Isogenies of Oriented Elliptic Curves

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Some Preliminary Notes

Conventions/Terminology

- Stack = étale sheaf of ∞ -groupoids on CAlg_R
- DM-stack (algebraic space) = spectral Deligne-Mumford stack (spectral algebraic space), not necessarily connective
- Formal DM-stack = formal filtered colimit of DM-stacks along closed immersions; called "honest" if actual DM-stack.
 Formal algebraic spaces defined similarly
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Related Work

Xuecai Ma and Yifei Zhu have a paper in the works ([MZ25]) which approaches this from a different perspective, defining level structures in terms of classical divisors. It isn't clear whether this is equivalent to my definition. A draft can be found on Professor Zhu's website.

Over $\mathbb C$

$$\left\{ \begin{array}{c} \mathcal{M}(\Gamma) \\ \downarrow_{\text{\'et}} \\ \mathcal{M}_{ell} \end{array} \right\} \longleftrightarrow \left\{ \text{Congruence subgroups } \Gamma \subset GL_2(\mathbb{Z}) \right\}$$



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Warning

It is not known whether Isog is an honest DM-stack.



Factorization Theorem (GN)

There is an orthogonal factorization system $(\mathscr{C}onn, \mathscr{E}t)$ on $\mathrm{Ell}^{\mathrm{or}}_{\mathrm{Isog}}$ such that

- ullet $\mathscr{C}onn$ is the class of connected isogenies, and
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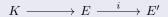


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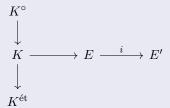


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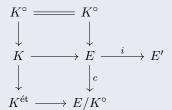


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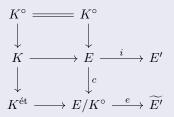


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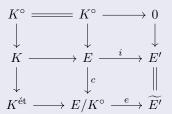


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Elliptic Rigidity Theorem, classical version ([KM85])

Zariski-locally on the base, every morphism of classical elliptic curves is either 0 or an isogeny.



Elliptic Rigidity Theorem, spectral version (GN)

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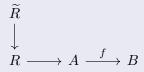
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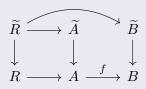




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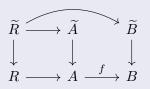




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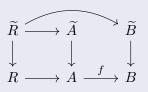


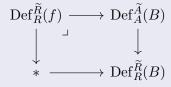


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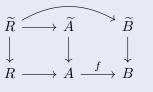




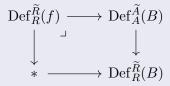
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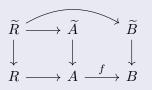


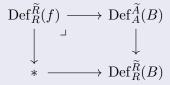
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Main idea: 0 map deforms uniquely through square-zero extensions \Rightarrow through Postnikov tower.





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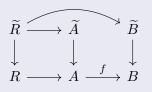
Digression: Why are the components isogenies?

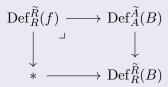
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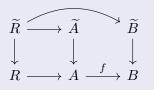
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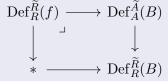
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$$\Rightarrow B\otimes_A L_{A/R} \text{ vanishes}$$

$$\Rightarrow \operatorname{Def}_{R}^{\widetilde{R}}(f) = 0.$$



Connected-Étale Factorization

Corollary

We have a pullback of functors

$$\begin{array}{ccc} \operatorname{Isog} & \longrightarrow & \operatorname{Isog}^{\operatorname{\acute{e}t}} \\ \downarrow & & \downarrow^s \\ \operatorname{Isog}^{\operatorname{conn}} & \xrightarrow{t} & \mathcal{M}_{\operatorname{ell}}^{\operatorname{or}}. \end{array}$$



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Just need to show that $Isog^{\acute{e}t}, Isog^{conn}$ are formal DM-stacks.



$\overline{\mathsf{Identifying}}\ \overline{\mathsf{Isog}}^{\mathrm{\acute{e}t}}$

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 $\rm Isog^{\acute{e}t}$ is a DM-stack.



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Proof sketch.

 $\textbf{0} \ \ [\mathsf{KM85}] \colon \{(E,K) \mid E \ \mathsf{elliptic} \ \mathsf{curve}, K \subset E \ \mathsf{finite}\} \to (\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}})^{\heartsuit} \\ \mathsf{relative} \ \mathsf{scheme}.$



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- $\bullet \text{ [KM85]: } \{(E,K) \mid E \text{ elliptic curve}, K \subset E \text{ finite}\} \rightarrow (\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}})^{\heartsuit}$ relative scheme.
- $\textbf{ 2} \ \{(E,K) \mid E \ \text{elliptic curve}, K \subset E \ \text{finite \'etale} \} \ \text{open substack}.$



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- **①** [KM85]: $\{(E,K) \mid E \text{ elliptic curve}, K \subset E \text{ finite}\} \rightarrow (\mathcal{M}_{ell}^{or})^{\heartsuit}$ relative scheme.
- $\ \ \, \textbf{ ($E,K$)} \mid E \text{ elliptic curve}, K \subset E \text{ finite \'etale} \textbf{) open substack}.$
- Leverage étaleness and use ([Lur18c], Theorem 18.1.0.2) to lift from classical to spectral.



