Lecture Notes on

Unit 01 Differential Equations

Aspen J. Johnson

`aspen.johnson @ my.pbsc.edu`

The contents of these pages constitute my authorized exam notes for Unit 01 of Differential Equations. This exam will take place at the Lake Worth campus under the supervision of Professor Tamara Johns on Wednesday, February 18, 2026, at 10:00 AM, in her physical office.

This is a retake opportunity granted after my initial Exam 01 Attempt 01 score of 25%, and these notes have been prepared to support a stronger performance on the retake. During the exam, I am permitted to reference these notes. The material included here is compiled directly from the following resources, which I consulted while preparing:

1. Lecture recordings and handouts from Prof. Johns
2. *Differential Equations* textbook by Blanchard, Hall, and Devaney
3. Online notes by Professor Lebl
4. Houston Math's YouTube channel

Contents

1 DEFINITIONS AND TERMINOLOGY, IVP, AND SLOPE FIELDS

1.1 Definitions and Terminology

1.2 Initial Value Problems

With initial value problems, we are given a differential equation and then given a point in the form of $y(x) = y_0$ to be substituted into the general solution (once found) and then used to solve for the arbitrary constant(s).

Example Problems and Solutions From Houston Math

1. Answer the following parts corresponding to each differential equation below:

$$\frac{dy}{dx} = 2x \tag{1}$$

$$y' = 6x^2 + 4 \tag{2}$$

$$x^2 y' = -1 \tag{3}$$

$$y'' = xe^x \tag{4}$$

   (a) Find the **general solution**.

   (b) Find the **particular solution** given that $y(1) = 4$.

   (c) Find the **particular solution** given that $y(1) = 3$.

2. Answer the following parts corresponding to the **second-order differential equation**:

$$\frac{d^2y}{dt^2} = \cos(t) \tag{5}$$

   (a) Find the **particular solution** given that $y'(0) = 0$ and $y(0) = 1$.

   (b) Find the **particular solution** given that $y(1) = 3$.

2 SEPARABLE EQUATIONS

2.1 (2.2) - (General Overview) Separable Equations

If a **differential equation** can be written with all of the dependent-variable expressions on one side (usually y), and all of the independent-variable expressions on the other side (usually x), then we say that the equation is **separable**.

The form we desire when identifying a **separable equation** is $\frac{dy}{g(y)} = f(x)\,d(x)$. If the form is not easy to spot at first glance, then this is the form that we wish to rewrite the given differential equation into.

It is also important that both sides are integrable. These antiderivatives may differ by a constant at most, but what we really care about is that the right-hand side of the equation is integrable.

Small examples where the equations are already in the form $\boldsymbol{h(y)\,dy = g(x)\,dx}$:

$$y^2\,dy = 4x\,dx \tag{6}$$
$$\frac{dy}{y} = te^t\,dt \tag{7}$$
$$\sec(t)\tan(t)\,dt = dx \tag{8}$$
$$\frac{x+1}{x-1}\,dx = \frac{dy}{y^2+1} \tag{9}$$

Here is an example where the equation is not in the form, but is separable (with a bit of work):

$$\frac{dy}{dx} - x = xy^2 \tag{10}$$

Here is an example of an equation that is **not** separable:

$$\frac{dy}{dx} - x = y^2 \tag{11}$$

The above **is not separable** because there is no way to create a product between x's and y's such that we attain the form $\boldsymbol{h(y)\,dy = g(x)\,dx}$ or $\boldsymbol{\frac{dy}{g(y)} = f(x)\,d(x)}$.

General Outline For Solving A Separable Equation:

1. Separate the variables, making sure each differential is on top.
2. Integrate both sides **with respect to their particular variable**. For example:
   (a) In $\frac{dy}{dx}$, y is the dependent variable, as it is **dependent** on x.
   (b) In $\frac{dy}{dt}$, y is the dependent variable, as it is **dependent** on t.
   (c) In $\frac{dx}{dt}$, x is the dependent variable, as it is **dependent** on t.
3. Solve for the dependent variable, if reasonable.

