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Encyclopedia of Analytical Surfaces

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Preface

This book is an encyclopedic edition on analytic and differential geometry of regular analytical surfaces, which has found application in some parts of mathematics or in different branches of techniques and building. The encyclopedia was formed along the same principle as the scientific edition *Encyclopedia of Analytical Surfaces*, Krivoshapko S.N., Ivanov V.N., Publishing House LIBROCOM, 2010, 560 pp. However, the present book is a supplemented and recasted edition.

The main demand for this book lies in its most complete account of materials on the geometry of each surface in one or two pages. A bibliography from several titles is presented at the end of each paper. These references may help to find information for an extended study of problems connected with the geometry of a presented surface, with strength analysis of a shell in the form of this surface and their application. At the beginning or at the end of most of the book parts, a one- or two-page list of the literature on geometry, application, and strength analysis of shells with middle surfaces in the form of corresponded surfaces is given.

Only the surfaces that can be pictured by means of descriptive geometry and computer graphics have been included in the encyclopedia.

The material of the book is grouped into 38 classes of surfaces. The indications of forms of generatrix and directrix lines and the laws of their location concerning the base planes and lines are taken as the principle of classification. The order of equations of surfaces, total curvature, and kinematics of generation of surfaces are also taken into consideration. Before the description of surfaces belonging to the same class, a one- or two-page general characteristic of surfaces of this class is given.

The encyclopedia contains both classical surfaces known to geometricians for several centuries and surfaces known only to a narrow range of specialists. The surfaces discovered and investigated by the authors are also included in the book.

Thin-walled smooth structures are the most economical structures. From the geometrical point of view, shells are described by the form of their middle surface. However, traditionally a limited circle of these structures, such as spherical, cylindrical, conical, shallow translation shells, and some shells of revolution, constitute a small percentage in comparison to those consisting a variety of geometrical forms presented by geometricians but unknown to architects and civil and mechanical engineers.

The main aim of the encyclopedia is to help in exposure and decision of scientific and technical problems connected with the theory of forming of thin-walled structures on the basis of geometrical investigation of the middle surfaces of shells.

The generation of nontraditional effective constructive forms of large-span space for achieving the maximum level of manufacturing resourcefulness shall favor the fulfillment of complex fundamental and applied problems raised for science in architectural and -building spheres. The availability of a wide choice of different forms and surfaces gives an opportunity to solve some problems in machine-building sphere too.

The authors consider that they kept off the reiteration of some mistakes passing from one edition to another and eliminated wrong variant readings in the definition of some surfaces. The chapter with the most formulas presented in the book was tested by the authors.

The authors tried not to include in the book questionable formulas or formulas giving rise to doubts.

The materials in the encyclopedia will be interesting and useful to mathematicians, engineers, architects, postgraduate students, lecturers, and specialists dealing with geometry of surfaces, and for specialists working in other fields of knowledge but using geometrical images in their work.

The encyclopedia also contains a dictionary of geometrical terms in Russian, English, French, and German languages.

There is an Index at the end of the edition.

Pages 152–155, 156–158 were written by Ph.D. Ya.S. Pul'pinskiy; pp. 145–148, 331–337 were written by Ph.D. V.A. Nikityuk, pp. 606–612 were written by Ph.D. G.S. Rachkovskaya; D.Sc. Professor V.N. Ivanov prepared pp. 19, 88, 97–98, 190–195, 197–202, 201–203. 210–211, 240–252, 266–278, 281–290, 307–314, 339–340, 344–357, 356–357, 364–372, 376–378, 222–223, 385–386, 394–395, 400–407, 412–413, 425–426, 434–435, 441–442, 443–444, 463–469, 492–496, 501–502, 515–518, 522–526, 532–533, 541–542, 547–550, 558–559, 563, 564–565, 569–570, 570–581, 634–635, 642–646, 662. All the rest were written by D.Sc. Professor S.N. Krivoshapko.

The authors express thanks to assistant I. Kushnarenko for help in preparation of this edition.

Contents

Information About Structure of the Encyclopedia

This encyclopedia is not a textbook on geometry or strength analysis of shell-type spatial structures. It is not intended for reading all materials contained in it on end. The inquiry information on geometry of curves and surfaces is given only in the first parts "Plane", "Surfaces" and "Extract of Basic Formulas from Differential Geometry of Space Curve and Surfaces," which may be recommended for reading or preview at first.

The encyclopedia is meant for the trained reader having an idea of differential geometry. It can be of interest to readers easily carried away along construction of different space geometrical forms or looking for mathematical analogies for modeling natural objects of animate or inanimate nature.

The encyclopedia can help to find a geometrical image for further thorough mathematical analysis. It gives an opportunity to choose a thin-walled object in the form of a presented surface for research of its stress–strain state under action of external loading for the purpose of its further introduction to the suitable sphere of human activity.

Inquiry and scientific literature supplementing each chapter in the book will make easier the search for the necessary literature on geometry, application, or strength analysis of shells of the chosen form.

The family names of researchers presented at the end of the encyclopedia emphasize a circle of people who have researched or are now researching problems connected with theory of surfaces.

The materials in the encyclopedia are divided into 38 chapters, which are denoted by a single numerical cipher. The number of the chapters is equal to the number of the classes of presented surfaces. When necessary, the chapters are divided into sections, which are denoted by two numerical ciphers, for example, 1.2 (Chapter 1 and Section 2). In turn, the sections contain subsections noted by three numerical ciphers, for example, 1.2.1 (Chapter 1, Section 2, and Subsection 1). Lastly, the papers on the presented particular surfaces are marked by points (■).

The content of the book is given in the part "The Surfaces Presented in the Encyclopedia." Sometimes this part can help at once to find the necessary material. If only a surface name is known to the reader, then it is sometimes not rational to look for this surface in this part because it can be written under another name (synonym). In this case, it is better to begin searching from the "Index" where all names of curves and surfaces presented in the encyclopedia are given. Next to the name of a surface, in the same line the numbers of pages where this surface is mentioned are given. A reader who wants to know about the contribution of a definite scientist to the theory of surface must use the "Name Index." Here the numbers of pages, where surnames of researchers are mentioned, are given after their surnames. But if a reader does not find any surname in "Name Index," it does not mean that this researcher did not concern with theory of surfaces, because he may be mentioned in the references presented at the end of each page or on the individual pages of the encyclopedia.

The bibliography containing references of cited researchers as a rule is not enumerated in the encyclopedia. Only a quantity of names of the literature present in references is pointed out in brackets. This permitted the authors to avoid increasing the volume of the encyclopedia and to enable the concerned person to find tens or hundreds of names of published works devoted to the definite theme.

The authors very carefully used the materials on the geometry of surfaces available on the Internet. First, these materials contain many errors. In addition, it is difficult to confirm the authority of the texts. Finally, the materials are short-lived and permit corrections after putting them on the Internet. Reference to the Internet is only made where absolutely necessary.

Conventional Signs

The following notations are used in the encyclopedia if additional explanations are not given:

x, y, z	Cartesian coordinates;
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	The unit vectors in the direction of coordinate axes x, y, z, relatively;
k	Curvature of the space curve;
κ	Torsion of the space curve;
u, v	Curvilinear nonorthogonal coordinates of the surface;
$\boldsymbol{r}(u, v)$	A radius-vector of any point of a surface;
$\boldsymbol{\rho}(u)$	A radius-vector of the space or plane curve;
k_u, k_v	Curvatures of the curvilinear coordinate u, v;
R_u, R_v	Radii of curvatures of the curvilinear coordinate lines u, v;
χ	An angle between two intersecting curvilinear coordinates u and v on the surface;
k_1, k_2	Principal curvatures of the surface;
R_1, R_2	Principal radii of curvature of the surface;
E, F, G	Coefficients of the first fundamental form in the theory of surfaces;
L, M, N	Coefficients of the second fundamental form in the theory of surfaces;
$A^2 = E;\ B^2 = G$	Gaussian quantities of the first order in the theory of surfaces (Lame coefficients in the theory of curvilinear coordinates);
$K = k_1 k_2$	The Gaussian curvature of the surface;
$H = (k_1 + k_2)/2$	The mean curvature of the surface;
$\mathrm{tg}\ \alpha = \tan\ \alpha$	Tangent of an angle α;
$\mathrm{ctg}\ \alpha = \mathrm{cotan}\ \alpha$	Cotangent of an angle α;
$\mathrm{sh}\, x = \sinh x = (\mathrm{e}^x - \mathrm{e}^{-x})/2$	Hyperbolic sine;
$\mathrm{ch}\, x = \cosh x = (\mathrm{e}^x + \mathrm{e}^{-x})/2$	Hyperbolic cosine, and
$\mathrm{th}\, x = \tanh x = (\mathrm{e}^x - \mathrm{e}^{-x})/(\mathrm{e}^x + \mathrm{e}^{-x})$	Hyperbolic tangent.

Plane and Surfaces

Plane

A surface in the space is *a plane* only when it is the first-order algebraic surface. The contrary assertion is right too, i.e., any surface of the first order is a plane. Plane is infinite. Another definition of plane: a plane is a two-dimensional, doubly ruled surface spanned by two linearly independent vectors.

The main properties of plane are taken axiomatically, i.e., without proofs, for example:

(a) if two points of a straight line belong to a plane, then every point of this straight line belongs to the plane;
(b) if two planes have a common point, then they intersect along the straight line passing through this point;
(c) only one plane may be drawn through any three points not lying on one straight line; and
(d) always in the space, four points exist not belonging to the same plane.

For three planes in the space, eight essentially different types of their mutual disposition are possible.

When a line passes through a plane, the point of intersection is called a "trace." When a plane passes through another plane, the line of intersection is also called a "trace."

The Forms of Definition of a Plane

(1) The general equation of a plane P:

$$F(x, y, z) = Ax + By + Cz + D = 0,$$

where even if one of the quantities A, B, or C is different from zero, a vector $\boldsymbol{N}(A, B, C)$ is perpendicular to the plane. The components of the unit normal vector $\boldsymbol{n}(\boldsymbol{n}_x, \boldsymbol{n}_y, \boldsymbol{n}_z) = \boldsymbol{N}/|\boldsymbol{N}|$ are

$$\boldsymbol{n}_x = \frac{A}{\sqrt{A^2 + B^2 + C^2}}, \ \boldsymbol{n}_y = \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \ \boldsymbol{n}_z = \frac{C}{\sqrt{A^2 + B^2 + C^2}}.$$

A vector $\boldsymbol{a}(l, m, n)$ is parallel to a plane P only when $Al + Bm + Cn = 0$. The equality to zero of one or several coefficients A, B, C, D points out the special position of the plane relative to the coordinate axes. For example, if $A = 0$ then the plane is parallel to an axis Ox; if $D = 0$ then the plane is crossing the point $O(0, 0, 0)$.

Two planes P and P_1 will be parallel to each other if the ratios

$$\frac{A}{A_1} = \frac{B}{B_1} = \frac{C}{C_1}$$

are fulfilled.

Parallel planes coincide even if they have one common point. In spite of this, the following ratios are carried out:

$$\frac{A}{A_1} = \frac{B}{B_1} = \frac{C}{C_1} = \frac{D}{D_1}.$$

An angle θ between the planes P and P_1 can be determined with the help of the formula:

$$\cos\,\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

The angle θ between two intersecting planes is known as *the dihedral angle*.

A plane P divides the space into two half-spaces. One of them consists of all points $M = (x, y, z)$ for which $F(x, y, z) > 0$, but the other consists of all points $M = (x, y, z)$ for which $F(x, y, z) < 0$.

The first half-space is called positive; the second half-space is called negative relative to the given equation of the plane P. A vector $\boldsymbol{n}(A, B, C)$ is directed to a positive side of the plane P.

(2) An intercept equation of a plane:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

A plane intersects a coordinate axis Ox in the point $(a, 0, 0)$, an axis Oy in the point $(0, b, 0)$, and an axis Oz in the point $(0, 0, c)$. Here, $a = -D/A$, $b = -D/B$, $c = -D/C$.

(3) A normal equation of a plane:

$$x\cos\alpha_x + y\cos\alpha_y + z\cos\alpha_z - p = 0,$$

where $p > 0$ is the length of a perpendicular dropped from the beginning of coordinates to the plane; $\cos\,\alpha_x, \cos\,\alpha_y, \cos\,\alpha_z$ are direct cosines of a perpendicular to the plane the beginning of which is the beginning of the coordinates, but the end is a point of the plane.

(4) An equation of a plane crossing through a given point $P_1(x_1, y_1, z_1)$:

$$A(x - x_1) + B(y - y_1) + C(z - z_1) + D = 0.$$

(5) An equation of a plane crossing through three given points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ not lying on one straight line:

$$\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} x + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} z - \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

(6) Parametrical form of definition:

$$x = x_1 + a_x u + b_x v,\ y = y_1 + a_y u + b_y v,\ z = z_1 + a_z u + b_z v.$$

The definitions of planes given in the encyclopedia are given below. First, the definition of some planes in a theory of linear perspective are given, but later, in differential geometry.

If a given surface is cut by as many planes as you wish, and the intersection of this surface with each of the planes is always a straight line, then the surface in question will be *a plane.*

Picture plane is a vertical or inclined plane on which the objects are projected. *Plane of general position* is a plane inclined to all planes of projections.

Plane of projections or *projection plane* is a plane on which they receive the picture of the original in the process of design. A horizontal plane, on which projected objects are disposed, is called *object plane*. *Compartment of a plane* is a part of the plane limited by any contour. *Secant plane* intersects another plane or a surface. This plane is used for the realization of cut or cross-section as auxiliary plane. A plane parallel to the basic plane is called *a plane of level*.

Projecting plane is perpendicular to any plane of projections. The figures call symmetrical relatively the Q plane if their points relate in symmetrical pairs, but the plane Q itself calls *a plane of symmetry*.

Normal plane of the space curve in the M point is a plane passing through M perpendicular to the tangent straight to the same point. Normal plane contains all normals to the curve passing through a given point. *Osculating plane* of a curve in the point M passes through the tangent straight and the principal normal of the space curve in the point M. *Rectifying plane* of the curve contains the binormal and the tangent straight of the curve. Rectifying, normal, and osculating planes of a curve are mutually perpendicular.

Tangent plane to a surface S in the point M is the plane passing through the point M and having infinitesimal distance from this plane to the variable point M_0 of the surface S in comparison with a distance MM_0 when M_0 is moving to M. A tangent plane in singular point of a surface stops to be definite.

A given plane is *a plane of parallelism* if all generatixes of any surface are parallel to the given plane.

An oblique plane is a plane that is inclined to all of the principal planes. *An inclined plane* is inclined to two of the principal planes and is perpendicular to the third. The planes of the faces of the cube and all planes parallel to them are called *isometric planes.* Planes that are not parallel to any isometric plane are called *non-isometric planes*.

Every set P consisting of elements of two genuses called relatively “points” and “straights” and connected between themselves with the help of a ratio called the patio of incidence between any “point” and any “straight” is *a projective plane*. But it is necessary to keep mutually single-valued correspondence of incidence between “points” and “straights” of a projective plane P, on the one hand, and between projecting rays and planes of connectives, on the other hand. “Point” and “straight” of projective plane P are incidental between themselves only then when corresponding to them the ray and the plane of connective are incidental. Projective plane with singular straight chosen on it is *affine-projective* plane.

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Surfaces

Differential geometry considers a surface as a geometric locus defined by a vector equation

$$\boldsymbol{r} = \boldsymbol{r}(u, v),$$

where u and v are independent parameters. A surface given by a vector equation

$$\boldsymbol{r} = \boldsymbol{r}(u, v),$$

is a *smooth one* if the function $\boldsymbol{r}(u, v)$ has continuous derivatives $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ in a domain of definition of the parameters and in this domain, one has

$$\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0.$$

A surface can be determined with the help of three parametrical equations

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v),$$

or in *implicit form* $f(x, y, z) = 0$, or *in explicit form* $z = z(x, y)$. *A local-simple surface* is a connected set in a space E^3 and every point of it has a neighborhood which is *a simple surface*.

A metrical differential form or *the first fundamental form of surface* expressing square of a *linear element of surface* is written in the form

$$\begin{aligned} \mathrm{d}s^2 &= (\partial \boldsymbol{r}/\mathrm{d}u)^2 + 2(\partial \boldsymbol{r}/\mathrm{d}u)(\partial \boldsymbol{r}/\mathrm{d}v) + (\partial \boldsymbol{r}/\mathrm{d}v)^2 \\ &= E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\,\mathrm{d}v^2 = A^2\,\mathrm{d}u^2 + 2AB\cos\chi\,\mathrm{d}u\mathrm{d}v + B^2\,\mathrm{d}v^2. \end{aligned}$$

Surfaces having the same first general fundamental form and, that is why, having the same inner geometry are called *isometric surfaces*.

The second fundamental form of surface is written in the form

$$-\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{n} = L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\mathrm{d}v + N\,\mathrm{d}v^2,\ \text{where } \boldsymbol{n} = [\boldsymbol{r}_u \boldsymbol{r}_v] / \left[A^2B^2 - F^2\right]^{1/2}.$$

A surface clarifies itself with the accuracy to movement by its two fundamental forms.

Geometricians have tried repeatedly to create full classification of surfaces. L. Euler first divided surfaces into *algebraic* and *transcendental* and gave a classification of second-order algebraic surfaces.

According to the sign of Gaussian curvature $K = k_1 k_2$, surfaces are divided into *surfaces of positive, zero,* and *negative Gaussian curvature*. In the literature on descriptive geometry, they divide surfaces into *smooth* and *hipped.*

Two lines of *principal curvatures* pass through every non-umbilical point of a surface and they generate the only net on the surface orthogonal ($F = 0$) and conjugate ($M = 0$) simultaneously. *Geodesic line* may be a line of principal curvature only in the case when this line is plane. The coordinate net u, v in which the opposite sides of a curvilinear quadrangle, formed by it, have the same length is called a *Tchebychef's coordinate net.* A coordinate net is called a *half-Tchebychef's coordinate net* if every curvilinear quadrangle has two equal opposite sides but two other opposite sides are not equal.

A vector product $[\boldsymbol{r}_u \boldsymbol{r}_v]$ in *singular points* turns into zero. If this product is not equal to zero then the points are *ordinary points*. There are no asymptotical directions in *elliptic point* ($K > 0$) of the surface; *hyperbolic point* ($K < 0$) has two asymptotical directions and there is one asymptotical direction in *parabolic point* ($K = 0$). A surface may have both elliptical and hyperbolic points. A boundary between the region of elliptic and hyperbolic points will consist of parabolic points. Every direction in *an umbilical point* of the surface is a principal direction. Umbilical points are characterized by the equation $H^2 - K = 0$ (Euler difference).

In *a plane point* of the surface, we have $L = M = N = 0$. A surface is a plane or its part if K and H are equal to zero identically, but a point in which $K = H = 0$ is a plane point.

Further, the definitions of some classes of surfaces will be presented without any commentaries. These surfaces do not form separate chapters but are distributed among other chapters.

Liouville surface has a linear element of surface ds^2 in the following form:

$$\mathrm{d}s^2 = [\varphi(u) - \psi(v)]\left(\mathrm{d}u^2 + \mathrm{d}v^2\right).$$

Central surfaces of the second order belong to Liouville surfaces.

A polar surface of a given curve is an envelope surface of a family of the normal planes of this curve L. Under the definition, every curve L intersects all tangent planes of its polar surface at right angle. A polar surface of plane line is a cylindrical surface. A polar surface of every spherical line is a conical surface with a vertex coinciding with a center of the sphere. A developable surface developing a family of rectifying planes of a line L is called *a rectifying surface of the line L*. A curve L becomes a straight line under bending of a rectifying surface of the line L on a plane.

A surface of normals of a curve placed on a surface is formed by normals to the surface run out in all points of the curve. Under the definition, this surface is a ruled surface. A surface of normals of every line of principal curvature is a developable surface. A family of cuspidal edges of surfaces of normals of lines of principal curvatures of the S surface is called *an evolute surface* or *an evolute of the surface S* or *a surface of centers*. One chamber of a surface of centers gives birth to one family of lines of principal curvatures of the surface S. A surface S in direct of its evolute is called *an evolvent surface*.

A surface P' is *a parallel surface P* if a parameter z in its equation $\boldsymbol{r}' = \boldsymbol{r} + z\boldsymbol{n}$ has a constant value h. An absolute value of a number h is called a distance of the surface P' from P. Surfaces on which Darboux tensor turns into zero identically are called *surfaces of Darboux*.

A middle evolute surface Σ or *middle envelope surface* Σ of a given surface S is called an envelope of planes parallel to tangent planes of the surface S and passing through the middle M of segment between the centers of normal curvatures of lines of principal curvatures.

A middle envelope of *surfaces of Bonnet* degenerates into a plane. *Goursat surface* is connected with the conception of middle envelope surface. Minimal surface is a particular case of Goursat surface. A formula for the determination of Gaussian curvature of *Bianchi surfaces* (*the B surfaces*) was obtained in the following form:

$$K = -1\Big/[U(u) + V(v)]^2,$$

where u, v are asymptotical parameters of the surface. Sometimes, *surfaces with two families of plane lines of principal curvatures* and *surfaces with one family of plane lines of principal curvatures* are picked out in special classes. *Surfaces with one spherical family of lines of principal curvatures* and surfaces with one plane family of lines of principal curvatures have close connection permitting to get any of these surfaces by transformation of other one.

A general Weddle surface W in three-dimensional space is geometric locus of vertexes of cones of the second order containing six given points. This surface is the fourth-order surface.

The third-order spherical surface forms by perpendiculars dropped to the tangent planes of a paraboloid of revolution F from a point P. Niče Vilko has studied these surfaces and solved some positional problems on it.

N.S. Gumen has published series of papers devoted to *one-sheet* and *two-sheet elliptical paraboloids. A Voss net* is conjugate and geodesic net but a surface with such net is called *the Voss surface*. Minimal surface is the Voss surface. Non-degenerating surface of the second order cannot be the Voss surface. There is no Voss surface of constant negative curvature. The right helicoid is a surface with infinite number of Voss nets.

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Extract of Basic Formulas

Extract of Basic Formulas from Differential Geometry of Spatial Curves and Surfaces

In this book, definitions of plane and spatial curves are presented and formulas from differential geometry of spatial curves and surfaces are given. This information is necessary for understanding the main text of the encyclopedia and is used subsequently.

In order to locate points, lines, and surfaces their positions must first be referenced to some well-known position. The Cartesian coordinate system, commonly used in mathematics and graphics, locates the positions of geometric forms in 2-D and 3-D space. The right-hand rule is used to determine the positive direction of the axes. The right-hand rule defines the x, y, and z axes, as well as the positive and negative directions of rotation on each axis.

The Methods of Representation of Curves

C. Jordan presented the following definition: "A curve in the space is a set of points of the space coordinates of which x, y, z are continuous functions

$$x = x(t),\ y = y(t),\ z = z(t)$$

of some parameter t changing on the interval $[a, b]$ of numerical axis."

(1) Parametrical equations of a curve:

$$x = x(t),\ y = y(t),\ z = z(t),$$

where t is a variable parameter.

(2) Vector equation of a curve:

$$\boldsymbol{r} = \boldsymbol{r}(t) = \{x(t),\ y(t),\ z(t)\} = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}.$$

If a curve is projected reciprocally definitively on any segment of the coordinate axis x, then one can point out the simplest representation for a curve:

$$\boldsymbol{r} = \boldsymbol{r}(x) = \{x,\ y(x),\ z(x)\}.$$

(3) A curve clarifies itself by its curvature $k(s)$ and torsion $\kappa(s)$ definitively accurate to the movement in the space and that is why it is possibly to define a curve with the help of two equations

$$k = k(s) \text{ and } \kappa = \kappa(s).$$

These are *natural equations of curve*. This representation does not depend on the choice of axes of coordinates. If natural equations are given, then the determination of a curve consists of integration of the equation of Frenet.

(4) Representation of plane curve in polar coordinates:

$$\boldsymbol{r} = \boldsymbol{r}(\varphi) = r(\varphi)\cos\varphi\,\boldsymbol{i} + r(\varphi)\sin\varphi\,\boldsymbol{j}.$$

A polar system of coordinates on plane is defined by a point O (*a pole*) and a directed straight Ox (*a polar axis*); φ is *a polar angle*, $r(\varphi)$ is *a polar radius*.

(5) The line of intersection of two surfaces $F^{(1)}(x, y, z) = 0$ и $F^{(2)}(x, y, z) = 0$ is a curve the points of which satisfy each from two equations. A line of intersection can have more than one branch.

Ordinary and Singular Points of Plane Curve

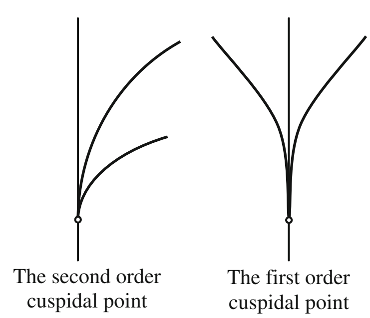
A point $M_0(x_0, y_0)$ is *an ordinary point* if values of partial derivatives $F_x(x_0, y_0)$, $F_y(x_0, y_0)$ do not reduce to zero simultaneously in the given point $M_0(x_0, y_0)$ of a plane curve $F(x, y) = 0$. So, the points of a curve $F(x, y) = 0$ in which $F_x(x_0, y_0) = F_y(x_0, y_0) = 0$, i.e., both partial derivatives reduce to zero simultaneously and are singular points. Two examples of double singular points of a plane curve are shown in Fig. 1. Near the singular point, a curve can be presented with the help of an equation $y = y(x)$.

A curve coincides with its tangent accurate to the second order at least in *a point of straightening* M_0 where $y_0'' = y''(x_0) = 0$. A point M_0 is called *an inflection point* if the curve is placed over its tangent on one side from the point M_0 but under its tangent on the other side. Sometimes, there are no points of a curve in the rather small vicinity of a point $M_0(x_0, y_0)$ except the point M_0 itself. In this case, a point M_0 is called *an isolated point of a curve*.

Asymptotes of Plane Curve

A straight is called *an asymptote of a curve* $x = x(t)$, $y = y(t)$, if the distance a point M on the curve from this straight tends to zero when a point M is moving to infinity. An angle coefficient k of an asymptote not parallel to an axis Oy is a limit of a relation $y(t)/x(t)$. If an asymptote is parallel to an axis Oy then an equation of the asymptote is of the following form: $x - a = 0$, where a = const. A parameter a is the limit of $x(t)$ if $t \to T$.

Fig. 1



Ordinary and Singular Points of Spatial Curve

A point $t = t_0$ is *an ordinary point* if the derivatives $x'(t_0), y'(t_0), z'(t_0)$ in the given point do not all go to zero simultaneously.

If a curve

$$\boldsymbol{r} = \boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

has the points where $\boldsymbol{r}''\|\boldsymbol{r}'$, then these points are *points of straightening*. Tangency of a curve with its tangent in the point of straightening is of the second order at least.

Tangent, Principal Normal, and Binormal of Curve

A straight line passing through a point $M[x(t), y(t), z(t)]$ in the direction of the vector

$$r'(\boldsymbol{t}) = \boldsymbol{x}'(\boldsymbol{t})i + \boldsymbol{y}'(\boldsymbol{t})j + \boldsymbol{z}'(\boldsymbol{t})k$$

is *a tangent* in a given point $M(t)$ of a curve and that is why a radius-vector of a point on the tangent can be written in the form

$$\boldsymbol{r}(\lambda) = \boldsymbol{r}(t_0) + \lambda\boldsymbol{r}'(t_0).$$

An equation of tangent straight of a spatial curve can also be written in the following form:

$$\frac{X - x(t)}{x'(t)} = \frac{Y - y(t)}{y'(t)} = \frac{Z - z(t)}{z'(t)},$$

where X, Y, Z are the current coordinates.

Binormal of a curve is directed along the vector $[\boldsymbol{r}', \boldsymbol{r}'']$.

Principal normal is given by the vector

$$[\boldsymbol{r}', [\boldsymbol{r}', \boldsymbol{r}'']] = \boldsymbol{r}'(\boldsymbol{r}', \boldsymbol{r}'') - \boldsymbol{r}''(\boldsymbol{r}')^2.$$

Curvature and Torsion of Curve

A twice differentiable curve $\boldsymbol{r} = \boldsymbol{r}(s)$ has the curvature k in every point and this curvature can be determined with the help of the formula:

$$k = |\boldsymbol{r}''(s)|.$$

If a curve is given in arbitrary parameterization $\boldsymbol{r} = \boldsymbol{r}(t)$ then its curvature k is defined by

$$k = \frac{\sqrt{\begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}^2 + \begin{vmatrix} x' & x'' \\ z' & z'' \end{vmatrix}^2 + \begin{vmatrix} y' & y'' \\ z' & z'' \end{vmatrix}^2}}{(x'^2 + y'^2 + z'^2)^{3/2}}.$$

Curvature k and a radius of curvature R of a curve is mutually inverse magnitudes, i.e.,

$$k = 1/K$$

Thrice differentiable curve $\boldsymbol{r} = \boldsymbol{r}(s)$ in every point with curvature $k \neq 0$ has torsion κ given by

$$\kappa = (\boldsymbol{r}'(s), \boldsymbol{r}''(s), \boldsymbol{r}'''(s))\big/k^2.$$

If a curve is given in arbitrary parameterization $\boldsymbol{r} = \boldsymbol{r}(t)$ then its torsion κ is determined as

$$\kappa = \frac{(\boldsymbol{r}'(t),\ \boldsymbol{r}''(t),\ \boldsymbol{r}'''(t))}{|[\boldsymbol{r}'(t),\ \boldsymbol{r}''(t)]|^2} = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2}.$$

Curvature of a curve given in polar coordinates r, φ is:

$$k = \left[r^2 + 2\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 - r\frac{\mathrm{d}^2 r}{\mathrm{d}\varphi^2}\right]\Bigg/\left[\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2}\right]^3.$$

Arc Length of Curve

Length of a broken line inscribed in a curve $\boldsymbol{r}(t)$ approaches to a limit

$$s = \int_a^b |\boldsymbol{r}'(t)|\,\mathrm{d}t.$$

under infinite reducing to fragments. This limit is called *a length of curve*.

An expression for determination of length of a curve can be written as

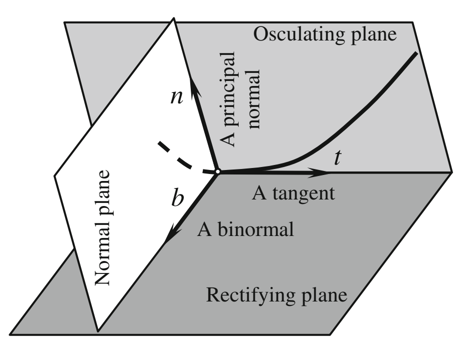
$$s = s(t) = \int_a^b \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\,\mathrm{d}t.$$

Frenet' s Formulas

If a curve $\boldsymbol{r} = \boldsymbol{r}(s)$ is referred to a parameter s (arc length of a curve), then unit vectors of moving trihedral (Fig. 2) will be definitively determinate functions of s:

$$\boldsymbol{t} = \boldsymbol{t}(s),\ \boldsymbol{n} = \boldsymbol{n}(s),\ \boldsymbol{b} = \boldsymbol{b}(s).$$

Fig. 2



An analytical sense of Frenet's formulas consists in decomposition of derivatives of vectors $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ with respect to the arc s along these vectors themselves:

$$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = k\boldsymbol{n},\ \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}s} = -\kappa\boldsymbol{n},\ \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}s} = \kappa\boldsymbol{b} - k\boldsymbol{t}.$$

Rectifying, Normal, and Osculating Planes of Curve (Fig. 2)

There is an innumerable set of normals in a given point of a curve. All of them are perpendicular to the tangent in the same point and lie in the normal plane of a curve:

$$x'(t)[X - x(t)] + y'(t)[Y - y(t)] + z'(t)[Z - z(t)] = 0.$$

Osculating plane of a curve passes through the vectors $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$. The osculating plane of a plane curve coincides with a plane in which the curve lies. An equation of osculating plane can be presented in the form of a mixed product of three vectors:

$$(\boldsymbol{R} - \boldsymbol{r}(t_0), \boldsymbol{r}'(t_0), \boldsymbol{r}''(t_0)) = \begin{vmatrix} X - x(t_0) & x'(t_0) & x''(t_0) \\ Y - y(t_0) & y'(t_0) & y''(t_0) \\ Z - z(t_0) & z'(t_0) & z''(t_0) \end{vmatrix} = 0.$$

An osculating plane of a curve passing through the M point has the highest possible order of tangency with the curve in this point M. A plane having the tangency of the second order in a given point M is called an osculating plane. A curve has one and only one osculating plane in each of its points.

Planes passing through a tangent in a point M have the first-order tangency with a given curve in the given point M. Such planes are called tangent planes to a curve.

Assume that a spatial curve is defined as a line of intersection of two surfaces $F^{(1)}(x, y, z) = 0$ and $F^{(2)}(x, y, z) = 0$. In this case, an equation of *a normal plane* of a curve of intersection of two surfaces is

$$\begin{vmatrix} X - x_0 & Y - y_0 & Z - z_0 \\ F_x^{(1)} & F_y^{(1)} & F_z^{(1)} \\ F_x^{(2)} & F_y^{(2)} & F_z^{(2)} \end{vmatrix} = 0.$$

An equation of a tangent straight line to the line of intersection of two surfaces is defined by

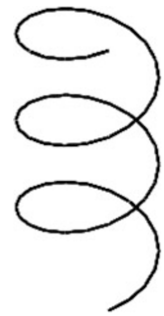
$$\frac{X - x_0}{\begin{vmatrix} F_y^{(1)} & F_z^{(1)} \\ F_y^{(2)} & F_z^{(2)} \end{vmatrix}} = \frac{Y - y_0}{\begin{vmatrix} F_z^{(1)} & F_x^{(1)} \\ F_z^{(2)} & F_x^{(2)} \end{vmatrix}} = \frac{Z - z_0}{\begin{vmatrix} F_x^{(1)} & F_y^{(1)} \\ F_x^{(2)} & F_y^{(2)} \end{vmatrix}}.$$

Cylindrical Helix

Spatial curves, all tangent lines that constitute the same angles with a definitive plane are called *lines of constant slope. Tangent developable surfaces* of these curves are *developable surfaces of equal slop*e.

A cylindrical helical line of equal slope (*a helix*) is placed on a cylinder and can be given by the vector equation:

Fig. 3



$$\boldsymbol{r} = \boldsymbol{r}(t) = a\cos t\boldsymbol{i} + a\sin t\boldsymbol{j} + bt\boldsymbol{k},$$

where a is a radius of the circular cylinder; $L = 2\pi b$ is a pitch of the helix (Fig. 3).

An equation of a tangent to helix is

$$\boldsymbol{r} = \boldsymbol{r}(u) = \{a\cos t - ua\sin t;\ a\sin t + ua\cos t;\ b(t+u)\}.$$

An equation of a normal plane is

$$ax\sin t - ay\cos t - bz + b^2 t = 0.$$

An equation of a binormal of helix is

$$\boldsymbol{r} = \boldsymbol{r}(u) = \{a\cos t + ub\sin t;\ a\sin t - ub\cos t;\ bt + au\}.$$

An equation of an osculating plane is

$$bx\sin t - by\cos t + az - abt = 0.$$

An equation of the principal normal of helix is

$$\boldsymbol{r} = \boldsymbol{r}(u) = \{(a+u)\cos t;\ (a+u)\sin t;\ bt\}.$$

An equation of a rectifying plane is

$$x\cos t + y\sin t + bz - \left(a + b^2 t\right) = 0.$$

Conical Spiral Lines

A conical spiral line (*a conical spiral*) is placed on *a circular cone*. Conical spiral lines with constant lead and conical spiral lines of constant slope (Fig. 4) are the most known curves.

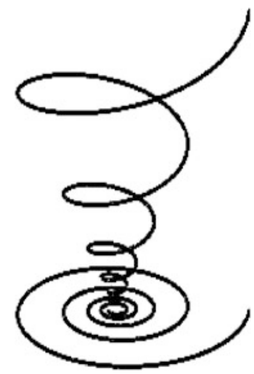
A magnitude of displacement of a point of conical spiral line in the direction of the cone axis per revolution around the cone axis is *a pitch of conical spiral line*.

A conical spiral line with constant pitch is projected on a plane, perpendicular to a cone axis, in the form of *a spiral of Archimedes*. A pole of a spiral of Archimedes is a projection of a vertex of the circular cone. A spiral of Archimedes in polar coordinates is defined by the equation

$$\rho = c_0\varphi + c_1,$$

where c_0 and c_1 are constants.

Fig. 4



A conical spiral line of constant slope (Fig. 4)

$$x = a\cos\varphi\, e^{k\varphi},\ y = a\sin\varphi\, e^{k\varphi},\ z = be^{k\varphi}$$

has a horizontal projection in the shape of *logarithmic spiral* with a pole in a point coinciding with the horizontal projection of vertex of a cone of revolution. Logarithmic spiral intersects all rays emerging from the pole, under the constant angle and

$$\sin\varphi_0 = 1\Big/\sqrt{k^2+1}.$$

Spherical Lines

Tangents to *a spherical line of equal slope* have a constant angle of slope to a plane perpendicular to an axis of the spherical line of constant slope.

A spherical loxodrome is a curve intersecting all meridians of the sphere under the same angle.

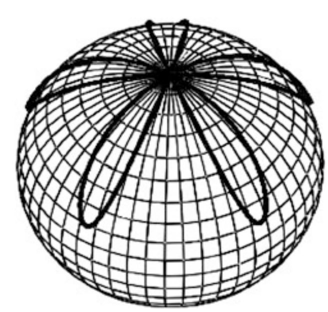
A condition for a spherical curve:

$$\frac{\kappa}{k} + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{k}\right)\right) = 0.$$

Assuming D as a closed curve with a length L lying on a sphere with radius R (Fig. 5), then the following inequality must exist:

$$L\int_{\Gamma} \kappa^2\, \mathrm{d}s \le R^2\left[\int_{\Gamma} k^2\, \mathrm{d}s \int_{\Gamma} \kappa^2\, \mathrm{d}s - \left(\int_{\Gamma} k\kappa\, \mathrm{d}s\right)^2\right].$$

Fig. 5



Equations of Some Plane Curves Used in the Encyclopedia

Neil's parabola (a semicubical parabola): $y = ax^{3/2}$;
Agnesi curl: $x^2y = 4a^2(2a - y)$; a logarithmic spiral: $\rho = ae^{b\varphi}$;
an astroid: $x^{2/3} + y^{2/3} = a^{2/3}$; a cycloid: $x = at - b\sin t$, $y = a - b\cos t$;
a catenary: $y = a\,\mathrm{ch}(x/a)$; a tractrix: $x = a[\cos t + \ln\mathrm{tg}(t/2)]$, $y = a\sin t$;
a spiral of Archimedes: $\rho = a\varphi$; an ellipse: $x = a\cos u$, $y = b\sin u$;
a parabolic spiral: $\rho^2 = 2p\varphi$; a hyperbolic spiral: $\rho = a/\varphi$;
an evolvent of the circle: $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$.

A Tangent Plane of the Surface

Assume an implicit equation of a surface as $F(x, y, z) = 0$. In this case, an equation of a tangent plane to the surface will be of the following form:

$$F_x(x,y,z)(X-x) + F_y(x,y,z)(Y-y) + F_z(x,y,z)(Z-z) = 0,$$

where X, Y, Z are the current coordinates. Having assumed X, Y, Z as current coordinates of the tangent plane to a surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$ one can write an equation of this plane in the form

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x_u & y_u & z_u \\ x_u & y_u & z_u \end{vmatrix} = 0.$$

A Normal to the Surface

A perpendicular to a tangent plane in the point of tangency is called *a normal to surface*. It is easy to obtain an equation of normal as an equation of a straight passing through a given point $M(x, y, z)$ in a given direction:

$$\frac{X-x}{F_x(x,y,z)} = \frac{Y-y}{F_y(x,y,z)} = \frac{Z-z}{F_z(x,y,z)},$$

where X, Y, Z are the current coordinates.

If a surface is given by its radius-vector $\boldsymbol{r} = \boldsymbol{r}(u, v)$ then a vector product

$$[r_u r_v] = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

shows a direction of the normal to a surface in a given point but a unit vector of the normal $\boldsymbol{m}$ to a surface is defined by the formula

$$\boldsymbol{m} = \frac{[\boldsymbol{r}_u \boldsymbol{r}_v]}{\sqrt{EG - F^2}}.$$

Formulas for the Determination of Coefficients of the Fundamental Forms in the Theory of Surfaces

Coefficients of the first fundamental form of a surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$:

$$E = A^2 = \boldsymbol{r}_u\boldsymbol{r}_u = x_u^2 + y_u^2 + z_u^2,$$
$$F = \boldsymbol{r}_u\boldsymbol{r}_v = x_u x_v + y_v y_v + z_v z_v,$$
$$G = B^2 = \boldsymbol{r}_v\boldsymbol{r}_v = x_v^2 + y_v^2 + z_v^2.$$

A coefficient $F = AB\cos\chi$, hence,

$$\cos\chi = \frac{F}{AB}.$$

So, if coefficient F is equal to zero then $\chi = 90°$.

Coefficients of the second fundamental form of surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$:

$$L = \frac{(\boldsymbol{r}_{uu}\boldsymbol{r}_u\boldsymbol{r}_v)}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{1}{\sqrt{A^2B^2 - F^2}}\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},$$
$$M = \frac{(\boldsymbol{r}_{uv}\boldsymbol{r}_u\boldsymbol{r}_v)}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{1}{\sqrt{A^2B^2 - F^2}}\begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},$$
$$N = \frac{(\boldsymbol{r}_{vv}\boldsymbol{r}_u\boldsymbol{r}_v)}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{1}{\sqrt{A^2B^2 - F^2}}\begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}.$$

Six coefficients of the fundamental forms in the theory of surfaces follow two equations of Peterson and Codazzi

$$\frac{\partial L}{\partial v} - \frac{\partial M}{\partial u} = \Gamma_1^{12}L - \left[\Gamma_1^{11} - \Gamma_2^{12}\right]M + \Gamma_2^{11}N;$$
$$\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v} = \Gamma_2^{12}N - \left[\Gamma_2^{22} - \Gamma_1^{12}\right]M - \Gamma_1^{22}L;$$

where Γ_i^{jk} are symbols of Christoffel, and they also follow the equation of Gauss

$$\frac{LN - M^2}{A^2B^2\sin^2\chi} = \frac{-1}{AB\sin\chi}\left\{\frac{\partial^2\chi}{\partial u\partial v} + \frac{\partial}{\partial u}\left[\frac{\frac{\partial B}{\partial u} - \frac{\partial A}{\partial v}\cos\chi}{A\sin\chi}\right] + \frac{\partial}{\partial v}\left[\frac{\frac{\partial A}{\partial v} - \frac{\partial B}{\partial u}\cos\chi}{B\sin\chi}\right]\right\}.$$

Lengths of Curves and Angles Between Curves on Surfaces

Coefficients of the first fundamental form of surface give an opportunity to determine lengths of curves on a surface. If a portion of any curve $u = u(t)$, $v = v(t)$, $t_1 \le t \le t_2$ on a surface is given, then accurate length of this portion is obtained from the formula

$$s = \int_{t_1}^{t_2} \sqrt{E(u,v)\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + 2F(u,v)\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t} + G(u,v)\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

The angles between arbitrary curves on a surface one can be determined with the help of the formula

$$\cos(\mathrm{d}\boldsymbol{r}, \delta\boldsymbol{r}) = \frac{E\,\mathrm{d}u\delta u + F(\mathrm{d}u\delta v + \mathrm{d}v\delta u) + G\,\mathrm{d}v\delta v}{\sqrt{E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\,\mathrm{d}v^2}\sqrt{E\,\delta u^2 + 2F\,\delta u\delta v + G\,\delta v^2}}.$$

If one seeks an angle χ between curvilinear coordinate lines u and v, then

$$\cos\chi = \pm\frac{F}{\sqrt{EG}} = \pm\frac{F}{AB}.$$

Length of the curvilinear coordinate line u belonging to a surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$:

$$s_u = \int_{u_1}^{u_2} A\,\mathrm{d}u.$$

Length of the curvilinear coordinate line v belonging to a surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$:

$$s_v = \int_{v_1}^{v_2} B\,\mathrm{d}v.$$

Area of Surface

For the determination of an area of full surface or its fragment one can use the formula:

$$S = \iint_D \sqrt{EG - F^2}\,\mathrm{d}u\mathrm{d}v = \iint_D \sqrt{A^2B^2 - F^2}\,\mathrm{d}u\mathrm{d}v.$$

Curvatures of Lines on a Surface and Curvatures of Surface

A curvature of the normal section of a surface in a point M:

$$k_m = \frac{L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\mathrm{d}v + N\,\mathrm{d}v^2}{E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\,\mathrm{d}v^2}.$$

If principal curvatures k_1 and k_2 of a surface are known, then for determination of normal curvature k_m in arbitrary direction, they use the formula of Euler:

$$k_m = k_1\cos^2\zeta + k_2\sin^2\zeta.$$

Curvatures of the curvilinear coordinate lines u and v on a surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$:

$$k_u = \frac{L}{E} = \frac{L}{A^2}, \quad k_v = \frac{N}{G} = \frac{N}{B^2}.$$

The Gaussian curvature of a surface:

$$K = k_1 k_2 = \frac{LN - M^2}{EG - F^2} = \frac{LN - M^2}{A^2B^2 - F^2}.$$

A sign of the Gaussian curvature is defined by the expression $LN - M^2$.

The Gaussian curvature is positive in *elliptical points*, negative in *hyperbolic points*, and equal to zero in *parabolic points*.

The mean curvature of a surface is:

$$H = \frac{k_1 + k_2}{2} = \frac{1}{2}\frac{LB^2 - 2MF + NA^2}{A^2B^2 - F^2}.$$

The surface integral of the Gaussian curvature over some region of a surface is called *the total curvature*.

The principal curvatures k_1, k_2 of the surface are roots of the quadratic equation:

$$\begin{vmatrix} L - kA^2 & M - kF \\ M - kF & N - kB^2 \end{vmatrix} = 0.$$

A differential equation of lines of principal lines of a surface are:

$$\begin{vmatrix} \mathrm{d}v^2 & -\mathrm{d}u\mathrm{d}v & \mathrm{d}u^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0.$$

Directrix and Generatrix Curves

Just as a line represents the path of a moving point, a surface represents the path of a moving line called *a generatrix*. A generatrix can be a straight or a curved line. The path that the generatrix travels is called *the directrix*. A directrix can be a point, a straight line, or a curved line.

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Ruled Surfaces

1

A surface formed by the continuous movement of a straight line is called *a ruled surface* or a ruled surface, also known as *a scroll surface S*, is the result of movement of a straight line along a curve.

A surface is called *an elementary ruled surface* if a straight line passing through every point P of this surface has the common straight-line segment with a surface containing the point P. But the ends of this straight-line segment may not belong to the surface. Straight lines belonging to a ruled surface are called *rectilinear generatrixes* or *rectilinear generators* or *rulings*. A curve intersecting all rectilinear generatrixes of the surface is called *a directrix* (*a directing curve*, or *a directrix curve*, or *a director curve*).

A vector equation of a ruled surface can be written in the following form:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{a}(v) + u\boldsymbol{b}(v),$$

where $\boldsymbol{a}(v)$ is the radius vector of a directrix curve, $\boldsymbol{b}(v)$ is the directrix vector of a rectilinear generatrix. The conditions $b''(v) \neq 0$ and $(b,\, b',\, b'') = 0$ hold true for *Catalan's surfaces*. Ruled surface cannot be a surface of positive Gaussian curvature. The Gaussian curvature of ruled surface is negative or equal to zero. Ruled surfaces cannot have constant mean curvature with the exception of plane and right helicoid. Rectilinear generatrixes are asymptotic lines. If all points of a ruled surface are parabolic points, then rectilinear generatrixes are the lines of principal curvatures. Thus, rectilinear generatrix of a not developable surface will not be a line of principal curvature of ruled surface.

The surfaces F_1 and F_2 will be ruled surfaces if not trivial bending of any surface F_1 into a surface F_2 is *parabolic bending*.

Ruled surfaces of zero Gaussian curvature (developable surfaces)

1. *Not degenerated developable surfaces* (*torse surfaces*, or *torsal ruled surfaces*, or *torses*) are the surfaces of tangent lines to their edges of regression,
2. *Cylindrical surfaces*,
3. *Conical surfaces*,
4. *A plane* is the only minimal developable surface.

Ruled surfaces of negative Gaussian curvature

1. *Oblique ruled surfaces of the second order* (twice ruled surfaces):
   - *Hyperbolic paraboloids*,
   - *One-sheet hyperboloids*,
   - *One-sheet hyperboloids of revolution*.
2. *Oblique ruled surfaces of more than the second order* and *not algebraic oblique ruled surfaces* including
   - *Conoids* (They are ruled surfaces from a family of Catalan's surfaces),
   - *Oblique* and *right helicoids* (Right helicoid is the only ruled surface from a class of minimal surfaces),
   - *Cylindroids* are ruled surfaces from a family of Catalan's surfaces,
   - *Surfaces of an oblique transition*,
   - *Surface of an oblique wedge*,
   - *Surface of a double oblique conoid*,
   - *Surfaces of an oblique cylinder* and *surfaces of a double oblique cylindroid.*

The only ruled surfaces of revolution are a one-sheet hyperboloid of revolution, right circle cylinder, and right circle cone. The last two surfaces are the only developable surfaces of revolution. *The geodesic orthogonal system of coordinates* can be put only on developable surfaces.

Practically, all methods of forming ruled surfaces are based on movement of a straight line on the two given contour curves.

In several papers, machining a ruled surface with a conical tool is discussed. The main goal of the spatial tangent points shift method is to minimize the errors between the given surface and the machined surface. A three-step-optimization is applied. In each step, the cutting tool is tangential to two guiding rails.

In computer graphic, we are interested in modeling and animating objects seen in everyday life, and many objects can be approximated by ruled surfaces and especially by piecewise continuous developable surfaces. But some

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researchers consider that while developable surfaces have been widely used in engineering, design, and manufacture, they have been less popular in computer graphics, despite the fact that their isometric properties make them ideal primitives for texture mapping, some kinds of surface modeling, and computer animation. In practice, ruled surfaces are of value in avant-garde architecture.

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1.1 Ruled Surfaces of Zero Total Curvature

Cylindrical, conical, *torse surfaces*, a plane, and surfaces of polyhedrons are called surfaces of *zero Gaussian curvature* or *developable surfaces*.

A surface is called a developable surface if it can be developed on a plane without any lap fold or break. During this process, the length of the curves and the angles between two curves belonging to the developable surface remain unchanged. Cylinders and cones are the simplest developable surfaces, being degenerate. All surfaces having zero Gaussian (total) curvature $K = k_1 k_2 = 0$ may be produced by bending a part of a plane.

Let us put down some well-known theorems and definitions for surfaces of zero total curvature without proofs.

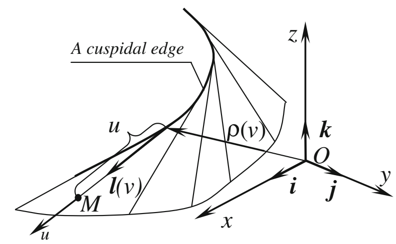

Fig. 1

Any developable surface is either *a cylindrical surface*, or *a conical surface*, or else *a surface of tangent lines* of arbitrary curve. *Any surface of tangent lines* is a developable surface. A developable surface is generated by tangent lines of *the edge regression* (*cuspidal edge*) of the developable surface (Fig. 1). Any spatial curve can be taken as an edge of regression and the tangent lines of this curve will generate the particular developable surface. The cuspidal edge of a cone is a point (vertex of the cone).

A surface of principal normals and that of *binormals* of any spatial curve cannot be a developable surface. A surface of binormals of a curve can be a surface of zero total curvature only when a given curve is a plane curve. In that case, a ruled surface will be a cylindrical surface.

A surface of tangent lines of a plane curve is *a plane*. For a given spatial closed curve, only two surfaces of zero total curvature, rested upon it, exist and their every generatrix intersects the given curve in two points.

Only a degenerated developable surface in the shape of a plane may be *a minimal surface*.

If a surface formed by a continuous family of lines along which a tangent plane remains unaltered, then this surface is either a cylindrical surface, or a conical surface, or else a surface of tangents and each from mentioned lines is a straight generatrix. If two developable surfaces touch each other along any line, then this line is a common generatrix.

Developments of developable surfaces of general type differ from developments of cylinders and cones. Rectilinear generatrixes of developable surfaces of general type do not intersect themselves in one point as in a development of a cone and are not parallel as in a development of a cylinder.

Reducing the torsion of a cuspidal edge but keeping constant its curvature, it is possible to generate the developable surface on a plane. The rectilinear generatrix of a developable surface will remain rectilinear and tangent to the degenerated plane cuspidal edge. Bending with preservation of generatrixes of developable surfaces is *parabolic bending* and this is necessary and sufficient condition for parabolic bending.

The Lamé coefficients of a developable surface undergoing parabolic bending do not change. According to Love's formulas, one obtains

$$\kappa_1 = \frac{1}{R_1} - \frac{1}{R_1^*} + \frac{\varepsilon_2}{R_1^*} = 0, \quad \kappa_{12} = 0,$$
$$\kappa_2 = \frac{1}{R_2} - \frac{1}{R_2^*} + \frac{\varepsilon_2}{R_2^*} = \frac{1}{R_2} - \frac{1}{R_2^*},$$

where R_1^*, R_2^* are the radii of principal curvatures.

Rectilinear generatrixes of a developable surface are *asymptotical lines*. The single asymptotic line (a rectilinear generatrix), passing through every point of the surface, will be conjugate to any other line, passing through the same point. All geodesic lines of a developable surface turn into straight lines after bending. One can pass a geodesic line through every point in any direction on every developable surface. Only ruled surfaces of zero total curvature accommodate bending under which all asymptotic lines go to asymptotic lines.

Normals to a surface along the lines of principal curvature and only along the lines of principal curvature form a developable surface but the corresponding center of curvature form the cuspidal edge.

If a straight line and an orthogonal curve a intersecting it are given, then the only surface Φ with parabolic points containing these lines exists and these two lines are the lines of principal curvature. If one turns the generatrixes of a developable surface Φ of normals of a line a in the corresponding normal planes through a constant angle, then the new surface will be developable too.

If one of the normal sections not coinciding with the principal direction has a singular point with curvature equal to zero, then all normal cross-sections in this point have the curvature equal to zero.

Since Archimedes time till present, developable surfaces attract the attention of geometricians and engineers. Some developable surfaces have been named in honor of the scientists.

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1.1.1 Torse Surfaces (Torses)

Torse or *torsal surfaces* are nondegenerated ruled developable surfaces with cuspidal edges and they can be developed on a plane without any lap fold or break. All developable surfaces have only parabolic points in which $k_1k_2 = 0$. So, an equality of Gaussian curvature to zero ($K = 0$) is the sufficient and necessary condition for a developable surface.

Every single-parametric system of planes (with the exception of a pencil of planes passing through any axis or parallel to each other) has an envelope surface, which is a developable surface. The planes of the system are osculating planes of the cuspidal edge, which contain the tangent lines and the principal normals to the cuspidal edge. A tangent plane along the rectilinear generatrixes of developable surfaces does not change its position. Nondegenerated developable surface is generated by tangent lines of *the edge regression* (*cuspidal edge*) of the developable surface. Sometimes, a surface of tangent lines is called *a tangent torse*. Any spatial curve can be taken as a cuspidal edge and its tangent lines will generate a developable surface.

The method of designing a developable surface with the help of the known cuspidal edge is widely used in graphical design. Those who want to use analytical method may apply the equation of a developable surface in the vector form

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{a}(v) + u\boldsymbol{l}(v),$$

where $\boldsymbol{a}(v)$ is the radius vector of a cuspidal edge,

$$\boldsymbol{a}(v) = x(v)\boldsymbol{i} + y(v)\boldsymbol{j} + z(v)\boldsymbol{l};$$

$x(v)$, $y(v)$, and $z(v)$ are the parametric coordinates of the cuspidal edge, $\boldsymbol{l}(v)$ is a unit tangent vector, given at every point of the cuspidal edge by

$$\boldsymbol{l}(v) = \boldsymbol{a}(v)'/\left|\boldsymbol{a}(v)'\right|.$$

The coordinate lines u coincide with the straight generatrixes.

In 1805, G. Monge established that one can form a developable surface by movement of a straight line on two arbitrary curves. Using this postulate and having the equations of two curves, it is possible to find the equation of the single-parametric system of the tangent planes $M(x, y, z, v) = 0$ and after this, the equation of the torse in an implicit form or the equation of the cuspidal edge:

$$\boldsymbol{a}(v) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = x(v)\boldsymbol{i} + y(v)\boldsymbol{j} + z(v)\boldsymbol{k}.$$

If we have parametric equations of a cuspidal edge, then coefficients of fundamental forms in the theory of surfaces can be defined by formulas:

$$A = 1; \quad F = \sqrt{x'^2 + y'^2 + z'^2},$$
$$B^2 = F^2 + u^2\left[F^2\left(x''^2 + y''^2 + z''^2\right) - \left(x'x'' + y'y'' + z'z''\right)^2\right]/F^4,$$
$$L = M = 0, \quad N = u^2\begin{vmatrix} x''' & y''' & z''' \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}\left(F^3\sqrt{B^2 - F^2}\right),$$
$$\text{or } B^2 = F^2(1 + u^2k^2), \quad N = uF^2k\kappa,$$

where k is the curvature and κ is the torsion of the edge of regression of examined torse surface.

If we take as the v parameter the length s of a cuspidal edge, then we have the following parametric equation of the cuspidal edge: $x = x(s)$, $y = y(s)$, and $z = z(s)$. In this case, we have

$$\boldsymbol{r} = \boldsymbol{r}(u, s) = \boldsymbol{a}(s) + u\boldsymbol{a}'(s).$$

Using the last formula SP Finikov has obtained

$$A^2 = 1;\ \ F = 1;\ \ B^2 = 1 + u^2k^2;$$
$$L = M = 0;\ \ N = uk\kappa.$$

The equation of a developable surface can be obtained without preliminary determination of the equation of the cuspidal edge. If we have two directrix curves $\boldsymbol{r}_1 = \boldsymbol{r}_1(u)$ and $\boldsymbol{r}_2 = \boldsymbol{r}_2(v)$, we may write the vector equation of a developable surface as

$$\boldsymbol{r}(u,\ \lambda) = \boldsymbol{r}_1(u) + \lambda[\boldsymbol{r}_2(v) - \boldsymbol{r}_1(u)] = \boldsymbol{r}_1(u) + \lambda\boldsymbol{m}(u),$$

where

$$0 \le \lambda \le 1,\ \boldsymbol{r}_2 = \boldsymbol{r}_2(v) = \boldsymbol{r}_2[v(u)] = \boldsymbol{R}(u),$$
$$\boldsymbol{m}(u) = \boldsymbol{R}(u) - \boldsymbol{r}_1(u).$$

The angle between the coordinate lines u, λ does not depend on the λ parameter.

For the developable surface with curvilinear coordinates u, λ Bhattacharya has derived

$$A^2 = [\boldsymbol{m}(u)]^2,\ \ B^2 = \boldsymbol{r}_u\boldsymbol{r}_u = [\mathrm{d}\boldsymbol{r}_1/\mathrm{d}u + \lambda\boldsymbol{m}(u)]^2,$$
$$F = \boldsymbol{m}(u)\mathrm{d}\boldsymbol{r}_1/\mathrm{d}u + \lambda\boldsymbol{m}(u)\boldsymbol{m}_u(u),$$
$$N = \boldsymbol{n}\boldsymbol{r}_{uu} = \boldsymbol{n}[\mathrm{d}^2\mathbf{r}_1/\mathrm{d}u^2 + \lambda\boldsymbol{m}_{uu}(u),\ \ L = M = 0.$$

A developable surface of equal slope may be designed as an envelope surface of the single parametric family of cones of revolution. A method of design of torses, rested on the spatial closed curves, is known too.

A problem of analytical presentation of a torse surface in the space and on the drawing is one of the important problems coming into existence in the process of design, investigation, and machining of technological surfaces.

Developable shells are attractive due to their ability to form different configurations in a plane and in space. These shells can satisfy the various requirements of designers and geometricians who work in civil and industrial engineering, road building, aircraft construction, and shipbuilding.

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P.S.: Additional literature is presented on the corresponding pages of the Sect. "1.1. Ruled Surfaces of Zero Total Curvature".

■ Open Evolvent Helicoid

An open evolvent helicoid or *a torse helicoid* is a developable surface formed by the tangent lines to the helix of constant slope on a circular cylinder with radius a.

Forms of definition of an open evolvent Helicoid

(1) Parametric equations (Fig. 1):

$$x = x(u, v) = a\cos v - au\sin v/m,$$
$$y = y(u, v) = a\sin v + au\cos v/m,$$
$$z = z(u, v) = bv + bu/m,$$

where $m = \sqrt{a^2 + b^2}$, b is the lead of a helix $u = 0$ which is the cuspidal edge, v is an angle measured from an axis Ox, a is a radius of a cylinder on which the helical cuspidal edge is lying.

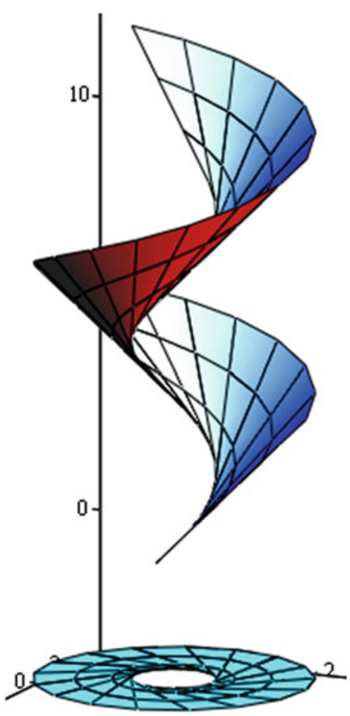

Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1, \quad F = m,$$
$$B^2 = m^2 + u^2a^2/m^2, \quad N = uab/m^2,$$
$$L = M = 0, \quad k_u = k_1 = 0,$$
$$k_v = N/B^2, \quad k_2 = b/(au).$$

The coordinate lines u coincide with the straight generatrixes of the helicoid. The coordinate lines v are the co-axis helixes. Conjugated nonorthogonal system of curvilinear coordinates u, v is used.

(2) Parametric equations (Fig. 1):

$$x = x(u, s) = a\cos\frac{s}{m} - \frac{au}{m}\sin\frac{s}{m},$$
$$y = y(u, s) = a\sin\frac{s}{m} + \frac{au}{m}\cos\frac{s}{m},$$
$$z = z(u, s) = (s + u)b/m,$$

where s is a length of the arc of the helical cuspidal edge, $s = mv$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = F = 1, \quad B^2 = 1 + u^2a^2/m^4,$$
$$N = abu/m^4, \quad L = M = 0,$$
$$k_u = k_1 = 0, \; k_s = N/B^2, \quad k_2 = b/(au).$$

(3) Parametric equations (Fig. 1):

$$x = x(u, v) = a\cos v - u\cos\varphi\,\sin v,$$
$$y = y(u, v) = a\sin v + u\cos\varphi\,\cos v,$$
$$z = z(u, v) = av\,\mathrm{tg}\varphi + u\sin\varphi,$$

where φ is the angle of the slope of rectilinear generators to a plane xOy, but

$$\mathrm{tg}\varphi = b/a.$$

Coefficients of the fundamental forms of the surface:

$$A = 1,\ \ F = a/\cos\varphi,\ \ B^2 = F^2 + u^2\cos^2\varphi,$$
$$B^2 - F^2 = u^2\cos^2\varphi,$$
$$L = M = 0,\ \ N = u\sin\varphi\cos\varphi.$$

(4) Parametric form of definition:

$$x = x(u,s) = a_0\cos^2\varphi\left(\cos\frac{s}{m} - \frac{u}{m}\sin\frac{s}{m}\right),$$
$$y = y(u,s) = a_0\cos^2\varphi\left(\sin\frac{s}{m} + \frac{u}{m}\cos\frac{s}{m}\right),$$
$$z(u,s) = (s+u)\sin\varphi,$$

where

$$m = a_0\cos\varphi,\quad a = a_0\cos^2\varphi,\quad b = a_0\sin\varphi\cos\varphi,$$

a_0 is the radius of the development of the helical cuspidal edge of an open evolvent helicoid on a plane, $a = a_0\cos^2\varphi$.

Let us take an annulus with the inside radius a_0 and cut it along a straight line passing through the point with coordinates $x = a_0$, $y = 0$ and parallel to the y axis. This straight line is the tangent line to the inside contour. Parabolic bending transforms the annulus into an open helicoid with a cuspidal edge in the form of a helix, lying on the cylinder with a radius a. Hence, taking an annulus cut along a tangent to the inside contour, we can write the equation of all class of open helicoids with the same annulus. If we shall take $x = x(u,\varphi), y = y(u,\varphi), x = x(u,\varphi)$ but s = const, then we shall construct a ruled surface of trajectory of the chosen rectilinear generatrix with s = const.

Coefficients of the fundamental forms of the surface:

$$A = F = 1,\ \ B^2 = 1 + u^2/a_0^2 = (a_0^2 + u^2)/a_0^2,$$
$$N = utg\varphi/a_0^2,\ \ L = M = 0.$$

If we intersect an open evolvent helicoid by circular cylinders with radii R_{int} and R_{ext}, the axes of which coincide with the z axis of the open evolvent helicoid then their intersections will be cylindrical helical lines. The surface between these lines is called *Archimedes' screw* or *a helical torse*. If we shall intersect an open evolvent helicoid by a plane perpendicular to its axis, then *an evolvent of the circle* will be in the cross-section.

Open evolvent helicoidal surfaces are formed by a single parametric system of planes and that is why the approximation of open evolvent helicoids by *polyhedral surfaces* is an easy process.

First, Euler determined that an evolvent helicoid is a developable surface.

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■ Monge's Ruled Surface with the Circular Cylindrical Directing Surface

Monge's ruled surface with the circular cylindrical directing surface is formed by a straight line belonging to a plane P when rounding without slip on a directing circular cylinder with the radius r (Fig. 1).

In any arbitrary position, the plane P touches the circle cylinder. The formed surface will be a surface of zero Gaussian curvature. This surface is a part of an open helicoid limited by lines of principal curvatures. A family of straight lines is called *meridians*; the orthogonal trajectories of the points of a generatrix straight line are called *parallels* (Fig. 2). The meridians and parallels are lines of principal

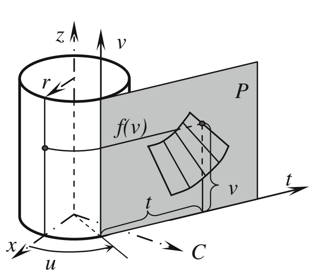

Fig. 1

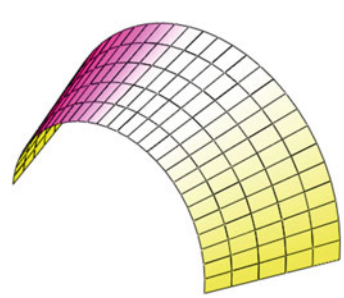

Fig. 2

curvatures. In some papers, this surface is called *a cylindrical ruled rotational limaçon*.

Two families of plane lines of principal curvatures of Monge's ruled surfaces are *geodesic lines* (Fig. 2). G. Monge was the first who investigated these surfaces. When the directing circle cylinder with the radius r degenerates into the z axis, i.e., when $r = 0$, then the orthogonal trajectories of the points of a generatrix straight line, called parallels, will represent themselves the circumferences. But the Monge's surface will be *a surface of revolution*.

Forms of definition of Monge's ruled surface

(1) Parametric form of definition (Fig. 2):

$$x = x(u, v) = r\cos u - t\sin u,$$
$$y = y(u, v) = r\sin u + t\cos u, \quad z = z(v) = v,$$

where u is the angle between the x axis and the normal to the plane P, measured in the direction against hand if to look from the positive direction of the z axis; *t and v* are the right-angled coordinates of arbitrary point of a directrix straight line ($t = cv + b - ru$), the v axis coincides with the line of tangency of the cylinder and plane P.

An equation of a tangent plane corresponding to this form of definition will be as

$$M(x, y, z, u) = x\sin u - y\cos u + cz + b - ru = 0.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = cv + b - ru, \quad F = 0,$$
$$B^2 = 1 + c^2,$$
$$L = A/B, \quad M = N = 0,$$
$$k_u = k_1 = 1/(AB), \quad k_v = k_2 = 0.$$

Parametric equations of the cuspidal edge of the examined torse surface have the following form:

$$x = x(u) = r\cos u; \quad y = y(u) = r\sin u;$$
$$z = z(u) = (ru - b)/c,$$

which are equations of a helix on a circle director cylinder with a radius r.

(2) Equations of Monge's ruled surface in generalized cylindrical coordinates v, u, t:

$$x = x(u, t) = r\cos u - t\sin u,$$
$$y = y(u, t) = r\sin u + t\cos u, \quad z = z(u, t) = v,$$

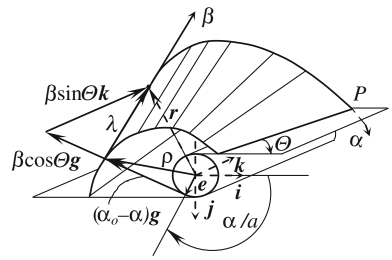

Fig. 3

where $v = (t - b + ru)/c$ and these equations are equivalent to the parametrical equations.

