HIGHER LOCAL CLASS FIELD THEORY

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Contents

1. Galois Extensions of Rings

Let $R \to T$ be a map of commutative rings, and let G be a finite group acting on T via R-algebra homomorphisms. Then we have natural maps

$$i : R \to T^G$$

and

$$h : T \otimes_R T \to \prod_G T,$$

with the latter given by the formula

$$t_1 \otimes t_2 \mapsto \{g \mapsto t_1 g(t_2)\}.$$

Definition 1.1. We say that $R \to T$ is a *Galois extension with Galois group G* (or just *G-Galois extension*) if both i and h are isomorphisms.

A ring with no nontrivial finite Galois extensions is called *separably closed*.

Roughly speaking, Galois extensions of rings should be thought of as analogous to unramified extensions of fields—though, of course, this reduces to the usual definition of Galois extension when R is a field. This interpretation is justified by the following theorem of [1].

Theorem 1.2 (Auslander–Buchsbaum). *Let L/K be a finite Galois extension of local or global fields. Then $\mathcal{O}_L/\mathcal{O}_K$ is Galois if and only if every finite prime of K is unramified in L.*

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Since every finite extension of $\mathbb{Q}$ ramifies at a finite prime, it follows that $\mathbb{Z}$ is separably closed. (A full proof is given in [9], where this result is Proposition 10.3.2.) This is problematic if we wish to study extensions of $\mathbb{Z}$, but fortunately, we can get around this by localizing. There are many unramified extension of $\mathbb{Q}_p$, and thus many Galois extensions of $\mathbb{Z}_p$.

Before moving on, let's make a few more useful definitions.

Definition 1.3. Let $R \to T$ be a Galois extension with Galois group G. A *Galois subextension* of $R \to T$ is $R \to T^N$ for any $N \lhd G$. (This will always be a Galois extension.)

Definition 1.4. Let $(R \to T_\alpha)$ be a directed system of Galois subextensions with corresponding inverse system of finite groups (G_α). Then we say that $R \to T = \operatorname{colim}_\alpha T_\alpha$ is a *pro-Galois extension* with Galois group $G = \lim_\alpha G_\alpha$ (carrying the profinite topology).

So, for example, $\mathbb{Z}_p \to \mathcal{O}(\mathbb{Q}_p^{\mathrm{ur}})$ is a pro-Galois extension with Galois group $\widehat{\mathbb{Z}}$.

2. $\mathbb{E}_\infty$-Rings

As a homotopy theorist, I like to work with objects which are similar to the ones you're familiar with, except things are defined up to "coherent homotopy". For example, an ∞-category is like an ordinary category in that it has objects and morphisms, but it also has 2-morphisms (morphisms between morphisms), 3-morphisms (morphisms between 2-morphisms), and so on. The higher morphisms are invertible and should be thought of as being like homotopies. Composition of morphisms is defined up to a 2-morphism, and that 2-morphism is unique up to a 3-morphism, etc. This infinite ascending tower is much stronger than just "unique up to homotopy", and is what is meant by "coherent homotopy".

The homotopical version of a commutative ring is an $\mathbb{E}_\infty$*-ring*. You should think of it as a ring where the laws of unitality, associativity, and commutativity only hold up to coherent homotopy. The "E" stands for "everything", as it has all the operations one might want. Formally, $\mathbb{E}_\infty$ is an *operad*, an object which categorically encodes certain kinds of operations. The classical analogue is the commutative operad Comm, whose algebras are commutative monoids. $\mathbb{E}_\infty$ is in some sense a "resolution" of Comm. We can use it to build the analogue of commutative rings by (mostly) copying the process used to build them from sets, but starting with spaces instead.

$$
\begin{tikzcd}
\mathrm{Set} \arrow[d, squiggly, "\text{Comm-algebras}"] & \mathcal{S} \arrow[d, squiggly, "\mathbb{E}_\infty\text{-algebras}"] \\
\mathrm{CMon} \arrow[d, squiggly, "\text{grouplike objects}"] & \mathbb{E}_\infty(\mathcal{S}) \arrow[d, squiggly, "\text{grouplike objects}"] \\
\mathrm{Ab} \arrow[dd, squiggly, "\text{Comm-algebras}"] & \mathrm{Sp}^{\mathrm{cn}} \arrow[d, squiggly, "\text{don't require boundedness}"] \\
 & \mathrm{Sp} \arrow[d, squiggly, "\mathbb{E}_\infty\text{-algebras}"] \\
\mathrm{CRing} & \mathrm{CAlg}
\end{tikzcd}
$$

