ELLLIPTIC COHOMOLOGY AND CONFORMAL FIELD THEORIES

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Abstract. One of the main goals of contemporary quantum field theory research is to compute and understand the path integrals associated to conformal field theories. In the 1980s, Witten and some others discovered that these values could be interpreted as certain numerical invariants called *elliptic genera*, cohomological objects which contain deep information about geometric structures on manifolds. In this talk, I will describe the theory of elliptic genera and how this suprising relationship sheds light on the underlying phenomena of conformal field theory.

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Introduction

The main sources for this talk are [5] and [7].

This story begins with the theory of genera, very classical topological invariants of manifolds. These objects admit both analytic and cohomological descriptions, and we will need both of them to understand the subclass of so-called *elliptic* genera. Equipped with this Atiyah-Singer-type relationship and the associated correspondence between elliptic objects and spin geometry, we'll enter the slightly shaky (but highly promising) grounds of conformal field theory, which will be described as a kind of equivariant cohomology. Finally, I'll explain how this correspondence can be categorified, lending it a lot more structure and motivating the Stolz-Teichner program.

1. Elliptic Genera

I'll start with a definition.

Definition 1.1. A *genus* is a ring homomorphism $\Phi : \Omega \to \mathbb{C}$, where Ω is the oriented cobordism ring.

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A priori, this might seem like a very complicated object. By taking our homomorphism to be valued in a field, however, we have effectively rationalized the problem, which greatly simplifies it. Because $\Omega \otimes \mathbb{Q}$ is a free polynomial algebra on $\{\mathbb{CP}^{2n} \mid n \in \mathbb{Z}^+\}$, a genus is the same as a choice of complex number for each positive even integer. This sequence can be expressed by its *logarithmic generating function* $\log_\Phi(x) = \sum_{n\in\mathbb{Z}^+} \frac{1}{2n+1}\Phi(\mathbb{CP}^{2n})x^{2n+1}$, so genera correspond precisely to formal power series over $\mathbb{C}$ whose even-degree terms are 0.

We will be interested in the genera whose logarithm is, roughly speaking, the functional inverse of an elliptic function. More precisely:

Definition 1.2. A genus Φ is called *elliptic* if its logarithm has the form $\log_\Phi(x) = \int_0^x (1 - 2\delta t^2 + \varepsilon t^4)^{-1/2}\, dt$.

This is called an "elliptic integral of the first kind", and it extends to a holomorphic function on an appropriate domain. If its discriminant $\varepsilon^2(\delta^2 - \varepsilon)$ is nonzero, then it is (locally) invertible, with the inverse function being an odd elliptic function with a unique order-two zero ω. This is called an odd Jacobi elliptic function, and it is uniquely characterized by ω together with its period lattice L. (In the degenerate cases, the two periods fail to be linearly independent and thus give rise to sin or tanh.)

It can be shown that elliptic genera admit a homogeneity property with respect to the parameters: multiplying δ by λ^2 and ε by λ^4 sends L to $\lambda^{-1}L$ and $\Phi(M)$ to $\lambda^{\frac{1}{2}\dim M}\Phi(M)$ (for any manifold M). This has two important implications. First, it shows that certain elliptic genera are equivalent up to scaling, and so it is no loss of generality to only consider genera up to normalization; and second, it shows that $\Phi(M)$ is a *modular form*! Specifically, it is a modular form of weight $1/2$ under the subgroup $\Gamma_0(2) \subset SL_2(\mathbb{Z})$ of the (non-normalized) modular group that preserves ω under the natural action associated to L. So we can think of $\Phi(M)$ as being a "modular form with respect to L and ω". Whichever parameters we use, this shows that we have a *universal elliptic genus* which includes all elliptic genera and is valued in the ring of modular forms rather than $\mathbb{C}$. This is also called the *Witten genus.*

As a consequence of this discussion (and because complex elliptic curves are classified by their lattice up to scaling), a nondegenerate elliptic genus corresponds to an elliptic curve with a choice of degree-two point. But since tori are parallelizable, the canonical divisor of an elliptic curve is its identity element, and hence a theta characteristic (square root of the canonical bundle) is a point of order two. A Serre spectral sequence argument ([3]) shows that theta characteristics correspond bijectively to spin structures; roughly speaking, a spin structure is a special kind of double cover of the tangent bundle, which corresponds to a square root in cohomology by some basic obstruction theory. Thus, normalized nondegenerate elliptic genera correspond precisely to *elliptic curves with spin structure!* What's more, compactifying the coarse moduli space $\mathbb{H}/\Gamma_0(2)$ of such objects (or, indeed, the actual moduli stack) adds in two cusps which correspond to the two kinds of degenerate elliptic genus: the signature and the $\widehat{A}$-genus.