(3) Vector form of definition (Fig. 3):

$$\boldsymbol{r}(\alpha, \beta) = a\boldsymbol{e} + (\alpha_0 - \alpha + \beta\cos\Theta)\boldsymbol{g} + \beta\sin\Theta\boldsymbol{k},$$

where $\boldsymbol{e}$, $\boldsymbol{g}$ are vector circle functions;

$$\boldsymbol{\rho} = a\boldsymbol{e} + (\alpha_0 - \alpha)\boldsymbol{g}$$

is the equation of a parallel which is *an evolvent of the circle* with the radius $r = a$; α is a natural parameter, $\beta = |\lambda|$, Θ is the angle between the vectors $\boldsymbol{g}$ and $\boldsymbol{\lambda}$;

$$\boldsymbol{\lambda} = \beta\cos\Theta\boldsymbol{g} + \beta\sin\Theta\boldsymbol{k}$$

is the equation of a meridian in arbitrary position. The curvilinear coordinate α coincides with the parallels of the surface; the β lines are the rectilinear generatrixes of the surface.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = (\alpha_0 - \alpha + \beta\cos\Theta)/a = C_0 + C_1\alpha + C_2\beta; \quad B = 1,$$
$$L = -A\sin\Theta/a, \quad F = M = N = 0,$$
$$k_1 = \sin\Theta/(aA), \quad k_2 = 0.$$

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■ Developable Conical Helicoid

A developable conical helicoid has a conical spiral

$$x = x(\varphi) = r_0 \sin\lambda \cos\varphi \cdot e^{k\varphi},$$
$$y = y(\varphi) = r_0 \sin\lambda \sin\varphi \cdot e^{k\varphi},$$
$$z = z(\varphi) = r_0 \cos\lambda \cdot e^{k\varphi}$$

as the cuspidal edge (Fig. 1). This conical spiral lies on a circular cone, where λ is the angle between an axis Oz and a directrix of the cone, φ is the angle between a plane xOz and a mobile plane of the axis cross-section; k is any positive or negative constant number; r_0 is a constant value. If any explanations are not given, then they consider that *a conical helicoid of equal slope* is under consideration.

Forms of definition of the surface of a developable conical helicoid

(1) Parametrical equations (Fig. 1):

$$x = x(u,\varphi) = \sin\lambda\left(r_0 e^{k\varphi}\cos\varphi + u\frac{k\cos\varphi - \sin\varphi}{\sqrt{k^2+\sin^2\lambda}}\right),$$
$$y = y(u,\varphi) = \sin\lambda\left(r_0 e^{k\varphi}\sin\varphi + u\frac{k\sin\varphi + \cos\varphi}{\sqrt{k^2+\sin^2\lambda}}\right),$$
$$z = z(u,\varphi) = r_0 e^{k\varphi}\cos\lambda + uk\cos\lambda/\sqrt{k^2+\sin^2\lambda}.$$

Coefficients of the fundamental forms of the surface:

$$A = 1,\quad F = r_0 e^{k\varphi}\sqrt{k^2+\sin^2\lambda},$$
$$B^2 = F^2 + u^2\sin^2\lambda(1+k^2)/F^2,$$
$$L = M = 0,$$
$$N = u\sin\lambda\cos\lambda k r_0 e^{k\varphi}\sqrt{1+k^2}/(k^2+\sin^2\lambda).$$

The torse with the cuspidal edge (u = 0), limited by the conical spiral u = 0.35 m and by the straight generatixes $\varphi = 0$, $\varphi = 2.5\pi$; $\lambda = 0.05\pi$ is shown in Fig. 1. The same torse but limited by the conical spirals u_1 and u_2 ($u_1 = 0.2\,m \le u \le 0.5\,m = u_2$) is presented in Fig. 2.

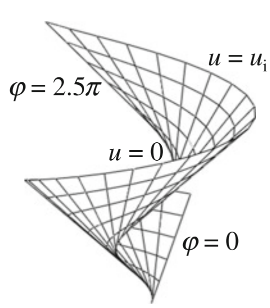


Fig. 1

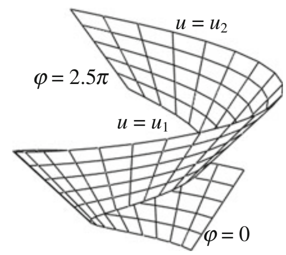


Fig. 2

(2) Parametrical equations (Fig. 1):

$$x = x(u,s) = r_0(as+1)\sin\lambda\cos(\ln|as+1|/k) + u\sin\lambda[k\cos(\ln|as+1|/k) - \sin(\ln|as+1|/k)]/\sqrt{k^2+\sin^2\lambda},$$
$$y = y(u,s) = r_0(as+1)\sin\lambda\sin(\ln|as+1|/k) + u\sin\lambda[k\sin(\ln|as+1|/k) + \cos(\ln|as+1|/k)]/\sqrt{k^2+\sin^2\lambda},$$
$$z = z(u,s) = r_0(as+1)\cos\lambda + uk\cos\lambda/\sqrt{k^2+\sin^2\lambda},$$

where s is the length of the conical spiral (the edge of regression); $|u|$ is a distance from the edge of regression along its tangent till arbitrary point on the surface of the

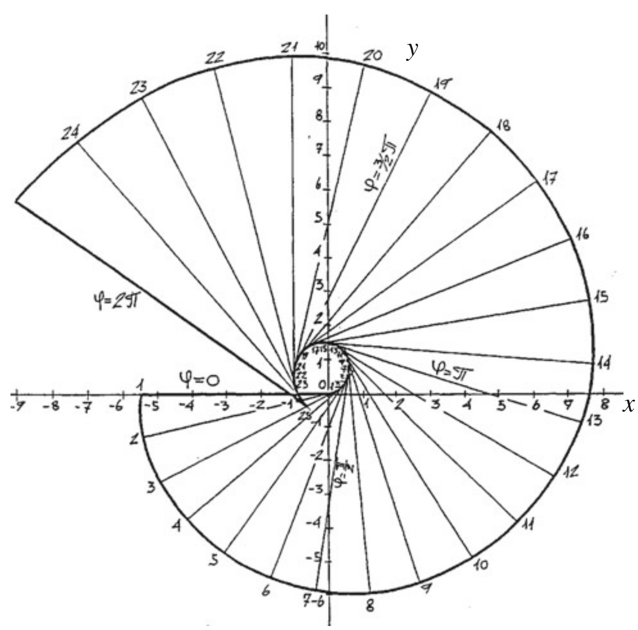


Fig. 3

conical helicoid. The coordinate line $u = 0$ is the edge of regression.

Yu.G. Kardashevskaya derived parametrical equations of the plane cuspidal edge ($u = 0$) after developing of a conical helicoid on a plane (Fig. 3):

$$\bar{x} = \frac{\mu(as+1)}{a(1+\mu^2)}\left[\frac{1}{\mu}\cos(\mu \ln|as+1|) + \sin(\mu \ln|as+1|)\right] - \frac{1}{a(1+\mu^2)},$$
$$\bar{y} = \frac{\mu(as+1)}{a(1+\mu^2)}\left[\frac{1}{\mu}\sin(\mu \ln|as+1|) - \cos(\mu \ln|as+1|)\right] + \frac{1}{a(1+\mu^2)},$$

where

$$\mu = \frac{r_0 a \sin\lambda}{k^2}\sqrt{1+k^2} = \text{const}.$$

In Fig. 3, the development of the developable conical helicoid limited by the cuspidal edge and the line of intersection of the helicoid by the plane $z = 0$ is shown.

(3) Parametrical equations:

$$x = x(\varphi, t) = v \sin\lambda \cos\varphi - t\sin\varphi;$$
$$y = y(\varphi, t) = v \sin\lambda \sin\varphi + t\cos\varphi;$$
$$z = z(\varphi, t) = v\cos\lambda;$$

where $v = kt + ce^{k\sin\lambda\varphi} = kt + ce^{au}$; t is a distance from the line of tangency of the cone and its tangent plane to a point on a straight directrix of the conical helicoid lying in this tangent plane; φ, t are hyperbolic coordinates of the developable conical helicoid formed with the help of a kinematic method.

Coefficients of the fundamental forms of the surface:

$$A^2 = c^2e^{2au}\sin^2\lambda(1+k^2) + t^2(1+k^2\sin^2\lambda);$$
$$F = [ce^{au}(1+k^2) + t(k-1)]\sin\lambda;$$
$$B^2 = 1+k^2;\quad L = t^2k(1+k^2\sin^2\lambda)\cos\lambda/\sqrt{A^2B^2-F^2},$$
$$M = N = 0.$$

Additional Literature

Kardashevskaya YG. Applied problems of bending torses. Prikl. Geom. i Ingen. Grafika. 1965; 3, p. 96-103.

■ Developable Helicoid with a Cuspidal Edge on the Paraboloid of Revolution

A developable helical surface of equal slope with different law of change of lead is formed by tangent lines of a helical curve

$$x = x(t) = a\cos t + at\sin t,$$
$$y = y(t) = a\sin t - at\cos t,$$
$$z = z(t) = (at^2/2)\tan\beta,$$

where β = const is the slanting angle of the rectilinear generators to a plane xOy; t is any parameter. A projection of a cuspidal edge on a plane xOy is an evolvent of the circle with a radius a. The cuspidal edge lies on *a paraboloid of revolution* the axial section of which by the plane yOz may be described by an equation

$$z = \tan\beta\left[y^2/(2a) - a/2\right),$$

where a is the radius of the circumference lying in the section of the paraboloid of revolution by the plane $z = 0$.

A top of the paraboloid of revolution has the coordinates

$$z = -(a/2)\tan\beta,\ x = y = 0.$$

Forms of definition of the studied helicoid

(1) Parametrical form of definition (Fig. 1):

$$x = x(u, t) = a(\cos t + t\sin t + ut\cos t),$$
$$y = y(u, t) = a(\sin t - t\cos t + ut\sin t),$$
$$z = z(u, t) = a(0.5t^2 + ut)\tan\beta.$$

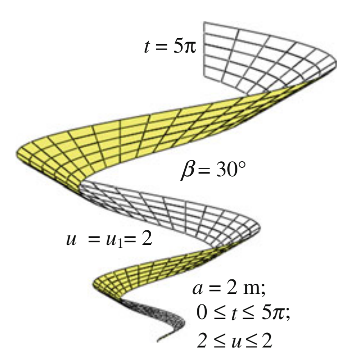


Fig. 1

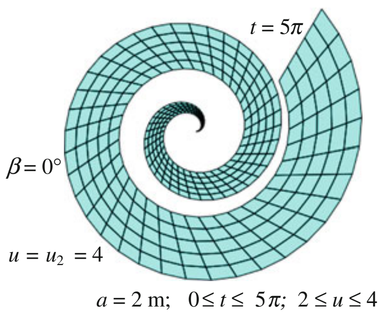


Fig. 2

Parametrical equations of the plane development of the studied helicoid of equal slope:

$$X_p = \frac{a\cos(t\cos\beta)}{\cos\beta}\left(\frac{1}{\cos^2\beta} + ut\right) + \frac{at\sin(t\cos\beta)}{\cos^2\beta},$$
$$Y_p = \frac{a\sin(t\cos\beta)}{\cos\beta}\left(\frac{1}{\cos^2\beta} + ut\right) - \frac{at\cos(t\cos\beta)}{\cos^2\beta}.$$

A development of the helicoid with the cuspidal edge on the paraboloid of revolution on a plane is presented in Fig. 2. This development is limited by the coordinate lines $u = u_i = \mathrm{const}$ ($i = 1; 2$) and by rectilinear generatrixes $t = 0$, $t = 5\pi$.

Coefficients of the fundamental forms of the surface:

$$A = at/\cos\beta, \quad F = a^2 t(t + u)/\cos^2\beta,$$
$$B^2 = a^2\left[(t + u)^2/\cos^2\beta + u^2 t^2\right],$$
$$L = M = 0, \quad N = aut\sin\beta.$$

(2) Parametrical form of definition:

$$x = x(t,\rho) = at\sin t \pm \cos t\sqrt{\rho^2 - a^2t^2},$$
$$y = y(t,\rho) = -at\cos t \pm \sin t\sqrt{\rho^2 - a^2t^2},$$
$$z = 0.5a\mathrm{tg}\beta(t^2 - 2) \pm \mathrm{tg}\beta\sqrt{\rho^2 - a^2t^2},$$

where ρ is the distance from a point on the generatrix to the axis Oz,

$$u = (-a \pm \sqrt{\rho^2 - a^2t^2})/(at).$$

The presence of two signs is in agreement with two sheets of the torse.

Coefficients of the first fundamental form of the surface:

$$A^2 = \left(1 \mp \frac{a}{\sqrt{\rho^2 - a^2t^2}}\right)^2 (a^2t^2\mathrm{tg}^2\beta + \rho^2);$$
$$B = \frac{\rho}{\cos\beta\sqrt{\rho^2 - a^2t^2}}$$
$$F = \frac{-at\rho}{\sqrt{\rho^2 - a^2t^2}}\left(\frac{a}{\sqrt{\rho^2 - a^2t^2}} \mp 1\right)\frac{1}{\cos^2\beta}.$$

(3) Parametrical form of definition:

$$x = x(v,t) = a\cos t + at\sin t + v\cos t\cos\beta;$$
$$y = y(v,t) = a\sin t - at\cos t + v\sin t\cos\beta;$$
$$z = z(v,t) = \left(at^2\mathrm{tg}\beta\right)/2 + v\sin\beta.$$

Coefficients of the fundamental forms of the surface given by these parametrical equations may be taken in a Subsect. "1.1.1. Torse Surfaces (Torses)".

Additional Literature

Pilipaka SF. Design of helical surfaces from torses of equal slope. Prikl. Geom. i Ingen. Grafika. 1987; 43, p. 39-41.

Krivoshapko SN. Geometry of Ruled Surfaces with Cuspidal Edge and Linear Theory of Torse Shells. Moscow: Izd-vo RUDN,2009; 357 p.

■ Parabolic Torse

A torse of the forth order formed by revolving of two parabolas

$$x = \frac{(z-q)^2}{2p_2},\ y = 0 \quad \text{and} \quad x = 0,\ y = \frac{z^2}{2p_1},$$

lying in two mutually perpendicular planes will be *parabolic* because any tangent plane

$$M = z(v-q) - xp_2 - yp_1(v-q)/(v+q) - (v^2 - q^2)/2 = 0$$

to the both director parabolas contains a parabola. Having been based on this principal proposition, VS Obukhova and RI Vorobkevich suggested calling this developable surface as *a parabolic torse*. An equation of a cuspidal edge of a parabolic torse was obtained in the following form

$$x = x(v) = -\frac{(v-q)^3}{4qp_2},$$
$$y = y(v) = \frac{(v+q)^3}{4qp_1},$$
$$z = z(v) = \frac{3}{2}v + \frac{q}{2},$$

where $v = z$ of the parabola lying in the plane xOz; $\gamma = v + q$; $\gamma = z$ of the parabola lying in the plane yOz. Parametrical equations of the edge of regression of a parabolic torse are of the third order. Assuming $q = 0$, we shall have *an oblique parabolic cylinder*.

Forms of definition of parabolic torse

(1) Implicit form of definition:

$$\begin{aligned}&4q^2z^4 - 4p_1qz^3y + 4p_2qz^3x + 2p_1p_2z^2xy + p_1^2z^2y^2\\&+ p_2^2z^2x^2 - 8q^3z^3 - 2p_1^3y^3 - 2p_2^3x^3 - 4p_1q^2z^2y\\&+ 8p_1^2qzy^2 - 16p_2q^2z^2x - 10p_2^2qzx^2 - 6p_1^2p_2xy^2\\&- 6p_1p_2^2yx^2 - p_1p_2qxyz - 8p_1^2q^2y^2 + 4q^4z^2 + p_2^2q^2x^2\\&+ 16p_1q^3zy + 4p_2q^3xz + 20p_1p_2q^2xy - 8p_1q^4y = 0.\end{aligned}$$

A tangent plane intersects the parabolic torse along a parabola and a double straight line.

An equation of a tangent plane can be written as

$$\begin{vmatrix} x & y & z & 1\\ 0 & 0 & z_i & 1\\ \frac{2(z_i-q)^2}{p_2} & 0 & 2z_i - q & 1\\ 0 & \frac{2z_i^2}{p_1} & 2z_i & 1\end{vmatrix} = 0.$$

(2) An equation of the continuous skeleton of rectilinear generatrixes of a parabolic torse (Fig. 1):

$$y = -\frac{(v+q)^2p_2}{(v-q)^2p_1}x + \frac{(v+q)^2}{2p_1} = k(v)x + l(v),$$
$$z = -\frac{2qp_2}{(v-q)^2}x + (v+q) = m(v) + n(v).$$

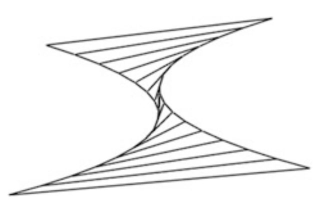


Fig. 1

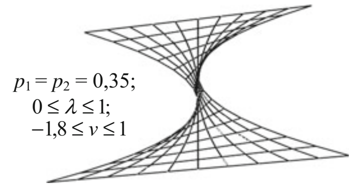


Fig. 2

(3) Vector form of definition:

$$\boldsymbol{r}(u,v) = -\frac{(v-q)^3}{4qp_2}\boldsymbol{i} + \frac{(v+q)^3}{4qp_1}\boldsymbol{j} + (\frac{3}{2}v + \frac{q}{2})\boldsymbol{k} + u\frac{-(v-q)^2p_1\boldsymbol{i} + (v+q)^2p_2\boldsymbol{j} + 2qp_1p_2\boldsymbol{k}}{\left[(v-q)^4p_1^2 + (v+q)^4p_2^2 + 4q^2p_1^2p_2^2\right]^{1/2}}.$$

Coefficients of the fundamental forms of the surface $A = 1$, $L = M = 0$ show that the coordinate lines u coincide with rectilinear generatrixes of the surface.

(4) Parametrical equations (Fig. 2):

$$\begin{aligned}x &= x(v,\lambda) = (1-\lambda)(v-q)^2/(2p_2),\\ y &= y(v,\lambda) = \lambda(v+q)^2/(2p_1),\\ z &= z(v,\lambda) = v + \lambda q,\end{aligned}$$

where $0 \le \lambda \le 1$, $A = A(v)$, $N = M = 0$, amd $v = z$ of the parabola lying in the plane yOz. Coordinate lines $\lambda = 0$ and $\lambda = 1$ coincide with the director parabolas.

Additional Literature

Obukhova VS., *Vorobkevich RI*. Parabolic torse. Prikl. Geom. i Ingen. Grafika (Kiev). 1981; 31, p. 22-26.
Obukhova VS, *Vorobkevich RI*. Analytical description of parabolic torses of the forth order. Prikl. Geom. i Ingen. Grafika (Kiev). 1982; 33, p. 16-19.
Krivoshapko SN. Geometry of Ruled Surfaces with Cuspidal Edge and Linear Theory of Torse Shells. Moscow: Izd-vo RUDN,2009; 357 p.

■ Torse with an Edge of Regression on the Ellipsoid of Revolution

A torse with an edge of regression

$$\begin{aligned}x &= x(u) = 2n(n\cos u\cos nu + \sin u\sin nu)/m,\\ y &= y(u) = 2n(n\sin u\cos nu - \cos u\sin nu)/m,\\ z &= z(u) = -(2\,m^2/n)\cos nu\end{aligned}$$

on the ellipsoid of revolution (Fig. 1)

$$\frac{x^2+y^2}{(2n/m)^2} + \frac{z^2}{(2\,m/n)^2} = 1$$

is *a surface of constant slope*. All tangent lines of the edge of regression form a constant angle γ with a coordinate axis Oz and

$$n = \sin\gamma;\quad m = \cos\gamma;\quad n^2 + m^2 = 1.$$

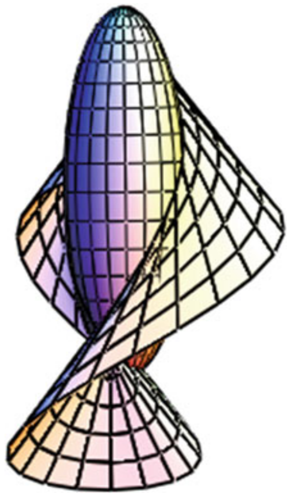


Fig. 1

The expressions for the determination of curvature k and torsion κ of the edge of regression take the form:

$$k = n/(2m \sin nu), \quad \kappa = 1/(2 \sin nu).$$

The last formulas show that

$$\kappa/k = m/n = \text{const}.$$

Hence, the cuspidal edge is *a line of slope*.

Form of definition of the studied developable surface

(1) A vector equation (Fig. 1):

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = x(u)\boldsymbol{i} + y(u)\boldsymbol{j} + z(u)\boldsymbol{k} + v\boldsymbol{l}(u),$$

where $x(u)$, $y(u)$, and $z(u)$ are the coordinates of the cuspidal edge presented above; $|v|$ is a distance from a point of the cuspidal edge to arbitrary point on the torse taken along a straight generatrix of the surface; $\boldsymbol{l}(u)$ is the unit tangent vector of the cuspidal edge, so that

$$\boldsymbol{l}(u) = [x'(u)\boldsymbol{i} + y'(u)\boldsymbol{j} + z'(u)\boldsymbol{k}]/\sqrt{x'^2(u) + y'^2(u) + z'^2(u)}$$
$$= n \cos u\boldsymbol{i} + n \sin u\boldsymbol{j} + m\boldsymbol{k}.$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 4m^2 \sin^2 nu + v^2n^2, \quad F = 2m \sin nu,$$
$$B = 1, \quad A^2B^2 - F^2 = v^2n^2,$$
$$L = nmv, \quad M = N = 0,$$
$$k_u = nmv/(m^2 \sin^2 nu + v^2n^2), \quad k_v = 0, \quad K = 0.$$

W. Wunderlich adduced interesting properties of the represented developable surface.

Reference

Wunderlich Walter. Kurven konstanter ganzer Krümmung und fester Hauptnormalenneigung. Monatsh. ath. 1973; 77, No. 2, p. 158-171 (12 ref.).

■ Torse with an Edge of Regression on One Sheet Hyperboloid of Revolution

Wunderlich Walter has studied a torse with an edge of regression

$$x = x(u) = \frac{(1-n)m}{2(1+n)}\cos(1+n)u + \frac{(1+n)m}{2(1-n)}\cos(1-n)u,$$
$$y = y(u) = \frac{(1-n)m}{2(1+n)}\sin(1+n)u + \frac{(1+n)m}{2(1-n)}\sin(1-n)u,$$
$$z = z(u) = -(m^2/n)\cos nu$$

on one sheet hyperboloid of revolution

$$\frac{x^2 + y^2}{(2n/m)^2} - \frac{z^2}{4} = 1,$$

where $n^2 + m^2 = 1$; $n = \sin\gamma$; $m = \cos\gamma$.

Form of definition of the studied developable surface

(1) A vector form of definition (Fig. 1):

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = x(u)\boldsymbol{i} + y(u)\boldsymbol{j} + z(u)\boldsymbol{k} + v\boldsymbol{l}(u),$$

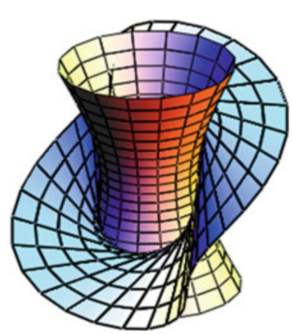


Fig. 1

where

$$\boldsymbol{l}(u) = (n\cos u\sin nu - \sin u\cos nu)\boldsymbol{i} + (\cos u\cos nu + n\sin u\sin nu)\boldsymbol{j} + m\sin nu\boldsymbol{k};\quad \text{but } |\text{l(u)}| = 1.$$

References

Wunderlich Walter. Kurven konstanter ganzer Krümmung und fester Hauptnormalenneigung. Monatsh. ath. 1973; 77, No. 2, p. 158-171 (12 ref.).

Kirischiev RI. Lines of slope on the second order surfaces of revolution. Matematika, nekotorie eyo prilozheniya i metodika prepodavaniya. Rostov-na-Donu, 1972; p. 80-94.

■ Torse with an Edge of Regression Given as $x = e^{-t}\cos t$; $y = e^{-t}\sin t$, $z = e^{-t}$

A spatial curve

$$x = e^{-t}\cos t,\quad y = e^{-t}\sin t,\quad z = e^{-t}$$

is *a conic spiral* lying on the circle cone with $\lambda = 45°$, where λ is the angle between an axis Oz and a generatrix of the cone. Assume this curve for an edge of regression of the developable surface.

In the presence of cuspidal edge, a developable surface is constructed definitively and its coefficients of fundamental forms are easily obtained. The analogous torse surface is presented in the Chap. "Developable conic helicoid".

The length of the arc of the edge of regression can be calculated due to the formula:

$$s = \sqrt{3}(1 - e^{-t}).$$

An expression for the determination of curvature of the edge of regression can be written as

$$k(s) = \sqrt{\frac{2}{3}}\frac{1}{\sqrt{3} - s}.$$

Parametrical equations of the development of the edge of regression on a plane are

$$x_p = \sqrt{\frac{6}{5}}\left[\sin\varphi - \sqrt{\frac{3}{2}}\cos\varphi\right]e^{-\sqrt{3/2}\varphi},$$
$$y_p = \sqrt{\frac{6}{5}}\left[\cos\varphi - \sqrt{\frac{3}{2}}\sin\varphi\right]e^{-\sqrt{3/2}\varphi},$$

where are assumed

$$\varphi = -\sqrt{\frac{2}{3}}\ln(\sqrt{3} - s).$$

Having used a polar system of coordinates, one can obtain an equation of a logarithmic spiral

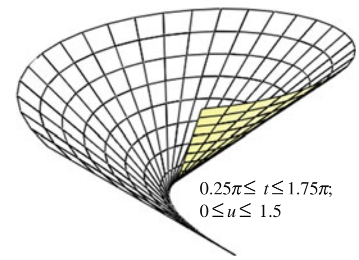


Fig. 1

$$\rho = \sqrt{3/5}e^{-\sqrt{3/2}\varphi}.$$

A form of the definition of the torse surface

(1) Parametrical equations (Fig. 1):

$$x = x(u,t) = e^{-t}\cos t - u\frac{\sin t + \cos t}{\sqrt{3}},$$
$$y = y(u,t) = e^{-t}\sin t + u\frac{\cos t - \sin t}{\sqrt{3}},$$
$$z = z(u,t) = e^{-t} - \frac{u}{\sqrt{3}}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1,\quad F = \sqrt{3}e^{-t},\quad B^2 = 3e^{-2t} + \frac{2u^2}{3},$$
$$L = M = 0,\quad N = -\frac{\sqrt{2}}{3}u,$$
$$k_u = 0,\quad k_t = -\frac{\sqrt{2}u}{3B^2},\quad k_1 = 0,\quad k_2 = -\frac{1}{u\sqrt{2}},$$
$$K = 0.$$

The system of curvilinear coordinates u, t is nonorthogonal but conjugate. An angle between two intersecting curvilinear coordinate lines u and t is calculated as

$$\cos\chi = 3e^{-t}/\sqrt{9e^{-2t} + 2u^2}.$$

This torse surface is *a surface of constant slope* with the sloping angle β and $\mathrm{tg}\beta = -\sqrt{2}$. A projection of the edge of regression on a plane $z = 0$ is a logarithmic spiral $\rho = e^{-t}$. Hence, *the director plane curve of the surface of constant slope* is *an evolvent of the logarithmic spiral:*

$$x = -e^{-t}\sin t, \ y = e^{-t}\cos t.$$

Reference

Kardashevskaya YG., *Gorbatovich JN.* A design of the surface of the plough blade with using of a given development by a method of bending. In: Tehn. Mehanika v selskohoz. proizvodstve. 1976; 13 (9), MIISP, p. 9-14.

■ Torse with an Edge of Regression Given as $x = v - \frac{v^3}{3}$, $y = v^2$, $z = a(v + \frac{v^3}{3})$

Assume a spatial curve

$$x = v - \frac{v^3}{3}, \quad y = v^2, \quad z = a(v + \frac{v^3}{3})$$

as an edge of regression of a torse surface. The length of the arc of the edge of regression can be calculated due to the formula:

$$s = \sqrt{1 + a^2}(v + v^3/3).$$

Assume $a = \mathrm{tg}\alpha$, then the length of the arc of the spatial curve taken as the edge of regression may be expressed as

$$s = \frac{z(v)}{\sin\alpha},$$

where $z = z(v)$ is a coordinate of the edge of regression. The curvature k and torsion κ of the edge of regression are

$$k = \frac{2}{(1 + a^2)(1 + v^2)^2},$$
$$\kappa = ak = \frac{2a}{(1 + a^2)(1 + v^2)^2}.$$

A form of the definition of the torse surface

(1) Parametrical equations (Figs. 1, 2 and 3):

$$x = x(u, v) = v - \frac{v^3}{3} + \frac{u(1 - v^2)}{(1 + v^2)\sqrt{1 + a^2}},$$
$$y = y(u, v) = v^2 + \frac{2uv}{(1 + v^2)\sqrt{1 + a^2}},$$
$$z = z(u, v) = a(v + \frac{v^3}{3}) + \frac{au}{\sqrt{1 + a^2}}.$$

Having known parametrical equations of an edge of regression of a torse surface, it is possible to write the equations of the torse surface and to obtain its coefficients of fundamental forms. This is considered in a Subsect. "1.1.1. Torse Surfaces (Torses)".

A parameter $|u|$ is a distance from the edge of regression to arbitrary point taken along a straight generatrix. So, coordinate lines u coincide with rectilinear generatrixes of the torse surface but coordinate lines v are the lines equally spaced (*equidistant lines*) from the edge of regression.

The same torse is shown in Figs. 1, 2 and 3. But in Fig. 1, the torse is designed within the limits $0 \le u \le u_0$, consequently $0 \le v \le v_0$; in Fig. 2, it is shown within the limits $-u_0 \le u \le u_0$ and $-v_1 \le v \le v_1$; and in Fig. 3 it is shown within the limits $0 \le u \le u_0$ and $-v_1 \le v \le v_1$.

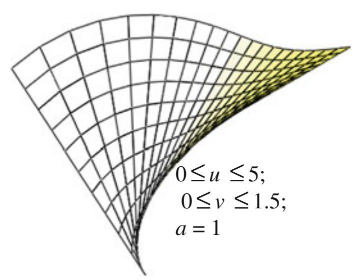


Fig. 1

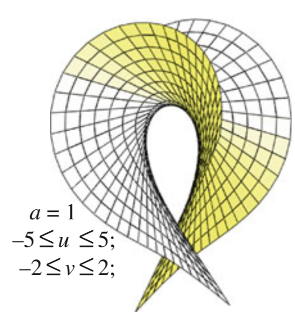


Fig. 2

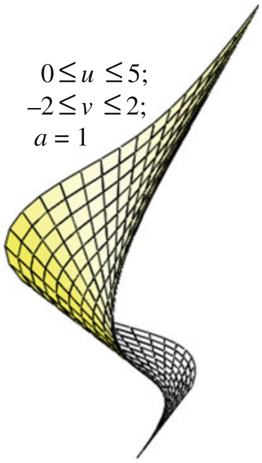


Fig. 3

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1, \quad F = (1 + v^2)\sqrt{1 + a^2},$$
$$B^2 = F^2 + \frac{4u^2}{(1 + a^2)(1 + v^2)^2},$$
$$L = M = 0, \quad N = \frac{4au}{(1 + a^2)(1 + v^2)^2},$$
$$k_u = k_1 = 0, \quad k_2 = a/u.$$

Thus, we have the curvilinear nonorthogonal, conjugate system of coordinates u, t ($F \neq 0$, $M = 0$).

This torse surface is *a surface of constant slope*. A formula for the determination of the angle φ of the slope of rectilinear generators is

$$\mathrm{tg}\varphi = \frac{1}{a}.$$

Additional Literature

Wunderlich W. Über die von der kubischen Böschungstorse algeleitete Pirondini-Schar windschiefer Regelflächen. Sitzungsberichte. Österreichische Akademie der Wissenschaften. Math.-naturwiss. Klasse. 1980; Abt. 2, Bd. 189, No. 4-7, p. 149-169.
Krivoshapko SN. Developable Surfaces and Shells. Moscow: Izd-vo UDN. 1991; 287 p.

■ Torse with an Edge of Regression in the Form of a Line of Intersection of Two Cylinders with the Perpendicular Axes

A line of intersection of two cylinders is the fourth-order curve:

$$\left.\begin{aligned} x^2 + y^2 = r^2 \\ x^2 + z^2 = R^2 \end{aligned}\right\}.$$

Assume this spatial curve as an edge of regression of a torse surface. In this case, we derive the torse surface in the section of which by a plane $z = 0$, the fourth-order curve is lying

$$y^2(r^2x^2 - R^4) - x^2(r^2 - R^2)^2 = 0.$$

This curve is *Lame's curve*. The curve has four asymptotes.

Forms of the definition of the torse surface

(1) Parametrical equations presented by A.N. Voronina (Fig. 1):

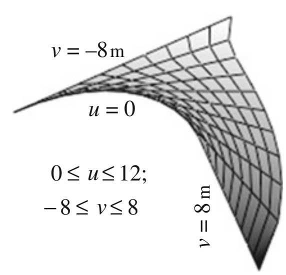


Fig. 1

$$x = x(u, v) = v + u,$$
$$y = y(u, v) = \pm\sqrt{r^2 - v^2} \mp \frac{uv}{\sqrt{r^2 - v^2}},$$
$$z = z(u, v) = \sqrt{R^2 - v^2} - \frac{uv}{\sqrt{R^2 - v^2}},$$

where $-r < v < r$ and $|v| < R$.

A parameter v shows an abscissa x of the edge of regression. Coordinate lines u coincide with the tangents to the line of intersection of two given cylinders.

The torse surface with r = 10 m and R = 20 m; $0 \le u \le 12$ m; -8 m $\le v \le 8$ m is shown in Fig. 1. The studied surface will degenerate into two planes if one shall assume $r = R$.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{v^2}{r^2 - v^2} + \frac{v^2}{R^2 - v^2},$$
$$F = A^2 + uv\left[\frac{r^2}{(r^2 - v^2)^2} + \frac{R^2}{(R^2 - v^2)^2}\right],$$
$$B^2 = 1 + \frac{1}{r^2 - v^2}\left(v + \frac{ur^2}{r^2 - v^2}\right)^2 + \frac{1}{R^2 - v^2}\left(v + \frac{uR^2}{R^2 - v^2}\right)^2,$$
$$L = M = 0,$$
$$N = \frac{3u^2 v r^2 R^2 (r^2 - R^2)}{\sqrt{A^2B^2 - F^2}(r^2 - v^2)^{5/2}(R^2 - v^2)^{5/2}},$$
$$k_u = 0, \quad K = 0.$$

(2) Vector form of definition:

$$\boldsymbol{r} = \boldsymbol{r}(t, v) = v\boldsymbol{i} \pm \sqrt{r^2 - v^2}\boldsymbol{j} + \sqrt{R^2 - v^2}\boldsymbol{k} + t\frac{\left(\boldsymbol{i} \mp \frac{v}{\sqrt{r^2 - v^2}}\boldsymbol{j} - \frac{v}{\sqrt{R^2 - v^2}}\boldsymbol{k}\right)}{f(v)}$$

where $f(v) = \sqrt{1 + \frac{v^2}{(r^2 - v^2)} + \frac{v^2}{(R^2 - v^2)}}$ is a length of the vector, tangent to the edge of regression; t is the distance the edge of regression from arbitrary point of the surface taken along its rectilinear generatrix.

(3) Parametrical equations (Figs. 2 and 3):

$$0 \le \alpha \le 2\pi; \; -\infty \le u \le \infty;$$

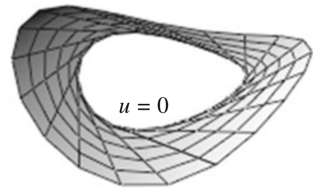


Fig. 2

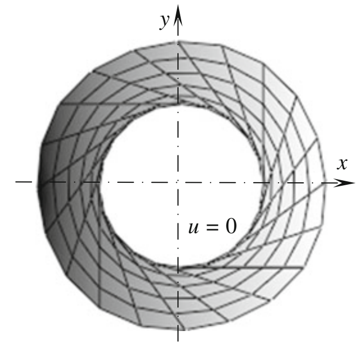


Fig. 3

$$x = x(u, \alpha) = r\cos\alpha - u\sin\alpha\sqrt{\frac{R^2 - r^2\cos^2\alpha}{R^2 - r^2\cos^4\alpha}},$$
$$y = y(u, \alpha) = r\sin\alpha + u\cos\alpha\sqrt{\frac{R^2 - r^2\cos^2\alpha}{R^2 - r^2\cos^4\alpha}},$$
$$z = z(u, \alpha) = \sqrt{R^2 - r^2\cos^2\alpha} + ur\sin 2\alpha\left[R^2 - r^2\cos^4\alpha\right]^{-1/2}/2,$$

Reference

Voronina AN. Construction of torses with the cuspidal edge in the form of the fourth order curve. Prikl. Geom. i Ingen. Grafika. 1967; 5, p. 103-105.

■ Torse with an Edge of Regression in the Form of Hyperbolic Helical Line

The same torse with a cuspidal edge in the form of *a hyperbolic helical line*

$$x = x(t) = a\cosh t;$$
$$y = y(t) = a\sinh t;$$
$$z = z(t) = at$$

is presented in Figs. 1 and 2.

A torse surface is formed by tangents of its cuspidal edge. In the case in question, the cuspidal edge is a hyperbolic

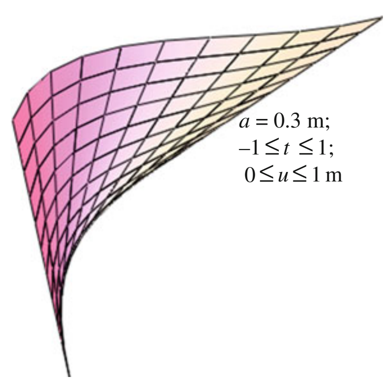
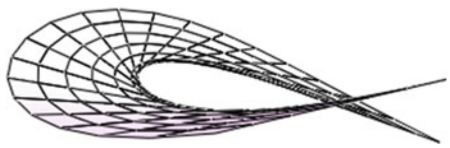


Fig. 1



$a = 0.3$ m; $-2 \le t \le 2$; $0 \le u \le 1$ m

Fig. 2

helical line. The length of the arc of the cuspidal edge between points 0 and t is

$$s = a\sqrt{2}\sinh t.$$

Curvature k and torsion κ of the hyperbolic helical line can be determined with the help of the following formula:

$$k = \kappa = \frac{1}{2a\cosh^2 t} = \frac{a}{2a^2 + s^2}.$$

Forms of definition of the studied developable surface

(1) Parametric form of definition (Fig. 1):

$$x = x(u,t) = a\cosh t + \frac{\mathrm{sh}t}{\sqrt{2}\cosh t}u,$$
$$y = y(u,t) = a\sinh t + \frac{u}{\sqrt{2}},$$
$$z = z(u,t) = at + \frac{u}{\sqrt{2}\mathrm{coch}t},$$

where u is the length of the rectilinear generatrix of the torse taken from the cuspidal edge until arbitrary point; $-\infty \le u \le \infty$; $-\infty \le t \le \infty$.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = 1,\quad F = a\sqrt{2}\cosh t,\quad B^2 = F^2 + \frac{u^2}{2\cosh^2 t},$$
$$A^2B^2 - F^2 = \frac{u^2}{2\cosh^2 t},$$
$$L = M = 0,\quad N = \frac{u}{2\cosh^2 t},$$
$$k_u = k_1 = 0,\quad k_t = \frac{u}{4a^2\cosh^4 t + u^2},$$
$$k_2 = \frac{1}{u},\quad K = 0,\quad H = \frac{1}{2u}.$$

The torse surface is given in curvilinear nonorthogonal, conjugate coordinates u, t. The coordinate line $u = 0$ is the edge of regression. Coordinate lines u (t = const) coincide with the rectilinear generatrixes of the surface.

In the section of the torse by the planes $z = z_0$ = const, curves

$$u = u(t) = (z_0 - at)\sqrt{2}\cosh t$$

are lying.

(2) Parametrical equations (Fig. 1):

$$x = x(u,s) = \frac{\sqrt{2a^2 + s^2}}{\sqrt{2}} + \frac{us}{\sqrt{2}\sqrt{2a^2 + s^2}},$$
$$y = y(u,s) = \frac{s}{\sqrt{2}} + \frac{u}{\sqrt{2}},$$
$$z = z(u,s) = a\mathrm{Arsinh}\frac{s}{a\sqrt{2}} + \frac{au}{\sqrt{2a^2 + s^2}},$$

where s is the length of the arc of the cuspidal edge; $s = a\sqrt{2}\sinh t$.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = F = 1,\quad B^2 = 1 + \frac{u^2a^2}{(2a^2 + s^2)^2},$$
$$A^2B^2 - F^2 = \frac{u^2a^2}{(2a^2 + s^2)^2},$$
$$L = M = 0,\quad N = \frac{ua^2}{(2a^2 + s^2)^2},$$
$$k_u = k_1 = 0,\quad k_s = \frac{ua^2}{(2a^2 + s^2)^2B^2},$$
$$k_2 = \frac{1}{u},\quad K = 0,\quad H = \frac{1}{2u}.$$

In this case, the surface is given in curvilinear nonorthogonal, conjugate coordinates u, s.

■ Torse with a Given Line of Curvature in the Form of the Second-Order Parabola

Assume a line of principal curvature of a torse surface in the form of the second-order parabola

$$x = 0,\quad y = v,\quad z = -av^2.$$

Parametrical equations of the cuspidal edge of designed torse surface can be written in the following form

$$x = -\frac{(4a^2v^2 + 1)^{3/2}}{c},$$
$$y = -4a^2v^3,\quad z = -3av^2 - \frac{1}{2a}.$$

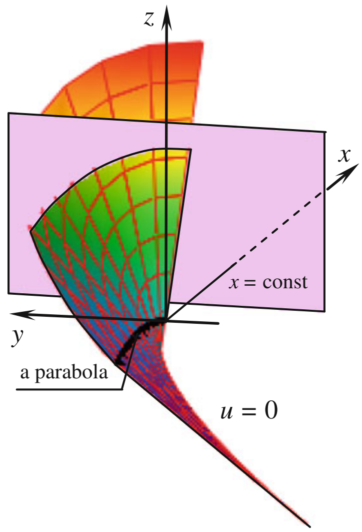


Fig. 1

Having chosen an arbitrary constant c appearing in the parametrical equations of the cuspidal edge, it is possible to define a family of the torse surfaces incidental to the given in advance line of principal curvature. The parameter c is connected functionally with the angle φ between the principal normal of the parabola and the generatrix of the torse passing through the top of the parabola;

$$c = 2a\,\text{cotan}\,\varphi.$$

A fragment of the torse for a concrete value of the constant $c = 2a\ \text{cotan}(\pi/6)$ is pictured in Fig. 1. The represented developable surface is a torse of constant slope in the respect of the plane x = const. This surface is studied too in a Subsect. "Surfaces of Constant Slope" under the name "Torse of constant slope with directrix parabola".

Forms of definition of the studied developable surface

(1) An equation of the continuous skeleton of rectilinear generatrixes of a torse (Fig. 1):

$$y = kx + l = \frac{vc}{\sqrt{1 + 4a^2v^2}}x + v,$$
$$z = mx + n = \frac{c}{2a\sqrt{1 + 4a^2v^2}}x - av^2,$$

where the parameter v is equal to the ordinate y of the parabola lying in a plane $x = 0$.

Parametrical equations of the cuspidal edge of the torse given by *the continuous skeleton of its rectilinear generatrixes* were obtained by R.U. Alimov:

$$x = -(1 + 4a^2v^2)^{3/2}/c, \quad y = -4a^2v^3,$$
$$z = -3av^2 - 1/(2a).$$

R.U. Alimov used parametrical equations of a cuspidal edge of a torse derived by N.N. Ryzhov:

$$x = -\frac{\mathrm{d}l}{\mathrm{d}k} = -\frac{\mathrm{d}n}{\mathrm{d}m}, \quad y = kx + l, \quad z = mx + n.$$

(2) A parametrical form of the definition of the torse with the given cuspidal edge (Fig. 1):

$$x = x(u, v) = -\frac{(4a^2v^2 + 1)^{3/2}}{c} - \frac{2au}{\sqrt{4a^2 + c^2}},$$
$$y = y(u, v) = -4a^2v^3 - \frac{2acuv}{\sqrt{(4a^2 + c^2)(1 + 4a^2v^2)}},$$
$$z = z(u, v) = -3av^2 - \frac{1}{2a} - \frac{cu}{\sqrt{(4a^2 + c^2)(1 + 4a^2v^2)}}.$$

In the presence of cuspidal edge, a developable surface is constructed definitively and its coefficients of fundamental forms are easily obtained.

A line $u = 0$ on the torse surface is the cuspidal edge of this surface.

In the section of the torse by a plane $x = 0$, a curve

$$u = -\frac{(4a^2v^2 + 1)^{3/2}\sqrt{4a^2 + c^2}}{2ac}$$

is lying. This curve is a parabola (Fig. 1)

$$y = v; \quad z = -av^2.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1, \;\; F = \frac{6av}{c}\sqrt{(4a^2 + c^2)(1 + 4a^2v^2)},$$
$$B^2 = F^2 + \frac{4a^2c^2u^2}{(c^2 + 4a^2)(1 + 4a^2v^2)^2},$$
$$L = M = 0, \;\; N = -\frac{16a^3cu}{(c^2 + 4a^2)(1 + 4a^2v^2)},$$
$$k_1 = k_u = 0, \;\; k_2 = -\frac{4a(1 + 4a^2u^2)}{cu}.$$

Additional Literature

Alimov RU. Algorithmization of design and developing of torse surfaces as applied to automation of construction of developments of fragments of pipelines. PhD Thesis, Moscow: MAI. 1984; 19 p.
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Ryzhov NN. Algorithmization of the determination of equations of ruled surfaces with taken into account the given condition. Prikl. Geom. i Ingen. Grafika. 1972; 14, p. 3-8.

■ Torse with Generating Straight Lines Lying in the Normal Planes of a Spherical Curve

A torse surface with a directrix spherical line

$$\boldsymbol{E}_0(u) = a\boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\sin\omega,$$
$$\omega = pu; \quad p = \text{const},$$

rested on the spherical surface with a radius a may be formed if the generatrix straight lines to set out in the normal planes of the spherical line in a specific manner. A torse surface designed in such manner is called *a torse with generating straight lines lying in the normal planes of a spherical line*.

Forms of definition of the studied developable surface

(1) A vector form of definition:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = a\boldsymbol{e}_0(u) + v\boldsymbol{l}(u)$$
$$= a\boldsymbol{e}_0(u) + v[\cos\theta\boldsymbol{e}_0(u) + \sin\theta\boldsymbol{g}_0(u)],$$

where

$$\boldsymbol{g}_0(u) = \left[\boldsymbol{e}_0'(u)/s \times \boldsymbol{e}_0(u)\right] = \left[\sin\omega\cos\omega\boldsymbol{h} + p\boldsymbol{n} - \cos^2\omega\boldsymbol{k}\right]/s;$$
$$\boldsymbol{h} = \boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u;$$
$$\boldsymbol{n} = \boldsymbol{n}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u; \; s = \left[\omega^{1/2} + \cos^2\omega\right]^{1/2}$$
$$= \left[p^2 + \cos^2\omega\right]^{1/2}.$$

A unit vector $\boldsymbol{e}_0(u)$ is the normal of the sphere on which a directrix line is rested.

(2) A parametrical form of definition (Figs. 1, 2 and 3):

$$x = x(u,v) = (a + v\cos\theta)\cos\omega\cos u$$
$$+ v\sin\theta(\sin\omega\cos\omega\cos u - p\sin u)/s,$$
$$y = y(u,v) = (a + v\cos\theta)\cos\omega\sin u$$
$$+ v\sin\theta(\sin\omega\cos\omega\sin u - p\cos u)/s,$$
$$z = z(u,v) = (a + v\cos\theta)\sin\omega - (v/s)\sin\theta\cos^2\omega.$$

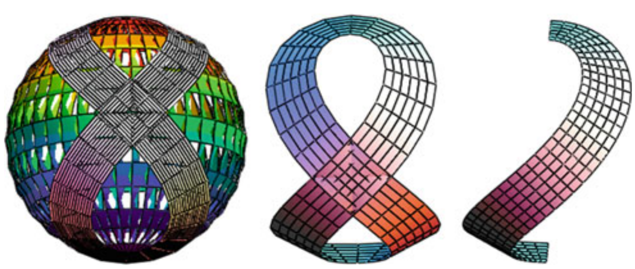

Fig. 1 $a = 10;\ -3 \le v \le 3;\ p = 1$

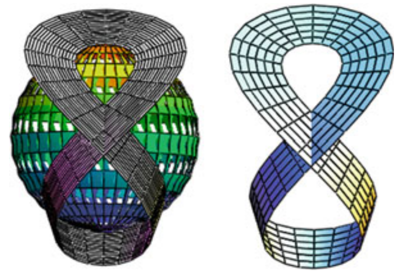

Fig. 2 $a = 10;\ -3 \le v \le 3;\ p = 1$

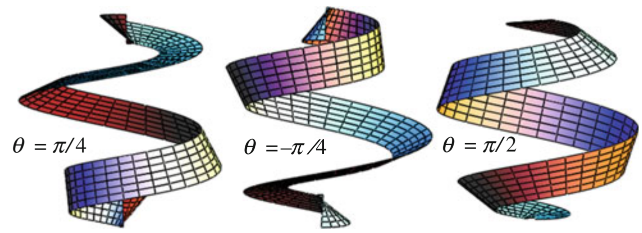

Fig. 3 $a = 10;\ p = 1/5;\ -3 \le v \le 3;\ -2.5\pi \le u \le 2.5\pi$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = s(a + v\cos\theta) + \left(1 + \frac{p^2}{s^2}\right)v\sin\omega\sin\theta,$$
$$B = 1, \quad L = A\left[s\sin\theta - \left(1 + \frac{p^2}{s^2}\right)\sin\omega\cos\theta\right],$$
$$F = M = N = 0,$$
$$k_1 = k_u = \frac{L}{A^2}, \quad k_2 = k_v = 0, \quad K = 0.$$

Having used formulas of differential geometry, V.N. Ivanov obtained a vector equation of the cuspidal edge of the studied torse surface:

$$\boldsymbol{r}(u) = a\boldsymbol{e}_0(u) - \frac{a\boldsymbol{e}_0'(u)\boldsymbol{l}'(u)}{\boldsymbol{l}'^2(u)}\boldsymbol{l}(u)$$
$$= a\boldsymbol{e}_0(u) - as\left[s\cos\theta + \left(1 + \frac{p^2}{s^2}\right)\sin\omega\sin\theta\right]$$
$$\times (\cos\theta\boldsymbol{e}_0 + \sin\theta\boldsymbol{g}_0).$$

Assume $\theta = \pi/2$. In this case, the straight generatrixes of the torse will be lying in tangent planes of the sphere and in the normal planes of the spherical lines simultaneously, i.e., they will coincide with the lines of intersection of these planes.

Reference

Ivanov VN. Spherical curves and geometry of surfaces rested on the sphere. Modern Problems of Geometrical Modeling: Proc. of Ukraine-Russia Scientific Conf., Kharkov, April 19-22, 2005; p. 114-120.

■ Developable Surfaces with Two Plane Directrix Curves

At least, one developable surface may be constructed by the movement of a straight line on two arbitrary curves. Assume two spatial curves:

$$\boldsymbol{r}_1(z) = f_1(z)\boldsymbol{i} + F_1(z)\boldsymbol{j} + z\boldsymbol{k} \quad \text{and}$$
$$\boldsymbol{r}_2(z) = f_2(z) + F_2(z)\boldsymbol{j} + z\boldsymbol{k}.$$

Later on, we shall denote a point on the first curve by $z = \beta$, a point on the second curve by $z = \gamma$. So, we shall write the equations cited above in the following form:

$$\boldsymbol{r}_1(\beta) = f_1(\beta)\boldsymbol{i} + F_1(\beta)\boldsymbol{j} + \beta\boldsymbol{k} \quad \text{and}$$
$$\boldsymbol{r}_2(\gamma) = f_2(\gamma) + F_2(\gamma)\boldsymbol{j} + \gamma\boldsymbol{k}.$$

If a tangent plane touches two curves,

$$\boldsymbol{r}_1 = \boldsymbol{r}_1(z) = \boldsymbol{r}_1(\beta) \text{ and } \boldsymbol{r}_2 = \boldsymbol{r}_2(z) = \boldsymbol{r}_2(\gamma)$$

simultaneously then it is obvious that

$$\left(\boldsymbol{r}_1'(\beta), \boldsymbol{r}_2'(\gamma), \boldsymbol{r}_1(\beta) - \boldsymbol{r}_2(\gamma)\right) = 0,$$

or

$$\begin{vmatrix} f_1'(\beta) & F_1'(\beta) & 1 \\ f_2'(\beta) & F_2'(\gamma) & 1 \\ f_1(\beta) - f_2(\gamma) & F_1(\gamma) - F_2(\gamma) & \beta - \gamma \end{vmatrix} = 0. \qquad (1)$$

Derivatives with respect to the corresponding parameters are denoted by the strokes. So *the condition of developable surface's uniqueness* can be obtained. It follows from the formula (1) that $\gamma = \varphi(\beta)$ or $\beta = \Phi(\gamma)$.

Having the equations of two curves, it is possible to find the equation of the single parametric system of the planes

$$M(x, y, z, \gamma) = 0$$

after elimination of three parameters from four algebraical equations obtained by Monge. The equation of a developable surface in an implicit form is determined after the elimination γ from two equations

$$M(x, y, z, \gamma) = 0 \text{ and } \partial M/\partial\gamma = 0.$$

Solving three equations

$$M(x, y, z, \gamma) = 0, \quad \partial M/\partial\gamma = 0, \quad \partial^2 M/\partial\gamma^2 = 0$$

jointly one can obtain the equation of the cuspidal edge as

$$x = x(\gamma), \quad y = y(\gamma), \quad z = z(\gamma).$$

Having two directrix plane curve

$$x = f_1(\beta), \quad y = a, \text{ and } x = f_2(\gamma), \quad y = b.$$

lying in the planes parallel to the coordinate plane xOz, one can obtain the condition of developable surface's uniqueness

(1) in the following form:

$$f_1'(\beta) = f_2'(\gamma). \qquad (2)$$

It follows from the Eq. (2) that a rectilinear generatrix of a torse passes through two corresponding points of the plane curves the tangent lines in which are parallel.

If two directrix curves are given by the parametric equations

$$x = x_1(t_1); \quad y = y_1(t_1); \quad z = z_1(t_1) \quad \text{and}$$
$$x = x_2(t_2); \quad y = y_2(t_2); \quad z = z_2(t_2)$$

then the condition of developable surface's uniqueness can be written in the following form:

$$\begin{vmatrix} x_1(t_1) - x_2(t_2) & y_1(t_1) - y_2(t_2) & z_1(t_1) - z_2(t_2) \\ x_1'(t_1) & y_1'(t_1) & z_1'(t_1) \\ x_2'(t_2) & y_2'(t_2) & z_2'(t_2) \end{vmatrix} = 0. \qquad (3)$$

Having two directrix plane curves

$$x = x_1(t_1), \quad y = a, \quad z = z_2(t_2) \text{ and}$$
$$x = x_2(t_2), \quad y = b, \quad z = z_2(t_2)$$

lying in the planes parallel to the coordinate plane xOz, one can obtain the condition of developable surface's uniqueness

(3) in the following form:

$$z_1'(t_1)/x_1'(t_1) = z_2'(t_2)/x_2'(t_2).$$

A cuspidal edge of the torse constructed on two convex or concave directrix curves lies outside the range of the fragment limited by these curves. If one of the directrix curves is convex but other curve is concave, then the cuspidal edge lies inside the fragment limited by directrix curves.

For the application *of a skeleton-and-parametrical method of design of developable surfaces*, it is necessary to take equations of the continuous skeleton of rectilinear generatrixes of a developable surface in the form

$$y = kx + l; \quad z = mx + n,$$

where k, l, m, *and* n are the continuous functions only of one parameter satisfying a condition of developable surface's uniqueness:

$$l'/k' = n'/m'$$

where l', k', n' and m' are the derivatives of the function with the respect to the one parameter.

A condition of uniqueness of developable surfaces resting on two curves lying in two mutually perpendicular planes shows that a rectilinear generatrix of the torse passes through two corresponding points of the plane curves and the tangent lines of both directrix curves in these points intersect on the line of intersection of the planes containing two given curves

Additional Literature

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Liu Yong-Jin, *Lai Yu-Kun*, and *Hu Shi-Min*. Developable strip approximation of parametric surfaces with global error bounds. Computer Graphics and Applications, PG '07', 15th Pacific Conf., Oct. 29, 2007, Maui, USA. 2007; p. 441-444.

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■ Torse with Two Parabolas with Intersecting Axes

This developable surface contains two parabolas of the second order as the directrixes (Fig. 1):

$$\text{I: } x = f_1(z) = \frac{\cos\varphi_1}{2p_1}z^2,$$
$$y = F_1(z) = \frac{\sin\varphi_1}{2p_1}z^2$$
$$\text{II: } x = f_2(z) = \frac{\cos\varphi_2}{2p_2}z^2 - d,$$
$$y = F_2(z) = l - \frac{\sin\varphi_2}{2p_2}z^2.$$

The axes of the parabolas are intersecting in the point C (Fig. 2). The rectilinear generatrixes of the torse pass through the points z = β of the first parabola and $z = \gamma$ of the second parabola and

$$\gamma = \frac{\beta}{2} + \frac{a}{\beta} \pm \sqrt{\frac{a^2}{\beta^2} + \frac{\beta^2}{4} + b},$$

where $a = \frac{p_1(l\cos\varphi_2 - d\sin\varphi_2)}{\sin(\varphi_1+\varphi_2)}$, $b = a - \frac{2p_2(l\cos\varphi_1 + d\sin\varphi_1)}{\sin(\varphi_1+\varphi_2)}$.

It is necessary to take (+) before the square root if $\beta < 0$ and (–) if при $\beta > 0$.

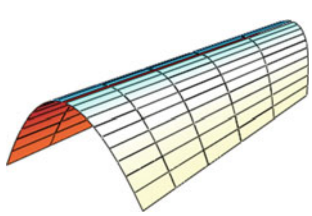

Fig. 1

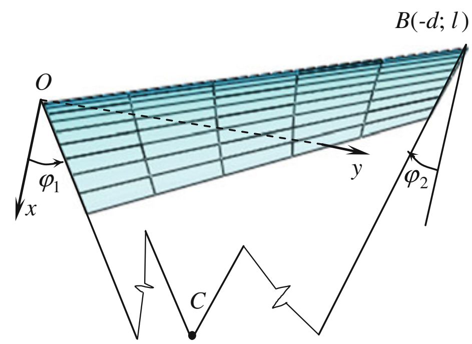

Fig. 2

Parametrical equations of the cuspidal edge of the presented developable surface are

$$x = \frac{\beta^2\gamma^2(\sin\varphi_1 + \sin\varphi_2) - 2y(p_1\cos\varphi_2\gamma^2 - p_2\beta^2\gamma'\cos\varphi_1)}{2p_1\gamma^2\sin\varphi_2 + 2p_2\beta^2\gamma'\sin\varphi_1},$$
$$y = \frac{2p_1\gamma^3 + \beta^3\sin\varphi_1(2\gamma'^2 - \gamma\gamma'')p_2}{2p_1p_2(2\beta\gamma'^2 - 2\gamma\gamma' - \gamma\beta\gamma'')},$$
$$z = \frac{\beta}{2} + \frac{p_2\sin\varphi_1(x - \mathrm{ctg}\varphi_1 y)}{\gamma\sin(\varphi_1+\varphi_2)} + \frac{p_1\sin\varphi_2(x + \mathrm{ctg}\varphi_2 y)}{\beta\sin(\varphi_1+\varphi_2)}.$$

The cuspidal edge of the torse has the first-order cuspidal point which lies on a straight line joining the vertexes of the parabolas.

An equation of single parametric system of tangent planes of the presented developable surface can be written in the following form:

$$p_2(x - y\mathrm{ctg}\varphi_1) - \left[\frac{\sin(\varphi_1+\varphi_2)}{\sin\varphi_1}\left(z - \frac{\beta}{2}\right) - \frac{p_1(x + y\mathrm{ctg}\varphi_2)}{\beta\sin\varphi_1}\sin\varphi_2\right]\gamma = 0.$$

Forms of the definition of the torse surface

(1) Parametrical form of definition of a torse with given cuspidal edge.

If we have a cuspidal edge, then a developable surface is constructed definitively and its coefficients of fundamental forms are easily obtained.

A coordinate plane $z = 0$ is a plane of symmetry of the developable surface.

(2) Parametrical form of definition (Fig. 1):

$$x = x(\lambda, \beta) = \frac{\cos\varphi_1}{2p_1}\beta^2(1-\lambda) + \lambda\left(\frac{\cos\varphi_2}{2p_2}\gamma^2 - d\right),$$
$$y = y(\lambda, \beta) = \frac{\sin\varphi_1}{2p_1}\beta^2(1-\lambda) + \lambda\left(l - \frac{\sin\varphi_2}{2p_2}\gamma^2\right),$$
$$z = z(\lambda, \beta) = \beta(1-\lambda) + \lambda\gamma, \quad 0 \le \lambda \le 1.$$

The relation $\gamma = \gamma(\beta)$ is presented above. The coordinate lines $\lambda = 0$ and $\lambda = 1$ coincide with the directrix parabolas. A system of curvilinear coordinates β, λ is nonorthogonal but conjugate because

$$A = A(\beta), \quad F \ne 0, \quad L = 0, \quad M = 0.$$

In addition,

$$\frac{\partial(B\cos\chi)}{\partial\lambda} = \frac{\partial A}{\partial\beta}$$

where χ is an angle between the coordinate lines β and λ.

Additional Literature

Krivoshapko SN. Developable Surfaces and Shells. Moscow: Izd-vo UDN. 1991; 287 p.

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Williams Orlan G, *Skaggs Robert L*. Method of forming a parabolic trough. United States Patent 4236399, 12.02.1980. Primary Class 72/295, B21D11/20.

■ Torse with Two Parabolas Lying in Intersecting Planes but with Parallel Axes

This developable surface contains two parabolas of the second order as the directrixes (Fig. 1):

$$\text{I:}\ x = f_1(z) = \sqrt{2pz},\ y = 0 \ \text{ and}$$
$$\text{II:}\ x = f_2(z) = \sqrt{2p_1(z+d)}\sin\varphi,$$
$$y = F_2(z) = l - \sqrt{2p_1(z+d)}\cos\varphi.$$

The rectilinear generatrixes of the torse pass through the points z = β of the first parabola and $z = \gamma$ of the second parabola, $\gamma \ge -d$, and

$$\sqrt{\beta} = \frac{l\tan\varphi}{\sqrt{2p}} \pm \sqrt{\frac{l^2\tan^2\varphi}{2p} + 2d + \gamma - \frac{\sqrt{2}\,l\sqrt{\gamma+d}}{\sqrt{p_1}\cos\varphi}}.$$

The last relation shows that two torses can be designed on two given parabolas. Parametrical equations of the edge of regression have the following form:

$$x = x(\gamma) = -\frac{\sqrt{2p_1}\sin\varphi(\sqrt{\gamma+d} - n/2)^3}{\eta} + l\tan\varphi \pm \frac{\sqrt{2p}}{\eta}f^3(\gamma);$$
$$y = y(\gamma) = \frac{\sqrt{2p_1}}{\eta}\cos\varphi(\sqrt{\gamma+d} - \frac{n}{2})^3;$$
$$z = z(\gamma) = \frac{2f^4(\gamma) + ef^3(\gamma) - 2\sqrt{\gamma+d}(\sqrt{\gamma+d} - n/2)^3}{\eta} - \sqrt{\gamma+d}(\sqrt{\gamma+d} - n) - d,$$

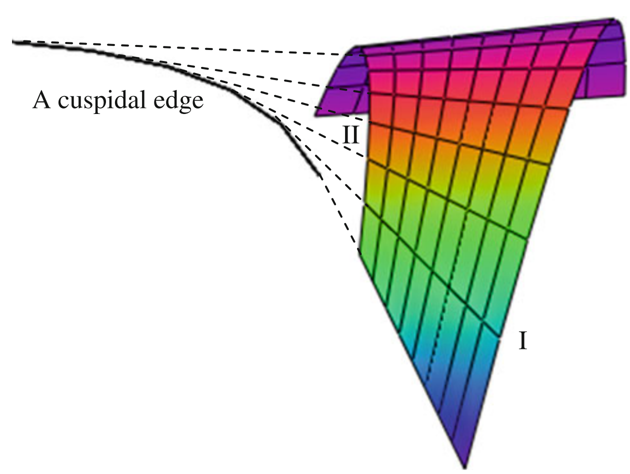

Fig. 1

where

$$f(\gamma) = \sqrt{\gamma + d - n\sqrt{\gamma + d} + m}; \quad e = \pm l \tan\varphi\sqrt{\frac{2}{p}};$$
$$\eta = m - \frac{n^2}{4}; \quad m = \frac{e^2}{4} + d; \quad n = \frac{\sqrt{2}\,l}{\sqrt{p_1}\cos\varphi}.$$

Forms of definition of the developable surface

(1) Vector form of definition (Fig. 1):

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{r}(v,\,\lambda) = \boldsymbol{r}_1(\beta)(1-\lambda) + \lambda\boldsymbol{r}_2(v)\\ &= (1-\lambda)\left(\sqrt{2p\beta}\boldsymbol{i} + \beta\boldsymbol{k}\right)\\ &\quad + \lambda[\sqrt{2p_1}v\sin\varphi\boldsymbol{i} + \left(l - \sqrt{2p_1}v\cos\varphi\right)\boldsymbol{j}\\ &\quad + (v^2 - d)\boldsymbol{k}],\end{aligned}$$

where $0 \le \lambda \le 1; v = \sqrt{\gamma + d}$; then

$$\beta = \left(l/2 + \sqrt{v^2 - nv + m}\right)^2.$$

(2) Vector form of definition:

$$\boldsymbol{r} = \boldsymbol{r}(u,\gamma) = \boldsymbol{a}(\gamma) + u\boldsymbol{l}(\gamma),$$

where $\boldsymbol{a}(\gamma)$ is a radius-vector of the cuspidal edge, $\boldsymbol{l}(\gamma)$ is the unit tangent vector given at every point of the cuspidal edge. The coordinate lines u coincide with the rectilinear generatrixes.

An equation of the first directrix parabola lying in the plane $y = 0$ and an equation of the second directrix parabola can be expressed as

$$u_1(\gamma) = -\frac{F(\sqrt{\gamma + d} - 0.5n)}{3} \text{ and } u_2(\gamma) = -\frac{\gamma + d - n\sqrt{\gamma + d} + m}{3(\sqrt{\gamma + d} - 0.5n)}F,$$

accordingly.

The lengths L of the rectilinear generatrixes of the torse taken from one parabola till other parabola are calculated with the help of a formula:

$$L = u_1 - u_2 = \sqrt{\Phi}$$

where

$$\begin{aligned}\Phi &= (2p_1 + 2p + e^2 + n^2)(\gamma + d)\\ &\quad - 2n(p_1 + p + 2d + e^2)\sqrt{\gamma + d} + p_1 n^2/2\\ &\quad + 2pm + 4m^2 + e^2 m + f(\gamma)\\ &\quad (\mp 4\sqrt{pp_1}\sin\varphi - 2\ en)\sqrt{\gamma + d} \pm 2\sqrt{pp_1}n\sin\varphi\\ &\quad - 4\ em).\end{aligned}$$

Additional Literature

Krivoshapko SN. Developable Surfaces and Shells. Moscow: Izd-vo UDN. 1991; 287 p.

■ Torse with Two Ellipses Lying in Parallel Planes and with Parallel Axes

This developable surface contains two ellipses as directrix curves

$$\frac{(y-m)^2}{c^2} + \frac{(z-n)^2}{d^2} = 1,\ x = l \text{ and } \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1,\ x = 0.$$

The rectilinear generatrixes of the torse pass through the points z = β of the ellipse lying in the plane $x = l$ and through the point $z = \gamma$ of the second ellipse lying in the coordinate plane yOz,

$$\gamma = \frac{a^2 c(\beta - n)}{\sqrt{b^2 d^4 + (\beta - n)^2(a^2c^2 - b^2d^2)}}.$$

Parametrical equations of the cuspidal edge can be written as

$$x = \frac{b^2d^4a^2cl}{b^2d^4a^2c - \left[b^2d^4 + \mu(\beta - n)^2\right]^{3/2}};$$
$$y = \frac{d^3a^2mc + \mu\left[d^2 - (\beta - n)^2\right]^{3/2}}{d^3a^2cl};$$
$$z = \frac{\left[b^2d^4n - \mu(\beta - n)^3\right]x}{b^2d^4l}$$

where $\mu = a^2c^2 - b^2d^2$.

