

Isogenies of Oriented Elliptic Curves

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Some Preliminary Notes

Conventions/Terminology

- Stack = étale sheaf of ∞ -groupoids on CAlg_R for some \mathbb{E}_∞ -ring R
- DM-stack = spectral Deligne-Mumford stack, not necessarily connective
- Formal DM-stack = formal filtered colimit of DM-stacks along closed immersions; called “honest” if actual DM-stack
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Related Work

Xuecai Ma and Yifei Zhu have a paper in the works ([MZ25]) which approaches this from a different perspective, defining level structures in terms of classical divisors. It isn't clear whether this is equivalent to my definition. A draft can be found on Professor Zhu's website.

Background: Classical \mathcal{M}_{ell}

Over \mathbb{C}

$$\left\{ \begin{array}{c} \mathcal{M}(\Gamma) \\ \downarrow \text{ét} \\ \mathcal{M}_{\text{ell}} \end{array} \right\} \longleftrightarrow \{ \text{Congruence subgroups } \Gamma \subset GL_2(\mathbb{Z}) \}$$



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Solution: Work with moduli interpretation directly!

The Moduli of Isogenies

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Main Theorem (GN)

Isog is a formal DM-stack.



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Warning

It is not known whether Isog is an honest DM-stack.



Connected-Étale Factorization

Factorization Theorem (GN)

There is an orthogonal factorization system $(\mathcal{C}onn, \mathcal{E}t)$ on $\mathbf{Ell}_{\text{Isog}}^{\text{or}}$ such that

- $\mathcal{C}onn$ is the class of connected isogenies, and
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This factorization system is natural with respect to change of base.



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Digression: Why are the components isogenies?



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Elliptic Rigidity Theorem, classical version ([KM85])

Zariski-locally on the base, every morphism of classical elliptic curves is either 0 or an isogeny.



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$\Rightarrow \mathrm{Def}_{\tilde{R}}^{\tilde{R}}(f) = 0. \quad \square$



Corollary

We have a pullback of functors

$$\begin{array}{ccc} \mathrm{Isog} & \longrightarrow & \mathrm{Isog}^{\mathrm{ét}} \\ \downarrow & & \downarrow s \\ \mathrm{Isog}^{\mathrm{conn}} & \xrightarrow{t} & \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}. \end{array}$$



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Need to show that $\mathrm{Isog}^{\mathrm{ét}}, \mathrm{Isog}^{\mathrm{conn}}$ are formal DM-stacks.



Theorem (GN)

$\mathrm{Isog}^{\mathrm{\acute{e}t}}$ is a DM-stack.



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Theorem (GN)

$\text{Isog}^{\text{ét}}$ is a DM-stack.

Proof sketch.

- 1 [KM85]: $\{(E, K) \mid E \text{ elliptic curve}, K \subset E \text{ finite}\} \rightarrow (\mathcal{M}_{\text{ell}}^{\text{or}})^{\heartsuit}$
relative scheme.



Theorem (GN)

$\text{Isog}^{\text{ét}}$ is a DM-stack.

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- 2 $\{(E, K) \mid E \text{ elliptic curve}, K \subset E \text{ finite étale}\}$ open substack.



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relative scheme.
- 2 $\{(E, K) \mid E \text{ elliptic curve}, K \subset E \text{ finite étale}\}$ open substack.
- 3 Leverage étaleness and use ([Lur18c], Theorem 18.1.0.2) to lift from classical to spectral. □



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$$\begin{array}{c} \{E \xrightarrow{\text{conn}} E'\} \\ \updownarrow \\ \{K \subset E \text{ closed, proper, connected}\} \\ \updownarrow \\ \{K \subset \widehat{\mathbb{G}}_R^Q \text{ honest subgroup}\} \end{array}$$



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$$\updownarrow$$

$$\{K \subset E \text{ closed, proper, connected}\}$$

$$\updownarrow$$

$$\{K \subset \widehat{\mathbb{G}}_R^Q \text{ honest subgroup}\}$$

$$\Rightarrow \text{Isog}^{\text{conn}} \simeq \mathcal{M}_{\text{ell}}^{\text{or}} \times \text{QuilIsog}$$



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Proof sketch (ctd).

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\Rightarrow Enough to show that $\mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q)$ is formal DM.



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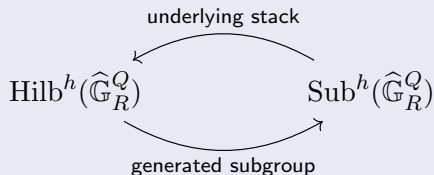


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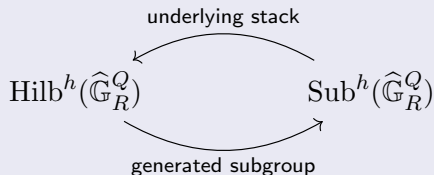


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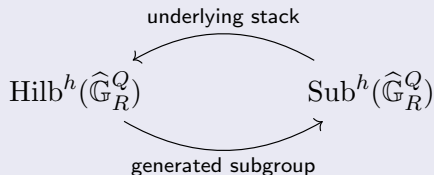


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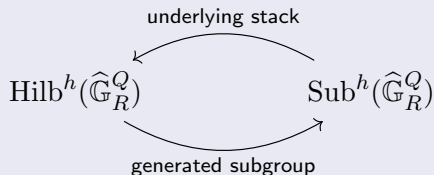


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[Lur04]: Hilb is DM.

$\Rightarrow \text{Hilb}^h$ is formal DM.

$\Rightarrow \text{Sub}^h$ is formal DM.



Thank you!

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