2. The analytic and topological descriptions

Historically, the interest in elliptic genera arose from the following theorem.

Theorem 2.1. *A genus is elliptic if and only if it is multiplicative with respect to spin bundles having compact connected structure group.*

Viewing genera as decategorified cohomology, this enhanced multiplicativity is analogous to how the Leray-Hirsch theorem generalizes the Künneth theorem. In fact, this is the right point of view: as we will see, this multiplicativity arises from orientation in generalized cohomology. To get there, though, we first have to describe genera as characteristic numbers.

Theorem 2.2. *Each genus Φ can be written as $\Phi(M) = \langle \varphi(TM), [M] \rangle$ for a unique stable multiplicative characteristic class $\varphi \in H^*(B\,\mathrm{O}; \mathbb{C})$ concentrated in even degree; and every such class gives rise to a genus in this way.*

(*Stable* means it sends trivial bundles to $1 \in H^*(M; \mathbb{C})$, and *multiplicative* means that the associated formal power series satisfies an analogue of the Whitney product formula.)

Some straightforward computations with the canonical bundles over projective spaces show that, identifying $H^*(\mathbb{CP}^\infty; \mathbb{C}) \cong \mathbb{C}[[c_1]]$ with the ring of power series in one variable, the inverse power series to $\log_\Phi(x)$ is $c_1/\varphi(\gamma)$ (where γ is the underlying real bundle of universal complex line bundle). Thus, nondegenerate elliptic genera are classified as those characteristic classes φ such that $c_1/\varphi(\gamma)$ is a Jacobi elliptic function.

Examples.

(1) The signature of a $4k$-manifold is defined to be the signature of the quadratic form induced on H^{2k} by the cup product and Poincarè duality. (It is defined to be zero on manifolds of dimension not divisible by 4.) This defines an elliptic genus, and in fact this genus corresponds to the degenerate case $\omega = 0$.

(2) The $\widehat{A}$-genus is defined abstractly to be the map on homotopy groups induced by a certain map $M\,\mathrm{Spin} \to K\,\mathrm{O} \otimes \mathbb{Q}$ of E_∞-rings called the Atiyah-Bott-Shapiro orientation (originally proven for the underlying homotopy-commutative ring spectra in [2]). Heuristically, this map is given by sending a spin manifold to the Hilbert space of L^2 sections of its spin bundle; the details can be found in ([1]). Of course, this only defines the $\widehat{A}$-genus for spin manifolds. However, the natural reduction from spin cobordism to oriented cobordism becomes an equivalence of rings $MSpin \otimes \mathbb{Q} \simeq M\,\mathrm{SO} \otimes \mathbb{Q}$ upon rationalization (in fact, this only requires the inversion of 2). Consequently, the ABS orientation extends to all manifolds, defining the degenerate elliptic genus corresponding to the case $\omega = \infty$.

(3) The Witten genus, as defined above, arises similarly from a map of E_∞-rings $M\,\mathrm{String} \to \mathrm{tmf}$ called the "topological Witten genus" ([1]).

If you're a topologist, these descriptions should already have you thinking of chromatic homotopy theory. (The last two should, at least. The signature also admits a cohomological refinement as an E_1 orientation in "L-theory", the K-theory of quadratic forms, although it is not known whether this orientation is E_∞.) The genera also have analytic descriptions, however, due to the Atiyah-Singer index theorem.

(1) The signature of a $4k$-manifold is equivalently the index of the *signature operator*, defined as follows. Fix a metric on M, which yields a Hodge star operator and thus allows us to define the adjoint d^* of the exterior

derivative d. Then the Kähler-Dirac operator $d + d^*$ swaps the $+1$ and -1-eigenspaces of the operator τ which acts on $\Omega^p(M)$ by $(-1)^{k+p(p-1)/2}\star$, and we take the map $D : \Omega_+ \to \Omega_-$ to be the signature operator. It is standard Hodge theory that D is elliptic.

(2) The $\widehat{A}$-genus of a spin manifold can be constructed similarly by replacing the Kähler-Dirac operator with the Dirac operator intertwining the two chiral summands of the spin bundle. (In fact, if we do this construction on an arbitrary manifold with the spinor bundle replaced by the square root of the exterior algebra of the complexification, we get the signature.)

