Errata for *Foundations of Stable Homotopy Theory*

Doron Grossman-Naples

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This document contains all the mathematically significant errors I have noted in the first edition of Drs. Barnes and Roitzheim's text *Foundations of Stable Homotopy Theory*, together with some corrections. The chapter and page numberings in this errata are based on the PDF copy of the text formerly available on Dr. Roitzheim's website. Thus, Chapter 1 refers to the introduction, and the first "proper" chapter is Chapter 2, "Basics of Stable Homotopy Theory", which begins on page 1. I cover Section 2.1 through Section 7.6 as well as Appendix A. This document is not officially endorsed by Drs. Barnes and Roitzheim or by Cambridge Studies in Advanced Mathematics, and is solely a product of my own notes.

Chapter 2

- Corollary 2.1.9 seems to use the definition of h-cofibration as a homotopy cofibration in the Serre model structure on Top_*, which is counter to the previously stated convention that it refers to cofibrations in the Hurewicz model structure. (The difference is that the former cares about the basepoint while the latter does not.)
- Property 8 of Section 2.1.4 has a sign error in the grading (which is correct in property 7).
- Section 2.2 says twice that suspension is an equivalence on the non-shifted Spanier-Whitehead category, which is false.

Chapter 3

- The proof of Proposition 3.1.9 uses cofibrant generation of Top_* to show that a cofibrant spectrum is levelwise homotopy equivalent to a CW complex, but in doing so, it assumes that the pushouts used to construct a cofibration will be along cellular maps, without giving any reason why this ought to be the case. The proof can be completed using the Whitehead theorem for model categories, since the CW approximation map from a CW complex to a cofibrant object is a weak equivalence between fibrant and cofibrant objects.
- The proof of Theorem 3.3.12 says that the codomains of the generating (acyclic) cofibrations are small, but while this is true, the recognition theorem for cofibrantly generated model structures requires the observation that the domains, not the codomains, are small.
- Definition 3.4.4 (of connected and connective spectra) is not only nonstandard but also inconsistent with the use of the term "connective" in Section 6.5.2.
- Definition 3.5.2 omits the requirement of stability for the cohomology operations in the Steenrod algebra.
- In Lemma 3.5.7, there is an error in the computation of the Bockstein homomorphism. In particular, the wrong value is computed for $H^1(M, \mathbb{Z}/4)$.
- In the statement and proof of Theorem 3.5.15, the counit ϵ is never given.

Chapter 4

- In the proof of Proposition 4.1.3, the first line says "path objects" where it should say "cylinder objects".
- In definition 4.3.1, $\mathscr{C}$ should be a pointed model category.
- The statement preceding Theorem 4.3.3 should say "coaction and action".
- In the equation at the bottom of page 103 (towards the end of the section "The operation $\odot$ is well-defined" in the proof of Theorem 4.3.3), the first i should be i_1.
- In the section "Comparison of Fibres" in the proof of Lemma 4.5.1, κ is not defined. It is presumably supposed to be the pullback projection $E \times_{PB} B \to E$.
- At the end of the proof of Lemma 4.5.1, the text says certain maps are in the same homotopy class when they have neither the same domain nor the same codomain; in each case, one is a left homotopy and the other is a right homotopy. It should say they correspond.

Chapter 5

- In the analogy to the Third Isomorphism Theorem on page 125 (following Definition 5.1.1), the equation should be $(U/X)/(Y/X) = U/Y$.
- Lemma 5.1.12 is formulated incorrectly: the stated result $Y \cong X \coprod Z$ only holds when f_3 is zero. It does not follow from f_1 or f_2 being zero, although these imply splitting formulae for shifted triangles.
- The sequence of equations at the bottom of page 134 (the end of the proof of Lemma 5.2.2) is out of order. The first term should be last.
- At the beginning of the proof of Lemma 5.2.5, the first diagram should say ΩZ rather than ΩX and the second should say $\Sigma\Omega Z$ rather than Z. Moreover, at the top of page 137, the rightmost vertical map should be $\Sigma\epsilon_X$ rather than $\Sigma\epsilon_Z$.
- In the second part of the proof of Lemma 5.2.5, there is a sign error: the boundary map of the suspended cofiber sequence should be the additive inverse of the suspension of the original boundary map.
- At the beginning of section 5.6, the map γ should have target W, not ΣW.
- In the proof of Lemma 5.6.2, the double equation saying $\tau' \circ j = \gamma \circ \bar{\beta} = \tau \circ j$ should be amended to the two equations $\tau' \circ j = \gamma \circ \bar{\beta}$ and $\tau \circ j = \gamma \circ \bar{\beta}'$.
- In Lemma 5.6.6 and the surrounding discussion, η is presumably supposed to denote the Hopf map, but this is never explicitly stated.
- The proof of Proposition 5.7.8 should say that L is a Quillen left adjoint, not just a left Quillen functor, in order to be consistent with the definitions in section A.4.
- In the proof of Proposition 5.7.10, the phrase "f factors over the cofibre of i" should be replaced by "f factors through $2\,\mathrm{Id}_A$": this is not the homotopy category of a model category, and even if it were, $2\,\mathrm{Id}_A$ would be the fiber, not the cofiber.

