Due date: 11:59 pm on Wednesday, April 12, 2023.

Logistical questions:

1. Did you collaborate with anyone? If so, list their names here. **No**
2. How long did this assignment take you? **24** hours
3. Do you have any comments on the assignment? Good.

Problem 1: Matching Markets [25 pts]

(a) (10 pts) Prove that there exists no mechanism for bipartite matching with two-sided preferences that is both stable and strategy-proof.

Hint: Consider an instance with two students and two courses with preference orderings $t_1 \succ_{s_1} t_2$, $t_2 \succ_{s_2} t_1$, $s_2 \succ_{t_1} s_1$, and $s_1 \succ_{t_2} s_2$, and possible manipulation via *truncation strategies* such as $t_1 \succ_{s_1} \phi$ instead of $t_1 \succ_{s_1} t_2$.

Solution. Consider the case of two students and two courses with preferences P given by

$$P(s1) = t_1, t_2; P(s2) = t_2, t_1; P(t_1) = s_2, s_1; P(t_2) = s_1, s_2$$

Then there are only two stable matching, $\mu_s = [(s_1, t_1); (s_2, t_2)]$, and $\mu_t = [(s1, t_2), (s_2, t_1)]$. We can easily observe that any stable matching mechanism will contradict the strategy-proof for one side of agents. □

(b) (15 pts) Suppose that there are three students, s_1, s_2, and s_3, and three schools, t_1, t_2, and t_3. Each school has a capacity of one (i.e., only one student can be matched to each school).

$s_1 : t_2 \succ t_1 \succ t_3$	$t_1 : s_1 \succ s_3 \succ s_2$
$s_2 : t_1 \succ t_2 \succ t_3$	$t_2 : s_2 \succ s_1 \succ s_3$
$s_3 : t_1 \succ t_2 \succ t_3$	$t_3 : s_2 \succ s_1 \succ s_3$

Use student-proposing DA to find the student-optimal, stable matching, and show your work. Is there a matching, perhaps unstable, that Pareto dominates this matching for students? If it is unstable, identify a blocking pair.

Solution. **Student-proposing DA**
Round 1: $(s_1, t_2), (s_3, t_1)$
Round 2: (s_2, t_3)
Therefore, we can easily check and conclude that $\{(s_1, t_2), (s_2, t_3), (s_3, t_1)\}$ is the student-optimal and stable matching.
No, there isn't. In two-sided matching markets, stability implies Pareto optimal. □

Problem 2: Implementing TTC [25 pts]

Implement the top trading cycles (TTC) algorithm. See the Python file (ttc.py) for a more detailed description of the input and desired output, but here are the evaluation metrics.

1. We should be able to run your code via a command line statement like

   ```
   python ttc.py infile
   ```

   where 'infile' is an input file that contains preferences for each agent (where line i consists of the ordered preferences of agent i). The file has a function for reading an input file already implemented.

2. The output should state all the trades that happen (in the form, 'agent $i \leftarrow$ agent j's item').

3. The output for test.txt is included at the bottom of the Python file. I would *highly* recommend writing your own test cases and verifying by hand!

Problem 3: Social Choice [20 pts]

(a) (10 pts) What is the outcome of running STV on the following profile? Interpret STV as a SRF and report a ranking.

$$2 : a_1 \succ a_3 \succ a_2 \qquad 2 : a_2 \succ a_3 \succ a_1 \qquad 1 : a_3 \succ a_2 \succ a_1.$$

By changing each of the first two votes (i.e., the $a_1 \succ a_3 \succ a_2$) in the same way, show a failure of the independence of irrelevant alternatives (IIA) under STV. Show your work.

Solution. STV PROCESS is as following,
Round 1: Eliminate a_3; $\quad 2 : a_1 \succ a_2 \quad 2 : a_2 \succ a_1 \quad 1 : a_2 \succ a_1$
Round 2: Eliminate a_1.
Final social rank order: $a_2 \succ a_1 \succ a_3$ □

(b) (10 pts) For this part, consider the setting of single-peaked preferences (SPP). Prove that there is a Condorcet winner in any single-peaked domain when there is an odd number of votes.

Solution. Suppose that we have 2n+1 candidates: 1,2,3,4,...., 2n+1. Now we can put them on the number line as lecture slides shows. Let's assume that 1 is the most left one and 2n+1 is the most right one. Then, due to SPP property, we know that if alternative n+1 beats n, than n+1 beats 1,2,3...n. Analogously if n+1 beats n+2, than it also beats n+3, n+4, ... 2n+1. Therefore, we can start by comparing two most middle candidates, if the left one win, we can compare the two most middle alternatives of the left half of all alternatives; if the right one wins, then we can compare two most middle alternatives of the right half of all alternatives; repeat the above process and we can eventually find a Condorcet winner, also at most in log(2n+1) steps. □

Problem 4: Evaluating Voting Rules [30 pts]

For this question, you will evaluate various voting rules on a particular profile.

For all parts in this question, consider the profile $\succ$ consisting of 30 votes over 4 alternatives: 11 voters report $a_1 \succ a_2 \succ a_4 \succ a_3$, 8 voters report $a_2 \succ a_3 \succ a_1 \succ a_4$, 6 voters report $a_3 \succ a_2 \succ a_1 \succ a_4$, and 5 voters report $a_4 \succ a_3 \succ a_2 \succ a_1$.

(a) (5 pts) Interpreting plurality as a SRF, what is the final rank order returned by the plurality rule? Explain.