(3) We would like to similarly exhibit the Witten genus as the index of a Dirac operator with some kind of "thickened" structure corresponding to the choice of (δ, ε). This is where conformal field theory enters the picture: we will find that, granted some theory which is still being developed, Φ_W is the equivariant index of a certain operator on the loop space of a manifold.

Explicitly, these constructions arise as follows. We have an isomorphism $K^0 \otimes \mathbb{C} \cong H^{\text{even}}(-;\mathbb{C})$ given by the Chern character. (This isomorphism can be checked on spheres, where it follows from the clutching construction and Bott periodicity.) Using this, we can represent an even-dimensional characteristic class φ as a morphism $\Lambda_\varphi : K\,\mathrm{O} \to K^0 \otimes \mathbb{C}$, and so applying the Pontryagin-Thom construction ("fiber integration in generalized cohomology") gives us an equivalent way of representing genera via characteristic classes (originally due to Hirzebruch):

Theorem 2.3. *Each genus Φ can be represented uniquely as $\Phi(M) = \pi_!^M(\Lambda_\varphi(TM))$, where φ is as above and $\pi_!^M : K^0(M)\otimes\mathbb{C} \to \mathbb{C}$ is the Gysin map on K-theory induced by $M \to *$.*

This is, by definition, the topological index of any Dirac operator lifting the characteristic class. Applying this gives us an explicit formula for the Witten genus in terms of $q = \exp(\omega)$ ([5]), which shows that it is an integral power series in q for any spin manifold. In fact, writing this out shows that it is the character of a virtual projective unitary representation $E_M^+ - E_M^-$ of $\mathrm{Diff}(S^1)$, which looks suspiciously like a difference of chiral spin representations (but we'll return to that later). It also allows us to lift this theory to the equivariant context:

Theorem 2.4. *Fix a Lie group G. Then each genus Φ induces a G-equivariant genus Φ_G valued in the character ring $R(G) \otimes \mathbb{C}$, and Φ is elliptic iff Φ_G is valued in the subring of constant characters.*

This brings us to the relationship to field theory.

3. Conformal Field Theory

In the geometry underlying field theory, there is a fundamental sequence of more and more restrictive symmetry structures. Beyond the structure of a smooth manifold, one can impose orientability, and beyond that one can impose spin structure, string structure, and so on. Mathematically, these structures are given by the Whitehead tower for the classifying space of the orthogonal group: $\cdots \to B\,\mathrm{Fivebrane} \to B\,\mathrm{String} \to B\,\mathrm{Spin} \to B\,\mathrm{SO} \to B\,\mathrm{O}$. (These are actually all infinite loop spaces, so this lifts to a Whitehead tower of spectra.) Segal relates the structure of an oriented Riemannian manifold M to the structure of its loop manifold $\mathcal{L}M = C^\infty(S^1, M)$. (Note that $\mathcal{L}M$ admits a canonical effective S^1-action whose fixed point set is the canonical embedding of M; and, moreover, the tangent space to a loop $\gamma \in \mathcal{L}M$ is just the space of ambient vector fields on the image of γ.)

Theorem 3.1. *$\mathcal{L}M$ is orientable iff M is spin, and $\mathcal{L}M$ is spin iff $p_1(M) = 0$.*

Proof. By polarizing the tangent bundle of $\mathcal{L}M$ using $\frac{\partial}{\partial\theta}$, we reduce its structure group to $O_{res}(T\mathcal{L}M) \simeq \varinjlim_n O(2n)/U(n) \simeq \Omega \varinjlim_n SO(2n)$. Then the first statement follows by a computation with the Serre spectral sequence.

To prove the second, we use the projective spin representation of O_{res} described by Segal and Pressley in [6], whose corresponding central extension is not along $\mathbb{Z}/2$ (as in the finite-dimensional case) but rather S^1. The cohomology class classifying this extension is $p_1(M)$. □

Sadly, the vanishing of the first Pontryagin class is not always equivalent to the existence of a string structure (although it is in the case that M is spin and of dimension at least 6; see [4]). Witten proposed ([7]) that the existence of a string structure on M should be equivalent to the existence of an *equivariant* spin structure on $\mathcal{L}M$. In fact, applying the localization formula for equivariant K-theory in this case (assuming that the Dirac operator exists), one finds the following incredible fact.

