Assignment 2

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Part A

Equation 1

$$x\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - t$$

Separatable first-order linear ODE:

$$\begin{gathered}
x\mathrm{d}x = (1-t)\mathrm{d}t \\
\int x\mathrm{d}x = \int (1-t)\mathrm{d}t \\
\frac{x^2}{2} = t - \frac{t^2}{2} + C_0 \\
x^2 = 2t - t^2 + C_1 \\
x = \pm\sqrt{2t - t^2 + C_1}
\end{gathered}$$

Answer:

$$x = \pm\sqrt{2t - t^2 + C}$$

Equation 2

$$y' - y = 2x - 3$$

Inseparable non-homogeneous first-order linear ODE. Let's solve a homogeneous ODE:

$$\begin{gathered}
y' - y = 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} = y \\
\frac{\mathrm{d}y}{y} = \mathrm{d}x \\
\int \frac{\mathrm{d}y}{y} = \int \mathrm{d}x \\
\ln y = x + C_0 \\
y = C_0\mathrm{e}^x
\end{gathered}$$

Go back to the initial ODE with $y = \varphi(x)\mathrm{e}^x$:

$$\begin{gathered}
\left(\varphi(x)\mathrm{e}^x\right)' - \varphi(x)\mathrm{e}^x = 2x - 3 \\
\varphi'(x)\mathrm{e}^x + \varphi(x)\mathrm{e}^x - \varphi(x)\mathrm{e}^x = 2x - 3 \\
\varphi'(x)\mathrm{e}^x = 2x - 3 \\
\varphi'(x) = (2x-3)\mathrm{e}^{-x} \\
\varphi(x) = \int (2x-3)\mathrm{e}^{-x}\mathrm{d}x
\end{gathered}$$

Let's compute the latter integral:

$$u = (2x - 3) \qquad \mathrm{d}u = 2\mathrm{d}x$$
$$\mathrm{d}v = \mathrm{e}^{-x} \qquad v = -\mathrm{e}^{-x}$$
$$\int u\mathrm{d}v \equiv uv - \int v\mathrm{d}u$$
$$\int (2x-3)\mathrm{e}^{-x}\mathrm{d}x = (3-2x)\mathrm{e}^{-x} + 2\int \mathrm{e}^{-x}\mathrm{d}x = (3-2x)\mathrm{e}^{-x} - 2\mathrm{e}^{-x} + C_0$$

So, φ:

$$\varphi(x) = (3-2x)\mathrm{e}^{-x} - 2\mathrm{e}^{-x} + C_0$$
$$y = \varphi(x)\mathrm{e}^{x} = (3-2x) - 2 + C_0\mathrm{e}^{x}$$
$$y = C_0\mathrm{e}^{x} - 2x + 1$$

Answer:

$$y = C\mathrm{e}^{x} - 2x + 1$$

Equation 3

$$y^2 + x^2y' = xyy'$$

Inseparable non-homogeneous first-order linear ODE. Let's brush it up:

$$\left(\frac{y}{x}\right)^2 + y' = \frac{y}{x}y'$$
$$t = \frac{y}{x} \qquad y = tx \qquad y' = t'x + t$$
$$t^2 + t'x + t = t(t'x + t)$$
$$t^2 + t'x + t = t'tx + t^2$$
$$t'x + t = t'tx$$
$$\frac{\mathrm{d}t}{\mathrm{d}x}x(t-1) = t$$
$$\frac{\mathrm{d}t}{t}x(t-1) = \frac{\mathrm{d}x}{x}$$
$$\int \frac{\mathrm{d}t}{t}(t-1) = \int \frac{\mathrm{d}x}{x}$$
$$\int \left(1 - \frac{1}{t}\right)\mathrm{d}t = \ln x$$
$$t - \ln t = \ln x + C$$

This cannot be simplified, this is the solution. Let's use x and y:

$$\frac{y}{x} - \ln\frac{y}{x} = \ln x + C$$

Potential solution is $x = 0$. Let's check:

$$y^2 = 0$$

Generally, no. Answer:

$$\frac{y}{x} - \ln\frac{y}{x} = \ln x + C$$

Equation 4

$$(2-9xy^2)x\mathrm{d}x+(4y^2-6x^3)y\mathrm{d}y=0$$

Let's check if this is an ODE in full differentials:

$$\underbrace{(2x-9x^2y^2)}_{F'_x}\mathrm{d}x+\underbrace{(4y^3-6x^3y)}_{F'_y}\mathrm{d}y=0$$

$$F'_{xy}=-18x^2y$$
$$F'_{yx}=-18x^2y$$

ODE in full differentials:

$$F'_x=(2x-9x^2y^2)$$
$$F(x,y)=\int 2x\mathrm{d}x-9\int x^2y^2\mathrm{d}x$$
$$F(x,y)=x^2-3x^3y^2+\varphi(y)$$
$$F'_y=(4y^3-6x^3y)\qquad(*)$$
$$F'_y=\Big(x^2-3x^3y^2+\varphi(y)\Big)'_y$$
$$F'_y=-6x^3y+\varphi'_y(y)\qquad(**)$$
$$(*),(**)\implies 4y^3-6x^3y=-6x^3y+\varphi'_y(y)$$
$$\varphi'_y(y)=4y^3$$
$$\varphi(y)=y^4+C_0$$

