Due date: 11:59 pm on Wednesday, March 29, 2023.

Notion: 1 Late days were user for this assignment.

Logistical questions:

1. Did you collaborate with anyone? If so, list their names here. No
2. How long did this assignment take you? 24 hours
3. Do you have any comments on the assignment? Good, but proof is a little hard

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Problem 1: Mechanism Design [25 pts]

Consider the following variant of the ascending-clock auction, which proceeds as follows.

Initialize the price p to zero. The auction will proceed in rounds. While at least two bidders remain in the auction, increase the price by a predetermined $\epsilon > 0$. Sell to the bidder who is the last to drop out, charging the last price at which one or more bidders were in the auction. If the auction closes with multiple bidders dropping out in the last round, select one uniformly at random as the winner.

In this setting, a *straightforward bidding strategy* for a bidder with value v_i is to remain active while the ask price $p \leq v_i$ and drop out otherwise.

(a) (10 pts) It is possible to simulate the clock auction described above together with straightforward bidding in a direct-revelation mechanism (DRM). Describe the simulated bidding strategy, along with the allocation and payment rules for the direct-revelation mechanism (DRM) obtained by simulating the clock auction described above together with straightforward bidding.

Solution. The simulated bidding strategy for each bidder is to report their true value v_i in the auction. In each round of the auction, the auctioneer announces the current ask price p, and each bidder reports whether they would like to stay in the auction or drop out based on their straightforward bidding strategy. Bidders who report dropping out are eliminated from the auction. The auction continues until only one or no bidder remains, and the item is allocated to the remaining bidder at the last price p at which at least two bidders were still in the auction. To be concrete, initially set $k = 0$, iterate $k = k + 1$ every time until only one bidder remained, which means only one bid price $v_i \geq k\epsilon$ at that time or no bidder remained, which means at least two bid price $(k-1)\epsilon \leq v \leq k\epsilon$ at that time.
Allocation rule: First case: if only one bidder remained in the end, then the item is allocated to him/her; Second case: no bidder remained in the end, then the item is allocated randomly to the bidders dropping out in the last round.
Payment rule: First case: one bidder remained in the end, the bidder need to pay his/her true value v_i; Second case: no bidder remained in the end with the ask price $p = k\epsilon$, the winner need to pay $(k-1)\epsilon$, which is the last price at which one or more bidders were in the auction. □

(b) (15 pts) Suppose the bid increment is $\epsilon = 5$ and there are three bidders with values $v_1 = 14$, $v_2 = 13$, and $v_3 = 2$. What is the outcome (i.e., who wins, how much do they pay, and what is their utility)? Does agent 1 have a useful manipulation? Justify your answer.

Solution. Outcome: 1. Either bidder 1 or bidder 2 will win; 2. The winner need to pay 10, others pay 0; 3. If bidder 1 win, his/her utility is 4, others 0; if bidder 2 win, his/her utility is 3, others 0.
No, agent 1 has no useful manipulation. Consider three cases: 1. if he/her pays below 10, he/her will lose; 2. if he/her pays between 10 and 14, he/her always has fifty percent to win, when he/her pays 14, the utility is the largest if winning; 3. if he/her pays above or equal to 15, he/her utility $u_1 \leq 0$. In conclusion, agent 1 has no manipulation in this auction. □

Problem 2: Vickrey-Clarke-Groves Mechanism [25 pts]

(a) (25 pts) Consider the weighted VCG mechanism, which is a slight modification to the VCG mechanism in which the choice rule maximizes a positive affine function on bidder evaluations, i.e.,

$$x(\hat{v}) \in \arg\max_{a \in A} \left[\sum_{i \in N} \alpha_i \cdot \hat{v}_i(a) + \beta_a \right]$$

for non-negative (per-agent) scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_{\geq 0}$ and a (per-allocation) scalar $\beta_\alpha \in \mathbb{R}$ for each $a \in A$.

By extending the proof of strategyproofness of the VCG mechanism presented in class, give a payment rule, $t(\hat{v})$, for which the weighted VCG mechanism is strategy-proof and prove that it is indeed strategy-proof.

