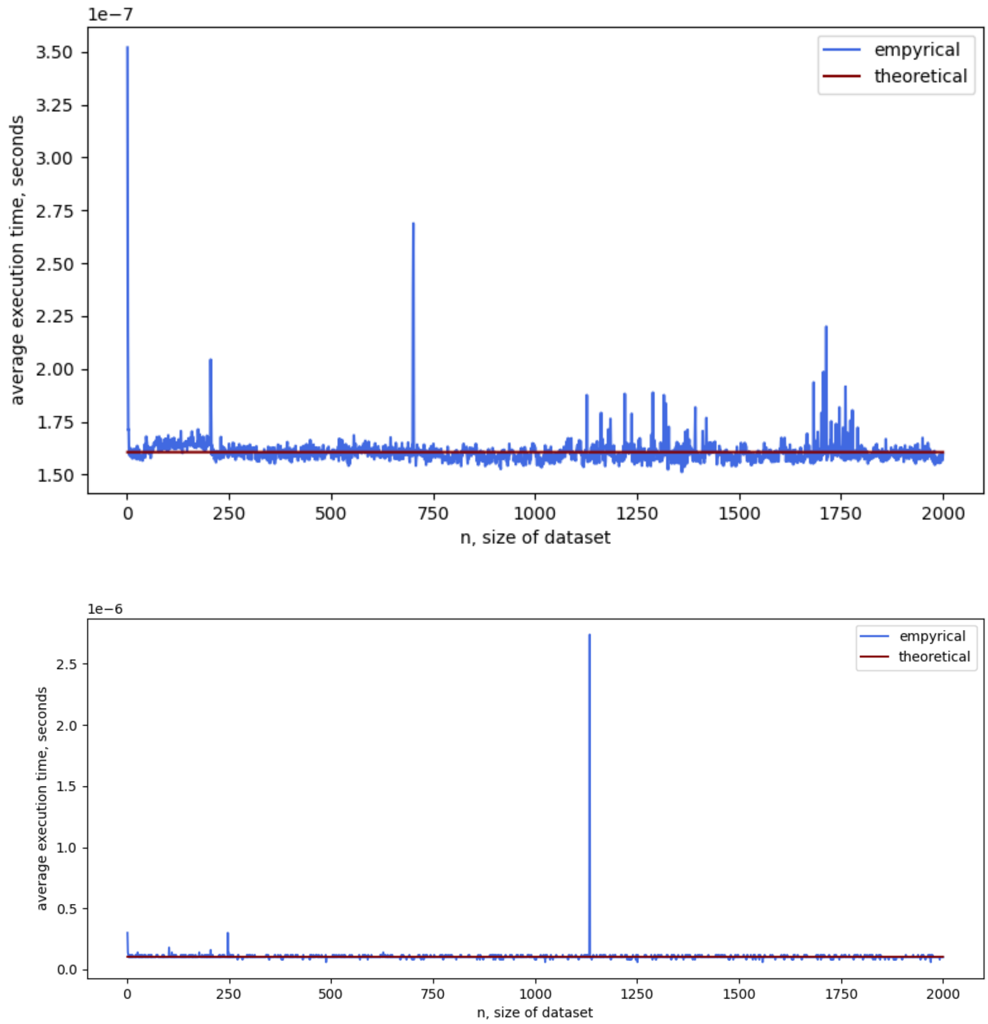
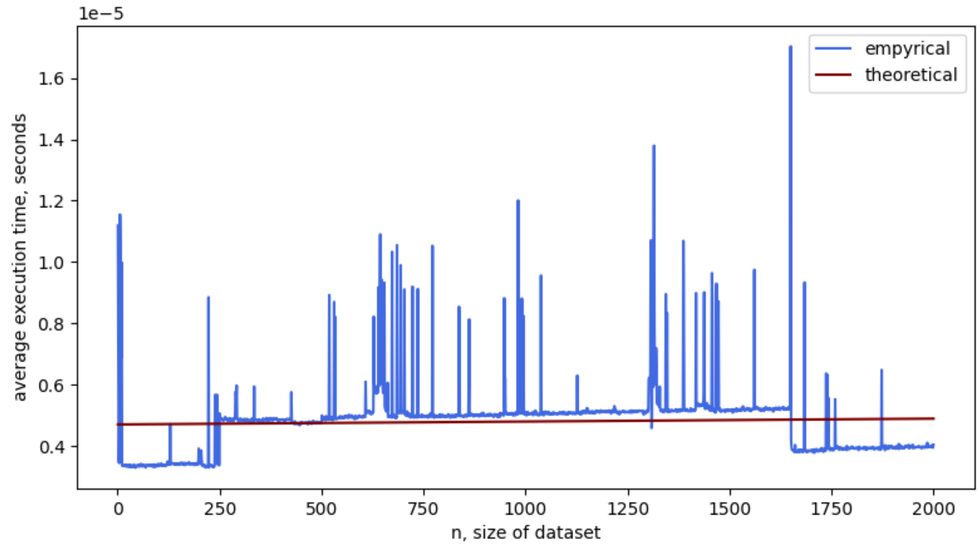