Example Problems On Separable Equations From Houston Math

1. Solve the following separable differential equations:

$$y' = \frac{x}{y} \tag{12}$$

$$\frac{dy}{dx} - x = xy^2 \tag{13}$$

$$x^2y' = -1 \tag{14}$$

$$y'' = xe^x \tag{15}$$

   (a) Find the **general solution**.
   (b) Find the **particular solution** given that $y(1) = 4$.
   (c) Find the **particular solution** given that $y(1) = 3$.

2. Answer the following parts corresponding to the **second-order differential equation**:

$$\frac{d^2y}{dt^2} = \cos(t) \tag{16}$$

   (a) Find the **particular solution** given that $y'(0) = 0$ and $y(0) = 1$.
   (b) Find the **particular solution** given that $y(1) = 3$.

2.2 Exponential Change

The point of this section is to show how exponential change can be written and interpreted as a differential equation, specifically a **separable differential equation**.

$$\frac{dy}{dt} = ky \tag{17}$$

As shown above, the equation represents the rate of change of some quantity $\left(\frac{dy}{dt}\right)$ being equal to some multiple of itself (k), which is exponential change.

We can solve the exponential separable equation as follows:

$$
\begin{aligned}
\frac{dy}{dx} &= xy \\
\frac{1}{y}\,dy &= x\,dx \\
\int \frac{1}{y}\,dy &= \int x\,dx \\
\ln|y| &= \frac{x^2}{2} + C \\
y &= Ce^{x^2/2}
\end{aligned}
$$

- y = final amount
- C = constant determined by the initial condition
- When $k > 0$, we have **growth**
- When $k < 0$, we have **decay**

2.3 Initial Value Problems (continued)

Problems On Initial Value Problems

1. Solve the following **separable equations**:

$$
\begin{aligned}
\frac{dy}{dx} &= 3x^2, \text{ where y(0) = 2} && (18)\\
\frac{dy}{dx} &= xy, \text{ where y(1) = 4} && (19)\\
y' &= \frac{y}{1+x^2}, \text{ where y(0) = 5} && (20)\\
y'' &= xe^x, \text{ where y(0) = 0} && (21)\\
y'' &= xe^x, \text{ where y(2) = 3} && (22)
\end{aligned}
$$

2. Solve the following **first-order linear equations**:

$$
\begin{aligned}
y' + 2y &= e^{-x}, \text{ where } y(0) = 1 && (23)\\
y' - \frac{1}{x}y &= x^2, \text{ where } y(1) = 0 && (24)\\
y' + (\tan x)\,y &= \sec x, \text{ where } y(0) = 2 && (25)\\
y' + \frac{2}{1+x}y &= (1+x)^2, \text{ where } y(0) = 1 && (26)
\end{aligned}
$$

3 LINEAR EQUATIONS

3.1 (2.3) - (General Overview) Linear Equations

A **first-order linear differential equation** is one that can be written in the standard form $y' + P(x)y = q(x)$, where $P(x)$ and $q(x)$ are known functions of x.

These equations are important because they have a consistent solution process: we compute an integrating factor to rewrite the left-hand side as the derivative of a product, integrate both sides, and then apply an initial condition (if given) to find the particular solution.

3.2 Integrating Factor Method

The integrating factor method is a reliable technique for solving **first-order linear** differential equations. Its importance is that it takes an equation where the unknown function (y) and its derivative (y') are mixed together and turns it into a form that can be integrated directly.

We do this by multiplying the entire equation by a carefully chosen function (the **integrating factor**) so that the left-hand side becomes the derivative of a single product.

Once that happens, the problem becomes straightforward: integrate both sides and then use the initial condition to determine the constant and produce the specific solution.

