COMPLEX ORIENTATIONS AND STRICT ELEMENTS

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Abstract. The theory of complex orientations is typically phrased in geometric language, but they are crucial to the algebraic project of understanding the category of spectra. In this talk, I give an algebraic description of complex orientations and propose a generalization thereof.

0. Introduction

Definition 0.1. Let R be a ring spectrum. A *complex orientation* of R is a choice of $c \in R^2(\mathbb{CP}^\infty)$ which goes to 1 under the composite $R^2(\mathbb{CP}^\infty) \to R^2(\mathbb{CP}^1) \cong R^2(S^2) \cong R^0(S^0) = \pi_0(R)$. This data is equivalent to the data of a map $MU \to R$ in $\mathrm{Mon}(\mathrm{Ho}(\mathrm{Sp}))$.

Geometric meaning. This is a generalized (first) Chern class for R-cohomology: it is a characteristic class for complex line bundles, and setting it to be 1 on the tautological bundle over $\mathbb{CP}^1$ "calibrates" it to behave properly under $\otimes$.

Algebraic meaning? Chromatic homotopy tells us that complex orientations parameterize the algebraic structure of Sp, so they should have some algebraic interpretation. Practically speaking, we would also like this to come with an algebraic obstruction theory. I give this using "strict elements".

1. Strict Elements

Recall the loopspace recognition principle.

Theorem 1.1 (May)**.** *For* $1 \leq n \leq \infty$, $\Omega^n : \mathrm{Spaces}_* \to \mathrm{Spaces}_*$ *induces an equivalence* $(\mathrm{Spaces}_{\geq n})_* \simeq E_n(\mathrm{Spaces}_*)^{gp}$*, where we take the convention that* $(\mathrm{Spaces}_{\geq \infty})_* = \mathrm{Sp}_{\geq 0}$.

Proposition 1.2. $\Omega^n S^n$ *is the free grouplike* E_n*-space on a point.*

Proof. By the diagram

$$\begin{tikzcd}
E_n(\mathrm{Spaces}_*)^{gp} \arrow[r, phantom, "\simeq"] \arrow[d, "F"] & (\mathrm{Spaces}_{\geq n})_* \arrow[dl, bend left, "\Omega^n"] \\
\mathrm{Spaces} \arrow[ur, bend left, "\Sigma^n_+"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd},$$

we see that the free object monad for the forgetful functor F is $\Omega^n\Sigma^n_+$; and $\Omega^n\Sigma^n_+(*) = \Omega^n\Sigma^n S^0 = \Omega^n S^n$. $\square$

Corollary 1.3. $\mathbb{Z}$ *is the free grouplike* E_1*-space on one element.*

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Proof. $\Omega^1 S^1 = \mathrm{End}_{\mathrm{Spaces}_*}(S^1) \simeq \mathbb{Z}$. □

In classical algebra, we have the pleasant coincidence that the free group on one element is also the free abelian group on one element. This makes it easy to say what an element of an abelian group is (a map out of $\mathbb{Z}$). In the homotopical context, though, things are a little more complicated. The free ∞-group on one element, $\mathbb{Z}$, does indeed support an E_∞ structure; but since this is a structure rather than a property, it is not the *free* E_∞ structure.[1]

Question: Let $X \in \mathrm{Sp}$. What is an "element" of X?

Answer 1: An element of $\pi_0 X$.

This is the *wrong* answer. The problem is, this doesn't take into account any of the structure of X that isn't also detected by its infinite loop space $\Omega^\infty X$. In particular, if we consider the case of connective spectra, this is essentially saying that an element of a grouplike E_∞-space is just a generalized element of its underlying space. This also leads to practical issues.