Figure 1: Empyrical and theoretical time complexities of an algorithm implementing the function $f(v) = 1$ (on two different machines).

Deviations from the constant are minimal and can be caused by system processes interrupting the program, as it is utilizing only one processor core.

Summation of all items within a random vector indeed was executed in linear time, as shown in the graph below.



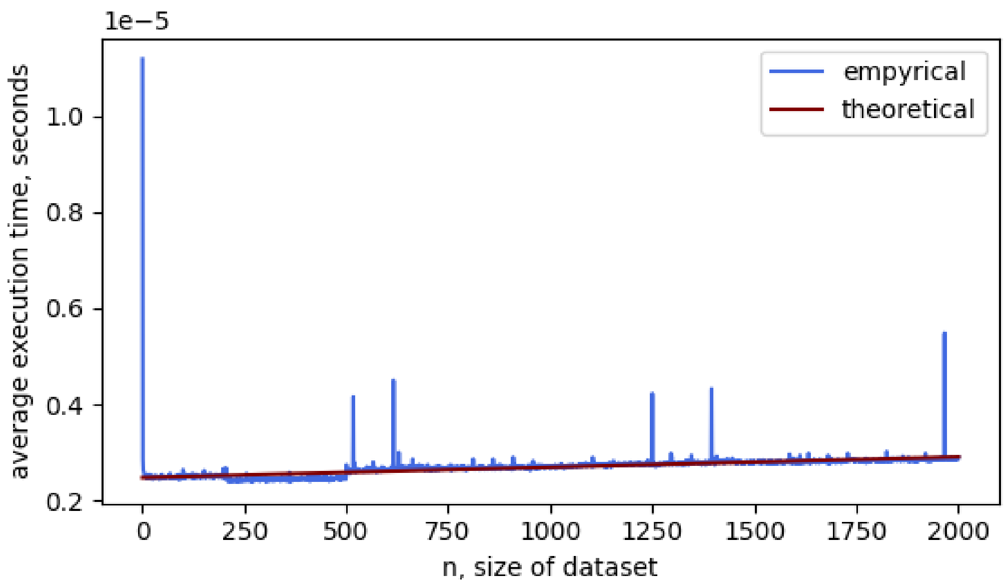
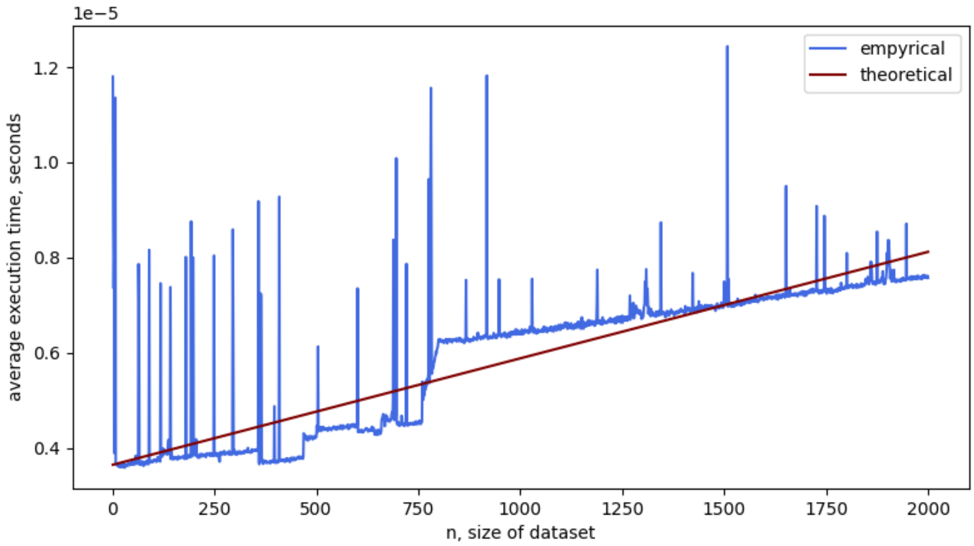