Assume $a = b$ and $d = c$. In this case, the cuspidal edge degenerates into a point, vertex of a cone. Having assumed $a = d$ and $b = c$, we shall have a cylinder.

A cuspidal edge of the represented torse is a closed curve and it has four cuspidal points of the first order (Fig. 1).

Forms of definition of the developable surface

(1) Parametrical form of definition of a torse with given cuspidal edge.

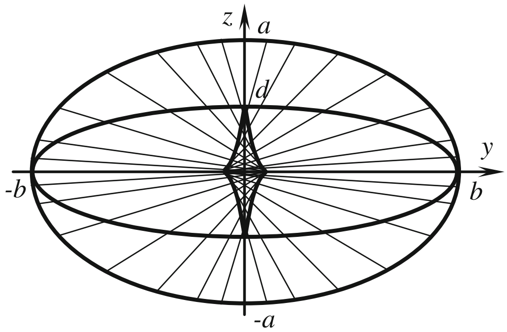

Fig. 1

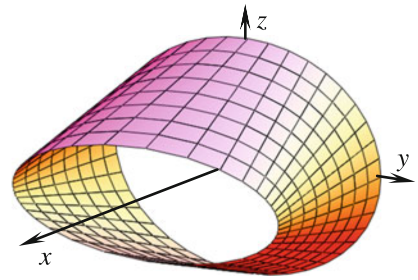

Fig. 2

If we have a cuspidal edge, then a developable surface is constructed definitively and its coefficients of fundamental forms are easily obtained.

An equation of single parametrical system of tangent planes was obtained by S.N. Krivoshapko

$$M = x\left(n + \frac{md\sqrt{d^2 - v^2} + cd^2 - \sqrt{b^2d^4 + \mu v^2}}{cv}\right) + \frac{l\sqrt{b^2d^4 + \mu v^2}}{cv} - lz - \frac{ld\sqrt{d^2 - v^2}}{cv}y = 0,$$

where $v = \beta - n$.

In Fig. 1, a projection of the torse with its cuspidal edge on the plane $x = 0$ for the case of $m = n = 0$, $c = b$ is shown.

(2) Parametrical form of definition:

$$x = x(\lambda) = \lambda l,\ y = y(\lambda, u) = b\sin u[1 - \lambda + \lambda a/f(u)],$$
$$z = z(\lambda, u) = \cos u\left[a - a\lambda + \lambda d^2/f(u)\right],$$

where $f^2(u) = d^2\cos^2 u + a^2\sin^2 u,\ 0 \le \lambda \le 1$.

The coordinate lines $\lambda = 0$ and $\lambda = 1$ coincide with the directrix ellipses (Fig. 2) given in the form

$$x = 0,\quad y = b\sin u,\quad z = a\cos u \text{ and}$$
$$x = l,\quad y = b\sin v,\quad z = d\cos v.$$

These ellipses in comparison with ellipses of general type have $c = b$, $m = n = 0$.

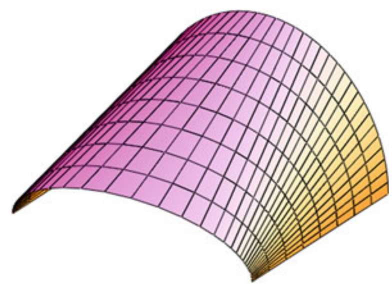

Fig. 3

The condition of developable surface's uniqueness is $d\mathrm{tg}v = a\mathrm{tg}u$.

A system of curvilinear coordinates u, λ is nonorthogonal but conjugate because

$$A = A(\beta),\quad F \ne 0,\quad L = 0,\quad M = 0.$$

In addition,

$$\frac{\partial(B\cos\chi)}{\partial\lambda} = \frac{\partial A}{\partial u}$$

where χ is an angle between the coordinate lines u and λ.

A rectangular plane with dimensions of $2b \times l$ lying in the coordinate plane $z = 0$ can be covered by this developable surface (Fig. 3).

Reference

Krivoshapko SN. Developable Surfaces and Shells. Moscow: Izd-vo UDN. 1991; 287 p.

■ Torse with Two Parabolas Having One Common Axis and Lying in Intersecting Planes

A developable surface with two parabolas having one common axis and lying in the intersecting planes contains two parabolas

$$\text{I: } x = az^2,\ y = bz$$
$$\text{and II: } x = dz^2 + c,\ y = 0$$

as directrix curves.

A coordinate axis Ox coincides with the line of intersection of two planes with the directrix parabolas. Rectilinear generatrixes of the developable surface get through the points $z = \beta$ of the first parabola and through the points $z = \gamma$ of the second parabola placed in the coordinate plane xOz,

$$\gamma = \pm\sqrt{\frac{a}{d}\beta^2 + \frac{c}{d}}.$$

Parametrical equations of the cuspidal edge can be written in the following form (Fig. 1)

$$x = 3a\beta^2 + 2c, \quad y = -\frac{ab}{c}\beta^3,$$
$$z = \pm\frac{d}{c}\left(\frac{a}{d}\beta^2 + \frac{c}{d}\right)^{3/2} - \frac{a}{c}\beta^3.$$

Forms of definition of the studied developable surface

(1) Parametrical form of definition of a torse with given cuspidal edge.

If we have a cuspidal edge, then a developable surface is constructed definitively and its coefficients of fundamental forms are easily obtained.

A developable surface with the cuspidal edge is presented in Fig. 1. The torse is limited by the director parabola $x = az^2$, $y = bz$ and coordinate lines $u = 0$, $\beta = 0$, and $\beta = a$. The coordinate line $u = 0$ is the cuspidal edge.

The coordinate lines β (u = const) do not coincide with the director parabolas I and II (Fig. 1).

(2) Vector form of the definition:

$$\boldsymbol{r} = \boldsymbol{r}(u, \beta) = \boldsymbol{a}(\beta) + u\boldsymbol{l}(\beta),$$

where $\boldsymbol{a}(\beta)$ is the radius vector of the cuspidal edge; $\boldsymbol{l}(\beta)$ is a unit tangent vector given at every point of the cuspidal edge. Coordinate lines u coincide with the rectilinear generatrixes

Equations of the first and second directrix parabolas are

$$u_1 = (c + a\beta^2)\sqrt{4 + \frac{b^2\beta^2}{c^2} + \frac{1}{c^2}\left(\beta \mp \sqrt{\frac{a}{d}\beta^2 + \frac{c}{d}}\right)^2},$$
$$u_2 = \frac{a\beta^2}{c}\sqrt{4c^2 + b^2\beta^2 + \left(\beta \mp \sqrt{\frac{a}{d}\beta^2 + \frac{c}{d}}\right)^2}.$$

(3) Parametrical form of the definition (Fig. 2):

$$x = x(\lambda, \beta) = a\beta^2 + 2c\lambda,$$
$$y = y(\lambda, \beta) = b\beta(1 - \lambda),$$
$$z = z(\lambda, \beta) = \pm\lambda\sqrt{\frac{a}{d}\beta^2 + \frac{c}{d}} + \beta(1 - \lambda),$$
$$0 \le \lambda \le 1.$$

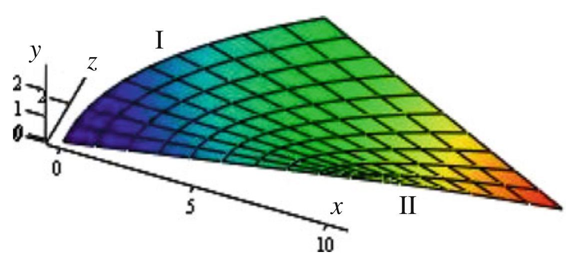

Fig. 2

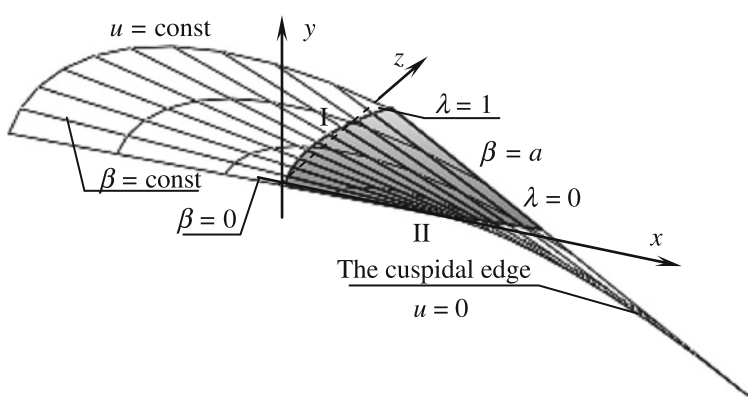

Fig. 1

A system of curvilinear coordinates β, λ is nonorthogonal and conjugate because

$$A = A(\beta), \quad F \neq 0, \quad L = 0, \quad M = 0.$$

In addition,

$$\frac{\partial(B\cos\chi)}{\partial\lambda} = \frac{\partial A}{\partial\beta}$$

where χ is an angle between the coordinate lines β and λ.

A coordinate line $\lambda = 0$ coincides with the first parabola and a line $\lambda = 1$ coincides with the second parabola, i.e., with the parabola placed in the coordinate plane $y = 0$.

Additional Literature

Krivoshapko SN. Developable Surfaces and Shells. Moscow: Izd-vo UDN. 1991; 287 p.
Krivoshapko SN. Static analysis of shells with developable middle surfaces. Applied Mechanics Reviews. 1998; Vol.51, No. 12, Part 1, p. 731-746.
Park FC, *Yu Junghyun*, *Chun Changmook*. Design of developable surfaces using optimal control. Journal of Mechanical Design. 2002; 124(4), p. 602-608.

■ Torse with Two Parabolas of the Second and Forth Order Placed in Parallel Planes and with Parallel Axes

A developable surface contains two parabolas

$$x = 0,\ y = az^4 \ \text{ and } \ x = l,\ y = bz^2$$

as directrix curves. The rectilinear generatrixes of the torse pass through the points z = β of the fourth-order parabola and through the point $z = \gamma$ of the second-order parabola,

$$\gamma = \frac{2a\beta^3}{b}.$$

Parametric equations of the cuspidal edge are

$$x = \frac{bl}{b - 6a\beta^2}, \quad y = \frac{-2a^2\beta^6}{b - 6a\beta^2}, \quad z = -\frac{4a\beta^3}{b - 6a\beta^2}.$$

The cuspidal edge has the first-order cuspidal edge with coordinates: $x = l$, $y = z = 0$ (Fig. 1). Discontinuity of the cuspidal edge takes a place under the following value of the β parameter:

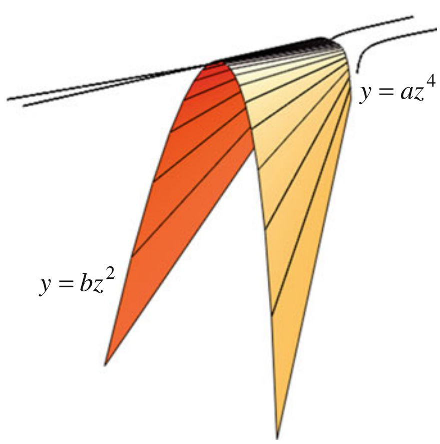

Fig. 1

$$\beta = \pm\sqrt{\frac{b}{6a}}.$$

The coordinate plane $z = 0$ is a plane of symmetry of the developable surface.

Forms of definition of the studied developable surface

(1) Parametrical form of definition of a torse with given cuspidal edge.

If we have a cuspidal edge, then a developable surface is constructed definitively and its coefficients of fundamental forms are easily obtained as it is shown in Subsect. "1.1.1. Torse Surfaces (Torses)".

An equation of single parametrical system of tangent planes was obtained by S.N. Krivoshapko:

$$M = 3ab(l - x)\beta^4 + 4a^2x\beta^6 - 4ablz\beta^3 + bly = 0.$$

(2) Vector form of the definition:

$$\boldsymbol{r} = \boldsymbol{r}(u, \beta) = \boldsymbol{a}(\beta) + u\boldsymbol{l}(\beta),$$

where $\boldsymbol{a}(\beta)$ is the radius-vector of the cuspidal edge; $\boldsymbol{l}(\beta)$ is a unit tangent vector given at every point of the cuspidal edge. Coordinate lines u coincide with the rectilinear generatrixes of the torse

Equations of the directrix parabolas of the fourth and second order can be presented in the following form:

$$u_1 = \mp\frac{\sqrt{b^2l^2 + (ab\beta^4 - 4a^2\beta^6)^2 + (b\beta - 2a\beta^3)^2}}{b - 6a\beta^2},$$

$$u_2 = \mp\frac{\sqrt{36a^2l^2\beta^4 + (6a^2\beta^6 - 24a^3\beta^8/b)^2 + (6a\beta^3 - 12a^2\beta^5/b)^2}}{b - 6a\beta^2}.$$

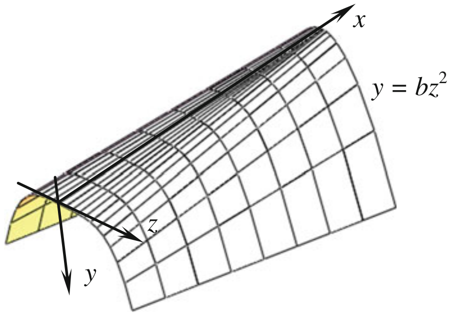

Fig. 2

(3) Parametrical form of the definition (Fig. 2):

$$x = x(\lambda) = l\lambda,$$
$$y = y(\lambda, \beta) = a\beta^4\left(1 - \lambda + \frac{4a\beta^2}{b}\lambda\right),$$
$$z = z(\lambda, \beta) = \beta(1 - \lambda) + \frac{2a}{b}\beta^3\lambda,$$
$$0 \le \lambda \le 1.$$

A system of curvilinear coordinates β, λ is nonorthogonal and conjugate because

$$A = A(\beta), \quad F \neq 0, \quad L = 0, \quad M = 0.$$

In addition,

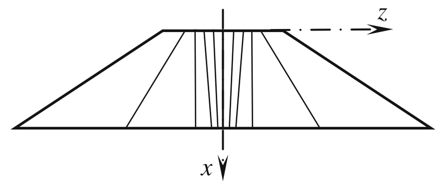

Fig. 3

$$\frac{\partial(B\cos\chi)}{\partial\lambda} = \frac{\partial A}{\partial\beta}$$

where χ is an angle between the coordinate lines β and λ. A coordinate line $\lambda = 0$ coincides with the fourth-order parabola and a line $\lambda = 1$ coincides with the second-order parabola placed in the plane $x = l$.

An approximate development of the studied developable surface is shown in Fig. 3.

Additional Literature

Krivoshapko SN. Developable Surfaces and Shells. Moscow: Izd-vo UDN. 1991; 287 p.
Krivoshapko SN. Developable surfaces for covering of given rectangular plane. Vestnik RUDN: “Engineering Research”. 2002; No. 1, p. 47-51.

■ Torse with Parabola and Circle in Parallel Planes

A developable surface contains circle and parabola (Figs. 1 and 2)

$$x = -\sqrt{R^2 - z^2}, \quad y = 0 \text{ and } x = \frac{z^2}{2p}, \quad y = h$$

as directrix curves.

The rectilinear generatrixes of the torse pass through the points z = β of a circle with a radius R lying in a coordinate plane xOz and through the correspondent point $z = \gamma$ of a parabola, (Fig. 1)

$$\gamma = \frac{p\beta}{\sqrt{R^2 - \beta^2}}, \quad \beta < R.$$

Parametric equations of the cuspidal edge are

$$x = \frac{p}{2}\sqrt{R^2 - \beta^2}\left[\frac{\beta^2 + 2R^2}{(R^2 - \beta^2)^{3/2} - pR^2}\right],$$
$$y = \frac{h}{1 - \frac{pR^2}{(R^2 - \beta^2)^{3/2}}}, \quad z = -\frac{p\beta^3}{(R^2 - \beta^2)^{3/2} - pR^2}.$$

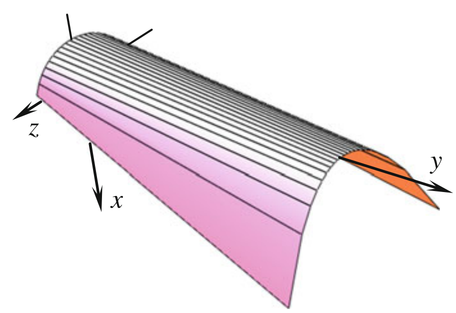

Fig. 1

A point with the coordinate

$$x = \frac{pR}{R - p}, \quad y = \frac{hR}{R - p}, \quad z = 0$$

on the edge of regression ($\beta = 0$) is the first-order cuspidal point (Fig. 3).

Discontinuity of the cuspidal edge takes a place in two points with the β parameter:

$$\beta = \pm\sqrt{R^2 - (pR^2)^{2/3}}.$$

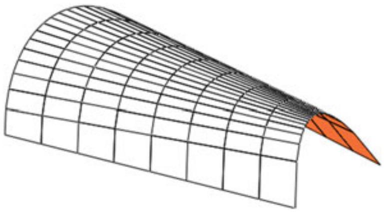

Fig. 2

The torse surface has a plane of symmetry with $z = 0$.

Forms of definition of the studied developable surface

(1) Parametrical form of definition of a torse with given cuspidal edge.

If we have a cuspidal edge, then a developable surface is constructed definitively and its coefficients of fundamental forms are easily obtained as it is shown in Subsect. "1.1.1. Torse Surfaces (Torses)".

A projection of the torse surface with its cuspidal edge on the coordinate plane $y = 0$ is shown in Fig. 3.

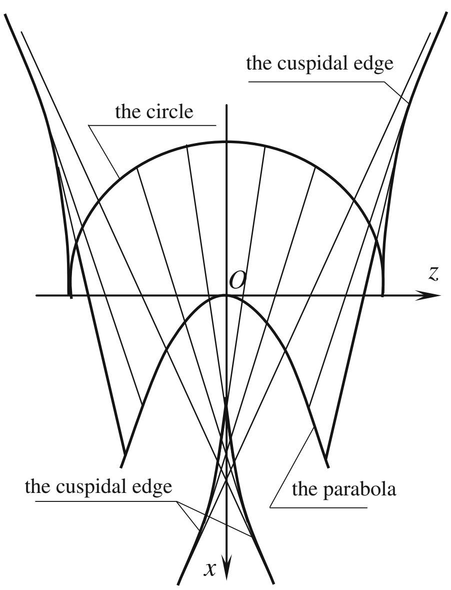

Fig. 3

An equation of single parametrical system of tangent planes was obtained in the following form:

$$M(x, y, z, \beta) = z\beta - x\sqrt{R^2 - \beta^2} + y\left(\frac{R^2}{h} - \frac{p\beta^2}{2h\sqrt{R^2 - \beta^2}}\right) = 0.$$

(2) An equation of the continuous skeleton of rectilinear generatrixes of the torse (Fig. 1):

$$y = \frac{2p\tan u + 2R\sin u}{2R\cos u - p\tan^2 u}x - \frac{p\tan^2 u(p\tan u + R\sin u)}{2R\cos u - p\tan^2 u} - p\tan u,$$
$$z = \frac{2hx}{p\tan^2 u - R\cos u} - \frac{2hR\cos u}{p\tan^2 u - 2R\cos u}.$$

Equations of the circle and parabola taken as directrix curves must be presented in the following form:

$$x = R\cos u, \; y = 0, \; z = R\sin u$$
$$\text{and } x = v^2/(2p), \; z = v, \; y = h.$$

One-to-one correspondence between the parameters u and v has the following form:

$$v = -p\tan u.$$

(3) Parametric form of definition (Fig. 2):

$$x = x(\lambda, \beta) = (\lambda - 1)\sqrt{R^2 - \beta^2} + \frac{\lambda p\beta^2}{2(R^2 - \beta^2)},$$
$$y = y(\lambda) = \lambda h,$$
$$z = z(\lambda, \beta) = \beta\left(1 - \lambda + \frac{p\lambda}{\sqrt{R^2 - \beta^2}}\right),$$
$$0 \le \lambda \le 1.$$

A coordinate line $\lambda = 0$ coincides with the circle but the line $\lambda = 1$ coincides with the parabola. A system of curvilinear coordinates β, λ is nonorthogonal and conjugate.

Additional Literature

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■ Torse with Parabola and Ellipse in Parallel Planes

A developable surface contains an ellipse and a parabola of the second-order placed in parallel planes (Figs. 1 and 2)

$$\boldsymbol{r}_1(u) = a\cos u\boldsymbol{i} + b\sin u\boldsymbol{k} \quad \text{(ellipse)},$$
$$\boldsymbol{r_2}(v) = 2cv\boldsymbol{i} + l\boldsymbol{j} + \left(b - cv^{\mathbf{2}}\right)\boldsymbol{k} \quad \text{(parabola)}.$$

The condition of developable surface's uniqueness gives an opportunity to find one-to-one correspondence between the parameters v and u in the form:

$$v = \frac{b}{a}\operatorname{cotan} u.$$

Canonical equations of the directrix ellipse and parabola are

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1,\ y = 0 \text{ (ellipse)};$$
$$y = l,\ z = b - \frac{x^2}{4c} \text{ (parabola)}.$$

Forms of definition of the studied developable surface

(1) Vector equation (Fig. 1):

$$\boldsymbol{r}(\lambda, u) = \left[a\cos u(1-\lambda) + \frac{2bc}{a}\lambda\operatorname{cotan} u\right]\boldsymbol{i} + \lambda l\boldsymbol{j} + b\left[(1-\lambda)\sin u + \lambda\left(1 - \frac{cb}{a^2}\operatorname{cotan}^2 u\right)\right]\boldsymbol{k},$$
$$0 \le \lambda \le 1.$$

A coordinate line $\lambda = 0$ coincides with the ellipse but a line $\lambda = 1$ coincides with the parabola of the second order.

A system of curvilinear coordinates u, λ is nonorthogonal and conjugate because

$$A = A(u), \quad F \neq 0, \quad L = 0, \quad M = 0.$$

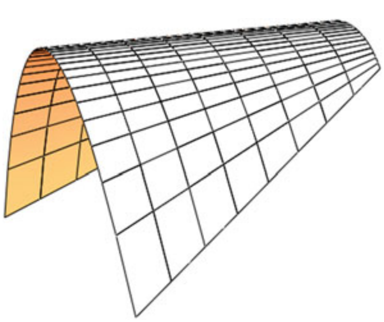

Fig. 1

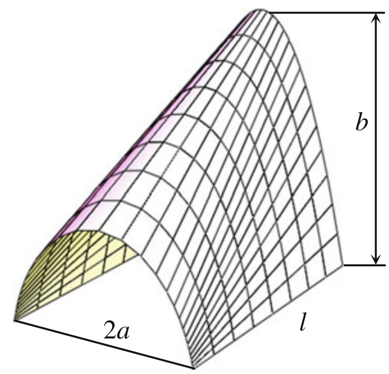

Fig. 2

In addition,

$$\frac{\partial(B\cos\chi)}{\partial\lambda} = \frac{\partial A}{\partial u}$$

where χ is an angle between the coordinate lines u and λ.

(2) A developable surface with the directrix parabola and ellipse (Fig. 2)

$$x = 0, \quad y = -\frac{bz^2}{a^2} \text{ and}$$
$$x = l, \quad y = \sqrt{\frac{c^2(2b-c)^2}{4(b-c)^2} - \frac{bc^2z^2}{a^2(b-c)}} - \frac{c^2}{2(b-c)}$$

covers a given rectangular plan $2a \times l$ lying in the plane $y = 0$ and two contour rectilinear generatrixes ($y = 0$, $z = \pm a$) of the developable surface are parallel to an axis Ox and lie in a horizontal plane $y = 0$.

The rectilinear generatrixes of the torse pass through the points $z = \beta$ of the parabola lying in a coordinate plane yOz and through the correspondent point $z = \gamma$ of the ellipse placed in the plane $x = l$.

One-to-one correspondence between the parameters β and γ has the following form:

$$\gamma = \frac{(2b-c)\beta}{\sqrt{c^2 + \frac{4b(b-c)}{a^2}\beta^2}}.$$

Parametrical form of the definition (Fig. 2):

$$x = x(\beta, \lambda) = \lambda l,$$
$$y = y(\beta, \lambda) = \left(b - \frac{b}{a^2}\beta^2\right)(1-\lambda) + \lambda\left(\sqrt{\frac{c^2(2b-c)^2}{4(b-c)^2} - \frac{bc^2\gamma^2}{a^2(b-c)}} - \frac{c^2}{2|b-c|}\right),$$
$$z = z(\beta, \lambda) = \beta(1-\lambda) + \lambda\gamma,$$
$$0 \le \lambda \le 1.$$

So, having assumed independent geometrical parameters a, l (Fig. 2) and b, c (rises of the parabola and ellipse) it is possible to design a torse surface covering the given rectangular plan $2a \times l$ and containing a parabola and ellipse on the opposite parallel faces.

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■ Torse with Hyperbola and Parabola in Parallel Planes

Assume a hyperbola and a quadratic parabola

$$\text{I: } x = 0,\ \frac{(y-d)^2}{k^2} - \frac{z^2}{m^2} = 1 \text{ (hyperbola);}$$
$$\text{II: } x = l,\ \ y = pz^2 + c \text{ (parabola)}$$

as directrix curves of a developable surface.

Assume that two rectilinear generatrixes lie on the opposite horizontal sides $z = \pm a$ of the rectangular plan $2a \times l$ (Fig. 1) then we must fulfill the following conditions:

$$k = -\frac{b}{1 - \sqrt{1 + a^2/m^2}}.\ d = b + k,\ p = -c/a^2,$$

In this case, equations of directrix curves must be taken as

$$\text{I}:\ x = 0,\ y = b + k - k\sqrt{1 + z^2/m^2};$$
$$\text{II}:\ \ x = l,\ y = -(c/a^2)z^2 + c,$$

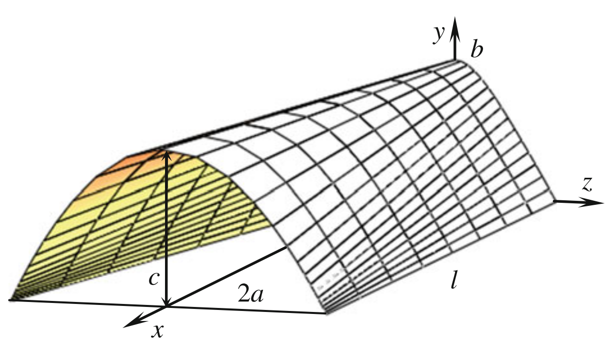

Fig. 1

where

$$m^2 = \frac{a^2 c}{b-c}\left(1 - \frac{b}{2c}\right)^2,\ \text{then } k = \frac{b|b-2c|}{b - |b-2c|} > 0,\ b > c.$$

Hence, it is necessary to assume four geometrical parameters a, b, c, and l of eight parameters (a, b, c, l, d, k, m, and p). The condition of developable surface's uniqueness gives

$$\gamma = ka^2\beta \Big/ \left(2\,cm^2\sqrt{1 + \beta^2/m^2}\right),$$

where $\beta = z$ of the hyperbola and $\gamma = z$ of the parabola.

Parametrical equations of the studied developable surface (Fig. 1) are

$$x = x(\beta, \lambda) = l\lambda,$$
$$y = y(\beta, \lambda) = \left(b + k - k\sqrt{1 + \frac{\beta^2}{m^2}}\right)(1 - \lambda)$$
$$+ \lambda\left(c - \frac{c}{a^2}\gamma^2\right),$$
$$z = z(\beta, \lambda) = \beta(1 - \lambda) + \lambda\gamma.$$

It is known the equation of single parametrical family of the tangent planes:

$$\left(b + k - c - \frac{k}{\mu(\beta)} - \frac{k^2 a^2 \beta^2}{4\,cm^4\mu^2(\beta)}\right)x + yl$$
$$+ \frac{lk}{\mu(\beta)}\left(1 + \frac{z\beta}{m^2}\right) - l(b + k) = 0;$$
$$\mu(\beta) = \sqrt{1 + \beta^2/m^2}.$$

■ Torse with Two Ellipses Placed in Mutually Perpendicular Planes

This developable surface rested on two directrix ellipses

$$x = l + d\sqrt{1 - z^2/c^2},\ y = 0 \quad \text{and}$$
$$x = 0,\ y = b\sqrt{1 - z^2/a^2}$$

placed in mutually perpendicular coordinate planes. Rectilinear generatrixes pass through a point $z = \beta$ of the ellipse placed in the plane $y = 0$ and through a point $z = \gamma$ of the second ellipse lying in the plane yOz,

$$\gamma = \beta d a^2/(c^2 d + cl\sqrt{c^2 - \beta^2}),\ \text{where}$$
$$-c \le \beta \le c,\ -a \le \gamma \le a.$$

Parametric equations of the cuspidal edge are

$$y = \frac{b\left[f^2(\beta) - a^2\beta^2 d^2\right]^{3/2}}{f^3(\beta) - (lc + d\sqrt{c^2 - \beta^2})lca^2\beta^2 d - a^2 f^2(\beta)d},$$
$$x = \frac{d\sqrt{c^2 - \beta^2}}{c} - \frac{y}{bc}\frac{f(\beta)(lc + d\sqrt{c^2 - \beta^2})}{\sqrt{f^2(\beta) - a^2\beta^2 d^2}} + l,$$

$$z = -\frac{c\sqrt{c^2 - \beta^2}}{\beta d}(x - l) + \frac{c^2}{\beta^2} - \frac{y}{b}\sqrt{\frac{f^2(\beta)}{\beta^2 d^2} - a^2}$$
where $f(\beta) = c^2 d + lc\sqrt{c^2 - \beta^2}$

Parametric equations of the studied developable surface are (Figs. 1 and 2):

$$x = x(\lambda, \beta) = \left(l + d\sqrt{c^2 - \beta^2}/c\right)(1 - \lambda),$$
$$y = y(\lambda, \beta) = \lambda b\sqrt{1 - \gamma^2/a^2},$$

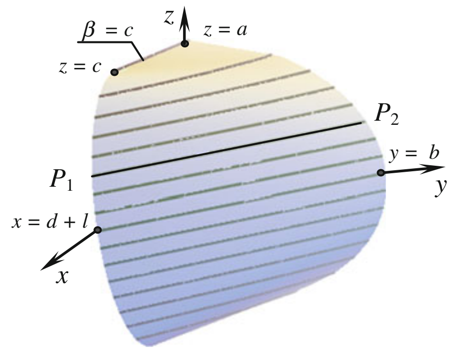

Fig. 1 $a = c = l = 2d;\ b = 3d$

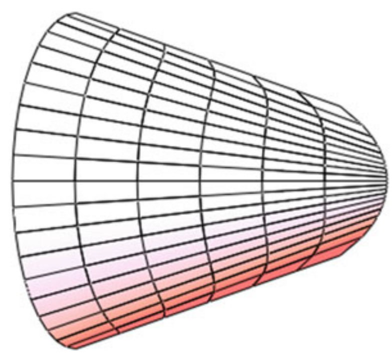
Fig. 2 $c = l = 2d;\ a = 1.5d;\ b = 3d$

$$z = z(\lambda, \beta) = (1 - \lambda)\beta + \lambda\gamma;$$
$$c(cd + l\sqrt{c^2 - \beta^2}) \ge ad\beta,\ 0 \le \lambda \le 1.$$

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■ The Third- and the Fourth-Order Developable Surfaces

The developable surfaces are the simplest developable surfaces if do not take into consideration degenerated developable surfaces in the form of cones and cylinders of the second order.

A developable surface of the third order can be only by conic or cylindrical surface. For example, five types of conical surfaces with directrix diverging parabolas given by equations

$$y^2 = ax^3 + bx^2 + cx + d$$

exist. Newton has presented a classification containing 72 types of third-order curves. All of them can be obtained in the sections of one (but not any) of the cones, the directrixes of which are five diverging parabolas.

Developable surfaces of the fourth order can be of two types. These are cones and cylinders with directrixes in the form of curves and torses of the fourth order of the third class T_3^4. The T_3^4 torses are surfaces of more complex formation. Their cuspidal edges are spatial curves of the third order. These curves can be designed as the lines of intersection of ruled surfaces of the second order having one common straight.

Two parabolas placed in parallel planes are also the base for creation of a torse surface of the fourth order. This

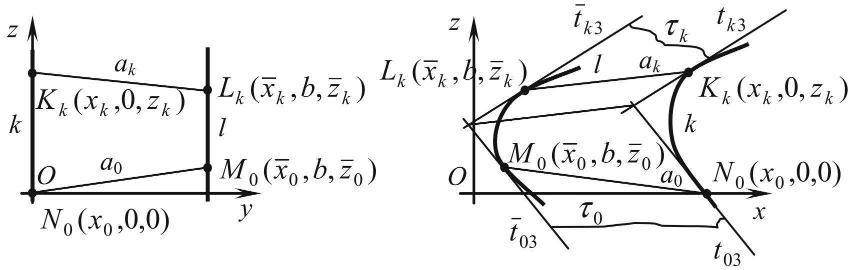

Fig. 1

problem was considered by V.Ya. Bulgakov. He took two planes $y = 0$ and $y = b$ with two directrix parabolas:

$$k:\ a_{11}x^2 + 2a_{12}xz + a_{22}z^2 + 2a_{13}x + 2a_{23}z + a_{33} = 0,\ y = 0;$$
$$l:\ b_{11}x^2 + 2b_{12}xz + b_{22}z^2 + 2b_{13}x + 2b_{23}z + b_{33} = 0,\ y = b,$$

where the coefficients a_{ij} and b_{ij} are unknown for the presence. Assume that equations of two tangent planes τ_0, τ_k to the torse and equations of two rectilinear generatrixes of the torse lying in the planes τ_0, τ_k and passing through the correspondent points (Fig. 1)

$$N_0(x_0, 0, 0) \text{ and } M_0(\bar{x}_0, b, \bar{z}_0),\ K_k(x_k, 0, z_k) \text{ and } L_k(\bar{x}_k, b, \bar{z}_k)$$

are known too.

The taken considerations give an opportunity to obtain an equation of the fourth order of the developable surface which contains generatrix straight line a_0 and a_k and touches two planes τ_0, τ_k. If we shall intersect the torse by the planes $y = 0$ and $y = b$, then we shall have two given plane parabolas.

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2. *Obukhova VS*, *Bulgakov VYa.* On one design method for the torses of the fourth order. Prikl. Geom. i Ingen. Grafika, Kiev. 1972; 15, p. 76-81 (6 ref.).
3. *Martirosov AL.* On developments of torses of the fourth order. Prikl. Geom. i Ingen. Grafika Kiev 1976; 22, p. 93-97.
4. *Bulgakov VYa.* On one method of design of the torses of the fourth order. Prikl. Geom. i Ingen. Grafika, Kiev. 1971; 13, p. 37-40

■ Torse with an Edge of Regression in the Form of the Line of Intersection of Circular Cylinder and Circular Cone

A torse with an edge of regression in the form of the of intersection of a circular cylinder

$$(x-3)^2+(y+4)^2=5^2$$

and a circular cone with the vertex (0; 0; 0) and with a directrix circle $(x-3)^2 + (y-4)^2 = 5^2$ in the plane $z = -8$ will be a torse surface of the 4th order. So, two equations determine the cuspidal edge of the studied torse:

$$(x-3)^2+(y+4)^2=5^2, \quad 4x^2+4y^2+3xz+4yz=0.$$

A form of the definition of the torse surface

(1) Parametrical equations (Fig. 1):

$$X = X(x, \lambda) = x + \lambda,$$
$$Y = Y(x, \lambda) = \frac{16 - x^2 + 6x + \lambda(3 - x)}{\pm e} - 4,$$
$$Z = Z(x, \lambda) = -\frac{4x^2 + 4(\pm e - 4)^2}{3x - 16 \pm 4e}$$
$$+\lambda\frac{4(\pm 3e - 4x + 12)\left[2x^2 + 2(\pm e - 4)^2\right] - (3x - 16 \pm 4e)[32x + 24(\pm e - 4)]}{\pm e(3x - 16 \pm 4e)^2},$$

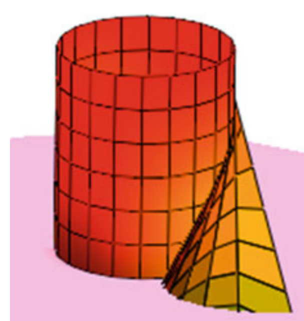

Fig. 1

where $e = \sqrt{16 - x^2 + 6x}$, λ is a parameter representing itself a projection of a line segment on the x axis from a point of the cuspidal edge of the torse along the tangent taken in the same point; x is the second independent parameter, $0 \le x \le 6$. In Fig. 1, the cylinder is limited by the planes $z = 8$ and $z = -8$.

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■ Surfaces of Constant Slope

Surfaces of constant slope are ruled surfaces having the constant angle α of their rectilinear generatrixes with the corresponding principal normals of a plane directrix curve (Fig. 1). Surfaces of constant slope are surfaces of zero Gaussian (total) curvature, in the general case, with cuspidal edges.

Tangent lines to the cuspidal edge coincide with the rectilinear generatrixes of a surface of constant slope. A right circle conical surface is also a surface of constant slope with a directrix circle. A cylindrical surface of arbitrary cross-section can be attached to surfaces of constant slope with $\alpha = \pi/2$.

Assume that a plane directrix curve

$$x = x(v),\ y = y(v)$$

lies in a horizontal plane then rectilinear generatrixes of the surface of constant slope will be disposed in vertical planes passing through the principal normals of the directrix curve and will constitute a constant angle α with the horizontal plane.

In this case, a vector equation of a surface of constant slope can be written in the form:

$$\begin{aligned} \boldsymbol{r} = \boldsymbol{r}(u,v) &= x(v)\boldsymbol{i} + y(v)\boldsymbol{j} \\ &\quad + u\frac{\cos\alpha}{\sqrt{x'^2(v)+y'^2(v)}}[y'(v)\boldsymbol{i} - x'(v)\boldsymbol{j}] \\ &\quad + u\sin\alpha\boldsymbol{k}. \end{aligned}$$

A cuspidal edge of a surface of constant slope is projected on the plane of directrix curve into the evolute of this directrix. The ratio of torsion to curvature of a cuspidal edge is a constant value and is equal to tangent of the slope angle α of the surface. A cuspidal edge of a surface of constant slope is *a sloping line*.

Any point of a cuspidal edge of the surface of constant slope with a plane generatrix curve orthogonal to the generatrix straight line is projected together with a center of principal curvature not equal to zero into the center of curvature of the directrix curve in its corresponding point.

For the determination of *Lame's coefficients* (coefficients of the first fundamental form) and principal curvatures of a surface of constant slope, the following formulas are obtained

$$\begin{aligned} &A = 1,\quad F = 0, \\ &B = \left\{1 + u\cos\alpha\frac{x'(v)y''(v) - x''(v)y'(v)}{[x'^2(v)+y'^2(v)]^{3/2}}\right\}\sqrt{x'^2(v)+y'^2(v)} \\ &\quad = \{1 + u\cos\alpha\cdot t(v)\}\sqrt{x'^2(v)+y'^2(v)}, \\ &k_u = k_1 = 0,\quad k_v = k_2 = \frac{\sin a}{B}t(v)\sqrt{x'^2(v)+y'^2(v)} \end{aligned}$$

where

$$t(v) = \frac{x'(v)y''(v) - x''(v)y'(v)}{[x'^2(v)+y'^2(v)]^{3/2}}.$$

So, the surface is given in lines of principal curvatures u, v where a coordinate line $u = 0$ coincides with a directrix curve.

Having known the vector equation of a surface of constant slope, it is easy to write the parametrical equations of the same surface:

$$\begin{aligned} X &= X(u,v) = x(v) + u\frac{y'(v)\cos\alpha}{\sqrt{x'^2+y'^2}}, \\ Y &= Y(u,v) = y(v) - u\frac{x'(v)\cos\alpha}{\sqrt{x'^2+y'^2}}, \\ Z &= Z(u) = u\sin\alpha. \end{aligned}$$

Surfaces of constant slope can be formed as envelope of all position of a circular cone with the vertex moving along given directrix line but the cone's axis must keep invariable direction. If we have a directrix line in the form of a cylindrical helical line of constant lead and the axis of the cone is parallel to the axis of the helix, then an open helicoid will be generated. An open helicoid is the most known surface of constant slope. An open helicoid can be formed if one will take an evolvent of the circumference as a directrix curve.

At present time, 10 surfaces of constant slope are known and some of them were presented before.

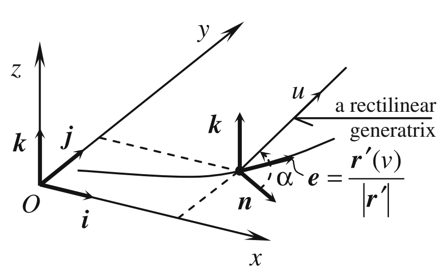

Fig. 1

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■ Torse of Constant Slope with Directrix Parabola

Let us study *a torse surface of constant slope with a directrix parabola*

$$y = ax^2,\ z = 0,$$

where a coefficient a is a given value.

Assume $k = \mathrm{tg}\alpha$; α is a slope angle, i.e., the angle of the rectilinear generatrixes with the horizontal plane $z = 0$.

Forms of definition of the developable surface of constant slope with the directrix parabola

(1) Implicit equation:

$$\begin{aligned}16a^4[k^2(x^2+y^2) - z^{2}]^3 - 8a^2k^2(1-2ay)^2[k^2(x^2+y^2) - z^2]^2 \\ + k^4(1-2ay)[k^2(x^2+y^2) - z^2][36a^2x^2 + (1-2ay)^3] \\ - k^6x^2[27a^2x^2 + (1-2ay)^3] = 0.\end{aligned}$$

A surface of the torse is an algebraic surface of the six order. Parameters x and y have even power. It means that the torse surface is symmetrical relative to the coordinate planes yOz and xOy.

(2) Parametrical equations (Fig. 1):

$$\begin{aligned}x &= x(u,v) = v + u\cos\alpha\frac{2av}{\sqrt{1+4a^2v^2}},\\ y &= y(u,v) = av^2 - \frac{u\cos\alpha}{\sqrt{1+4a^2v^2}},\\ z &= z(u) = -u\sin\alpha.\end{aligned}$$

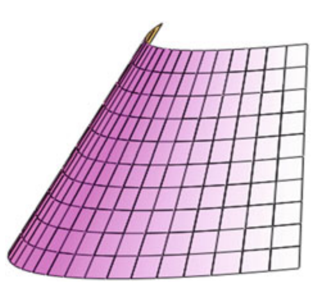


Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned}&A = 1,\ \ F = 0,\\ &B = \sqrt{1+4a^2v^2}\left[1 + \frac{2au\cos\alpha}{(1+4a^2v^2)^{3/2}}\right],\\ &L = M = 0,\ \ N = -\frac{2a\sin\alpha}{\sqrt{1+4a^2v^2}}\left[1 + \frac{2au\cos\alpha}{(1+4a^2v^2)^{3/2}}\right],\\ &k_1 = k_u = 0,\ \ k_2 = k_v = -\frac{2a\sin\alpha}{(1+4a^2v^2)^{3/2} + 2au\cos\alpha},\\ &K = 0.\end{aligned}$$

The torse surface is given in lines of principal curvatures u, v. The coordinate line $u = 0$ coincides with the given directrix parabola.

(3) Parametrical equations:

$$\begin{aligned}x = x(u,s) = x_p(s) + ux_p'(s), y = y(u,s) = y_p(s) + uy_p'(s), z \\ = z(u,s) = z_p(s) + uz_p'(s),\end{aligned}$$

where s is the length of the cuspidal edge. Parametrical equations of the edge of regression can be presented in the following form:

$$\begin{aligned}&x_p(s) = \frac{1}{\gamma}\left[\gamma\left(s + \frac{C}{\sin\alpha}\right)^{2/3} - \beta\right]^{3/2},\\ &y_p(s) = \frac{3}{2}\sqrt{\beta}\left(s + \frac{C}{\sin\alpha}\right)^{2/3} + C_y,\\ &z_p(s) = s\sin\alpha + C,\\ &\quad\gamma = 1 - C/\sin\alpha,\ \beta = \sqrt[3]{\left(1 - C^2/\sin^2\alpha\right)^2 \Big/ (4a^2)}.\end{aligned}$$

Here, C is determined from contour conditions on the surface; C_y is an arbitrary constant of integration. A sign …′ shows integration with respect to a parameter s.

It is possible to find the equation of the projection of the cuspidal edge on a coordinate plane $z = 0$ in Cartesian coordinates after elimination of the s parameter from two parametrical equations of the cuspidal edge in the following form:

$$x^{2/3} = \frac{2\sqrt[3]{2a}}{3}\left(y - \frac{1}{2a}\right).$$

So, the cuspidal edge is placed on the cylinder with a directrix curve in the form of *a semicubical parabola*.

The same surface is presented in Chap. "Torse with a given line of curvature in the form of the second order parabola" but that surface was designed under other conditions of the problem.

A.G. Varvaritza proposed to use torse surfaces of constant slope with a directrix parabola for approximation of topographical surfaces. An average angle of slope on country can be calculated by a formula

$$\alpha^0 = \frac{Lh}{0.175\,s}$$

where L is a length of all horizontals on the given lot (km); h is the height of section of the relief (m); and s is the area of the lot (hectare).

Additional Literature

Varvaritza AG. An approximation of topographical surface by a surface of constant slope. Prikl. Geom. i Ingen. Grafika, Kiev. 1976; 21, p. 39-42.

■ Torse of Constant Slope with Directrix Catenary

All rectilinear generatrixes of torse of constant slope with directrix catenary

$$y = a\cosh\frac{x}{a}$$

are inclined with respect to a plane, on which the catenary lies, under constant angle α.

Forms of definition of a developable surface of constant slope

(1) Parametrical equations (Fig. 1):

$$z = z(u) = -u\sin\alpha,$$
$$x = x(u,v) = v + u\cos\alpha\tanh\frac{v}{a},$$
$$y = y(u,v) = a\cosh\frac{v}{a} - u\frac{\cos\alpha}{}\cosh\frac{v}{a}.$$

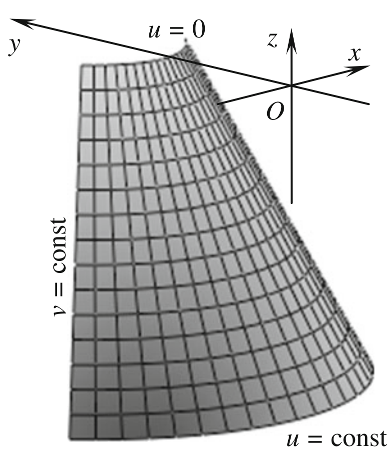


Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1,\quad F = 0,\quad B = \cosh\frac{v}{a} + \frac{u\cos\alpha}{a\cosh\frac{v}{a}},$$
$$L = M = 0,\quad k_u = k_1 = 0,$$
$$N = \frac{\left(a\cosh^2\frac{v}{a} + u\cos\alpha\right)\sin\alpha}{a^2\cosh^2\frac{v}{a}},$$
$$k_v = k_2 = \frac{\sin\alpha}{a\cosh^2\frac{v}{a} + u\cos\alpha},\quad K = 0.$$

The surface is given in lines of principal curvatures u, v. The coordinate line $u = 0$ coincides with the directrix catenary placed in a coordinate plane $z = 0$. A generatrix straight line

$$y = a + z\ \text{cotan}\ \alpha$$

lies in the section of the surface by a plane $x = 0$. Lines of intersection of the studied surface with the planes $z = z_0 = \text{const}$ are the plane curves:

$$x = x(v) = v - z_0\tanh(v/a)\text{cotan}\,\alpha;$$
$$y = y(v) = a\cosh(v/a) + z_0\,\text{sech}(v/a)\text{cotan}\,\alpha.$$

Additional Literature

Timoshin MA. Developable surface of constant slope with a directrix catenary. In: Vserossiyskaya vystavka NTTM: Sb. mat. Jily 7-10, 2004; p. 19-20.

■ Torse of Constant Slope with an Edge of Regression on One Sheet Hyperboloid of Revolution

Assume that one sheet hyperboloid of revolution is given in implicit form:

$$x^2 + y^2 - a^2z^2 = c^2,$$

where c is a radius of waist radius;

$$a = \tan\varphi;$$

φ is an angle between the axis of the hyperboloid and its straight lines.

Three types of sloping lines can be placed on a hyperboloid depending on the angle ω between the tangent lines to the sloping line and the axis of the hyperboloid (the angle of slope):

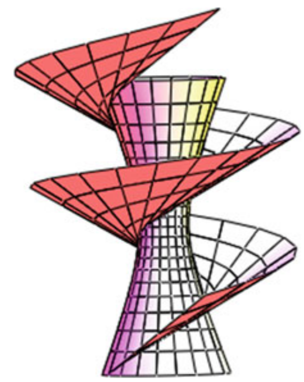
(1) straight generatrixes of a hyperboloid when $\varphi = \omega$;
(2) when $\varphi > \omega$; and
(3) when $\varphi < \omega$.

Parametrical equations of the sloping lines placed on the one sheet hyperboloid of revolution when $\varphi < \omega$ ($\mathrm{tg}\omega > \mathrm{tg}\varphi$) have the following form:

$$x(u) = c(m\sinh mu\cos u + \cosh mu\sin u),$$
$$y(u) = c(m\sinh mu\sin u - \cosh mu\cos u),$$
$$z(u) = \frac{c}{a}\sqrt{1+m^2}\sinh mu,$$

where $m = \frac{a}{\sqrt{\tan^2\omega - a^2}}$.

Having assumed this line as a cuspidal edge of the developable surface one can form *a torse of constant slope with an edge of regression on one sheet hyperboloid of revolution*.



$c = 5$; $\omega = \pi/3$; $\varphi = \pi/8$;
$0 \le v \le 30$; $-2\pi \le u \le 2\pi$

Fig. 1

A form of definition of a developable surface of constant slope

(1) Vector form of definition:

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{r}(u,v)\\ &= x(u)\boldsymbol{i} + y(u)\boldsymbol{j} + z(u)\boldsymbol{k}\\ &\quad + v\frac{a\sqrt{1+m^2}(\cos u\boldsymbol{i} + \sin u\boldsymbol{j}) + m\boldsymbol{k}}{\sqrt{a^2(1+m^2)+m^2}},\end{aligned}$$

u, v are curvilinear nonorthogonal conjugate coordinates (Fig. 1).

Additional Literature

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Wunderlich Walter. Kurven konstanter ganzer Krümmung und fester Hauptnormalenneigung. Monatsh. ath. 1973; 77, No. 2, p. 158-171 (12 ref.).

■ Torse of Constant Slope with Directrix Ellipse

A torse of constant slope with a directrix ellipse is a developable surface having the constant angle α of the rectilinear generatrixes with the corresponding principal normals of the directrix ellipse.

Forms of definition of a developable surface of constant slope

(1) Parametrical equations (Fig. 1):

$$z = z(u) = -u\sin\alpha,$$
$$x = x(u,v) = a\cos v + \frac{ub\cos\alpha\cos v}{\sqrt{a^2\sin^2 v + b^2\cos^2 v}},$$
$$y = y(u,v) = b\sin v + \frac{ua\cos\alpha\sin v}{\sqrt{a^2\sin^2 v + b^2\cos^2 v}}.$$

For this case, the directrix ellipse is given in the parametrical form:

$$x = x(v) = a\cos v, \quad y = y(v) = b\sin v$$

or in the implicit form:

$$x^2/a^2 + y^2/b^2 = 1.$$

The coordinate line $u = 0$ coincides with the directrix ellipse but a family of the coordinate lines u is the rectilinear generatrixes of the torse of constant slope.

Coefficients of the fundamental forms of the surface:

$$A = 1, \quad F = 0,$$
$$B = \sqrt{a^2 \sin^2 v + b^2 \cos^2 v} + u \frac{ab \cos \alpha}{\left(a^2 \sin^2 v + b^2 \cos^2 v\right)},$$
$$L = M = 0, \quad N = -\frac{ab \sin \alpha}{\left(a^2 \sin^2 v + b^2 \cos^2 v\right)} B,$$
$$k_u = k_1 = 0, \quad k_v = k_2 = -\frac{ab \sin \alpha}{B\left(a^2 \sin^2 v + b^2 \cos^2 v\right)}, \quad K = 0.$$

In the section of the torse surface by the planes $z = z_0 = \text{const}$, closed curves lie:

$$x = x(v) = \left(a - \frac{z_0 b \cos \alpha}{\sin \alpha} \frac{1}{\sqrt{a^2 \sin^2 v + b^2 \cos^2 v}}\right) \cos v,$$
$$y = y(v) = \left(b - \frac{z_0 a \cos \alpha}{\sin \alpha} \frac{1}{\sqrt{a^2 \sin^2 v + b^2 \cos^2 v}}\right) \sin v.$$

Torse surfaces designed in the limits (Figs. 1 and 2)

$$0 \le v \le 2\pi; \; 0 \le u \le (b^2/a)/\cos \alpha, \text{ if } b < a,$$
$$\text{i.e. under } H_{\max} < \left(b^2/a\right) \text{tg}\alpha,$$
$$0 \le v \le 2\pi; \; 0 \le u \le (a^2/b)/\cos \alpha, \text{ if } b > a,$$
$$\text{i.e. under } H_{\max} < \left(a^2/b\right) \text{tg}\alpha$$

can be the most interesting for practical needs.

The cuspidal edge is a spatial closed curve which is a line of slope. The directrix ellipse ($u = 0$) is an evolvent of the projection of the cuspidal edge on the plane xOy. Parametrical equations of the projection of the cuspidal edge on the plane xOy have the following form:

$$x = x(v) = \left[\left(a^2 - b^2\right)/a\right] \cos^3 v,$$
$$y = y(v) = -\left[\left(a^2 - b^2\right)/b\right] \sin^3 v.$$

(2) Parametrical equations (Fig. 1):

$$x = x(u, \beta) = r(\beta) \cos \beta + \frac{ub^2 \cos \alpha \cos \beta}{\sqrt{a^4 \sin^2 \beta + b^4 \cos^2 \beta}},$$
$$y = y(u, \beta) = r(\beta) \sin \beta + \frac{ua^2 \cos \alpha \sin \beta}{\sqrt{a^4 \sin^2 \beta + b^4 \cos^2 \beta}},$$
$$z = z(u) = -u \sin \alpha,$$

where $r = r(\beta) = \frac{ab}{\sqrt{a^2 \sin^2 \beta + b^2 \cos^2 \beta}}$, β is the angle taken from an axis Ox in the direction of an axis Oy, $\beta \ne v$, $r = r(\beta)$ is the distance from the center of the directrix ellipse till arbitrary point on it.

For this case, the directrix ellipse of the surface of constant slope is given in the following parametrical form

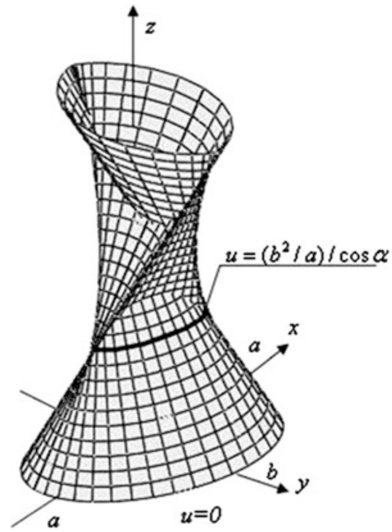


Fig. 1

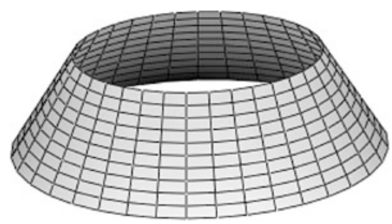


Fig. 2

$$x = x(\beta) = r \cos \beta,$$
$$y = y(\beta) = r \sin \beta \text{ or in implicit form } x^2/a^2 + y^2/b^2 = 1.$$

The line $u = 0$ coincides with the directrix ellipse

Additional Literature

Krutov AV. Some lines of surfaces of constant slope bound up with plastic deforming. Inform. Tehnologii i Sistemy, Voronezh: Mezhd. Akad. Informatizatzii. 2001; 4, p. 167-171 (9 ref.).

Kopytko MF, *Savula YaG*. On one possible expansion of class of shells of zero Gaussian curvature. Problemy Mashinostroiniya. Kiev: Naukova Dumka. 1982; 17, p. 61-65.

1.1.2 Cylindrical Surfaces

A cylindrical surface is formed by the movement of a generatrix straight remaining parallel to some given straight line and intersecting the given directrix curve (Fig. 1). Hence, a ruled surface having a family of the rectilinear generatrixes

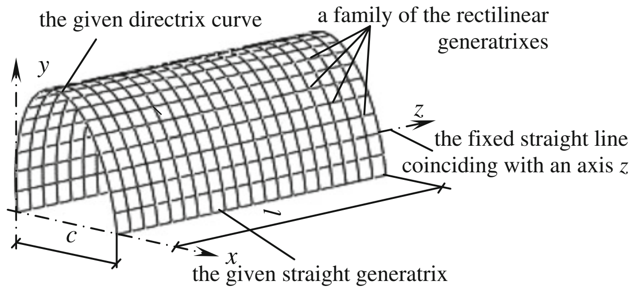


Fig. 1 Horizontal cylindrical surface

parallel to any fixed straight line is called a cylindrical surface. The direction of the fixed straight line is called *an axial direction of the cylinder*.

A cylindrical surface is given by a vector equation

$$\boldsymbol{r}(s,\lambda) = \boldsymbol{\rho}(s) + \lambda \boldsymbol{e},$$

where $\boldsymbol{\rho}(s)$ is the radius-vector of a director curve; $\boldsymbol{e}$ is a unit vector coinciding with the axial direction of the cylinder.

Cylindrical surfaces are surfaces of zero Gaussian curvature. A cuspidal edge of cylindrical surface moves off to infinity.

A cylinder is a volume limited by the cylindrical side surface and by two *base*. Cylinders are subdivided in oblique and right cylinders. *A right cylinder* is a cylinder having generatrixes of the side surface perpendicular to the bases. Generatrixes of a side surface of *the oblique cylinder* are not perpendicular to the bases.

Equations in the form

$$f(x,y) = 0, \quad f(x,z) = 0, \quad f(y,z) = 0$$

determine the side surfaces of right cylinders, the straight generatrixes of which are parallel to the axes Oz, Oy, and Ox correspondently.

The tangent plane of a cylindrical surface $f(x, y) = 0$ will have an equation:

$$f_x(\zeta - x) + f_y(\eta - y) = 0$$

where ζ, η are coordinates x, y of a point of tangency. This plane is parallel to an axis Oz and its position does not depend on the coordinate z of the point of tangency. *A line of slope* lying on the cylindrical surface transforms into the straight line in the time of developing of this surface. A spatial curve is called *a sloping line* (a line of slope) if torsion–curvature ratio does not change along the curve:

$$\kappa/k = \text{const} \neq 0.$$

A tangent vector of the sloping line forms a constant angle with the unchanged plane and that is why a sloping line is called *a line of constant angle of slope* also. A ruled surface formed by tangents of the sloping line is called a surface of constant slope. These surfaces are described in a Sect. "Surfaces of Constant Slope". A line placed on the cylinder is called a directrix of the cylinder if any rectilinear generatrix of the cylinder intersects it in one point only. A cylindrical surface with a straight directrix is *a plane*. *Geodesic lines* on a cylindrical surface are lines of slope.

In *cylindrical surfaces of the second order*, directrix curves are the second-order curves. Right cylinders with the side surfaces of the second order can be subdivided into three types of cylinders: (1) elliptical cylinder; (2) hyperbolical cylinder; and (3) parabolic cylinder.

A surface of the second order

$$\begin{aligned} &a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz \\ &+ 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0, \end{aligned}$$

will be a cylindrical surface if

$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ and}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0.$$

The wall surfaces and glass surfaces of the windows of The First University of Nations in Canada (Fig. 2) is the striking example of the application of cylindrical surfaces in real structures of buildings. This building was formed completely from cylindrical surfaces of general type. *Cylindrical surface of general type* has plane complex director curves not described by analytical equations.

Fig. 2 The first University of Nations, Canada, Regina, Saskatchewan

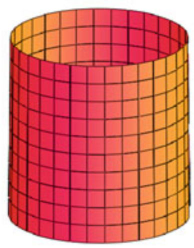
Additional Literature

Soldatos KP. Mechanics of cylindrical shells with non-circular cross-section: A survey. Applied Mechanics Reviews. 1999; Vol. 52 (8), p. 237-274.

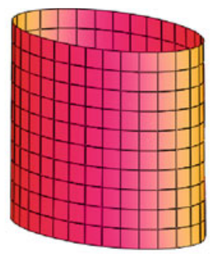
Soldatos KP. Review of three dimensional analysis of circular cylinders and cylindrical shells. Applied Mechanics Reviews. 1994; Vol. 47 (10), p. 501-516.

Simitses JG. Buckling and postbuckling of imperfect cylindrical shells. Applied Mechanics Reviews. 1986; Vol. 39 (10), p. 1517-1524.

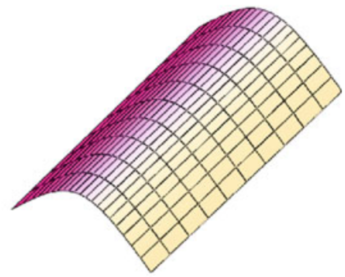
■ Cylindrical Surfaces Presented in the Encyclopedia



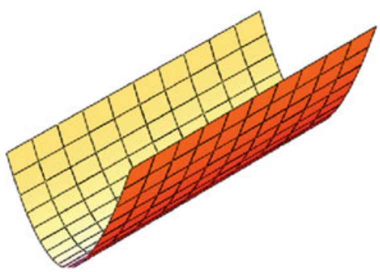
The cylindrical surface of revolution



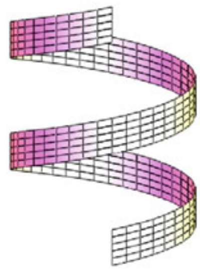
The elliptical cylinder



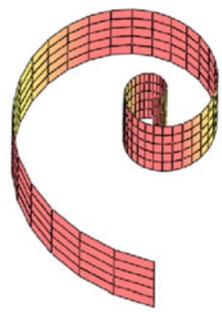
The hyperbolic cylinder



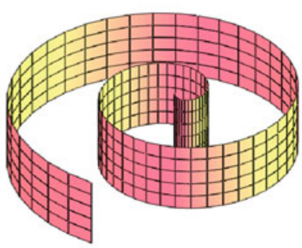
The parabolic cylinder



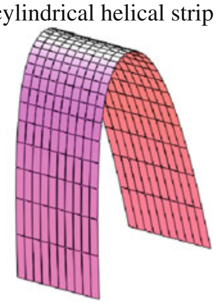
The cylindrical helical strip



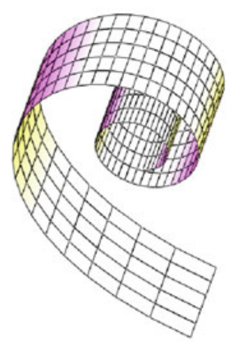
The right cylinder with a directrix logarithmic spiral



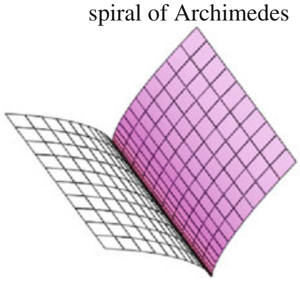
The right cylinder with a directrix spiral of Archimedes



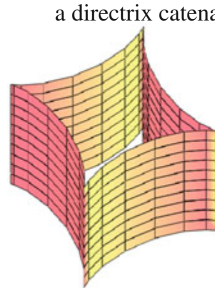
The right cylinder with a directrix catenary



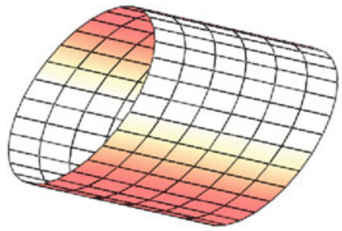
The right cylinder with a directrix hyperbolic spiral



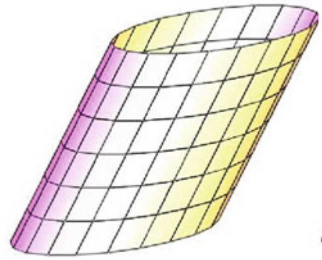
The right cylindrical surface with a directrix semi-cubical parabola



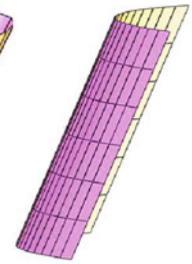
The right astroidal cylindrical surface



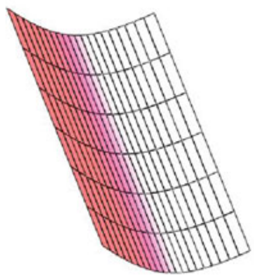
The oblique circle cylinder



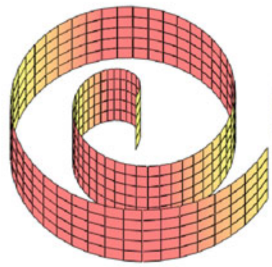
The oblique elliptic cylinder



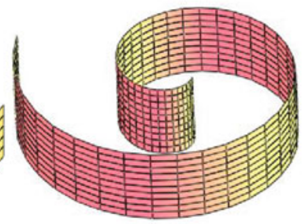
The oblique parabolic cylinder



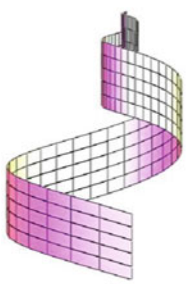
The oblique hyperbolic cylinder



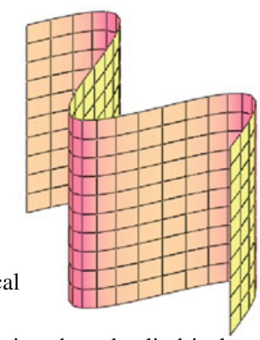
The right cylinder with a directrix of spiral of Fermat



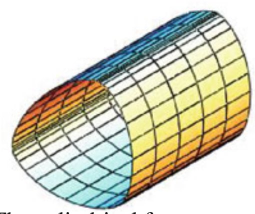
The evolvent cylindrical surfaces



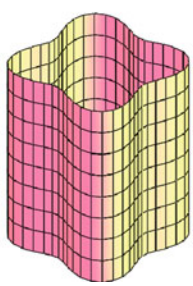
The cylindrical-and-conical strip



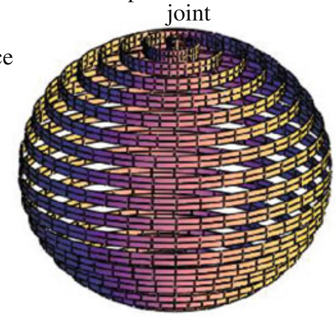
The sine-shaped cylindrical surface



The cylindrical fragment of the surface of the shaft for a profile detachable joint



The right wave cylindrical surface



The cylindrical-and-spherical spiral-shaped strip

The most known cylindrical surfaces are presented on these two pages. Every plane curve can be taken as a directrix curve of the cylindrical surface and on this base, it is possible to design a right or oblique cylindrical surface. Spatial curves can be taken as directrix curves too. It will extend a class of cylindrical surfaces still more.

■ Cylindrical Helical Strip

A cylindrical helical surface (*strip*) is formed by movement of a straight generatrix of constant or variable length along the helical directrix and the straight generatrix in all positions must parallel to an axis of the helical directrix. The lengths of the generatrixes must be less then pitch of the helix. So, a cylindrical helical strip is a part of the cylindrical surface limited by two co-axial helixes and that is why this is a surface of zero total curvatures.

Forms of definition of a cylindrical helical strip

(1) Parametrical equations (Fig. 1):

$$x = x(v) = a\cos v, \quad y = y(v) = a\sin v,$$
$$z = z(u, v) = cv + u.$$

The coordinate lines v = const are the rectilinear generatrixes of the helical strip and the coordinate lines u = const are the helixes with the constant pitch lying on a right circle cylinder with a radius a. Curvilinear contours of a strip of constant width coincide with the helical coordinate lines $u = 0$ and $u = b$, where b is the width of the strip; $b < 2\pi c$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1, \quad F = c, \quad B^2 = a^2 + c^2,$$
$$L = M = 0, \quad N = a,$$
$$k_u = k_1 = 0, \quad k_v = a/B^2, \quad k_2 = 1/a.$$

Parametrical equations of a plane development of the strip can be written as

$$x_p = x_p(v) = av, \; z_p = z_p(v) = cv + b.$$

The development of the cylindrical helical strip of constant width of b on a plane is shown in Fig. 2;

$$\tan\varphi = \frac{c}{a}.$$

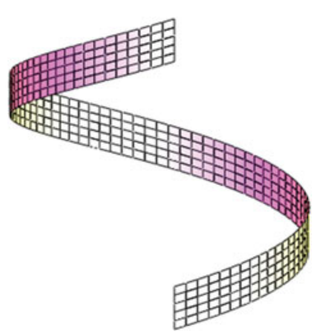


Fig. 1

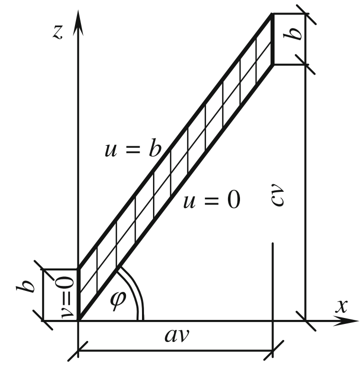


Fig. 2

(2) Parametrical equations (Fig. 3):

$$x = x(v) = a\cos v, \quad y = y(v) = a\sin v,$$
$$z = z(u, v) = cv + tvu + u.$$

The coordinate lines v = const are the rectilinear generatrixes of the helical strip and the coordinate lines u = const are the helical lines with the different pitch lying on a right cylinder with a radius a.