Solution. The plurality rule on preference profile $\succ$ is an SCF f that is ranked first by the highest number of agents. Final rank order is: $a_1 \succ a_2 \succ a_3 \succ a_4$. Explanation: $a_1 : 11; a_2 : 8; a_3 : 6; a_4 : 5$ and $11 > 8 > 6 > 5$ □

(b) (5 pts) Interpreting the Borda rule as a SRF, what is the final rank order returned by the Borda rule? You must show your work (i.e., calculate the Borda score for each alternative).

Solution. By definition, we can calculate as following,
$a_1 scores : 3 * 11 + 1 * 8 + 1 * 6 + 0 * 5 = 47$
$a_2 scores : 2 * 11 + 3 * 8 + 2 * 6 + 1 * 5 = 63$
$a_3 scores : 0 * 11 + 2 * 8 + 3 * 6 + 2 * 5 = 44$
$a_4 scores : 1 * 11 + 0 * 8 + 0 * 6 + 3 * 5 = 26$
Final rank order: $a_2 \succ a_1 \succ a_3 \succ a_4$ □

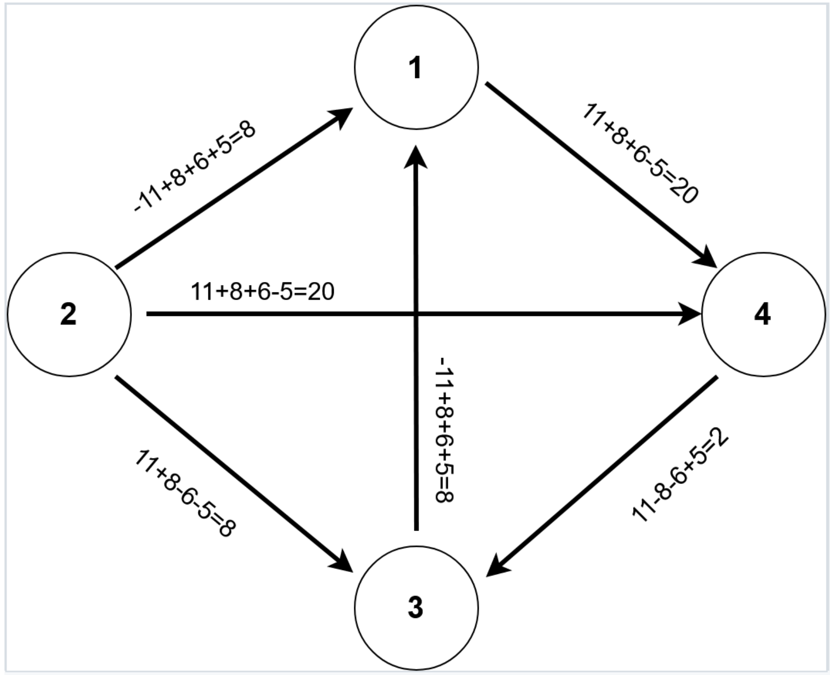
(c) (5 pts) What is the final rank order returned by STV? You must show your work (i.e., explain the order of eliminations).

Solution. By definition, the process is as following,
First round: eliminate a_4, pushing 5 votes toward a_3
Second round: eliminate a_2, pushing 8 votes toward a_3
Third round: eliminate a_1
Final rank order: $a_3 \succ a_1 \succ a_2 \succ a_4$ □

(d) (5 pts) What is the set of winners under the Copeland rule? You must show your work (i.e., calculate the Copeland score for each alternative).

Solution. By definition,
WMG wedges: $a_1 \leftarrow a_2$; $a_1 \leftarrow a_3$; $a_1 \rightarrow a_4$; $a_2 \rightarrow a_3$; $a_2 \rightarrow a_4$; $a_3 \leftarrow a_4$
Scores:
$a_1 : 1$ $a_2 : 3$ $a_3 : 1$ $a_4 : 1$, so a_2 is the Copeland winner. □

(e) (5 pts) Draw the weighted majority graph (WMG) representing this profile. Identify the Condorcet winner, if it exists. If it does not exist, report that there is no Condorcet winner for this profile.



Solution. Therefore, a_2 is the Condorcet winner by definition. □

(f) (5 pts) What is the final rank order returned by the Kemeny rule? Explain your reasoning and show your work. (Hint: parts (d) and/or (e) may be helpful in narrowing the search space.)

Solution. According to parts (d) and (e), since The Kemeny rule is Condorcet consistent, and the only Condorcet winner is a_2, so $f(\succ) = a_2$, which means that the first alternative in the final rank order must be a_2. Then we consider the remaining 3 positions, there are 6 cases to be considered, shown as following,

1. $a_2 \succ a_1 \succ a_3 \succ a_4$: $11 \cdot 2 + 8 \cdot 1 + 6 \cdot 2 + 5 \cdot 5 = 67$
2. $a_2 \succ a_1 \succ a_4 \succ a_3$: $11 \cdot 1 + 8 \cdot 2 + 6 \cdot 3 + 5 \cdot 4 = 65$
3. $a_2 \succ a_3 \succ a_1 \succ a_4$: $11 \cdot 3 + 8 \cdot 0 + 6 \cdot 1 + 5 \cdot 4 = 59$
4. $a_2 \succ a_3 \succ a_4 \succ a_1$: $11 \cdot 4 + 8 \cdot 1 + 6 \cdot 2 + 5 \cdot 3 = 79$
5. $a_2 \succ a_4 \succ a_1 \succ a_3$: $11 \cdot 2 + 8 \cdot 3 + 6 \cdot 4 + 5 \cdot 3 = 85$
6. $a_2 \succ a_4 \succ a_3 \succ a_1$: $11 \cdot 3 + 8 \cdot 2 + 6 \cdot 3 + 5 \cdot 2 = 77$

Therefore, we can easily observe that 59 is minimum KT distance.
The final rank order: $a_2 \succ a_3 \succ a_1 \succ a_4$ □