Figure 2: Empyrical and theoretical time complexities of an algorithm implementing the function $f(v) = \sum_{k=1}^{n} v_k$ (on two different machines).

The computation was done in linear time, although the slope of the line seems low. This can be caused by simple mathematical operators such as ”+” to be processed really fast.

Similar trends were observed with the product function. The slope, however, is higher as "*" operator is processed slower than "+". Several slopes can be calculated because of system interferences.



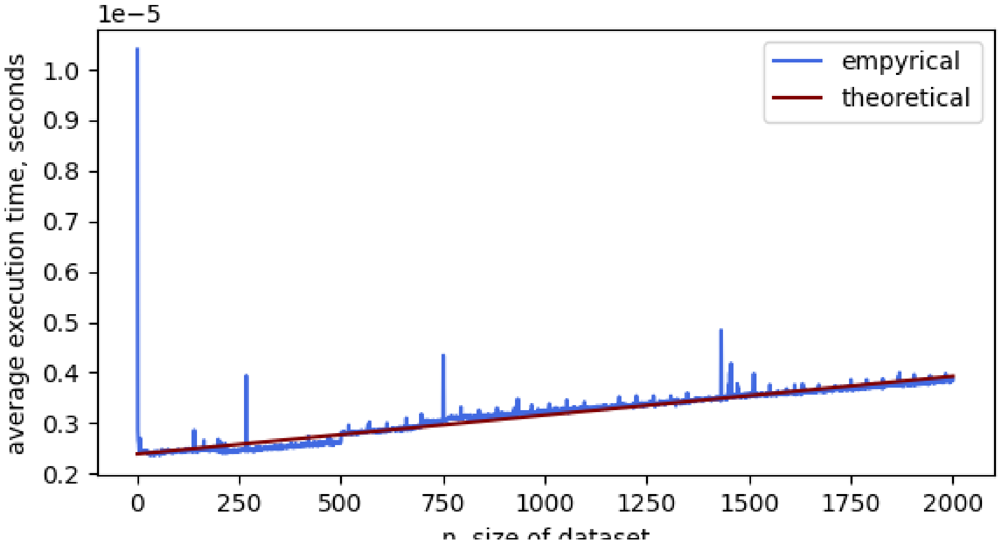
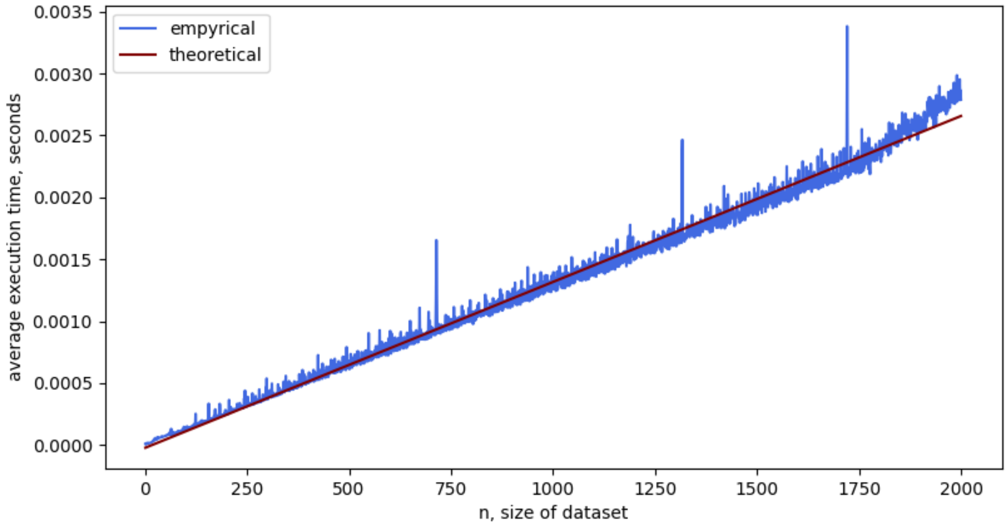


Figure 3: Empyrical and theoretical time complexities of an algorithm implementing the function $f(v) = \prod_{k=1}^{n} v_k$ (on two different machines).