Curvilinear contours of the strip shown in Fig. 3 coincide with the helical coordinate lines $u = 0$ and $u = b$. A development of surface of the cylindrical helical strip of variable width is shown in Fig. 4; b is a width of the strip if $v = 0$, tg$\varphi = c/a$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1 + tv, \quad F = (1 + tv)(c + tu),$$
$$B^2 = a^2 + (c + tu)^2,$$
$$L = M = 0, \quad N = a,$$
$$k_u = k_1 = 0, \quad k_v = a/B^2, \quad k_2 = 1/a.$$

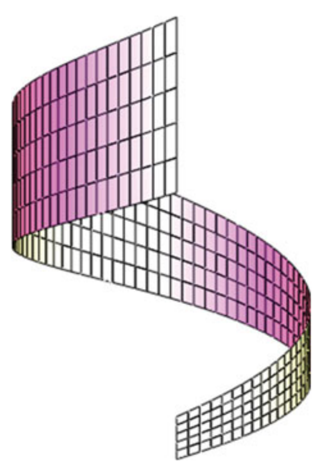


Fig. 3

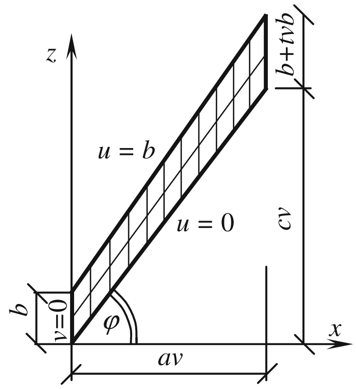


Fig. 4

Fig. 5 The helical ramp, Moscow, 2011

Cylindrical helical strip can be seen in the form of side surfaces of barrier of a helical ramp (Fig. 5).

Additional Literature

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■ Elliptical Cylinder

Elliptical cylindrical surface is a side surface of a right elliptical cylinder. *Elliptical cylinder* is a body limited by an elliptical cylindrical surface and by two bases formed in the sections of this body by two planes perpendicular to its straight generatrixes (Fig. 1). Very often in the scientific literature, they simply call the elliptical cylindrical surface by *an elliptical cylinder*. Any cross-section of an elliptical cylinder by a plane not parallel to its generatrix straights is ellipse. Every elliptical cylinder possesses a plane directrix circle. On the basis of this assertion, elliptical cylinders are sometimes called as *oblique or inclined circle cylinders*. Elliptical cylindrical surfaces belong to a class of *algebraic surfaces of the second order* and are surfaces of zero Gaussian curvature.

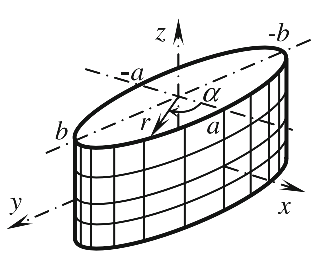

Fig. 1

Forms of definition of the studied cylindrical surface

(1) Implicit canonical equation:

$$x^2/a^2 + y^2/b^2 = 1.$$

An elliptical cylinder will be a right circle cylinder if $a = b$.

(2) Explicit equation (Fig. 1):

$$y = \pm b\sqrt{1 - x^2/a^2};$$
$$-a \le x \le a; \quad -b \le y \le b; \quad -\infty < z < \infty.$$

(3) Parametrical equations (Fig. 1):

$$x = x(\beta) = a\cos\beta; \quad y = y(\beta) = b\sin\beta; \quad z = z,$$

where β is an angle parameter taken from an axis Ox in the direction of an axis Oy; $0 \le \beta \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = a^2\sin^2\beta + b^2\cos^2\beta, \;\; F = 0, \;\; B = 1,$$
$$L = -ab/A, \;\; M = N = 0,$$
$$k_\beta = k_1 = -ab/A^3, \;\; k_z = k_2 = 0, \;\; K = 0.$$

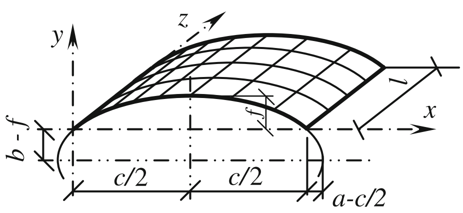

Fig. 2

(4) Parametrical equations (Fig. 1):

$$x = x(\alpha) = r\cos\alpha;\ y = y(\alpha) = r\sin\alpha;\ z = z,$$

where

$$r = \frac{ab}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}.$$

The surface is given in lines of principal curvatures α, z.

(5) Explicit equation (Fig. 2):

$$y = f - b + \frac{b}{a}\sqrt{a^2 - \left(x - \frac{c}{2}\right)^2},$$

$$\mathrm{tg}\gamma = y_x = \frac{b}{a}\frac{(c/2 - x)}{\sqrt{a^2 - (x - c/2)^2}},\quad a = \frac{cb}{2\sqrt{f(2b - f)}},$$

where γ is the angle of slope of the tangent to the elliptic contour taken from the axis Ox toward the positive direction of the axis Oy.

The presented form of definition is used if it is necessary to carry out a strength analysis of shallow shells in the form of an elliptic cylindrical surface on the rectangular plan $c \times l$.

Fig. 3 Silin bridge, Karpovka river, SPb, Russia

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{b^2(x - c/2)^2}{a^2[a^2 - (x - c/2)^2]},\quad B = 1,\quad F = 0,$$

$$L = -\frac{ab}{[a^2 - (x - c/2)^2]^{3/2}},\quad M = N = 0,$$

$$k_x = k_1 = L/A^2,\quad k_z = k_2 = 0,\quad K = 0.$$

The first major bridge "Silin bridge" over the river Karpovka in Saint Petersburg was erected in 1936. Its reinforced concrete vault was designed in the form of the elliptical cylindrical surface (Fig. 3).

Additional Literature

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■ Parabolic Cylinder

A right parabolic cylindrical surface is formed by movement of a straight generatrix which remains parallel to any fixed straight and intersects a given directrix parabola placed in a plane perpendicular to the fixed straight line (Fig. 1). Very often in the scientific literature, they simply call the parabolic cylindrical surface by *a parabolic cylinder*.

Any plane directrix of a parabolic cylinder is a parabola, i.e., in any cross-section of the cylinder by a plane, we have a parabola. But we must remember that the secant plane must not be parallel to *the axial direction of the cylinder*.

Forms of definition of parabolic cylindrical surface

(1) Explicit equation (Fig. 1):

$$y^2 = 2px,\ \text{where } p > 0.$$

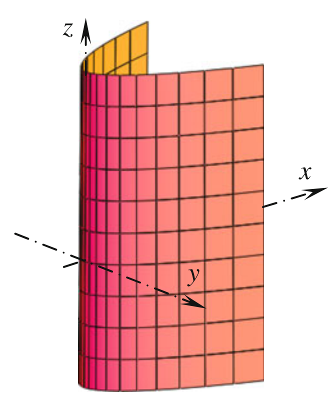

Fig. 1

A canonical equation of a parabolic cylindrical surface coincides with a canonical equation of its parabolic directrix. The value p is called *a parameter of parabolic cylinder*. This

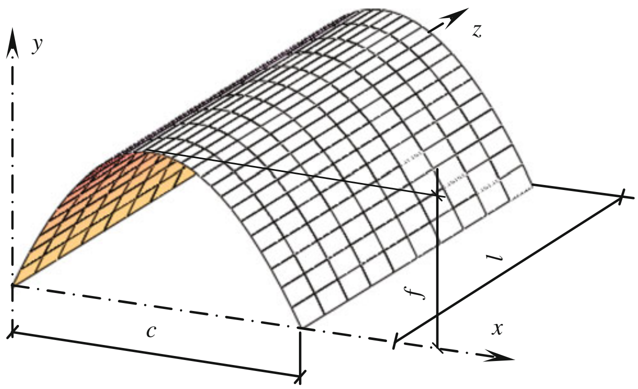

Fig. 2

value is determined by the cylinder itself and in one's turn, it defines the parabolic cylinder to within the position in the space.

(2) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(x, z) = x\boldsymbol{i} + \sqrt{2px}\boldsymbol{j} + z\boldsymbol{k}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{p}{2x}, \quad F = 0, \quad B = 1,$$
$$L = \frac{p^2}{(2px)^{3/2}A}, \quad M = N = 0,$$
$$k_x = k_1 = \frac{p^2}{(2px)^{3/2}A^3}, \quad k_z = k_2 = 0, \quad K = 0.$$

(3) Explicit equation (Fig. 2):

$$y = \frac{4fx(c - x)}{c^2}, \quad \mathrm{tg}\gamma = y_x = \frac{4f(c - 2x)}{c^2},$$

where f is a rise of a surface, c is its span, and γ is the angle of slope of the tangent to the parabolic contour taken from the axis Ox toward the positive direction of the axis Oy. The presented form of definition of the surface is used if it is necessary to carry out a strength analysis of shallow shells in the form of a parabolic cylindrical surface on the rectangular plan $c \times l$.

Fig. 3 The hangar no. 2 at present time, California, USA

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{16f^2}{c^4}(c - 2x)^2, \quad F = 0, \quad B = 1,$$
$$L = \frac{8f}{c^2A}, \quad M = N = 0;$$
$$k_x = k_1 = \frac{8f}{c^2A^3}, \quad k_z = k_2 = 0, \quad K = 0.$$

A length of a parabolic directrix of the surface within the limit of $0 \le x \le c$:

$$s = \sqrt{c^2 + 16f^2} + \frac{c^2}{4f}\ln\left|\frac{-c}{\sqrt{c^2 + 16f^2} - 4f}\right|.$$

As an example of using of parabolic cylindrical surface, we can offer the hangar No. 2 erected in California (USA) in 1943 (Fig. 3). This hangar is the National Property of the USA. It is one of the biggest spatial structures made of wooden elements.

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Andrushkov VI. Development of A.R. Rzhanitzin's method as applied to an analysis of shells rectangular in plan. Soprotivleniye mater. i teoriya soor.: Sb., Kiev. 1981; Vol. 39, p. 89-90.
Minakawa Couichi, *Maehata Tatumi*. Linear analysis of shallow translational shells with point support. Res. Repts. Eng. Kagoshima Univ. 1985; No 27, p. 103-117.
Hangar 2 (Building No. 46). NASA Ames Research Center, Moffett Field, California. 2006; 150p.

■ Hyperbolic Cylinder

A cylindrical surface is called *a right hyperbolic cylindrical surface* if it has the following canonical equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in some system of Cartesian coordinates x, y, z (Fig. 1) and

$$a > 0, \quad b > 0.$$

Very often in the scientific literature, they simply call the hyperbolic cylindrical surface by *a hyperbolic cylinder*.

A hyperbolic cylinder together with some point M_0 contains all straight line passing through this point parallel to an axis Oz. A family of rectilinear generatrixes will be parallel to some fixed straight called the axial direction of cylinder. Any line lying on a hyperbolic cylinder is called its generatrix if any rectilinear generatrix intersects it only in one point. Every plane directrix of a hyperbolic cylinder is a hyperbola.

Forms of definition of hyperbolic cylindrical surface

(1) A parametrical form of definition of one space of hyperbolic cylindrical surface:

$$x = x(\varphi) = \frac{p}{1 - e\cos\varphi}\cos\varphi,$$
$$y = y(\varphi) = \frac{p}{1 - e\cos\varphi}\sin\varphi,$$
$$z = z,$$

where $p/(1 - e\cos\varphi) = r$ is a polar radius of a point M_0 of the directrix plane hyperbola which is perpendicular to the axial direction of the hyperbolic cylinder; $p = b^2/a$ is a focal parameter of the director hyperbola; φ is an angle taken from the axis Ox at the pointer; $\theta < \phi < \pi + \theta$, $\cos\theta = 1/e$, 2θ is an angle between asymptotes of the hyperbola,

$$e = \sqrt{1 + b^2/a^2}$$

is a linear eccentricity of the hyperbola; a and b are real and imaginary half-axes. The axis Oz passes through *a focus of the directrix hyperbola* if one uses this method of definition.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = p^2\frac{1 - 2e\cos\varphi + e^2}{(1 - e\cos\varphi)^4}, F = 0, \quad B = 1,$$
$$L = -\frac{p^2}{A(1 - e\cos\varphi)^3}, \quad M = N = 0,$$
$$k_\varphi = k_1 = \frac{L}{A^2}, \quad k_2 = 0.$$

(2) Having taken two identical equilateral hyperbolas

$$z = 0, \quad y = \frac{a}{x} \text{ and } z = l, \; y = \frac{a}{x}.$$

as directrix curves, we can obtain an equation of the single-parametric family of planes

$$M(x, y, \beta) = ax - 2a\beta + \beta^2 y = 0$$

forming a cylindrical surface.

Having solved a system of two equations

$$M(x, y, \beta) = 0 \text{ and } \partial M/\partial\beta = 0$$

it is possible to find an explicit equation of the surface:

$$y = \frac{a}{x}.$$

So, parametrical equations of a side surface of the hyperbolic cylinder can be written in the following form:

$$x = x, \quad y = y(x) = \frac{a}{x}, \quad z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2}{x^4}, \quad F = 0, \quad B = 1,$$
$$L = -\frac{2a}{Ax^3}, \quad M = N = 0,$$
$$k_1 = k_x = -\frac{2a}{A^3x^3}, \quad k_2 = k_z = 0,$$
$$K = 0.$$

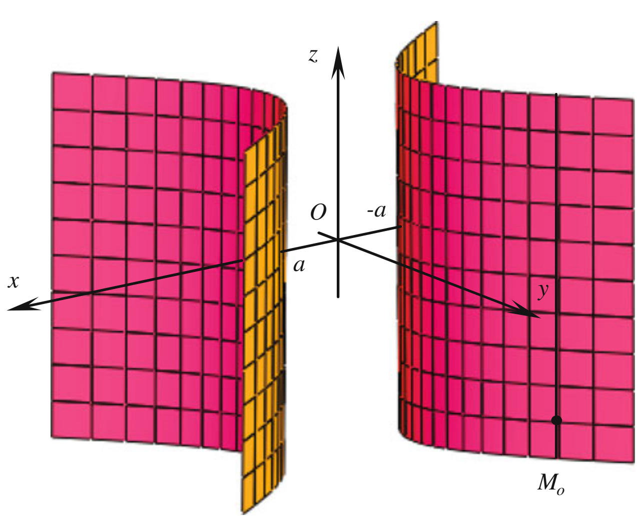

Fig. 1

Additional Literature
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Münch Mechthild. Beleuchtungsgebiete auf dem hyperbolischen Zylinder bei geometrischer Zentralbeleuchtung aus mehreren Lichtquellen. Wiss. Beitr. M. Luther-Univ. Halle, Wittenberg. 1986; M., No. 42, p. 95-102.
Soldatos KP. Mechanics of cylindrical shells with non-circular cross-section: A survey. Applied Mechanics Reviews. 1999; Vol. 52 (8), p. 237-274.

■ Right Cylindrical Surface with Directrix Semi-cubical Parabola

All points of this surface on the Ox axis are *singular points*. The singular straight line belonging to this cylindrical surface and lying on the Ox axis is called *a cuspidal edge of a surface* (Fig. 1).

Every plane perpendicular to the cuspidal edge intersects the surface along a semicubical parabola and the point of the cuspidal edge of the surface is a singular point of the curve of the intersection, i.e., *a cuspidal point* (*a cusp*).

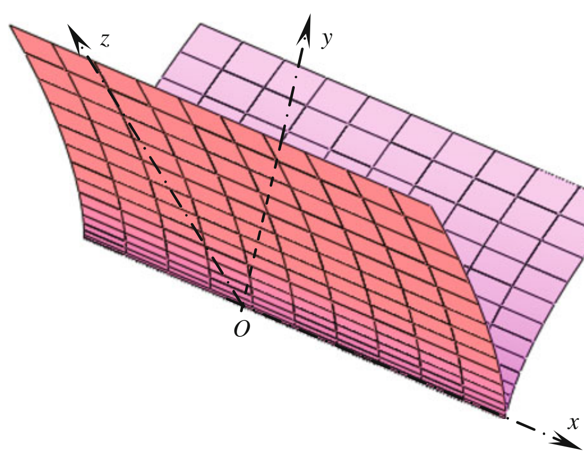

Fig. 1

Forms of definition of the cylindrical surface

(1) An explicit form of the definition (Fig. 1):

$$z = \pm a y^{3/2}.$$

The equation of the studied cylindrical surface is identical to an explicit equation of a semicubical parabola, i.e., *Neil's parabola*. In works on algebraic geometry, a semicubical parabola is sometimes called *a cuspidal cubic*.

(2) Parametrical form of the definition:

$$x = u, \quad y = a^2 v^2, \quad z = a^4 v^3.$$

Coefficients of the fundamental forms of the surface:

$$A = 1, \ \ F = 0, \ \ B^2 = 4a^4 v^2 + 9a^8 v^4,$$
$$L = M = 0, \ \ N = 6a^4 v / \sqrt{4 + 9a^4 v^2}.$$

Additional Literature
Pogorelov AV. Differential Geometry. Moscow: «Nauka», 1974; 176 p.
Gorbatovich JN. The second order curves as plane cross-sections of torse surfaces of constant slope. Belorus. technol. in-t, Minsk. 1991; 7p., dep v VINITI 04.01.91, No. 107-B91.

■ Right Cylinder with a Directrix Catenary

The least distance a right cylindrical surface with a directrix catenary given in the form

$$y = a\mathrm{ch}(x/a),$$

from an axis Ox is equal to a.

In the cross-section of the studied surface by planes z = const, catenaries

$$y = a\mathrm{ch}(x/a)$$

will be the length of which will be equal to $s = a\mathrm{sh}(x/a)$ within the limit $0 \leq x \leq x$.

The equation of catenary was obtained by G.W. Leibniz, Ch. Huygens and J. Bernoulli nearly.

A homogeneous elastic nonextended heavy filament hung by the ends has a form of catenary.

Forms of definition of the surface

(1) An explicit equation (Fig. 1):

$$y = a\mathrm{ch}(x/a).$$

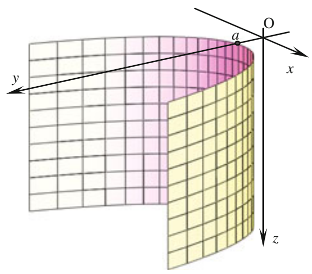

Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \mathrm{ch}(x/a), \quad F = 0, \quad B = 1,$$
$$L = -1/a, \quad M = N = 0,$$
$$k_x = k_1 = -\frac{1}{a\mathrm{ch}^2(x/a)},$$
$$k_z = k_2 = 0, \quad K = 0.$$

(2) An explicit equation: $y = a\left(\mathrm{ch}\frac{x}{a} - 1\right)$.

In this case, one of rectilinear generatrixes of the cylinder coincides with the coordinate axis Oz and the cylindrical surface is tangent to the coordinate plane xOz along this rectilinear generatrix.

(3) Parametrical equation:

$$x = x(s) = a\mathrm{Arsh}(s/a), \quad y = y(s) = \sqrt{a^2 + s^2}, \quad z = z,$$

where s is the length of a fragment of the catenary from a point $x = 0$ till $x = x$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = B = 1, \quad F = 0, \quad L = -\frac{a}{a^2 + s^2},$$
$$M = N = 0, \quad k_1 = k_x = L, \quad k_2 = k_z = 0.$$

(4) Parametrical form of definition:

$$x = x(s) = a\ln\left(\frac{s}{a} + \sqrt{1 + \frac{s^2}{a^2}}\right),$$
$$y = y(s) = \sqrt{a^2 + s^2},$$
$$z = z,$$

where s is the length of a fragment of the catenary from a point $x = 0$ till $x = x$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = B = 1, \quad F = 0, \quad L = -\frac{a}{a^2 + s^2},$$
$$M = N = 0, \quad k_1 = k_x = L, \quad k_2 = k_z = 0.$$

Additional Literature

Merkin DR. Introduction into Mechanics of Elastic filament. Moscow: "Nauka". 1980; 240 p.

■ Right Astroidal Cylindrical Surface

A right astroidal cylindrical surface has four singular straights (Fig. 1). Every plane perpendicular to the axis of the cylindrical surface intersects this surface along *an astroid* which is a plane algebraic curve of the sixth order. This curve is formed by a point of a circle with a radius of r rolling on the inner side of the circle with the radius of $a = 4r$. An astroid is also called a *quadrangle hypocycloid.* Astroid envelopes a family of ellipses which have constant sum of semiaxes.

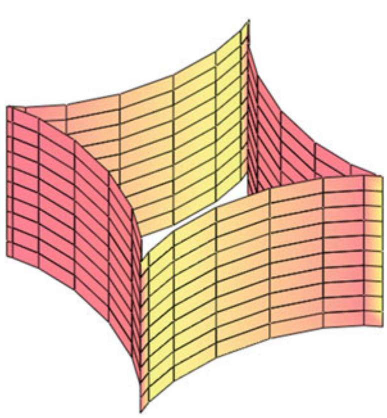

Fig. 1

Forms of definition of the cylindrical surface

(1) Implicit equation:

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

(2) Parametrical form of definition (Fig. 1):

$$x = x(t) = a\cos^3 t, \quad y = y(t) = a\sin^3 t, \quad z = z.$$

Coefficients of the fundamental forms of the surface:

$$A = 3a\sin t\cos t, \quad F = 0, \quad B = 1,$$
$$L = 3a\sin t\cos t, \quad M = N = 0, \quad k_t = 2/(3a\sin 2t).$$

The coordinate lines z coincide with the rectilinear generatrixes of the cylinder. The coordinate lines $t = 0$; $t = \pi/2$; $t = \pi$;

and $t = 3\pi/2$ are the cuspidal edges of the cylindrical surface. The lengths of all four arcs of the astroid are equal to $6a$.

An area limited by an astroid is equal to $3\pi a^2/8$.

Additional Literature

Hilbert D., *Cohn-Vossen S.* Anschauliche Geometrie. 1932; Berlin, 344 p.

■ Evolvent Cylindrical Surface

An evolvent of the circle is given in the form

$$x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t)$$

where t is an angle, taken from an axis Ox in the direction of an axis Oy, is a curvilinear directrix of *an evolvent cylindrical surface* (Fig. 1); $0 \le t < \infty$.

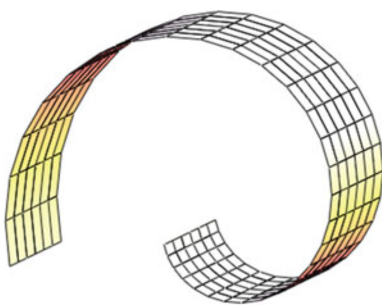

Fig. 1

Forms of definition of the surface

(1) Parametrical equations:

$$x = x(t, u) = a(\cos t + t\sin t) + u\cos\varphi\cos\beta,$$
$$y = y(t, u) = a(\sin t - t\cos t) + u\cos\varphi\sin\beta,$$
$$z = z(u) = u\sin\varphi,$$

where φ is the slope angle of straight generatrixes of the cylindrical surface to a plane $z = 0$; β is an angle between an axis Ox and the projection of the rectilinear generatrix of the cylindrical surface on the plane $z = 0$. We can design *an inclined evolvent cylindrical surface* if we shall take $\varphi \ne \pi/2$. Having assumed $\varphi = \pi/2$, we can design *a right evolvent cylindrical surface* (Fig. 1).

Coefficients of the fundamental forms of the surface:

$$A = at, \quad F = at\cos\varphi\cos(t - \beta), \quad B = 1,$$
$$L = -\,at\sin\varphi/\sqrt{1 - \cos^2\varphi\cos^2(t - \beta)}, \quad M = N = 0.$$

The side surface of the teeth of a *cylindrical gear with right teeth* is limited by a right evolvent cylindrical surface.

(2) Parametrical equations if $\varphi = \pi/2$ (Fig. 1):

$$x = x(t, z) = a(\cos t + t\sin t) + z\cos\beta\mathrm{ctg}\varphi,$$
$$y = y(t, z) = a(\sin t - t\cos t) + z\sin\beta\mathrm{ctg}\varphi, \ z = z.$$

■ Lemniscate Cylinder

Lemniscate cylindrical surface (*Der Lemniskate Zylinder*) has a plane directrix algebraic curve of the fourth order

$$(x^2 + z^2)^2 - r^2(x^2 - z^2) = 0$$
$$\text{or } x = x(u) = \frac{r\cos u}{1 + \sin^2 u},$$
$$z = z(u) = \frac{r\sin u\cos u}{1 + \sin^2 u}$$

with *the node point* at the beginning of coordinates.

So, a lemniscate cylindrical surface can be given in the following form (Fig. 1):

$$x = x(u) = \frac{r\cos u}{1 + \sin^2 u}, \quad y = y(v) = Hv,$$
$$z = z(u) = \frac{r\sin u\cos u}{1 + \sin^2 u}$$

$-\pi \le u \le \pi$, $0 \le v \le 1$; r, H are arbitrary constants but H may be called *a scale constant*.

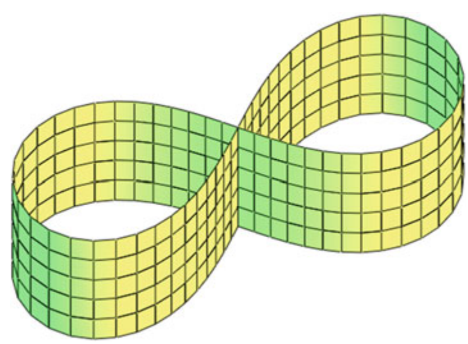

Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{r^2}{1 + \sin^2 u}, \quad F = 0, \quad B = H,$$
$$L = \frac{3r\cos u}{(1 + \sin^2 u)^{3/2}}, \quad M = N = 0.$$

Reference

Parametrische Flächen und Körper: http://www.3d-meier.de/tut3/Seite153.html

■ Right Cylinder with a Directrix Logarithmic Spiral

A right cylindrical surface with a directrix logarithmic spiral given in the polar coordinates as

$$\rho = ae^{m\varphi}$$

has the *cylindrical- and conical helical-shaped line of slope*:

$$\boldsymbol{r} = \boldsymbol{r}(\varphi) = e^{m\varphi}[a\cos\varphi\boldsymbol{i} + a\sin\varphi\boldsymbol{j} + b\boldsymbol{k}],$$

b = const, e is the natural number, φ is the azimuth.

Logarithmic spiral intersects all its radius-vectors under a constant angle θ:

$$\mathrm{ctg}\theta = m.$$

A length of a fragment of logarithmic spiral is

$$s = a\sqrt{1+m^2}(e^{m\varphi} - e^{m\varphi_0}).$$

The equiangular, or logarithmic, spiral was discovered by the French scientist René Descartes in 1638. In 1692, the Swiss mathematician Jakob Bernoulli named it *spira mirabilis* (“miracle spiral”) for its mathematical properties.

Parametrical equations of the studied surface can be written in the following form:

$$x = x(\varphi) = ae^{m\varphi}\cos\varphi, \quad y = y(\varphi) = ae^{m\varphi}\sin\varphi, \quad z = z.$$

If m has a positive magnitude then having assumed $\varphi \to \infty$ we have $\rho \to \infty$, but having taken $\varphi \to -\infty$, we will have $\rho \to 0$. Hence, the cylindrical surface twists around an axis Oz but never reaching it.

In Fig. 1, the cylindrical surface is shown for the case of

$$0 < m < 1; \; 0 < \varphi < 4\pi.$$

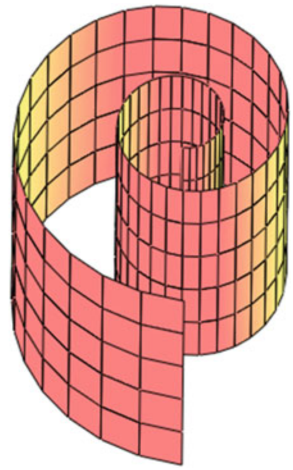

Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = ae^{m\varphi}\sqrt{1+m^2}, \quad F = 0, \quad B = 1,$$
$$L = -A, \quad M = N = 0,$$
$$k_\varphi = k_1 = -\frac{1}{A} = \frac{-e^{-m\varphi}}{a\sqrt{1+m^2}},$$
$$k_z = k_2 = 0, \quad K = 0.$$

Additional Literature

Girsh AG. Construction of logarithmic and Archimedes spirals. In: Nauchn. tr. Omsk. selhoz. in-t. 1974; 127, p. 112-115.

Logarithmic Spirals. Wolfram Demonstrations Project: Wolfram Mathematica. 2013.

■ Right Cylinder with a Directrix Hyperbolic Spiral

A directrix curve of this right cylinder surface is a hyperbolic spiral which is a plane transcendent curve with an equation in the polar coordinates:

$$\rho = \frac{a}{\varphi}.$$

If we shall assume $\varphi > 0$, then we shall have the right cylinder shown in Fig. 1.

A pole of this surface is *an asymptotical point*.

A length of the arc of this spiral between the two points $M_1(\rho_1, \varphi_1)$ and $M_2(\rho_2, \varphi_2)$ can be calculated with the help of the following formula:

$$s = a\left[-\frac{\sqrt{1+\varphi^2}}{\varphi} + \ln\left(\varphi + \sqrt{1+\varphi^2}\right)\right]_{\varphi_1}^{\varphi_2}.$$

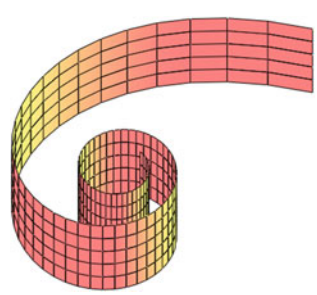

Fig. 1

Hyperbolic spiral is a particular case of *algebraic spirals*. A hyperbolic spiral is also known as *a reciprocal spiral*. A hyperbolic spiral is the opposite of an Archimedean spiral and is a type of *Cotes' spiral*.

Pierre Varignon has studied the curve as first, in 1704. Later Johann Bernoulli and Roger Cotes worked on the curve.

Forms of definition of the cylindrical surface

(1) Parametrical equations:

$$x = x(\varphi) = \frac{a}{\varphi}\cos\varphi, \quad y = y(\varphi) = \frac{a}{\varphi}\sin\varphi,$$
$$z = z.$$

A directrix hyperbolic spiral lying in cross-sections of the cylindrical surface by a plane z = const contains of two branches symmetrical relatively to an axis Oy. A hyperbolic spiral has *an asymptote* that is a straight line parallel to *the polar axis* (an axis Ox) and being away from it at the distance a.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{a}{\varphi^2}\sqrt{1+\varphi^2}, \quad F = 0, \quad B = 1,$$
$$L = -\frac{a^2}{A\varphi^2}, \quad M = N = 0,$$
$$k_1 = k_\varphi = -\frac{\varphi^2}{(1+\varphi^2)A},$$
$$k_2 = k_z = 0, \quad K = 0.$$

Additional Literature

Savelov AA. Plane Curves: Systematization, Properties, Applications. 1960; Moscow: Fizmatgiz, 293 p.

■ Right Cylinder with a Directrix Spiral of Fermat

A directrix curve of this right cylindrical surface is *a parabolic spiral* which is a plane transcendent curve with an equation in the polar coordinates in the form:

$$\rho = \pm a\sqrt{\varphi} + l, \quad l > 0.$$

If l = 0 then a parabolic spiral is called *a spiral of Fermat*. Every value of $\sqrt{\varphi}$ has positive or negative magnitude.

In Fig. 1, the cylindrical surface with a directrix curve in the form of a spiral of Fermat with only positive magnitudes of $\sqrt{\varphi}$ is shown. The cylindrical surface with only negative magnitudes of $\sqrt{\varphi}$ is presented in Fig. 2. The cylindrical surface with a parabolic directrix spiral $\rho = +a\sqrt{\varphi} + l$ where $l > 0$ is given in Fig. 3. Parabolic spiral is a particular case of *algebraic spirals*.

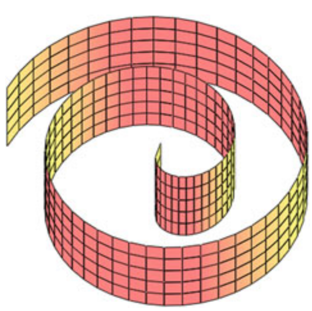

Fig. 1

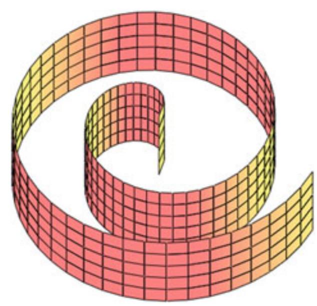

Fig. 2

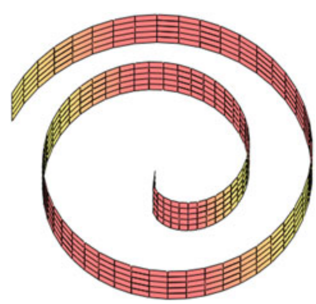

Fig. 3

Forms of definition of the cylindrical surface

(1) Parametrical form of the definition of the surface (Fig. 1):

$$x = x(\varphi) = \sqrt{2p\varphi}\cos\varphi,$$
$$y = y(\varphi) = \sqrt{2p\varphi}\sin\varphi,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = p\left(2\varphi + \frac{1}{2\varphi}\right), \quad F = 0, \quad B = 1,$$
$$L = -\frac{p(3+4\varphi^2)}{2\varphi A,}, \quad M = N = 0,$$
$$k_1 = k_\varphi = -\frac{3+4\varphi^2}{A(1+4\varphi^2)}, \quad k_2 = k_z = 0.$$

(2) Parametrical form of the definition of the surface (Fig. 3):

$$x = x(\varphi) = (a\sqrt{\varphi} + l)\cos\varphi,$$
$$y = y(\varphi) = (a\sqrt{\varphi} + l)\sin\varphi,$$
$$z = z.$$

■ Right Cylinder with a Directrix Spiral of Archimedes

A directrix curve of *a right cylindrical surface with a directrix spiral of Archimedes* is *a spiral of Archimedes*, i.e., a plane transcendent curve, an equation of which is

$$\rho = a\varphi.$$

Archimedes spiral is formed by a point P moving uniformly on a straight which rotates about a point O belonging to the same straight. A point P coincides with the center of rotation if $\varphi = 0$. The length of the arc between two points $P_1(\rho_1, \varphi_1)$ and $P_2(\rho_2, \varphi_2)$ can be calculated with the help of a formula:

$$s = a\left[\varphi\sqrt{1+\varphi^2} + \ln\left(\varphi + \sqrt{1+\varphi^2}\right)\right]_{\varphi_1}^{\varphi_2}/2.$$

Archimedes spiral belongs to algebraic spirals. *Neoida* an equation of which in polar coordinates can be written as

$$\rho = a\varphi + l$$

is a generalization of Archimedes spiral.

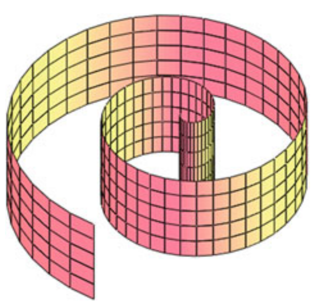

Fig. 1

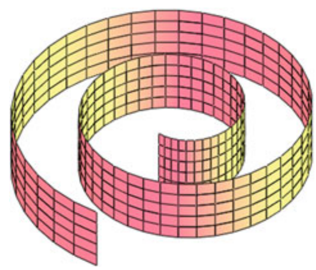

Fig. 2

Forms of definition of the studied surface

(1) Parametrical equations (Fig. 1):

$$x = x(\varphi) = a\varphi\cos\varphi, \quad y = y(\varphi) = a\varphi\sin\varphi, \quad z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = a\sqrt{1+\varphi^2}, \quad F = 0, \quad B = 1,$$
$$L = -\frac{2+\varphi^2}{\sqrt{1+\varphi^2}}a, \quad M = N = 0,$$
$$k_1 = k_\varphi = -\frac{2+\varphi^2}{a(1+\varphi^2)^{3/2}},$$
$$k_2 = k_z = 0, \quad K = 0.$$

(2) Parametrical equations (Fig. 2):

$$x = x(\varphi) = (a\varphi + l)\cos\varphi,$$
$$y = y(\varphi) = (a\varphi + l)\sin\varphi,$$
$$z = z.$$

Now, the neoida is taken as a plane directrix of the right cylindrical surface.

■ Surface of Cylindrical Flexible Hopper-Type Bin for the Keeping of Dry Materials

This flexible hopper-type bin is made in the form of a cylindrical nonclosed thin-walled shell with a horizontal axis which is hung up from two lengthwise bearing beams placed on the columns of building or on the detached posts. These flexible hopper-type bins are the most economical capacity for the storage of free-flowing bulk materials because of less expense of steel.

The directrix curve of the shell of flexible bunker is chosen from the condition of maximum conformity to the form of thin walls of the bunker under its full loading.

These are many formulas for the determination of the form of bunker but the following relation has the most distribution:

$$y = 2f\left[3\frac{x^2}{b^2} - 2\frac{|x|^3}{b^3}\right],$$

where b is a maximum width of a bunker (Fig. 1);

$$-\frac{b}{2} \le x \le \frac{b}{2};\quad 0 \le y \le f.$$

It is usually taken that $b \le 4$ m.

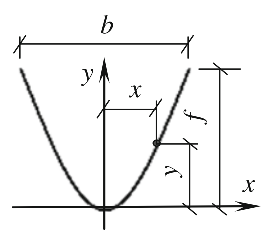

Fig. 1

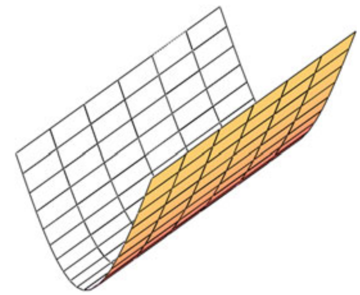

Fig. 2

The area A of the cross-section of the bunker and its volume V are

$$A = \frac{5fb}{8},\quad V = \frac{5fbL}{8},$$

where L is the length of the bunker.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + 144f^2x^2\frac{(1 - x/b)^2}{b^4},$$
$$F = 0,\quad B = 1,$$
$$L = -12f\frac{(1 - 2x/b)}{b^2A},\quad M = N = 0.$$

In Fig. 2, a surface of a flexible bunker is shown on condition that

$$b = 4\,\text{m};\quad f = 3.2\,\text{m};\quad L = 12\,\text{m}.$$

Reference

Metal Structures: Reference Book of Designer. The 2nd edition. Edited by NP Mel'nikov. 1980; Moscow: "Stroy-izdat", p. 465-466.

■ Cylindrical Surface "Eight"

Cylindrical surface "*Eight*" has a closed self-intersecting curve in the form of an "eight" figure

$$z = \frac{c}{b^2}y\sqrt{b^2 - y^2}$$

as a plane directrix curve and $-b \le y \le b$ (Fig. 1). The directrix curve lies in the coordinate plane yOz and the straight generatrixes are parallel to an axis x.

The parametrical equations of this cylindrical surface are

$$x = x(u, v) = a\cos u \sin v,$$
$$y = y(v) = b\cos v,$$
$$z = z(v) = c\sin v\cos v,$$

where $0 \le u \le \pi$, $-\pi/2 \le v \le \pi/2$; a, b, c are arbitrary constants. Coordinate lines u coincide with straight generatrixes of the cylindrical surface.

In the German scientific literature, this cylindrical surface is called "Die Schleife".

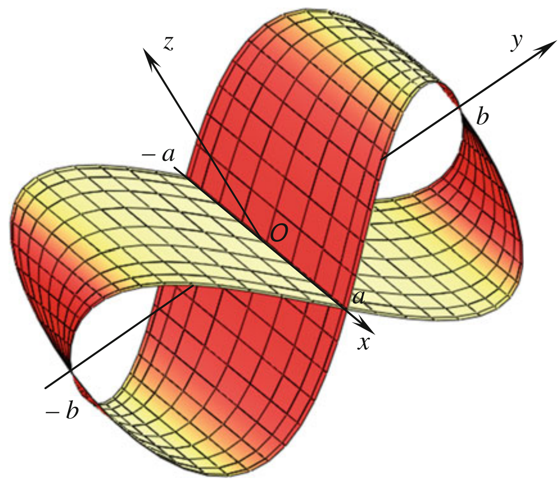

Fig. 1

If we begin to rotate the curve taken as generatrix of the studied cylindrical surface around the axis Oy, then we shall produce *a surface of revolution "Eight"* which is presented in Chap. "2. Surfaces of Revolution".

Reference

Parametrische Flächen und Körper. – http://www.3d-meier.de/tut3/Seite66.html

■ Cylindrical and Conical Spiral Strips

A cylindrical and conical spiral strips are formed by the movement of a straight generatrix of constant length along *a conical spiral* but the rectilinear generatrix in all position remains parallel to the axis of the conical spiral

$$x = x(\varphi) = r_0 \sin\lambda \cos\varphi \cdot e^{k\varphi},$$
$$y = y(\varphi) = r_0 \sin\lambda \sin\varphi \cdot e^{k\varphi},$$
$$z = z(\varphi) = r_0 \cos\lambda \cdot e^{k\varphi}$$

(Fig. 1), lying on the circle cone. Here, λ is an angle between the axis Oz and a generatrix of the cone; a longitude φ is the angle between a plane xOz and a mobile plane of the axial section; k is any positive or negative constant number; and r_0 is a constant value. A cylindrical and conical spiral strip is a fragment of *a right cylindrical surface with the directrix logarithmic spiral* presented in a Subsect. "Right cylinder with a Directrix Logarithmic Spiral" because a projection of conical spiral on a plane xOy is a logarithmical spiral

$$\rho = r_0 e^{k\varphi} \sin\lambda.$$

A conical spiral is a line of slope on a surface of a right cylinder with the directrix logarithmic spiral. An angle β between a tangent line to the conical spiral and a plane xOy perpendicular to the axis of the spiral can be determined with the help of a formula:

$$\sin\beta = \frac{k\cos\lambda}{\sqrt{k^2 + \sin^2\lambda}}.$$

The forms of the definition of a cylindrical and conical spiral strip

(1) Parametrical equations (Fig. 1):

$$x = x(\varphi) = r_0 \sin\lambda \cos\varphi \cdot e^{k\varphi},$$
$$y = y(\varphi) = r_0 \sin\lambda \sin\varphi \cdot e^{k\varphi},$$
$$z = z(\varphi, u) = r_0 \cos\lambda \cdot e^{k\varphi} + u.$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = r_0 e^{k\varphi}\sqrt{k^2 + \sin^2\lambda},\quad F = kr_0 e^{k\varphi}\cos\lambda,\quad B = 1,$$
$$L = -r_0 \sin\lambda e^{k\varphi}\sqrt{1 + k^2},\quad M = N = 0,$$
$$k_\varphi = \frac{\sqrt{1 + k^2}\sin\lambda}{r_0 e^{k\varphi}(\sin^2\lambda + k^2)},\quad k_u = k_2 = 0,$$
$$k_1 = \frac{1}{r_0\sqrt{1 + k^2}\sin\lambda \cdot e^{k\varphi}},\quad K = 0.$$

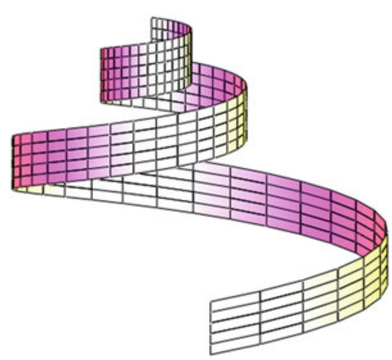

Fig. 1

(2) Parametrical equations (Fig. 1):

$$x = x(s) = r_0(as+1)\sin\lambda\cos\left(\frac{\ln|as+1|}{k}\right),$$
$$y = y(s) = r_0(as+1)\sin\lambda\sin\left(\frac{\ln|as+1|}{k}\right),$$
$$z = z(s,u) = r_0(as+1)\cos\lambda + u,$$

where s is a length of the arc of the conical spiral;

$$a = \frac{k}{r_0\sqrt{k^2+\sin^2\lambda}} = \frac{\sin\beta}{r_0\cos\lambda};$$

β is an angle of slope of the tangents of the conical directrix spiral to a plane xOy;

$$s = \frac{e^{k\varphi}-1}{a}.$$

A coordinate line $u = 0$ coincides with a directrix conical spiral.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = B = 1, \quad F = ar_0\cos\lambda = \sin\beta,$$
$$L = -\frac{a^3r_0^2(k^2+1)\sin^2\lambda}{k^3(as+1)\cos\beta}, \quad M = N = 0,$$
$$k_s = L, \quad k_u = k_2 = 0, \quad k_1 = \frac{L}{\cos^2\beta}, \quad K = 0.$$

Fig. 2 Spiral Minaret, Samarra, Iraq, 836

A cylindrical and conical spiral strip can be seen in the shape of spiral Minaret in Samarra, Iraq (836) (Fig. 2). At our time, this form was chosen by arch. F.L. Wright for the outer wall of Guggenheim Museum in New York, USA (1943) and by arch. I.M. Pei for the German Historical Museum in Berlin, Germany (2001).

Additional Literature

Krasic Sonja. Geometrijske Površi u Arhitekturi. Univerzitet u Nišu. 2012; 236p.

Filiz Ertem Kaya, Y. Yayli, H. Hilmi Hacısalihoglu The conical helix strip in E[3]. International Journal of Pure and Applied Mathematics. 2011; Vol. 66, No. 2, p. 145-156

■ Oblique Circular Cylinder

An oblique circular cylindrical surface formed by straight generatrixes intersecting *a directrix base circle* but remaining parallel to the axial direction of the cylinder. This direction forms an acute angle φ with the basis of the cylinder. An oblique cylinder has the top and bottom surfaces displaced from one another. A solid volume limited by a cylindrical lateral surface and by two circular bases is called *an oblique circular cylinder*. But some engineers call an oblique circular cylindrical surface as an oblique circular cylinder (Fig. 1).

An oblique circular cylindrical surface can be formed as an envelope of single parametrical system of planes touching simultaneously two directrix circles with a radius a:

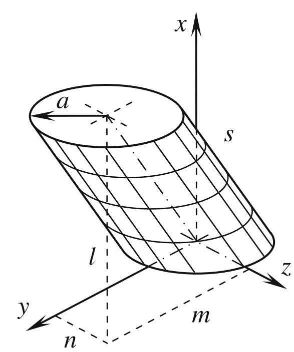

Fig. 1

$$(y-m)^2 + (z-n)^2 = a^2, \; x = l \text{ and } y^2+z^2 = a^2, \; x = 0.$$

These circles lie in parallel planes. The height l is the perpendicular distance between the circular bases. In this case, an equation of single parametrical system of tangent planes has the following form:

$$M(x,y,z,v) = x\left[nv + m\sqrt{a^2-v^2}\right] + la^2 - lvz - ly\sqrt{a^2-v^2} = 0,$$

where $v = z$ of the circle lying in the plane $x = 0$; $-a \le v \le a$, $v = \beta - n$ where $\beta = z$ of other directrix circle.

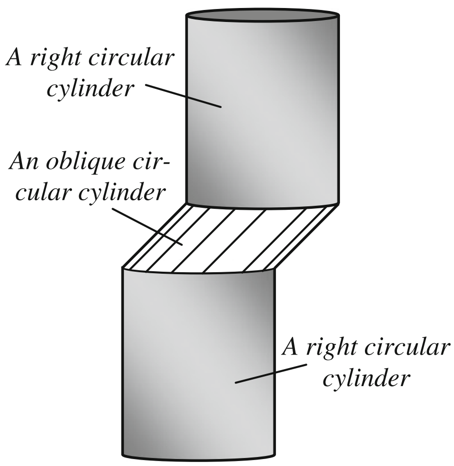

Fig. 2

Volume of the cylinder is

$$V = \pi a^2 l.$$

The volume of a right or oblique circular cylinder depends only on its base radius a and height l.

The formula for the area of an oblique circular cylinder is slightly different to the formula for the area of a right circular cylinder, because we must replace the height l with the length s, as follows:

$$A = 2\pi a^2 + 2\pi a s = 2\pi a(a + s).$$

The studied cylindrical surface can find the application in machine building, for example, it is a geometrical model of oblique cylindrical segment of bunker apparatus. The joint of two pipelines of equal diameters is presented in Fig. 2.

Forms of the definition of an oblique circular cylindrical surface

(1) Implicit form of definition:

$$(lz - xn)^2 + (ly - mx)^2 = a^2 l^2.$$

A center of the lower circle basis lies in the point with coordinates (0, 0, 0). A center of the upper circle basis lies in the point with coordinates (l, m, n). The sloping angle of the straight generatrixes with the coordinate plane $x = 0$ is obtained from a formula:

$$\tan\varphi = \frac{l}{\sqrt{m^2 + n^2}}.$$

The length of straight generatrixes of the cylinder between two circular bases is

$$s = \sqrt{l^2 + m^2 + n^2}.$$

There is an ellipse with the half-axes a and

$$b = \frac{al}{\sqrt{l^2 + m^2 + n^2}} = a\sin\varphi$$

in the cross-section of an oblique circle cylinder by a plane perpendicular to the axial direction of the cylinder.

(2) Parametrical form of definition (Fig. 1):

$$x = x, \quad y = y(x, \alpha) = a\sin\alpha + \frac{m}{l}x,$$
$$z = z(x, \alpha) = a\cos\alpha + \frac{n}{l}x,$$

where α is an angle taken from an axis Oz in the direction of an axis Oy.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{\sqrt{l^2 + m^2 + n^2}}{l} = \frac{1}{\sin\varphi} = \frac{s}{l},$$
$$F = \frac{a}{l}(m\cos\alpha - n\sin\alpha), \quad B = a,$$
$$L = M = 0, \quad N = -\frac{a^2}{\sqrt{A^2a^2 - F^2}},$$
$$k_x = k_1 = 0,$$
$$k_\alpha = -\frac{1}{\sqrt{A^2a^2 - F^2}}, \quad k_2 = \frac{-a^2 s^2}{l^2(A^2a^2 - F^2)^{3/2}},$$
$$K = 0, \quad H = \frac{k_2}{2}.$$

Additional Literature

Nikolaevskiy GK, Panov OV, Sitniov VS, Tomarevskaya ES. Compulsory Practical Work on Descriptive Geometry. Harkov: Izd-vo HGU. 1963; 124 p.

Cylinder. TechnologyUK: Geometry. http://www.technologyuk.net/mathematics/geometry/cylinders.shtml

■ Oblique Elliptical Cylinder

An oblique elliptical cylindrical surface formed by straight generatrixes intersecting *a directrix base ellipse* but remaining parallel to *the axial direction of the cylinder*. This direction forms an acute angle φ with the basis of the cylinder. An oblique cylinder has the top and bottom surfaces displaced from one another. A solid volume limited by a cylindrical lateral surface and by two elliptical bases is called *an oblique elliptical cylinder*. But some engineers call an oblique elliptical cylindrical surface by an oblique elliptical cylinder.

An oblique elliptical cylindrical surface can be formed as an envelope of single parametrical system of planes touching simultaneously two directrix ellipses:

$$\frac{(y-m)^2}{b^2}+\frac{(z-n)^2}{a^2}=1,\ x=l \text{ and } \frac{y^2}{b^2}+\frac{z^2}{a^2}=1,\ x=0.$$

These ellipses lie in parallel planes. The height l is the perpendicular distance between the elliptical bases. In this case, an equation of single parametrical system of tangent planes has the following form:

$$M(x,y,z,v)=x\left[bnv+ma\sqrt{a^2-v^2}\right] + bla^2-lbvz-lay\sqrt{a^2-v^2}=0,$$

where $v=\beta-n$; $-a\le v\le a$, $\beta=z$ of an ellipse placed in the plane $x=l$; $v=z$ of the ellipse lying in the plane $x=0$.

Forms of the definition of an oblique elliptical cylindrical surface

(1) Implicit form of definition:

$$b^2(lz-xn)^2+a^2(ly-mx)^2=a^2b^2l^2.$$

Having assumed $b=a$, we can obtain an implicit equation of *an oblique circle cylindrical surface*. Having taken $m=n=0$, we can obtain *a right elliptical cylindrical surface*. A center of the lower elliptical basis lies in the point with coordinates (0, 0, 0). A center of the upper elliptical basis lies in the point with coordinates (l, m, n).

The sloping angle of the straight generatrixes with the coordinate plane $x=0$ is obtained from a formula:

$$\tan\varphi=\frac{l}{\sqrt{m^2+n^2}}.$$

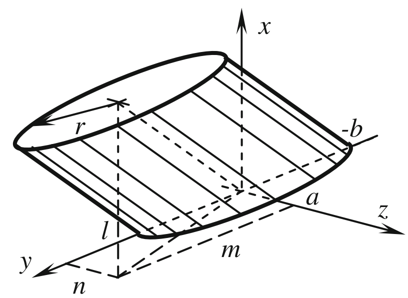

Fig. 1

The length of straight generatrixes of the cylinder between two elliptical bases is

$$s=\sqrt{l^2+m^2+n^2}.$$

In the cross-sections of the cylinder by planes x = const, ellipses will be with the half-axes of directrix ellipse.

(2) Parametrical equations (Fig. 1):

$$x=x,\ \ y=y(\alpha,x)=r\sin\alpha+\frac{m}{l}x,$$
$$z=z(\alpha,x)=r\cos\alpha+\frac{n}{l}x,$$

where

$$r=r(\alpha)=\frac{ab}{\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}}.$$

Coefficients of the fundamental forms of the surface:

$$A^2=\frac{a^2b^2\left(b^4\cos^2\alpha+a^4\sin^2\alpha\right)}{\left(b^2\cos^2\alpha+a^2\sin^2\alpha\right)},$$
$$B^2=\frac{l^2+m^2+n^2}{l^2}=\frac{s^2}{l^2}=\sin^2\varphi,$$
$$F=\frac{ab(b^2m\cos\alpha-a^2n\sin\alpha)}{l\left(b^2\cos^2\alpha+a^2\sin^2\alpha\right)^{3/2}},$$
$$L=\frac{a^3b^3l\left[l^2\left(a^4\sin^2\alpha+b^4\cos^2\alpha\right)+\left(ma^2\sin\alpha+nb^2\cos\alpha\right)^2\right]^{-\frac{1}{2}}}{\left(a^2\sin^2\alpha+b^2\cos^2\alpha\right)^{3/2}},$$
$$M=N=0,\ \ K=0.$$

(3) Parametrical equations (Fig. 1):

$$x = x,$$
$$y = y(\lambda, x) = b \sin \lambda + \frac{m}{l} x,$$
$$z = z(\lambda, x) = a \cos \lambda + \frac{n}{l} x$$

where λ is an angular parameter, $0 \le \lambda \le 2\pi$.

Additional Literature

Gulyaev VI, *Bazhenov VA*, *Gotzulyak EA*, *Gaydaychuk VV*. Analysis of Shells of Complex Form. Kiev: Budivelnik. 1990; 192 p.

■ Oblique Parabolic Cylinder

An oblique parabolic cylindrical surface formed by straight generatrixes intersecting *a directrix base parabola* but remaining parallel to *the axial direction of the cylinder*. This direction forms an acute angle φ with the basis of the cylinder.

Some engineers call an oblique parabolic cylindrical surface as an oblique parabolic cylinder.

An oblique parabolic cylindrical surface can be formed as an envelope of single parametrical system of planes touching simultaneously two directrix parabolas:

$$x = 0, \quad z = \frac{y^2}{2a} \text{ and } x = l, \quad z = \frac{(y-m)^2}{2a} + n.$$

These parabolas lie in parallel planes. The height l is the perpendicular distance between the parabolic bases. In this case, an equation of single parametrical system of tangent planes has the following form:

$$M(x, y, z, \beta) = x[\beta m - an] + zal + ly\beta + l\beta^2/2 = 0,$$

where $\beta = y$ of the parabola placed in the plane $x = 0$;

$$\gamma = \beta + m;$$

$\gamma = y$ of the parabola placed in the plane $x = l$.

Forms of definition of an oblique parabolic cylindrical surface

(1) Implicit form of definition:

$$(ly - xm)^2 + 2al(nx - lz) = 0.$$

Vertex of one directrix parabola is placed at the point with coordinates (0, 0, 0) but vertex of another parabola lies in the point with coordinates (l, m, n).

The sloping angle of the straight generatrixes with the coordinate plane $x = 0$ is obtained from a formula:

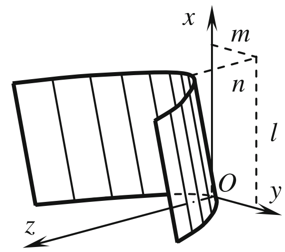

Fig. 1

$$\tan \varphi = \frac{l}{\sqrt{m^2 + n^2}}.$$

The length of straight generatrixes of the cylinder between two parabolic bases is

$$s = \sqrt{l^2 + m^2 + n^2}.$$

Parabolas with the parameter a will be in the cross-sections of the cylinder by planes x = const. The directrix parabolas have the same parameter a.

Having assumed $m = n = 0$, we shall design *a right parabolic cylindrical surface*.

(2) Parametrical equations (Fig. 1):

$$x = x,$$
$$y = y(x, \beta) = \beta + \frac{mx}{l},$$
$$z = z(x, \beta) = \frac{\beta^2}{2a} + \frac{nx}{l}.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{l^2 + m^2 + n^2}{l^2}, \quad F = \frac{1}{l}\left(m + \frac{n}{a}\beta\right),$$
$$B^2 = \frac{a^2 + \beta^2}{a^2},$$
$$L = M = 0,$$
$$N = \frac{1}{a\sqrt{A^2B^2 - F^2}}, \quad K = 0.$$

(3) Parametrical equations (Fig. 1):

$$x = x(u) = u\sin\varphi,$$
$$y = y(u,\beta) = \beta + u\cos\varphi\sin\lambda,$$
$$z = z(u,\beta) = \frac{\beta^2}{2a} + u\cos\varphi\cos\lambda,$$

where u is a length of a fragment of the rectilinear generatrix taken from the parabola lying in a plane $x = 0$ till a certain point of the cylindrical surface; λ is an angle between an axis z and the projections of the rectilinear generatrixes on a plane $x = 0$ read from an axis z in the direction of an axis y.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1,\quad F = \cos\varphi\left[\sin\lambda + \frac{\beta}{a}\cos\lambda\right],\quad B^2 = \frac{a^2+\beta^2}{a^2},$$
$$L = M = 0,\quad N = \frac{\sin\varphi}{a\sqrt{B^2-F^2}},$$
$$k_1 = k_u = 0,\quad k_\beta = \frac{N}{B^2},$$
$$k_2 = \frac{\sin\varphi}{a\left(B^2-F^2\right)^{3/2}},\quad K = 0.$$

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■ Oblique Hyperbolic Cylinder

An oblique hyperbolic cylindrical surface formed by straight generatrixes intersecting *a directrix base hyperbola* but remaining parallel to *the axial direction of the cylinder*. This direction forms an acute angle φ with the basis of the cylinder.

Some engineers call an oblique hyperbolic cylindrical surface as an oblique hyperbolic cylinder.

An oblique hyperbolic cylindrical surface can be formed as an envelope of single parametrical system of planes touching simultaneously two directrix hyperbolas:

$$x = \frac{a}{b}\sqrt{z^2+b^2},$$
$$y = 0$$

and

$$x = \frac{a}{b}\sqrt{(z-m)^2+b^2} + n,$$
$$y = l.$$

These hyperbolas lie in parallel planes. The height l is the perpendicular distance between the hyperbolic bases. In this case, an equation of single parametrical system of tangent planes has the following form:

$$M(x,y,z,\beta) = xbl\sqrt{\beta^2+b^2} + y\left(am\beta - nb\sqrt{\beta^2+b^2}\right) - alv(z-\beta) - al\left(\beta^2+b^2\right) = 0,$$

where $\beta = z$ of the hyperbola placed in the plane $y = 0$;

$$\gamma = \beta + m;$$

$\gamma = z$ of the hyperbola lying in the plane $y = l$.

Forms of definition of an oblique hyperbolic cylindrical surface

(1) Implicit form of definition:

$$b^2(lx-yn)^2 - a^2(lz-my)^2 - a^2b^2l^2 = 0.$$

Vertex of one directrix hyperbola is placed at the point with coordinates (a, 0, 0) but vertex of another hyperbola lies in the point with coordinates ($a + n$, l, m).

The sloping angle of the straight generatrixes with the coordinate plane

$$y = 0$$

is obtained from a formula:

$$\tan\varphi = \frac{l}{\sqrt{m^2+n^2}}.$$

The length of straight generatrixes of the cylinder between two hyperbolic directing curves is

$$s = \sqrt{l^2+m^2+n^2}.$$

Having assumed $m = n = 0$, we shall design *a right hyperbolic cylindrical surface*.

Hyperbolas identical to the directrix hyperbolas can be constructed if we cut the cylindrical surface by planes y = const.

(2) Parametrical equations (Fig. 1):

$$x = x(y,\beta) = \frac{a}{b}\sqrt{\beta^2+b^2} + \frac{n}{l}y,$$
$$y = y,$$
$$z = z(y,\beta) = \beta + \frac{m}{l}y.$$

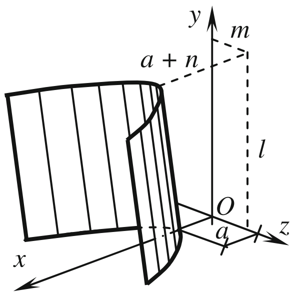

Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \frac{l^2 + m^2 + n^2}{l^2},$$
$$F = \frac{1}{l}\left(m + \frac{n a \beta}{b\sqrt{\beta^2 + b^2}}\right),$$
$$B^2 = 1 + \frac{a^2 \beta^2}{b^2(\beta^2 + b^2)},$$
$$L = M = 0, \quad N = \frac{ab}{(\beta^2 + b^2)^{3/2}\sqrt{A^2B^2 - F^2}},$$
$$k_y = k_1 = 0, \quad k_\beta = \frac{N}{B^2},$$
$$k_2 = \frac{(l^2 + m^2 + n^2)ab}{l^2(\beta^2 + b^2)^{3/2}(A^2B^2 - F^2)^{3/2}},$$
$$K = 0.$$

(3) Parametrical equations (Fig. 1):

$$x = x(y, t) = a\frac{1 + t^2}{1 - t^2} + \frac{n}{l}y,$$
$$y = y,$$
$$z = z(y, t) = b\frac{2t}{1 - t^2} + \frac{m}{l}y, \text{ where } -1 < t < 1.$$

1.1.3 Conical Surfaces

A conical surface is formed by the movement of a straight generatrix that passes through the given point (a vertex of the conical surface) and intersects the given directrix line. In general, a conical surface consists of *two identical unbounded halves* (*nappes*) joined by the vertex (Fig. 1) and positioned symmetrically about the vertex. However, in some cases, these two halves may intersect or even coincide.

Every conic surface is ruled and developable. The directrix does not contain the apex (the vertex). Each half is called *a nappe* and is the union of all the half-lines that start at the apex and pass through a point of some fixed space or plane curve that is called a directrix.

Every algebraic surface consisting of straight lines passing through one point is a conical surface. According to this assertion, one can say that any *plane* is a cone with a vertex in arbitrary point. Conical surface is determined by the basic curve (*director* or *directrix*) and a main vertex, which is a real point. If the point M_0 lies on a cone, then the straight line OM_0 also lies on the cone (Fig. 1).

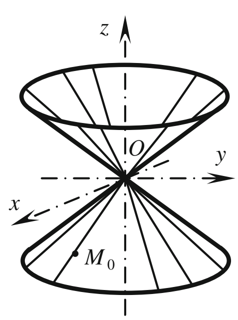

Fig. 1

Conical surfaces are surfaces of *zero Gaussian curvature* (K = 0) and that is why they can be developed on a plane without any lap fold or break.

A cone is a solid body or a volume limited by segment of a conical surface disposed at one direction from the vertex and placed between this vertex and the plane intersecting all generatrixes on the same side from the vertex. A part of the plane lying inside conical surface is called a *base* of the cone and a segment of the conical surface involved between the vertex and the base is called *a lateral surface of the cone*.

Subset of cones placed between two parallel planes is called *truncated cones* or *conical layers*.

Real conical surface of the second order may be regarded as degeneration of *one-sheet* or *two-sheet hyperboloid*.

Conical surface of revolution, oblique circular conical surface, right and oblique elliptical conical surfaces and imaginary conical surface an equation of which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

are belong to the second-order conical surfaces. The only real point of an imaginary conical surface is the point with coordinates (0, 0, 0).

Every cone of the second order has a plane directrix being a circumference.

If the height of the cone coincides with its axis, the cone is called a *right cone*. In opposite case, a cone is called *an oblique cone*. So, a right cone is a cone with its vertex above the center of its base. However, when the term “cone” is used without qualification, it often means "right cone." *A height of the cone* is a length of the perpendicular dropped to the plane of the base from the vertex.

A straight line passing through a vertex of the cone and the point of intersection of axes of the base is called *an axis of the cone*.

The union of all straight lines that intersect the axis at a fixed point and at a fixed angle θ forms *a right circular conical surface*. The aperture of this cone is the angle 2θ.

Assume a vertex of the cone at the beginning of coordinates but a directrix curve is placed in a plane $z = 1$ and has an equation $f(x, y) = 0$ then an equation $F(x, y, z) = 0$ will be an equation of the conical surface if $F(x, y, 1) = f(x, y)$.

If an equation of a surface of the second order is given in the form

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0,$$

then it will define a conical surface only if

$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \text{ and}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0.$$

An algebraic surface given in Cartesian coordinates x, y, z with the help of an equation $f(x, y, z) = 0$ where $f(x, y, z)$ is an homogeneous polynomial of degree n for the variable x, y, z is called *a conical surface of the n-order*. A vertex of the cone will be at the beginning of coordinates O.

A conical surface S can be described parametrically as

$$\boldsymbol{S}(t, u) = \boldsymbol{v} + u\boldsymbol{q}(t),$$

where $\boldsymbol{v}$ is the radius vector of a vertex and $\boldsymbol{q}$ is the radius vector of a directrix.

We can say also that if the tangent planes of a surface pass a fixed point, then it is a conical surface.

A conical surface can be designed as a particular case of a torse surface, a cuspidal edge of which degenerates into a point which will be the vertex of a cone. *A cylindrical surface* can be viewed as a limiting case of a conical surface whose vertex is moved off to infinity in a particular direction.

A pyramid is a special case of a cone with a polygonal base.

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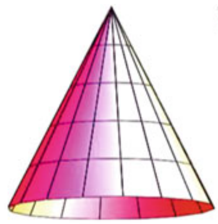
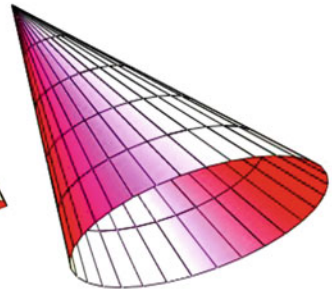
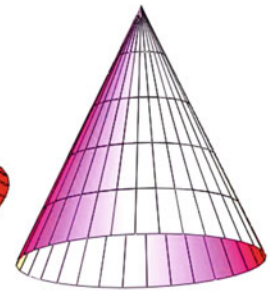
■ Key Conical Surfaces Presented in the Encyclopedia



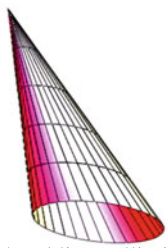
The conical surface of revolution



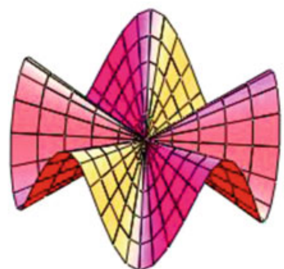
The oblique circular conical surface



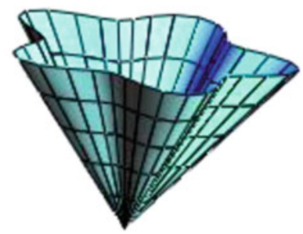
The elliptical conical surface



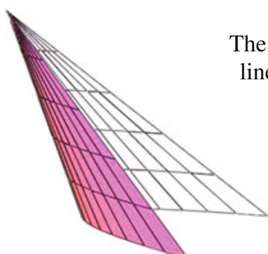
The oblique elliptical conical surface



The waving conical surface in lines of principal curvatures with inner vertex

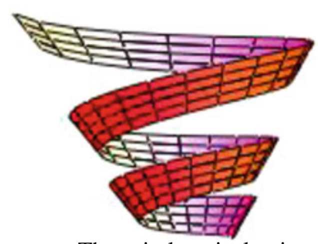


The right conical surface with the directrix curve in the form of circular sinusoid

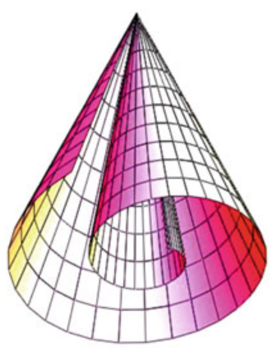


The parabolic conical surface

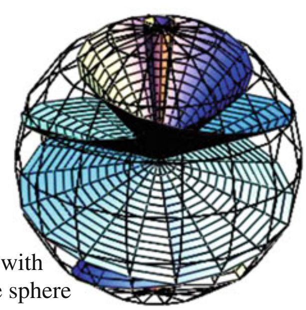
The right conical sinusoidal waved surface



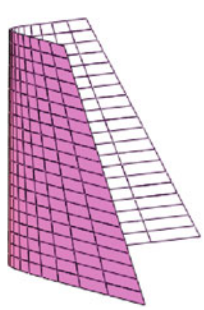
The spiral conical strip



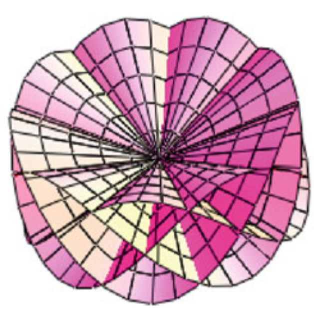
The evolvent conical surface



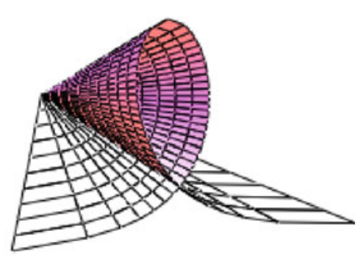
The conical surface with the directrix line on the sphere



The hyperbolic conical surface



The honeycomb conical surface



The conical surface with Cartesian folium

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■ Elliptical Conical Surface

An elliptical conical surface is a real cone of the second order that is formed by a moving straight line passing through the given point and intersecting a director ellipse (Fig. 1). A perpendicular dropped from the vertex of an elliptical cone passes through the point of intersection of axes of directrix ellipse and therefore an elliptical conical surface is called *a right elliptical conical surface*.