Many of the things that we can do with commutative rings can also be done with $\mathbb{E}_\infty$-rings. There's also a comparison between the two. There is a conservative functor $\pi_* : \mathrm{Sp} \to \mathrm{Gr}_{\mathbb{Z}}\mathrm{Ab}$, and an embedding $H : \mathrm{Ab} \to \mathrm{Sp}$ which is right inverse to π_0. Both of these extend to rings.

One big difference is that the base ring is no longer $\mathbb{Z}$, but instead the *sphere spectrum* $\mathbb{S}$. Much as abelian groups are $\mathbb{Z}$-modules, spectra are $\mathbb{S}$-modules; and much as commutative rings are commutative $\mathbb{Z}$-algebras, $\mathbb{E}_\infty$-rings are commutative $\mathbb{S}$-algebras. Modules over $H\mathbb{Z}$ correspond to chain complexes of abelian groups.

3. Galois Extensions of $L_{K(n)}\mathbb{S}$

Theorem 3.1. *$\mathbb{S}$ is separably closed.*

Proof. Let B be a finite Galois extension of $\mathbb{S}$. Then B is a finite spectrum, so $H_*(B)$ is finitely generated; that is, each $H_n(B)$ is finitely generated, and $H_n(B) = 0$ for all but finitely many n. I claim that $H_n(B) = 0$ for *all* $n \neq 0$.

Let n be minimal such that $H_n(B) \neq 0$, and suppose towards contradiction that $n < 0$. Then we have

$$\begin{aligned} H_n(B) \otimes H_n(B) &\cong H_{2n}(B \otimes B) && \text{by the Künneth theorem} \\ &\cong \prod_G H_{2n}(B) && \text{because } \mathbb{S} \to B \text{ is Galois} \\ &= 0. \end{aligned}$$

So $H_n(B) = 0$, which is a contradiction. The $n > 0$ case is similar.

Now, write $T = H_0(B)$. By the Hurewicz theorem, B is connective and $\pi_0(B) = T$. Combining this with the Künneth theorem, we find that $T \otimes T \cong \prod_G T$ and $\mathrm{Tor}_1(T, T) = 0$. Thus, T is a free abelian group of rank $[B : \mathbb{S}]$.

By the definition of the homology of a spectrum, $B \otimes_{\mathbb{S}} H\mathbb{Z} = HT$. This means that $H\mathbb{Z} \to HT$ is a pushout of $\mathbb{S} \to B$ in CAlg, which implies (skipping some details) that $\mathbb{Z} \to T$ is a Galois extension of degree $[B : \mathbb{S}]$. Since $\mathbb{Z}$ is separably closed, this means $[B : \mathbb{S}] = 1$, i.e. $B = \mathbb{S}$. □

To fix this, we should do the same thing as the classical case: localize. But it isn't enough anymore to localize at integer primes.

3.1. **The Balmer Spectrum.** Roughly speaking, we can think of a prime as being "something we can localize at". Over $\mathbb{Z}$, for example, we can localize at 0 or any integer prime. These localizations correspond to the prime fields, which are of course $\mathbb{Q}$ and $\mathbb{F}_p$. This is formalized as the *Balmer spectrum* of the category $\mathcal{D}(\mathbb{Z})^{\mathrm{perf}}$. The Balmer spectrum of the sphere, $\mathrm{Spc}\,\mathrm{Sp}^\omega$, looks like this:

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & K(2,2) \arrow[u] & K(3,2) \arrow[u] & K(5,2) \arrow[u] & \\
\text{chromatic height} & K(2,1) \arrow[u] & K(3,1) \arrow[u] & K(5,1) \arrow[u] & \cdots \\
 & & K(0) = H\mathbb{Q} \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & & \\
 & & \text{prime} & &
\end{tikzcd}
$$

At each prime p, we have a tower of *Morava K-theories* $K(n)$. These are the analogue to prime fields, and come with analogous localization functors on the category of spectra. We denote these functors by $L_{K(n)}$; so, for instance, the base ring of $L_{K(n)}\mathrm{Sp}$ (read:"$K(n)$-local spectra") is $L_{K(n)}\mathbb{S}$ (the $K(n)$-local sphere). Height n Morava K-theory is an associative ring spectrum with $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm}]$, where $|v_n| = 2(p^n - 1)$. Localizing at $\mathbb{F}_p$ is one way to algebraically describe p-completion; we should think of $K(n)$-localization as a refinement of this in the derived context. But if this is our analogue to $\mathbb{F}_p$, what is our analogue to $\mathbb{Z}_p$?

