Isogenies of Oriented Elliptic Curves

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Some Preliminary Notes

Conventions/Terminology

- Stack = étale stack of ∞-groupoids on CAlg_R for some $\mathbb{E}_\infty$-ring R
- DM-stack = spectral Deligne-Mumford stack, not necessarily connective
- Formal DM-stack = formal filtered colimit of DM-stacks; called "honest" if actual DM-stack
- Isogeny = strict abelian variety map which is finite, flat, and locally almost of finite presentation

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Related Work

Xuecai Ma and Yifei Zhu have a paper in the works ([MZ25]) which approaches this from a different perspective, defining level structures in terms of classical divisors. It isn't clear whether this is equivalent to my definition. A draft can be found on Professor Zhu's website.

Background: Classical $\mathcal{M}_{\text{ell}}$

Over $\mathbb{C}$

$$\left\{ \begin{array}{c} \mathcal{M}(\Gamma) \\ \Big\downarrow \text{ét} \\ \mathcal{M}_{\text{ell}} \end{array} \right\} \longleftrightarrow \{\text{Congruence subgroups } \Gamma \subset GL_2(\mathbb{Z})\}$$

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Solution: Work with moduli interpretation directly!

The Moduli of Isogenies

Definition

The *moduli stack of isogenies* over R is the functor $\mathrm{Isog} : \mathrm{CAlg}_R \to \mathcal{S}$ given by

$$A \mapsto \left\{ i : E \to E' \,\middle|\, \begin{array}{l} E, E' \in \mathcal{M}^{\mathrm{or}}_{\mathrm{ell}}(A), \\ i \text{ isogeny.} \end{array} \right\} \tag{1}$$

(The isogenies are not required to preserve the orientation.)

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Main Theorem (GN)

Isog is a formal DM-stack.

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Warning

It is not known whether Isog is an honest DM-stack.

Connected-Étale Factorization

Factorization Theorem (GN)

There is an orthogonal factorization system $(\mathscr{C}onn, \acute{\mathscr{E}}t)$ on $\mathrm{Ell}^{\mathrm{or}}_{\mathrm{Isog}}$ such that

- $\mathscr{C}onn$ is the class of connected isogenies, and
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This factorization system is natural with respect to change of base.

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$$E \xrightarrow{i} E'$$

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K^\circ \arrow[d] & & \\
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$$\begin{tikzcd}
K^\circ \arrow[r, equal] \arrow[d] & K^\circ \arrow[r] \arrow[d] & 0 \arrow[d] \\
K \arrow[r] \arrow[d] & E \arrow[r, "i"] \arrow[d, "c"] & E' \arrow[d, equal] \\
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$\square$

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Proof sketch.

True for $\mathbf{P}$-divisible groups ([Lur18a])
$\Rightarrow$ True for abelian varieties.

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So $\operatorname{coker}(f : G \to H) = (\ker(f^\vee))^\vee$.

Digression: Why are the components isogenies?

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Elliptic Rigidity Theorem, classical version ([KM85])

Zariski-locally on the base, every morphism of classical elliptic curves is either 0 or an isogeny.

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Proof sketch.

Main idea: 0 map deforms uniquely through square-zero extensions $\Rightarrow$through Postnikov tower.

Connected-Étale Factorization

Corollary

We have a pullback of functors

$$\begin{tikzcd} \mathrm{Isog} \arrow[r] \arrow[d] & \mathrm{Isog}^{\text{ét}} \arrow[d, "s"] \\ \mathrm{Isog}^{\text{conn}} \arrow[r, "t"] & \mathcal{M}_{\text{ell}}^{\text{or}}, \end{tikzcd}$$

where s is the source map and t the target map.

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where s is the source map and t the target map.

If we can show that $\mathrm{Isog}^{\text{ét}}$ and $\mathrm{Isog}^{\text{conn}}$ are formal DM-stacks, it will follow that Isog is one as well.

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$$\begin{tikzcd} \widehat{E} \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & \widehat{E'} \arrow[d, phantom, "\wr"] \\ \widehat{\mathbb{G}}_R^Q \arrow[r, "\sim"] & \widehat{\mathbb{G}}_R^Q \end{tikzcd}$$

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Reduce to studying automorphisms of $\widehat{\mathbb{G}}_R^Q$.

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Untangling these two parts of an isogeny allows us to classify it much more easily.

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Theorem (GN)

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1. [KM85]: $\{(E, K) \mid E$ elliptic curve, $K \subset E$ finite$\} \to (\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}})^{\heartsuit}$ relative scheme.

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2. $\{(E, K) \mid E \text{ elliptic curve}, K \subset E \text{ finite étale}\}$ open substack.
3. Leverage étaleness and use ([Lur18c], Theorem 18.1.0.2) to lift from classical to spectral. $\square$

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$$\{K \subset E \text{ closed, proper, connected}\}$$

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$\Rightarrow \mathrm{Isog}^{\mathrm{conn}} \simeq \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}} \times \mathrm{QuilIsog}$

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Proof sketch (ctd).

$$\begin{tikzcd} \mathrm{QuilIsog} \arrow[d] \\ \mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q) \end{tikzcd}$$

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\mathrm{OrDat}(\widehat{\mathbb{G}}_R^Q/K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{QuilIsog} \arrow[d] \\
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[Lur18b]: $\mathrm{OrDat}(\widehat{\mathbb{G}}^{Q}_{R}/K)$ is an affine DM-stack.

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[Lur18b]: $\mathrm{OrDat}(\widehat{\mathbb{G}}_R^Q/K)$ is an affine DM-stack.
$\Rightarrow$ Enough to show that $\mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q)$ is formal DM.

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$$\begin{tikzcd} \mathrm{Hilb}^h(\widehat{\mathbb{G}}_R^Q) \arrow[rr, bend right, "\text{generated subgroup}"'] & & \mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q) \arrow[ll, bend right, "\text{underlying stack}"'] \end{tikzcd}$$

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Theorem (GN)

$\mathrm{Isog}^{\mathrm{conn}}$ is a formal DM-stack.

Proof sketch (ctd).

We have a retract:

$$\begin{tikzcd} \mathrm{Hilb}^h(\widehat{\mathbb{G}}_R^Q) \arrow[rr, bend right, "\text{generated subgroup}"'] & & \mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q) \arrow[ll, bend right, "\text{underlying stack}"'] \end{tikzcd}$$

[Lur04]: Hilb is DM.
$\Rightarrow \mathrm{Hilb}^h$ is formal DM.
$\Rightarrow \mathrm{Sub}^h$ is formal DM. $\square$

Thank you!

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