Forms of definition of elliptical conical surface

(1) Implicit equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \text{ (a canonical equation)}.$$

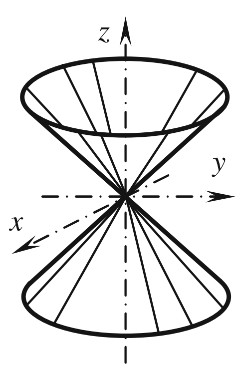

Fig. 1

A plane $z = h \neq 0$ intersects a cone along an ellipse with the half-axes equal to $a\,|h|\,/c$ and $b\,|h|\,/c$ (Fig. 1). A plane $z = 0$ intersects a cone at only one point (0, 0, 0). Planes $y = h \neq 0$ and $x = h \neq 0$ intersect a cone along the hyperbolas (Fig. 2). Planes $y = 0$ and $x = 0$ intersect a cone along a pair of intersecting straights. An elliptical cone will be a right circle cone if $a = b$.

(2) Explicit equation (Fig. 2):

$$z^2=\frac{x^2}{p^2}+\frac{y^2}{q^2}$$

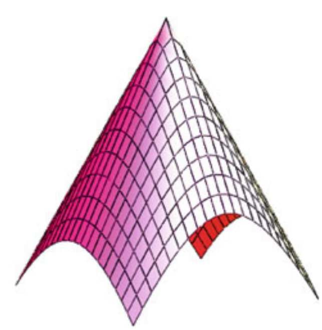

Fig. 2

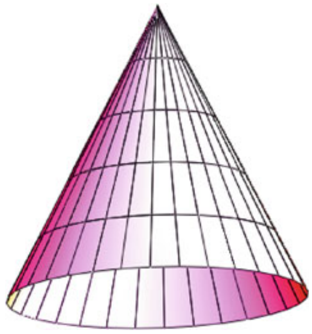

Fig. 3

where is taken

$$p^2 = \frac{a^2}{c^2}, \quad q^2 = \frac{b^2}{c^2}.$$

(3) Parametrical equations (Fig. 3):

$$x = x(u, v) = au\cos v;$$
$$y = y(u, v) = bu\sin v;$$
$$z = z(u) = cu,$$

where $u = z/c$ is a dimensionless value equal to the height of the cone divided by a parameter c, $0 \le v \le 2\pi$.

The elliptic cone of height c with semiaxis a and semiaxis b has volume

$$V = \pi abc/3.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = a^2\cos^2 v + b^2\sin^2 v + c^2,$$
$$F = \frac{1}{2}u\sin 2v(b^2 - a^2),$$
$$B^2 = a^2u^2\sin^2 v + b^2u^2\cos^2 v,$$
$$L = M = 0, \quad N = \frac{abcu^2}{\sqrt{A^2B^2 - F^2}};$$
$$k_u = k_1 = 0, \quad k_v = \frac{N}{B^2},$$
$$k_2 = \frac{A^2N}{A^2B^2 - F^2}, \quad K = 0.$$

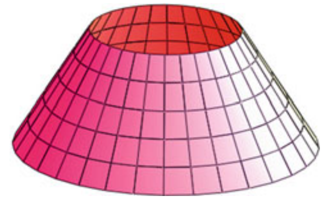

Fig. 4 The truncated elliptical conical surface

(4) Assume two similar ellipses placed in parallel planes (Fig. 4):

$$\frac{y^2}{c^2} + \frac{z^2}{d^2} = 1, \quad x = l \text{ and}$$
$$\frac{y^2}{b^2} + \frac{z^2}{a^2} = 1, \quad x = 0$$

where $ac = bd$.

In that case, a system of single-parametrical planes tangent to an elliptical cone can be written in the following form:

$$M = M(x, y, z, \beta) = (c - b)\frac{x}{l} - c\beta\frac{z}{d^2} - \sqrt{1 - (\frac{\beta}{d})^2}y + b = 0$$

where $\beta = z$ is an ordinate of the ellipse lying in a plane $x = l$. A vertex of the cone is at the point with coordinates

$$x = \frac{bl}{b - c}, \ y = z = 0.$$

An implicit equation of the lateral surface of the elliptical cone is

$$\left[x\frac{(c - b)}{l} + b\right]^2 - y^2 - c^2\frac{z^2}{d^2} = 0.$$

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■ Oblique Circular Conical Surface

An oblique circular conical surface is a conical *surface of the second order* and it is formed by a moving straight that passes through a given point and intersects a directrix circle (Fig. 1). A perpendicular dropped from the vertex of the conical surface does not pass through a center of the directrix circle.

An oblique circular cone is a volume limited by its lateral conical surface and a circular base but the height of the cone does not coincide with the axis of the cone. *The axis of the cone* is the segment whose endpoints are the vertex and the center of the base. So, if the axis is perpendicular to the plane of the circle, the cone is *a right cone* otherwise it is *an oblique cone*.

An envelope of single parametrical system of planes tangent simultaneously to two directrix circles with radii a and c

$$(y-m)^2+(z-n)^2=c^2,\quad x=l \text{ and } y^2+z^2=a^2,\quad x=0$$

where l is a distance between the planes with the directrix circles is an oblique circular conical surface (Fig. 2). In this case, an equation of the single parametrical system of planes can be represented in the following form:

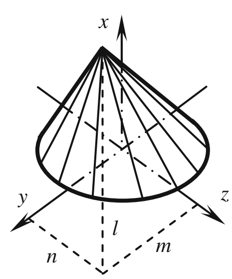

Fig. 1

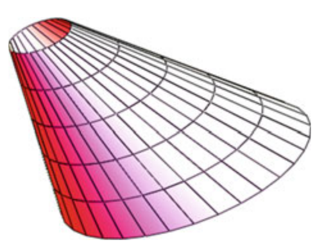

Fig. 2

$$M(x,y,z,v)=x[nv+m\sqrt{c^2-v^2}+c(c-a)]+lac-lzv - l\sqrt{c^2-v^2}y=0,$$

where $v=\beta-n$; $-c\le v\le c$; $\beta=z$ of a directrix circle placed in the plane $x=l$;

$$\gamma=a\frac{(\beta-n)}{c}=\frac{av}{c}$$

where $\gamma=z$ of a directrix circle lying in the plane $x=0$.

Coordinates of a vertex of the oblique circular cone can be expressed as

$$x_v=\frac{al}{a-c},\quad y_v=\frac{am}{a-c},\quad z_v=\frac{an}{a-c}.$$

A slope of an axis of the cone to a plane $x=0$ becomes

$$\tan\varphi=\frac{l}{\sqrt{m^2+n^2}}.$$

Forms of the definition of the surface

(1) Implicit form of definition:

$$(lz-xn)^2+(ly-mx)^2=[al+x(c-a)]^2.$$

There are circles with radii

$$r=a-(a-c)\frac{h}{l}$$

in the cross-sections $x=h$ of the oblique cone.

(2) Parametrical form of definition (Fig. 2):

$$x=x,$$
$$y=y(x,\lambda)=\frac{\cos\alpha}{\tan\varphi}x+r\cos\lambda=\frac{m-(a-c)\cos\lambda}{l}x+a\cos\lambda,$$
$$z=z(x,\lambda)=\frac{\sin\alpha}{\tan\varphi}x+r\sin\lambda=\frac{n-(a-c)\sin\lambda}{l}x+a\sin\lambda$$

where $0\le\lambda\le 2\pi$, $\alpha=\text{const}$ is an angle between the projection of the axis of the cone to a plane $x=0$ and an axis Oy:

$$\sin\alpha=\frac{n}{\sqrt{m^2+n^2}},\quad \cos\alpha=\frac{m}{\sqrt{m^2+n^2}}.$$

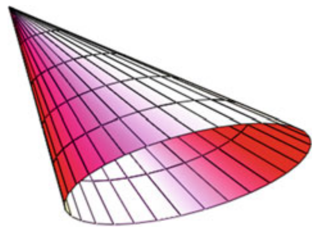

Fig. 3

$$A^2 = 1 + \frac{m^2 + n^2 + (a-c)^2 - 2(a-c)(m\cos\lambda + n\sin\lambda)}{l^2},$$
$$B = -x\frac{a-c}{l} + a = r, \quad F = -B\frac{m\sin\lambda - n\cos\lambda}{l},$$
$$L = M = 0, \quad N = \frac{B^2}{\sqrt{A^2B^2 - F^2}},$$
$$k_x = k_1 = 0, \quad k_2 = \frac{A^2B^2}{(A^2B^2 - F^2)^{3/2}}.$$

In Fig. 2, the truncated oblique circular conical surface is shown for the case $0 \le x \le l$, $l < x_v$. Having assumed $0 \le x \le x_v = al/(a-c)$,, we can obtain a conical surface with the vertex (Fig. 3).

Coefficients of the fundamental forms of the surface and its principal curvatures:

Additional Literature

Varshavskiy IP, *Tarasov AG*. The characteristics of vault-formation in conical bunkers with a vertical wall. Stroit. Mech. i raschet soor. 1981; No. 5, p. 12-15.

Krivoshapko SN, *Mamiyeva IA*. Possibilities of conical surfaces as applied to the architecture of buildings and structures. Montazh. i spetz. raboty v stroit. 2011; No. 9, p. 2-8.

■ Oblique Elliptical Conical Surface

An oblique elliptical conical surface is a conical *surface of the second order* and it is formed by a moving straight that passes through a given point and intersects a directrix ellipse (Fig. 1). A perpendicular dropped from the vertex of the conical surface does not pass through a center of the directrix ellipse.

An oblique elliptical cone is a volume limited by its lateral conical surface and an elliptical base but the height of the cone does not coincide with the axis of the cone. *The axis of the cone* is the segment whose endpoints are the vertex and the center of the base. So, if the axis is perpendicular to the plane of the circle, the cone is *a right cone* otherwise it is *an oblique cone*.

An oblique elliptical conical surface can be designed as an envelope of single parametrical system of planes tangent simultaneously to two directrix ellipses with half-axes a, b and d, c. If we want to have a conical elliptical surface then it is necessary to carry out the condition (Fig. 2):

$$\frac{a}{b} = \frac{d}{c}.$$

Having taken the parameters a, b, and c as independent values, we must assume that $d = ac/b$.

In this case, equations of two directrix ellipses may be written as

$$x = l, \ \frac{(y-m)^2}{c^2} + \frac{b^2(z-n)^2}{a^2c^2} = 1 \text{ and } x = 0, \ \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1,$$

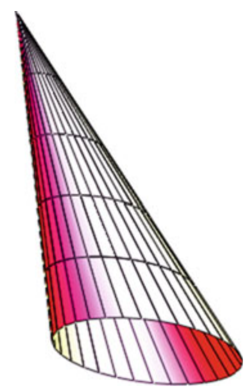

Fig. 1

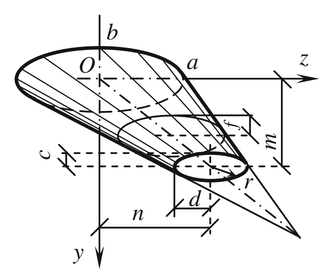

Fig. 2

where l is a distance between the planes with the directrix ellipses. The equation of single parametrical system of planes forming the surface has the following form:

$$M(x, y, z, v) = x[cnv + md\sqrt{d^2 - v^2} + d^2(c - b)] + lbd^2 - clzv - ld\sqrt{d^2 - v^2}y = 0$$

where $v = \beta - n$; $-d \leq v \leq d$, $\beta = z$ of the director ellipse placed in the plane $x = l$; $d = ac/b$,

$$\gamma = b\frac{\beta - n}{c} = b\frac{v}{c}$$

where $\gamma = z$ of the directrix ellipse lying in the plane $x = 0$.

Coordinates of a vertex of the oblique elliptical cone can be expressed as

$$x_v = \frac{bl}{b - c}, \quad y_v = \frac{bm}{b - c}, \quad z_v = \frac{bn}{a - c}.$$

A slope of an axis of the cone to a plane $x = 0$ becomes

$$\tan\varphi = \frac{l}{\sqrt{m^2 + n^2}}.$$

Forms of the definition of the surface

(1) Implicit form of definition:

$$b^2(lz - xn)^2 + a^2(ly - mx)^2 = a^2[bl + x(c - b)]^2.$$

There are ellipses with half-axes

$$p = \frac{a[bl - x(b - c)]}{bl} \text{ and } f = \frac{[bl - x(b - c)]}{l}.$$

in the cross-sections $x = h$ of the oblique cone.

(2) Parametrical form of definition (Fig. 2):

$$x = x, \quad y = y(\alpha, x) = \frac{mx}{l} + r\sin\alpha,$$
$$z = z(\alpha, x) = \frac{nx}{l} + r\cos\alpha,$$

where α ia an angle taken from an axis Oz in the direction of an axis Oy, $0 \leq \alpha \leq 2\pi$,

$$r = r(\alpha, x) = \frac{a[bl - x(b - c)]}{l\sqrt{b^2\cos^2\alpha + a^2\sin^2\alpha}}.$$

(3) Parametrical form of definition (Fig. 2):

$$x = x, \quad y = y(\lambda, x) = \frac{m}{l}x + \frac{bl - x(b - c)}{l}\sin\lambda,$$
$$z = z(\lambda, x) = \frac{n}{l}x + a\frac{bl - x(b - c)}{bl}\cos\lambda$$

where λ is an angular parameter, $0 \leq \lambda \leq 2\pi$,

Additional Literature

Kuramin VP. Distribution of the pressure of free-flowing materials along the depth of conical bunkers of special form. Stroit. Mech. i raschet soor. 1980; No. 3, p. 48-52 (5 ref.).

■ Conical Surface with a Directrix Agnesi Curve

Conical surface with a directrix in the form of Agnesi curve

$$z = \frac{2B^2T}{4y^2 + B^2} - T,$$

where $-B/2 \leq y \leq B/2$, $0 \leq z \leq T$, is formed by the movement of a straight line that passes through a point $(L; 0; 0)$ and intersects a given directrix plane curve of Agnesi (Fig. 1). The directrix curve is placed in the coordinate plane yOz.

Forms of definition of the surface

(1) Explicit equation:

$$z = \frac{2B^2T(L - x)^3}{L[4y^2L^2 + B^2(L - x)^2]} - \frac{T(L - x)}{L},$$

where L is a height of the conical surface. A conical surface with a directrix Agnesi curve is the third-order algebraic surface.

(2) Parametrical equations (Fig. 1):

$$x = x, \ y = y(x, \alpha) = (L - x)\mathrm{tg}\alpha,$$
$$z = z(x, \alpha) = \frac{T(L - x)}{L}\left(\frac{2B^2}{4L^2\mathrm{tg}^2\alpha + B^2} - 1\right)$$

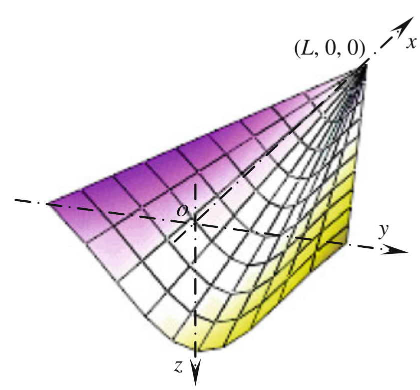

Fig. 1

where α is an angle in a plane xOy with a vertex in the point (L; 0; 0) taken from an axis Ox; $-B/(2L) \le \mathrm{tg}\alpha \le B/(2L)$.

Additional Literature

Avdon'ev EA, *Protod'yakonov SM*. Equations and characteristics of some algebraic curves of the highest orders. Prikl. Geom. i Ingen. Grafika. Kiev. 1976; 3, p. 108-120.

■ Conical Surface with a Directrix Curve in the Form of Cartesian Folium

The third-order algebraic surface formed by straight lines passing through a point A with coordinates (L, 0, $\sqrt{3}T/3$) and through a directrix curve in the form of Cartesian folium

$$y = \pm 2.5426\frac{B}{T}z\sqrt{\frac{3(T-z)}{T+3z}},$$

is called *a conical surface with a directrix curve in the form of Cartesian folium* (Figs. 1 and 2).

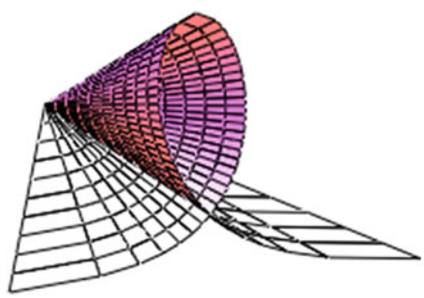

Fig. 1

Forms of definition of the surface

(1) Explicit equation:

$$y = \pm 2.5426\frac{B}{T}\frac{(Lz - \sqrt{3}xT/3)}{L}\sqrt{\frac{3\left(TL - Tx - Lz + \sqrt{3}xT/3\right)}{\left(TL - xT + 3Lz - \sqrt{3}xT\right)}}$$

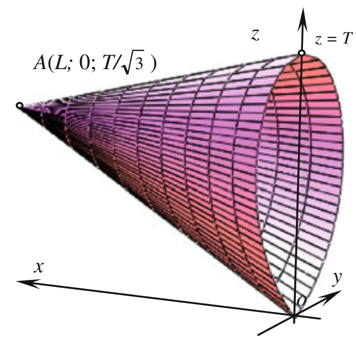

Fig. 2

where L is a length of the perpendicular dropped from a vertex of the cone to a plane yOz; $0 \le x \le L$; $y_{\max} = B$ if

$$z = T/\sqrt{3}.$$

(2) Parametrical equations (Figs. 1 and 2):

$$x = x(u) = Lu,$$

$$y = y(u,v) = \pm 2.5426B(1-u)\sqrt{\frac{1-v}{1/3+v}},$$

$$z = z(u,v) = Bu/\sqrt{3} + B(1-u),$$

where $0 \le z \le T; -B \le y \le B$; $0 \le x \le L$.

Having assumed L = 4 m; T = 2 m; B = 0.5 m, $0 \le u \le 1$; $-0.3 \le v \le 1$, we can construct the surface shown in Fig. 2.

■ Evolvent Conical Surface

An evolvent conical surface has a directrix curve in the form of an evolvent of the circle

$x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$, where t is an angle taken from an axis Ox in the direction of an axis Oy (Fig. 1);

$$0 \le t < \infty.$$

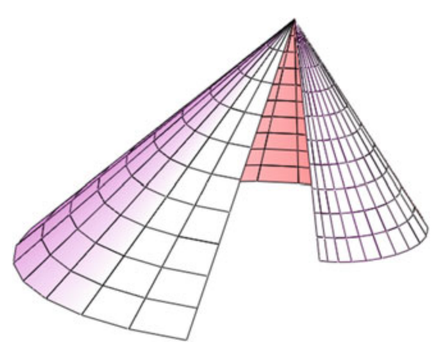

Fig. 1

Parametrical equations of an evolvent conical surface are

$$x = x(t, z) = \frac{x_v - a(\cos t + t \sin t)}{z_v} z + a(\cos t + t \sin t),$$
$$y = y(t, z) = \frac{y_v - a(\sin t - t \cos t)}{z_v} + a(\sin t - t \cos t); \quad z = z$$

where x_v, y_v, *and* z_v are coordinates of the vertex of the conical surface.

■ Hyperbolic Conical Surface

A hyperbolic conical surface is a nonclosed conical *surface of the second order* and formed by a moving straight that passes through a given point and intersects a directrix hyperbola (Fig. 1).

A hyperbolic conical surface can be designed as an envelope of single parametrical system of planes tangent simultaneously to two directrix hyperbolas

$$x = a\sqrt{z^2 + b^2}/b,$$
$$y = 0 \text{ and } x = n + c\sqrt{(z-m)^2 + d^2}/d, \quad y = l$$

but with a condition that $cb = ad$; l is the distance between two planes with directrix hyperbolas (Fig. 2).

It is assumed that directrix hyperbolas have parallel axes. A perpendicular to a coordinate plane $y = 0$ dropped from a top of the hyperbola placed in a plane $y = l$ passes through a point C with coordinates $C(c + n; l; m)$.

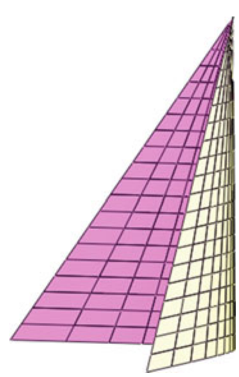

Fig. 1

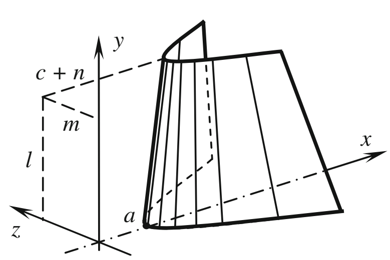

Fig. 2

An envelope of single parametrical system of planes tangent simultaneously to two hyperbolas will be a developable surface with a cuspidal edge if $c \neq ad/b$. But this developable surface degenerates into a conical surface if $c = ad/b$. The condition of developable surface's uniqueness was obtained in the form:

$$\gamma = m + \frac{a\beta d^2}{\sqrt{\beta^2(c^2b^2 - a^2d^2) + c^2b^4}},$$

which degenerates for conical surface into a relation

$$\gamma = m + \frac{\beta d}{b},$$

where $\beta = z$ of the hyperbola placed in a plane $y = 0$; but $\gamma = z$ of the hyperbola lying in a plane $y = l$.

The formulas for the determination of coordinates of a vertex of the conical surface give

$$x_v = \frac{bn}{b-d}, \quad y_v = \frac{bl}{b-d}, \quad z_v = \frac{bm}{b-d}.$$

Forms of definition of the surface

(1) Parametrical equations (Fig. 2):

$$x = x(\beta, y) = \frac{a}{bl}\sqrt{\beta^2 + b^2}\left(\frac{d}{b}y - y + l\right) + \frac{n}{l}y;$$
$$y = y;$$
$$z = z(\beta, y) = \left(\frac{d}{b}\beta - \beta + m\right)\frac{y}{l} + \beta.$$

Coordinate lines β = const are the rectilinear generatrixes of the conical surface, the lines y = const coincide with the lines of cross-sections of the conical surface by planes y = const.

(2) Parametrical surface (Fig. 1):

$$x = x(\beta, y) = \frac{a(y_v - y)}{by_v}\sqrt{\beta^2 + b^2} + \frac{x_v}{y_v}y,$$
$$y = y,$$
$$z = z(\beta, y) = \frac{z_v}{y_v}y + \frac{y_v - y}{y_v}\beta.$$

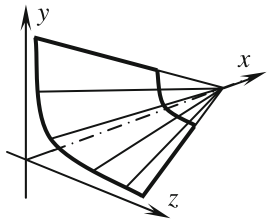

Fig. 3

This form of definition is very convenient if one knows the coordinates of vertex of the conical surface and parameters a and b of the directrix hyperbola placed in a plane $x = 0$.

(3) A conical surface is obtained if one takes two equilateral hyperbolas (Fig. 3)

$$x = 0; \quad y = \frac{a}{z} \text{ and } x = l; \quad y = \frac{b}{z}$$

as directrix curves. A system of single parametrical planes tangent simultaneously to these two directrix hyperbolas can be written in the following form:

$$M = M(x, y, z, \beta) = zl + 2\beta\left(x - l - x\sqrt{b/a}\right) + ly\beta^2/a = 0,$$

where $\beta = z$ of the hyperbola placed in the plane $x = 0$; $\beta = \gamma\sqrt{a/b}$, $\gamma = z$ of the hyperbola placed in the plane $x = l$.

Coordinates of a vertex of the conical surface can be expressed as

$$x_v = \frac{al}{a - \sqrt{ab}}; \quad y_v = z_v = 0.$$

The conical surface degenerates into a cylindrical surface if a is equal to b ($a = b$).

Reference

Krivoshapko SN. Developable Surfaces and Shells. Moscow: Izd-vo UDN. 1991; 287 p.

■ Right Conical Surface with a Plane Director Curve in the Form of Circular Sinusoid

This right conical surface is formed by a moving straight line, one point of which is stationary but the opposite end describes a sinusoid relatively to the base circle with a radius $r = z\text{tg}\theta_0$ in a plane perpendicular to the axis of the conical surface (Fig. 1).

Forms of definition of the conical surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = vtS(u)\cos u,$$
$$y = y(u, v) = vtS(u)\sin u, \quad z = v,$$
$$S(u) = 1 + \mu\sin(mu);$$

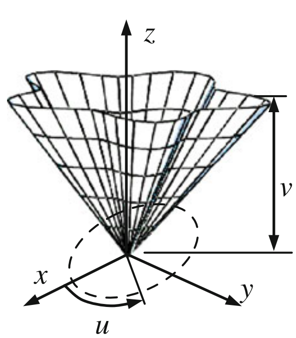

Fig. 1

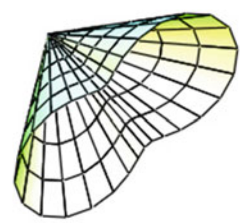

Fig. 2

μ is an amplitude of the sinusoid relative to the unit circle placed in the cross-section $z = \text{ctg}\theta_0$; m is a number of waves of the sinusoid; $0 \le u \le 2\pi$; $t = \text{tg}\theta_0$. A directrix straight has an angle θ_0 with the axis of the conical surface if $u = 0$.

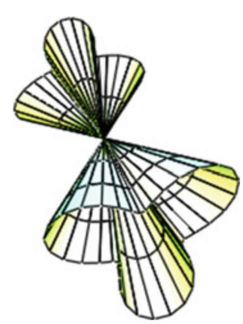

Fig. 3

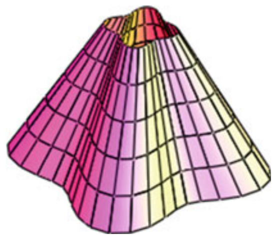

Fig. 4

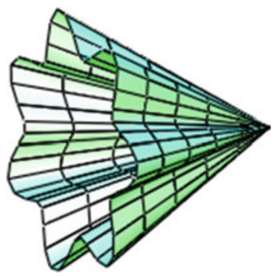

Fig. 5

Coefficients of the fundamental forms of the surface:

$$
\begin{aligned}
&A^2 = t^2v^2\left[S^2(u) + \mu^2m^2\cos^2(mu)\right],\\
&F = vt^2S(u)\mu m\cos(mu),\\
&B^2 = 1 + t^2S^2(u),\\
&L = -\frac{tv}{\sigma}\left\{\left[S(u) + \mu m^2\sin(mu)\right]S(u) - 2\mu^2m^2\cos(mu)\right\},\\
&M = 0;\ \ N = 0;\\
&\sigma^2 = \left(A^2B^2 - F^2\right)/(tv)^2 = S^2(u) + t^2S^4(u) + m^2\mu^2\cos^2(mu).
\end{aligned}
$$

In Fig. 1, the right conical surface with $m = 5$ is shown. The surface with $m = 2$ is presented in Fig. 2. In Fig. 3, the surface has $m = 3$. The truncated conical surface given in Fig. 4 has $m = 4$; in Fig. 5, the surface is with $m = 7$.

■ Spiral Conical Strip

A spiral conical strip is placed on *a conical surface of revolution*. This is a segment of a lateral surface of a right circular cone. A spiral conical strip is limited by two *conical spirals* lying on a directrix cone.

Forms of the definition of spiral conical strip

(1) Parametrical form of definition (Fig. 1):

$$
\begin{aligned}
x &= x(u,v) = (r_0e^{mu} + v)\sin\theta\cos u,\\
y &= y(u,v) = (r_0e^{mu} + v)\sin\theta\sin u,\\
z &= z(u,v) = (r_0e^{mu} + v)\cos\theta,
\end{aligned}
$$

where θ is the angle of rectilinear generatrixes of the base conical surface with its axis of revolution; r_0 and m are constants.

Coefficients of the fundamental forms of the surface and its curvatures:

$$
\begin{aligned}
&A^2 = r_0^2m^2e^{2mu} + (r_0e^{mu} + v)^2\sin^2\theta,\\
&F = r_0me^{mu},\ B = 1,\\
&L = -\sin(2\theta)(r_0e^{mu} + v)/2,\ \ M = N = 0,\\
&k_u = -\frac{L}{A^2},\ \ k_v = k_2 = 0,\\
&k_1 = -\frac{\mathrm{cotan}\theta}{r_0e^{mu} + v},\ \ K = 0.
\end{aligned}
$$

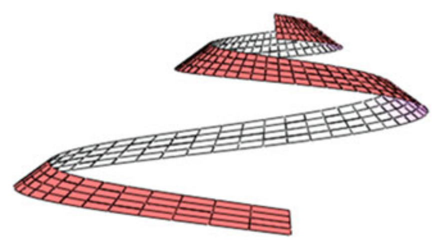

Fig. 1

A studied surface is given in curvilinear nonorthogonal conjugate coordinates u, v. Projections of coordinate lines u on a plane xOy are *logarithmic spirals*. Coordinate lines v coincide with the rectilinear generatrixes of the spiral conical strip and, hence, with rectilinear generatrixes of the base conical surface of revolution. An area of the spiral conical strip limited by the coordinate lines u_1, u_2 and v_1, v_2 can be found from the formula:

$$
\begin{aligned}
S = \frac{r_0}{m}&\sin\theta(e^{mu_1} - e^{mu_2})(v_1 - v_2)\\
&+ \frac{1}{2}(u_1 - u_2)\left(v_1^2 - v_2^2\right)\sin\theta.
\end{aligned}
$$

Assuming $\theta = \pi/2$ in the parametrical equations, we can design a plane area limited by two logarithmic spirals (Fig. 2).

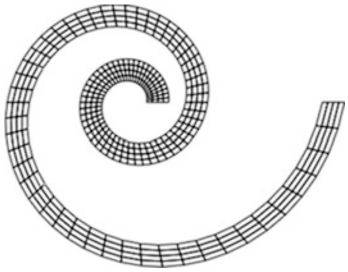

Fig. 2

(2) Explicit form of definition:

$$x^2 + y^2 = z^2 \mathrm{tg}^2 \theta.$$

■ Conical Surface with a Directrix Curve on a Sphere

A vertex of a conical surface with a directrix curve on a sphere lies in the center of the base sphere. An arbitrary directrix curve is placed on surface of the same sphere with a radius a and is given by a vector equation (Figs. 1, 2 and 3):

$$\boldsymbol{e}_0 = \boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\sin\omega,$$

where $\omega = pu$; $p = \text{const}$.

A parametrical form of definition is

$$x = x(u, v) = (a + v)\cos\omega\cos u,$$
$$y = y(u, v) = (a + v)\cos\omega\sin u,$$
$$z = z(u, v) = (a + v)\sin\omega.$$

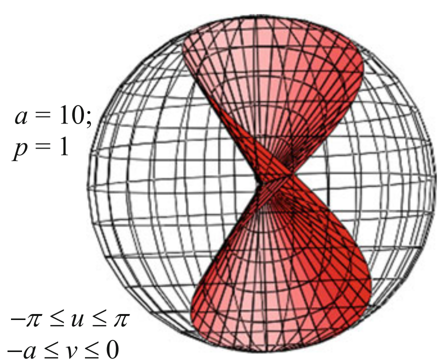

Fig. 1

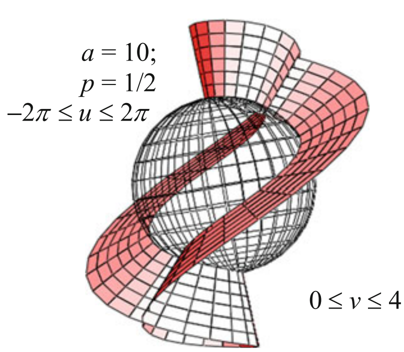

Fig. 2

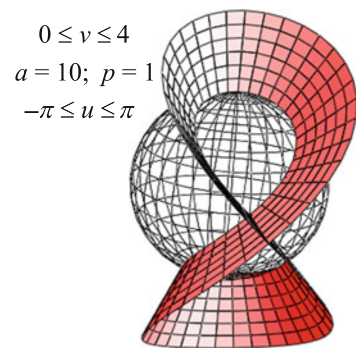

Fig. 3

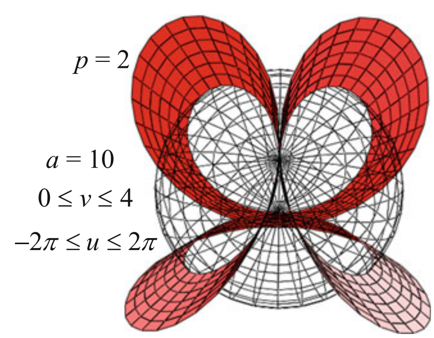

Fig. 4

A conical surface will be inside the base sphere if $v < 0$ (Fig. 1) but a conical surface will be outside the base sphere if $v > 0$ (Figs. 2, 3 and 4). In both cases, a directrix curve is placed on the sphere with radius a.

The surface is given in lines of principal curvatures u, v.

Reference

Ivanov VN. Spherical curves and geometry of surfaces on a supporting sphere. Proc.: Contemporary Problems of Geometric Design. Ukraine-Russia Scientific-and-Practical Conf., Kharkov, 19-22 April 2005. Kharkov. 2005; p. 114-120.

■ Helical Cone

A helical cone is formed by a radius vector of the helix

$$\boldsymbol{\rho}(u) = a\boldsymbol{h}(u) + bu\,\boldsymbol{k},$$

where $\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u$ is the vector function of the circle of unit radius; $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the unit orthogonal vectors; a, b are constants.

Forms of the definition of the surface

(1) Vector equation:

$$\boldsymbol{r}(u, v) = \boldsymbol{\rho}(u)v.$$

(2) Parametric form of definition:

$$x(u, v) = av\cos u, \quad y(u, v) = av\sin u, \quad z(u, v) = buv.$$

The vertex of the helical cone is the initial coordinates of the helical line.

The surfaces with parameters $a = 1$, $b = 0.3$ are shown in Figs. 1 and 2, but $u = (0 \div 4\pi)$ is in Fig. 1 and $u = (-3\pi \div 3\pi)$ is in Fig. 2. The movement of the vertex of the cone along the axis of the helix does not change the shape of the surface.

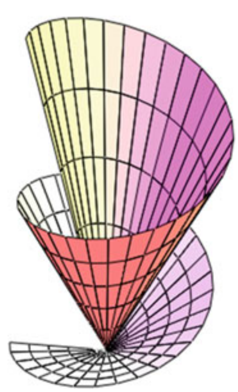

Fig. 1

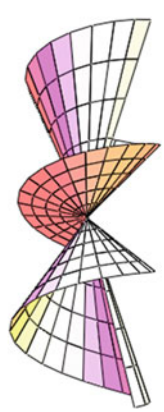

Fig. 2

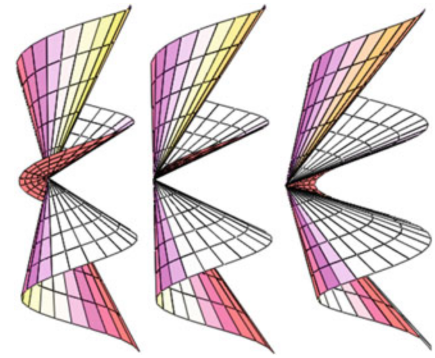

Fig. 3

Coefficients of the Fundamental Forms of the Surface

$$A = v\sqrt{a^2 + b^2},$$
$$B = \sqrt{a^2 + b^2u^2},$$
$$F = b^2uv,\; L = -a^2buv,$$
$$M = 0,\; N = 0.$$

At Fig. 3, the surfaces with the vertices of the cones placed at the horizontal plane xOy are presented in three positions when $x_0 = a/2$, $x_0 = a$, and $x_0 = 1.5a$.

For these cases, the parametric equations of the surface can be written in the form:

$$x(u, v) = x_0(1 - v) + av\cos u,$$
$$y(u, v) = av\sin u, \quad z(u, v) = buv.$$

1.2 Ruled Surfaces of Negative Gaussian Curvature

A ruled surface of negative Gaussian curvature is a two-dimensional ruled surface of 3D Euclid space having negative Gaussian curvature ($K < 0$) in its every point. Ruled surfaces of negative Gaussian curvature are called *oblique ruled surfaces* or *skew ruled surfaces*, or *ruled saddle-shaped surfaces* too. We also can meet the following name of ruled surface of negative Gaussian curvature as *general ruled surface* or *general double-curved ruled surface*. Conoids, hyperbolic paraboloids, one-sheet hyperboloids are *twice-ruled surfaces*. One-sheet hyperboloid of revolution is the only *ruled surface of revolution of negative Gaussian curvature*. Conoids and cylindroids belong also to a family of *Catalan's surfaces*. Hyperbolic paraboloids and one-sheet hyperboloids are *the second-order surfaces*. Right helicoid is the only *ruled minimal surface*.

There are no ruled surfaces of *constant negative Gaussian curvature*.

Ruled surfaces of the third and fourth orders are the most studied ruled surfaces after the second-order surfaces. Two types of the third-order ruled surfaces exist. The first type has two skew lines intersecting all generatrixes and one of them is double straight line. The second type has one double straight line. The both types of surfaces have two parametrical sets of conical sections lying in the pencils of planes passing through every generatrix. The fourth-order ruled surfaces have on themselves, in general case, a double curve of the third order. Twelve types of the fourth-order surfaces are known. Six types of them have single parametrical sets of conical sections. Spatial curves on the surface or plane curves in the form of curves of the third or fourth orders can be chosen as contours of segments of oblique ruled surfaces of the third and fourth orders.

Let us give the definitions of some ruled surfaces which can be seen in the forms of some industrial products.

Surface of oblique transition place is a ruled surface with three directrixes, two of them are the arcs of circles of equal radius lying in parallel planes but the third directrix is a straight line perpendicular to planes of the circles and passing through the middle of the straight line segment connecting the centers of these arcs of the circles. Surface of oblique transition place is used in architecture and construction. *Surface of an oblique wedge* is formed by the movement of rectilinear generatrix on three directrixes placed in parallel planes but two curvilinear directrixes are smooth curves and the third directrix is a straight line. This surface is used for the creation of the wing of the aircraft.

Surface of a double oblique conoid contains three directrixes: one of them is a curvilinear line but two others are straight lines.

Surface of an oblique cylinder is formed by the movement of a rectilinear generatrix on three curvilinear directrixes. *Surface of a double oblique cylinder* is a ruled surface formed on three directrixes, two of them are the curves but the third directrix is a straight line.

Methods of design of approximate developments of oblique ruled surfaces are more complex in comparison with design of developments of developable surfaces. Developments of oblique ruled surfaces are not precise and subsequent corrections and adjustments of finished articles will be necessary. It is necessary to make a development from several sheets and after, these sheets join between themselves by using any method. Sometimes, it is possible to change a designed oblique ruled sheet structure by elements of developable surfaces.

Many production processes are based on ruled surfaces and it is very comfortable that ruled surface is formed by a continuous family of straight line segments because it gives an opportunity to approximate free-form shapes by ruled surface to take advantage of cost-effective fabrication options. For example, one can approximate entire facades of buildings by single patches of ruled surfaces and—if it does not suffice—by multiple strips of ruled surfaces glued together. At a smaller scale, S. Flöry and H. Pottmann have shown how to provide NC data for the cost-effective production of free-form molds; for example, on heated wire cutting machines. They formulated the basic ruled surface approximation algorithm: if one family of asymptotic curves in a region is not curved too much, this indicates a good possibility for approximating that region by a ruled surface.

It is known that large parts of the facade of the Cagliari Contemporary Arts Center by Zaha Hadid Architects have been rationalized with ruled surfaces.

A.L. Podgorny and V.S. Obukhova also studied the opportunities of skew ruled surfaces as applied to architecture of public buildings. They used the segments of skew ruled surfaces of the third and fourth orders.

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P.S.: Additional literature is presented on the corresponding pages of the Sect. 1.2 "Ruled Surfaces of Negative Gaussian Curvature", a Subsect. "7.1.1. Ruled Helical Surfaces" and of the Chap. "35. The Second Order Surfaces".

■ Oblique Helicoid

An oblique helicoid is a helical ruled surface formed by a director straight line that intersects the axis of the helicoid under constant angle α not equal to 90° and rotates with constant angular speed around this axis and moves simultaneously with constant speed along the same axis (Fig. 1). The speeds of these movements are proportional.

Rectilinear generatrixes of an oblique helicoid are parallel to the generators of its *director cone*. Due to the given definition, we can make a conclusion: if the angular speed of the straight generatrix is equal to zero, then this generatrix will form a plane and if speed of translation is equal to zero then the generating straight line will form *a conical surface of revolution*. In general case, every point of a generatrix straight line forms a helix. Helices of the same lead will be in the intersections of coaxial circular cylinders with an oblique helicoid.

We can carry out approximate developing of flight of an oblique helicoid with the help of bending on the surface of one-sheet hyperboloid of revolution. The helixes of the helicoid will be superimposed on the parallels of the hyperboloid but the rectilinear generatixes will be put in the rectilinear generatrixes of the one-sheet hyperboloid of revolution. The axis of the helicoid wraps a waist circle of the hyperboloid.

Forms of definition of the surface of oblique helicoid

(1) Explicit form of definition:

$$z = c \arctan\frac{y}{x} + k\sqrt{x^2+y^2}$$

where c is the displacement of a generatrix straight line after its rotation at one radian; k = cotan α.
(2) Parametrical equations (Fig. 1):

$$x = x(r,v) = r\cos v, \quad y = y(r,v) = r\sin v, \quad z = cv + kr.$$

Coefficients of the fundamental forms of the surface:

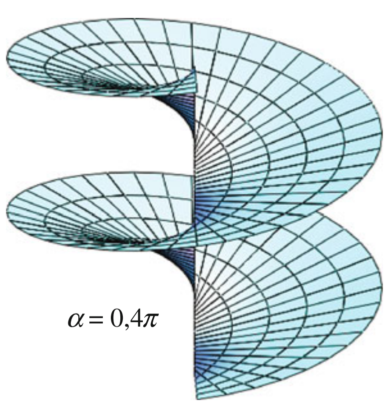

Fig. 1

$$A^2 = 1 + k^2, \quad F = ck, \quad B^2 = r^2 + c^2,$$
$$L = 0, \quad M = -\frac{c}{\sqrt{B^2 - F^2}}, \quad N = \frac{kr^2}{\sqrt{B^2 - F^2}},$$
$$k_r = 0, \quad k_v = \frac{N}{B^2}, \quad K < 0, \quad \cos\chi = \frac{ck}{\sqrt{r^2 + c^2}}.$$

Here k is an angular coefficient, k = cotan α; χ is an angle between coordinate lines r and v. A section of an oblique helicoid by a plane $z = 0$ gives

$$\rho = cv\tan\alpha,$$

i.e. we have *spiral of Archimedes*.

(3) Parametrical equations (Fig. 1):

$$x = x(u,v) = u\sin\alpha\cos v,$$
$$y = y(u,v) = u\sin\alpha\sin v,$$
$$z = cv + u\cos\alpha,$$

where α is an acute angle of the axis of the helicoid (an axis z) with the straight generatrix.

Coefficients of the fundamental forms of the surface:

$$A = 1, \ F = c\cos\alpha, \ B^2 = u^2\sin^2\alpha + c^2,$$
$$L = 0, \ M = -c\frac{\sin\alpha}{\sqrt{u^2+c^2}}, \ N = \frac{u^2\sin\alpha\cos\alpha}{\sqrt{u^2+c^2}}б$$
$$k_u = 0, \ k_v = \frac{N}{B^2}, \ K = -\frac{c^2}{(u^2+c^2)^2} < 0, \ H = \frac{u^2+2c^2}{2(u^2+c^2)^{3/2}}\operatorname{cotan}\alpha.$$

(4) Vector equation:

$$\boldsymbol{r}(u,\varphi) = p(u)\boldsymbol{e}_r + [lp(u) + a\varphi]\boldsymbol{e}_z.$$

The value a is connected with the lead L of the helix with the help of relation:

$$L = 2\pi a,$$

l = cotan α determines an angle of slope of the generatrix,

$$p(u) = mu.$$

An angle χ between coordinate lines u and φ one can determine from a formula:

$$\cos\chi = \frac{lap(u)}{AB} = \frac{lam}{AB}.$$

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■ Ruled Surface with Straight Generatrixes Passing Through a Logarithmic Spiral and Intersecting the Fixed Axis Under Constant Angle

A ruled surface with straight generatrixes passing through a logarithmic spiral and intersecting the fixed axis under constant angle (Fig. 1) can be given by parametrical equations:

$$x = x(u,v) = (ae^{mu} + v\sin\theta)\cos u,$$
$$y = y(u,v) = (ae^{mu} + v\sin\theta)\sin u,$$
$$z = z(v) = v\cos\theta.$$

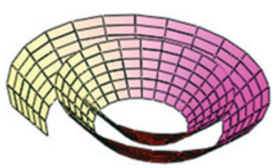

Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2m^2e^{2mu} + (ae^{mu} + v\sin\theta)^2,$$
$$F = ame^{mu}\sin\theta, \quad B = 1,$$
$$A^2B^2 - F^2 = a^2m^2e^{2mu}\cos^2\theta + (ae^{mu} + v\sin\theta)^2,$$
$$M = -F\cos\theta\Big/\sqrt{A^2B^2 - F^2}, \quad N = 0,$$
$$L = \frac{-\cos\theta}{\sqrt{A^2B^2 - F^2}}\left[(ae^{mu} + v\sin\theta)^2 + am^2e^{mu}(ae^{mu} - v\sin\theta)\right],$$
$$k_v = 0, \quad K = \frac{-M^2}{A^2B^2 - F^2} < 0.$$

This ruled surface is given by nonorthogonal, nonconjugate curvilinear coordinates u, v. It is a special case of *a spiral surface with straight generatrixes in the pencil of planes* is presented in the Chapter. "Spiral Surfaces".

■ Spherical Helicoid

A spherical helicoid has a directrix *spherical line*

$$\boldsymbol{E}_0(u) = a\boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\sin\omega,$$
$$\omega = pu; \; p = \text{const},$$

placed on a surface of the sphere with a radius a. Having assumed

$$p = 1/n$$

where n is integers, one can design the spherical line of a spiral-shaped form (Fig. 1). Straight generatrixes of a spherical helicoid intersect a plane z = const under a constant angle θ and is placed in the planes of pencil with the fixed straight coinciding with the helicoidal axis Oz.

$a = 20; \; p = 1/40,$

Fig. 1 $-20\pi \le u \le 20\pi$

Forms of definition of spherical helicoid

(1) Vector form of definition:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = a\boldsymbol{e}_0(u) + v\cos\theta\boldsymbol{h}(u) + v\sin\theta\boldsymbol{k}$$
$$= (a\cos\omega + v\cos\theta)\boldsymbol{h}(u) + (a\sin\omega + v\sin\theta)\boldsymbol{k}$$

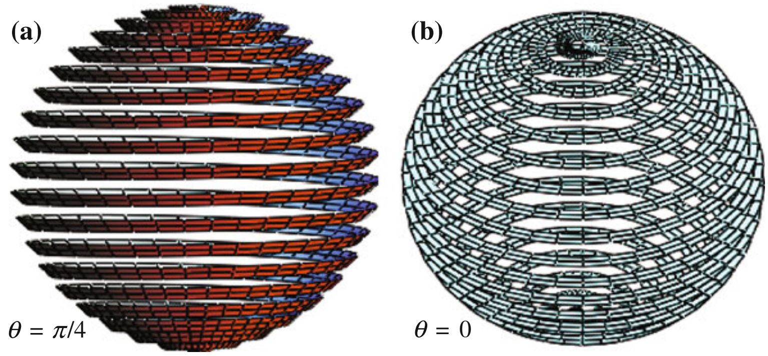

Fig. 2

where

$$\boldsymbol{h} = \boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u.$$

The unit vector $\boldsymbol{e}_0(u)$ is a normal of the sphere with the directrix curve placed on it.

(2) Parametrical form of definition (Fig. 2a):

$$x = x(u,v) = (a\cos\omega + v\cos\theta)\cos u,$$
$$y = y(u,v) = (a\cos\omega + v\cos\theta)\sin u,$$
$$z = z(u,v) = a\sin\omega + v\sin\theta.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2p^2 + (a\cos\omega + v\cos\theta)^2,\ B = 1,$$
$$F = ap\sin(\theta - \omega),$$
$$L = \{-2a^2p^2\sin\omega\cos(\omega-\theta) - (a\cos\omega + v\cos\theta)\left[ap^2\sin(\theta-\omega) + (a\cos\omega + v\cos\theta)\sin\theta\right]\}\Big/\sqrt{A^2 - F^2},\quad N = 0,$$
$$M = ap\cos\theta\cos(\omega-\theta)\Big/\sqrt{A^2B^2 - F^2},$$
$$K = -M^2\big/\left(A^2 - F^2\right) < 0.$$

The studied surface is a surface of *strictly negative curvature*. A spherical helicoid degenerates into *a cylindrical-and-spherical spiral-shaped strip* with $K = 0$ if $\theta = \pi/2$. But if we take $\theta = 0$ then a spherical helicoid becomes *a right spherical helicoid* (Fig. 2b).

1.2.1 Catalan Surfaces

Catalan surfaces are ruled surfaces with rectilinear generatrixes parallel to any fixed plane which is called *a plane of parallelism* or *the directrix plane of the surface*. Catalan's surface has a plane *restriction line*. A vector equation of Catalan's surfaces can be written in the following form

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + v\boldsymbol{l}(u)$$

where

$$\boldsymbol{l}''(u) \neq 0,\ (\boldsymbol{l}, \boldsymbol{l}', \boldsymbol{l}'') = 0.$$

If all straight generatrixes (rulings) of Catalan's surface intersect the same straight line, then this surface is *a conoid*. *Cylindroids* are also belonging to Catalan surfaces because rectilinear generatrixes of cylindroid intersecting two given curves remain parallel to any plane of parallelism.

A Catalan surface is *an oblique ruled surface* because it is a surface of negative Gaussian curvature. The definition keeps out cylindrical surfaces from this class of surfaces because torsal rulings of cylindrical surfaces are not only parallel to a constant plane but additionally are parallel to a fixed straight.

A Catalan surface was named after the Belgian mathematician Eugène Charles Catalan. Catalan proved that the helicoid and the plane were the only *ruled minimal surfaces*.

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■ Whitney Umbrella (Cartan Umbrella)

A Whitney umbrella is a ruled self-intersecting surface of negative Gaussian curvature containing *a double straight line*. Whitney umbrella has also another name that is *a Cartan umbrella*. Having bent a square with the cut and then having closed the edges of the cut through the sheet of the paper, one can make an applicable paper model of the umbrella (Fig. 1).

Forms of definition of the umbrella

(1) Implicit form of definition:

$$zy^2 - x^2 = 0 \text{ (a canonical equation)}.$$

There are parabolas $z = x^2/b^2$ in the cross-sections of the umbrella by the planes $y = -b$ = const. When the parameter y gets over zero then two branches of the parabola turn out passing through the double line and after open again as the plane $y = b$ moves along the y axis.

Two straights $y = \pm x/\sqrt{c}$ intersecting at an axis z lie in the cross-section of a Whitney umbrella made by the plane $z = c > 0$. It should be noted that the straight rotates around the axis Oz in the process of movement of this plane along the Oz axis until the position $z = 0$ and after that the straight rotates again around the Oz axis but in other direction remaining all the time parallel to a plane z = const. So, a Whitney umbrella belongs to a family of Catalan surfaces and can be called *a right conoid* or more precisely *a right conoid with a directrix parabola the axis of which is parallel to the axis of the conoid*.

The end of the line of self-intersection (the "plus" shape) is *a pinch point* also called *a Whitney singularity* or *branch point*.

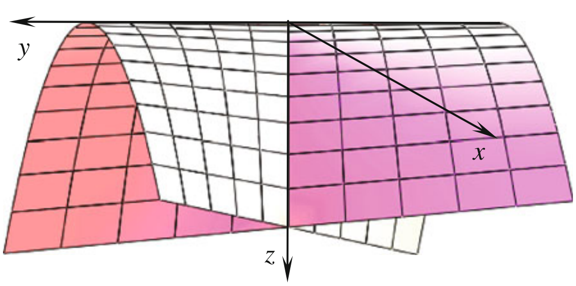

Fig. 1

(2) Parametrical equations (Fig. 1):

$$x = x(r,t) = rt, \quad y = y(r) = r, \quad z = z(t) = t^2.$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 1 + t^2, \quad F = rt,$$
$$B^2 = r^2 + 4t^2,$$
$$L = 0, \quad M = \frac{2t}{\sqrt{r^2 + 4t^2 + 4t^4}},$$
$$N = -\frac{2r}{\sqrt{r^2 + 4t^2 + 4t^4}},$$
$$A^2B^2 - F^2 = r^2 + 4t^2 + 4t^4,$$
$$k_r = 0, \ k_t = -\frac{2r}{\sqrt{r^2 + 4t^2 + 4t^4}},$$
$$K = -\frac{4t^2}{(r^2 + 4t^2 + 4t^4)} < 0,$$
$$H = -\frac{r(1 + 3t^2)}{(r^2 + 4t^2 + 4t^4)^{3/2}} \neq 0.$$

This geometrical object was first studied by Hassler Whitney in the 1940.

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■ Pseudodevelopable Helicoid

Assume a helix of constant lead $L = 2\pi b$ on a cylinder with a radius a as a director curve l. Let a coordinate axis Oz coincides with an axis of the cylinder. Assume a horizontal straight line, one end of which is placed on the curve l but the straight line itself must be parallel to the projection of the corresponding tangent line to the helix on the plane xOy, for a rectilinear generatrix.

The rectilinear generatrix and the corresponding tangent line to the helix intersect in a point belonging to the helix. In this case, the rectilinear generatrixes will be parallel to the plane xOy and in the process of its movement will not intersect the axis of a studied helicoid.

A ruled surface formed by the method described before is called a *pseudodevelopable helicoid* (Fig. 1) or *an open right helicoid*. Hence, a pseudodevelopable helicoid is formed by projections of tangent lines of the helix of the constant pitch on the plane perpendicular to the axis of the helix. This surface is a particular case of *a convolute helicoid*. The least distance between a generatrix straight and an axis of the helicoid is called *an eccentricity of helicoid*.

An angle φ of the tangent to a helix l to a rectilinear generatrix of the surface can be determined as

$$\text{tg}\varphi = b/a.$$

Surface of pseudodevelopable helicoid is used in design of drills for wooden products.

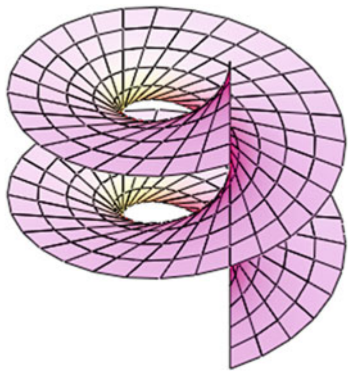

Fig. 1

Forms of definition of the studied surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = a\cos v - u\sin v,$$
$$y = y(u, v) = a\sin v + u\cos v,$$
$$z = z(v) = bv,$$

where $|u|$ is a distance from a helical directrix till the corresponding point on the surface taken along a rectilinear; v is an angle from the axis Ox in the direction of the axis Oy; a is an eccentricity of the helicoid. Coordinate lines u (v = const) coincide with rectilinear generatrixes of the surface but the lines v (u = const) are equidistant helices in the relation of the directrix helix (u = 0). In Fig. 1, a pseudodevelopable helicoid is shown when $0 \le v \le 4\pi$.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = 1,\ F = a,$$
$$B^2 = a^2 + b^2 + u^2,$$
$$L = 0,\ M = -\frac{b}{\sqrt{b^2 + u^2}},$$
$$N = -\frac{ab}{\sqrt{b^2 + u^2}} = aM,\ k_u = 0,$$
$$k_v = -\frac{ab}{B^2\sqrt{b^2 + u^2}},$$
$$k_1 = \frac{a + \sqrt{a^2 + 4b^2 + 4u^2}}{2(b^2 + u^2)^{3/2}}b,$$
$$k_2 = \frac{a - \sqrt{a^2 + 4b^2 + 4u^2}}{2(b^2 + u^2)^{3/2}}b,$$
$$K = -\frac{b^2}{(b^2 + u^2)^2} < 0,$$
$$H = \frac{ab}{(b^2 + u^2)^{3/2}} \neq 0.$$

The coefficients of the fundamental forms of the surface show that the studied ruled surface is given in nonorthogonal ($F \neq 0$), nonconjugated ($M \neq 0$) system of curvilinear coordinates u, v. The mean curvature H of the surface confines that pseudodevelopable helicoid is not minimal surface by contrast to right helicoid.

An angle χ between coordinate lines u and v is given by

$$\cos\chi = \frac{a}{B}.$$

(2) Implicit form of definition:

$$x\cos\frac{z}{b} + y\sin\frac{z}{b} = a.$$

(3) General parametrical equations of *an open oblique helicoid*:

$$x = a\cos v - u\cos\varepsilon\sin v,$$
$$y = a\sin v + u\cos\varepsilon\cos v,$$
$$z = bv + u\sin\varepsilon,$$

where ε is an angle between a straight generatrix of the surface and a horizontal plane. If we put $\varepsilon = 0$, then we shall have *an open right helicoid*, i.e., a *pseudodevelopable helicoid.*

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■ Ruled Rotor Cylindroid

A surface of *a ruled rotor cylindroid* is a trajectory of sum of two helical motions the axes of which are perpendicular to each other and cross but the speeds of rotation are characterized by parameters u and nu. The surfaces of trajectories are closed and algebraic if n is a rational number [1].

Forms of definition of surface

(1) Parametrical equations (Figs. 1, 2 and 3):

$$x = x(u, v) = (a + b \sin nu) \cos u - v \sin u,$$
$$y = y(u, v) = (a + b \sin nu) \sin u + v \cos u,$$
$$z = z(u) = b \cos nu,$$

where $0 \le u \le 2\pi$. Curvilinear coordinate lines v are straight lines that are parallel to a coordinate plane xOy. Hence, a presented surface belongs to *Catalan surfaces* but more precisely to *cylindroids*.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = b^2n^2 + v^2 - 2bnv \cos nu + (a + b \sin nu)^2,$$
$$F = a + b \sin nu, \quad B = 1,$$
$$A^2B^2 - F^2 = A^2 - F^2 = b^2n^2 + v^2 - 2bnv \cos nu,$$
$$L = -\frac{bn}{\sqrt{A^2 - F^2}}\left[bn^2 + (a + b \sin nu) \sin nu - nv \cos nu\right],$$
$$M = -\frac{bn \sin nu}{\sqrt{A^2 - F^2}}, \quad N = 0,$$
$$K = \frac{-b^2n^2 \sin^2 nu}{(b^2n^2 + v^2 - 2bnv \cos nu)^2} \le 0,$$
$$H = \frac{bn}{2(A^2 - F^2)^{3/2}}\left[(a + b \sin nu) \sin nu - bn^2 + nv \cos nu\right] \ne 0.$$

(2) Implicit equation: $(xz - ab)^2 - (b^2 - z^2)(b - y)^2 = 0$ for $n = 1$.

A ruled rotor cylindroid is the fourth-order algebraic surface (Fig. 1) if $n = 1$.

(3) Implicit equation:

$$\left[\frac{x^2(z + b)}{2b} + \frac{y^2(b - z)}{2b} - a^2 - (b^2 - z^2)\right]^2 - (b^2 - z^2)\left(2a - \frac{xy}{b}\right)^2 = 0 \text{ if } n = 2.$$

A ruled rotor cylindroid is the sixth-order algebraic surface (Fig. 2) if $n = 2$.

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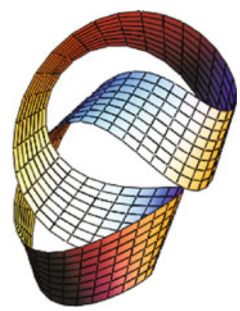
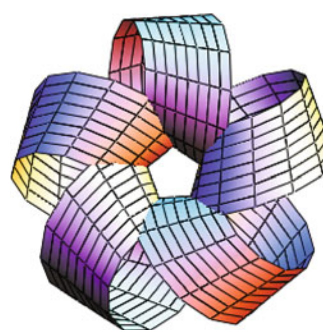


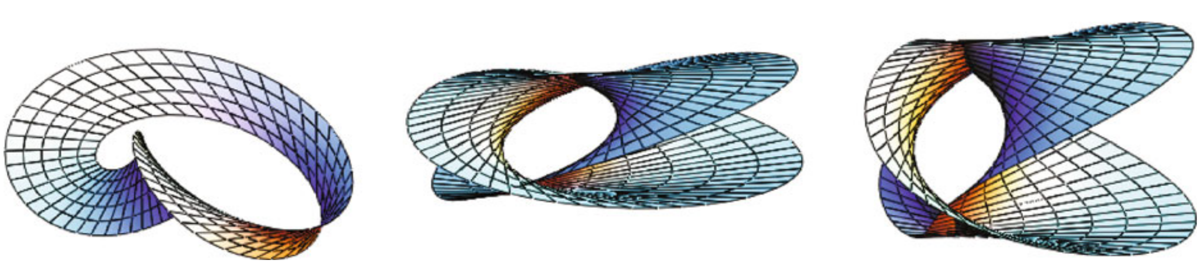
$a = 1; b = 2; n = 2;$

Fig. 2



$a = 2; b = 1{,}5; n = 5;$

Fig. 3



$a = 2;\ b = 2;\ n = 1;\ 0 \le v \le 2$ $\quad a = 1;\ b = 1;\ n = 1;\ -3 \le v \le 3$ $\quad a = 1;\ b = 2;\ n = 1;\ -3 \le v \le 3$

Fig. 1

■ Saddle in the Drum

A surface "*Saddle in the Drum*" is *a hyperbolic paraboloid* given in polar coordinates (Fig. 1) r and t:

$$x = x(r,t) = r\cos t,$$
$$y = y(r,t) = r\sin t,$$
$$z = z(r,t) = 0.5r^2\sin(2t).$$

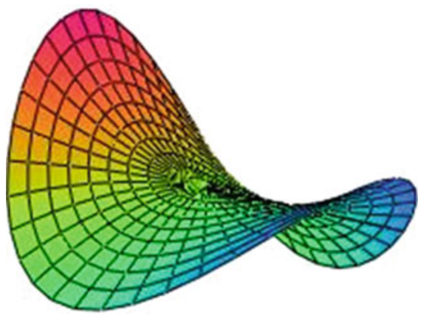


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + r^2\sin^2(2t),$$
$$F = 0.5r^3\sin(4t),$$
$$B^2 = r^2\left[1 + r^2\cos^2(2t)\right],$$
$$A^2B^2 - F^2 = r^2(1+r^2),$$
$$L = \frac{\sin(2t)}{\sqrt{1+r^2}},$$
$$M = \frac{r\cos(2t)}{\sqrt{1+r^2}},$$
$$N = \frac{-r^2\sin(2t)}{\sqrt{1+r^2}},$$
$$K = -\frac{1}{(1+r^2)^2} < 0.$$

This surface is a surface of strictly negative Gaussian curvature and is given in nonorthogonal, nonconjugated curvilinear coordinates r, t.

■ Conoids

Conoids belong to a family of *Catalan surfaces.* All rectilinear generatrixes of catenoid intersect a fixed straight line which is called *a conoidal axis*. Straight generatrixes of ruled Catalan surface are parallel to one and the same plane of parallelism. So, a conoid is a ruled surface (Fig. 1) formed by a moving straight line that is parallel to the fixed plane P, intersects an immovable straight AB (an axis of the conoid) and an immovable director curve C.

It is assumed that a straight AB does not intersect a curve C but intersects a plane P. *A hyperboloidal paraboloid* is the simplest conoid. It is formed by a straight line moving on two skew straight lines and remains parallel to the fixed plane. In some publications, conoids are called *wedge-shaped surfaces*.

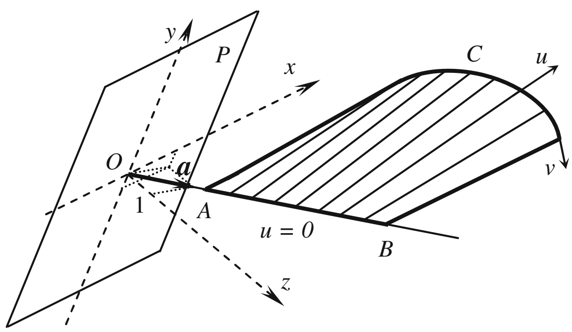


Fig. 1

Assume a point of intersection of a plane P with a straight line AB as a point of intersection of coordinate axes Ox, Oy, *and* Oz and the plane P as a coordinate plane $z = 0$, and assume a directrix vector of the straight line AB in the form $\boldsymbol{a} = \{x_0, y_0, 1\}$, but assume an equation of the director curve C in the form $\boldsymbol{r}(v) = \{f(v), g(v), h(v)\}$, then parametrical equations of the conoidal surface can be written as

$$x = x(u,\ v) = x_0h(v) + u[f(v) - x_0h(v)];$$
$$y = y(u,\ v) = y_0h(v) + u[g(v) - y_0h(v)];$$
$$z = z(v) = h(v).$$

Coordinate lines v = const coincide with rectilinear generatrixes of the conoid and the coordinate line $u = 0$ is the axis of conoid AB.

Conoids can be given also with the help of parametrical equations:

$$x = x(u,v) = u\cos v + \alpha f(v),$$
$$y = y(u,v) = u\sin v + \beta f(v),$$
$$z = z(v) = \gamma f(v),$$

where $\{\alpha, \beta, \gamma\}$ is the unit vector having the direction of the conoidal axis; $f(v)$ is any function. *Right conoids* can be constructed if one will take $\alpha = \beta = 0$, $\gamma = 1$. A right conoid has a fixed straight that is perpendicular to the plane of parallelism. The axis of a right conoid is *a line of striction* (*waist line*). A right conoid with $f(v) = av$ is *a right helicoid*.

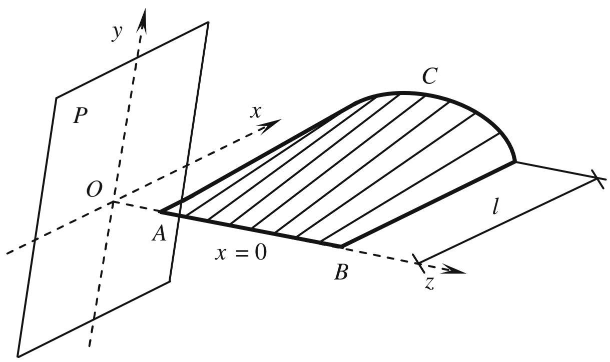


Fig. 2

Let us suppose that a straight AB coincides with a coordinate axis Oz and a curve C lies at the plane $x = l$ (Fig. 2), i.e., l is the distance from the axis of the conoid till the plane with the directrix curve C. If $f(z)$ is an equation of the directrix curve C then surface of a conoid can be given in an explicit form:

$$y = xf(z)/l.$$

First, shell covering in the form of the conoid was designed by Eugène Freyssinet in France. Fauconnier M. was the first one who has carried out the experimental researches of thin conoidal reinforced concrete shells under action of external loading. Pilarski I. began to use theoretical method for the investigation of membrane stress state of conoidal shells with the parabolic directrix.

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■ Parabolic Conoid

A parabolic conoid is a ruled surface formed by a moving straight line that intersects the fixed straight and the fixed director curve (Fig. 1). A generatrix straight line is parallel to the fixed plane but the fixed plane is perpendicular to a plane with the directrix parabola and passes through the axis of the parabola. The director straight is perpendicular to the fixed plane.