Polynomial evaluation takes more time than the former algorithms, so random system interferences do not impact the measurements as much. Clear linear trend can be observed in direct polynomial evaluation.



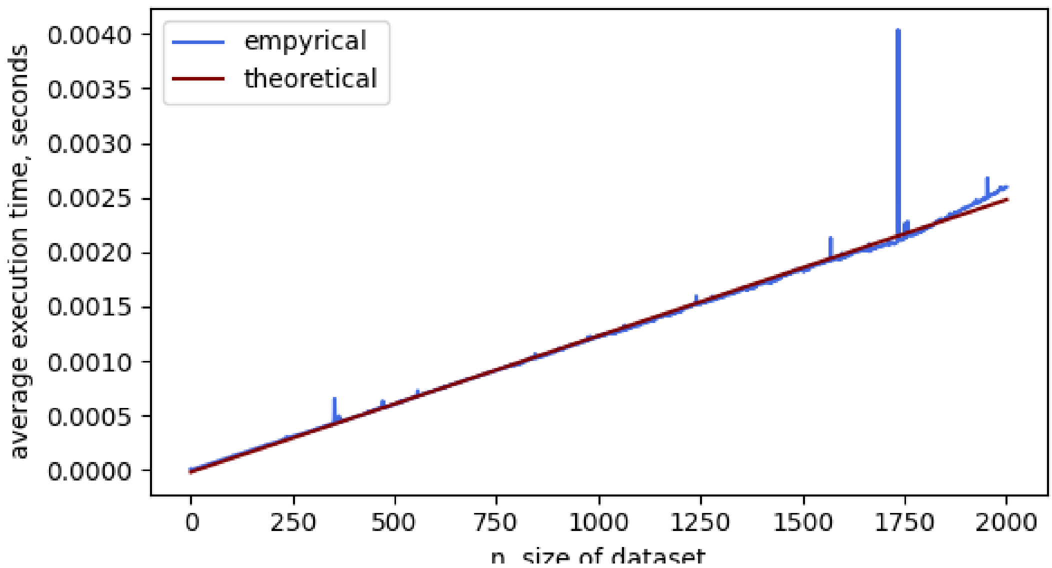
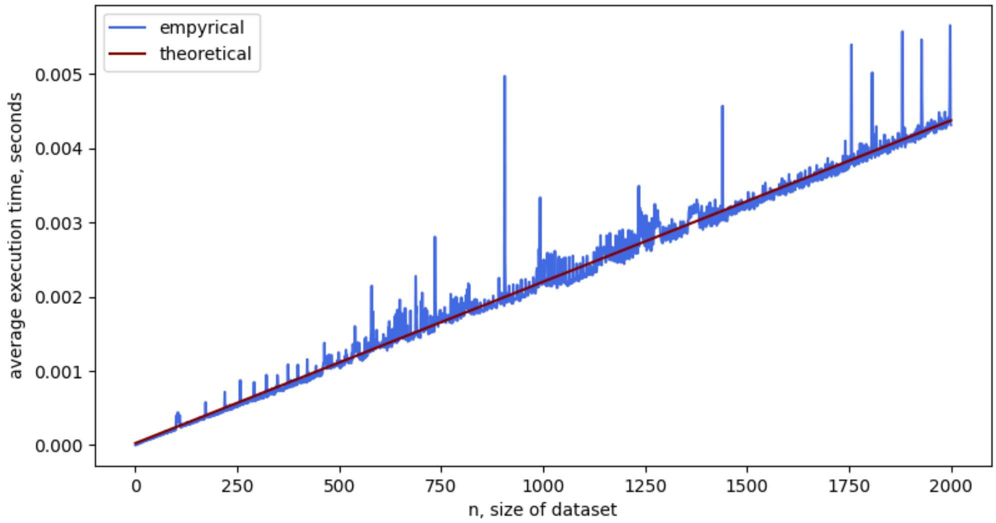


Figure 4: Empyrical and theoretical time complexities of an algorithm implementing direct term-by-term evaluation of a polynomial (on two different machines).