First-Order Linear Equations (Integrating Factor Form). A first-order linear differential equation can be written in the standard form

$$y' + P(x)\,y = q(x).$$

In this form, $P(x)$ and $q(x)$ are known functions of x, and the integrating factor method applies directly.

Examples (identify $\boldsymbol{P(x)}$ and $\boldsymbol{q(x)}$):

$$y' + 3y = 6$$

- $P(x) = 3$
- $q(x) = 6$

$$y' + \frac{1}{x}y = e^x$$

- $P(x) = \frac{1}{x}$
- $q(x) = e^x$

$$xy' + 3y = x$$

Dividing both sides by x (assuming $x \neq 0$) gives the standard form:

$$y' + \frac{3}{x}y = 1$$

- $P(x) = \frac{3}{x}$
- $q(x) = 1$

Problem: Solve the linear differential equation

$$x^2y' + 5xy = x.$$

Step 1: Put in standard linear form. Divide both sides by x^2 (assuming $x \neq 0$):

$$y' + \frac{5}{x}y = \frac{1}{x}.$$

Step 2: Compute the integrating factor.

$$\mu(x) = e^{\int \frac{5}{x}\,dx} = e^{5\ln|x|}.$$

Inverse property (log/exponential). Since $e^{\ln(a)} = a$ for $a > 0$,

$$e^{5\ln|x|} = \left(e^{\ln|x|}\right)^5 = |x|^5.$$

On an interval where $x > 0$, this simplifies to $\mu(x) = x^5$.

Step 3: Multiply the differential equation by $\boldsymbol{\mu(x)}$. (Using $x > 0$ so $\mu(x) = x^5$.)

$$x^5y' + x^5\left(\frac{5}{x}\right)y = x^5\left(\frac{1}{x}\right)$$
$$x^5y' + 5x^4y = x^4.$$

Step 4: Recognize the product rule form.

$$\frac{d}{dx}\left(x^5y\right) = x^5y' + 5x^4y,$$

so the equation becomes

$$\frac{d}{dx}\left(x^5y\right) = x^4.$$

Step 5: Integrate both sides and solve for $\boldsymbol{y}$.

$$\int \frac{d}{dx}\left(x^5y\right)\,dx = \int x^4\,dx$$
$$x^5y = \frac{x^5}{5} + C$$
$$y = \frac{1}{5} + \frac{C}{x^5}.$$

Final solution:

$$\boxed{y = \frac{1}{5} + \frac{C}{x^5}.}$$

4 Exact Equations

An equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is **exact** if there exists a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$. Equivalently, the exactness test is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

4.1 Example 1

Solve: $(x^2 + xy^2)\,dx + (yx^2 - y^3)\,dy = 0$

Step 1: Identify M and N.

$$M(x, y) = x^2 + xy^2, \qquad N(x, y) = yx^2 - y^3.$$

Step 2: Test for exactness.

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + xy^2\right) = 2xy,\\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(yx^2 - y^3\right) = 2xy.\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is **exact**.

Step 3: Find a potential function $\psi(x, y)$. Integrate M with respect to x:

$$\begin{aligned}\psi(x, y) &= \int \left(x^2 + xy^2\right)\,dx\\ &= \frac{x^3}{3} + \frac{x^2}{2}y^2 + h(y) \qquad (h(y) \text{ is constant with respect to } x).\end{aligned}$$

Step 4: Differentiate ψ with respect to y and match N.

$$\begin{aligned}\psi_y(x, y) &= \frac{\partial}{\partial y}\left(\frac{x^3}{3} + \frac{x^2}{2}y^2 + h(y)\right)\\ &= x^2y + h'(y).\end{aligned}$$

Set $\psi_y = N$:

$$x^2y + h'(y) = yx^2 - y^3 \quad\Longrightarrow\quad h'(y) = -y^3 \quad\Longrightarrow\quad h(y) = -\frac{y^4}{4}.$$