Example. Take $R \in \mathrm{CAlg}$. We define the space of units of R as the pullback

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R). \end{array}$$

Because R is an E_∞-ring, $GL_1(R)$ carries a grouplike E_∞ structure and thus lifts to a connective spectrum $gl_1(R)$. This defines a functor $\mathrm{CAlg} \to \mathrm{Sp}$ which is corepresentable by the Laurent E_∞-ring in one variable, $\mathbb{S}\{t^\pm\}$. But we have a problem: this ring is not flat over $\mathbb{S}$! Not to mention that its homotopy groups are terribly unpleasant. For both of these reasons, its completion is not a formal group in the sense of Lurie, which inhibits our ability to do chromatic homotopy theory with it. We need something better.

Answer 2: A map of spectra $H\mathbb{Z} \to X$. This will take into account the additional structure we want.

Returning to our example, define $\mathbb{G}_m(R) = \tau_{\geq 0}\mathrm{Map}_{\mathrm{Sp}}(H\mathbb{Z}, gl_1(R))$ (see [4]). This defiines a functor $\mathrm{CAlg} \to \mathrm{Mod}_{\mathbb{Z}}$ which is corepresentable by the *smooth* Laurent polynomial ring in one variable, $\mathbb{S}[t^\pm] = \Sigma^\infty_+ \mathbb{Z}$, which is flat over $\mathbb{S}$ and has nice homotopy groups ($\pi_*(\mathbb{S}[t^\pm]) = (\pi_*\mathbb{S})[t^\pm]$). We call $\mathbb{G}_m(R)$ the *strict units* of R. In that spirit, I propose the following definition.

Definition 1.4. A *strict element* of $X \in \mathrm{Sp}$ is an element of $\pi_0 \mathrm{Map}_{\mathrm{Sp}}(H\mathbb{Z}, X)$.

Theorem 1.5. *A strict element of an E_∞-ring R is equivalent to an E_∞ map $\mathbb{S}[t] \to R$.*

Proof. By the loop space recognition principle and symmetric monoidality of suspension-loop adjunction, we have isomorphisms $\mathrm{Map}_{\mathrm{Sp}}(H\mathbb{Z}, R) \simeq \mathrm{Map}_{E_\infty}(\mathbb{Z}, \Omega^\infty R) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\Sigma^\infty_+ \mathbb{Z}, R)$, and the domain of this last mapping space is the definition of $\mathbb{S}[t]$. □

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[1] Moreover, confusingly, the free E_n-group on one element ($1 < n < \infty$) does *not* admit an E_∞ structure, since $\Omega^n S^n$ is not an infinite loop space.

Remark 1.6. The weak elements are also corepresentable in both Sp and CAlg, corepresented by $\mathbb{S}$ in the first category and $\mathbb{S}\{t\} = \Sigma^\infty_+\Omega^\infty\mathbb{S}$ in the second. The evident forgetful map from strict elements to weak elements is corepresented by the unit map $\mathbb{S} \to H\mathbb{Z}$ in spectra and by the canonical map $\mathbb{S}\{t\} \to \mathbb{S}[t]$ in CAlg. This implies in particular that weak and strict units coincide for rational spectra.

2. Higher Complex Orientations

To place this in a broader context, observe that we have $\mathrm{Map}_{\mathrm{Sp}}(H\mathbb{Z}, X) \simeq \mathrm{Map}_{E_\infty}(\mathbb{Z}, \Omega^\infty X)$. We have towers of similar mapping spaces:

$$\begin{tikzcd}
 & & & & \mathrm{Map}_{E_\infty}(\mathbb{Z}, \Omega^\infty X) \arrow[d] \\
\vdots \arrow[d] & \simeq & \vdots \arrow[d] & \simeq & \vdots \arrow[d] \\
\mathrm{Map}_{E_1}(\mathbb{CP}^\infty, \Omega^{\infty-2}X) \arrow[d] & \simeq & \mathrm{Map}_{E_2}(S^1, \Omega^{\infty-1}X) \arrow[d] & \simeq & \mathrm{Map}_{E_3}(\mathbb{Z}, \Omega^\infty X) \arrow[d] \\
\mathrm{Map}(\mathbb{CP}^\infty, \Omega^{\infty-2}X) & \simeq & \mathrm{Map}_{E_1}(S^1, \Omega^{\infty-1}X) \arrow[d] & \simeq & \mathrm{Map}_{E_2}(\mathbb{Z}, \Omega^\infty X) \arrow[d] \\
 & & \mathrm{Map}(S^1, \Omega^{\infty-1}X) & \simeq & \mathrm{Map}_{E_1}(\mathbb{Z}, \Omega^\infty X).
\end{tikzcd}$$