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Isog^{conn} is a formal algebraic space.



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Identifying $\mathrm{Isog}^{\mathrm{conn}}$

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$$\{E \xrightarrow{\operatorname{conn}} E'\}$$

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$$\{K \subset \widehat{\mathbb{G}}_R^Q \text{ honest subgroup} \qquad \}$$



Identifying Isogconn

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$$\{E \xrightarrow{\mathrm{conn}} E'\}$$

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$$\{K \subset E \text{ closed, proper, connected AND equiv } \widehat{E/K} \simeq \widehat{\mathbb{G}}_R^Q\}$$

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$$\{K \subset \widehat{\mathbb{G}}_R^Q \text{ honest subgroup AND equiv } \widehat{\mathbb{G}}_R^Q/K \simeq \widehat{\mathbb{G}}_R^Q\}$$



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$$\Rightarrow \mathrm{Isog}^{\mathrm{conn}} \simeq \mathcal{M}_{\mathrm{oll}}^{\mathrm{or}} \times \mathrm{QuilIsog}$$



Theorem (GN)

 $\operatorname{Isog^{conn}}$ is a formal algebraic space.

Proof sketch (ctd).

QuilIsog $\downarrow \\ \operatorname{Sub}^h(\widehat{\mathbb{G}}_R^Q)$

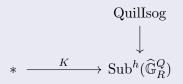


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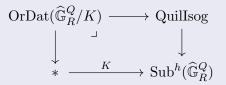


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$$\begin{array}{ccc} \operatorname{OrDat}(\widehat{\mathbb{G}}_R^Q/K) & \longrightarrow & \operatorname{QuilIsog} \\ & & & \downarrow & & \downarrow \\ & * & \stackrel{K}{\longrightarrow} & \operatorname{Sub}^h(\widehat{\mathbb{G}}_R^Q) \end{array}$$

[Lur18b]: $OrDat(\widehat{\mathbb{G}}_R^Q/K)$ is an affine DM-stack.



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[Lur18b]: $\operatorname{OrDat}(\widehat{\mathbb{G}}_R^Q/K)$ is an affine DM-stack.

 \Rightarrow Enough to show that $\mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q)$ is formal algebraic space.



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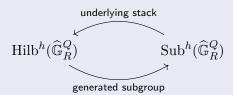
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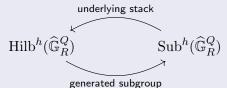


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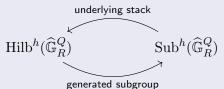


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 $\Rightarrow \operatorname{Hilb}^h(\widehat{\mathbb{G}}_R^Q)$ is formal algebraic space.



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$$\operatorname{Hilb}^h(\widehat{\mathbb{G}}_R^Q) \qquad \operatorname{Sub}^h(\widehat{\mathbb{G}}_R^Q)$$
 generated subgroup

[Lur04]: Hilb of separated algebraic space is algebraic space.

- $\Rightarrow \operatorname{Hilb}^h(\widehat{\mathbb{G}}_R^Q)$ is formal algebraic space.
- $\Rightarrow \operatorname{Sub}^h(\widehat{\mathbb{G}}_R^{\widehat{Q}})$ is formal algebraic space.



Thank you!

References [KM85] Nicholas M. Katz and Barry Mazur. Arithmetic Moduli of Elliptic Curves. 108. Princeton University Press, 1985. [Lur04] Jacob Lurie. "Derived Algebraic Geometry". PhD thesis. Massachusetts Institute of Technology, 2004, URL: http://oastats.mit.edu/handle/1721.1/30144. [Lur18a] Jacob Lurie. Elliptic Cohomology. 2018. URL: https://www.math.ias.edu/~lurie/papers/Elliptic-I.pdf. Pre-published. [Lur18b] Jacob Lurie. Elliptic Cohomology II: Orientations. Apr. 2018. URL: https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf. Pre-published. [Lur18c] Jacob Lurie. Spectral Algebraic Geometry. 2018. URL: https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf. [MZ25] Xuecai Ma and Yifei Zhu. Spectral Moduli Problems for Level Structures and an Integral Jacquet-Langlands Dual of Morava E-theory. 2025. URL: https://vifeizhu.github.io/sagreal.pdf. Pre-published.

