Due date: 11:59 pm on Wednesday, February 15, 2023.

Logistical questions:

1. Did you collaborate with anyone? If so, list their names here. **No.**
2. How long did this assignment take you? **12 hours**
3. Do you have any comments on the assignment? **Very good, help me a lot**

Problem 1: Minimax and Maximin [20 pts]

Player 2

Player 1		L	R
	U	4, -4	-3, 3
	M	2, -2	-2, 2
	D	-3, 3	1, -1

Table 1: Payoff matrix.

(a) (10 pts) Given the game in Table 1, write the linear program (LP) corresponding to the minimax solution for player 1. Explain in your own words what each of your constraints represents. (Hint: Your objective should be to minimize player 2's value, v_2.)

A minmax strategy $\underline{x}$ for Player 1 in a two-player game is a strategy that satisfies

$$\underline{x} \in \arg\min_{x \in \Delta(A_1)} \left[\max_{a_2 \in A_2} u_2(x, a_2)\right] = \arg\min_{x \in \Delta(A_1)} \left[\max[u_2(x, L), u_2(x, R)]\right]$$

where $\Delta(A_1)$ is set of all probability distribution over actions that player 1 can take,

By Table 1 we know: $u_2(x, L) = -4x - 2y + 3(1-x-y)$, $u_2(x, R) = 3x + 2y - (1-x-y)$

where we assume x is the probability player 1 play U, y the probability Player 1 play M.

$$\Rightarrow \underline{x} \in \arg\min_{x \in \Delta(A_1)} \left[\max[-4x - 2y + 3(1-x-y), 3x + 2y - (1-x-y)]\right]$$

So, for Player 1 we can formulate a LP as follows:

$$\min_{v_2, x} \quad v_2$$

$$\text{s.t.} \sum_{j \in A_1} u_2(j, k) \cdot x_j \le v_2 \quad \cdots\cdots ①$$

$$\sum_{j \in A_1} x_j = 1 \quad \cdots\cdots ②$$

$$x_j \ge 0, \quad \forall j \in A_1 \quad \cdots\cdots ③$$

Constraint ① means with knowledge of Player 1's strategy, selects an action $a_2 \in A_2$ to maximize their own expected utility v_2;

Constraint ② and ③ are valid probability constraints.

Solution. □

(b) (10 pts) Prove that in any two-player, zero-sum game, for each player, the set of maximin strategies is equal to the set of minimax strategies. (You may recall that this is the first part of the Minimax Theorem.)

as 1.1 discussed, the LP for Player 1 minmax strategy as follows

$$\min_{v_2, x} \quad v_2$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{A}_1} u_2(j,k) \cdot x_j \leq v_2 , \quad \forall k \in \mathcal{A}_2$$

$$\sum_{j \in \mathcal{A}_1} x_j = 1$$

$$x_j \geq 0 , \quad \forall j \in \mathcal{A}_1$$

since it is a two-player, zero-sum game $\Rightarrow$ $v_1 = -v_2$, $u_1(j,k) = -u_2(j,k)$, $\forall j \in \mathcal{A}_1, \forall k \in \mathcal{A}_2$

Therefore we can rewrite the above LP as

$$\max_{v_1, x} \quad v_1$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{A}_1} u_1(j,k) \cdot x_j \geq v_1 , \quad \forall k \in \mathcal{A}_2$$

$$\sum_{j \in \mathcal{A}_1} x_j = 1$$

$$x_j \geq 0 , \quad \forall j \in \mathcal{A}_1$$

which is exactly the LP for Player 1 maxmin strategy.

Similarly, the proof can be done analogously for Player 2.

Solution. □

Problem 2: Potential Games [20 pts]

		Player 2: W	Player 2: G
Player 1	W	1, 0	0, 2
	G	3, 0	-4, -4

Table 2: Modified Chicken payoff matrix.

(a) (10 pts) Consider a slightly asymmetric version of Chicken depicted in Table 2. Show that this version of Chicken is a potential game by constructing a valid potential function $R : A \to \mathbb{R}$. Demonstrate that your potential function is correct.

$$P = \begin{pmatrix} 2 & 4 \\ 4 & 0 \end{pmatrix}$$

By definition, valid $P : A \to \mathbb{R}$ s.t. $u_i(j, y) - u_i(j', y) = P(j, y) - P(j', y)$ for every action $j \in A_i$, $j' \in A_i$, every action profile $y \in A_{-i}$ and every agent i