Final implicit solution:

$$\boxed{\frac{x^3}{3} + \frac{x^2y^2}{2} - \frac{y^4}{4} = C}$$

4.2 Example 2

Solve: $\dfrac{dy}{dx} = \dfrac{3ye^x - 4xe^y}{2x^2e^y - 3e^x}$

Step 1: Rewrite in the form $M\,dx + N\,dy = 0$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3ye^x - 4xe^y}{2x^2e^y - 3e^x} \\
\left(2x^2e^y - 3e^x\right)\,dy &= \left(3ye^x - 4xe^y\right)\,dx \\
\left(4xe^y - 3ye^x\right)\,dx + \left(2x^2e^y - 3e^x\right)\,dy &= 0.
\end{aligned}$$

Step 2: Identify M and N.

$$M(x,y) = 4xe^y - 3ye^x, \qquad N(x,y) = 2x^2e^y - 3e^x.$$

Step 3: Test for exactness.

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4xe^y - 3ye^x\right) = 4xe^y - 3e^x, \\
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x^2e^y - 3e^x\right) = 4xe^y - 3e^x.
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is **exact**.

Step 4: Find a potential function $\psi(x,y)$. Integrate N with respect to y:

$$\begin{aligned}
\psi(x,y) &= \int \left(2x^2e^y - 3e^x\right)\,dy \\
&= 2x^2e^y - 3ye^x + g(x) \qquad (g(x) \text{ is constant with respect to } y).
\end{aligned}$$

Step 5: Differentiate ψ with respect to x and match M.

$$\begin{aligned}
\psi_x(x,y) &= \frac{\partial}{\partial x}\left(2x^2e^y - 3ye^x + g(x)\right) \\
&= 4xe^y - 3ye^x + g'(x).
\end{aligned}$$

Set $\psi_x = M$:

$$4xe^y - 3ye^x + g'(x) = 4xe^y - 3ye^x \quad \Longrightarrow \quad g'(x) = 0,$$

so $g(x)$ is constant and can be absorbed into C.

Final implicit solution:

$$\boxed{2x^2e^y - 3ye^x = C}$$

4.3 General Outline for Solving an Exact Equation

General Outline For Solving An Exact Equation:

1. Rewrite the equation in the form $M(x,y)\,dx + N(x,y)\,dy = 0$.
2. Compute $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$.
3. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the equation is **exact**.
4. Find a potential function $\psi(x,y)$:
   (a) Integrate $M(x,y)$ with respect to x to get $\psi(x,y) = \int M\,dx + h(y)$, **or**
   (b) Integrate $N(x,y)$ with respect to y to get $\psi(x,y) = \int N\,dy + g(x)$.
5. Differentiate ψ with respect to the other variable and match it to the remaining function ($\psi_y = N$ or $\psi_x = M$) to determine $h(y)$ or $g(x)$.
6. Write the implicit solution as $\psi(x,y) = C$.

---

5 SOLUTIONS BY SUBSTITUTION

5.1 (2.5) - (General Overview) Solutions By Substitutions

Solutions by substitution are used when a differential equation does not appear separable or linear at first, but can be transformed into a familiar form by introducing a new variable.

The main idea is to recognize a pattern in the equation and choose a substitution that simplifies the algebra and reduces the problem to a standard method you already know.

In this section, we will focus on **common substitution-based techniques**, including homogeneous equations (often handled by substituting $v = y/x$), **Bernoulli equations** (which become linear after an appropriate substitution), and numerical approaches such as **Euler's method**, which approximates solutions when an exact formula is difficult or impossible to obtain.

5.2 Homogeneous Equations

If the equation is in the form

$$\frac{dy}{dx} = f(x,y),$$

and it is true that

$$f(tx,ty) = f(x,y),$$

then the equation is homogeneous and we can deploy the substitution

$$y = vx.$$

If we differentiate $y = vx$, we obtain an expression that eliminates y:

$$dy = v\,dx + x\,dv.$$

5.3 Bernoulli Equations

A Bernoulli equation is a first-order differential equation that looks almost linear, except for a power of the unknown function.