Horner's method of evaluation of a polynomial at a certain point gave similar results with a linear trend.



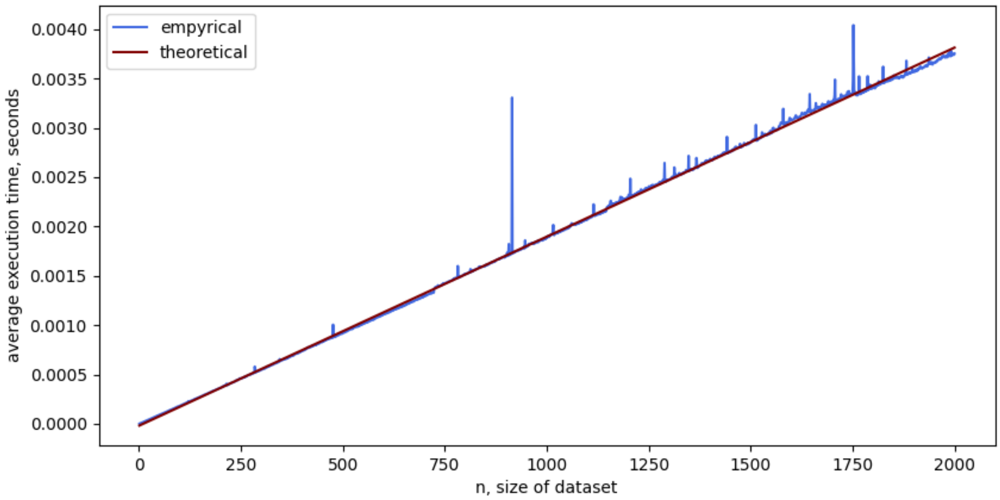


Figure 5: Empyrical and theoretical time complexities of an algorithm implementing Horner's method of evaluation of a polynomial (on two different machines).

A vastly different picture was observed with bubble sort. It gave a generally parabolic trend, however the deviation of execution time grew as datasets increased in size. This could be caused by bubble sorting algorithm being really dependent on the starting condition of the input vector.