$\Rightarrow$ $P(G, W) - P(W, W) = u_1(G, W) - u_1(W, W) = 2$

$P(W, G) - P(G, G) = u_1(W, G) - u_1(G, G) = 4$

$P(W, G) - P(W, W) = u_2(W, G) - u_2(W, W) = 2$

$P(G, W) - P(G, G) = u_2(G, W) - u_2(G, G) = 4$

Therefore, $P = \begin{pmatrix} 2 & 4 \\ 4 & 0 \end{pmatrix}$ is a valid potential function

Solution. □

(b) (10 pts) What is the time complexity for checking that all 2-by-2 cycles have zero value in an n-player game with m actions per player? Explain your answer.

since there are n players $\Rightarrow$ total $[(n-1)+(n-2)+\cdots+1] = \frac{n(n-1)}{2}$ combinations of 2 agents;

Then we focus on one combination, the others are the same

since each player has m actions, by definition of 2-by-2 cycle, total $m^2(m-1)^4$ cycles needs to check for any two agents

So total time complexity is $\frac{n(n-1)}{2} \cdot m^2 \cdot (m-1)^4$

Solution. □

Problem 3: Bargaining Game 2.0 [20 pts]

Consider a variant of the Bargaining game (Figure 1) in which *me*, *even*, and *you* by player 1 correspond to offers of (2, 0), (1, 1), and (0, 2). As in the original example, if player 2 rejects an offer, then the players get (0, 0).

(a) (10 pts) Write out the normal-form representation of the game and determine the set of pure-strategy Nash equilibria. (Hint: There should be nine.)

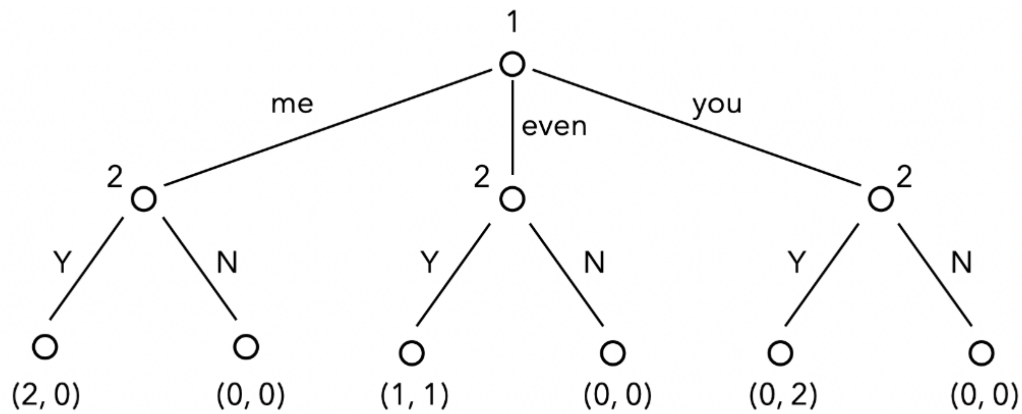

Figure 1: Variant of the Bargaining game.

Normal-Form — Player 2

Player 1	<Y,Y,Y>	<Y,Y,N>	<Y,N,Y>	<Y,N,N>	<N,Y,Y>	<N,Y,N>	<N,N,Y>	<N,N,N>
me	$\boxed{(2,0)}$	$\boxed{(2,0)}$	$\boxed{(2,0)}$	$\boxed{(2,0)}$	(0,0)	(0,0)	$\boxed{(0,0)}$	$\boxed{(0,0)}$
even	(1,1)	(1,1)	(0,0)	(0,0)	$\boxed{(1,1)}$	$\boxed{(1,1)}$	(0,0)	(0,0)
you	(0,2)	(0,0)	(0,2)	(0,0)	(0,2)	(0,0)	$\boxed{(0,2)}$	(0,0)

Solution. All PSNE is drawn with red cycle as above figure shows, they are (me, $< Y, Y, Y >$), (me, $< Y, Y, N >$), (me, $< Y, N, Y >$), (me, $< Y, N, N >$), (me, $< N, N, Y >$), (me, $< N, N, N >$), (even, $< N, Y, Y >$), (even, $< N, Y, N >$), (you, $< N, N, Y >$), total 9 PSNES ☐

(b) (10 pts) Determine the set of pure-strategy subgame-perfect equilibria (SPE). Show your work. (Hint: There is more than one SPE.)

