Continuous 2

Solution in full differentials

$$\underbrace{2xy}_{F'_x}\mathrm{d}x + \underbrace{(x^2 - y^2)}_{F'_y}\mathrm{d}y = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}y} 2xy = 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - y^2) = 2x$$

$$F(x, y) = \int 2xy\mathrm{d}x = yx^2 + \varphi(y)$$

$$F'_y = x^2 - y^2 = \frac{\mathrm{d}}{\mathrm{d}y}\Big(yx^2 + \varphi(y)\Big)$$

$$x^2 + \varphi'_y = x^2 - y^2$$

$$\varphi'_y = -y^2$$

$$\varphi = -\int y^2\mathrm{d}y = -\frac{1}{3}y^3$$

$$F(x, y) = yx^2 - \frac{1}{3}y^3$$

Answer:

$$yx^2 - \frac{1}{3}y^3 = C$$

First-order linear ODEs

Reminder:

$$y' + a(x)y = \underbrace{b(x)}_{\text{assume } 0}$$

FOLODE 1

$$xy' - 2y = 2x^4$$

Assume:

$$xy' - 2y = 0$$

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$$

$$\frac{\mathrm{d}y}{y} = \frac{2\mathrm{d}x}{x}$$

$$\int \frac{\mathrm{d}y}{y} = \int \frac{2\mathrm{d}x}{x}$$

$$\ln y = 2\ln x + C_0$$

$$y = x^2\mathrm{e}^{C_0} = Cx^2$$

Go back:

$$\underbrace{C'(x)x^3 + 2C(x)x^2}_{xy'} - \underbrace{2C(x)x^2}_{2y} = 2x^4$$
$$C'(x)x^3 = 2x^4$$
$$C'(x) = 2x$$
$$C(x) = x^2 + M$$

Answer:

$$y = (x^2 + M)x^2 = x^4 + Mx^2$$

Modeling task

A population of field mice inhabits a certain rural area. In the absence of predators, the mice population increases so that each month (30 days), the population increases by 50%.

However, several owls live in the same area and they kill 15 mice per day. Find an equation describing the population size and use it to predict the long-term behavior of the population.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0.5y - \underbrace{30 \cdot 15}_{450}$$
$$\dot{y} - 0.5y = -450$$

Assume:

$$\dot{y} - 0.5y = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2}$$
$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}x}{2}$$
$$\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{2}$$
$$\ln y = 0.5x + C_0$$
$$\mathrm{e}^{\ln y} = \mathrm{e}^{0.5x + C_0}$$
$$y = \mathrm{e}^{0.5x} \cdot \underbrace{\mathrm{e}^{C_0}}_{C}$$
$$y = C\mathrm{e}^{x/2}$$

Go back:

$$y = \varphi(x)\mathrm{e}^{x/2}$$
$$\dot{y} - 0.5y = -450$$
$$\left(\varphi(x) \cdot \mathrm{e}^{x/2}\right)' - 0.5\varphi(x) \cdot \mathrm{e}^{x/2} = -450$$
$$\underbrace{\varphi'(x) \cdot \mathrm{e}^{x/2} + \varphi(x) \cdot \mathrm{e}^{x/2}\frac{1}{2}}_{\left(\varphi(x) \cdot \mathrm{e}^{x/2}\right)'} - \varphi(x) \cdot \mathrm{e}^{x/2}\frac{1}{2} = -450$$
$$\varphi'(x) \cdot \mathrm{e}^{x/2} = -450$$
$$\varphi'(x) = -450 \cdot \mathrm{e}^{-x/2}$$
$$\varphi(x) = -450 \int \mathrm{e}^{-x/2} \mathrm{d}x$$

Exchange variables:

$$u = -x/2 \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -0.5$$

$$\mathrm{d}u = -0.5\mathrm{d}x$$

$$\mathrm{d}x = -2\mathrm{d}u$$

$$\varphi(x) = -450 \int \mathrm{e}^{-x/2} \underbrace{\mathrm{d}x}_{-2\mathrm{d}u}$$

$$\varphi(u) = 900 \int \mathrm{e}^{u}\mathrm{d}u$$

$$\varphi(u) = 900\mathrm{e}^{u} + C$$

$$\varphi(x) = 900\mathrm{e}^{-x/2} + C$$

Go back:

$$y = \varphi(x)\mathrm{e}^{x/2}$$

$$y = \mathrm{e}^{x/2}\left(900\mathrm{e}^{-x/2} + C\right) = 900 + C\mathrm{e}^{x/2}$$

Answer:

$$y = 900 + C\mathrm{e}^{x/2},$$

where x is time measured in months.