

# Isogenies of Oriented Elliptic Curves

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# Some Preliminary Notes

## Conventions/Terminology

- Stack = étale sheaf of  $\infty$ -groupoids on  $\mathrm{CAlg}_R$  for some  $\mathbb{E}_\infty$ -ring  $R$
- DM-stack = spectral Deligne-Mumford stack, not necessarily connective
- Formal DM-stack = formal filtered colimit of DM-stacks along closed immersions; called “honest” if actual DM-stack
- Isogeny = strict abelian variety map which is finite, flat, and locally almost of finite presentation



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## Related Work

Xuecai Ma and Yifei Zhu have a paper in the works ([MZ25]) which approaches this from a different perspective, defining level structures in terms of classical divisors. It isn't clear whether this is equivalent to my definition. A draft can be found on Professor Zhu's website.

# Background: Classical $\mathcal{M}_{\text{ell}}$

Over  $\mathbb{C}$

$$\left\{ \begin{array}{c} \mathcal{M}(\Gamma) \\ \downarrow \text{ét} \\ \mathcal{M}_{\text{ell}} \end{array} \right\} \longleftrightarrow \{ \text{Congruence subgroups } \Gamma \subset GL_2(\mathbb{Z}) \}$$



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**Solution:** Work with moduli interpretation directly!

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## Main Theorem (GN)

$\mathrm{Isog}$  is a formal DM-stack.

## Warning

It is not known whether  $\mathrm{Isog}$  is an honest DM-stack.



# Connected-Étale Factorization

## Factorization Theorem (GN)

There is an orthogonal factorization system  $(\mathcal{C}onn, \mathcal{E}t)$  on  $\mathbf{Ell}_{\mathbf{Isog}}^{\text{or}}$  such that

- $\mathcal{C}onn$  is the class of connected isogenies, and
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$$\begin{array}{ccccc} K^\circ & \xlongequal{\quad} & K^\circ & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & E & \xrightarrow{i} & E' \\ \downarrow & & \downarrow c & & \parallel \\ K^{\text{ét}} & \longrightarrow & E/K^\circ & \xrightarrow{e} & \widetilde{E'} \end{array}$$



## Digression: Why are the components isogenies?



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Elliptic Rigidity Theorem, classical version ([KM85])

Zariski-locally on the base, every morphism of classical elliptic curves is either 0 or an isogeny.



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 $\Rightarrow$  through Postnikov tower.



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$\Rightarrow \mathrm{Def}_{\tilde{R}}^{\tilde{R}}(f) = 0. \quad \square$



## Corollary

We have a pullback of functors

$$\begin{array}{ccc} \mathrm{Isog} & \longrightarrow & \mathrm{Isog}^{\mathrm{ét}} \\ \downarrow & & \downarrow s \\ \mathrm{Isog}^{\mathrm{conn}} & \xrightarrow{t} & \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}. \end{array}$$



## Corollary

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Just need to show that  $\mathrm{Isog}^{\mathrm{\acute{e}t}}, \mathrm{Isog}^{\mathrm{conn}}$  are formal DM-stacks.





## Theorem (GN)

$\mathrm{Isog}^{\mathrm{\acute{e}t}}$  is a DM-stack.



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- 1 [KM85]:  $\{(E, K) \mid E \text{ elliptic curve}, K \subset E \text{ finite}\} \rightarrow (\mathcal{M}_{\text{ell}}^{\text{or}})^{\heartsuit}$   
relative scheme.



## Theorem (GN)

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- 2  $\{(E, K) \mid E \text{ elliptic curve}, K \subset E \text{ finite étale}\}$  open substack.



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relative scheme.
- 2  $\{(E, K) \mid E \text{ elliptic curve}, K \subset E \text{ finite \acute{e}tale}\}$  open substack.
- 3 Leverage \acute{e}taleness and use ([Lur18c], Theorem 18.1.0.2) to lift from classical to spectral. □



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$\{K \subset E \text{ closed, proper, connected} \quad \quad \quad \}$



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$$\begin{array}{c} \{E \xrightarrow{\text{conn}} E'\} \\ \updownarrow \\ \{K \subset E \text{ closed, proper, connected} \} \\ \updownarrow \\ \{K \subset \widehat{\mathbb{G}}_R^Q \text{ honest subgroup} \} \end{array}$$



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$$\Rightarrow \text{Isog}^{\text{conn}} \simeq \mathcal{M}_{\text{ell}}^{\text{or}} \times \text{QuilIsog}$$



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## Proof sketch (ctd).

$$\begin{array}{c} \text{QuilIsog} \\ \downarrow \\ \text{Sub}^h(\widehat{\mathbb{G}}_R^Q) \end{array}$$



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$$\begin{array}{ccc} \mathrm{OrDat}(\widehat{\mathbb{G}}_R^Q/K) & \longrightarrow & \mathrm{QuilIsog} \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{K} & \mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q) \end{array}$$

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$\Rightarrow$  Enough to show that  $\mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q)$  is formal DM.



# Identifying $\mathrm{Isog}^{\mathrm{conn}}$

## Theorem (GN)

$\mathrm{Isog}^{\mathrm{conn}}$  is a formal DM-stack.

## Proof sketch (ctd).

We have a retract:

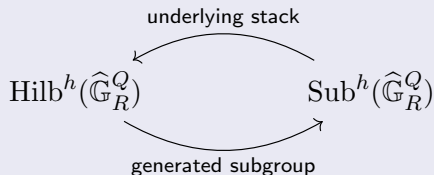


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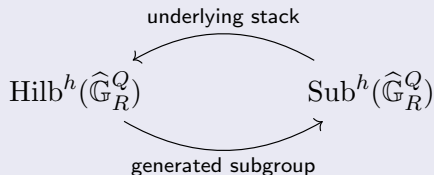


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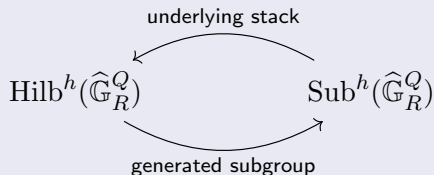


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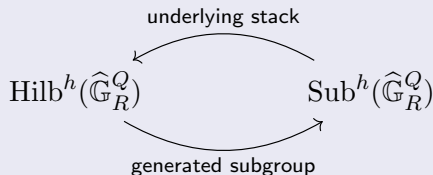


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$\Rightarrow \text{Sub}^h(\hat{\mathbb{G}}_R^Q)$  is formal DM. □



# Thank you!

## References

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