Forms of definition of the studied surface

(1) Explicit equation (Figs. 1, 2 and 3):

$$z = \frac{c}{ab^2}(y^2 - b^2)(a - x), \quad a, b, c \neq 0.$$

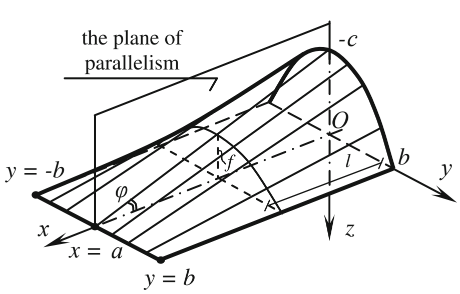

Fig. 1

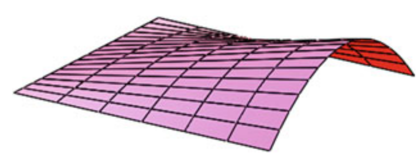

Fig. 2

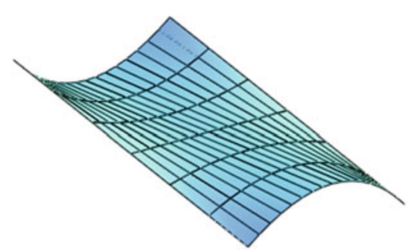

Fig. 3

A plane of parallelism is given by an equation: $y = y_0 = \text{const}$. The cross-sections of a parabolic conoid by a plane $x = l$ contain parabolas:

$$z = \frac{c(a-l)}{ab^2}(y^2 - b^2) = \frac{f}{b^2}(y^2 - b^2)$$

where f is the distance from a peak of the parabola placed in the cross-section by a plane $x = l$ till a plane $z = 0$ (*a rise*). This conoid covers a rectangular plan with dimensions of $2b \times a$.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{c^2}{a^2b^4}(y^2 - b^2)^2,$$
$$F = -\frac{2c^2y}{a^2b^4}(y^2 - b^2)(a - x),$$
$$B^2 = 1 + \frac{4c^2y^2}{a^2b^4}(a - x)^2, L = 0,$$
$$M = -\frac{2cy}{\sqrt{a^2b^4 + c^2(y^2 - b^2)^2 + 4c^2y^2(a - x)^2}},$$
$$N = \frac{2c(a - x)}{\sqrt{a^2b^4 + c^2(y^2 - b^2)^2 + 4c^2y^2(a - x)^2}}.$$

(2) Explicit equation (Fig. 4):

$$z = -c\left(1 - \frac{gx}{l}\right)\left(1 - \frac{y^2}{b^2}\right), \quad \text{where } g = 1 - \frac{f}{c}, \ c > f.$$

Here, a conoid is given by two directrix parabolas with coordinates of the peaks equal to (0; 0; –c) and (l; 0; –f), where l is a distance between planes with directrix parabolas. A parabolic conoid covers a rectangular plan with dimensions of $2b \times l$.

(3) Explicit equation:

$$z = -\frac{xc}{a}\left(1 - \frac{4y^2}{d^2}\right).$$

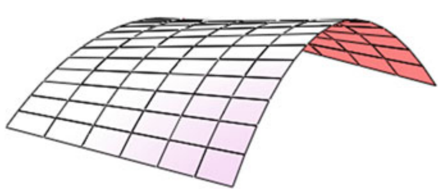

Fig. 4

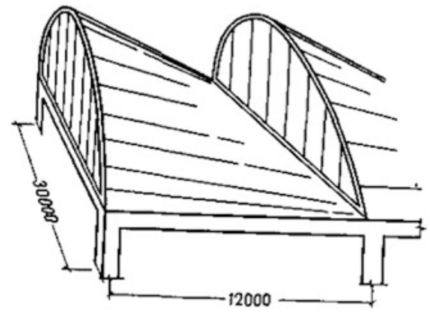

Fig. 5 The type section of right conoidal shells with the parabolic directrix

In this case, the beginning of Cartesian system of coordinates is placed at the directrix straight line in contrast to Fig. 1 but $d = 2b$. A parabolic conoid covers a rectangular plan with dimensions of $d \times a$.

A slope angle of a ruling that passes through the peak of a directrix parabola one can find by a formula:

$$\tan\varphi = -\frac{c}{a} = \frac{c - f}{l}.$$

Conoidal roofs are made in the form of single-wave and multiwave shells (Fig. 5).

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■ Conoid with a Directrix Circle

A conoid with a directrix circle is a ruled surface formed by a moving straight line that intersects the fixed straight and the fixed director circle (Fig. 1). A generatrix straight line is parallel to the fixed plane but the fixed plane is perpendicular to a plane with the directrix circle and perpendicular to the axis of conoid. So, this conoid with a director circle is *a right conoid*.

Forms of definition of a conoid with a directrix circle

(1) Explicit equation (Fig. 1):

$$z = \frac{x}{l}\left(a - f - \sqrt{a^2 - y^2}\right),$$

where a is a radius of the directrix circle; $y \le a$; f is the distance from the peak of the circle place in the plane

$x = l$ till the plane $z = 0$ (a *rise*). This conoid covers a rectangular plan with dimensions of

$$l \times 2\sqrt{f(2a-f)};\; -\sqrt{f(2a-f)} \le y \le \sqrt{f(2a-f)}.$$

A plane of parallelism is given by an equation $y = y_0 = \text{const}$. In Fig. 1, the conoid with $f < a$, $0 \le x \le l$ is shown; in Fig. 2 the conoid has $f = a$, $z = \mp x\sqrt{a^2 - y^2}/l$, $0 \le x \le l$. In Fig. 3, the conoid with – c $f < a$, $-l \le x \le l$ and in Fig. 4, the conoid with $f = a$, $z = \mp x\sqrt{a^2 - y^2}/l$, $-l \le x \le l$ are presented.

If we intersect the conoid with the director circle shown in Fig. 2 by a plane $x = x_0$, then we shall produce the ellipses

$$\frac{z^2 l^2}{a^2 x_0^2} + \frac{y^2}{a^2} = 1$$

with semi-axes a and ax_0/l. There are rulings

$$z = \frac{x}{l}\left(a - f - \sqrt{a^2 - y_0^2}\right)$$

in the cross-sections $y = y_0 = \text{const}$.

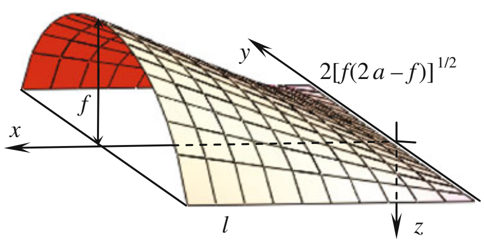

Fig. 1

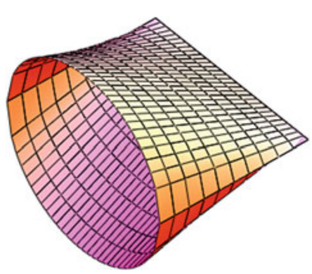

Fig. 2

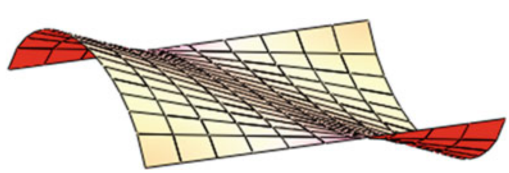

Fig. 3

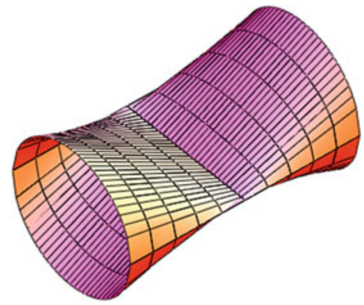

Fig. 4

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{1}{l^2}\left(a - f - \sqrt{a^2 - y^2}\right)^2,$$

$$F = \frac{xy}{l^2}\left(\frac{a-f}{\sqrt{a^2-y^2}} - 1\right),$$

$$B^2 = 1 + \frac{x^2y^2}{l^2(a^2-y^2)},$$

$$A^2B^2 - F^2 = A^2 + B^2 - 1,\; L = 0,$$

$$M = \frac{y}{l\sqrt{a^2-y^2}\sqrt{A^2+B^2-1}},$$

$$N = \frac{a^2x}{l(a^2-y^2)^{3/2}\sqrt{A^2+B^2-1}},$$

$$K = \frac{-y^2}{l^2(a^2-y^2)(A^2+B^2-1)^2} < 0.$$

So, a right conoid with a directrix circle is a surface of negative Gaussian curvature and is given in curvilinear nonorthogonal, nonconjugated coordinates.

In Fig. 5, the structure consisting of four circular conoids is presented.

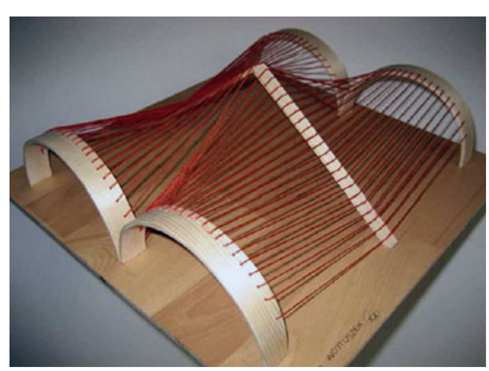

Fig. 5 A model of the shell consisting of four circular conoids [*Jolanta Tofil*]

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■ Conoid with a Directrix Catenary

A conoid with a directrix catenary is a ruled surface formed by a moving straight line that intersects the fixed straight and the fixed director catenary (Fig. 1). Rulings are parallel to the fixed plane but the fixed plane is perpendicular to a plane with the directrix catenary and is perpendicular the axis of conoid. So, this conoid with a directrix catenary is a right conoid.

An explicit equation of a conoid with a directrix catenary can be written in the following form (Fig. 1):

$$z = \frac{x}{l}\left(a\mathrm{ch}\frac{y}{a} - a - f\right).$$

A conoid with a catenary covers a rectangular plan with dimensions equal to

$$l \times 2a\mathrm{Arch}(1 + \frac{f}{a}).$$

Every plane $y = y_0 = \mathrm{const}$ can be a plane of parallelism. In the cross-section of a studied right conoid by a plane $x = x_0 = \mathrm{const}$, lines

$$z = z(y) = \frac{ax_0}{l}\mathrm{ch}\frac{y}{a} - \frac{a+f}{l}x_0$$

will be placed.

In the cross-section of a studied right conoid by planes $y = y_0 = \mathrm{const}$, rectilinear generatrixes

$$z = z(x) = \frac{x}{l}\left(a\mathrm{ch}\frac{y_0}{a} - a - f\right) = -x\mathrm{tg}\alpha$$

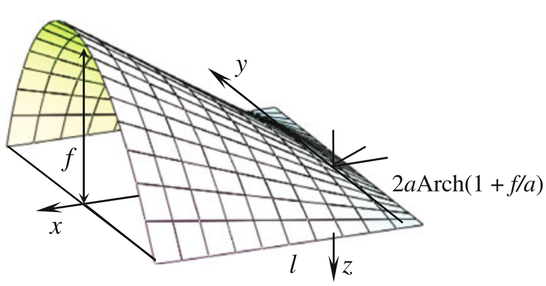

Fig. 1

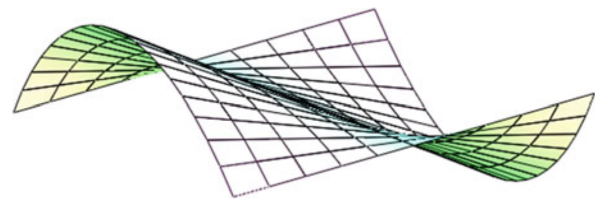

Fig. 2

are placed, where α is the slope angle of the ruling with a plane $z = 0$.

In Fig. 1, the conoid was designed in the limits $0 \le x \le l$; in Fig. 2, it is in the limits $-l \le x \le l$; in Fig. 3 the conoid was designed in the limits $c \le x \le l$, but $c < l$.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{1}{l^2}\left(a\mathrm{ch}\frac{y}{a} - a - f\right)^2,$$
$$F = \frac{x}{l^2}\left(a\mathrm{ch}\frac{y}{a} - a - f\right)\mathrm{sh}\frac{y}{a},$$
$$B^2 = 1 + \frac{x^2}{l^2}\mathrm{sh}^2\frac{y}{a},$$
$$A^2B^2 - F^2 = A^2 + B^2 - 1,$$
$$L = 0,\ M = \frac{1}{l\sqrt{A^2 + B^2 - 1}}\mathrm{sh}\frac{y}{a},$$
$$N = \frac{x}{al\sqrt{A^2 + B^2 - 1}}\mathrm{ch}\frac{y}{a},\ k_x = 0,$$
$$K = \frac{-1}{l^2(A^2 + B^2 - 1)^2}\mathrm{sh}^2\frac{y}{a} < 0.$$

So, a right conoid with a directrix catenary is a surface of negative total curvature and is given in curvilinear nonorthogonal, nonconjugated coordinates x, y.

The Cathedral erected in Canada with the application of conoidal shells of large dimensions was described by Hahn L.

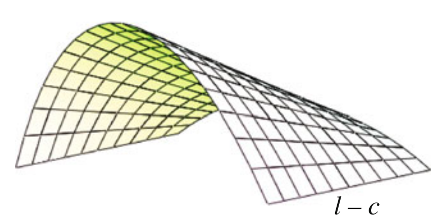

Fig. 3

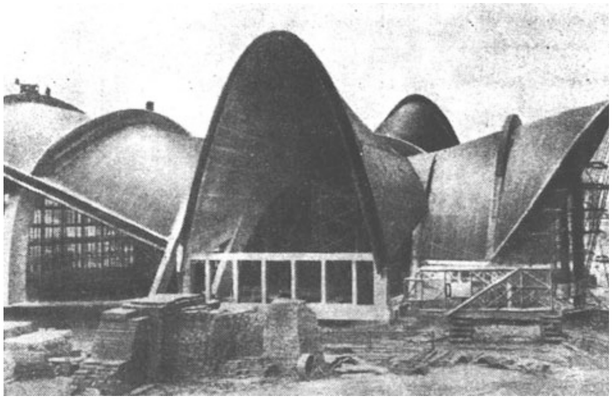

Fig. 4 The Cathedral in Canada

The directrix curves of this Cathedral have the form similar to a catenary (Fig. 4).

Additional Literature

Soare M. Zur Membrantheorie der Konoidschalen. Der Bauingenieur. 1958; h. 7, p. 256-265 (18 ref.).
Hahn L. Ruled vaults having been used for cathedral in Canada. Large-span shells: Int. Cong. in Leningrad. Vol. 2. Moscow: Stroyizdat, 1969; p. 379–402.
Krivoshapko SN., *Gil-oulbe Mathieu*. Geometrical and strength analysis of thin pseudo-spherical, epitrochoidal, catenoidal shells, and shells in the form of Dupin's cyclides. Shells in Architecture and Strength Analysis of Thin-Walled Civil- Engineering and Machine-Building Constructions of Complex Forms: Proc. Int. Conf., June 4-8, 2001, Moscow, Russia. 2001; p. 183-192.

■ Right Sinusoidal Conoid

A right sinusoidal conoid is formed by a moving straight line that intersects the fixed straight s and the fixed director sinusoid S (Fig. 1). A generatrix straight line (a ruling) is parallel to the fixed plane, i.e., to the plane of parallelism but the fixed plane is perpendicular to a plane with the directrix sinusoid and perpendicular to the axis of conoid.

An explicit form of definition of a right sinusoidal conoid (Figs. 1, 2, 3 and 4) is

$$z = \frac{l-x}{l}\left[a\sin\left(\frac{\pi}{2}+\frac{n\pi y}{b}\right)+c\right] = \frac{l-x}{l}\left(a\cos\frac{n\pi y}{b}+c\right)$$

where a is an amplitude of the sinusoid; n is a number of integer half-waves of the sinusoid placed on a line segment with length of b; l is the distance the axis of the conoid from a plane with a director sinusoid.

The director sinusoid is placed in a plane $x = 0$, the axis of the conoid coincides with a coordinate line $x = l$.

In the cross-section of the right sinusoidal conoid by a plane $x = d =$ const, the sinusoid with an amplitude $a(l - d)/l$ is and it degenerates into a straight line, which is an axis of conoid, in the cross-section $x = d = l$.

The plane of parallelism is given by an equation: $y = y_0 =$ const.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{1}{l^2}\left(a\cos\frac{n\pi y}{b}+c\right)^2,$$
$$F = \frac{(l-x)an\pi}{l^2 b}\sin\frac{n\pi y}{b}\left(c + a\cos\frac{n\pi y}{b}\right),$$
$$B^2 = 1 + \frac{(l-x)^2a^2n^2\pi^2}{l^2b^2}\sin^2\frac{n\pi y}{b},$$
$$A^2B^2 - F^2 = A^2 + B^2 - 1,$$
$$L = 0,\ M = \frac{an\pi}{lb\sqrt{A^2+B^2-1}}\sin\frac{n\pi y}{b},$$
$$N = \frac{-an^2\pi^2(l-x)}{b^2 l\sqrt{A^2+B^2-1}}\cos\frac{n\pi y}{b}.$$

Coordinate lines $y =$ const coincide with the rulings of the surface and coordinate lines $y = 0 + ib/n$, where i is integer, are lines of principal curvatures of the conoid. The axis of a right sinusoidal conoid intersects all coordinate lines x at right angle but it is not a line of principal curvature.

In Fig. 1, the right sinusoidal conoid covering the rectangular plan $2b \times l$ is shown. The surface is bounded on one side by the axis of the conoid s and on the opposite side by the director sinusoid S; $c > a$, $n = 3$; $0 \le x \le l$.

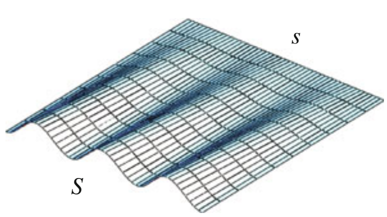

Fig. 1

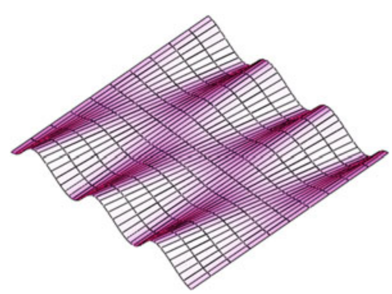

Fig. 2

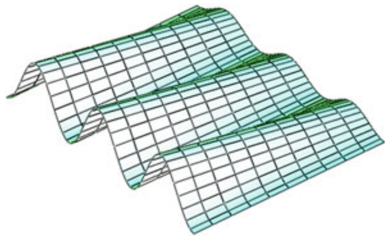

Fig. 3

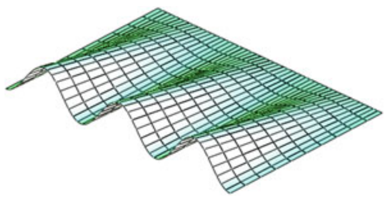

Fig. 4

In Fig. 2, two sheets of the sinusoidal conoid are shown and $0 \le x \le 2l$, $c > a$, $n = 3$.

The sinusoidal conoid bounded on both sides by two sinusoids lying in the parallel planes $x = 0$ and $x = d < l$ is presented in Fig. 3.

There is a conoid with $c = 0$ in Fig. 4.

In 1908, architect Antoni Gaudi has decided to design an original but very cheap school for the children of the Sagrada Familia workers with a cover in the form of a right sinusoidal conoid (Fig. 5). The works were finished in autumn of 1909 and quickly the classes were initiated. Later, this building was called the brilliant erection.

Additional Literature

Krivoshapko SN., *Mamieva IA*. Analytical Surfaces in the Architecture of Buildings, Structures and Products. Moscow: LIBROKOM, 2012; 328 p.

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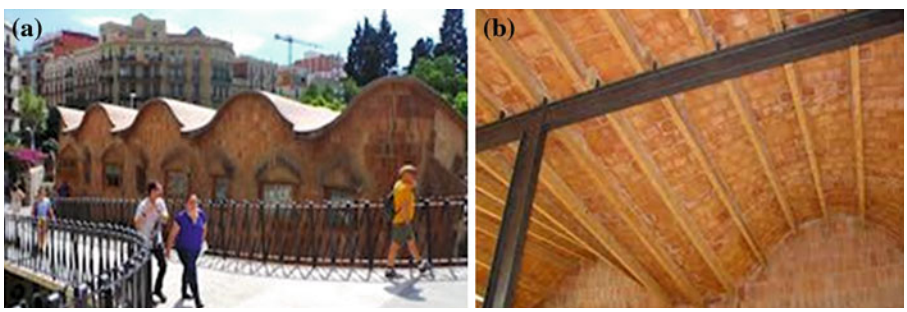


Fig. 5 **a** The Gaudi school for the children (Spain) (http://www.simplezenguy.com). **b** The rulings and the axis of the conoid (http://www.gaudiallgaudi.com)

■ Right Conoid with a Directrix Parabola the Axis of Which is Parallel to the Axis of Conoid

A right conoid with a directrix parabola the axis of which is parallel to the axis of conoid can be formed if one will take any coordinate plane as a plane of parallelism, and a straight line perpendicular to this coordinate plane but placed in another coordinate plane as a fixed straight, *i.e.*, as the axis of conoid, and a parabola lying in the third coordinate plane as the director parabola. The axis of the parabola must be parallel to the fixed straight line.

Forms of the definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(\lambda) = a\lambda,$$
$$y = y(\lambda, v) = v(1 - \lambda),$$
$$z = z(v) = v^2/(2p),$$

where $v = y$ of the director parabola; a is the distance a coordinate axis Oz from the axis of conoid; λ is a dimensionless parameter.

This method of definition supposes that the fixed straight line is given by equations $x = a$, $y = 0$, i.e., the straight is placed in the coordinate plane xOz. The coordinate plane

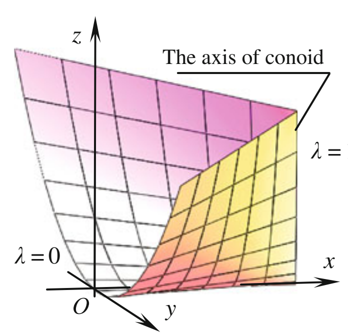


Fig. 1

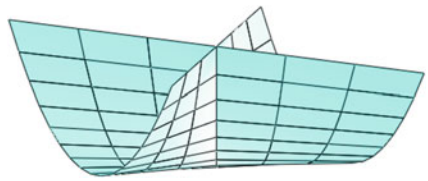


Fig. 2

xOy ($z = 0$) is assumed as a plane of parallelism. The director parabola $x = 0$, $y^2 = 2pz$ is located in a coordinate plane zOy ($x = 0$).

The conoid pictured in the limits of $0 \le \lambda \le 1$, $-b \le v \le b$ is shown in Fig. 1, b = const. The conoid constructed for $0 \le \lambda \le 2$ is presented in Fig. 2. The coordinate line $\lambda = 1$ coincides with an axis of conoid but the coordinate line $\lambda = 0$ coincides with the directrix parabola.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = a^2 + v^2,\ F = -v(1-\lambda),$$
$$B^2 = (1-\lambda)^2 + \frac{v^2}{p^2},\ L = 0,$$
$$M = \frac{av}{p\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{a(1-\lambda)}{p\sqrt{A^2B^2 - F^2}},$$
$$k_\lambda = 0,$$
$$K = \frac{-a^2v^2}{p^2(A^2B^2 - F^2)^2} \le 0.$$

A surface of the studied right conoid is given in a nonorthogonal, nonconjugate system of curvilinear coordinates. A surface of a right conoid with a directrix parabola the axis of which is parallel to the axis of conoid is a surface of negative Gaussian curvature ($K < 0$) but only along the line $v = 0$, i.e., along a coordinate line Ox, parabolic points are arranged in line ($K = 0$).

(2) An implicit equation: $a^2y^2 = 2pz(x-a)^2$.

There are parabolas $y^2 = 2pz(b-a)^2/a^2$ in the cross-sections of the conoid by the planes $x = b$ = const.

Having assumed $p = 0.5$; $a = 1$, we can obtain a canonical equation of *the Whitney umbrella:* $y^2 = z(x-1)^2$.

■ Evolvent Conoid

An evolvent conoid is a ruled surface formed by a moving straight line that intersects the fixed straight $x = z = 0$ and the fixed director evolvent of a circumference with an a radius (Fig. 1). Rulings are parallel to the fixed coordinate plane $y = 0$.

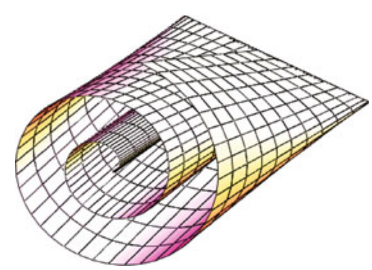
The parametrical equations of an evolvent conoid are written in the form:

$$x = ul,$$
$$y = y(v) = a(\cos v + v\sin v),$$
$$z = z(u,v) = au(\sin v - v\cos v)$$

where l is the distance the axis of conoid from the plane with the evolvent of circumference (Figs. 1 and 2), $0 \le u \le 1$; $0 \le v \le \infty$.

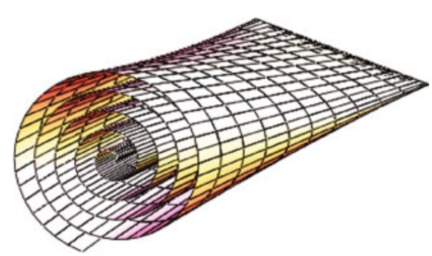
Coefficients of the fundamental forms of the surface:

$$A^2 = l^2 + a^2(\sin v - v\cos v)^2,$$
$$F = a^2uv(\sin v - v\cos v)\sin v,$$
$$B^2 = a^2v^2(\cos^2 v + u^2\sin^2 v),$$
$$L = 0,\quad M = \frac{a^2lv^2}{2\sqrt{A^2B^2 - F^2}}\sin 2v,$$
$$N = \frac{a^2luv^2}{\sqrt{A^2B^2 - F^2}}.$$



$a = 0{,}02\,l;\ 0 \le v \le 5\pi$

Fig. 1



$a = 0{,}005\,l;\ 0 \le v \le 10\pi$

Fig. 2

■ Plücker Conoid

Having assumed a coordinate plane xOy as a plane of parallelism and an axis Oz as the axis of conoid but a spatial closed curve on a round plan $x = x(\theta) = t\cos\theta$, $y = y(\theta) = t\sin\theta$, $z = z(\theta) = \sin(n\theta)$ as the director curve, we can carry out the conoids shown in Figs. 1 and 2.

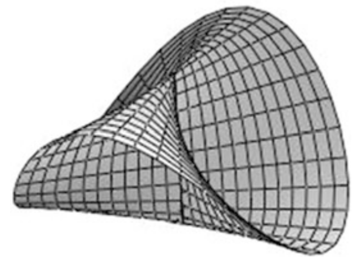


Fig. 1

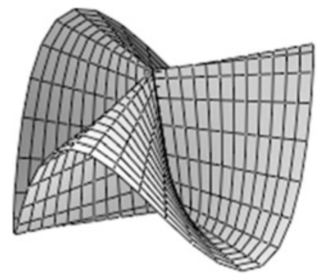


Fig. 2

Forms of definition of the surface

(1) Parametrical equations:

$$x = x(r,\theta) = r\cos\theta,$$
$$y = y(r,\theta) = r\sin\theta,$$
$$z = z(\theta) = \sin(n\theta)$$

where $0 \le z \le 2\pi$, $0 \le r \le \infty$, n is any integer; $-1 \le z \le 1$. The right conoid with $n = 2$ shown in Fig. 1 is called *Plucker conoid*. In Fig. 2, the right conoid with $n = 3$ is presented.

Coefficients of the fundamental forms of the surface:

$$A = 1, \quad F = 0,$$
$$B^2 = r^2 + n^2\cos^2(n\theta),$$
$$L = 0, \; M = -\frac{n\cos(n\theta)}{B},$$
$$N = -\frac{rn^2\sin(n\theta)}{B}, \; k_r = 0, \; k_\theta = \frac{N}{B^2},$$
$$K = -\frac{n^2\cos(n\theta)}{B^4}, \quad H = -\frac{rn^2\sin(n\theta)}{2B^3}.$$

The studied conoids are surfaces of negative Gaussian curvatures, only along the coordinate lines $\theta = \pi/(2n) + k\pi/n$ ($k = 1; 2; \ldots, k$), parabolic points are arranged in line with $K = 0$. Two intersecting straight lines $y = x\mathrm{tg}\theta_0$ are placed in the cross-sections of a Plucker conoid by planes z = const and it is equivalent to $\theta = \theta_0$ = const.

(2) An implicit equation of Plucker conoid (Fig. 1):

$$\left(x^2 + y^2\right)z = 2xy.$$

Two intersecting straight lines $y = x\left(1 \pm \sqrt{1-d^2}\right)/d$ lie in the cross-sections of Plucker conoid by the planes $z = d$, $-1 \le d \le 1$

Additional Literature

Gray A. Modern Differential Geometry of Curves and Surfaces with Mathematica (2nd ed.). Boca Raton, FL: CRC Press, 1998; 1053 p.

■ Wallis's Conical Edge

Wallis's conical edge (Figs. 1 and 2) is a right conoid and its axis is a coordinate axis Oz, any plane parallel to the coordinate plane $z = 0$ is the plane of parallelism, but a spatial closed waving line

$$x = x(u) = t\cos u,$$
$$y = y(u) = t\sin u,$$
$$z = z(u) = c\left(a^2 - b^2\cos^2 u\right)^{1/2}$$

on the circular plan is the director curve.

Forms of definition of the surface

(1) Parametrical equations:

$$x = x(u,v) = v\cos u,$$
$$y = y(u,v) = v\sin u,$$
$$z = c\sqrt{a^2 - b^2\cos^2 u}$$

where $0 \le u \le 2\pi$; $0 \le v \le \infty$, a, b, c are constants; $c\sqrt{a^2-b^2} \le z \le ac$, $a > b$.

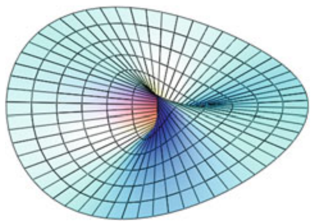


Fig. 1

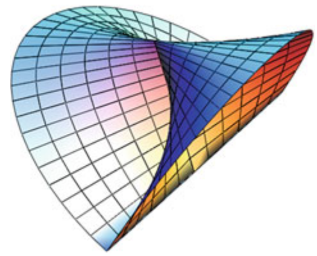


Fig. 2

Coefficients of the fundamental forms of the surface:

$$A^2 = v^2 + \frac{c^2b^4 \sin^2 u \cos^2 u}{(a^2 - b^2 \cos^2 u)},$$
$$F = 0, \quad B = 1,$$
$$M = \frac{cb^2 \sin u \cos u}{A\sqrt{a^2 - b^2 \cos^2 u}},$$
$$L = -\frac{cb^2 v[a^2 \cos^2(2u) - b^2 \cos^4 u]}{A(a^2 - b^2 \cos^2 u)^{3/2}},$$
$$N = 0, \quad K = -\frac{c^2b^4 \sin^2(2u)}{4A^4(a^2 - b^2 \cos^2 u)} \le 0.$$

Wallis's conical edge is a surface of negative Gaussian curvature but along the lines $u = 0 + k\pi/2$, where $k = 1; 2; \ldots, k$, parabolic points are arranged in line with $K = 0$.

(2) An implicit equation:

$$c^2b^2x^2 - \left(a^2c^2 - z^2\right)\left(x^2 + y^2\right) = 0.$$

Wallis's conical edge is named after the English mathematician John Wallis, who was one of the first to use Cartesian methods to study conic sections. He is also credited with introducing the symbol ∞ for infinity.

Additional Literature

Surfaces: Wallis conical edge from Differential Geometry Library. http://digi-area.com/DifferentialGeometryLibrary/Surfaces/Wallis-Conical-Edge.php

Weisstein, Eric W. "Wallis's Conical Edge." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/WallissConicalEdge.html

■ Zindler's Conoid

Zindler's conoid is a right helical conoid of changing pitch. A coordinate axis Oz is an axis of conoid. Any plane parallel to a plane xOy is *a plane of parallelism*.

Forms of the definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = u \cos v;$$
$$y = y(u, v) = u \sin v;$$
$$z = a \tan 2v,$$

where $-\infty \le u \le \infty$; $-\pi/4 < v < \pi/4$; a is an arbitrary constant. Coordinate lines u coincide with rulings of the conoid, the line $u = 0$ is a straight axis of conoid.

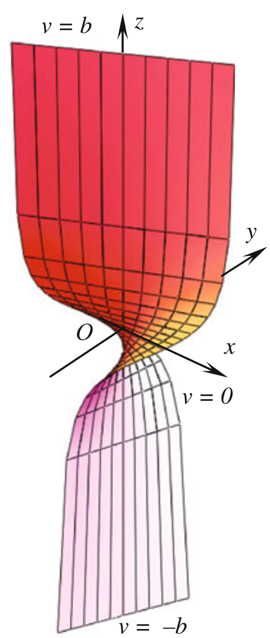


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A = 1; F = 0;$$
$$B^2 = u^2 + 4a^2/\cos^4 2v;$$
$$\sqrt{A^2B^2 - F^2} = B;$$
$$L = 0;\ M = -\frac{2a}{B\cos^2 2v};$$
$$N = \frac{8au\sin 2v}{B\cos^3 2v};\ k_u = 0;$$
$$k_v = \frac{8au\sin 2v}{B^3\cos^3 2v};$$
$$K = -\frac{4a^2\cos^4 2v}{(u^2\cos^4 2v + 4a^2)^2} < 0.$$

(2) An implicit equation: $2axy = z(x^2 - y^2)$.

A Zindler's conoid is an algebraic surface of the third order.

The Zindler conoid has many properties which are similar to those of the Plucker conoid. This comes from the fact that the complex extensions of the two surfaces are the same.

Additional Literature

Pottmann H. Wallner J. Computational Line Geometry. Springer. 2010; 562 p.

■ Continuous Topographic Ruled Surface with Distributing Ellipse

Let us distribute a single-parametrical family of curves n_i of the same type along a given distributing curve $z = f_2(x)$. Assume that equations of projections of a single parametrical family of curves n_i of the same type on a coordinate plane yOz can be written in the form: $z = k_i f_1(y)$. Thus, an explicit equation of a continuous topographic surface can be presented as $z = k_i f_2(x) f_1(y)$.

A continuous topographic ruled surface with distributing ellipse has an ellipse

$$z = f_2(x) = b\sqrt{a^2 - (x-c)^2}/a,$$

as the distributing curve and a pencil of planes

$$z = k_i f_1(y) = (z_{el}/y_0)(y_0 - y)$$

as a single-parametrical family of curves n_i of the same type. All geometrical parameters are shown in Fig. 1.

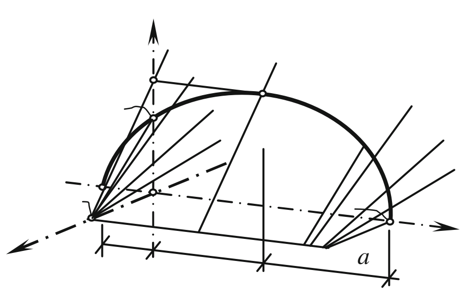


Fig. 1

The studied surface has an explicit equation:

$$z = \frac{b}{a}\left(\frac{y_0 - y}{y_0}\right)\sqrt{a^2 - (x-c)^2}$$

where $0 \le x \le a + c$; $-\infty \le y \le \infty$. The surface with geometrical parameters $a = 2$ m; $b = 1$ m; $c = 1.3$ m; $y_0 = 2$ m; $0 \le x \le 3.3$ m; $0 \le y \le y_0$ is presented in Fig. 2. The distributing ellipse $z^2 = b^2\left[a^2 - (x-c)^2\right]/a^2$ lies at the cross-section of the surface by the plane $y = 0$. In the cross-section of the surface by the plane $x = 0$, a generatrix curve

$$z = \frac{b}{a}\sqrt{a^2 - c^2}\left(\frac{y_0 - y}{y_0}\right) = z_0\left(\frac{y_0 - y}{y_0}\right)$$

lies. In the cross-section of the surface by the planes $y = y_c$, the ellipses

$$\frac{(x-c)^2}{a^2} + \frac{z^2 y_0^2}{b^2(y_0 - y_c)^2} = 1$$

are placed. The straight lines

$$z = (b/a)[(y_0 - y)/y_0]\sqrt{a^2 - (x_c - c)^2},$$

in the parallel cross-sections of the surface by the planes $x = x_c$ pass through the straight line $z = 0$; $y = y_0$ lying at the plane xOy. So, the studied surface is *a conoid* and belongs to

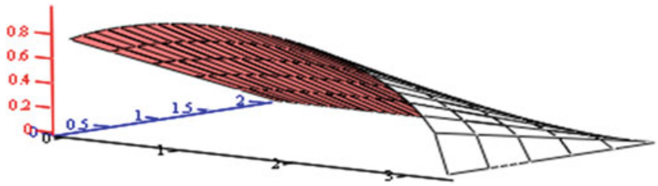


Fig. 2

a family of *Catalan surfaces*. The studied ruled surface is an algebraic surface of the fourth order with the plane yOz as a plane of parallelism.

Additional Literature

Kirillov SV. On one method of design of continuous topographic surfaces. Kibernetika grafiki i prikl. geom. poverhnostey. Moscow: MAI, 1972; Vol. IX, Iss. 243, p. 69-75 (3 ref.).

■ Cylindroids

A cylindroid is a ruled surface formed by the movement of rectilinear generatrix along two curvilinear directrixes and in all positions, the generating straight line is parallel to any *plane of parallelism*. Cylindroids are surfaces of negative Gaussian curvature ($K < 0$) and that is why, they cannot be developed on a plane without lap folds or rupture. They belong to a family of *Catalan surfaces*. Cylindroids with the exception of *a right helicoid* (*a helical cylindroid*) cannot have *a constant mean curvature* ($H \neq$ const). Cylindrical surfaces ($K = 0$) may become cylindroids in certain cases. For example, *a cylindroid with two directrix circumferences lying in mutually perpendicular planes* may be called simultaneously a cylindroid and a cylinder.

A cylindroid having one of two directrix curves in the form of a straight line is called *a conoid*. So, a conoid is a particular case of a cylindroid.

For example, assume two parabolas

$$x = 0,\ z = c(y^2 - b^2)/b^2 \text{ and } x = l,\ z = (y^2 - b^2)f/b^2,$$

in the capacity of the directrix curves, where c, f are the distances the peaks of the parabolas from the plane $z = 0$ (rises), and a plane $y = 0$ as a plane of parallelism, then *a parabolic conoid* will be.

A helical cylindroid is a surface formed by a straight line moving in the space parallel to a plane of parallelism and all time, intersecting with a helix and touching with surface of a right circular cylinder. An axis of the helix coincides with an axis of the cylinder. The ruling and the axis are *skew lines*. So, the plane of parallelism is perpendicular to the axis.

If two directrix curves

$$\boldsymbol{r}_1 = \boldsymbol{r}_1(u) = x_1(u)\boldsymbol{i} + y_1(u)\boldsymbol{j} + z_1(u)\boldsymbol{k} \text{ and}$$
$$\boldsymbol{r}_2 = \boldsymbol{r}_2(v) = x_2(v)\boldsymbol{i} + y_2(v)\boldsymbol{j} + z_2(v)\boldsymbol{k},$$

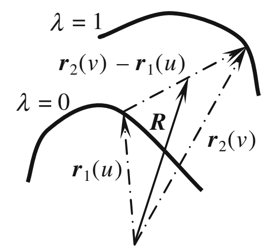


Fig. 1

are given, then a vector equation of a cylindroid designed on these two directrix curves can be written in the following form (Fig. 1):

$$\boldsymbol{R} = \boldsymbol{R}(u, \lambda) = \boldsymbol{r}_1(u) - \lambda[\boldsymbol{r}_1(u) - \boldsymbol{r}_2(v = f(u))]$$

where $0 \leq \lambda \leq 1$. Having chosen a plane of parallelism, we must find the dependence $v = f(u)$. For example, if we choose a coordinate plane xOy as the plane of parallelism, then the condition $z_1(u) - z_2(v) = 0$ must be fulfilled and we have an opportunity to obtain the dependence $v = f(u)$.

If we choose a coordinate plane yOz ($x = 0$) as the plane of parallelism then the condition $x_1(u) - x_2(v) = 0$ must be fulfilled and after, one can obtain the dependence $v = f(u)$.

An approximate development of a cylindroid can be constructed with the help of a method of triangulation. An approximate development will be bounded by a closed broken line and divided into triangles. Segments of broken line will be the sides of the triangles and the ends of these segments will be vertexes of not more than four triangles (*ND Haustova*).

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■ Cylindroid with Two Directrix Ellipses

Assume two ellipses

$$\mathrm{I}:\ x = a,\ \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } \mathrm{II}:\ x = 0,\ \frac{y^2}{d^2} + \frac{z^2}{c^2} = 1$$

in the capacity of the directrix curves of a cylindroid. Let a coordinate plane xOy is a plane of parallelism.

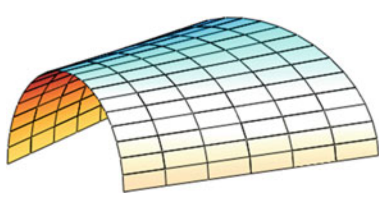

Fig. 1

Forms of the definition of the surface

(1) Parametrical equations (Fig. 1):

$$\begin{aligned} x &= x(\lambda) = a - a\lambda, \\ y &= y(u, \lambda) = [b - \lambda(b - d)]\cos u, \\ z &= z(u) = c \sin u \end{aligned}$$

where $0 \le \lambda \le 1$, $0 \le u \le 2\pi$.

In this case, ellipses are given by vector equations:

$$\mathrm{I}:\ \boldsymbol{r}_1 = \boldsymbol{r}_1(u) = (a;\ b\cos u;\ c\sin u) \ \text{ and } \ \mathrm{II}:\ \boldsymbol{r}_2 = \boldsymbol{r}_2(v) = (0;\ d\cos v;\ c\sin v).$$

(2) Implicit equations:

$$a^2c^2y^2 = \left(c^2 - z^2\right)[ab + (x - a)(b - d)]^2.$$

So, *a cylindroid with two directrix ellipses* is the fourth-order algebraic surface.

■ Cylindroid with Two Directrix Circles Lying in Mutually Perpendicular Planes

Having assumed two circumferences with the same radiuses

$$y = 0,\ (x - a)^2 + z^2 = a^2 \ \text{ and } \ (y - a)^2 + z^2 = a^2,\ x = 0$$

as directrix curves and a coordinate plane xOy in the capacity of the plane of parallelism, we shall construct *a cylindroid with two directrix circles lying in mutually perpendicular planes*.

The circumferences with a radius equal to a at the beginning of the system of coordinate have the common tangent line.

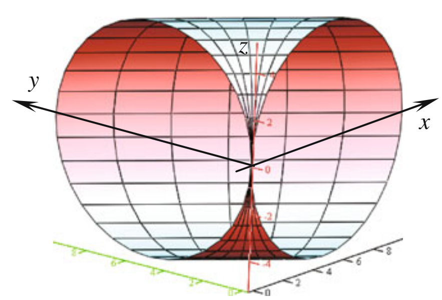

Fig. 1

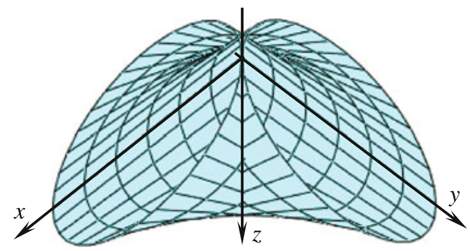

Fig. 2

Forms of the definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, \lambda) = a(1 + \cos u)(1 - \lambda),$$
$$y = y(u, \lambda) = a\lambda(1 + \cos u),$$
$$z = z(u) = a \sin u$$

where $0 \le \lambda \le 1$; $0 \le u \le 2\pi$. In this case, the circumferences are given by equations:

$$\mathrm{I}: \boldsymbol{r}_1 = \boldsymbol{r}_1(u) = [a(1 + \cos u);\ 0;\ a \sin u] \text{ and}$$
$$\mathrm{II}: \boldsymbol{r}_2 = \boldsymbol{r}_2(v) = [0;\ a(1 + \cos v);\ a \sin v].$$

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2[2\lambda(\lambda - 1)\sin^2 u + 1],$$
$$F = a^2(1 - 2\lambda)(1 + \cos u)\sin u,$$
$$B = a(1 + \cos u)\sqrt{2},$$
$$L = a/\sqrt{2 + (4\lambda - 5)\sin^2 u},$$
$$M = N = 0,\ K = 0.$$

So, a cylindroid with two directrix circles lying in mutually perpendicular planes is *an elliptical cylindrical surface* with $K = 0$.

(2) A cylindroid with the same circumferences can be also given as (Fig. 2):

$$x = x(u, \lambda) = a(1 - \lambda)(1 + \cos u),$$
$$y = y(u, \lambda) = a\lambda(1 - \cos u),$$
$$z = z(u) = a \sin u,$$

where $0 \le \lambda \le 1$; $0 \le u \le 2\pi$.

The cylindroid has a negative total curvature, i.e., $K < 0$.

Additional Literature

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■ Ball's Cylindroid

Ball's cylindroid is of great importance in the screw theory being a trajectory of movement of momentary position of helical axes of two bodies when mutual movement. In 1875, R.S. Ball has constructed a model of this cylindroid from steel wires and this model he placed on the box-tree cylinder as it is shown in Fig. 1.

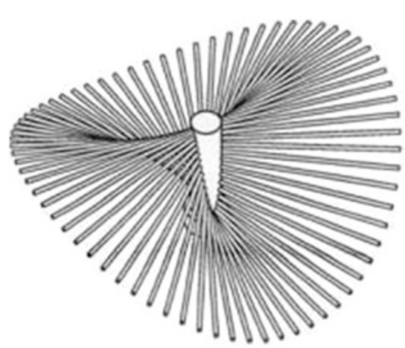

Fig. 1

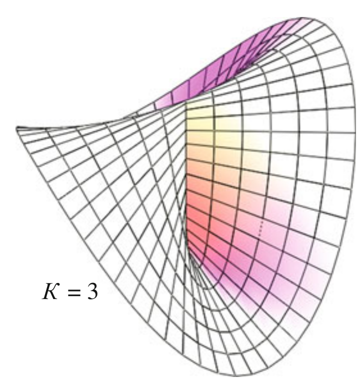

Fig. 2

Forms of definition of the surface

(1) Explicit equation:

$$z = Kxy/\left(x^2 + y^2\right)$$

where K is a constant depending on the parameters of two screws $(p_2 - p_1)$, on the shortest distance between these screws and on the angle between the axes of the screws.

(2) Parametrical equations (Fig. 2):

$$x = x(r,\theta) = r\cos\theta;$$
$$y = y(r,\theta) = r\sin\theta;$$
$$z(r,\theta) = K\sin\theta\cos\theta.$$

A Ball's cylindroid becomes *a Plucker conoid* if $K = 2$.

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■ Cylindroid with a Parabola and a Sinusoid Lying on the Parallel Ends

This conoid contains a parabola $z = b - ay^2$, $x = 0$ lying in a plane yOz and a sinusoid $z = d + c\cos[n\pi y/(2b/a)]$, $x = l$ as directrix curves.

Parametrical equations of *a cylindroid with a parabola and a sine curve placed in the parallel ends* are written in the following form:

$$x = x(\lambda) = \lambda l, \quad y = y(u) = u,$$
$$z = z(u,\lambda) = (1-\lambda)(b - au^2) + \left[d + c\cos\frac{n\pi u}{2(b/a)}\right]\lambda$$

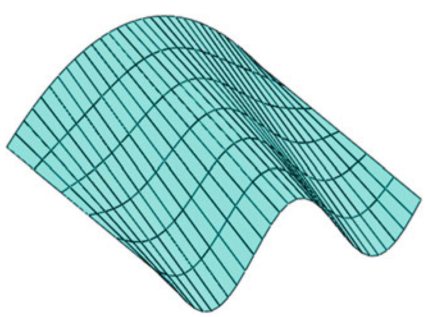

Fig. 1

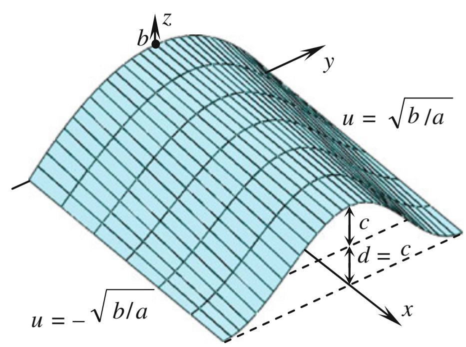

Fig. 2

where $0 \le \lambda \le 1$; $-\sqrt{b/a} \le u \le \sqrt{b/a}$; d is the distance the sine axis from the plane xOy; c is the amplitude of directrix sinusoid lying in the plane $x = l$; b is a rise of the directrix parabola; n is a number of whole half-waves on the segment $-\sqrt{b/a} \le y \le \sqrt{b/a}$; $2\sqrt{b/a} \times l$ are the dimensions of the rectangular plan covered by the cylindroid. The rectilinear generatrixes $u = -\sqrt{b/a}$ and $u = \sqrt{b/a}$ are placed at the coordinate plane xOy. The coordinate plane xOz is taken as *a plane of parallelism*. The cylindroid with $n = 3$, $b = 2$ m, $d = 0$, $c = 1$ m, $a = (2/9)$ m^{-1}, $l = 6$ m is presented in Fig. 1. The cylindroid shown in Fig. 2 has $n = 2$, $b = 2$ m, $c = d = 1$ m, $a = (2/9)$ m^{-1}, $l = 6$ m.

■ Frezier's Cylindroid

Frezier's cylindroid has two semi-circumferences k_1 and k_2:

$$k_1 : \boldsymbol{r}_1(x) = \left(x,\ 0,\ \sqrt{r^2 - (x-p)^2}\right),$$
$$p - r \le x \le p + r,\ \ p - r > 0;$$
$$k_2 : \boldsymbol{r}_2(y) = \left(0,\ y,\ \sqrt{r^2 - (y-p)^2} + q\right),$$
$$p - r \le y \le p + r,\ \ p - r <; 0;\ \ q \ne 0$$

with the same radiuses r. These circumferences lie in mutually perpendicular planes (Fig. 1). A plane $x + y = 0$ is *a plane of parallelism.*

Parametrical equations of a Frezier's cylindroid are

$$x = x(u,t) = t(1+u)/2;$$
$$y = y(u,t) = t(1-u)/2;$$
$$z = z(u,t) = \sqrt{r^2 - (t-p)^2} + q(1-u)/2,$$

where $p - r \le t \le p + r$; $-1 \le u \le 1$. Coordinate lines t = const are the rulings of the cylindroid.

A cross-section of the cylindroid by the plane $x = 0$ coincides with a coordinate line $u = -1$ coinciding with the circle k_2. A cross-section of the cylindroid by the plane $y = 0$ coincides with a coordinate line $u = 1$ coinciding with the circle k_1.

Coefficients of the fundamental forms of the surface:

$$A^2 = t^2/2 + q^2/4;$$
$$F = \frac{ut}{2} + \frac{q(t-p)}{2\sqrt{r^2 - (t-p)^2}},$$
$$B^2 = \frac{1+u^2}{2} + \frac{(t-p)^2}{r^2 - (t-p)^2};$$
$$L = 0;\ \ M = \frac{q}{4\sqrt{A^2B^2 - F^2}};$$
$$N = \frac{-tr^2}{2\sqrt{A^2B^2 - F^2}\left[r^2 - (t-p)^2\right]^{3/2}}.$$

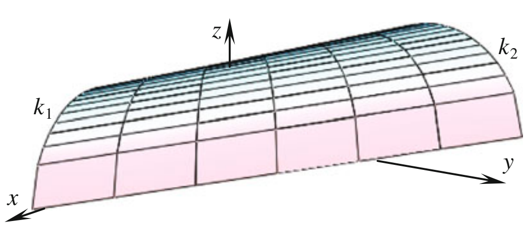

Fig. 1

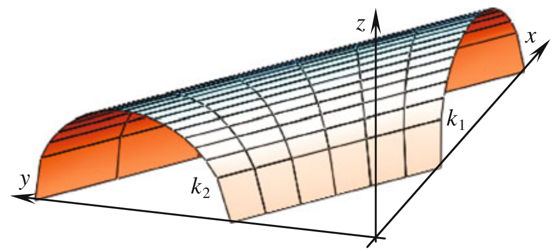

Fig. 2

Assume $q = 0$, then parametrical equations of the cylindroid will become:

$$x = x(u,t) = t(1+u)/2;$$
$$y = y(u,t) = t(1-u)/2;$$
$$z = z(u,t) = \sqrt{r^2 - (t-p)^2}$$

and a Frezier's cylindroid degenerates into a cylindrical surface (Fig. 2).

Assume $q = 0$ and $p = r$, then parametrical equations of the cylindroid give:

$$x = x(u,t) = t(1+u)/2;$$
$$y = y(u,t) = t(1-u)/2;.$$
$$z = z(u,t) = \sqrt{2tr - t^2}$$

As a result, we have *an elliptical cylindrical surface* with the directrix circumferences of radius r lying in the perpendicular planes and having the common tangent line coinciding with the coordinate axis z. This surface is studied in "*A cylindroid with two directrix circles lying in mutually perpendicular planes.*" Its implicit equation is $z^2 + (x+y)^2 - 2r(x+y) = 0$.

Additional Literature

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1.2.2 Twice Oblique Cylindroids

A twice oblique cylindroid is a ruled surface formed with the help of three directrixes, two of them are curves but the third one is a straight line (Fig. 1).

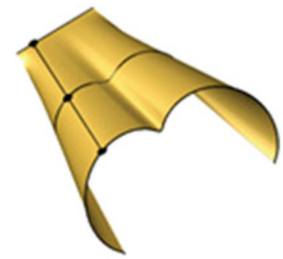

Fig. 1

Additional Literature

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■ Twice Oblique Trochoid Cylindroid

A twice oblique trochoid cylindroid is formed by rotation of a rectilinear generatrix l around the axis J with the simultaneous translational movement of this axis along a straight line (an axis Ox). The ruling l intersects the axis of rotation J at the point S (Fig. 1) under the angle α. The axis of rotation J is placed in the coordinate plane xOz and, in general case, it is not perpendicular to the line of translational movement coinciding with an axis Ox.

A twice oblique trochoid cylindroid is formed in the process of cutting of gear wheels with an arch meshing.

A twice oblique trochoid cylindroid can be given by the following parametrical equations (Fig. 1):

$$
\begin{aligned}
x &= x(u,v) = cu + v(\sin\varphi + tg\alpha\cos\varphi\cos\theta),\\
y &= y(u,v) = vtg\alpha\sin\theta,\\
z &= z(u,v) = H + v(tg\alpha\sin\varphi\cos\theta - \cos\varphi)
\end{aligned}
$$

where $\theta = \theta(u) = pu + \theta_0$, $p = k\pi$; k is a integer; $-1 \le u \le 1$; a parameter θ_0 defines a initial position of the ruling. The translational movement of the axis of the rotation is characterized by a parameter c.

Coefficients of the fundamental forms of the surface:

$$
\begin{aligned}
A^2 &= c\left[1 - \cos^2\varphi\sin^2\theta + \left(\frac{vbp}{c} - \cos\varphi\sin\theta\right)^2\right]\\
&= c[1 - 2pbv\cos\varphi\sin\theta] + v^2b^2p^2;\\
F &= c(b\cos\varphi\cos\theta + \sin\varphi),\\
B^2 &= 1 + b^2 = \frac{1}{\cos^2\alpha}, \quad \text{where} \quad b = tg\alpha,
\end{aligned}
$$

$$
A^2B^2 - F^2 = c^2\left[\left(\frac{vbp}{c} - \cos\varphi\sin\theta\right)\frac{1}{\cos^2\alpha} + (\cos\varphi\cos\theta - b\sin\varphi)^2\right],
$$

$$
L = \frac{bp^2v}{\sqrt{A^2B^2 - F^2}}(vpb - c\cos\varphi\sin\theta),
$$

$$
M = \frac{bpc}{\sqrt{A^2B^2 - F^2}}(\cos\varphi\cos\theta - b\sin\varphi), \quad N = 0,
$$

$$
K = -\frac{(bpc)^2}{(A^2B^2 - F^2)^2}(\cos\varphi\cos\theta - b\sin\varphi)^2.
$$

A twice oblique trochoid cylindroid is a surface of negative Gaussian curvature. Coordinate lines v coincide with rectilinear generatrixes of the cylindroid. The cylindroids with different geometrical parameters are shown in Figs. 2 and 3.

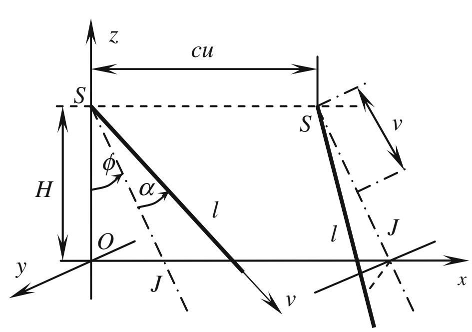
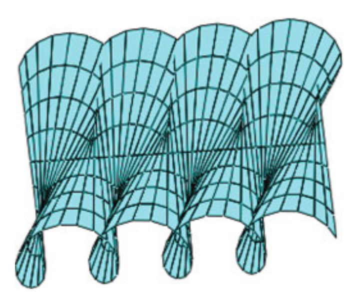
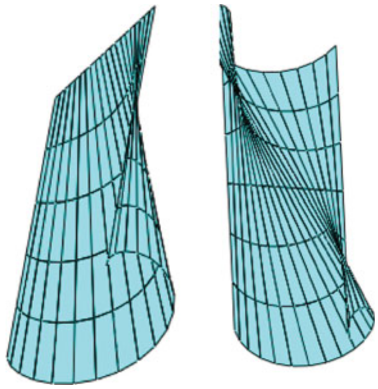


Fig. 1



$H = 1;\ c = 5;\ \alpha = 0.033\pi;\ p = 8\pi;$
$0 \le u \le 1;\ \varphi = 0; -5 \le v \le 5$

Fig. 2



$H = 1$; $c = 2$; $\alpha = 0.05\pi$; $0 \le u \le 1$; $p = 2\pi$:
$\varphi = 0.3\pi$; $0 \le v \le 4$; $\varphi = 0$; $-4 \le v \le 4$

Fig. 3

Additional Literature

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2 Surfaces of Revolution

A surface of revolution is generated by rotation of a plane curve $z = f(x)$ about an axis Oz called *the axis of the surface of revolution*. The resulting surface therefore always has *azimuthal symmetry*. Hence, an explicit equation of a surface of revolution can be presented in the following form:

$$z = f(r) = f(\sqrt{x^2 + y^2}),$$

where $r = \sqrt{x^2 + y^2}$ is the distance a point of the surface from the axis of rotation. Right cylindrical and conical surfaces are examples of surfaces generated by a straight line when the line is coplanar with the axis, as well as hyperboloids of one sheet when the line is skew to the axis. *A sphere* is a surface of revolution of a circle around an axis which runs through the center of the circle. If the circle is rotated about *a coplanar axis*, not crossing the circle, then it generates *a torus*.

Meridians are the lines of intersections of a surface of revolution with planes passing through an axis of rotation. All meridians of one surface of revolution are congruent to the rotated curve. A plane passing through the axis of the surface of revolution is called *the meridian plane*. It is *the plane of symmetry* of the surface. Any surface of revolution has the infinite number of planes of symmetry. *Parallels* are the lines of intersection of the surface with planes orthogonal to an axis of rotation. Meridians and parallels of a surface of revolution are the lines of principal curvatures. Any normal of surfaces of revolution intersects its axis of rotation. A surface of revolution having more than one axis of rotation is *a sphere* or *a plane*.

Tangents to all meridians in the points located on one parallel circle are lines on *the tangent conical* (or *cylindrical*) *surface of revolution*, which is created by the revolution of the tangent about the axis of the rotation. A vertex of the tangent conical surface is located on the axis of revolution.

A parallel is called *the neck circle*, if tangent planes to the surface of revolution in the points on this circle are parallel to the axis of revolution and the tangent cylindrical surface is located inside the surface of revolution. A parallel is called *the equator circle*, if tangent planes to the surface of revolution in the points on this circle are parallel to the axis of revolution and the tangent cylindrical surface is located outside the surface of revolution. A parallel is called *the crater circle*, if tangent plane to the surface of revolution in the points on this circle is perpendicular to the axis of revolution and normal to the surface of revolution in the points of this parallel are parallel to the axis of revolution and form the normal cylindrical surface.

Umbilical points of a surface of revolution are placed on those latitudes on which *a center of curvature* of a meridian is located on the axis of rotation. *Sphere* is umbilical surface. Under Alexis-Claude Clairaut theorem, the product of a radius of a parallel into cosines of an angle of intersection of the geodesic line with the parallel is constant along the geodesic line.

A surface of revolution admits *bending* into another surface of revolution and a net of lines of principal curvatures is remained.

Parametrical equations of arbitrary surface of revolution are

$$\boldsymbol{r} = \boldsymbol{r}(r, \beta) = r\cos\beta\boldsymbol{i} + r\sin\beta\boldsymbol{j} + f(r)\boldsymbol{k}.$$

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_2,

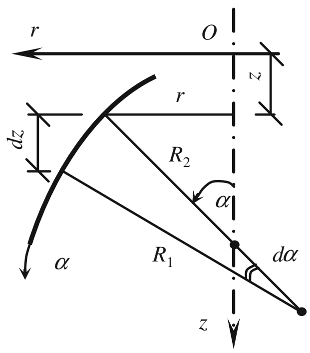


Fig. 1

Assume an equation of a meridian in the form $r = r(\alpha)$ where α is the angle of the normal to the surface passing through a given point with the axis of rotation (Fig. 1) then $r = R_2 \sin\alpha$. Coefficients of the fundamental forms of the surface of revolution can be obtained with the help of formulas:

$$A = A(\alpha) \ = R_1\alpha,\ B = B(\alpha) \ = r = R_2 \sin\alpha,\ F = \ 0;$$
$$L = R_1(\alpha),\ M = \ 0,\ N = R_2 \sin\alpha,$$

where R_1 is the principal radius of curvature of the meridian that is the coordinate line of α, R_2 is the principal radius of curvature of the parallel. The lines α = const are parallels and the lines β = const are meridians.

If an equation of a meridian is given in the form $r = r(z)$ (Fig. 1) then an equation of a surface of revolution can be written with the help of three scalar equations:

$$x = r\sin\beta, \quad y = r\cos\beta, \quad z = z$$

where $r = r(z)$ is a function that determines the shape of the meridian (*a profile curve*); β is the angle of rotation of the plane of the meridian and then

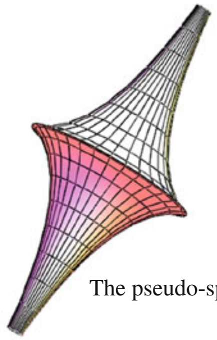
$$A = \sqrt{1 + r'^2}, \quad F = 0, \quad B = r(z),$$
$$k_1 = \frac{1}{R_1} = -\frac{r''}{\left(1 + r'^2\right)^{3/2}}, \quad k_2 = \frac{1}{R_2} = \frac{1}{r\sqrt{1 + r'^2}},$$

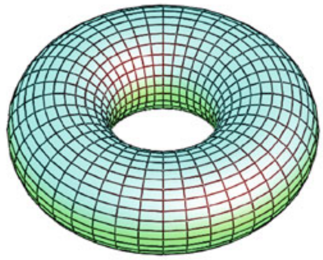
where the derivatives with respect to z are denoted by primes; k_1, k_2 are principal curvatures of the surface. A normal curvature of a surface in the direction of the meridian is equal to a curvature of the meridian, i.e., k_1. Meridians of surface of revolution are geodesic lines.

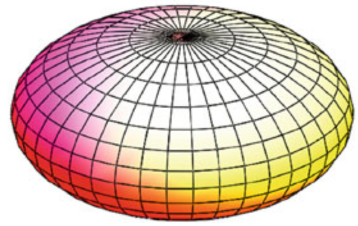
Catenoid is the only one *minimal surface of revolution*. *One-sheet hyperboloid of revolution, right circular cylinder* and *right circular cone* are *the only ruled surfaces*. The last two surfaces are the only developable surface of revolution. If a beginning and an end of unclosed rotated line are placed on an axis of rotation then the surface of revolution will be the closed one.

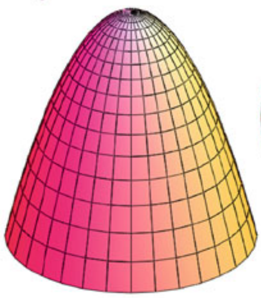
A great deal of surfaces of revolution exists and is studied in different scientific publications. Tens of surfaces of revolution are presented in this encyclopedia and shown on pages 101–104. Such surfaces of revolution as “Lochdiskus”, “Jet Surface”, “Apple Surface”, “Kidney Surface”, “Fish Surface”, “Limpet Torus”, Darwin-de Sitter spheroid, and others are known but used less and may be found in other original sources.

