Isogenies of Oriented Elliptic Curves

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Some Preliminary Notes

Conventions/Terminology

- Stack = étale sheaf of ∞-groupoids on CAlg_R
- DM-stack (algebraic space) = spectral Deligne-Mumford stack (spectral algebraic space), not necessarily connective
- Formal DM-stack = formal filtered colimit of DM-stacks along closed immersions; called "honest" if actual DM-stack. Formal algebraic spaces defined similarly
- Isogeny = strict abelian variety map which is finite, flat, and locally almost of finite presentation

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Related Work

Xuecai Ma and Yifei Zhu have a paper in the works ([MZ25]) which approaches this from a different perspective, defining level structures in terms of classical divisors. It isn't clear whether this is equivalent to my definition. A draft can be found on Professor Zhu's website.

Background: Classical $\mathcal{M}_{\mathrm{ell}}$

Over $\mathbb{C}$

$$\left\{ \begin{array}{c} \mathcal{M}(\Gamma) \\ \Big\downarrow \text{ét} \\ \mathcal{M}_{\mathrm{ell}} \end{array} \right\} \longleftrightarrow \{\text{Congruence subgroups } \Gamma \subset GL_2(\mathbb{Z})\}$$

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Solution: Work with moduli interpretation directly!

The Moduli of Isogenies

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Main Theorem (GN)

Isog is a formal DM-stack.

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Warning

It is not known whether Isog is an honest DM-stack.

Connected-Étale Factorization

Factorization Theorem (GN)

There is an orthogonal factorization system $(\mathscr{C}onn, \acute{\mathscr{E}}t)$ on $\mathrm{Ell}^{\mathrm{or}}_{\mathrm{Isog}}$ such that

- $\mathscr{C}onn$ is the class of connected isogenies, and
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This factorization system is natural with respect to change of base.

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$$\begin{tikzcd}
K^\circ \arrow[d] & & \\
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K^{\text{ét}} & &
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$$\begin{tikzcd}
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$\square$

Digression: Why are the components isogenies?

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Elliptic Rigidity Theorem, classical version ([KM85])

Zariski-locally on the base, every morphism of classical elliptic curves is either 0 or an isogeny.

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$$\begin{tikzcd} \widetilde{R} \arrow[d] & & \\ R \arrow[r] & A \arrow[r, "f"] & B \end{tikzcd}$$

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f factors through R

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$\Rightarrow \mathrm{Def}_R^{\widetilde{R}}(f) = 0.$ $\square$

Connected-Étale Factorization

Corollary

We have a pullback of functors

$$\begin{tikzcd} \mathrm{Isog} \arrow[r] \arrow[d] & \mathrm{Isog}^{\text{ét}} \arrow[d, "s"] \\ \mathrm{Isog}^{\mathrm{conn}} \arrow[r, "t"] & \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}. \end{tikzcd}$$

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Just need to show that $\mathrm{Isog}^{\text{ét}}, \mathrm{Isog}^{\text{conn}}$ are formal DM-stacks.

Identifying $\mathrm{Isog}^{\text{ét}}$

Theorem (GN)

$\mathrm{Isog}^{\text{ét}}$ is a DM-stack.

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1. [KM85]: $\{(E, K) \mid E$ elliptic curve, $K \subset E$ finite$\} \to (\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}})^{\heartsuit}$ relative scheme.

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2. $\{(E, K) \mid E \text{ elliptic curve}, K \subset E \text{ finite étale}\}$ open substack.
3. Leverage étaleness and use ([Lur18c], Theorem 18.1.0.2) to lift from classical to spectral. $\square$

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$\{K \subset E$ closed, proper, connected $\}$

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$$\{K \subset E \text{ closed, proper, connected} \qquad\qquad\qquad\qquad \}$$

$$\updownarrow$$

$$\{K \subset \widehat{\mathbb{G}}_R^Q \text{ honest subgroup} \qquad\qquad\qquad\qquad \}$$

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$$\{E \xrightarrow{\mathrm{conn}} E'\}$$

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$$\{K \subset E \text{ closed, proper, connected AND equiv } \widehat{E/K} \simeq \widehat{\mathbb{G}}_R^Q\}$$

$$\updownarrow$$

$$\{K \subset \widehat{\mathbb{G}}_R^Q \text{ honest subgroup AND equiv } \widehat{\mathbb{G}}_R^Q/K \simeq \widehat{\mathbb{G}}_R^Q\}$$

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$$\{K \subset \widehat{\mathbb{G}}_R^Q \text{ honest subgroup AND equiv } \widehat{\mathbb{G}}_R^Q/K \simeq \widehat{\mathbb{G}}_R^Q\}$$

$\Rightarrow \mathrm{Isog}^{\mathrm{conn}} \simeq \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}} \times \mathrm{QuilIsog}$

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Proof sketch (ctd).

$$\begin{array}{c} \mathrm{QuilIsog} \\ \big\downarrow \\ \mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q) \end{array}$$

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\mathrm{OrDat}(\widehat{\mathbb{G}}^Q_R/K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{QuilIsog} \arrow[d] \\
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[Lur18b]: $\mathrm{OrDat}(\widehat{\mathbb{G}}_R^Q/K)$ is an affine DM-stack.

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$$\begin{tikzcd}
\mathrm{OrDat}(\widehat{\mathbb{G}}^Q_R/K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{QuilIsog} \arrow[d] \\
* \arrow[r, "K"] & \mathrm{Sub}^h(\widehat{\mathbb{G}}^Q_R)
\end{tikzcd}$$

[Lur18b]: $\mathrm{OrDat}(\widehat{\mathbb{G}}^Q_R/K)$ is an affine DM-stack.

$\Rightarrow$ Enough to show that $\mathrm{Sub}^h(\widehat{\mathbb{G}}^Q_R)$ is formal algebraic space.

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Theorem (GN)

$\mathrm{Isog}^{\mathrm{conn}}$ is a formal algebraic space.

Proof sketch (ctd).

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$$\begin{tikzcd} \mathrm{Hilb}^h(\widehat{\mathbb{G}}_R^Q) \arrow[rr, bend right, "\text{generated subgroup}"'] & & \mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q) \arrow[ll, bend right, "\text{underlying stack}"'] \end{tikzcd}$$

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[Lur04]: Hilb of separated algebraic space is algebraic space.

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[Lur04]: Hilb of separated algebraic space is algebraic space.
$\Rightarrow \mathrm{Hilb}^h(\widehat{\mathbb{G}}_R^Q)$ is formal algebraic space.
$\Rightarrow \mathrm{Sub}^h(\widehat{\mathbb{G}}_R^Q)$ is formal algebraic space. $\square$

Thank you!

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