$$NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$CNOT_{12} = P_0 \otimes I + P_1 \otimes NOT, \quad CNOT_{21} = I \otimes P_0 + NOT \otimes P_1$$

$$M[CNOT_{12}] = M[P_0] \otimes M[I] + M[P_1] \otimes M[NOT]$$

$$M[CNOT_{21}] = M[I] \otimes M[P_0] + M[NOT] \otimes M[P_1]$$

$$\boldsymbol{M[CNOT_{12}]} = \begin{pmatrix} \boldsymbol{M[I]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M[NOT]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M[I]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[NOT]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[I]} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[NOT]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[I]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[NOT]} \end{pmatrix}$$

$$\boldsymbol{M[CNOT_{21}]} = \begin{pmatrix} \boldsymbol{M[P_0]} & \boldsymbol{M[P_1]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{M[P_1]} & \boldsymbol{M[P_0]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M[P_0]} & \boldsymbol{M[P_1]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M[P_1]} & \boldsymbol{M[P_0]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[P_0]} & \boldsymbol{M[P_1]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[P_1]} & \boldsymbol{M[P_0]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[P_0]} & \boldsymbol{M[P_1]} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[P_1]} & \boldsymbol{M[P_0]} \end{pmatrix}$$

$$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \quad Im\big(P(\phi)\big) = \begin{pmatrix} 0 & 0 \\ 0 & \sin(\phi) \end{pmatrix}, \quad Re\big(P(\phi)\big) = \begin{pmatrix} 1 & 0 \\ 0 & \cos(\phi) \end{pmatrix}$$

$$M[P(\phi)] = \begin{pmatrix} Re\big(P(\phi)\big) & 0 & 0 & Im\big(P(\phi)\big) \\ 0 & Re\big(P(\phi)\big) & Im\big(P(\phi)\big) & 0 \\ Im\big(P(\phi)\big) & 0 & Re\big(P(\phi)\big) & 0 \\ 0 & Im\big(P(\phi)\big) & 0 & Re\big(P(\phi)\big) \end{pmatrix}$$

$$CP(\phi) = P_0 \otimes I + P_1 \otimes P(\phi)$$

$$M[CP(\phi) = M[P_0] \otimes M[I] + M[P_1] \otimes M[P(\phi)]$$

$$\boldsymbol{M[CP(\phi)]} = \begin{pmatrix} \boldsymbol{M[I]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M[P(\phi)]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M[I]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[P(\phi)]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[I]} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[P(\phi)]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[I]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M[P(\phi)} \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \boldsymbol{M[H]} = \begin{pmatrix} \boldsymbol{H} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{H} \end{pmatrix}$$

$$SWAP = CNOT_{12} \cdot CNOT_{21} \cdot CNOT_{12}, \quad \boldsymbol{M}[\boldsymbol{SWAP}] = \boldsymbol{M}[\boldsymbol{CNOT_{12}}] \cdot \boldsymbol{M}[\boldsymbol{CNOT_{21}}] \cdot \boldsymbol{M}[\boldsymbol{CNOT_{12}}]$$

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \to \frac{1}{8}\begin{pmatrix}2\\1\\0\\1\\1\\1\\1\\1\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \to \frac{1}{8}\begin{pmatrix}1\\2\\1\\0\\1\\1\\1\\1\end{pmatrix} \qquad |k\rangle = c_0|0\rangle + c_1|1\rangle = \begin{pmatrix}c_0\\c_1\end{pmatrix} = \begin{pmatrix}x_0 + iy_0\\x_1 + iy_1\end{pmatrix} = \vec{x} + i\vec{y}$$

$$\varphi|k\rangle = \frac{1}{8}(\vec{u} + \vec{p}) \quad \vec{u} = \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}, \quad \vec{p} = \begin{pmatrix}\vec{x}\\-\vec{x}\\\vec{y}\\-\vec{y}\end{pmatrix}$$

entanglement: $s_{12} = \frac{1}{8^2}(\vec{u} \otimes \vec{u} + \vec{p}_1 \otimes \vec{p}_2), s_{12} = \frac{1}{2}\big(s_1 \otimes s_2 + \Pi(s_1) \otimes \Pi(s_2)\big) = \tau(s_1, s_2)$