3.2. **Lubin–Tate Spectra.** Let Γ be a formal group over a field k, and let A be a complete local ring with residue class field k. A *deformation* of Γ to A means a formal group $\widetilde{\Gamma}$ over A whose reduction to k is Γ. Geometrically, it's basically an extension of Γ from $\operatorname{Spec} k$ to $\operatorname{Spec} A$. Lubin and Tate's construction can be described as follows.

Theorem 3.2 (Lubin-Tate)**.** *Let k be a perfect field of characteristic $p > 0$ and Γ a formal group over k of finite height n. Then the ring $R_\Gamma = \mathbb{W}(k)[\![v_1, \ldots, v_{n-1}]\!]$ carries a* universal *deformation of Γ. That is, for any other complete local ring $A \to k$, a deformation of Γ to A is given by a map $R_\Gamma \to A$.*

R_Γ is called the *Lubin-Tate ring* of Γ, and the universal deformation over it is called the *Lubin-Tate formal group.* An important example is given by taking $k = \overline{\mathbb{F}}_p$ and Γ to be the unique height n formal group over it. Then $R_\Gamma = \mathcal{O}(\mathbb{Q}_p^{ur})[\![v_1, \ldots, v_{n-1}]\!]$. This is the case discussed by Khaled and Necef.

There is a higher algebra version of this theorem ([6],[8]).

Theorem 3.3 (Goerss–Hopkins–Miller, Lurie)**.** *Let k be a perfect field of characteristic $p > 0$ and Γ a formal group over k of finite height n. Then there is an $\mathbb{E}_\infty$-ring E_Γ carrying the universal deformation of Γ to an* oriented *formal group. E_Γ has the following properties:*

i) $\pi_0 E_\Gamma \cong R_\Gamma$
ii) E_Γ *is $K(n)$-local*
iii) E_Γ *is complex-periodic.*

In the case that $k = \overline{\mathbb{F}}_p$, there is only one possibility for Γ, and we denote the associated Lubin-Tate spectrum by E_n. We think of $K(n)$ as being the "residue class field" for E_n, and this is almost literally true.

Because E_Γ depends functorially on Γ, it has an action of $\mathrm{Aut}_k(\Gamma)$. We can augment this with an action of $\mathrm{Gal}(k/\mathbb{F}_p)$.

Definition 3.4. The nth (big) *Morava stabilizer group* is $\mathbb{G}_n = \mathrm{Aut}_k(\Gamma_n) \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

This group, which can be defined more generally for any formal group over a positive characteristic field, is the automorphism group of Γ_n as a formal group together with its base field. As described above, it has a natural action on E_n. Since E_n is $K(n)$-local, it has a unit map from the $K(n)$-local base ring $L_{K(n)}\mathbb{S}$.

Theorem 3.5 (Devinatz–Hopkins, [5])**.** *$L_{K(n)}\mathbb{S} \to E_n$ is a pro-Galois extension with Galois group $\mathbb{G}_n$.*

Theorem 3.6 (Baker–Richter, [2])**.** *E_n is the separable closure of $L_{K(n)}\mathbb{S}$.*

This fundamental result allows us to study chromatic homotopy theory using results from arithmetic geometry. We can take the fixed-point spectral sequence for the action of $\mathbb{G}_n$ to get the E_n-based Adams-Novikov spectral sequence

$$E_2^{s,t} = H^s_{\mathrm{cts}}(\mathbb{G}_n, \pi_t E_n) \Rightarrow \pi_{t-s} L_{K(n)}\mathbb{S}.$$

This spectral sequence is one of the central objects of study in computational stable homotopy theory. Some excellent recent examples of using arithmetic geometry to access information about it can be found in a recent pair of papers by Barthel–Schlank–Stapleton–Weinstein ([3],[4]).