Solution. $t_i(\hat{v}) = \frac{1}{\alpha_i} \left[(\sum_{j \neq i} \alpha_j \cdot \hat{v}_i(a_{-i}) + \beta_{a_{-i}}) - (\sum_{j \neq i} \alpha_j \cdot \hat{v}_i(a^*) + \beta_{a^*}) \right]$
Consider agent i with valuation function v_i, fix other agents' reports $\hat{v}_{-i}$, and consider a misreport $\hat{v}_i \neq v_i$
Let $a^* = x(v_i, \hat{v_{-i}})$, $\hat{a} = x(\hat{v}_i, \hat{v_{-i}})$, $a^{-i} \in \arg\max_{a \in A^{-i}} \sum_{j \neq i} \hat{v}_j(a)$
Utility to u from truthful report: $u_i = v_i(a^*) - \frac{1}{\alpha_i} \left[(\sum_{j \neq i} \alpha_j \cdot \hat{v}_i(a_{-i}) + \beta_{a_{-i}}) - (\sum_{j \neq i} \alpha_j \cdot \hat{v}_i(a^*) + \beta_{a^*}) \right]$,
and utility to i from misreport: $\hat{u}_i = v_i(\hat{a}) - \frac{1}{\alpha_i} \left[(\sum_{j \neq i} \alpha_j \cdot \hat{v}_i(a_{-i}) + \beta_{a_{-i}}) - (\sum_{j \neq i} \alpha_j \cdot \hat{v}_i(\hat{a}) + \beta_{\hat{a})} \right]$
Subtracting yields

$$u_i - \hat{u}_i = \frac{1}{\alpha_i} \left[\alpha_i(v_i(a^*)) + \sum_{j \neq i} \alpha_j \cdot \hat{v}_i(a^*) + \beta_{a^*} \right] - \frac{1}{\alpha_i} \left[\alpha_i(v_i(\hat{a})) + \sum_{j \neq i} \alpha_j \cdot \hat{v}_i(\hat{a}) + \beta_{\hat{a}} \right] \geq 0$$

because of the choice rule for VCG □

position	position effect	value-per-click	bid-per-click	GSP price
1	0.2	4	4	
2	0.18	10	2.1	
3	0.1	2	2	
-	0	1	1	

Table 1: Nash equilibrium of a GSP auction.

Problem 3: Ad Auctions [20 pts]

(a) (10 pts) There are four bidders with the same quality. Their per-click values are $w_1 = 10$, $w_2 = 4$, $w_3 = 2$, and $w_4 = 1$. They are competing for three ad slots whose position effects are $pos_1 = 0.2$, $pos_2 = 0.18$, and $pos_3 = 0.1$. Consider the bid profile depicted in Table 1 and verify that this bid profile is a Nash equilibrium in the GSP auction. Also, show that the outcome is not envy-free and explain why the Nash equilibrium property need not imply envy-freeness.

Solution. To verify that this bid profile is a Nash equilibrium, we nee to consider the 4 bidders respectively.
Bidder 4: $u_4 = 0(1-0) = 0 > max 0.1(1-2), 0.18(1-2.1), 0.2(1-4) = -0.1$
Bidder 3: $u_3 = 0.1(2-1) = 0.2 > max 0, 0.18(2-2.1), 0.2(2-4) = 0$
Bidder 2: $u_2 = 0.2(4-2.1) = 0.38 > max 0.18(4-2), 0.1(4-1), 0 = 0.36$
Bidder 1: $u_1 = 0.18(10-2) = 1.44 > max 0.2(10-4), 0.1(10-1), 0 = 1.2$
Therefore, no one has deviation, which means this bid profile is a Nash equilibrium in this GSP auction.
As discussed above, consider bidder 2, $u_2 = 0.38 < 0.2(10-2.1) = 1.58$, which means the outcome is not envy-free by definition.
Explanation: Nash equilibrium guarantees that no one has deviation in the auction, while Envy-Freeness ensures each bidder would not be happier with another bidder's outcome. It is possible for a game to have a Nash equilibrium that is not envy-free. This is because the Nash equilibrium only takes into account the players' strategies and their resulting payoffs. It does not take into account how the players feel about each other's outcomes. In other words, even if a player is not happy with their outcome, they may still be willing to play their current strategy if they believe that changing it would result in an even worse outcome. To be concrete, a bidder changes the position but not switch to the correct price-per-click in Envy-Freeness, but in Nash Equilibrium, the bidder need to change the price-per-click if changing positions. □

position	position effect	value-per-click	bid-per-click	GSP price
1	0.2	10	10	
2	0.18	4	1.7	
3	0.1	2	13/9	
-	0	1	1	

Table 2: Envy-free Nash equilibrium of a GSP auction.

(b) (10 pts) In the same setup as part (a), consider the bid profile depicted in Table 2 and verify that this bid profile is an envy-free Nash equilibrium (EFNE) in the GSP auction. You will have to argue that this is both a Nash equilibrium and envy-free.

