Due date: 11:59 pm on Wednesday, March 15, 2023.

Logistical questions:

1. Did you collaborate with anyone? If so, list their names here. No.
2. How long did this assignment take you? About 12 hours(including review time)
3. Do you have any comments on the assignment? Good, help me have a deeper understanding of auction concept.

Problem 1: Equilibria in Auctions [30 pts]

(a) (10 pts) An all-pay auction has the following allocation and payment rules. Given a bid profile b, the auction allocates the item to the bidder with the highest bid, i.e., $x_i(b) = 1$ if $b_i > b_j$ for all bidders $j \neq i$, and $x_i(b) = 0$ otherwise. However, the auction collects payment from all bidders equal to their bid regardless of whether or not they received the item, i.e., $t_i(b) = b_i$.

Consider the all-pay auction with three bidders, each of whom have values uniformly distributed on $[0, 1]$. Verify that bid strategy $s_i^*(v_i) = \frac{2}{3}v_i^3$ is a BNE. (Hint: Take the perspective of bidder 1 and calculate their expected utility for a given bid.)

Solution. Take the perspective of bidder 1, the expected utility $\pi(b_1) = P(win)(v_1 - b_1) + P(lose)(-b_1)$. And we can easily obtain that $P(win) = P(\frac{2}{3}(v_2)^3 \leq b_1)P(\frac{2}{3}(v_3)^3 \leq b_1) = P(v_2 \leq (\frac{3}{2}b_1)^{\frac{1}{3}})P(v_3 \leq (\frac{3}{2}b_1)^{\frac{1}{3}}) = (\frac{3}{2}b_1)^{\frac{2}{3}}$, $P(lose) = 1 - P(win) = 1 - (\frac{3}{2}b_1)^{\frac{2}{3}}$, then we can simplify and get $\pi(b_1) = (\frac{3}{2}b_1)^{\frac{2}{3}}v_1 - b_1$. We want the value b_1 that maximize the expected utility, solving $\frac{\partial}{\partial b_1}((\frac{3}{2}b_1)^{\frac{2}{3}}v_1 - b_1) = 0$, that is, $\frac{2}{3}(\frac{3}{2})^{\frac{2}{3}}b_1^{-\frac{1}{3}} = 1$, then we can get $b_1 = \frac{8}{27} \times \frac{9}{4}v_1^3 = \frac{2}{3}v_1^3$. So we can conclude that bid strategy $s_i^*(v_i) = \frac{2}{3}v_i^3$ is a BNE. □

(b) (10 pts) What is the interim payment in a FPSB auction with n bidders with IID values uniform on $[0, 1]$ if all bidders bid their true values? Note that in this case they are *not* following a BNE.

Solution. By definition, for leading bidder i with value v_i, the expected interim payment is $t_i^*(v_i) = v_i \times v_i^{n-1} = v_i^n$ □

(c) (10 pts) What is the interim payment in the truthful DSE of the SPSB auction in an auction with n bidders with IID values uniform on $[0, 1]$? Justify your answer.

Solution. By definition, for leading bidder i with value v_i, the expected interim payment is $t_i^*(v_i) = E(z_2|n\ (0, v_i)) \times v_i^{n-1} = \frac{n-2+1}{n+1}v_i \times v_i^{n-1} = \frac{n-1}{n+1}v_i^{n-1}$ □

Problem 2: Collusion and Shill Bids [20 pts]

(a) (10 pts) Consider an SPSB auction. Suppose bidders 1, 2, and 3 with values \$10, \$8, and \$4 collude (i.e., work together), while bidder 4's bid is distributed uniformly in the range $[\$5, \$7]$. How can the colluding bidders bid in order to collude, how might their proceeds be divided to keep all members of the coalition happy, and why can bidders 2 and 3 not usefully deviate from this play? You do not need to provide a formal equilibrium analysis, but you must clearly justify your answer.

Solution. Bidder 1 bid his/her true value, \$10; Bidder 2 and 3 bid 0. Bidder 2 and 3 cannot usefully deviate from this play because if they bid no more than their truthful value, bidder 1 always "win", their expected utility is 0; if they bid more than their truthful value, their utility will be negative, which is worse than 0. Therefore for bidder 2 and 3, the best utility is 0, so they can choose not to bid, only bidder 1 need to bid. □

(b) (10 pts) In shill bidding, the auctioneer can place its own bid with knowledge of the real bids, known as a shill bid. Why is shill bidding a problem in the SPSB auction? Given n bidders with IID values uniform on $[0, 1]$, what is the BNE of a SPSB auction where the auctioneer was known for always placing shill bids?