All in all:

$$F(x,y)=x^2-3x^3y^2+y^4+C_0$$

Answer:

$$x^2-3x^3y^2+y^4=C$$

Equation 5

$$(2x+1)y'=4x+2y$$
$$y'-\frac{2}{2x+1}y=\frac{4x}{2x+1}$$

Inseparable non-homogeneous first-order linear ODE. Let's solve a homogeneous ODE:

$$y'-\frac{2}{2x+1}y=0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{2x+1}y$$
$$\frac{\mathrm{d}y}{y}=\frac{2\mathrm{d}x}{2x+1}$$
$$\ln y=\ln(2x+1)+C_0$$
$$y=C_1(2x+1)$$

Go back to the initial ODE with $y=\varphi(x)(2x+1)$:

$$(2x+1)\Big(\varphi(x)(2x+1)\Big)' = 4x + 2\varphi(x)(2x+1)$$
$$(2x+1)\Big(\varphi'(x)(2x+1) + 2\varphi(x)\Big) = 4x + 2\varphi(x)(2x+1)$$
$$\varphi'(x)(2x+1)^2 = 4x$$
$$\varphi'(x) = \frac{4x}{(2x+1)^2}$$
$$\varphi(x) = 4\int \frac{x}{(2x+1)^2}\mathrm{d}x$$

Let's compute the integral:

$$\int \frac{x}{(2x+1)^2}\mathrm{d}x = \int \Big(\frac{1}{2(2x+1)} - \frac{1}{2(2x+1)^2}\Big)\mathrm{d}x =$$
$$= \int \frac{1}{2(2x+1)}\mathrm{d}x - \int \frac{1}{2(2x+1)^2}\mathrm{d}x =$$
$$= \frac{1}{4}\ln(2x+1) + \frac{1}{4(2x+1)} + C_2$$

Get back:

$$\varphi(x) = \ln(2x+1) + \frac{1}{(2x+1)} + C_3$$

Get back once more:

$$y = \varphi(x)(2x+1) = (2x+1)\ln(2x+1) + 1 + C_4$$

Potential solution is $x = -0.5$. Let's check:

$$0y' = -2 + 2y$$
$$y = 1$$

Not necessarily. Answer:

$$y = (2x+1)\ln(2x+1) + 1 + C$$

Part B

Task 1

A population of protozoa develops with a constant relative growth rate of 0.7 per member per day. Initially, the population consist of two members. Find the population size after six days.

Let n be the population size, t be time in days:

$$\dot{n} = 0.7n$$
$$\frac{\mathrm{d}n}{\mathrm{d}t} = 0.7n$$
$$\frac{\mathrm{d}n}{n} = 0.7\mathrm{d}t$$
$$\int \frac{\mathrm{d}n}{n} = 0.7\int \mathrm{d}t$$
$$\ln n = 0.7t + C_0$$
$$n = C\mathrm{e}^{0.7t}$$

Starting point:

$$n(t=0) = C\mathrm{e}^0 = C = 2$$
$$n = 2\mathrm{e}^{0.7t}$$
$$n(t=6) = 2\mathrm{e}^{4.2} \approx 133.37$$

Answer:

$$n(t=6) = 2\mathrm{e}^{4.2} \approx 133.37$$

Task 2

Consider an insect population whose size p is measured as biomass (mass of the population members) in kilograms. The population is increasing by 30% per year. However, the population is also controlled by a natural predator population that destroys 6 kg of insects per year.

(a) Find the model describing the population size p at any given time t;

(b) Find the population size 4 years after if the initial biomass is 15 kg.

$$\dot{p} = 0.3p - 6$$
$$\frac{\mathrm{d}p}{\mathrm{d}t} - 0.3p = -6$$

Non-homogeneous ODE, consider homogeneous:

$$\frac{\mathrm{d}p}{\mathrm{d}t} - 0.3p = 0$$
$$\frac{\mathrm{d}p}{\mathrm{d}t} = 0.3p$$
$$\frac{\mathrm{d}p}{p} = 0.3\mathrm{d}t$$
$$\int \frac{\mathrm{d}p}{p} = \int 0.3\mathrm{d}t$$
$$\ln p = 0.3t + C_0$$
$$p = C\mathrm{e}^{0.3t}$$

Return to the initial ODE with $p = \varphi(t)\mathrm{e}^{0.3t}$:

$$(\varphi(t)\mathrm{e}^{0.3t})' = 0.3\varphi(t)\mathrm{e}^{0.3t} - 6$$
$$\varphi'(t)\mathrm{e}^{0.3t} + 0.3\varphi(t)\mathrm{e}^{0.3t} = 0.3\varphi(t)\mathrm{e}^{0.3t} - 6$$
$$\varphi'(t)\mathrm{e}^{0.3t} = -6$$
$$\varphi'(t) = -6\mathrm{e}^{-0.3t}$$
$$\varphi(t) = -6\int \mathrm{e}^{-0.3t}\mathrm{d}t$$
$$\varphi(t) = 20\mathrm{e}^{-0.3t} + C_1$$

Get back:

$$p = \varphi(t)\mathrm{e}^{0.3t} = (20\mathrm{e}^{-0.3t} + C_1)\mathrm{e}^{0.3t} = 20 + C\mathrm{e}^{0.3t}$$

Initial condition:

$$p(t=0) = 20 + Ce^0 = 20 + C = 15$$
$$C = -5$$
$$p(t=4) = 20 - 5e^{0.3 \cdot 4} = 20 - 5e^{1.2} \approx 3.4$$

Answers:

(a) General model:

$$p = 20 + Ce^{0.3t}$$

(b) The population size 4 years after if the initial biomass is 15 kg:

$$p(t=4) = 20 - 5e^{1.2} \approx 3.4$$