The key idea is that even though the presence of y^n (with $n \neq 0, 1$) prevents it from being linear in y, it can be converted into a linear equation by an appropriate substitution.

After rewriting the equation in the standard Bernoulli form $y' + P(x)y = Q(x)y^n$, we substitute $v = y^{1-n}$. This choice is important because it transforms the nonlinear y^n term into something involving v and v', producing a first-order linear equation in v.

Once the equation is linear, we solve it using the integrating factor method and then substitute back to recover y.

Bernoulli Equation Example. Solve

$$\frac{dy}{dx} - \frac{1}{x}y = xy^2.$$

Step 1: Identify Bernoulli form and choose substitution.

$$y' + P(x)y = Q(x)y^n \quad \Rightarrow \quad P(x) = -\frac{1}{x},\ Q(x) = x,\ n = 2.$$

Let

$$v = y^{1-n} = y^{-1} = \frac{1}{y}.$$

Then

$$\frac{dv}{dx} = -\frac{y'}{y^2}.$$

Step 2: Divide the DE by y^2 and rewrite in terms of v.

$$\begin{aligned}
\frac{dy}{dx} - \frac{1}{x}y &= xy^2 \\
\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x}\frac{y}{y^2} &= x \\
\frac{y'}{y^2} - \frac{1}{x}\frac{1}{y} &= x \\
-\frac{dv}{dx} - \frac{1}{x}v &= x \qquad \left(\text{since } \frac{y'}{y^2} = -v' \text{ and } \frac{1}{y} = v\right) \\
\frac{dv}{dx} + \frac{1}{x}v &= -x \qquad (\text{multiply both sides by } -1)
\end{aligned}$$

Step 3: Integrating factor.

$$\mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln|x|} = |x|.$$

On an interval where $x > 0$, take $\mu(x) = x$.

Step 4: Multiply by $\mu(x) = x$ and use the product rule.

$$\begin{aligned} x\left(v' + \frac{1}{x}v\right) &= x(-x) \\ xv' + v &= -x^2 \\ \frac{d}{dx}(xv) &= -x^2 \qquad \text{(because } (xv)' = xv' + v) \end{aligned}$$

Step 5: Integrate and solve for v.

$$\begin{aligned} \int \frac{d}{dx}(xv)\,dx &= \int -x^2\,dx \\ xv &= -\frac{x^3}{3} + C \\ v &= -\frac{x^2}{3} + \frac{C}{x} \end{aligned}$$

Step 6: Substitute back $v = \frac{1}{y}$ and solve for y.

$$\begin{aligned} \frac{1}{y} &= -\frac{x^2}{3} + \frac{C}{x} \\ y &= \frac{1}{\frac{C}{x} - \frac{x^2}{3}} = \frac{1}{\frac{3C - x^3}{3x}} = \frac{3x}{3C - x^3}. \end{aligned}$$

Final solution:

$$\boxed{y = \frac{3x}{3C - x^3}.}$$

5.4 Homogeneous First-Order Differential Equation

Problem. Solve the differential equation

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}.$$

Step 1: Test for homogeneity. Let

$$f(x, y) = \frac{x}{y} + \frac{y}{x}.$$

Check whether $f(tx, ty) = f(x, y)$:

$$\begin{aligned} f(tx, ty) &= \frac{tx}{ty} + \frac{ty}{tx} \\ &= \frac{x}{y} + \frac{y}{x} \qquad \text{(the } t\text{'s cancel)} \\ &= f(x, y). \end{aligned}$$

Since $f(tx, ty) = f(x, y)$, the equation is **homogeneous** (degree 0), so we use the substitution $y = vx$.