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle, \quad \varphi|\psi\rangle = s_{12}, \quad \varphi|\psi_1\rangle = s_1, \quad \varphi|\psi_2\rangle = s_2$$

$$s_{12} = \frac{1}{64}\left(\begin{pmatrix}\vec{u}\\\vec{u}\\\vec{u}\\\vec{u}\\\vec{u}\\\vec{u}\\\vec{u}\\\vec{u}\end{pmatrix} + \begin{pmatrix}\vec{x}_1 \otimes \begin{pmatrix}\vec{x}_2\\-\vec{x}_2\\\vec{y}_2\\-\vec{y}_2\end{pmatrix}\\ -\vec{x}_1 \otimes \begin{pmatrix}\vec{x}_2\\-\vec{x}_2\\\vec{y}_2\\-\vec{y}_2\end{pmatrix}\\ \vec{y}_1 \otimes \begin{pmatrix}\vec{x}_2\\-\vec{x}_2\\\vec{y}_2\\-\vec{y}_2\end{pmatrix}\\ -\vec{y}_1 \otimes \begin{pmatrix}\vec{x}_2\\-\vec{x}_2\\\vec{y}_2\\-\vec{y}_2\end{pmatrix}\end{pmatrix}\right)$$

$$s_{12\ldots n} = \frac{1}{8^n}\big(\vec{u}^{\otimes n} + \vec{p}_1 \otimes \vec{p}_2 \otimes \ldots \otimes \vec{p}_n\big) = s_{1,n}$$

Pseudo algorithm for QFT without measurements:

- Take n Qbits
- k=n-1
- Phase = $\pi/2$
- while k≠0:
    - Implement H on q_k
    - i=n-1
    - While i>k-1:
        - Implement CP on q_i using q_{k-1} with phase of: $Phase/2^{i-k}$
        - i = i-1
    - k = k-1
- implement H on q_0
- if n is even
    - k=n
    - while k≠ $n/2$-1:
        - swap $q_{(n-k)}, q_{k-1}$
        - k=k-1
- if n is odd
    - k=n
    - while k≠ $\frac{n}{2} - \frac{1}{2}$:
        - swap $q_{(n-k)}, q_{k-1}$
        - k=k-1

example 2 Qbits:

q_0, q_1.

- H on q_1
- CP($\pi/2$) q_1, q_0
- H on q_0
- SWAP q_1, q_0
- Measurement

$$M[H]s_1 \Rightarrow M[H]s_0 \Rightarrow CP\left(\frac{\pi}{2}\right)\binom{s_0}{s_1} \Rightarrow M[SWAP]s_{01}$$

Example 4 Qbits:

q_0, q_1, q_2, q_3.

- H on q_3
- CP($\pi/2$) q_3, q_2
- H on q_2
- CP($\pi/4$) q_3, q_1
- CP($\pi/2$) q_2, q_1
- H on q_1
- CP($\pi/8$) q_3, q_0
- CP($\pi/4$) q_2, q_0
- CP($\pi/2$) q_1, q_0
- H on q_0
- SWAP q_3, q_0
- SWAP q_2, q_1
- Measurement

To implement CP correctly, we must expand it (since it is creating an entanglement)

$$M[H]s_3 \Rightarrow M[H]s_2 \Rightarrow M[H]s_1 \Rightarrow M[H]s_0 \Rightarrow CP\left(\frac{\pi}{2}\right)\binom{s_2}{s_3} \Rightarrow CP_{3,1}\left(\frac{\pi}{4}\right)\binom{s_1}{s_{23}} \Rightarrow CP_{2,1}\left(\frac{\pi}{2}\right)\binom{s_{123}}{s_{123}}$$