Solution. To verify that this bid profile is a Nash equilibrium, we nee to consider the 4 bidders respectively.
Bidder 4: $u_4 = 0(1-0) = 0 > max0.1(1-\frac{13}{9}), 0.18(1-1.7), 0.2(1-10) = -\frac{2}{45}$
Bidder 3: $u_3 = 0.1(2-1) = 0.2 > max0, 0.18(2-1.7), 0.2(2-10) = 0.054$
Bidder 2: $u_2 = 0.18(4-\frac{13}{9}) = \frac{23}{50} > max0.2(4-10), 0.1(4-1), 0 = 0.3$
Bidder 1: $u_1 = 0.2(10-1.7) = 1.66 > max0.18(10-\frac{13}{9}), 0.1(10-1), 0 = \frac{77}{50}$
Therefore, no one has deviation, which means this bid profile is a Nash equilibrium in this GSP auction.
To check envy-free, also consider 4 bidders, respectively.
Bidder 4: $u_4 = 0(1-0) = 0 \geq max0.1(1-1), 0.18(1-\frac{13}{9}), 0.2(1-1.7) = 0$
Bidder 3: $u_3 = 0.1(2-1) = 0.2 > max0, 0.18(2-\frac{13}{9}), 0.2(2-1.7) = 0.1$
Bidder 2: $u_2 = 0.18(4-\frac{13}{9}) = \frac{23}{50} > max0.2(4-1.7), 0.1(4-1), 0 = 0.3$
Bidder 1: $u_1 = 0.2(10-1.7) = 1.66 > max0.18(10-\frac{13}{9}), 0.1(10-1), 0 = \frac{77}{50}$
Therefore, this is envy-free.
In conclusion, we verify that this is both NE and Envy-free, thus this is EFNE. □

Problem 4: Implementing Ad Auctions [30 pts]

For this question, you will implement the VCG position auction, the GSP auction, and balanced bidding in Python.

Given n bidders and m ad slots, you will be given some combination of values-per-click $w = (w_1, \ldots, w_n)$, bids-per-click $b = (b_1, \ldots, b_n)$, ad qualities $Q = (Q_1, \ldots, Q_n)$, and position effects $pos = (pos_1, \ldots, pos_m)$.

The Python file 'adauctions.py' contains stubs for various functions you may find useful. Please read the following very closely; you will be evaluated according to criteria (1) and (2), and points (3) and (4) offer useful clarifications.

1. We should be able to run your code via a command line statement like

   ```
   python adauctions.py infile
   ```

   where 'infile' is an input file that contains w, b, Q, and pos. The file has a function for reading an input file already implemented.

2. The output should be threefold: (1) the VCG allocation and payments, (2) the GSP allocation and payments, and (3) the balanced bidding profile. There are some potentially helpful printing functions in 'adauctions.py'.

3. For the VCG and GSP portions, you may not assume that all ad qualities are the same. However, for the balanced bidding implementation, please assume that all ad qualities are the same for simplicity.

4. We have provided one test file (the expected output is commented out at the bottom of 'adauctions.py') but we **strongly** recommend writing your own test cases and making sure that your functions behave as expected.

For completeness, here are the specific instructions for each part.

(a) (10 pts) Implement the VCG position auction. In particular, given values-per-click $w = (w_1, \ldots, w_n)$, bids-per-click $b = (b_1, \ldots, b_n)$, ad qualities $Q = (Q_1, \ldots, Q_n)$, and position effects $pos = (pos_1, \ldots, pos_m)$, output the assignment $x_{vcg}(b)$ and price-per-click $p_{vcg}(b)$.

(b) (10 pts) Implement the GSP ad auction. In particular, given values-per-click $w = (w_1, \ldots, w_n)$, bids-per-click $b = (b_1, \ldots, b_n)$, ad qualities $Q = (Q_1, \ldots, Q_n)$, and position effects $pos = (pos_1, \ldots, pos_m)$, output the assignment $x_{gsp}(b)$ and price-per-click $p_{gsp}(b)$.

(c) (10 pts) Implement balanced bidding. In particular, given values-per-click $w = (w_1, \ldots, w_n)$ and position effects $pos = (pos_1, \ldots, pos_m)$, output the balanced bid profile $b = (b_1, \ldots, b_n)$. (Note: For only this part, we assume that all ad qualities are the same for simplicity.)

```
D:\Anaconda\python.exe H:\CSC289\HW4\test.py
VCG:
Position 0:      bidder 0
Position 1:      bidder 1
Position 2:      bidder 2
Position 3:      bidder 3
Bidder 0:    1.70
Bidder 1:    1.44
Bidder 2:    1.00
Bidder 3:    0.00

GSP:
Position 0:      bidder 0
Position 1:      bidder 1
Position 2:      bidder 2
Position 3:      bidder 3
Bidder 0:    4.00
Bidder 1:    2.00
Bidder 2:    1.00
Bidder 3:    0.00

Balanced bids:
Bidder 0:    10.00
Bidder 1:    1.70
Bidder 2:    1.44
Bidder 3:    1.00
```