Step 2: Substitute $y = vx$ and rewrite dy/dx.

$$\begin{aligned} y &= vx \qquad (\text{set } v = \frac{y}{x}) \\ \frac{dy}{dx} &= v + x\frac{dv}{dx} \qquad (\text{differentiate } y = vx \text{ using the product rule}) \end{aligned}$$

Step 3: Plug into the differential equation and separate variables.

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{x}{vx} + \frac{vx}{x} \qquad (\text{substitute } y = vx \text{ into the RHS}) \\ v + x\frac{dv}{dx} &= \frac{1}{v} + v \qquad (\text{simplify}) \\ x\frac{dv}{dx} &= \frac{1}{v} \qquad (\text{subtract } v \text{ from both sides}) \\ v\,dv &= \frac{1}{x}\,dx \qquad (\text{separate variables}) \end{aligned}$$

Step 4: Integrate.

$$\begin{aligned} \int v\,dv &= \int \frac{1}{x}\,dx \\ \frac{1}{2}v^2 &= \ln|x| + C \\ v^2 &= 2\ln|x| + C \qquad (\text{multiply both sides by 2}) \end{aligned}$$

Step 5: Resubstitute $v = \frac{y}{x}$ and solve for y. We solve for v in terms of x because $v = \frac{y}{x}$ is the substitution that connects back to the original variables.

$$\begin{aligned} \left(\frac{y}{x}\right)^2 &= 2\ln|x| + C \qquad (\text{since } v = \frac{y}{x}) \\ y^2 &= x^2\big(2\ln|x| + C\big) \qquad (\text{multiply both sides by } x^2) \\ y &= \pm x\sqrt{2\ln|x| + C} \qquad (\text{take square roots}) \end{aligned}$$

Final solution:

$$\boxed{y = \pm x\sqrt{2\ln|x| + C}\,.}$$

6 LINEAR MODELS

6.1 (3.1) - (General Overview) Modeling with 1st-Order Differential Equations

Many real-world models begin by identifying a changing quantity and writing

$$\frac{d(\text{amount})}{dt} = (\text{rate in}) - (\text{rate out}).$$

After the model is written, we solve the resulting differential equation (often separable or linear) and use initial conditions to determine constants.

6.2 Exponential Growth and Decay

Proportional Change (Growth/Decay). If the rate of change of a quantity is proportional to the quantity itself, then

$$\frac{dP}{dt} = kP,$$

where $k > 0$ represents growth and $k < 0$ represents decay.

Example: Population Growth (Exponential)

Suppose a population $P(t)$ grows at a rate proportional to its size:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0.$$

$$\begin{aligned} \frac{1}{P}\,dP &= k\,dt \\ \int \frac{1}{P}\,dP &= \int k\,dt \\ \ln|P| &= kt + C \\ P &= Ce^{kt}. \end{aligned}$$

Using $P(0) = P_0$ gives $P_0 = C$, so

$$\boxed{P(t) = P_0 e^{kt}}.$$

Radioactive Decay

Radioactive decay is modeled the same way, but with $k < 0$:

$$\frac{dN}{dt} = kN, \qquad k < 0.$$

Thus the amount remaining is

$$\boxed{N(t) = N_0 e^{kt}}.$$

General Outline For Exponential Growth and Decay:

1. Identify the changing quantity and name it $Y(t)$.
2. If the rate is proportional to the amount present, write

   $$\frac{dY}{dt} = kY,$$

   where $k > 0$ implies growth and $k < 0$ implies decay.
3. Separate variables and integrate:

   $$\frac{1}{Y}\,dY = k\,dt \quad \Rightarrow \quad \ln|Y| = kt + C.$$

4. Solve for $Y(t)$:

$$Y(t) = Ce^{kt}.$$

5. Use the initial condition $Y(0) = Y_0$ to get $C = Y_0$ and write

$$\boxed{Y(t) = Y_0 e^{kt}}.$$

6.3 Logistic Growth

With logistic growth, an example is population growth. A key feature of logistic growth is that there is a "limit" or cap on how far we can grow.