$$\Rightarrow CP_{3,0}\left(\frac{\pi}{8}\right)\binom{s_0}{s_{123}} \Rightarrow CP_{2,0}\left(\frac{\pi}{4}\right)\binom{s_{0123}}{s_{0123}} \Rightarrow CP_{1,0}\left(\frac{\pi}{2}\right)\binom{s_{0123}}{s_{0123}}$$

$$\Rightarrow M_{3,0}[SWAP]s_{0123} \Rightarrow M_{2,1}[SWAP]s_{0123}$$

חשוב לשים לב

הפעלת המטריצה במרחב הקלאסי היא הפעלה מהצורה הבאה:

$$T[\hat{U}](\vec{s}) = \frac{1}{8}\left(\vec{u} + \widetilde{M}[\hat{U}] \cdot \vec{p}\right) = \frac{1}{8}\left(\tilde{I} - \widetilde{M}[\hat{U}]\right)\vec{u} + \widetilde{M}[\hat{U}] \cdot \vec{s}$$

ניתן להרחיב את הנוסחא הזו לכל מספר קיוביטים שנרצה:

$$T_n[\hat{U}](\vec{s}_{1,n}) = \frac{1}{8^n}\left(\tilde{I} - \widetilde{M}[\hat{U}]\right) \cdot (\vec{u} \otimes \vec{u} \otimes \ldots . \otimes \vec{u}) + \widetilde{M}[\hat{U}] \cdot \vec{s}_{1,n}$$

עבור מקרים מהצורה (דוגמא עבור שני קיוביטים):

$$\widetilde{M}[\hat{U}] = \widetilde{M}[\hat{U}_1] \otimes \widetilde{M}[\hat{U}_2]$$

נקבל:

$$T_2[\hat{U}] = \frac{1}{8^2}\left(\vec{u} \otimes \vec{u} + \widetilde{M}[\hat{U}_1] \cdot \vec{p}_1 \otimes \widetilde{M}[\hat{U}_2] \cdot \vec{p}_2\right)$$

<u>דוגמא להמרה:</u>

$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,\ \ H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\varphi H|0\rangle = \frac{1}{8}\left(\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\\0\\0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\\0\\0\\0\\0\end{pmatrix}\right),\ \ \varphi H|1\rangle = \frac{1}{8}\left(\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\\0\\0\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\\0\\0\\0\\0\end{pmatrix}\right)$$

כאשר את ההמרה ביצענו לפי:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle,\ \ \varphi|\psi\rangle = \vec{s}(\psi) = \frac{1}{8}\left(\vec{u} + \vec{P}_0(c_0) + \vec{P}_1(c_1)\right)$$

$$\vec{P}_0(c_0) = \begin{pmatrix}Re(c_0)\\0\\-Re(c_0)\\0\\Im(c_0)\\0\\-Im(c_0)\\0\end{pmatrix},\ \ \vec{P}_1(c_1) = \begin{pmatrix}0\\Re(c_1)\\0\\-Re(c_1)\\0\\Im(c_1)\\0\\-Im(c_1)\end{pmatrix}$$

על מנת לחשב את ההסתברויות:

עבור קיוביט מהצורה $|z \dots dcba\rangle$, יש לקחת שני איברים במיקומים $x, x+4$:

$x = a\cdot 8^0 + b\cdot 8^1 + c\cdot 8^2 + d\cdot 8^3 + \cdots$

וההסתברויות יהיו:

$$P(|z \dots dcba\rangle = state) = \left(1 - 8^n \cdot s_{1,n}[x]\right)^2 + \left(1 - 8^n \cdot s_{1,n}[x+4]\right)^2$$

בחישוב שלנו ב$verilog$, אנחנו מזניחים את החלוקה ב-8 לכל אורך הקוד, על מנת לשמור כמה שיותר מידע (יש לנו רק 19 ביטים של מידע לאחר הנקודה, עבור כמות קיוביטים גדולה אנחנו עלולים לאבד הכל), לכן בקוד שלנו אין הכפלה ב8^n