Recall that we are supposed to think of Galois extensions of rings as being like unramified extensions of fields. From that perspective, these results tell us that the "maximal unramified extension" of the $K(n)$-local base ring is E_n, a ring whose π_0 is a power series ring over $\mathbb{Z}_p^{ur}$. Something very beautiful and very mysterious is going on here, but I feel confident at least in saying that we are studying a higher-algebraic analogue of local Galois theory. To truly have a version of local class field theory, though, we'll need to go a little bit further and look at the $K(1)$-local Picard group.

4. A Higher Local Reciprocity Theorem

Recall the main theorem of local class field theory.

Theorem 4.1. *Let L/K be a finite Galois extension of non-Archimedean local fields. Then we have a canonical reciprocity isomorphism*

$$K^\times/\mathrm{Nm}_{L/K} L^\times \cong \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

which sends the uniformizer π_K to the Frobenius element.

For the maximal unramified extension, this sends π_K to the standard topological generator of $\mathrm{Gal}(K^{ur}/K) \cong \widehat{\mathbb{Z}}$.

Now, consider the following theorem, which is Proposition 2.1 in [7].

Theorem 4.2. *Consider the short exact sequence of abelian groups*

$$0 \to M \to \mathrm{Pic}(L_{K(1)}\mathbb{S}) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

where the quotient map sends an invertible spectrum X to 0 if the generator of $(E_1)_*X$ is even-dimensional and 1 if it is odd-dimensional.[1] *Define a map* $\mathrm{ev} : M \to \mathbb{Z}_p^\times$ *which sends a spectrum X to the eigenvalue of ψ^γ on $(E_1)_0(X)$, where γ is a topological generator of $\mathbb{Z}_p^\times$. Then* ev *is an isomorphism.*

Remark 4.3. The above is the version of the theorem for odd primes. There is another version for $p = 2$, which replaces $\mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/8\mathbb{Z}$ and E_1 with its fixed points under a certain C_2 action. This corresponds to the fact that we need to replace complex K-theory, which is 2-periodic, with its C_2-fixed points KO, which is 8-periodic.

I claim that ev is a reciprocity isomorphism for the maximal unramified extension of the sphere spectrum. To start with, it is a fact that $\mathbb{G}_1 = \mathbb{Z}_p^\times$. Thus we have an isomorphism from an index 2 subgroup of the Picard group of $L_{K(1)}\mathbb{S}$, which generalizes the class group, to the Galois group of the maximal unramified extension. This is an isomorphism between the right things, at least.

The map itself is given by taking the eigenvalue of a certain Adams operation. Adams operations are power operations, which generalize the Frobenius morphism (as well as Hecke operators and whatnot—but at height 1, it's an analogue to the Frobenius). In fact, Adams operations literally reduce to Frobenius maps once you take π_0 and reduce mod p. So in essence, we're taking an element of the class group of our K, and seeing how it transforms under the Frobenius of K^{ur}. For a non-Archimedean local field, this is a way of describing the reciprocity isomorphism for K^{ur}/K, since the Frobenius generates its Galois group. So the result above really is a reciprocity isomorphism for the $K(n)$-local sphere!

If you want to take this further, you could interpret this as sending a topological $GL_1(\mathbb{Z}_p)$-newform to the character of its Hecke representation, which describes a $K(1)$-local Galois representation. This is the base example for my conjectured theory of *topological modular Galois representations*, which is central to my chromatic Langlands program. Alternately, if you want to globalize this, you can use the fact that E_1 is nothing more than p-completed complex K-theory. Thus an analogue of this result with E_1 replaced by KU ought to give us a global unramified reciprocity theorem at height 1.

My current research combines these two goals. It focuses on producing Hecke operators for elliptic cohomology, which is the higher-algebraic analogue to modular forms and is known to approximate $L_{K(2)}\mathbb{S}$ at all primes.

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[3] Tobias Barthel et al. *On Hopkins' Picard Group*. July 30, 2024. arXiv: 2407.20958. Pre-published.

[4] Tobias Barthel et al. *On the Rationalization of the K(n)-Local Sphere*. Sept. 9, 2025. arXiv: 2402.00960. Pre-published.

[1] Technically we don't want the E_1-homology, but rather a slight modification of it which agrees with E_1-homology when $(E_1)_*X$ is finitely-generated over $(E_1)_*$. I'll ignore this.

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