Chapter 6

- It is stated on page 163 that property (4) of the stable homotopy category follows due to Lemma 5.4.4; but this lemma concerns (co)products in C rather than Ho(C), and while it can be used to prove property (4) (by comparing homotopy (co)products in C to (co)products in Ho(C)), this property already holds in any additive category by the discussion following Definition 4.2.5.
- In the proof of Lemma 6.1.4, it is necessary to note not only that $F_d^{\mathbb{N}}$ is left Quillen, but also that the generating cofibrations of the stable model structure are images under $F_d^{\mathbb{N}}$ of cofibrations in Top_*.
- The grading for represented cohomology switches repeatedly between the correct $E^* = [-, E]_{-*}$ and the incorrect $E^* = [-, E]_*$ throughout the latter part of section 6.1, even changing when the properties from section 2.1.4 are restated.
- The balanced smash product is not defined prior to its use on page 170 as part of Examples 6.2.3.
- Lemma 6.2.4 neglects to mention that the forgetful functor satisfies the solution set condition.
- In the proof of Lemma 6.2.14, the total space of the second fiber bundle should be $O(a)$, not $O(n)$.
- Near the bottom of page 187, immediately prior to Lemma 6.3.9, the codomain of the deformation retraction should be $F_n^{\Sigma} S^0$.
- At the beginning of the proof of Lemma 6.3.17, f should be both a stable equivalence and an h-cofibration, while g should be neither (or the other way around) to fit the hypotheses of the lemma.
- The proof of part 3 of Lemma 6.3.17 is invalid: it invokes Corollary A.7.14, which is only valid in stable model categories. There is no obvious easy fix for this, since the given proof of this result relies heavily on the triangulated structure of Ho(C). The Milnor sequence should be rederived from scratch or taken as given (with an appropriate reference).
- The definition of equivariant homotopy groups in Section 6.5.3 is incorrect: it has the case $q \geq 0$ twice rather than $q \geq 0$ and $q < 0$.
- Throughout Section 6.6, the language switches back and forth between the terms "small coproducts" and "arbitrary coproducts"; it should be clarified that the arbitrary coproducts are small.
- In Lemma 6.6.9, the hom-sets should be graded. If they are not, the categories $\mathcal{T}_i$ in the proof will not be closed under suspension.
- Also in Lemma 6.6.9, the hypothesis that $F(\mathfrak{g}) = \mathfrak{g}$ is too strong. It is only necessary that $F(\mathfrak{g})$ be a set of compact generators. This is important because the stronger assumption need not hold in the setting of Corollary 6.6.10.
- In the second part of the proof of Lemma 6.6.9, it needs to be noted that $\tilde{\mathcal{T}}$ is closed under coproducts in order to justify the conclusion that $\tilde{\mathcal{T}} = \mathcal{T}$.
- In Corollary 6.6.10, the adjoint functors should be exact. If the convention is that functors between triangulated categories are assumed to be exact, this should be stated.
- The notation in the second part of the proof of Proposition 6.6.12 is inconsistent. X is used for both the parameter defining H and the argument of $\hat{H}$.
- In the proof of Theorem 6.6.13, the lower-left object in the pushout square should be the wedge of disks *without* disjoint basepoints (which is homotopy equivalent to the mapping cylinder of the map from the upper-left object to a point). Otherwise, it is not weakly contractible and the pushout is not a homotopy cofiber, so we do not get an exact triangle.

Chapter 7

- In Definition 7.1.3, it should be specified that $\mathscr{C}$ has pushouts (since this is not automatic). Likewise, in the discussion following Definition 7.1.5, it is not automatic that $\mathscr{C}$ has an initial object.
- At the beginning of the proof of Theorem 7.1.14, it should say that $\otimes Y$ preserves (good) cylinder objects *of* cofibrant objects, not between them.
- In the latter part of the proof of Theorem 7.1.14, the part concerning part (5) of the theorem, the second diagram (the one produced from the 3×3 lemma) is incorrect. There are some maps where e and e' are switched, some of the maps should be suspended, and the lower-right object has a Z where it should have an X.
- In Definition 7.1.25, the domain of the cotensor should be $\mathscr{D} \times \mathscr{C}^{op}$, not $\mathscr{C} \times \mathscr{D}^{op}$.
- In the second part of Definition 7.1.28, $\mathscr{C}$ should be the category of pointed topological spaces.
- In Lemma 7.3.3, a and b should be switched in the block permutation matrix.
- In Corollary 7.4.4, the first term in the isomorphism should have S^0 where it has S^n.
- The construction of $\psi_n(X, Y)$ in the proof of Proposition 7.4.11 is phrased misleadingly. The second sequence of maps of homotopy groups after the diagram comes from the diagram smashed with Y, not from the first sequence of homotopy groups. (In particular, this only works because Y is cofibrant.)
- At the top of page 272 (immediately after the proof of Proposition 7.4.11), it should say "homology theory", not "cohomology theory".
- The right side of the isomorphism in Corollary 7.4.12 should have $n + b$ where it currently has n.
- In Definition 7.5.1, the second composite should be the identity on DX, not on X.
- In Examples 7.6.2, within the subsection Dual Spectra, either $F_{\mathbb{S}}$ should be replaced with Top_* or (equivalently) A_+ should be replaced with $\Sigma^\infty A_+$.

Appendix A

- In the section after Definition A.2.5, it is stated that property (4) is not proven in Dwyer and Spalinski. It is, in fact, proven there, as Lemma 4.21.
- Corollary A7.10 assumes well-pointedness unnecessarily (just add tails). Its application to Lemma 3.2.13(3) does not assume well-pointedness, so this is an important omission.