Solution. Lack of credibility, the auctioneer adds shill bids to drive up payment, the leading bidder need to pay more, which decrease the expected utility for leading bidder.
Truthful bidding is still the BNE of a SPSB auction where the auctioneer was konw for always placing shill bids. □

Problem 3: Revenue Comparison [30 pts]

(a) (10 pts) Suppose there is one bidder in an auction with value $v_1 \sim U(0, 1)$ (i.e., uniform on $[0, 1]$). Now, consider running a take-it-or-leave-it auction, where the auctioneer proposes one price and the buyer has to decide whether to take it (i.e., purchase the item at that price) or leave it (i.e., not purchase the item). Derive the optimal take-it-or-leave-it price to offer to this bidder in order to maximize expected revenue.

Solution. Assume the price is x, the expected revenue is then $x(1-x) = -x^2 + x$. We can easily obtain that when x = $\frac{1}{2}$, the expected revenue reach the maximum. □

(b) (10 pts) Now, suppose there are two bidders with values uniformly distributed on $[0, 1]$. Consider an SPSB auction with a reserve price of $r = 1/2$, where a reserve price works by requiring the winning bid to be at least r, and the payment of the winning bidder is the maximum of r and the second-highest bid. If no bidder bids above the reserve price, the item is not allocated and neither bidder pays anything. What is the expected revenue for this auction?

Solution. There are three cases:

(a) both players bid below $\frac{1}{2}$, then the revenue is 0 since the item is not allocated. The probability for this event is $\frac{1}{4}$

(b) one player bids above $\frac{1}{2}$, the other below $\frac{1}{2}$, then the revenue is $\frac{1}{2}$. The probability for this event is: $\frac{1}{2}$

(c) both players bid above $\frac{1}{2}$, then the expected revenure is $\frac{2}{3}$. The probability for this event is $\frac{1}{4}$

Therefore, the expected revenue for this auction is $\frac{1}{4} \times 0 + \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{2}{3} = 0 + \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ □

(c) (10 pts) Compare the expected revenue in (b) with the expected revenue in the standard SPSB auction, i.e., without using a reserve price. What do you find? Explain why this observation is consistent with the revenue equivalence theorem.

Solution. In the standard SPSB auction, the expected revenue is $\frac{2-2+1}{2+1} = \frac{1}{3} according to the order statistics.$, which is smaller than the expected revenue in (b). They are not equal because their interim allocation are not identical. □

Problem 4: Multi-Round Auctions [20 pts]

(a) (10 pts) Consider an eBay auction with a starting price of \$8 and a secret reserve of \$15. Suppose the minimal bid increment is \$0.50 when the price is between \$8 and \$19.99 (inclusive), \$1 when the price is between \$20 and \$24.99 (inclusive), and \$2 when the price is at or above \$25. Consider four bidders, with the following bids placed in sequence: (bidder 1, \$8.39), (bidder 2, \$11.99), (bidder 1, \$17.49), (bidder 3, \$20.01), (bidder 1, \$26.90), (bidder 4, \$32.20), (bidder 1, \$32), and then the auction closes. Report the state of the auction after each bid, namely:

- the provisional winner, or "reserve not met",
- the new price, p_t, and
- the new ask price, $p_{\text{ask},t}$.

Solution. Sates are listed below.

- bidder 1-\$8.39: "reserve not met"; $p_t = \$8.89, p_{\text{ask},t} = \$9.39$
- bidder 2-\$11.99 "reserve not met"; $p_t = \$8.89, p_{\text{ask},t} = \$9.39$
- bidder 1-\$17.49 winner: bidder 1; $p_t = \$15, p_{\text{ask},t} = \$15.5$
- bidder 3-\$20.01 winner: bidder 3; $p_t = \$17.99, p_{\text{ask},t} = \$18.45$
- bidder 1-\$26.90 winner: bidder 1; $p_t = \$21.01, p_{\text{ask},t} = \$22.01$
- bidder 4-\$32.20 winner: bidder 4; $p_t = \$28.90, p_{\text{ask},t} = \$30.90$
- bidder 1-\$32 winner: bidder 4; $p_t = \$32.20, p_{\text{ask},t} = \$34.20$

□

(b) (10 pts) Say that you are running an eBay auction and you want to extract maximum revenue from the current highest bidder. Explain how eBay's pricing rule allows you to create a shill bidder that, given knowledge of ϵ and the ability to observe p_t or $p_{\text{ask},t}$, will drive up the price to the highest bidder without much risk of displacing the high bidder. (Hint: Think about what the shill bidder can learn about the highest bidder's bid.)

Solution. Since we know ϵ, with the knowledge of p_t or $p_{\text{ask},t}$, we can always get the value of p_t. Furthermore, $p_t = min(b_t^{(2)} + \epsilon, b_t^{(1)})$, we can always bid price p_t to drive up the price to the highest bidder without much risk of displacing the high bidder, or we can just bid price a little below p_t to ensure that the price is always strictly below leading bid. □