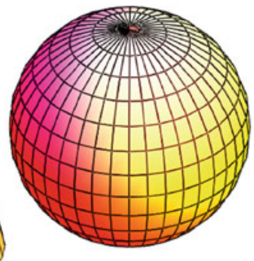
■ Surfaces of Revolution Presented in the Encyclopedia

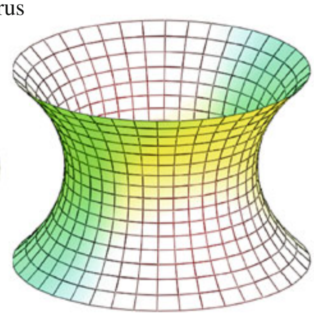

The pseudo-sphere

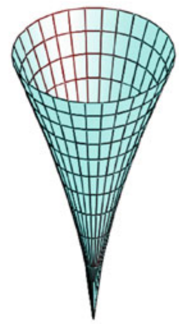

The circular torus

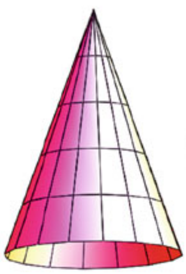

The ellipsoid of revolution

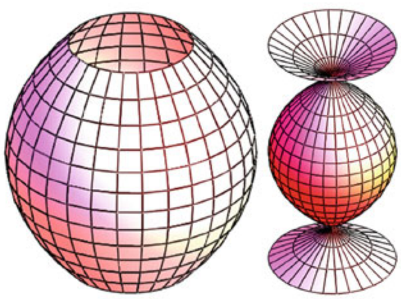

Paraboloid of revolution

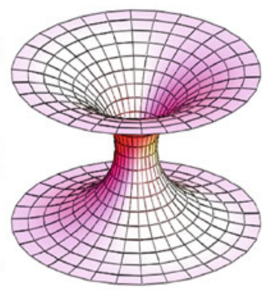

The spherical surface (sphere)

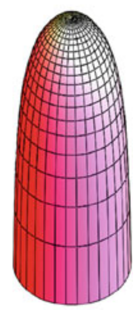

The catenoid

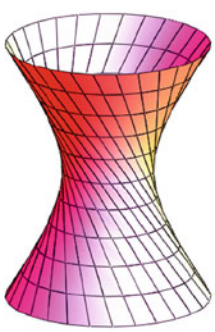

The surface of rotation of a Neil's parabola

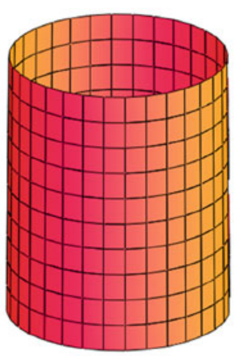

Conical surface of revolution

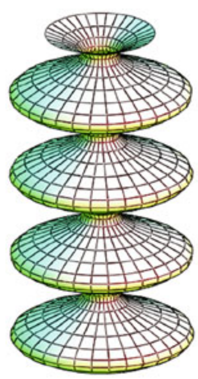

Surface of revolution of a parabola

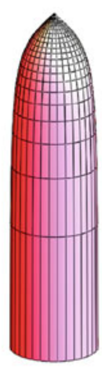

Globoid (toroid)

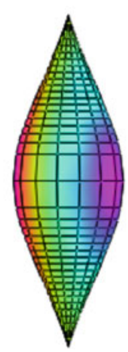

The Kappa surface

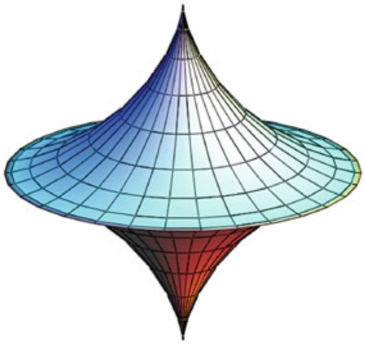

The one-sheet hyperboloid of revolution


The cylindrical surface of revolution

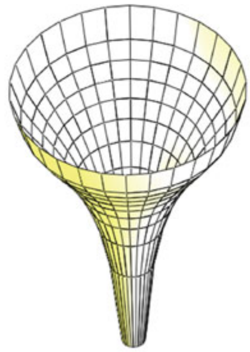

Corrugated surface of revolution of a common sinusoid


The bullet nose

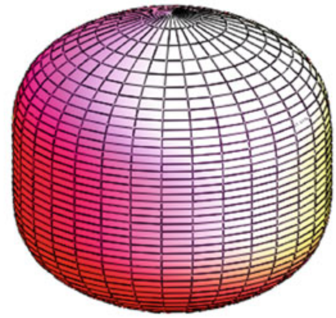

The surface of revolution of a Agnesi curl



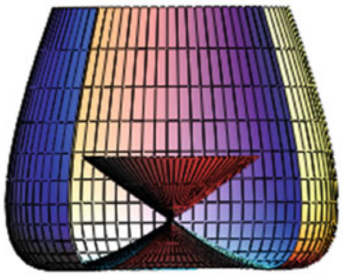
The surface of revolution of a astroid



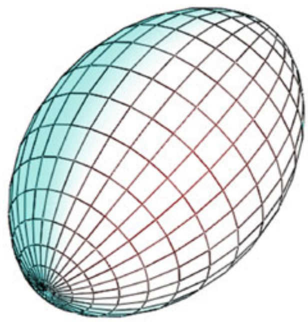
The surface of revolution of the Agnesi curl



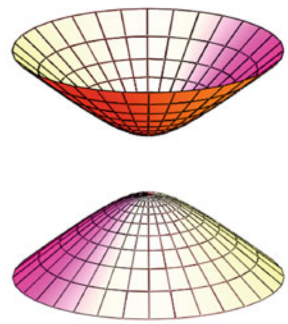
Surface of revolution of the biquadrate parabola



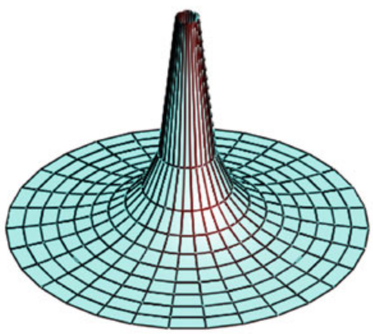
Surface of revolution of the parabola of arbitrary position



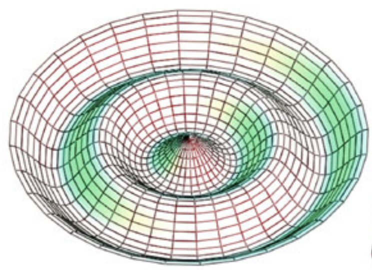
The surface of revolution of a cycloid



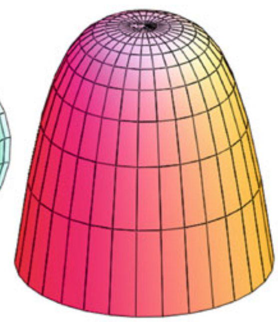
Two-sheeted hyperboloid of revolution



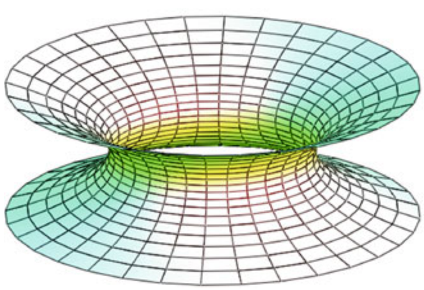
The surface of revolution of a hyperbola $z = b/x$ around the Oz axis



The surface of revolution of a sinusoid

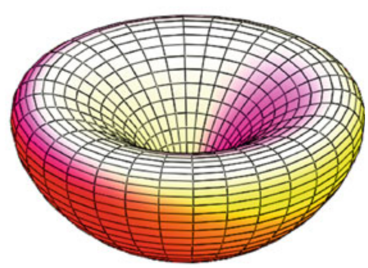


The fourth order paraboloid of revolution

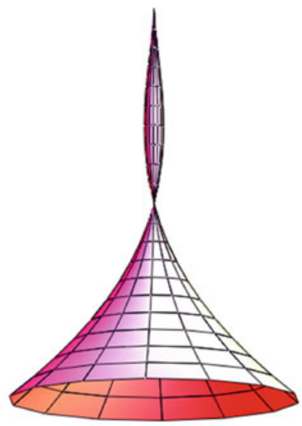


The pseudo-catenoid

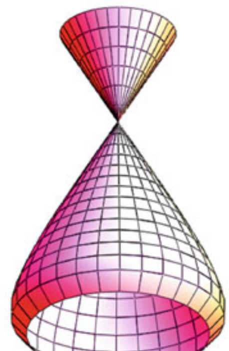
"Penka"



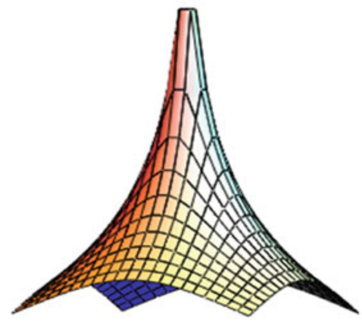
The elliptic torus



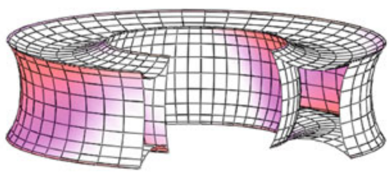
The hyperbolic-and-logarithmic surface of revolution



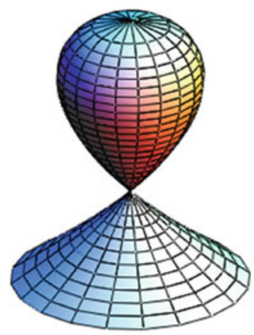
The parabolic-and-logarithmic surface of revolution



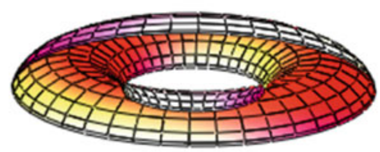
Surface of revolution given by a harmonic function $z = \ln[x^2 + y^2]^{1/2}$



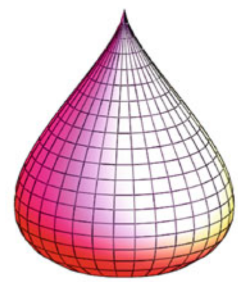
The astroidal torus



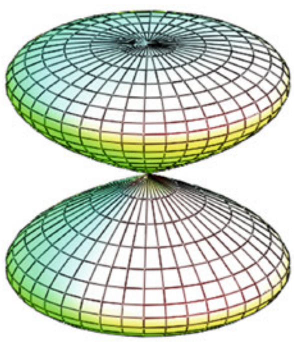
The Ding–Dong surface



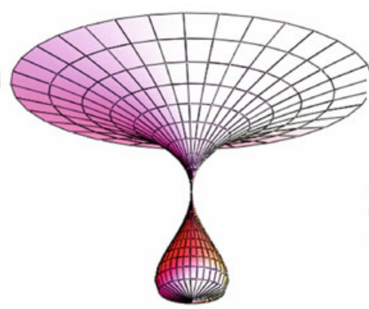
The cycloidal torus



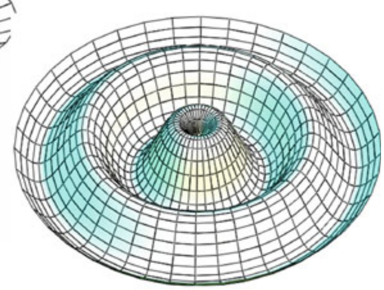
Surface of revolution «Pear»



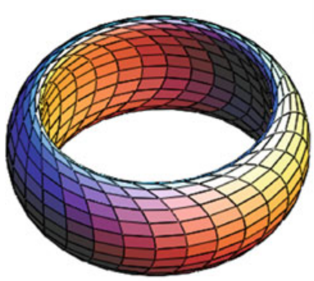
“Eight surface”



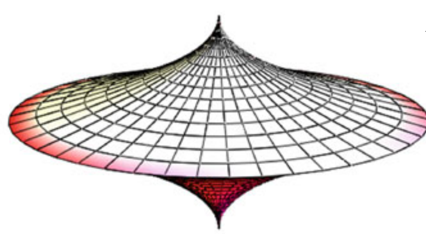
“Kiss surface”



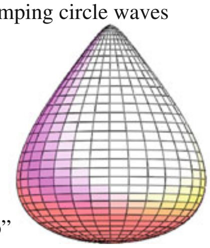
The surface of revolution with damping circle waves



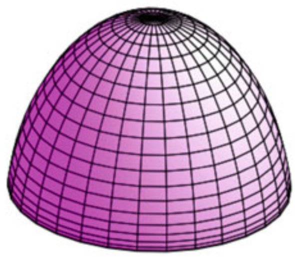
The cyclic surface of revolution “Wedding-ring”



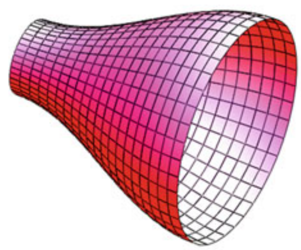
The parabolic humming-top



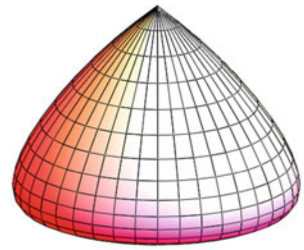
“Drop”



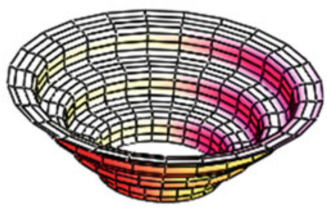
Fairing of cycloidal type



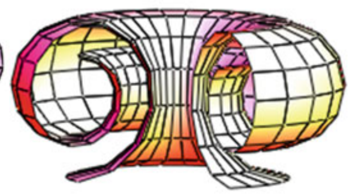
The surface of conjugation of two coaxial cylinders of different diameters



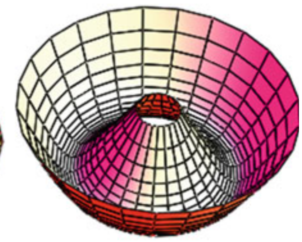
The deformed sphere



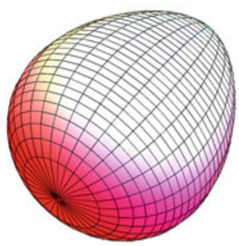
Surface of revolution of the inclined sinusoid



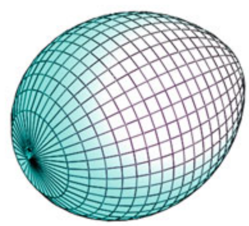
Surface of revolution of the evolvent of the circle



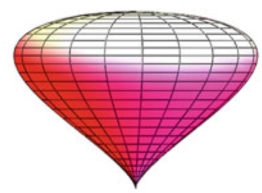
Surface of revolution of the hyperbola of arbitrary position



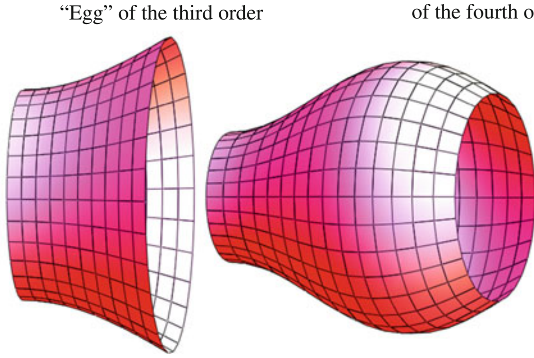
Surface of revolution "Egg" of the third order



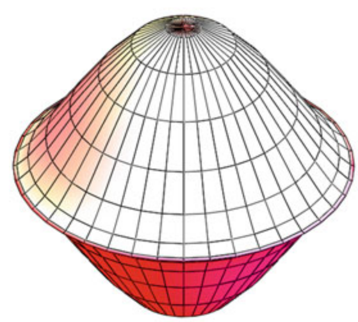
Surface of revolution "Egg" of the fourth order



The Piriform Surface



The surface of conjugation of the coaxial cylinder and the cone



Soucoupoid

Additional Sources

Parametrische Flächen und Körper. http://www.3d-meier.de/tut3/

http://www.wolframalpha.com/input/?i=surface+of+revolution (2014).

■ One-Sheet Hyperboloid of Revolution

One-sheet hyperboloid of revolution is generated by the rotation of a hyperbola

$$x^2/a^2 - z^2/c^2 = 1$$

about the Oz axis (Fig. 1). These are *twice ruled surfaces*. Through every point of the surface, two straight lines, lying on the hyperboloid, pass (Fig. 2). A hyperboloid can be constructed by rotation of a generatrix straight line about the Oz axis but the straight generatrix and the axis are skew lines (Figs. 3 and 4). The surface is the only one *ruled surface of revolution of negative Gaussian curvature*. The parallel lying in a plane $z = 0$ has a radius $r = a$ and is called *a waist circumference* that represents *a geodesic line*. All of the rest of the geodesic lines besides the equator go from infinity coming nearer to the equator. One of them intersects the equator and goes to other half of the surface but others do not reach the equator and touching the some parallel, turn back; the third geodesic lines come nearer asymptotically to the equator.

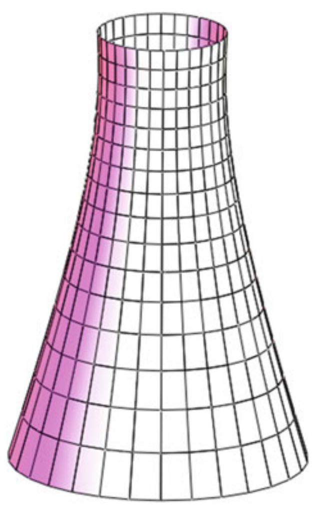


Fig. 1

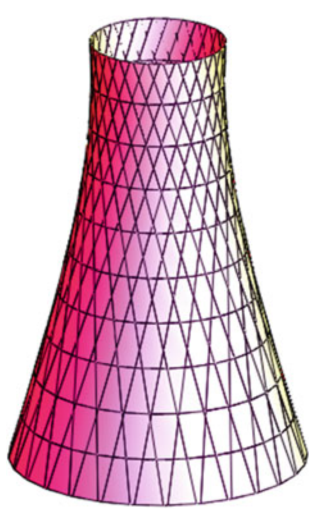


Fig. 2

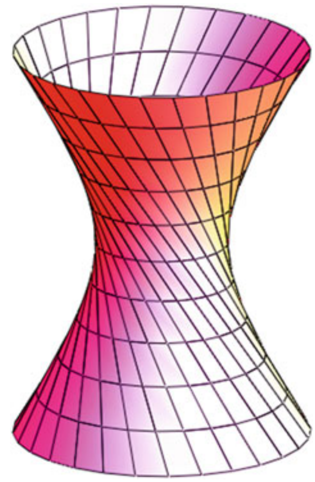


Fig. 3

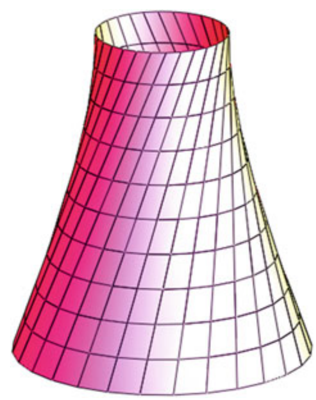


Fig. 4

Forms of definition of one-sheet hyperboloid of revolution

(1) Implicit equation (canonical equation):

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1.$$

If $a = c$, then a hyperboloid is called *a right hyperboloid*.

(2) Parametrical equations (Figs. 3 and 4):

$$x = x(u, v) = -a \sin u \pm av \cos u,$$
$$y = y(u, v) = a \cos u \pm av \sin u,$$
$$z = z(v) = \pm cv.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2(1+v^2),\ B^2 = a^2 + c^2,\ F = \mp a^2,$$
$$L = \mp ca^2(1+v^2)/(A^2B^2 - F^2)^{1/2},$$
$$M = a^2c/(A^2B^2 - F^2)^{1/2},\ N = 0.$$

Coordinate lines v (u = const) coincide with one system of straight lines but the lines u are the parallels of the hyperboloid of one sheet. In Fig. 3, the hyperboloid is shown with taking into consideration the upper signs in the parametrical equations of the surface. The lower signs are taken into account in Fig. 4.

(3) Parametrical equations (Fig. 1):

$$x = x(r,\beta) = r\cos\beta, \quad y = y(r,\beta) = r\sin\beta,$$
$$z = z(r) = c\sqrt{r^2 - a^2}/a.$$

Coordinate lines r and β (parallels and meridians) are the lines of principal curvatures.

(4) Parametrical equations (Fig. 1):

$$x = x(z,\beta) = \frac{a}{c}\sqrt{c^2 + z^2}\sin\beta,$$
$$y = y(z,\beta) = \frac{a}{c}\sqrt{c^2 + z^2}\cos\beta,$$
$$z = z.$$

Coordinate lines z and β (meridians and parallels) are the lines of principal curvatures.

(5) Parametrical equations (Fig. 1):

$$x = x(\beta,\alpha) = a\mathrm{ch}\alpha\cos\beta, \quad y = y(\beta,\alpha) = a\mathrm{ch}\alpha\ \sin\beta,$$
$$\mathrm{z} = \mathrm{z}(v) = c\mathrm{sh}\alpha.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = a\mathrm{ch}\alpha, \quad F = 0, \quad B^2 = a^2\mathrm{sh}^2\alpha + c^2\mathrm{ch}^2\alpha,$$
$$L = -ac\mathrm{ch}^2\alpha/B, \quad M = 0,\ N = ac/B,$$
$$k_1 = -c/(aB), \quad k_2 = ac/B^3.$$

Fig. 5 The planetarium in Saint Louis, USA

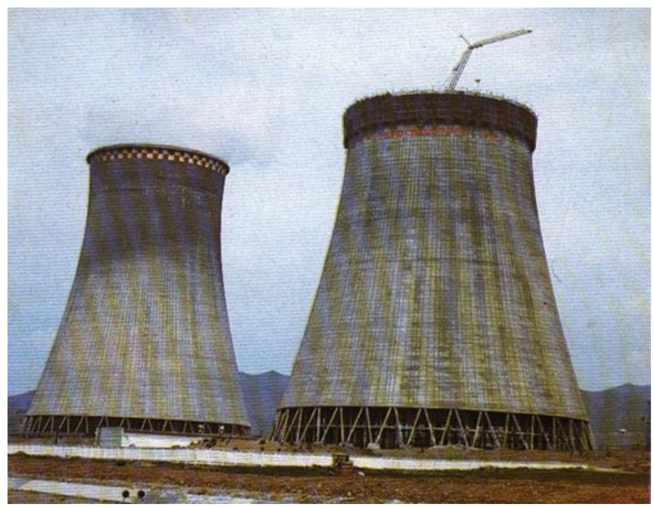


Fig. 6 The Cooling Towers, Uzbekistan

The surface is widely used in civil (Fig. 5) and industrial (Fig. 6) engineering.

Additional Literature

Krivoshapko SN. Static, vibration, and buckling analyses and applications to one-sheet hyperboloidal shells of revolution. Applied Mechanics Reviews. 2002; Vol. 55, No. 3, p. 241-270 (261ref.).

■ Fairing of Cycloidal Type

A surface of a fairing of cycloidal type is formed by the rotation of *a cycloidal curve*

$$x = x(t) = a(t + \sin t), \quad z = z(t) = c(1 + \cos t)$$

about an axis Oz (Fig. 1). If $a = c$, then a generatrix curve becomes a typical cycloid. The form of fairing is defined by a form of meridian that is given with the help of splines. Assume a curve generated by the trajectories of the points of an axis of symmetry of *a limaçon of Pascal* in the process of its rolling along a cycloid as a generatrix curve of a surface of revolution.

Forms of definition of the surface

(1) Parametrical equations (Figs. 1, 2 and 3):

$$x = x(z, \beta) = r(z) \sin \beta,$$
$$y = y(z, \beta) = r(z) \cos \beta,$$
$$z = z,$$

where

$$r = r(z) = a\left[\frac{\sqrt{z(2c - z)}}{c} + \arccos(\frac{z}{c} - 1)\right],$$

β is the angle counted off from the coordinate axis Oy in the direction of the axis Ox; $0 \le \beta \le 2\pi$; $0 \le z \le 2c$. In Fig. 1, it is assumed that $c = 2a$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2 z}{c^2(2c - z)}, \quad F = 0, \ B = r(z),$$
$$k_1 = k_z = -\frac{r''(z)}{A^3} = \frac{az}{A^3(2cz - z^2)^{3/2}},$$
$$M = 0, \ k_2 = k_\beta = \frac{1}{AB}.$$

The contour parallel $z = 0$ is the only geodesic parallel on the surface because the tangent lines to the meridians in its points are parallel to the axis of rotation. Choosing the

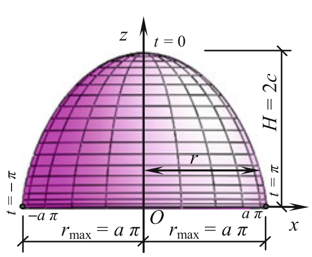


Fig. 1

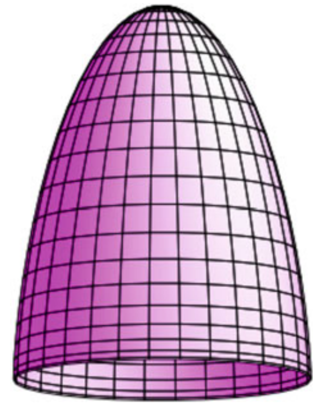


Fig. 2 $c = 4a$

parameters a и c, it is possible to seek necessary characteristics for a fairing. The ratio of maximum height H of the surface to the diameter ($2r_{max} = 2a\pi$) of the geodesic parallel and a radius of curvature of the meridian in the frontal point ($z = 2c$) are the main characteristics of the fairing.

A radius of curvature R of the meridians in the frontal point of the surface is defined by a formula:

$$R = \frac{4a^2}{c}.$$

(2) Parametrical equations (Figs. 1, 2 and 3):

$$x = x(t, \gamma) = a(t + \sin t) \cos \gamma,$$
$$y = y(t, \gamma) = a(t + \sin t) \sin \gamma,$$
$$z = z(t) = c(1 + \cos t),$$

where γ is the angle counted off from the coordinate axis Ox in the direction of the axis Oy; $0 \le \gamma \le 2\pi$; $0 \le t \le \pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = a^2(1 + \cos t)^2 + c^2 \sin^2 t, \quad F = 0, \quad B = a(t + \sin t),$$
$$L = -\frac{ac(1 + \cos t)}{A}, \ M = 0, \ N = -\frac{cB}{A} \sin t,$$
$$k_1 = k_t = -\frac{ac(1 + \cos t)}{A^3}, \ k_2 = k_\gamma = -\frac{c \sin t}{AB}.$$

(3) A particular case of parametrical equations (Fig. 3).

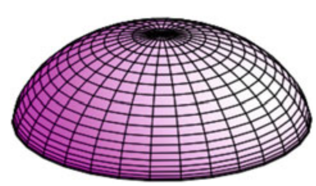


Fig. 3 $c = a$

If one takes $c = a$, then a surface of rotation of a typical cycloid about an axis of Oz will be:

$$x = x(t, \gamma) = a(t + \sin t)\cos\gamma,$$
$$y = y(t, \gamma) = a(t + \sin t)\sin\gamma,$$
$$z = z(t) = a(1 + \cos t).$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 2a^2(1 + \cos t), \quad F = 0, \quad B = a(t + \sin t),$$

$$L = -\frac{A}{2}, \quad M = 0, \ N = -\frac{aB}{A}\sin t,$$

$$k_1 = k_t = -\frac{1}{2A}, k_2 = k_\gamma = -\frac{a\sin t}{AB}, K = \frac{a\sin t}{2A^2B} > 0.$$

References

Krutov AV. On movement defined by centroid-and- trajectory pairs. Izv. vuzov. Mashinostroenie. 2001; No. 2-3, p. 3-6 (11 ref.).

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■ Pseudo-Sphere

Gaussian curvature ($K = k_1k_2$) is equal to a constant negative number, i.e.

$$K = -1/a^2,$$

in all points of *a pseudo-spherical surface* (Figs. 1 and 2). *A pseudo-sphere* or *Beltrami surface* is formed by rotation of a *tractrix* that is *trahere* in Latin, about an axis Oz. A tractrix is an evolvent of the catenary:

$$r = a\operatorname{ch}\frac{z}{a}.$$

Parametrical equations of a tractrix are written as

$$x = a\sin u,$$
$$z = a\left[\cos u + \ln\tan\frac{u}{2}\right],$$

where $0 < u < \pi$, u is the angle of the axis Oz with the tangent to the tractrix.

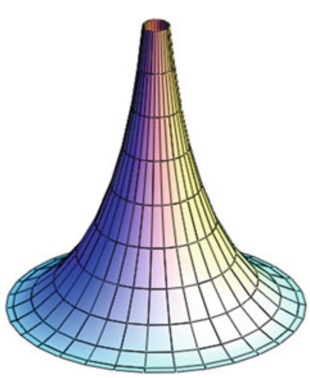


Fig. 1

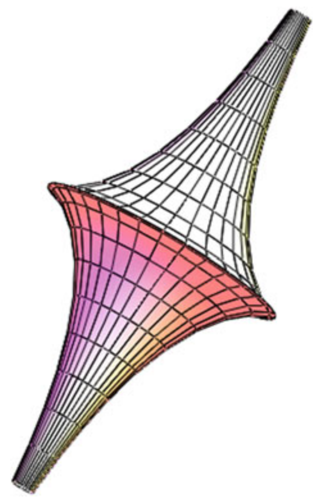


Fig. 2

A tractrix can be defined by an explicit equation:

$$z = a\ln\frac{a \pm \sqrt{a^2 - r^2}}{r} \mp \sqrt{a^2 - r^2},$$

where the upper signs concern the positive branch $z > 0$, lower signs concern the negative branch $z < 0$ (Fig. 2). A length of fragment of the tangent line to the tractrix from the point of tangency till the point of intersection with the Oz axis is constant and equal to $a > 0$. The line of the cross section of a pseudo-sphere by a plane xOy (*an edge of a pseudo-sphere*) is the circle with a radius a, all of the rest of parallels have a less radius r, that is $r < a$.

A volume of one part of a pseudo-sphere is

$$V = \frac{\pi a^3}{3}.$$

The inner geometry of pseudo-sphere coincides locally with the Lobachevski geometry.

Forms of definition of the surface

(1) Parametrical form of definition:

$$x = x(u,v) = a\sin u\cos v,$$
$$y = y(u,v) = a\sin u\sin v,$$
$$z = z(u) = a\left[\cos u + \ln\tan\frac{u}{2}\right],$$

where u is the angle of the axis Oz with the tangent to the meridian. An edge of a pseudo-sphere has $u = \pi/2$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = a\mathrm{co}\tan u, \quad F = 0, \quad B = a\sin u,$$
$$L = -ac\tan u, \quad M = 0, \quad N = a\sin u\cos u,$$
$$k_1 = -\tan u/a, \quad k_2 = \mathrm{co}\,\tan u/a.$$

Meridians u and parallels v except the edge of the pseudo-sphere ($u = \pi/2$) are the lines of principal curvatures.

(2) Parametrical equations:

$$x = x(r,\beta) = r\cos\beta, \quad y = y(r,\beta) = r\sin\beta,$$
$$z = z(r) = a\ln\left[\left(a + \sqrt{a^2 - r^2}\right)/r\right] - \sqrt{a^2 - r^2},$$

where r is the distance an axis of rotation from a corresponding point of the pseudo-sphere ($r < a$), the circumference $r = a$ is the edge of the pseudo-sphere.

An area of the fragment of a pseudo-sphere between the parallels $r = a$ and $r = r_o$ is

$$S = 2\pi a(a - r_o).$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{a}{r}, \; F = 0, \; B = r,$$
$$L = \frac{a}{r\sqrt{a^2 - r^2}}, \; M = 0, \; N = -\frac{r\sqrt{a^2 - r^2}}{a},$$
$$k_1 = \frac{r}{a\sqrt{a^2 - r^2}}, \; k_2 = -\frac{\sqrt{a^2 - r^2}}{ar}.$$

(3) Parametrical equations:

$$x = x(\gamma,t) = \frac{1}{\gamma}\cos at, \quad y = y(\gamma,t) = \frac{1}{\gamma}\sin at,$$
$$z = z(\gamma) = a\ln\left(a\gamma + \sqrt{a^2\gamma^2 - 1}\right) - \sqrt{a^2 - 1/\gamma^2}.$$

Coefficients of the fundamental forms of the surface:

$$A = B = \frac{a}{\gamma}, \; F = 0,$$
$$L = -\frac{a}{\gamma^2\sqrt{a^2\gamma^2 - 1}}, \; M = 0,$$
$$N = \frac{a\sqrt{a^2\gamma^2 - 1}}{\gamma^2},$$
$$K = -1/a^2 = \mathrm{const}.$$

Here, using the substitution $\gamma = 1/r$ and $t = \beta/a$, we reduced a linear element of the surface to *isothermal form* that is when $A = B$.

Additional Literature

Popov AG. Pseudo-spherical surfaces and some problems of mathematical physics. Fundamental and Applied Mathematics. 2005; Vol. 11, No. 1, p. 227-239.

■ Paraboloid of Revolution

A paraboloid of revolution is created by the rotation of a parabola

$$x^2 = 2pz$$

about an axis z (Fig. 1). The parabolic surface can be generated also by translation of a movable parabola $y^2 = 2pz$ along the fixed parabola $x^2 = 2pz$ (Fig. 2).

The peak of the movable parabola must slide along the fixed parabola but the plane and the axis of the moving parabola must remain parallel. The concavities of the both parabolas must be directed in one side.

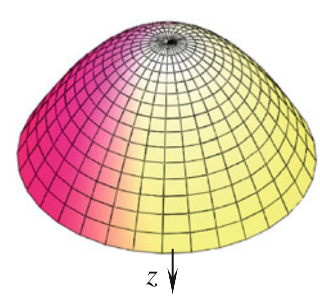


Fig. 1

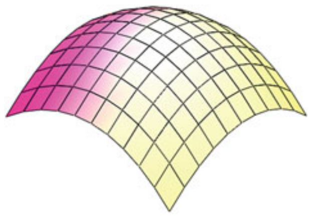


Fig. 2

Paraboloid of revolution possesses the interesting optical property. The light rays coming from the focus after the reflection of them from the surface of the paraboloid will go parallel to the axis of paraboloid of revolution.

Forms of definition of the surface

(1) Explicit form of definition (Fig. 2):

$$2z = (x^2 + y^2)/p.$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 1 + \frac{x^2}{p^2}, \quad F = \frac{xy}{p^2}, \quad B^2 = 1 + \frac{y^2}{p^2},$$
$$L = \frac{1}{\sqrt{p^2 + x^2 + y^2}} = N, \quad M = 0, \quad k_{>} = \frac{L}{A^2},$$
$$k_{—} = \frac{L}{B^2}, \, k_1 = L, \quad k_2 = p^2 L^3.$$

On the surface of a paraboloid of revolution, coordinate lines *x, y* generate *Tchebychef's net*, i.e., every quadrangle formed by the lines of curvilinear coordinate net has equal opposite sides. The coordinate net is non-orthogonal ($F \neq 0$) but conjugate ($M = 0$).

The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ are much less than one in strength analyses of real shallow shell objects and that is why it is possible to neglect squares of the derivatives in comparison with 1. So, the formulas obtained will take the simplified form for shallow middle surfaces of shells:

$$A = B = 1, \quad F = 0, \quad L = 1/p = N, \quad M = 0,$$
$$k_x = k_y = 1/p.$$

(2) Parametrical equations (Fig. 1):

$$x = x(r, \beta) = r\cos\beta, \quad y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = r^2/(2p).$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + r^2/p^2, \; F = 0, \; B = r,$$
$$L = 1/(pA), \; M = 0, \; N = r^2/(pA),$$
$$k_1 = 1/(pA^3), \; k_2 = L.$$

(3) Parametrical equations (Fig. 1):

$$x = x(u, v) = a\sqrt{u/h}\cos v,$$
$$y = y(u, v) = a\sqrt{u/h}\sin v,$$
$$z = z(u) = u \;\text{ where } u \geq 0; \; 0 \leq v \leq 2\pi.$$

The paraboloid has a radius $r = a$ at the height of $z = h$. An area of the lateral surface of a paraboloid of revolution is

$$S = \pi a\left[\left(a^2 + 4h^2\right)^{3/2} - a^3\right]/\left(6h^2\right).$$

A volume of a paraboloid of revolution is $V = \pi a^2 h/2$ if $0 \leq v \leq 2\pi, \; 0 \leq u \leq h$.

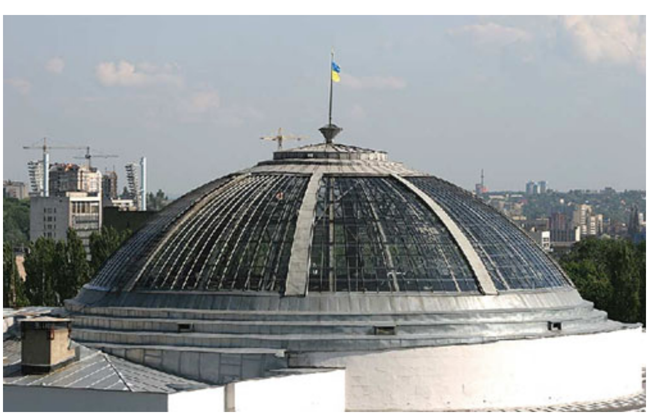


Fig. 3 The glass dome of museum, Kiev, Ukraine

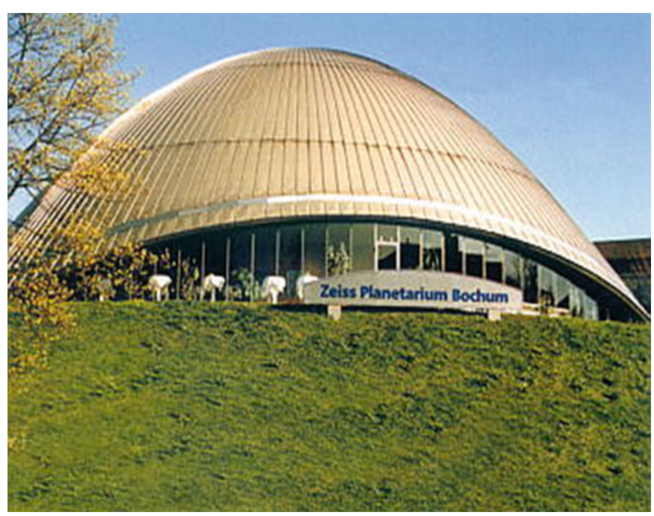


Fig. 4 A planetarium in Bochum, Germany

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2}{4uh},\ F = 0,\ B^2 = \frac{a^2u}{h},$$
$$L = \frac{a}{2u\sqrt{a^2 + 4uh}},\ M = 0,\ N = \frac{2au}{\sqrt{a^2 + 4uh}},$$
$$k_1 = \frac{L}{A^2},\ k_2 = \frac{N}{B^2}.$$

The surface is widely used in civil (Fig. 3) and industrial (Fig. 4) engineering.

Additional Literature

Krivoshapko SN. Parabolic shells of revolution. Montazhn. i spetz. raboty v stroitelstve. 1999; No. 12, p. 5-12 (63 ref.).

■ Circular Torus

A circular torus or *torus* in Latin is formed by rotation of a circumference

$$(x-a)^2 + z^2 = b^2$$

about an axis *Oz*. *An open torus* is a torus (Fig. 1) generated by rotation of a circumference about an axis lying outside limit of this circle ($a > b$). *A closed torus* (*Horn Torus*) is a torus generated by rotation of a circumference about an axis touching ($a = b$, Fig. 2) or intersecting ($a < b$, Figs. 3 and 4) the circle. The inner part of surface of an open torus is a surface of negative Gaussian curvature but the outer surface is a surface of positive Gaussian curvature (Figs. 1, 2 and 3).

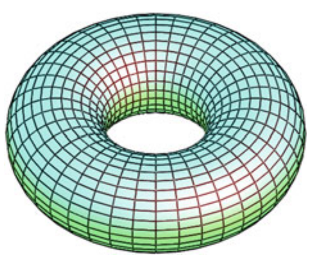


Fig. 1 The torus with $a > b$ (the open torus)

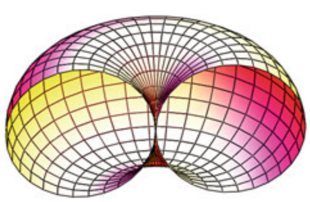


Fig. 2 The torus with $a = b$ (the closed torus)

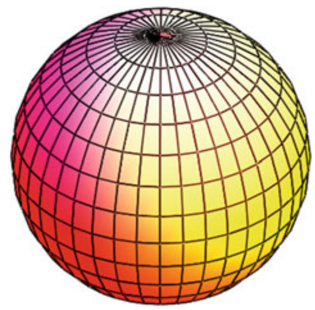


Fig. 3 The torus with $a = 0$ (a sphere)

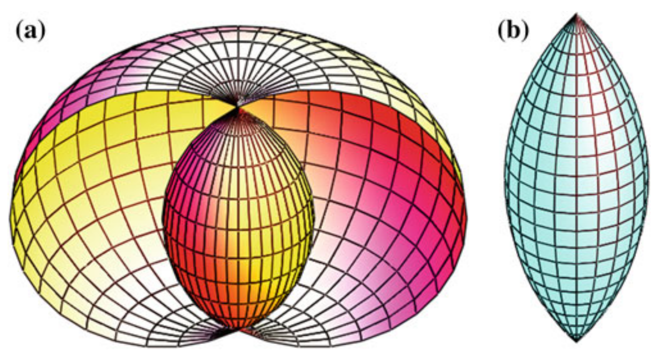


Fig. 4 The torus with $a < b$ (the closed torus)

Forms of definition of the surface

(1) Implicit equations:

$$\left(x^2 + y^2 + z^2 + a^2 - b^2\right)^2 = 4a^2\left(x^2 + y^2\right).$$

(2) Parametrical equations:

$$x = x(u, v) = (a + b\cos v)\cos u,$$
$$y = y(u, v) = (a + b\cos v)\sin u,$$
$$z = z(v) = b\sin v,$$

where a is a radius of the centers of generatrix circles, b is a radius of a generatrix circle, an angle u is called *an inner latitude* of a point of the torus; $0 \le u \le 2\pi$, $0 \le v \le 2\pi$; a ratio b/a is *an eccentricity of torus*. On a circular torus besides parallels and meridians, two families of plane circles, called *Villarceau circles*, exist. They can be seen in the cross sections of a torus by a plane touching the torus at two points. A radius of Villarceau circles is equal to a.

An area of the whole surface of a torus is $4\pi^2 ab$, its volume is $2\pi^2 ab^2$.

Coefficients of the fundamental forms of the surface:

$$A = a + b\cos v, \quad F = 0, \quad B = b,$$
$$L = -(a + b\cos v)\cos v, \quad M = 0, \quad N = -b,$$
$$K = \cos v/(bA).$$

Assume $a < b$ (Fig. 4), then the angle v changes in the limit of

$$-\arccos(-a/b) \le v \le \arccos(-a/b),$$

but if we want to have the torus (*the lemon*) shown in Fig. 4b then we must take

$$\arccos(-a/b) \le v \le 2\pi + \arccos(-a/b).$$

(3) Parametrical equations:

$$x = x(u, \beta) = \frac{a\left(\sqrt{a^2 + \beta^2} - b\right)}{\sqrt{a^2 + \beta^2}}\cos u,$$

$$y = y(u, \beta) = \frac{a\left(\sqrt{a^2 + \beta^2} - b\right)}{\sqrt{a^2 + \beta^2}}\sin u,$$

$$z = \frac{b\beta}{\sqrt{a^2 + \beta^2}}, \quad \beta = a\tan\alpha,$$

where α is the angle of the straight line, connecting the center of the generatrix circle of the radius b with arbitrary point of the torus, with a plane $z = 0$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{a(\sqrt{a^2 + \beta^2} - b)}{\sqrt{a^2 + \beta^2}}, \quad F = 0, \quad B = \frac{ab}{a^2 + \beta^2},$$

$$L = -\frac{a^2(\sqrt{a^2 + \beta^2} - b)}{a^2 + \beta^2}, \quad M = 0, \quad N = \frac{a^2 b}{(a^2 + \beta^2)^2},$$

$$k_1 = k_u = -1/(\sqrt{a^2 + \beta^2} - b), \quad k_2 = k_v = 1/b.$$

(4) Parametrical equations of a circular torus if $a = b$ (Fig. 2):

$$x = x(\gamma, u) = \frac{a(\mathrm{ch}\gamma \pm 1)}{\mathrm{ch}\gamma}\cos u,$$
$$y = y(\gamma, u) = \frac{a(\mathrm{ch}\gamma \pm 1)}{\mathrm{ch}\gamma}\sin u,$$
$$z = a\,\mathrm{th}\gamma.$$

Additional Literature

Gulyaev VI, Bazhenov VA, Gotzulyak EA, Gaydaychuk VV. An Analysis of Shells of Complex Form. 1990; Kiev: Budivelnik, 192 p.

Kutzenko GV. Axis-symmetrical deformation of a circular torus. PM. 1979; Vol. 15, No. 11, p. 46-51.

■ Elliptic Torus

An elliptic torus is generated by the rotation of an ellipse of arbitrary position (Fig. 1):

$$x = x(v) = a + r\cos v, \quad z = z(v) = r\sin v,$$

where $r = r(v) = \frac{cb}{\sqrt{b^2\sin^2\beta + c^2\cos^2\beta}}$, $\beta = v - \theta$, about an axis Oz; θ = const is the slope angle of the semi-axis of the ellipse ξ with the plane xOy.

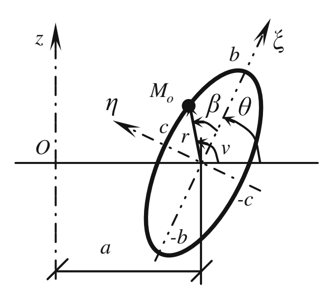


Fig. 1

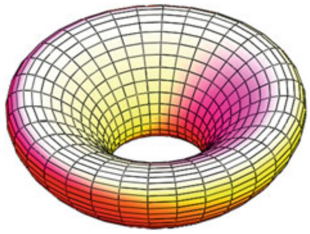


Fig. 2

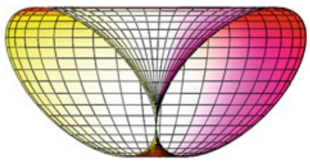


Fig. 3

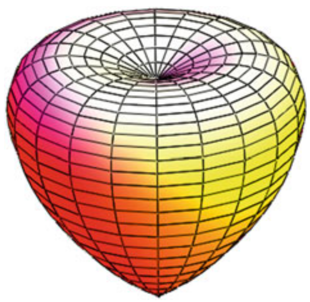


Fig. 4

An open elliptic torus is a torus formed by the rotation of an ellipse about an axis Oz lying outside of the limit of this ellipse (Figs. 1 and 2).

A closed torus is a torus generated by rotation of an ellipse about an axis Oz touching (Fig. 3) or intersecting (Fig. 4) the ellipse.

An ellipse touches an axis of rotation if the condition $\partial x/\partial v = 0$ carries out or

$$r^2\left(c^2 - b^2\right)\ \sin[2(v-\theta)] = 2c^2b^2\tan v.$$

Parametrical equations of the surface have the following form:

$$x = x(u,v) = (a + r\cos v)\cos u,$$
$$y = y(u,v) = (a + r\cos v)\sin u,$$
$$z = z(v) = r\sin v,$$

where a is the radius of the circle generated by the point of the intersection of the axes ξ and η of the generatrix ellipse (Fig. 1); r is the distance the point of the intersection of the ellipse's axes from an arbitrary point M_o belonging to the ellipse; b, c are the semi-axes of the ellipse; $0 \le u \le 2\pi$, $0 \le v \le 2\pi$; u is the angle of the axis Ox with the axis Oy.

If one of the axes of the generatrix ellipse, for example, the ξ axis, is parallel to the axis of rotation Oz, then it is necessary to assume $\theta = \pi/2$. If we take $b = c$, then we shall have $r = b$, $v = \beta$, but an elliptical torus will degenerate into *a circular torus* where a will be a radius of the centers of generatrix circles with the radius of b.

Coefficients of the fundamental forms of the surface:

$$A = a + r\cos v,\ F = 0,$$
$$B^2 = \frac{(b^4\sin^2\beta + c^4\cos^2\beta)r^6}{c^4b^4},$$
$$L = -\frac{A}{B}r\left[\frac{c^2-b^2}{2c^2b^2}r^2\sin 2\beta\sin v + \cos v\right],$$
$$M = 0,\ N = -\frac{r^6}{c^2b^2B}.$$

Having assumed $a = 0$, we can design *an oblique ellipsoid of revolution* (Fig. 5a, b and c).

Additional Literature

Clark RA, *Girloy TI. and Reissner E.* Stresses and deformation of toroidal shells of elliptical cross section. J. Appl. Mech. 1953; Vol. 20, No. 4.

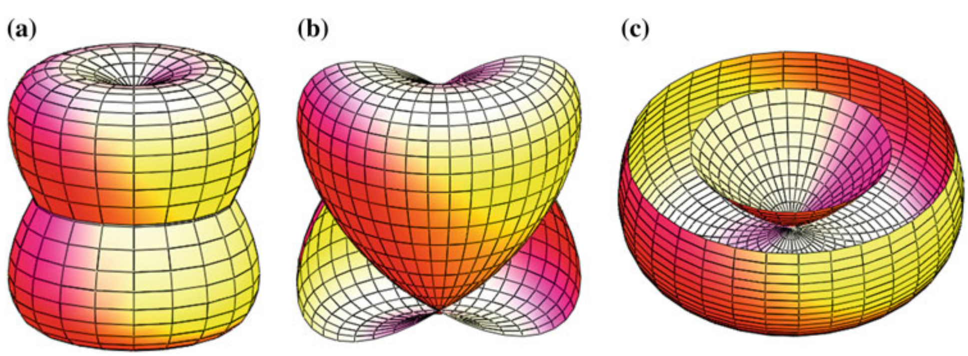


Fig. 5

■ Surface of Revolution of a Curve $z = b\ \exp(-a^2x^2)$ Around the Z Axis

The surface is formed by rotation of a curve $z = be^{-a^2x^2}$ about a coordinate axis z.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u) = u, \quad y = y(v) = v, \quad z = b\ \exp[-a^2(u^2 + v^2)].$$

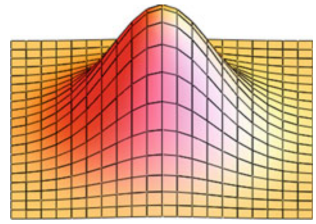
The surface is called «Die Glocke» in German.

(2) Parametrical equations (Fig. 2):

$$x = x(r, \beta) = r\ \cos\beta; \quad y = y(r, \beta) = r\ \sin\beta;$$
$$z = z(r) = be^{-a^2r^2},$$

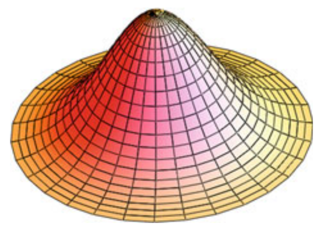
where $0 \le r < \infty$; $0 \le \beta \le 2\pi$; $z \le b$.

(3) An explicit equation (Fig. 1): $z = be^{-a^2(x^2+y^2)}$



$a = b = 1$;
$-2 \le u, v \le 2$m

Fig. 1



$a = b = 1$;
$0 \le r \le 2$m

Fig. 2

■ Two-Sheeted Hyperboloid of Revolution

Two-sheeted hyperboloid of revolution is formed by rotation of a hyperbola

$$-\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1,$$

about its *focal axis* (an axis Oz). The surface has two separate sheets when the axis of revolution is *the transverse axis.*

A section of a hyperboloid by a plane $z = h > c =$ const gives a circle with a radius $r = a\sqrt{h^2 - c^2}/c$ (Fig. 1). If we cut a hyperboloid by a plane $y = t =$ const, then hyperbolas $z = \pm c\sqrt{a^2 + t^2 + x^2}/a$ will be in the cross section (Fig. 2), but having intersected a hyperboloid by a plane $x = p =$ const, we can have hyperbolas $z = \pm c\sqrt{a^2 + p^2 + y^2}/a$ (Fig. 2).

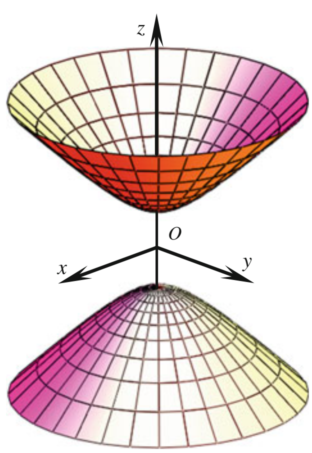


Fig. 1

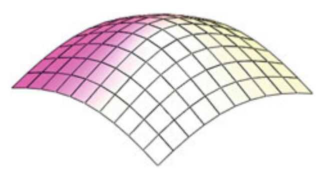


Fig. 2

The peaks of two sheets of hyperboloid are placed at the points with coordinates (0, 0, $\pm c$). The signs correspond two sheets of hyperboloid. Two-sheeted hyperboloid of revolution belongs to a class of *not closed central surfaces of the second order*. It is a particular case of *hyperboloid of two sheets* which is presented in Chap. "35. *Surfaces of the second order.*"

Forms of definition of the surface

(1) Implicit equation:

$$\frac{-x^2 - y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

where a and c are the semi-axes of a hyperboloid of revolution, $|z| \geq c$; $a^2/c = p$ is a focal parameter of meridian. A hyperboloid is called *a right hyperboloid of revolution* if $a = c$. It is formed by rotation of *an equilateral hyperbola*. *An asymptotical cone* of two-sheeted hyperboloid of revolution is defined by an implicit equation:

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 0.$$

A hyperboloid of revolution is *a quadric surface*.

(2) Explicit equation (Fig. 2):

$$z = \pm \frac{c}{a}\sqrt{a^2 + x^2 + y^2}$$

(3) Parametrical equations (Fig. 1):

$$x = x(u, v) = a\mathrm{sh}u \cos v, \quad y = y(u, v) = a\mathrm{sh}u \sin v,$$
$$z = \pm c\mathrm{ch}u.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = a^2\mathrm{ch}^2 u + c^2\mathrm{sh}^2 u,$$
$$F = 0, \quad B = a\mathrm{sh}u,$$
$$L = \pm\frac{ac}{A}, \quad M = 0,$$
$$N = \pm\frac{ac}{A}\mathrm{sh}^2 u,$$
$$k_1 = \pm\frac{ac}{A^3}, \quad k_2 = \pm\frac{c}{aA}.$$

Coordinate lines u, v are the lines of principal curvatures.

(4) Parametrical equations (Fig. 1):

$$x = x(z, \beta) = r\sin\beta,$$
$$y = y(z, \beta) = r\cos\beta,$$
$$z = z, \quad \text{where } r = \frac{a}{c}\sqrt{z^2 - c^2}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \sqrt{1 + r'^2}, \quad F = 0, \quad B = r(z),$$
$$k_1 = \frac{1}{R_1} = -\frac{r''}{\left(1 + r'^2\right)^{3/2}}, \quad k_2 = \frac{1}{R_2} = \frac{1}{r\sqrt{1 + r'^2}},$$

where the first and second derivatives of r with respect to parameter z are denoted by primes.

(5) A parametrical form of definition with the help of polar coordinates of the meridians (Fig. 1):

$$x = x(\varphi, \beta) = \rho\sin\varphi\sin\beta,$$
$$y = y(\varphi, \beta) = \rho\sin\varphi\cos\beta,$$
$$z = z(\varphi) = \rho\cos\varphi,$$

where

$$\rho = \frac{p}{1 - e\cos\varphi}, \quad p = \frac{a^2}{c}, \quad e = \sqrt{1 + \frac{a^2}{c^2}},$$
$$\theta \leq \varphi \leq \pi + \theta, \quad \cos\theta = \frac{1}{e}.$$

Additional Literature

Vasil'ev AN. Stability of anisotropic two-sheeted hyperboloid of revolution with filling material. Kazan: KFEI, 1991; 14 p., 6 ref., Dep. v VINITI 08.07.91, No. 2887-B91.

Gritskevich OV, *Meshcheryakov NA*, *Pod'yapol'skii YuV*, Precision laser processing of curved surfaces of revolution, QUANTUM ELECTRON. 1996; 26 (7), p. 644-646.

■ Surface of Conjugation of Two Coaxial Cylinders of Different Diameters

A surface of conjugation of two coaxial cylinders of different diameters may be included as a component of the two classes of surfaces. These are a class of cyclic surfaces and a class of surfaces of revolution.

The surface is formed by rotation of the sinusoid about a common axis of two conjugated cylinders (Fig. 1).

Parametrical equations of the surface of conjugation are (Figs. 1 and 2).

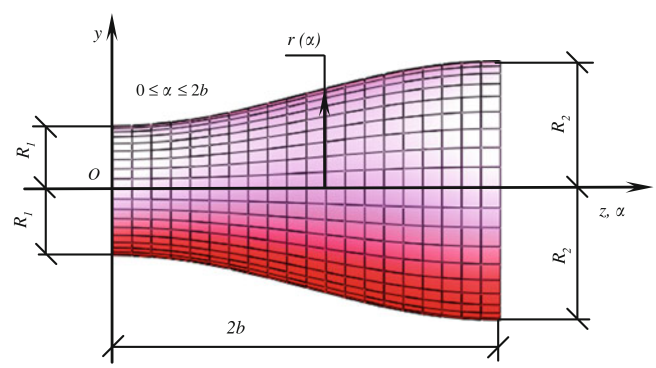


Fig. 1

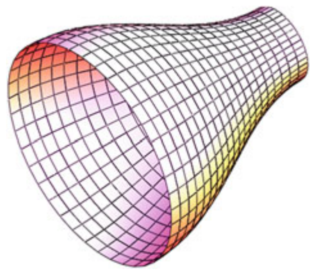


Fig. 2

$$x = x(\alpha, \beta) = r(\alpha)\cos\beta, \quad y = y(\alpha, \beta) = r(\alpha)\sin\beta,$$
$$z = \alpha,$$

where

$$r = r(\alpha) = \frac{R_2 - R_1}{2}\left(1 - \cos\frac{\pi\alpha}{2b}\right) + R_1$$
$$= (R_2 - R_1)\sin^2\frac{\pi\alpha}{4b} + R_1$$

is a law of change of a radius of the studied surface of conjugation along an axis Oz (an axis of rotation); $R_2 \geq R_1$; $0 \leq \alpha \leq 2b$; $2b$ is a length of a segment between two cylinders of different diameters; β is the angle in the planes of parallels taken from the axis Ox in the direction of the axis Oy; $0 \leq \beta \leq 2\pi$.

Two parallels placed in the cross sections $z = 0$ and $z = 2b$ are *geodesic lines*, because the tangent to the meridians at the points of these parallels are parallel to the axis of rotation.

All meridians of the surface of revolution are geodesic lines too.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{\pi^2}{16b^2}(R_2 - R_1)^2\sin^2\frac{\pi\alpha}{2b}, \quad F = 0, \quad B = r(\alpha),$$
$$L = -\frac{\pi^2(R_2 - R_1)}{8b^2A}\cos\frac{\pi\alpha}{2b}, \quad M = 0, \quad N = \frac{B}{A},$$
$$k_\alpha = k_1 = -\frac{\pi^2(R_2 - R_1)}{8b^2A^3}\cos\frac{\pi\alpha}{2b}, \quad k_\beta = k_2 = \frac{1}{AB},$$
$$K = -\frac{\pi^2(R_2 - R_1)}{8b^2A^4B}\cos\frac{\pi\alpha}{2b},$$
$$H = \frac{\pi^2(R_2 - R_1)\{R_2 - R_1 - (R_2 + R_1)\cos[\pi\alpha/(2b)]\} + 16b^2}{32b^2A^3B}.$$

A curvilinear coordinate net is given in lines of principal curvatures α, β. If $R_2 > R_1$, then the surface has a segment of negative Gaussian curvature if $0 \leq \alpha \leq b$ and of positive Gaussian curvature if $b \leq \alpha \leq 2b$. In Fig. 2, the surface of conjugation is shown with

$$R_2 = 3R_1; \quad b = 3R_1; \quad 0 \leq \alpha \leq 2b; \quad 0 \leq \beta \leq 2\pi.$$

The surface in issue is a component of subclass "*Corrugated surface of revolution of a common sinusoid*" contained also in a class "Surface of revolution." A surface of conjugation degenerates into a cylindrical surface of revolution if $R_1 = R_2$.

Additional Literature

Gulyaev VI, Bazhenov VA, Gotzulyak EA, Gaydaychuk VV. An Analysis of Shells of Complex Form. 1990; Kiev: Budivelnik, 192 p.

■ Surface of Revolution "Wellenkugel"

Information about a surface of revolution "Wellenkugel" is presented in sites given in References. This surface has parametrical equations:

$$x = u\cos(\cos u)\cos v;$$
$$y = u\cos(\cos u)\sin v;$$
$$z = u\sin(\cos u).$$

In Fig. 1, the surface with $0 \leq u \leq 14,5$ m; $0 \leq v \leq 1,5\pi$ is shown.

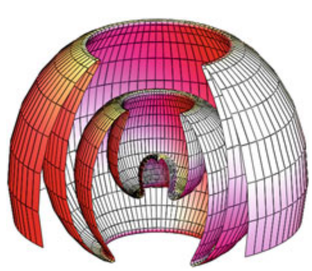


Fig. 1

References

1. Mathematics Museum (Japan). Introduction to Geometry, Ibaraki University, 2002, http://mathmuse.sci.ibaraki.ac.jp/MuseumE.html
2. Parametrische Flächen und Körper.—http://www.3d-meier.de/tut3/Seite63.html

■ Surface of Conjugation of Coaxial Cylinder and Cone

A surface of conjugation of coaxial cylinder and cone is a fragment of *a corrugated surface of revolution of a common sinusoid.* It is formed by rotation of a curve

$$y = a[1 - \cos(2\pi z/c)] + R_1$$

about an axis Oz. Having assumed two necessary conditions

$$\frac{2\pi a}{c}\sin\left(\frac{2\pi b}{c}\right) = \tan\varphi \quad \text{and}$$
$$a\left[1 - \cos\left(\frac{2\pi b}{c}\right)\right] + R_1 = R_2,$$

we may design a surface of conjugation of coaxial cylinder with a radius R_1 and circular cone with the angle φ at the vertex and with a base having a radius R_2 (Fig. 1). So, having six constants R_1, R_2, a, b, c, and φ, one may take four constants as desired but two remaining geometrical constants are derived from the system of two presented equations. Moreover, it is necessary to take $a < 0$ when $R_1 > R_2$.

For example, let us consider that R_1, R_2, c, and φ are given, then the rest two parameters a and b can be obtained with the help of formulas:

$$a = \frac{-1}{R_1 - R_2}\left[\frac{(R_1 - R_2)^2}{2} + \frac{c^2\tan^2\varphi}{8\pi^2}\right];$$
$$b = \frac{c}{2\pi}\arcsin\frac{c\tan\varphi}{2\pi a} \quad \text{if} \quad \varphi > 0,$$

$R_2 > R_1$ (Fig. 1) or $\varphi < 0$, $R_2 < R_1$ and

$$b = \frac{c}{2} - \frac{c}{2\pi}\arcsin\frac{c\tan\varphi}{2\pi a} \quad \text{if} \quad \varphi < 0, \quad R_2 > R_1 \quad \text{or} \quad \varphi > 0, \quad R_2 < R_1.$$

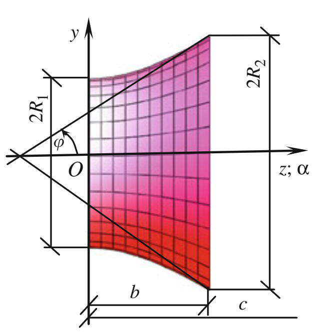


Fig. 1

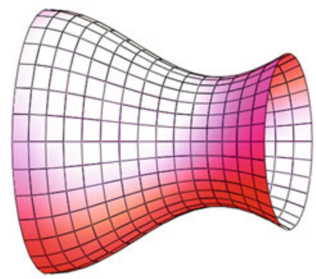


Fig. 2

Forms of definition of the surface

(1) Parametrical equations:

$$x = x(z,\beta) = r\cos\beta,$$
$$y = y(z,\beta) = r\sin\beta,$$
$$z = z$$

where

$$r = r(z) = a[1 - \cos(2\pi z/c)] + R_1;$$

$0 \le z \le b$; $b < c$; $0 \le \beta \le 2\pi$ (Figs. 1 and 2).

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{4\pi^2 a^2}{c^2}\sin^2\frac{2\pi z}{c}, \; F = 0, \; B = r(z),$$
$$L = -\frac{4a\pi^2}{c^2 A}\cos\frac{2\pi z}{c}, \; M = 0, \; N = \frac{r}{A}.$$
$$k_1 = k_z = -\frac{4a\pi^2}{c^2 A^3}\cos\frac{2\pi z}{c}, \; k_2 = k_\beta = \frac{1}{rA},$$
$$K = -\frac{4a\pi^2}{c^2 r A^4}\cos\frac{2\pi z}{c}.$$

All meridians and also the parallels $z = 0$, $z = c/2$, and $z = c$ on surface of a coaxial cylinder and a cone are geodesic lines. The surface of conjugation contains fragments of positive Gaussian curvature in the limits of $c/4 < z < 3c/4$ if $a > 0$ and fragments of negative Gaussian curvature in the limits of $0 < z < c/4$ and $3c/4 < z < c$ if $a > 0$.

The surface of conjugation shown in Fig. 1 has the following geometrical parameters: $R_2 = 1.5R_1$, $c = 4R_2$, and $\varphi = \pi/6$.

The surface of conjugation with $R_1 = 1.5R_2$, $c = 4R_2$, and $\varphi = \pi/6$ is presented in Fig. 2.

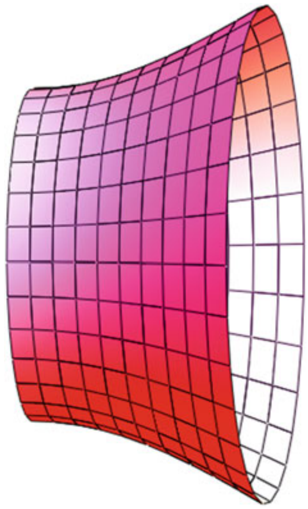


Fig. 3

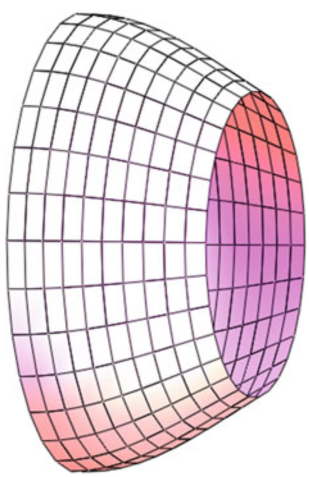


Fig. 4

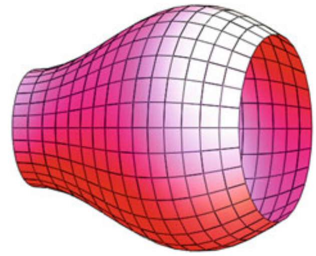


Fig. 5

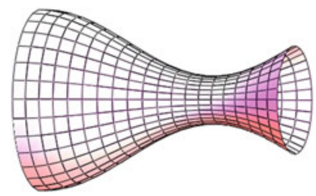


Fig. 6

(2) Parametrical equations:

$$x = x(z, \beta) = r\cos\beta,$$
$$y = y(z, \beta) = r\sin\beta,$$
$$z = z,$$
$$r = r(z) = a[1 - \cos(2\pi z/c)] + R_1;$$
$$a = R_2 - R_1,$$

where $b = c/4$; $c = 2\pi a/\tan\varphi$ if $\varphi > 0$, $a > 0$ (Fig. 3) or $\varphi < 0$, $a < 0$ (Fig. 4) and $b = 3c/4$; $c = -2\pi a/\tan\varphi$ if $\varphi < 0$, $a > 0$ (Fig. 5) or $\varphi > 0$, $a < 0$ (Fig. 6).

Coefficients of the fundamental forms of the surface are defined by the formulas given for the first variant.

The surfaces shown in Figs 1, 2, 3, 4, 5 and 6 are constructed when $|\varphi| = \pi/6$.

Reference

Krivoshapko SN. Model surfaces of connecting fragments of two pipe lines. Montazhn. i spetz. raboty v stroitelstve. 2005; No.10, p. 25-29.

■ Surface Formed by Rotation of a Meridian in the Form of Semicubical Parabola

A surface is generated by rotation of *a semicubical parabola* $z = bx^{2/3}$ (*Neil's parabola*) about an axis Oz. This surface of revolution has *a singular point* with coordinates (0, 0, 0).

Forms of definition of the surface

(1) Explicit equation:

$$z = b\sqrt[3]{x^2 + y^2}.$$

(2) Parametrical equations:

$$x = u^3, \quad y = v^3, \quad z = b\left(u^6 + v^6\right)^{1/3}.$$

(3) Parametrical equations (Fig. 1):

$$x = x(r, \beta) = r\cos\beta,$$
$$y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = br^{\frac{2}{3}}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{4b^2}{9} r^{-\frac{2}{3}}, \ F = 0, \ B = r,$$
$$L = -\frac{2b}{9A} r^{-\frac{4}{3}}, \ M = 0, \ N = \frac{2b}{3A} r^{\frac{2}{3}},$$

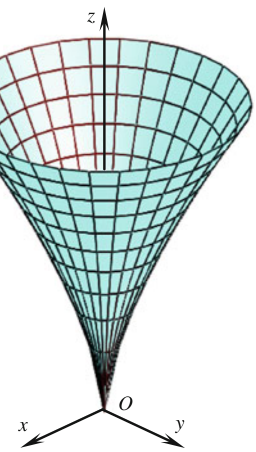


Fig. 1

$$k_1 = -\frac{2b}{9A^3} r^{-\frac{4}{3}}, \ k_2 = \frac{2b}{3A} r^{-\frac{4}{3}}.$$

This is a surface of negative total curvature, i.e., $K < 0$.

■ Surface of Revolution of a Hyperbola $z = b/x$ About the Oz Axis

Forms of Definition of the Surface

(1) Explicit equation:

$$z = \frac{b}{\sqrt{x^2 + y^2}}.$$

A surface of rotation of a hyperbola $z = b/x$ about the axis Oz can be reckoned also in *Tzitzéica's surface with central affine invariant equal to* $I = -4/(27b^2)$.

(2) Parametrical equations (Fig. 1):

$$x = x(r, \beta) = r\cos\beta, \quad y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = b/r,$$

where $x > 0$, $y > 0$, $r = b/r$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{b^2}{r^4}, \quad F = 0, \quad B = r,$$
$$L = \frac{2b}{Ar^3}, \quad M = 0, \quad N = -\frac{b}{Ar},$$
$$k_1 = \frac{2b}{r^3A^3}, \quad k_2 = -\frac{b}{r^3A}.$$

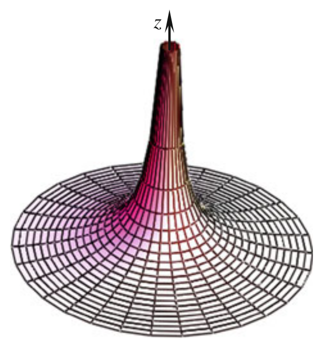


Fig. 1

The surface of rotation of a hyperbola is a surface of strictly negative Gaussian curvature. Not a single parallel will be a geodesic line.

If we assume $b = 1$, i.e., $z = 1/x$ on $[1, \infty]$, then we have *Gabriel's Horn*, or *Gabriel's Trumpet*, due to a highly unusual and paradoxical trait. The volume of Gabriel's Horn is equal to π on $[1, \infty]$ and the area of lateral surface is equal to infinity, i.e., $A = \infty$, on $[1, \infty]$. So, we have a surface with infinitive surface area enclosing a finite volume.

Additional Literature

Tzitzéica G. Sur une nouvelle classe de surface. Comptes Rendus, Acad. Sci. Paris. 1907; 144, p. 1257-1259.

■ Parabolic Humming-Top

A surface "*Parabolic humming-top*" has a parabola, as a meridian, the axis of which is perpendicular to the axis of rotation but a peak of the parabola is lying at the axis of rotation, i.e., on an axis z (Fig. 1).

This surface called also "Der Kreisel" can be given by parametrical equations (Fig. 2):

$$x = \frac{(|z| - h)^2}{2p}\cos\beta,$$
$$y = \frac{(|z| - h)^2}{2p}\sin\beta, \quad z = z,$$

where h is a height of one sheet of the surface; $h^2/(2p)$ is a radius of the equator of the surface of revolution (Fig. 1); $-h \le z \le h$; $0 \le \beta \le 2\pi$. The peaks of two generatrix parabolas are placed in the points with coordinates (0; 0; ± h). This surface contains two segments of *a surface of rotation of a parabola* (Page 123).