$$\tfrac{dP}{dt} = r\left(\tfrac{k-P}{k}\right)$$

- $\frac{dP}{dt}$ is the change in population
- r is the rate
- P is the current population
- $\frac{k-P}{k}$ is the carrying capacity (k), or the "limit" that we cannot grow beyond

How We Write Population Growth Depends On:

1. A rate
2. The size of the current population
3. The growth slowing as the population approaches a "carrying capacity"

After working this as a first-order separable equation, we get:

$$P = \tfrac{k}{1+ce^{-rt}}$$

Example: Logistic Growth With an Initial Condition

If

$$P(t) = \frac{k}{1+ce^{-rt}} \quad \text{and} \quad P(0) = P_0,$$

then

$$\begin{aligned} P_0 &= \frac{k}{1+c} \\ 1+c &= \frac{k}{P_0} \\ c &= \frac{k-P_0}{P_0}. \end{aligned}$$

So

$$\boxed{P(t) = \frac{k}{1 + \left(\frac{k-P_0}{P_0}\right) e^{-rt}}}.$$

General Outline For Logistic Growth:

1. Let $P(t)$ be the population and identify the carrying capacity k.
2. Write the logistic model:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right) \quad \text{(equivalently } \frac{dP}{dt} = rP\left(\frac{k-P}{k}\right)\text{)}.$$

3. Separate variables and integrate (partial fractions are typically required).
4. Solve for $P(t)$ and write the standard solution form:

$$P(t) = \frac{k}{1 + ce^{-rt}}.$$

5. Use $P(0) = P_0$ to determine $c = \dfrac{k - P_0}{P_0}$.

6.4 Newton's Law Of Cooling

Newton's Law of Cooling states that the rate of change in the temperature of an object is some multiple of the difference between its temperature and the temperature of the surrounding medium.

$$\tfrac{dT}{dt} = k(T - T_m), \text{ where}$$

1. $\boldsymbol{\frac{dT}{dt}}$ is the rate of change of temperature, T, over time
2. $\boldsymbol{k}$ is the constant multiple
3. $\boldsymbol{(T - T_m)}$ is the difference between:
   (a) the object (T)
   (b) the surrounding medium (T_m)

After working this as a first-order separable equation, we get:

$$T = ce^{kt} + T_m$$

Example: Cooling With Initial Temperature

If $T(0) = T_0$, then

$$\begin{aligned} T(0) &= ce^{k\cdot 0} + T_m \\ T_0 &= c + T_m \\ c &= T_0 - T_m. \end{aligned}$$

So

$$\boxed{T(t) = T_m + (T_0 - T_m)e^{kt}}.$$

General Outline For Newton's Law of Cooling (or Heating):

1. Let $T(t)$ be the temperature of the object and let T_m be the (constant) surrounding temperature.
2. Write the model:
$$\frac{dT}{dt} = k\left(T - T_m\right),$$
where typically $k < 0$ for cooling (object temperature approaches T_m over time).
3. Separate variables:
$$\frac{1}{T - T_m}\, dT = k\, dt.$$
4. Integrate:
$$\ln|T - T_m| = kt + C.$$
5. Solve for $T(t)$:
$$T - T_m = Ce^{kt} \quad \Rightarrow \quad T(t) = T_m + Ce^{kt}.$$
6. Use the initial condition $T(0) = T_0$ to get $C = T_0 - T_m$, so
$$\boxed{T(t) = T_m + (T_0 - T_m)e^{kt}}.$$

6.5 Mixing Problems

General Setup (Salt/Sugar Mixing). Let $A(t)$ be the amount of solute (e.g., salt) in the tank at time t. A standard mixing model is

$$\frac{dA}{dt} = (\text{rate in}) - (\text{rate out}).$$

Typically,

$$\text{rate in} = (\text{inflow concentration})\cdot(\text{inflow rate}), \qquad \text{rate out} = (\text{tank concentration})\cdot(\text{outflow rate}),$$

where tank concentration $= \dfrac{A(t)}{V(t)}$ and $V(t)$ is the volume.