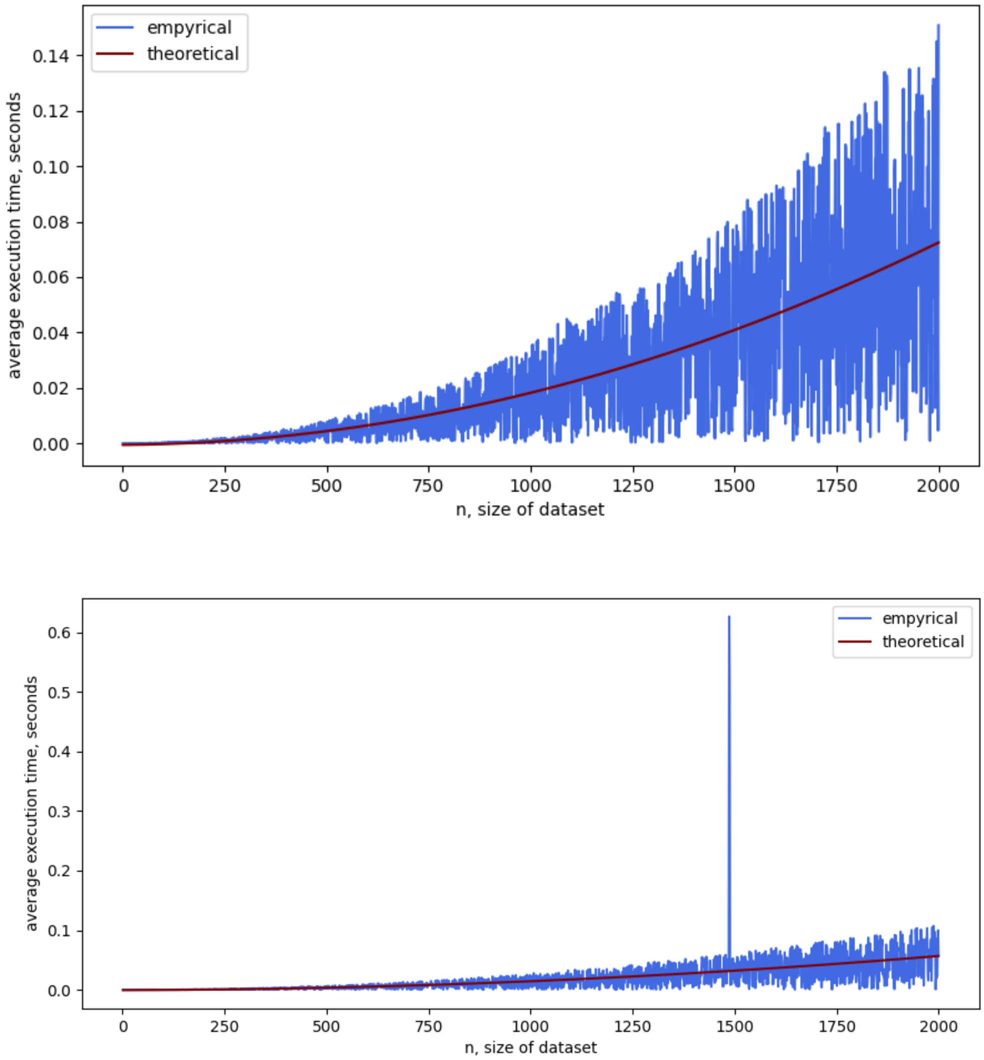


Figure 6: Empyrical and theoretical time complexities of a bubble sorting algorithm (on two different machines).

Clear $O(n \log n)$ time complexity can be observed in the graph below, where execution time of quicksort algorithm is depicted.