Comparing towers, we see that

i) $\pi_0\mathrm{Map}_{E_1}(\mathbb{Z}, \Omega^\infty X) \cong \pi_0 X$,
ii) $\pi_0\mathrm{Map}_{E_2}(\mathbb{Z}, \Omega^\infty X) \cong X^2(\mathbb{CP}^\infty)$, and
iii) The forgetful functor $E_2(\mathrm{Spaces}_*)^{gp} \to E_1(\mathrm{Spaces}_*)^{gp}$ induces a map $X^2(\mathbb{CP}^\infty) \to \pi_0 X$.

Theorem 2.1. *The induced map $X^2(\mathbb{CP}^\infty) \to \pi_0 X$ coincides with the map induced by the inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ as described in §0.*

Proof. The inclusion $S^2 \cong \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$ can be written as $\Sigma S^1 \to BS^1$. By suspension-loop adjunction, this is the transpose of a map $S^1 \to \Omega BS^1 \simeq S^1$, and in fact its transpose is the identity (since it induces the identity on H^1). So the induced map is given by upper horizontal arrow in

$$\begin{tikzcd}
\pi_0\mathrm{Map}(BS^1, \Omega^{\infty-2}X) \arrow[r] & \pi_0\mathrm{Map}(\Sigma S^1, \Omega^{\infty-2}X) \\
\pi_0\mathrm{Map}_{E_1}(S^1, \Omega^{\infty-1}X) \arrow[r] \arrow[u, "\cong"] & \pi_0\mathrm{Map}(S^1, \Omega^{\infty-1}X), \arrow[u, "\cong"]
\end{tikzcd}$$

and the lower horizontal arrow is the map described above. $\square$

This brings us full circle.

Definition 2.2. An *E_n-element* of $X \in \mathrm{Sp}$ is an E_n map $\mathbb{Z} \to \Omega^\infty X$. (In particular, an E_∞-element is the same as a strict element and an E_1-element is the same as a weak element.)

Theorem 2.3 (GN)**.** *A ring spectrum R is complex-orientable if and only if it has 1 as an E_2 element, and complex orientations are E_2-lifts of 1.*

There are some natural questions to ask at this point.

Question 1. Is there a "chromatic" interpretation of E_n-elements for $n \geq 3$?

Suppose, for example, that 1 lifts to an E_3-element for some $R \in \mathrm{CAlg}$. Call this a level 3 complex orientation. What does this tell about the structure of R as a complex-orientable E_∞-ring? Does this show up in the structure of its formal group? And what are examples of such "higher complex orientations?" (Finding examples would amount to computing $R^n(K(\mathbb{Z}, n))$ and its canonical map to $R^1(K(\mathbb{Z}, 1))$ for $n \geq 3$.)

Question 2. What is the obstruction theory for going from an E_n-element to an E_{n+1}-element?

By Dunn additivity (see [5]), $E_{n+1}(\mathrm{Spaces}_*) \simeq E_1(E_n(\mathrm{Spaces}_*))$, so we are really looking for obstructions to (unique?) E_1-structures on maps between E_1-objects of $E_n(\mathrm{Spaces})^{gp}$. Charles Rezk has pointed out to me that the obstruction theory for extending from E_1 to E_2 must coincide with the cellular obstruction theory for extending from $\mathbb{CP}^1$ to $\mathbb{CP}^\infty$. How does this generalize?

Question 3. Are strictifications of this type related to strictifying $MU \to R$ to a map of structured ring spectra?