Theorem 3.2. *Suppose $\mathcal{L}M$ has an S^1-equivariant spin structure. Then the equivariant index of the Dirac operator on $\mathcal{L}M$ is the Witten genus of M.*

Recall from earlier that we were able to exhibit ordinary elliptic genera as indices of Dirac operators associated to some kind of spin structure. This theorem tells us that we can do something similar with the *universal* elliptic genus; we just have to "thicken" M first (to $\mathcal{L}M$).

I'd like to digress for a moment, and use this as an opportunity to discuss the philosophy of conformal field theory. The main goal when studying any quantum field theory is to compute its partition function and operator insertions, which is an integral over some space of paths. The issue is the symmetry involved, which introduces lots of infinities and redundancies that need to be cleverly folded together. From a *topological* perspective, this is exactly a problem of *equivariant cohomology.* Indeed, the famous localization formula can be viewed as a special case of the general paradigm of Mackey functors in equivariant homotopy theory (functors parameterized by the subgroup lattice) where the structure collapses down to a single (improper) subgroup. As any equivariant algebraic topologist can attest, the general version of this problem is extremely difficult, and has not really been developed for actions of infinite groups yet.

From this point of view, then, it rather miraculous that Witten managed to come up with a concrete realization of S^1-equivariant cohomology in this way. In principle, at least, this also explains the modular structure of the Witten genus (and thus of elliptic genera in general): the index of the Dirac operator in the finite-dimensional case can be computed as an integral over the loop manifold, and so one would like to imagine that it works in the infinite-dimensional case as well, yielding an integral over the torus manifold $\mathcal{L}^2M$. Since complex elliptic curves are precisely tori, this should give us the modular form structure we know that elliptic genera have. This is, as Segal puts it, "obviously the right explanation", but it is phrased in terms of a mathematical theory which is not fully developed yet.

Actually computing this index as an integral is questionable, and even defining the Dirac operator $\mathcal{D}$ is difficult. If we can construct an appropriate Hilbert space of spinor fields on $\mathcal{L}M$, however, it will admit a natural projective representation of $\mathrm{Diff}(S^1)$, and this will split into (the diagonal of) a representation of $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$ with respect to which the chiral splitting is supersymmetric. This explains our character formula from earlier! While constructing this Hilbert space is problematic in general, Segal carries out an analogous construction for the normal bundle of $M \subset \mathcal{L}M$ and shows that it behaves properly. I'll skip the details, but suffice to say that it provides evidence for Witten's conjecture, at least heuristically.

4. Elliptic Cohomology

Before concluding this talk, I'd like to describe how this relates to actual cohomology, in the generalized Eilenberg-Steenrod sense. Recall that the inverse of $\log_\Phi$ is an elliptic function when Φ is a nondegenerate elliptic genus. We have a similar result for the logarithm of the universal elliptic genus Φ_W; call its inverse $s = s_{\delta,\varepsilon}$. We know from the discussion of genera as characteristic classes that this parameterized elliptic function can naturally be thought of as a power series in the generator $c_1 \in H^2(\mathbb{CP}^\infty; \mathbb{C}[\delta, \varepsilon])$ of the cohomology algebra of $\mathbb{CP}^\infty$. The group structure of $\mathbb{CP}^\infty$ (which arises from its role as the classifying space for the Picard group) then gives a formal group law on $\mathbb{Z}[1/2][\delta, \varepsilon, \varepsilon^{-1}]$, and a computation (which I'll skip) shows that it satisfies Landweber's flatness condition and thus lifts to a complex-oriented cohomology theory. This is one construction of the spectrum called topological modular forms, or tmf. In fact, it can be shown ([1]) that this is an E_∞ ring spectrum, and the Atiyah-Bott-Shapiro orientation of the $\widehat{A}$-genus lifts to a string orientation of tmf, $M\,\mathrm{String} \to \mathrm{tmf}$, which is a categorification of the Witten genus. In particular, this means that the Witten genus is defined not only on the spin cobordism ring, but the *string* cobordism ring! This is why, in the Stolz-Teichner program and the "cobordism category" approach to QFT, the relationship between ordinary cohomology and oriented cobordism describing conformal field theory in dimension 0 (given by the Conner-Floyd isomorphism) and between K-theory and spin cobordism describing conformal field theory in dimension 1 (given by the ABS orientation) extends to a relationship between tmf and string cobordism which should, in principle, describe conformal field theory in dimension 2.

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