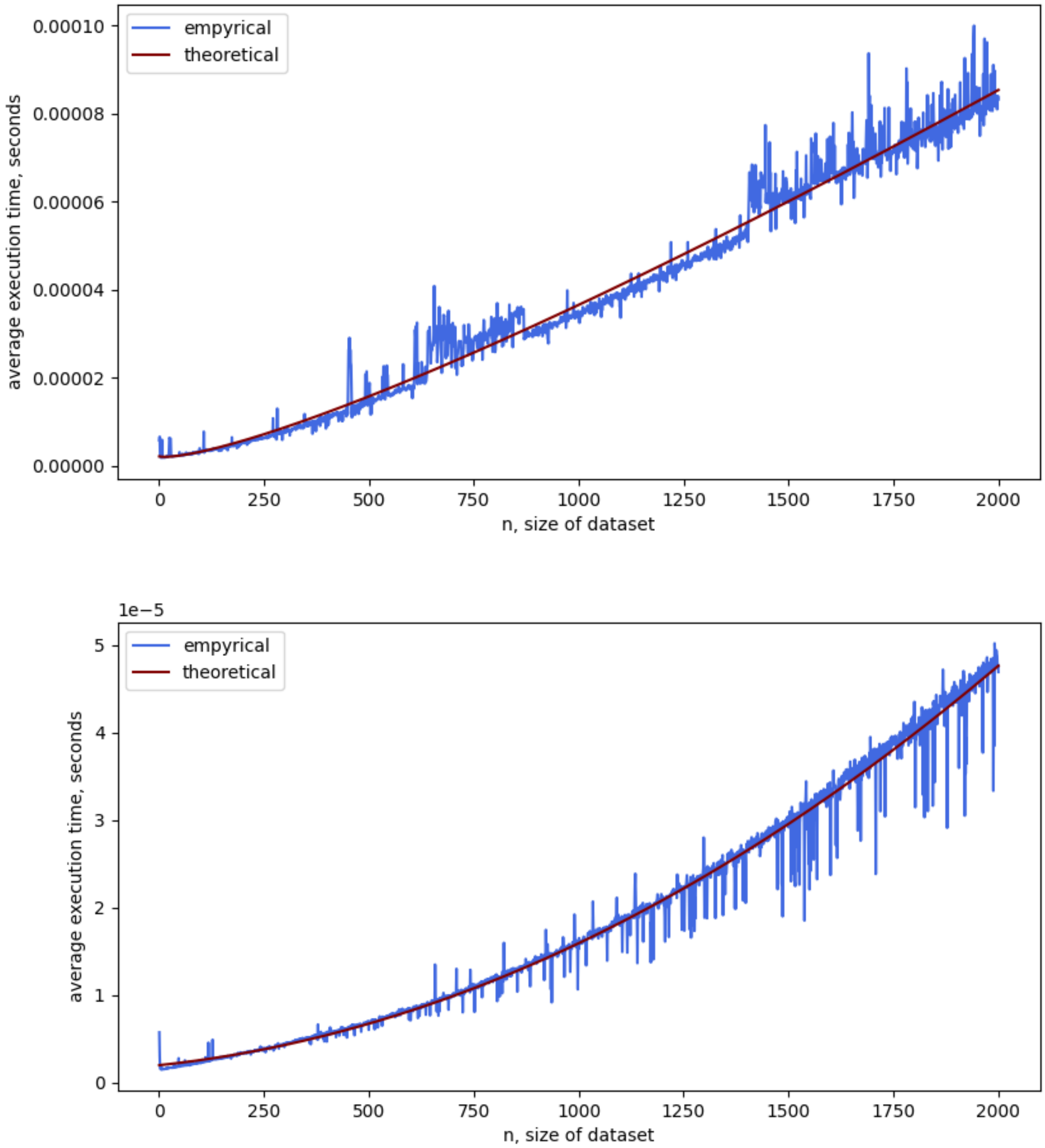
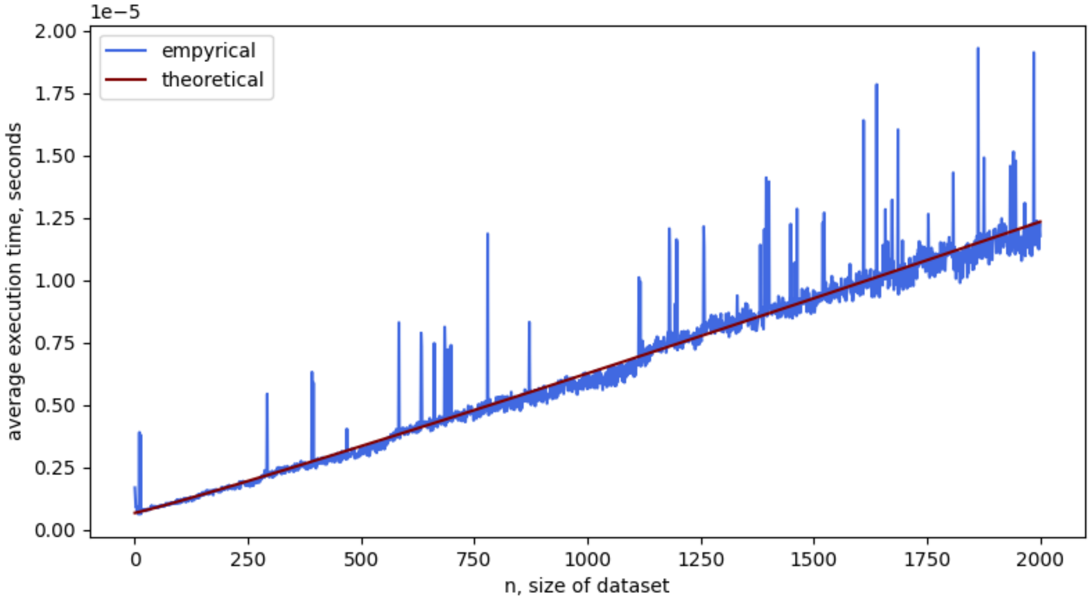


Figure 7: Empyrical and theoretical time complexities of a quicksort algorithm (on two different machines).

Deviation is minimal and seems caused by system interruptions as the deviation incresed and decreased in a step manner.

Timsort gave an $n \log n$ tendency, although visually it resembles linear time. Thus it can be stated, that timsort is significantly more time efficient than quicksort.