It is known ([2]) that there is an infinite sequence of E_∞-rings interpolating between $\mathbb{S}$ and MU which gives rise to an obstruction theory for lifting a map of homotopy-commutative ring spectra $MU \to R$ to a map of E_∞-rings. It is unclear whether there is a relation between higher complex orientations and this form of strictification; and if so, what the relationship is between the two obstruction theories. There is an analogous sequence of maps $\mathbb{S} \to \cdots \to \tau_{\leq 2}\mathbb{S} \to \tau_{\leq 1}\mathbb{S} \to \tau_{\leq 0}\mathbb{S} = H\mathbb{Z}$ (and similarly in CAlg), so there is at least some formal similarity. However, there is no universal example because $MU^*(H\mathbb{Z}) = 0$, and moreover $KU^*(H\mathbb{Z}) = 0$ as well. In fact, an argument of Nardin and Peterson ([6]) shows the following.

Theorem 2.4. *Let R be a ring spectrum with $\pi_0 R \cong \mathbb{Z}$. If 1 is a strict element of R, then $H\mathbb{Z}$ is a retract of $\tau_{\geq 0}R$. (That is, the first k-invariant of $\Omega^\infty R$ is zero.)*

Proof. By assumption, we have a map $H\mathbb{Z} \to R$ which is the identity on π_0, and we can factor through the connective cover to get a map $H\mathbb{Z} \to \tau_{\geq 0}R$. On the other hand, since $\pi_0 R = \mathbb{Z}$, we have a "cotruncation map" $\tau_{\geq 0}R \to H\mathbb{Z}$ which is also the identity on π_0. The composition of these is an endomorphism of $H\mathbb{Z}$ which is the identity on π_0, and thus it is the identity. □

It isn't too surprising that we have a strong obstruction to level ∞ complex orientations. We might hope to have more luck with finite level. This is a vain hope, however, because there are in some sense no "interesting" examples.

Theorem 2.5 (GN, Rezk)**.** *Let R be an E_∞-ring which has a nontrivial $T(n)$-localization for some $n > 0$. Then R does not have an E_3 lift of 1, so it does not admit any higher complex orientations.*

Proof. A corollary of the Chromatic Nullstellensatz ([1]) is that any nontrivial $T(n)$-local E_∞-ring admits an E_∞ map to E_n. Since orientations push forward, it is enough to show that E_n does not admit an E_3 lift of 1.

We have a tower

$$\mathbb{S} \xrightarrow{\simeq} \Sigma^{-1}\Sigma^\infty K(\mathbb{Z},1) \to \Sigma^{-2}\Sigma^\infty K(\mathbb{Z},2) \to \Sigma^{-3}\Sigma^\infty K(\mathbb{Z},3) \to \cdots$$

such that a level n orientation on R corresponds to a factorization of the unit $\mathbb{S} \to R$ through the $(n+1)$st term of the tower.[2] Therefore, we need to show that the unit map of E_n does not factor through $\Sigma^{-3}\Sigma^\infty K(\mathbb{Z},3)$.

Combining a result of Hovey-Strickland (Proposition 2.5 of [3]) with Ravenel-Wilson's computation of the Morava K-theory of Eilenberg-MacLane spaces ([7]), we find that the E_n-homology of $K(\mathbb{Z},3)$ is always concentrated in even degree. (This is true for any $K(\mathbb{Z},*)$, actually.) But this means that there are no non-trivial elements in degree 3, i.e. no nontrivial maps $\Sigma^{-3}\Sigma^\infty K(\mathbb{Z},3) \to E_n$, so no factorization exists. □

Certainly any ring of interest in chromatic homotopy theory would need to have some nontrivial telescopic localization, so chromatic homotopy doesn't see anything about higher complex orientations. Why this is, I don't know. If anyone can give an algebraic explanation of why the strictification process stops after the first step, I'd love to hear it.

References

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[2] Incidentally, this means that the free E_∞-ring on the $(n+1)$st term of the tower is the universal example of an E_∞-ring with level n orientation.