Solution. By single deviation principle, we can use the backward induction. For Plyer 2, $< Y, Y, Y >$, $< N, Y, Y >$ both satisfying the requirement, then we check for Player 1, we can get (me, $< Y, Y, Y >$) and (even, $< N, Y, Y >$) are SPEs we want. ☐

Problem 4: Tit-for-Tat (TfT) Strategies [20 pts]

All references to the (repeated) Prisoners' Dilemma in this question and the next assume the stage game is depicted in Table 3.

(a) (10 pts) Prove that Tit-for-Tat (TfT) is not a subgame-perfect equilibrium of the infinitely-repeated Prisoners' Dilemma when adopted by both players for any discount factor $\delta \in (0, 1)$. (Hint: Look at two classes of subgames and see what values of discount factor make TfT a SPE in each subgame.)

		Player 2	
		C	D
Player 1	C	3, 3	0, 5
	D	5, 0	1, 1

Table 3: Prisoners' Dilemma stage game payoff matrix.

proof:

① if on the equilibrium path, supposing that playing Tit-for-Tat is in fact in equilibrium, on that path every one would cooperate forever $\Rightarrow$ Payoff $= 3 + 3\delta + 3\delta^2 + \cdots$

On the other hand, if one tries to deviate in any given point by defecting, you would receive a slight bump in payoff - 5, for defecting while opponent cooperates, but then in the next period, ~~tit-for-Tat~~ Tit-for-Tat would issure this roll reversal where you would cooperate while opponent would defect so you only get payoff - 0 in that stage $\Rightarrow$ Payoff $= 5 + 5\delta^2 + 5\delta^4 \cdots$

Cooperate if $\Rightarrow 3 + 3\delta + 3\delta^2 + \cdots > 5 + 5\delta^2 + 5\delta^4 + \cdots$

$$\Rightarrow \frac{3}{1-\delta} > \frac{5}{1-\delta^2} \Rightarrow \delta \in \left(\frac{2}{3}, 1\right)$$

② keep Tit-for-Tat strategy if

$$5 + 5\delta^2 + 5\delta^4 + \cdots > 3 + \delta + 3\delta^2 + \cdots$$

$$\Rightarrow \delta \in \left(0, \frac{2}{3}\right)$$

Contradiction! We can't find such a discount factor that make TfT a SPE in each subgame.

Solution. □

(b) (10 pts) Prove that TFT is a Nash equilibrium of the infinitely-repeated Prisoners' Dilemma for any discount factor $\delta \in (2/3, 1)$. (Remark: Note that TfT is not an SPE of the infinitely-repeated Prisoners' Dilemma, and you will not be able to show this using the single-deviation principle.)

Solution. As discussed in (a), if $\delta \in (2/3, 1)$, everyone would cooperate forever, Tit-for-Tat is exactly in equilibrium. Therefore, TFT is a Nash equilibrium. □

Problem 5: Subgame-Perfect Equilibria [20 pts]

(a) (10 pts) Prove the following statement. If the stage game G has a unique (possibly mixed) Nash equilibrium, then the only SPE s^* of the finitely-repeated game G^T plays the Nash equilibrium of the stage game after every history. (Hint: You must prove (1) that s^* is an SPE of G^T and (2) that s^* is the only SPE of G^T.)

Solution. First prove that s^* is an SPE of finitely-repeated game G^T. By the definition of SPE, we can use backward induction working from the bottom of the game, since every stage has a unique Nash equilibrium, so s^* is a Nash equilibrium for every history h for sure.
Second prove that s^* is the only SPE of G^T. We can generalize the statement to: *If a finite multistage game consists of stage games that each have a unique Nash equilibrium, then the multistage game has a unique SPE.* We can also use the backward induction. In the last stage the players must play the unique Nash equilibrium of that game. In the stage before last they cannot condition future behavior on current stage outcomes, so here again they must play the unique Nash equilibrium of the stage game continuous by the Nash equilibrium of tha last stage game, and this induction argument continues till the first stage of the game.
Then we can conclude that the given statement is true. $\square$

(b) (10 pts) In the infinitely-repeated Prisoners' Dilemma, confirm that the grim trigger (i.e., the strategy that plays C until either player deviates and then plays D forever) is an SPE when played by both players for all discount factors $\delta \in [1/2, 1)$.

Solution. Similar to Problem 4, we can write:
$3 + 3\delta + 3\delta^2 + ... \geq 5 + \delta + \delta^2 + ...$
$\frac{3}{1-\delta} \geq \frac{1}{1-\delta} + 4$
$4(1-\delta) \leq 2$
$\delta \geq \frac{1}{2}$
So we can conclude that $\delta \in [\frac{1}{2}, 1)$. $\square$