Example: Constant Volume Mixing (Linear DE)

A tank has constant volume V gallons. Brine enters at rate r (gal/min) with concentration c_{in} (lb/gal) and leaves at the same rate r. Then

$$\frac{dA}{dt} = rc_{\text{in}} - r\frac{A}{V}.$$

This is linear:

$$\frac{dA}{dt} + \frac{r}{V}A = rc_{\text{in}}.$$

Solving gives

$$\boxed{A(t) = Vc_{\text{in}} + (A_0 - Vc_{\text{in}})\,e^{-(r/V)t}},$$

where $A(0) = A_0$.

General Outline For Solving a Mixing Problem:

1. Let $A(t)$ be the amount of solute in the tank at time t.
2. Write the model using
$$\frac{dA}{dt} = (\text{rate in}) - (\text{rate out}).$$
3. Compute rate in:
$$\text{rate in} = (\text{inflow concentration}) \cdot (\text{inflow rate}).$$
4. Compute rate out:
$$\text{rate out} = \left(\frac{A(t)}{V(t)}\right) \cdot (\text{outflow rate}),$$
where $V(t)$ is the volume in the tank.
5. Determine $V(t)$ (constant volume if inflow = outflow; otherwise $V(t)$ changes with time).
6. Solve the resulting differential equation (often linear in $A(t)$).
7. Apply the initial condition $A(0) = A_0$.

6.6 Motion in One Direction

General Idea. In one dimension, common modeling relationships are:

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt}.$$

If a force (or acceleration) depends on velocity, position, or time, we can create a differential equation.

Example 1: Constant Acceleration

If $a(t) = a$ (constant), then

$$\frac{dv}{dt} = a.$$

Integrate:

$$v(t) = at + C,$$
$$v(0) = v_0 \Rightarrow C = v_0,$$

so

$$\boxed{v(t) = v_0 + at}.$$

Since $v = \dfrac{dx}{dt}$,

$$\frac{dx}{dt} = v_0 + at,$$

and integrating again yields

$$\boxed{x(t) = x_0 + v_0 t + \frac{1}{2}at^2}.$$

Example 2: Linear Drag (Velocity-Dependent Resistance)

A simple drag model assumes acceleration proportional to velocity:

$$\frac{dv}{dt} = -kv, \qquad k > 0.$$

This is separable:

$$\begin{aligned} \frac{1}{v}\,dv &= -k\,dt \\ \ln|v| &= -kt + C \\ v &= Ce^{-kt}. \end{aligned}$$

If $v(0) = v_0$, then $C = v_0$, so

$$\boxed{v(t) = v_0 e^{-kt}}.$$

General Outline For Modeling Motion in One Dimension:

1. Choose variables:
   (a) $x(t) =$ position, $v(t) = \dfrac{dx}{dt} =$ velocity, $a(t) = \dfrac{dv}{dt} =$ acceleration.
2. Use the given information to write a differential equation for either $v(t)$ or $x(t)$.
3. Solve for $v(t)$ first if the model is written in terms of acceleration $a = \dfrac{dv}{dt}$.
4. If position is needed, use $v = \dfrac{dx}{dt}$ and integrate to obtain $x(t)$.
5. Apply initial conditions (such as $v(0) = v_0$ or $x(0) = x_0$) to determine constants.

6.7 Proportional Change Models

Many models in this course reduce to the proportional change form

$$\frac{dY}{dt} = kY \quad \Rightarrow \quad \boxed{Y(t) = Y_0 e^{kt}}.$$

Common applications include population growth (when $k > 0$), radioactive decay (when $k < 0$), interest models, and some simplified motion/resistance models.