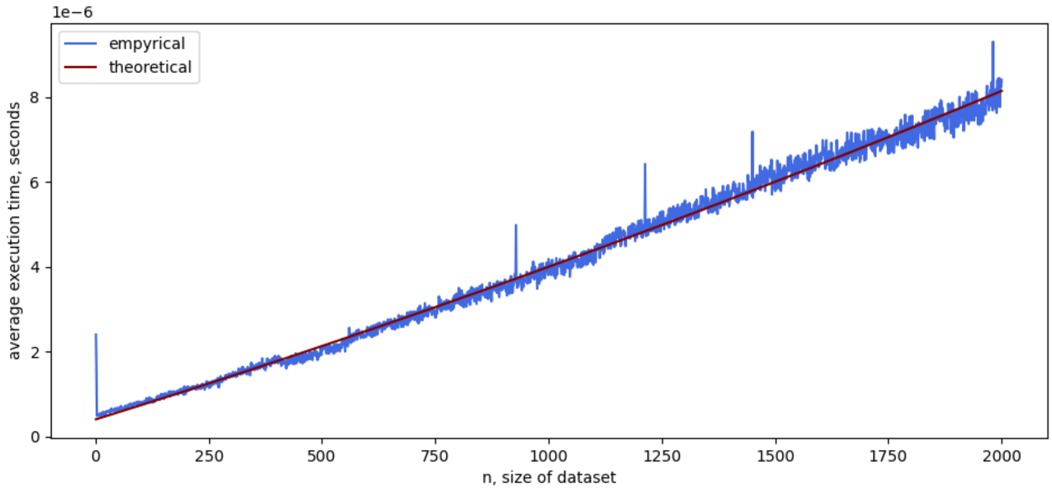


Figure 8: Empyrical and theoretical time complexities of a timsort algorithm (on two different machines).

Matrix multiplication is a time consuming task which usually has time complexity of $O(n^3)$. Constants decide, whether computations will take seconds or hours, in that case. As shown in the graph below, cubic time was measured for numpy matmul implementation of matrix multiplication.

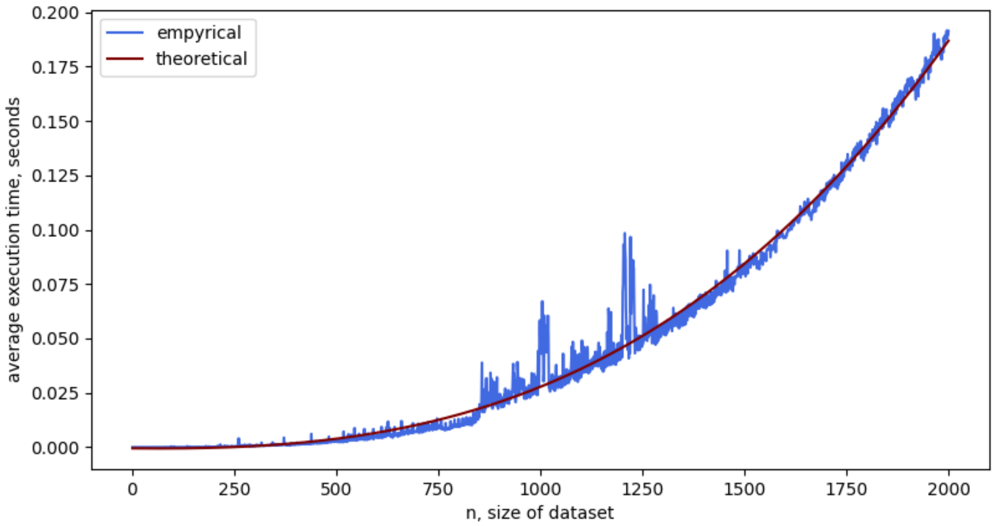


Figure 9: Empyrical and theoretical time complexities of an algorithm implementing matrix multiplication.

Conclusions

In this practical work average execution times of different algorithms and functions were measures. The results proved to correspond to theoretical time complexities. Deviations in execution times were supposedly caused by system processes interfering with the program running on one processor core.

Appendix

GitHub link: https://github.com/Dormant512/itmo_lab_listings/blob/main/lab1.py.