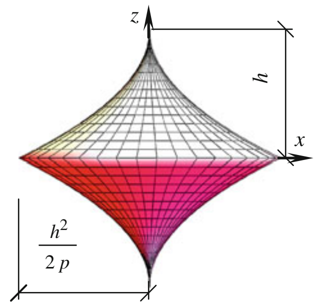


Fig. 1

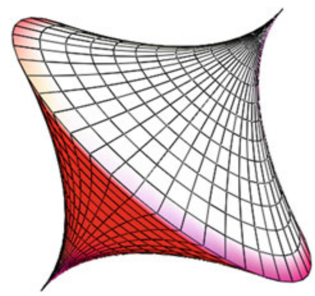


Fig. 2

■ Surface of Revolution of an Astroid

A surface of revolution of an astroid can be generated by the rotation of a astroid $x^{2/3} + z^{2/3} = a^{2/3}$ about its axis Ox or Oz (Fig. 1).

Forms of definition of the surface

(1) Explicit equation:

$$z = \pm\left[a^{2/3} - \left(x^2 + y^2\right)^{1/3}\right]^{\frac{3}{2}}.$$

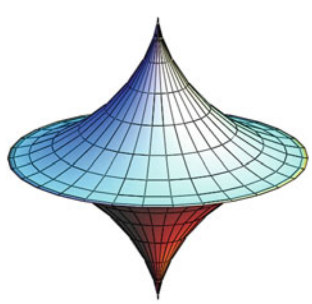


Fig. 1

The surface has two singular points in the poles of the surface with the coordinates $x = y = 0$, $z = \pm\grave{a}$ and an edge of regression that is the parallel $r = a$ when $z = 0$.

(2) Parametrical equations:

$$x = x(r, \beta) = r\cos\beta,$$
$$y = y(r, \beta) = r\sin\beta,$$
$$z = \pm\left(a^{2/3} - r^{2/3}\right)^{3/2},$$

where $0 \le r \le a$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \left(\frac{a}{r}\right)^{1/3}, \quad F = 0, \quad B = r,$$
$$L = \frac{a^{1/3}}{3r\sqrt{a^{2/3} - r^{2/3}}}, \quad M = 0, \quad N = -\frac{r\sqrt{a^{2/3} - r^{2/3}}}{a^{1/3}},$$
$$k_1 = \frac{1}{3(ar)^{1/3}\sqrt{a^{2/3} - r^{2/3}}}, \quad k_2 = -\frac{\sqrt{a^{2/3} - r^{2/3}}}{ra^{1/3}},$$
$$K = -\frac{1}{3r^{4/3}a^{2/3}} < 0.$$

(3) Parametrical equations:

$$x = x(t,\beta) = a\sin^3 t\cos\beta, \quad y = y(t,\beta) = a\sin^3 \sin\beta,$$
$$z = z(t) = a\cos^3 t.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 3a\sin t\cos t, \quad F = 0, \quad B = a\sin^3 t,$$
$$L = 3a\sin t\cos t, \quad M = 0, \quad N = -a\sin^3 t\cos t,$$
$$k_1 = \frac{2}{3a\sin 2t}, \quad k_2 = -\frac{\cos t}{a\sin^3 t}, \quad K < 0.$$

■ Astroidal Torus

A surface of the rotation of an astroid is formed by an astroid

$$x^{2/3} + z^{2/3} = a^{2/3}$$

rotating about any of two its axes Ox or Oz. If an astroid

$$x = x(u) = a\cos^3 u, \quad z = z(u) = a\sin^3 u$$

is placed at the r distant from the axis of rotation, then we will have *an astroidal torus*. An inner area bounded by an astroid is

$$A = \frac{3}{8}\pi a^2.$$

A length of full astroid is $6a$. It can be noted that an astroid is *an evolute of the ellipse*. *The evolute of an astroid* is another astroid.

An astroidal torus can be defined by parametrical equations:

$$X = X(u,v) = [r + x(u)\cos\theta - z(u)\sin\theta]\cos v;$$
$$Y = Y(u,v) = [r + x(u)\cos\theta - z(u)\sin\theta]\sin v;$$
$$Z = Z(u) = x(u)\sin\theta + z(u)\cos\theta,$$

where θ is the angle of rotation of local axes x, z of the generatrix astroid in the vertical plane containing the axis The local coordinate system is rotated counter-clockwise if the θ angle has positive value.

An astroidal torus degenerates into *an astroidal surface of revolution* when $r = 0$, $\theta = 0$ (Fig. 1).

In Fig. 1, the astroidal torus is given when $a = 1$ m, $r = 2$ m, $\theta = 0$, $0 \le v \le 2\pi$; $-\pi \le u \le \pi$.

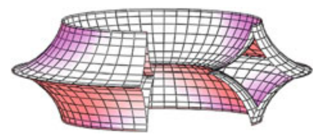


Fig. 1

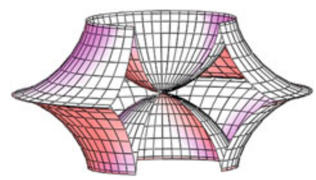


Fig. 2

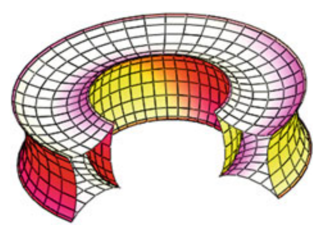


Fig. 3

The astroidal torus with $\theta = 0$, $0 \le v \le 2\pi$, $-\pi \le u \le \pi$, $a = r = 1$ m is given in Fig. 2.

The right astroidal torus is represented in Fig. 3 when $a = 1$ m, $r = 2$ m, $\theta = 0.25\pi$; $0 \le v \le 2\pi$; $-\pi \le u \le \pi$.

Additional Literature

Weisstein EW. Astroid from MathWorld.

■ Surface of Revolution of the Agnesi Curl

The meridians of *a surface of revolution of the Agnesi curl* about its asymptote intersect the plane $z = 0$, perpendicular to the rotation axis, at angle of 90° (Fig. 1). An implicit equation of an Agnesi curl is

$$z^2 y = 4a^2(2a - y).$$

The circle with a radius $2a$ lies in the cross section of this surface of revolution by the plane $z = 0$. This parallel is a geodesic line.

Forms of definition of the surface

(1) Implicit equation:

$$z^2 = 4a^2\left(\frac{2a}{\sqrt{x^2 + y^2}} - 1\right).$$

(2) Parametrical equations (Fig. 1):

$$x = x(r, \beta) = r\cos\beta, \quad y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = 2a(2a/r - 1)^{1/2}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

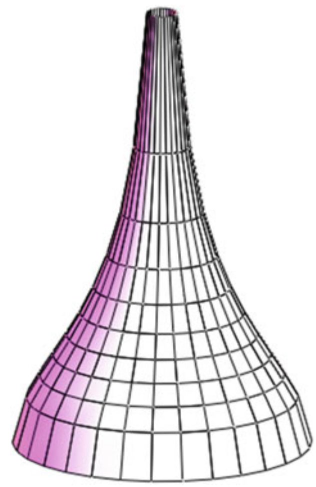


Fig. 1

$$A^2 = 1 + \frac{4a^4}{r^4(2a/r - 1)}, \quad F = 0, \quad B = r,$$
$$L = \frac{2a^2(3a - 2r)}{Ar^4(2a/r - 1)^{3/2}}, \quad M = 0, \quad N = -\frac{2a^2}{Ar\sqrt{2a - 1}},$$
$$k_1 = k_r = \frac{L}{A^2}, \quad k_2 = k_\beta = \frac{N}{B^2} < 0.$$

So, $K > 0$ if $r > 1.5a$; $K < 0$ if $r < 1.5a$ and $K = 0$ on the parallel $r = 1.5a$.

■ Deformed Sphere

Surface of revolution "*Deformed Sphere*" is a closed surface consisting of two parts one of which is a surface of positive Gaussian curvature but another one is of negative Gaussian curvature. These parts of the surface are jointed along the plane circle with parabolic points.

"Deformed Sphere" has the following parametrical equations (Fig. 1):

$$x = x(u, v) = \cos u \cos v,$$
$$y = y(u, v) = \cos u \sin v,$$
$$z = z(u) = \sin(u - a)$$

where a is a constant parameter, $-\pi/2 \le u \le \pi/2$, $0 \le v \le 2\pi$.

A "Deformed Sphere" is degenerated into a sphere when $a = 0$ and $a = \pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \sin^2 u + \cos^2(u - a), \quad F = 0, \; B = \cos u,$$
$$L = \frac{\cos a}{A}, \quad M = 0, \quad N = \frac{\cos u \cos(u - a)}{A},$$
$$k_1 = \frac{\cos a}{A^3}, \quad k_2 = \frac{\cos(u - a)}{A\cos u}.$$

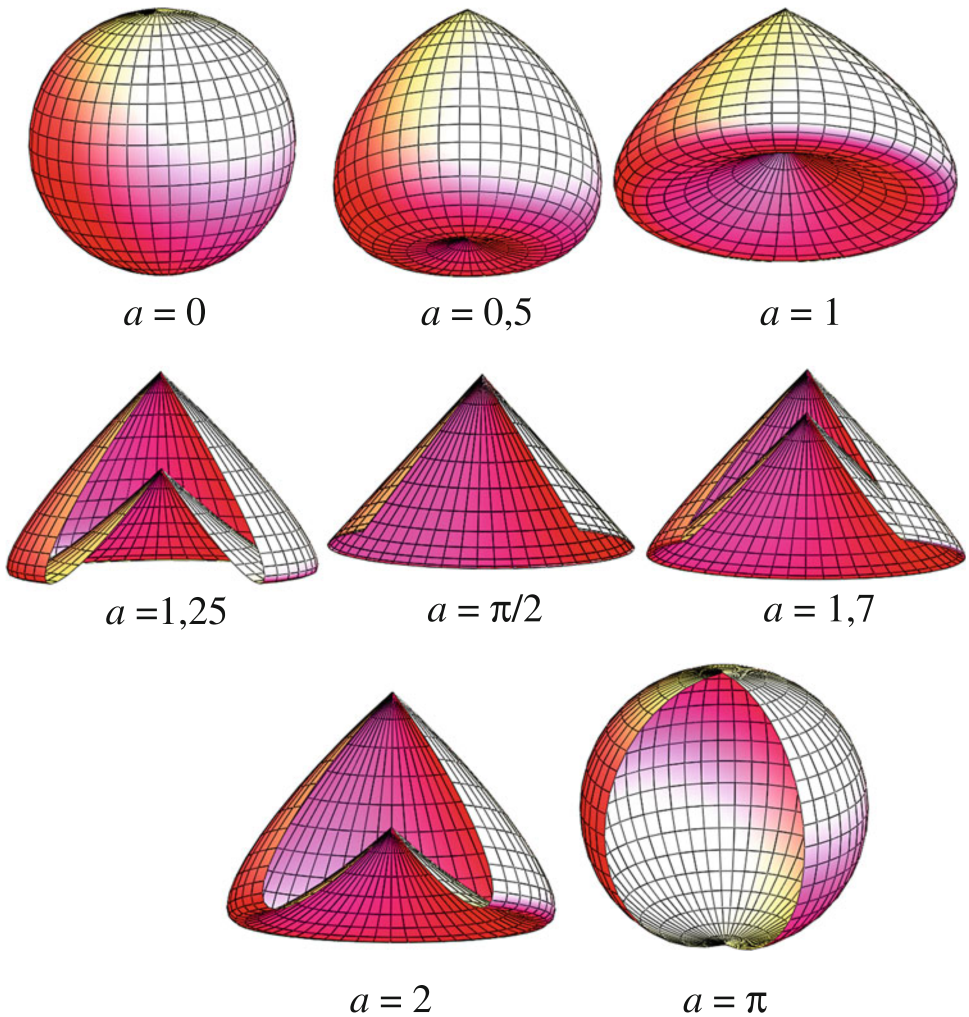


Fig. 1

■ Surface of Revolution of a Parabola

A paraboloid of revolution is formed by rotation of a parabola about its axis of symmetry, i.e., about the axis of the parabola. *A surface of revolution of a parabola* is generated by rotation of a parabola about a straight line that is perpendicular to the axis of the parabola, i.e., is parallel to the directrix of the parabola. A parabola has the only one directrix which is p away from its focus.

The general surface of revolution of a parabola is obtained when a parabolic arc is rotated about an arbitrary axis. In the encyclopedia, this surface is called *a surface of revolution of a parabola of arbitrary position*.

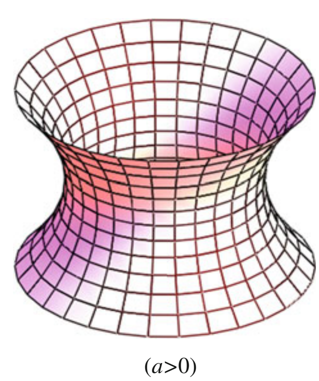
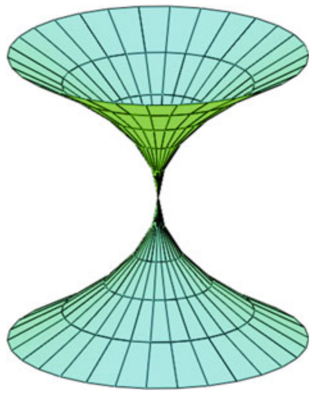


Fig. 1



$(a = b = 0)$

Fig. 2

Forms of the definition of the surface

(1) Parametrical equations (Fig. 1):

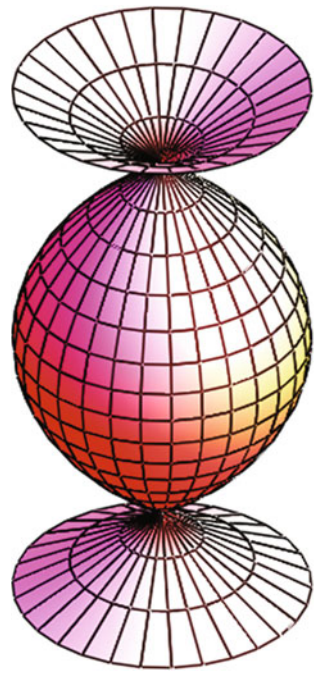
$$x = x(r, \beta) = r\cos\beta,$$
$$y = y(r, \beta) = r\sin\beta,$$
$$z = z(r) = \sqrt{2p(r-a)},$$

where $r = a$ is the radius of *the waist circle*, p is a distance the focus from the directrix of the parabolic meridian, $|x| \geq a,\ |y| \geq a,\ 0 \leq \beta \leq 2\pi$. The surface of revolution is formed by the rotation of a parabola $z^2 = 2p(x - a)$ about the z axis. The surface of revolution with $a > 0$ is shown in Fig. 1. If one takes $a = 0$, then he will design the surface represented in Fig. 2.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{p}{2(r-a)},\ F = 0,\ B = r,$$
$$L = -\frac{p^2}{A[2p(r-a)]^{3/2}},$$
$$M = 0,\ N = \frac{pr}{A\sqrt{2p(r-a)}},$$
$$k_1 = k_r = -\frac{p^2}{A^3[2p(r-a)]^{3/2}},$$
$$k_2 = k_\beta = \frac{p}{Ar\sqrt{2p(r-a)}},\quad K < 0.$$

A surface of revolution of a parabola belongs to surfaces of negative Gaussian curvature if $a \geq 0$. A directrix of the family of meridians becomes the axis of rotation when $a = p$.



$(b > 0)$

Fig. 3

(2) Parametrical equations (Figs. 1 and 2):

$$x = x(z, \beta) = \left[a + z^2/(2p)\right]\cos\beta,$$
$$y = y(z, \beta) = \left[a + z^2/(2p)\right]\sin\beta,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{z^2}{p^2},\quad F = 0,\quad B = r = a + \frac{z^2}{2p},$$
$$L = \frac{1}{pA},\quad M = 0\quad N = -\frac{B}{A},$$
$$k_1 = k_z = \frac{1}{pA^3},\quad k_2 = k_\beta = -\frac{1}{AB},$$
$$K = -\frac{1}{pA^4B} < 0.$$

(3) Parametrical equations (Figs. 3 and 4):

$$x = x(z, \beta) = \left[\frac{z^2}{2p} - b\right]\cos\beta,$$
$$y = y(z, \beta) = \left[\frac{z^2}{2p} - b\right]\sin\beta,$$
$$z = z,$$

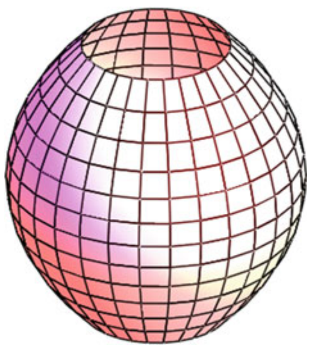


Fig. 4

where $b \geq 0$ is the distance the peak of the parabola from the axis of rotation. The surface shown in Fig. 2 is formed when $b = 0$. In Fig. 3, the surface with $b > 0$ is presented.

Having assumed $b > 0$ and $-\sqrt{2pb} < z < \sqrt{2pb}$, we can design *a barrel-shaped surface* of revolution (Fig. 4).

In several works, the surfaces shown in Figs 1, 2, 3 and 4 were called *a parabolic torus*.

Additional Literature

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■ Parabolic-and-Logarithmic Surface of Revolution

A parabolic-and-logarithmic surface of revolution of positive Gaussian curvature is formed by rotation of a plane curve

$$r = r(z) = a\sqrt{cz+b}\ln(cz+b)$$

about the z axis.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(z,\beta) = r(z)\sin\beta,$$
$$y = y(z,\beta) = r(z)\cos\beta,$$
$$z = z.$$

The indeterminacy in the form of $0 \cdot \infty$ existing at the point z_o ($cz_o + b = 0$) is disclosed and leads to an equality $r(z_o) = 0$. The parallel, lying in the plane $z = 0$, has a radius $r_o = ab^{1/2}\ln b$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2c^2}{cz+b}\left[\frac{\ln(cz+b)}{2} + 1\right]^2, \quad F = 0,$$
$$B = r(z) = a\sqrt{cz+b}\ln(cz+b),$$

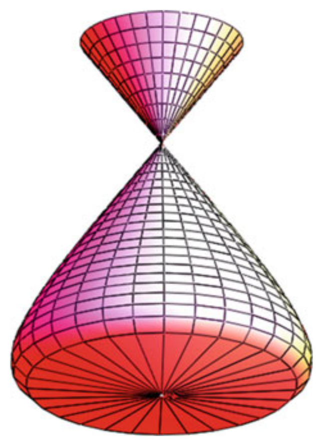

Fig. 1

$$L = \frac{ac^2\ln(cz+b)}{4A(cz+b)^{3/2}}, \quad M = 0, \quad N = \frac{r(z)}{a},$$
$$k_1 = \frac{ac^2\ln(cz+b)}{4A^3(cz+b)^{3/2}},$$
$$k_2 = \frac{1}{r(z)A}, \quad K = \frac{c^2}{4A^4(cz+b)^2} > 0.$$

Additional Literature

Nazarov GI, Puchkov AA. An equilibrium of a parabolic-and-logarithmic surface of revolution. Prikl. Mat. i Mehanika (Moscow). 1991; 55, No. 5, p. 867-869.

■ Hyperbolic-and- Logarithmic Surface of Revolution

A hyperbolic-and-logarithmic surface of revolution of negative Gaussian curvature has meridians:

$$r = r(z) = a(z+b)^2 \ln(z+b),$$

where $a > 0$ is a constant characterizing the form of the surface (Fig. 1). A constant b does not influence on the form of the surface but the position of the beginning of coordinates depends on the parameter b. The beginning of a system of Cartesian coordinates is placed at the peak of the surface of revolution when $b = 0$. The axis Oz is an axis of rotation. The indeterminacy in the form of $0 \cdot \infty$ existing at the peak when $z = -b$ is disclosed due to de l'Hopitale rule. So, one will obtain:

$$r = r(z = -b) = 0.$$

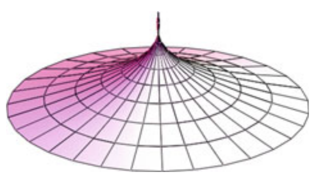

Fig. 1

Parametrical equations of the studied surface of revolution can be written as (Fig. 1):

$$x = x(z,\beta) = r(z)\sin\beta,$$
$$y = y(z,\beta) = r(z)\cos\beta,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + a^2(z+b)^2[1 + 2\ln(z+b)]^2,$$
$$F = 0,\ B = r(z) = a(z+b)^2\ln(z+b),$$
$$L = -\frac{a[2\ln(z+b)+3]}{A},\ M = 0,\ N = \frac{r(z)}{A},$$
$$k_1 = -\frac{a[2\ln(z+b)+3]}{A^3},\ k_2 = \frac{1}{r(z)A},$$
$$K = -\frac{3+2\ln(z+b)}{(z+b)^2\ln(z+b)A^4} < 0.$$

In Fig. 1, the hyperbolic-and-logarithmic surface of revolution is shown when $a = 0.5$; $b = 0$; $0.1 \le z \le 4$ m; $r_{max} = 11.09$ m if $z = 4$ m.

Additional Literature

Nazarov GI, *Puchkov AA*. An inverse problem for a shell of revolution of negative Gaussian curvature. Izv. Vuzov: Stroit. i Arhitectura. 1990; No. 12, p. 22-24.

■ Bullet Nose

"*Bullet Nose*" is formed by rotation of a curve: $x = \pm az/\sqrt{b^2+z^2}$ (Figs. 1 and 2) about a coordinate axis z.

Forms of definition of the surface

(1) Parametrical equations (рис. 3):

$$x = x(u,v) = a\cos v\cos u,$$
$$y = y(u,v) = a\cos v\sin u,$$
$$z = z(v) = -b/\tan v,$$
$$x < a;\ y < a;\ 0 \le u \le 2\pi;\ 0 < v \le \pi/2.$$

(2) Implicit equation

$$(b^2+z^2)(x^2+y^2) = a^2z^2$$

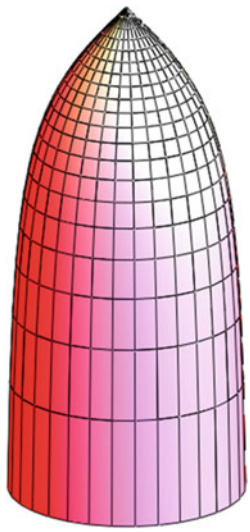

Fig. 1

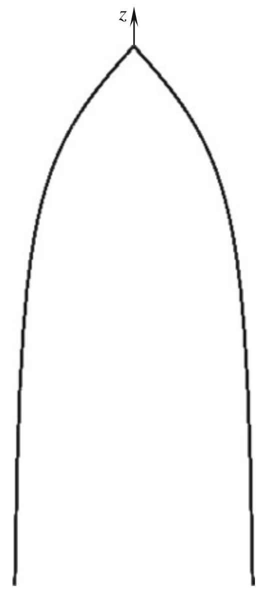

Fig. 2

■ The Fourth-Order Paraboloid of Revolution

The fourth-order paraboloid of revolution is formed by rotation of biquadratic parabola $x^4 = cz$ about an axis z (Fig. 1). This surface is also called *a quartoid*.

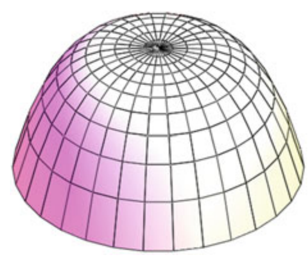

Fig. 1

Forms of definition of the surface

(1) Explicit equation:

$$cz = \left(x^2 + y^2\right)^2.$$

Having assumed $c = a^3$, we can get *a poweroid* (Jackway and Deriche).

In the cross section of the surface of revolution by the planes $z = h = \text{const}$, circles with radii

$$r = \sqrt[4]{hc}$$

are placed; $h > 0$.

(2) Parametrical equations (Figs. 1 and 2):

$$x = x(r,\beta) = r\cos\beta,$$
$$y = y(r,\beta) = r\sin\beta,$$
$$z = z(r) = r^4/c.$$

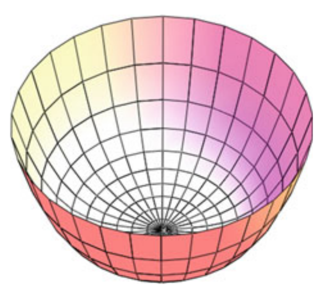

Fig. 2

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + 16\frac{r^6}{c^2},\ F = 0,\ B = r,$$
$$L = \frac{12r^2}{cA},\ M = 0,\ N = \frac{4r^4}{cA},$$
$$k_r = k_1 = \frac{12r^2}{cA^3},\ k_\beta = k_2 = \frac{4r^2}{cA},$$
$$K = \frac{48r^4}{c^2A^4} > 0,\ H = \frac{2r^2}{cA}\left(1 + \frac{3}{A^2}\right).$$

The studied surface of revolution is given in the lines of principal curvatures r and β. A paraboloid of revolution of the fourth order is a surface of positive total curvature. The surface has zero Gaussian and mean curvatures ($K = H = 0$) only at one point $r = 0$. So, the peak of a paraboloid of revolution of the fourth order is *a plane point*.

(3) Parametrical equations (Figs. 1 and 2):

$$x = x(z,\beta) = \sqrt[4]{cz}\cos\beta,$$
$$y = y(z,\beta) = \sqrt[4]{cz}\sin\beta,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{c^2}{16(cz)^{3/2}},\quad F = 0,\quad B = \sqrt[4]{cz},$$
$$L = \frac{3c^2}{16AB^7},\quad M = 0,\quad N = \frac{B}{A},$$
$$k_z = k_1 = \frac{3c^2}{16A^3B^7},\quad k_\beta = k_2 = \frac{1}{AB},$$
$$K = \frac{3c^2}{16A^4B^8} = \frac{48z}{(\sqrt{c} + 16z^{3/2})^2} > 0.$$

The obtained values of the coefficients of the fundamental forms of surface show that the surface of rotation of a biquadratic parabola is given in lines of principal curvatures z and β but the fourth-order paraboloid of revolution is a surface of positive total curvature and only in one point $z = 0$, the surface has zero Gaussian and mean curvatures.

Additional Literature

Sun Bo-Hua, Zhang Wei, Yeh Kai-Yuan, Rimrott FPJ. Exact displacement solution of arbitrary degree paraboloidal shallow shell of revolution made of linear elastic materials. Int. J. Solids and Struct. 1996; 33, No. 16, p. 2299-2308 (14 ref.).

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Jackway PT. and *Deriche M.* Scale-space properties of the multiscale morphological dilation-erosion. Trans. on Pattern Analysis and Machine Intelligence. 1996; **18**(1), p. 38-51.

Palm G. Robust segmentation of human cardiac contours from spatial magnetic resonance images. Diss. zur Erlangung des Doct. (Dr. rer.nat.), der Fakultät für Informatik der Universität Ulm.; 2004; 130 p.

■ Surface of Revolution with Damping Circular Waves

Having researched damped natural vibrations, one seeks the amplitude-time dependence in the form of a function

$$z = z(x) = ae^{-nx}\sin(\omega x + \varphi).$$

A surface of revolution with damping circular waves is traced by a curve $z = z(x)$ in the process of its rotation about an axis Oz.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(r,u) = r\cos u,$$
$$y = y(r,u) = r\sin u,$$
$$z = z(r) = ae^{-nr}\sin(\omega r + \varphi),$$

where $\omega = m\pi/b$, m is a number of integral half-waves, placed at the straight line segment with the b length; $\varphi = \text{const}$.

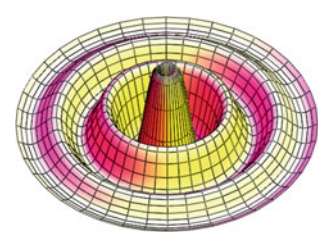


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + a^2e^{-2nr}[-n\sin(\omega r + \varphi) + \omega\cos(\omega r + \varphi)]^2,$$
$$F = 0,\quad B = r,$$
$$L = ae^{-nr}\left[(n^2 - \omega^2)\sin(\omega r + \varphi) - 2n\omega\cos(\omega r + \varphi)\right]/A,$$
$$M = 0, N = rae^{-nr}[-n\sin(\omega r + \varphi) + \omega\cos(\omega r + \varphi)]/A.$$

In Fig. 1, the surface of revolution with $m = 6$, $b = 6$ m; $a = 4$ m; $n = 0.5$; $0 \le r \le b$; $\varphi = 0$ is shown.

■ Kiss Surface

A "*Kiss Surface*" is *an algebraic surface of the fifth order* (Fig. 1). Sometimes this surface is called *a* "*Falling Drop*." It is traced by a curve $x = x(z) = z^2(1 - z)^{1/2}$ in the process of its rotation about an axis Oz.

Forms of definition of the surface

(1) Implicit form of the definition:

$$x^2 + y^2 = (1 - z)z^4, \text{ where } -\infty \le z \le 1.$$

(2) Explicit form of the definition:

$$x = \pm\sqrt{(1 - z)z^4 - y^2}.$$

(3) Parametrical equations (Fig. 1):

$$x = x(u, z) = z^2\sqrt{1 - z}\cos u, \quad y = y(u, z) = z^2\sqrt{1 - z}\sin u,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = z^4(1 - z), \quad F = 0,$$
$$B^2 = \frac{4(1 - z) + z^2(4 - 5z)^2}{4(1 - z)}, \quad M = 0,$$

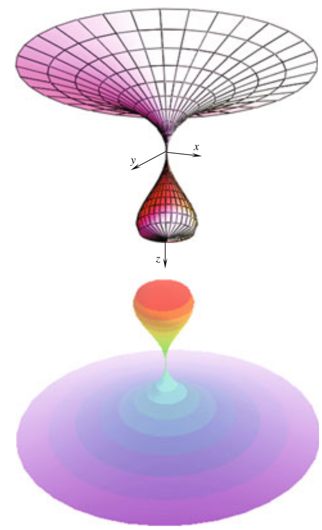


Fig. 1

$$L = \frac{2(1 - z)z^2}{\sqrt{4(1 - z) + z^2(4 - 5z)^2}},$$
$$N = \frac{15z^2 - 24z + 8}{2(1 - z)\sqrt{4(1 - z) + z^2(4 - 5z)^2}},$$
$$K = \frac{4(15z^2 - 24z + 8)}{z^2\left[4(1 - z) + z^2(4 - 5z)^2\right]^2}.$$

The surface contains the parts of positive and negative Gaussian curvatures. Parabolic points with $K = 0$ are placed at the cross section of the surface by a plane $z = 0.8 - 0.4(2/3)^{1/2} = 0.473$. In Fig. 2, the surface is shown when $-1 \le z \le 1$; $0 \le u \le 2\pi$.

■ Soucoupoid

Forms of Definition of the Surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = a\cos u\cos v, \quad y = y(u, v) = a\cos u\sin v,$$
$$z = z(u) = b\sin^3 u,$$

where coordinate lines *u, v* (meridians and parallels) are the lines of principal curvatures; *a, b* are constants; $-\pi/2 \le u \le \pi/2$, $0 \le v \le 2\pi$.

(2) Implicit equation: $z^2 = b^2\left(1 - \frac{x^2 + y^2}{a^2}\right)^3$.

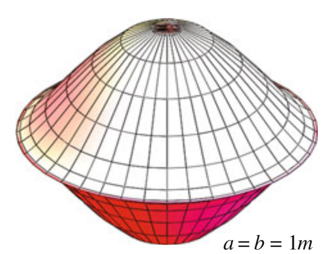


Fig. 1

Reference

Encyclopédie Des Formes Mathematiques Remarquables Surfaces.—http://mathcurve.com/surfaces/surfaces.shtml

■ Globoid (Toroid)

A globoid is a surface formed by rotation of an arc of the circle m about an axis z lying at the plane of this arc. A method of generation of a surface of a globoid shows that we have a segment of *the circular torus* which has a negative Gaussian curvature (Fig. 1). A line on the globoid generated by uniform motion of a point along the axis of the globoid with simultaneous steady rotation of the globoid about its axis is called *a globoidal helical line*.

A globoidal worm gearing is an example of application of globoid in the technique.

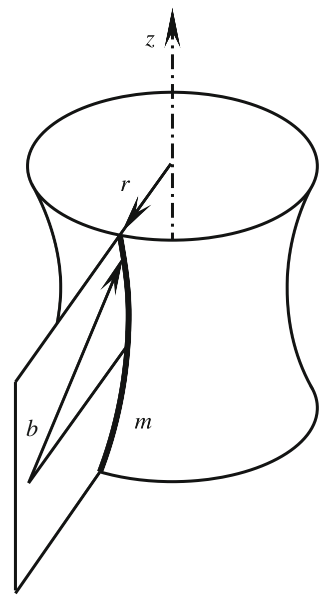


Fig. 1

Forms of definition of the surface

(1) Parametrical equations (Fig. 2):

$$x = x(u, v) = (a + b\cos v)\cos u,$$
$$y = y(u, v) = (a + b\cos v)\sin u,$$
$$z = z(v) = b\sin v,$$

where a is a radius of centers of generatrix circles; b is a radius of the generatrix circle, $0 \le u \le 2\pi$, $\pi/2 \le v \le (3/2)\pi$. In Fig. 3, a fragment of the surface bounded by the lines of principal curvatures is shown; $0 \le u \le \pi$ and $\pi \le v \le (3/2)\pi$.

Coefficients of the fundamental forms of the surface:

$$A = a + b\cos v, \quad F = 0, \quad B = b,$$
$$L = -(a + b\cos v)\cos v, \quad M = 0, \quad N = -b,$$
$$k_u = k_1 = -\frac{\cos v}{A}, \quad k_v = k_2 = -\frac{1}{b},$$
$$K = \frac{\cos v}{bA}.$$

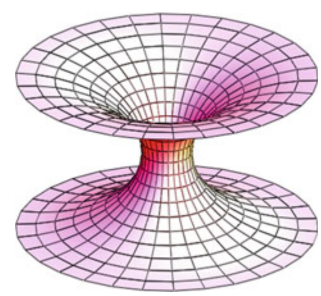


Fig. 2

(2) Parametrical equations:

$$x = x(u, \beta) = \frac{a\left(\sqrt{a^2 + \beta^2} - b\right)}{\sqrt{a^2 + \beta^2}}\cos u,$$
$$y = y(u, \beta) = \frac{a\left(\sqrt{a^2 + \beta^2} - b\right)}{\sqrt{a^2 + \beta^2}}\sin u,$$
$$z = \frac{b\beta}{\sqrt{a^2 + \beta^2}},$$

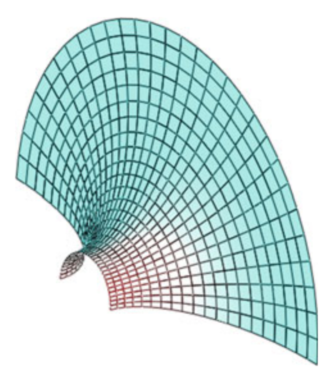


Fig. 3

where $\beta = a\tan\alpha$; α is the angle of a straight, connecting the center of generatrix circle with a radius b with an arbitrary point of the torus, with a plane $z = 0$. Positive direction is counted off anticlockwise; $-\pi/2 < \alpha < \pi/2$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{a(\sqrt{a^2 + \beta^2} - b)}{\sqrt{a^2 + \beta^2}}, \quad F = 0, \quad B = \frac{ab}{a^2 + \beta^2},$$
$$L = -\frac{a^2(\sqrt{a^2 + \beta^2} - b)}{a^2 + \beta^2}, \quad M = 0, \quad N = \frac{a^2 b}{(a^2 + \beta^2)^2},$$
$$k_1 = k_u = -\frac{1}{\sqrt{a^2 + \beta^2} - b}, \quad k_2 = k_v = \frac{1}{b}.$$

Coordinate lines u, v and u, β are the lines of principal curvatures. They coincide with the meridians and the parallels of surface of revolution.

(3) Parametrical equations:

$$x = x(\gamma, v) = \frac{a(\mathrm{ch}\gamma - 1)}{\mathrm{ch}\gamma}\cos v,$$
$$y = y(\gamma, v) = \frac{a(\mathrm{ch}\gamma - 1)}{\mathrm{ch}\gamma}\sin v,$$
$$z = a\,\mathrm{th}\gamma,\ -\infty < \gamma < +\infty,$$

The globoid has a degenerated point with coordinates (0, 0, 0) or when $\gamma = 0$; $a = b$.

Additional Literature

Blachut J and Jaiswal OR. Instabilities in torispheres and toroids under suddenly applied external pressure. Int. J. Impact. Eng. 1999; 22 (5), p. 511-530 (16 ref.).

■ Surface of Revolution of a Usual Cycloid

A surface of revolution of a usual cycloid is formed by the rotation of *an usual cycloid*

$$z_c = at - a\sin t,\ x_c = a - a\cos t$$

about the axis z_c, where t is *a real parameter*, corresponding to the angle through which the rolling circle has rotated, measured in radians. For given t, the circle's center lies at $z_c = at$, $x_c = a$.

A usual cycloid is generated by a point that is apart from a center of the circle with a radius a, rolling without sliding on the axis z_c, at the distance of a.

Let us study a general case when a cycloid is rotated about the axis z which is parallel to the axis z_c and is apart from it at the distance of c.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(t, \beta) = (a + c - a\cos t)\cos\beta,$$
$$y = y(t, \beta) = (a + c - a\cos t)\sin\beta,$$
$$z = z(t) = at - a\sin t.$$

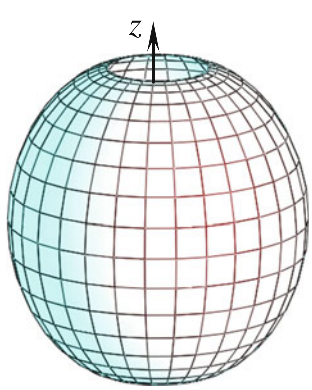


Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 2a\sin\frac{t}{2},\quad F = 0,\quad B = c + 2a\sin^2\frac{t}{2},$$
$$L = \frac{A}{2},\quad M = 0,\quad N = \frac{AB}{2a},$$
$$k_1 = k_t = \frac{1}{2A} = \frac{1}{4a\sin\frac{t}{2}},$$
$$k_2 = k_\beta = \frac{A}{2aB} = \frac{\sin\frac{t}{2}}{\left(c + 2a\sin^2\frac{t}{2}\right)},$$
$$K = \frac{1}{4aB} = \frac{1}{4a\left(c + 2a\sin^2\frac{t}{2}\right)} > 0.$$

Coordinate lines β and t (parallels and meridians) are the lines of principal curvatures.

A length of a meridian from a parallel $t = 0$ till a parallel $t = \text{const}$ is calculated by a formula:

$$s = 4a\left(1 - \cos\frac{t}{2}\right).$$

In Fig. 2, the fragment of the surface bounded by the parallels $t = 0$, $t = 2\pi$ and by the meridians $\beta = 0$, $\beta = \pi$ is presented.

In Fig. 3, three sections of the surface of the rotation of a usual cycloid with $c = 0$ are given; but in Fig. 4, the surface with $c > 0$ is shown, $0 \le t \le 5\pi$.

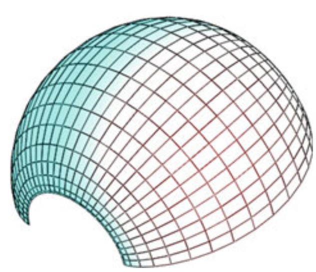


Fig. 2

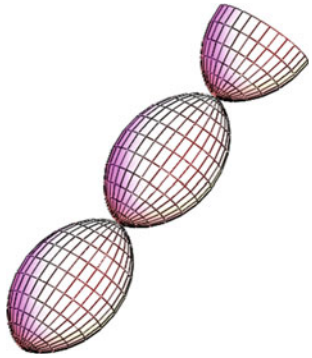


Fig. 3

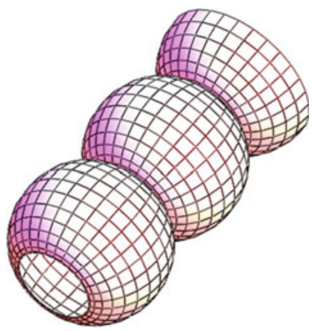


Fig. 4

Two sections of the surface presented in Fig. 3 belong to a category of *closed surfaces of revolution* because the beginning and the end of a not closed rotated usual cycloid is placed at the rotation axis.

An area of a surface of rotation of a segment of the meridian ($t_0 \le t \le t_1$) in the form of a usual cycloid can be defined by a formula:

$$A = 8a\pi \left[\frac{2a}{3} \cos^3 \frac{t}{2} \Big|_{t_0}^{t_1} - (c + 2a) \cos \frac{t}{2} \Big|_{t_0}^{t_1} \right], \quad 0 \le \beta \le 2\pi.$$

For example, an area of one closed section of the surface shown in Fig. 3 is

$$A_1 = \frac{64a^2\pi}{3}, \quad 0 \le t \le 2\pi, \ \ 0 \le \beta \le 2\pi.$$

Additional Literature

Barra Mario. The cycloid. Educ. Stud. Math. 1975; 6, No. 1, p. 93-98.

Churkin GM. The property of points of a cycloid. In-t him. Kinet. I goreniya SO AN SSSR, Novosibirsk, 1989; 10 p., 3 ref., Dep v VINITI 06.01.89, No. 156-B89.

Wells D. (1991). The Penguin Dictionary of Curious and Interesting Geometry. New York: Penguin Books. 1991; p. 445-47.

■ Pseudo-Catenoid

A catenoid is formed by the rotation of *a catenary*

$$x = a \cos\mathrm{h}(z/a)$$

about an Oz axis (Fig. 1). A catenoid is the only *minimal surface of revolution*, i.e., mean curvature of its surface is equal to zero at all points of the surface. It is the first minimal surface to be discovered.

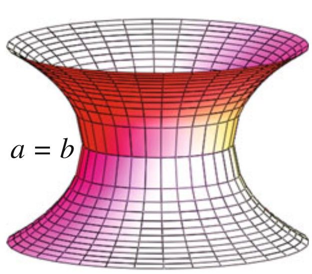


Fig. 1

A pseudo-catenoid is generated by the rotation of a curve

$$x = b \cos\mathrm{h}(z/a)$$

about an Oz axis. A pseudo-catenoid is a surface of rigorously negative Gaussian curvature but it is not a minimal surface.

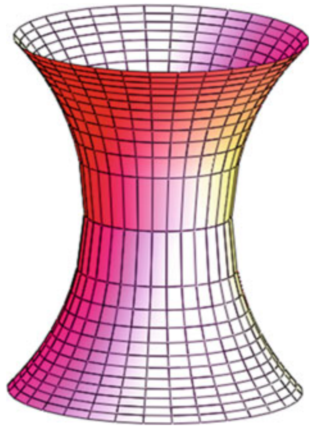
Forms of definition of the surface

(1) Explicit equation:

$$z = a\mathrm{Ar}\ \cos\mathrm{h}\sqrt{(x^2 + y^2)/b^2}.$$

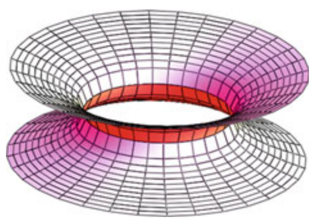
(2) Parametrical equations (Figs. 2 and 3):

$$x = x(r,\ \beta) = r \cos\beta,$$
$$y = y(r,\ \beta) = r\ \sin\beta,$$
$$z = z(r) = \pm\, a\mathrm{Ar} \cos\mathrm{h}(r/b),$$



$(a > b)$

Fig. 2



$(a < b)$

Fig. 3

where β is the angle taken from the axis Ox in the directions of the Oy axis.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \frac{r^2 - b^2 + a^2}{r^2 - b^2},\ F = 0,\ B = r,$$
$$L = \frac{-ar}{(r^2 - b^2)\sqrt{r^2 - b^2 + a^2}},$$
$$M = 0,\ N = \frac{ra}{\sqrt{r^2 - b^2 + a^2}},$$
$$k_1 = \frac{-ar}{(r^2 - b^2 + a^2)^{3/2}},\ k_2 = \frac{a}{r\sqrt{r^2 - b^2 + a^2}},$$
$$K = \frac{-a^2}{[r^2 - b^2 + a^2]^2} < 0,$$
$$H = \frac{a(a^2 - b^2)}{2r(r^2 - b^2 + a^2)^{3/2}} \neq 0.$$

Coordinate lines r and β (parallels and meridians) are the lines of principal curvatures (Figs. 1, 2 and 3). In Fig. 2, the pseudo-catenoid has $a > b$. The surface of revolution shown in Fig. 3 was created when $a < b$. And a pseudo-catenoid becomes a minimal surface if $a = b$ (Fig. 1) and this surface can be called a catenoid.

Substituting $a = b$ in the formulae for the determination of coefficients of the fundamental forms of surface, it is possible to obtain corresponding values of these coefficients for catenoid.

Additional Literature

Krivoshapko SN. On mistakes in the terminology on theory of surfaces and geometric modelling. Present Problems of Geometric Modelling: Proc. of Ukraine-Russian Scientific-and-Practical Conf. April 19-22, 2005. Kharkov, 2005; p. 82-87.

■ Surface of Revolution "Pear"

A surface of revolution called "*Pear*" is generated by rotating curve

$$b^2y^2 = z^3(a - z)$$

about its coordinate axis Oz.

Forms of definition of the surface

(1) Parametrical form of the definition (Fig. 1):

$$x = x(z, \beta) = r(z)\sin\beta;$$
$$y = y(z, \beta) = r(z)\cos\beta; \quad z = z,$$

where $r = r(z) = z\sqrt{z(a - z)}/b$; a and b are arbitrary constants; $0 \leq z \leq a$;

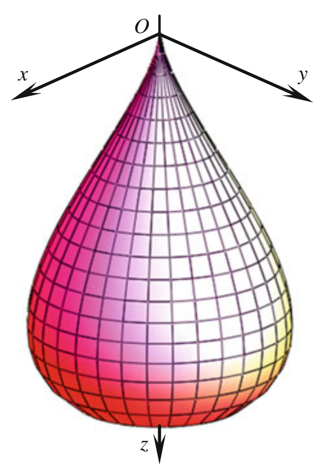


Fig. 1

$$0 \le r \le 3\sqrt{3}a^2/(16b).$$

A parallel $z = 3a/4$ with

$$r = r_{\max} = 3\sqrt{3}a^2/(16b)$$

is *a geodesic line*.

(2) Implicit equation:

$$z^3(a - z) - b^2(x^2 + y^2) = 0.$$

It means that the studied surface "Pear" is an algebraic surface of the fourth order.

Additional Literature

Gustavo Gordillo. A collection of famous plane curves. http://curvebank.calstatela.edu/famouscurves/famous.htm. August 14, 2001.

■ Surface of Revolution of a General Sinusoid

A surface of revolution of a general sinusoid

$$z = a\,\sin(n\pi x/R + \pi/2) = a\cos(n\pi x/R)$$

about an axis Oz is used in technics. *General sinusoid* in contrast to *usual sinusoid* ($z = \sin x$) is elongated $|a|$ times along the axis Oz and contracted $R/(n\pi)$ times along the axis Ox, where n is an integer, R is a dimension of an integer n of half-waves of the sinusoid, and is shifted to the left by a straight-line segment $R/(2n)$. A period of the function is $T = 2R/n$. The points of intersection of the sine function with the Ox axis have the coordinates $[(k + ½)R/n, 0]$. A surface of revolution of a general sinusoid has the parts of positive and negative Gaussian curvatures. This surface can be reckoned in a subclass of waving or corrugated surfaces.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(r,\beta) = r\cos\beta, \quad y = y(r,\beta) = r\sin\beta,$$
$$z = z(r) = a\cos\frac{n\pi r}{R}.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2n^2\pi^2}{R^2}\sin^2\frac{n\pi r}{R},$$
$$F = 0,\ B = r,$$
$$L = -\frac{an^2\pi^2}{AR^2}\cos\frac{n\pi r}{R},$$
$$M = 0,\ N = -\frac{an\pi}{AR}r\sin\frac{n\pi r}{R},$$
$$k_1 = k_r = -\frac{an^2\pi^2}{A^3R^2}\cos\frac{n\pi r}{R},$$
$$k_2 = k_\beta = -\frac{an\pi}{rAR}\sin\frac{n\pi r}{R},$$
$$K = \frac{a^2n^3\pi^3}{2rA^4R^3}\sin\frac{2n\pi r}{R}.$$

The curvilinear coordinate net is put down to lines of principal curvatures.

(2) Parametrical equations (Fig. 2):

$$x = x(r,\beta) = r\cos\beta, \quad y = y(r,\beta) = r\sin\beta,$$
$$z = z(r) = a\sin\frac{n\pi r}{R}.$$

The general generating sinusoid in contrast to *usual sinusoid* ($z = \sin x$) is elongated $|a|$ times along the axis Oz and contracted $R/(n\pi)$ times along the axis Ox, where n is an integer, R is a dimension of an integer n of half-waves of the sinusoid. A period of the function is $T = 2R/n$. The points of intersection of the sine function with the Ox axis have the coordinates $[kR/n, 0]$.

The presented surface of revolution can be given in an explicit form (Fig. 3):

$$z = a\sin\left(\frac{n\pi}{R}\sqrt{x^2 + y^2}\right).$$

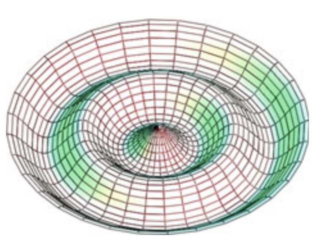


Fig. 1

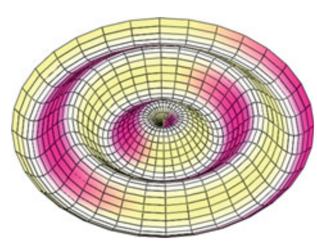


Fig. 2

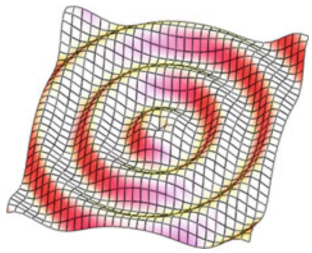


Fig. 3

The surface shown in Fig. 3 is called “Die Sinuswelle” in the German language scientific literature.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2n^2\pi^2}{R^2}\cos^2\frac{n\pi r}{R},$$
$$F = 0,\ B = r,$$
$$L = -\frac{an^2\pi^2}{AR^2}\sin\frac{n\pi r}{R},$$
$$M = 0, \quad N = \frac{an\pi}{AR} r\cos\frac{n\pi r}{R},$$
$$k_1 = k_r = -\frac{an^2\pi^2}{A^3R^2}\sin\frac{n\pi r}{R},$$
$$k_2 = k_\beta = \frac{an\pi}{rAR}\cos\frac{n\pi r}{R},$$
$$K = -\frac{a^2n^3\pi^3}{2rA^4R^3}\sin\frac{2n\pi r}{R}.$$

The parallels β and meridians r of the surface of revolution of a general sinusoid coincide with lines of principal curvatures.

(3) Explicit equation:

$$z = a\cos\left(\frac{n\pi}{R}\sqrt{x^2+y^2}\right).$$

Additional Literature

http://samoucka.ru/document22180.html

■ Corrugated Surface of Revolution of a General Sinusoid

A corrugated surface of revolution of a general sinusoid

$$x = a\sin\frac{n\pi z}{b} + c$$

about the axis Oz contains circular parts of both positive and negative curvatures.

General sinusoid in contrast to *usual sinusoid* ($x = \sin z$) is elongated $|a|$ times along the axis Ox and contracted $b/(n\pi)$ times along the axis Oz, where n is an integer, b is a dimension of an integer n of half-waves of the sinusoid.

A period of the function is $T = 2b/n$.

A volume of a body bounded by a surface of revolution of the half-wave of a usual sinusoid $x = \sin z$ is equal to $\pi^2/2$.

Forms of definition of the surface

(1) Explicit equation:

$$z = \frac{b}{n\pi}\arcsin\frac{\sqrt{x^2+y^2} - c}{a}.$$

(2) Implicit equation:

$$x^2 + y^2 - \left(a\sin\frac{n\pi z}{b} + c\right)^2 = 0.$$

(3) Parametrical equations (Fig. 1):

$$x = x(z,\beta) = r(z)\cos\beta,$$
$$y = y(z,\beta) = r(z)\sin\beta,$$
$$z(z) = z,$$

where $r = r(z) = a\sin\frac{n\pi z}{b} + c$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2n^2\pi^2}{b^2}\cos^2\frac{n\pi z}{b}, \quad F = 0, \quad B = r(z),$$
$$L = \frac{an^2\pi^2}{Ab^2}\sin\frac{n\pi z}{b}, \quad M = 0, \quad N = \frac{r(z)}{A},$$
$$k_1 = k_z = \frac{an^2\pi^2}{A^3b^2}\sin\frac{n\pi z}{b}, \quad k_2 = k_\beta = \frac{1}{r(z)A},$$
$$K = \frac{an^2\pi^2}{rA^4b^2}\sin\frac{n\pi z}{b}.$$

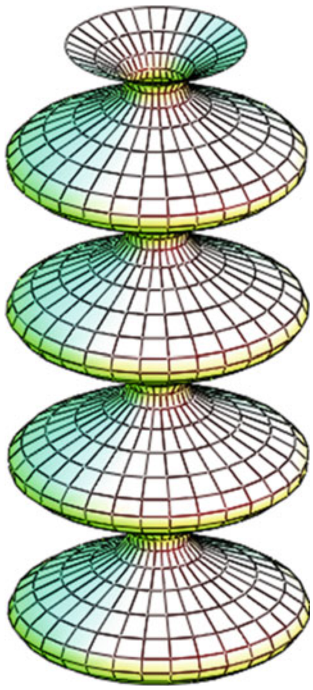


Fig. 1

Fig. 2

The curvilinear coordinate net is put down to lines of principal curvatures β and z.

In Fig. 1, the corrugated surface of revolution of a general sinusoid is shown when $a < c$. Having assumed $c \gg a$, we can obtain *a corrugated cylinder* (Wolfram Demonstrations Project) or *a sinusoidal cylinder* (SpringerImages).

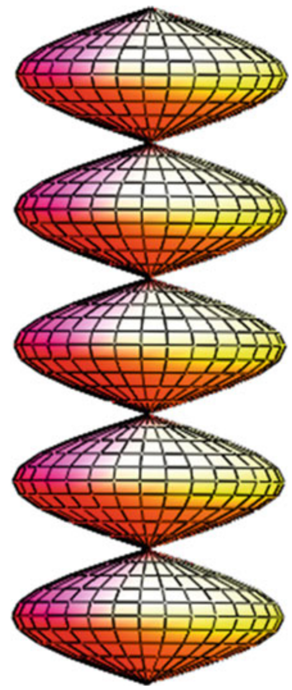


Fig. 3

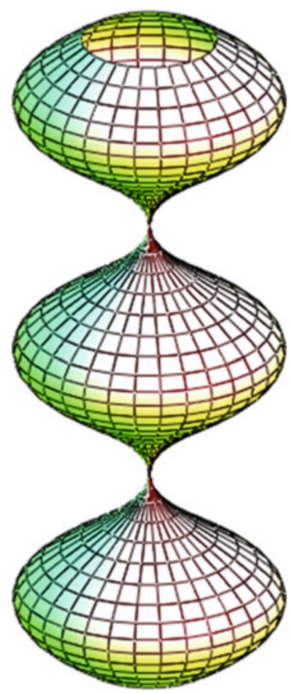


Fig. 4

In Fig. 2, the surface of revolution has $a > c$; in Fig. 3, it is $c = 0$, and in Fig. 4, the surface of revolution has $a = c$.

The surface of revolution represented in Fig. 1 is called "*Isolator*."

The surfaces of revolution shown in Figs. 1, 2, and 4 have the parts of both positive and negative Gaussian curvatures.

The surface of revolution represented in Fig. 3 is a surface of positive Gaussian curvature.

Additional Literature

Krivoshapko AN, *Halabi SM, Se Tsyan*. Analytical surfaces with a sine generatrix. Vestnik RUDN. "Engineering Researches". 2005; No. 1 (11), p. 115-120.

Zhulaev VP, *Sultanov BZ*. Screw pumping stations for recover of oil: Manual. Ufa: Izd-vo UShU, 1997; 43 p.

2014 Wolfram Demonstrations Project: http://demonstrations.wolfram.com/SinusoidalBellows/

SpringerImages: http://www.springerimages.com/Images/RSS/1-10.1007_s00348-005-0981-9-0

■ Surface of Revolution of a Parabola of Arbitrary Position

A surface of revolution of a parabola of an arbitrary position is formed by rotation of a parabola $Y(t) = ct^2$ with the axis Y, turned relatively to an axis of rotation Oz at the θ angle, about the axis Oz. A peak of the parabola lies at the distance a from the axis of rotation (Fig. 1).

Forms of definition of the studied surface

(1) Parametrical equations (Fig. 1):

$$x(u,t) = \left(a + t\cos\theta + ct^2\sin\theta\right)\cos u;$$
$$y(u,t) = \left(a + t\cos\theta + ct^2\sin\theta\right)\sin u;$$
$$z(u,t) = -t\sin\theta + ct^2\cos\theta.$$

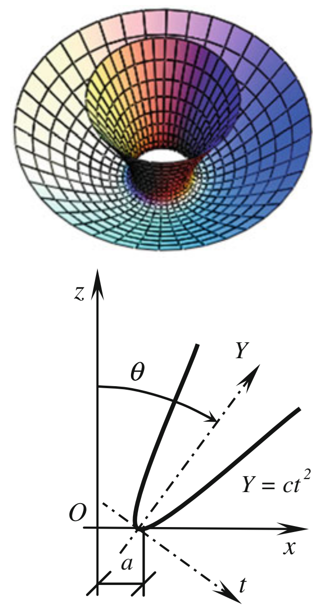
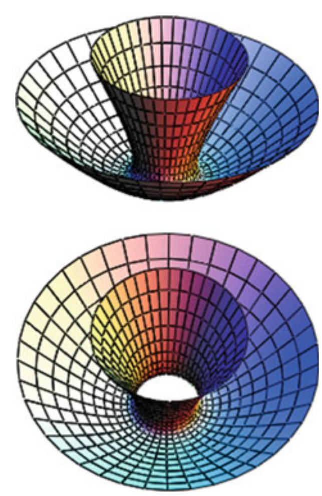


Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \left(a + t\cos\theta + ct^2\sin\theta\right);$$
$$F = 0; \quad B^2 = 1 + 4c^2t^2;$$
$$L = \left(a + t\cos\theta + ct^2\sin\theta\right)\frac{2ct\cos\theta - \sin\theta}{B};$$
$$M = 0; N = \left(a + t\cos\theta + ct^2\sin\theta\right)\frac{2c}{B};$$
$$k_u = k_1 = \frac{2ct\cos\theta - \sin\theta}{AB},$$
$$k_t = k_2 = \frac{2c}{B^3}.$$



$a = 0.8$ m; $c = 2$ m^{-1}; $\theta = 0.2\pi$

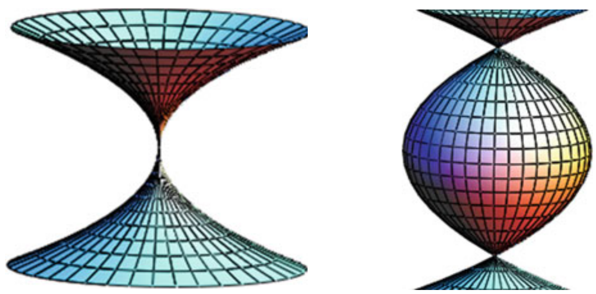
Fig. 2

In Fig. 2, the surface of revolution of positive Gaussian curvature is shown when $a = 0.8$ m; $c = 2$ m^{-1}; $\theta = 0.2\pi$.

In Fig. 3, the studied surfaces of revolution of negative Gaussian curvature are presented. Here, the surface given in Fig. 3a has $\theta = \pi/2$, $a = 0$, $c = 1$ m^{-1}, but the surface in Fig. 3b has $\theta = -\pi/2$, $a = 0.8$ m; $c = 1$ m^{-1}. These surfaces are studied in the section "*Surface of revolution of a parabola*" of the Chap. "2. Surfaces of revolution".

In Fig. 4, two types of the studied surfaces of revolution are presented some more.

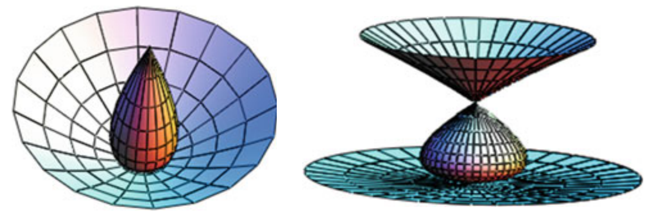
(a) (b)



$a = 0$; $c = 1\ \text{m}^{-1}$; $\theta = \pi/2$ $a = 0.8$ m; $c = 1\ \text{m}^{-1}$; $\theta = -\pi/2$

Fig. 3

Assume a slope angle of the axis of a parabola to an axis of rotation equal to zero (θ = 0) and the distance a peak of the parabola from the rotation axis equal to zero (a = 0) too, then the studied surface of revolution will degenerate into *a paraboloid of revolution* that is considered in section "Paraboloid of revolution".



$a = 0.5$ m; $c = 1\ \text{m}^{-1}$; $\theta = -\pi/5$ $a = 0.8$ m; $c = 1\ \text{m}^{-1}$; $\theta = -2\pi/5$

Fig. 4

Additional Literature

Ivanov VN. Geometry and design of shells on the base of surfaces with a system of curvilinear coordinate lines in the pencil of planes. Spatial Structures of Buildings and Erections: Collected articles. Moscow: OOO "Devyatka Print". 2004; vol. 9, p. 26-35 (13 ref.).

Weisstein Eric W. "Parabola". From MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com/Parabola.html

■ Surface of Revolution of a Biquadrate Parabola

A paraboloid of revolution of the fourth order is generated by a rotating biquadrate parabola about its axis of symmetry, i.e., about the axis of the parabola.

A surface of revolution of a biquadrate parabola is formed in the process of rotation of a biquadrate parabola about a straight that is perpendicular to the parabola axis.

Forms of definition of the surface of revolution

(1) Parametrical equations (Fig. 1):

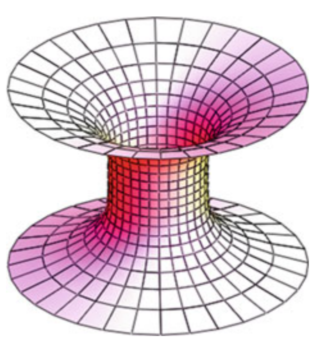
$$x = x(r,\beta) = r\cos\beta, \quad y = y(r,\beta) = r\sin\beta,$$
$$z = z(r) = \sqrt[4]{c(r-a)},$$

where $r = a$ is a radius of *the waist circle*, $|x| \geq a$, $|y| \geq a$, $0 \leq \beta \leq 2\pi$. The surface is formed by rotation of *a parabola of the fourth order*

$$z^4 = c(x - a)$$

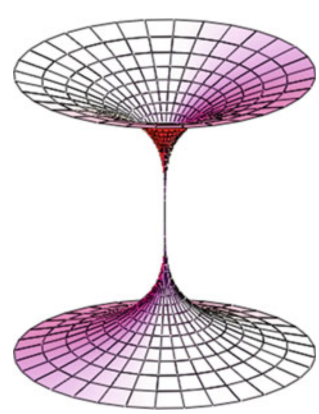
about the axis z. In Fig. 1, the surface of rotation of the biquadrate parabola is shown when $a > 0$.

Having assumed a = 0, we can design the surface of revolution presented in Fig. 2. If $a \geq 0$, then the surface of revolution of the biquadrate parabola belongs to a class of *surfaces of negative Gaussian curvature*.



$(a > 0)$

Fig. 1



$(a = b = 0)$

Fig. 2

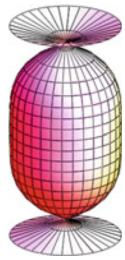

$(b > 0)$

Fig. 3

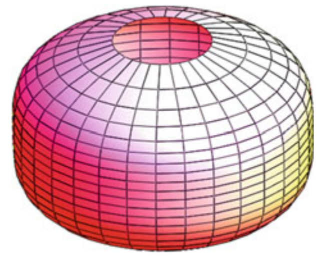

$(b > 0)$

Fig. 4

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{\sqrt{\;}}{16(r-a)^{3/2}}, \quad F = 0, \quad B = r,$$
$$L = -\frac{3-^{1/4}}{16A(r-a)^{7/4}}, \quad M = 0, \quad N = \frac{c^{1/4}r}{4A(r-a)^{3/4}},$$
$$k_1 = k_r = -\frac{3c^{1/4}}{16A^3(r-a)^{7/4}}, \quad k_2 = k_\beta = \frac{c^{1/4}}{4Ar(r-a)^{3/4}},$$
$$K = -\frac{3\sqrt{c}}{64rA^4(r-a)^{5/2}} < 0.$$

(2) Parametrical equations (Figs. 3 and 4):

$$x = x(z,\beta) = \left[\frac{z^4}{c} - b\right]\cos\beta,$$
$$y = y(z,\beta) = \left[\frac{z^4}{c} - b\right]\sin\beta,$$
$$z = z,$$

where $b \geq 0$ is a distance between a peak of the parabola and the axis of rotation.

If $b = 0$, then we can produce the surface shown in Fig. 2. In Fig. 3, the surface is shown when $b > 0$. Having assumed $b > 0$ and$-bc < z^4 < bc$, we can have *a barrel-shaped surface* of revolution of positive Gaussian curvature (Fig. 4). A surface of revolution of a biquadrate parabola has two conical points:

$$x = y = 0, \quad z = \pm(cb)^{1/4}.$$

If $z^4 > |bc|$, then a surface of revolution of a biquadrate parabola becomes a surface of negative Gaussian curvature.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + 16\frac{z^6}{c^2}, \quad F = 0, \quad B^2 = \left(\frac{z^4}{c} - b\right)^2,$$
$$L = -\frac{12z^2}{cA}, \quad M = 0, \quad N = \frac{B}{A},$$
$$k_1 = k_z = -12\frac{z^2}{cA^3}, \quad k_2 = k_\beta = \frac{1}{AB}, \; K = -\frac{12z^2}{cA^4B}.$$

■ Ellipsoid of Revolution

An ellipsoid of revolution is a surface formed by rotating of an ellipse

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$$

about its axis of symmetry Oz. An ellipsoid of revolution is *a closed quadric surface*. Older literature uses "*spheroid*" in place of "ellipsoid of revolution." *An oblate spheroid* (*oblate ellipsoid of revolution*) is formed by rotation of the ellipse about its minor axis (Fig. 1a). A special case arises when $a = b$, then the surface is *a sphere* and the intersection with any plane passing through it is a circle (Fig. 1b). *A prolate spheroid* (*prolate ellipsoid of revolution*) is formed by rotation of the ellipse about its major axis (Fig. 1c).

An ellipsoid of *revolution lies inside* the rectangular parallelepiped bounded by the sides $-a \leq x \leq a$; $-a \leq — \leq a$; $-b \leq z \leq b$. The geodesic line coincides with the equator parallel of an ellipsoid of revolution. The geodesic line passing through a pole point of an ellipsoid passes through an opposite pole point too. A volume contained inside the surface of ellipsoid of revolution is

$$V = \frac{4}{3}\pi a^2 b.$$

In cartography, the Earth is often approximated by an oblate spheroid instead of a sphere. The current World

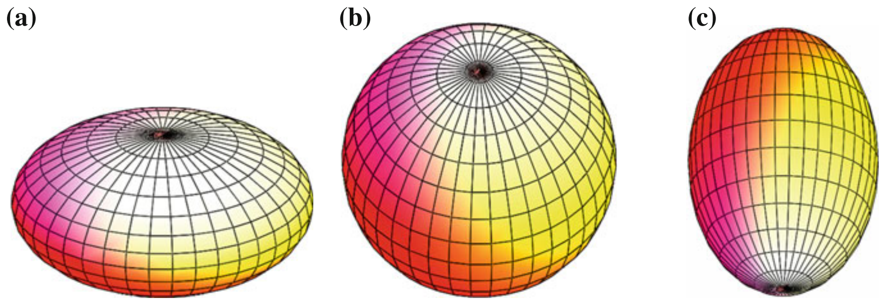


Fig. 1 **a** The oblate ellipsoid of revolution ($a > b$). **b** The sphere ($a = b$). **c** The prolate ellipsoid of revolution ($a < b$)

Geodetic System model uses a spheroid whose radius is 6,378.137 km at the equator and 6,356.752 km at the poles.

Forms of definition of the surface

(1) The standard equation of an ellipsoid of revolution centered at the origin of a Cartesian coordinate system and aligned with the axes is:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

(2) Parametrical equations (Fig. 1):

$$x = x(\alpha, \beta) = a \cos\alpha \cos\beta,$$
$$y = y(\alpha, \beta) = a \sin\alpha \cos\beta,$$
$$z = z(\beta) = b \sin\beta;$$
$$0 \le \alpha \le 2\pi;\ -\pi/2 \le \beta \le \pi/2.$$

Coefficients of the fundamental forms of the surface:

$$A = a \cos\beta, F = 0; B^2 = a^2 \sin^2\beta + b^2 \cos^2\beta;$$
$$L = ab \cos^2\beta/B; M = 0; N = -ab/B.$$

Coordinate lines α and β (parallels and meridians) are lines of principal curvatures.

(3) Parametrical equations (Fig. 2):

$$x = x(u, v) = \rho \sin u \cos v,$$
$$y = y(u, v) = \rho \sin u \sin v,$$
$$z = z(u) = \rho \cos u,$$

where $\rho = \dfrac{b}{\sqrt{1 + \omega \sin^2 u \cos^2 v}}$; $\omega = \dfrac{b^2}{a^2} - 1.$

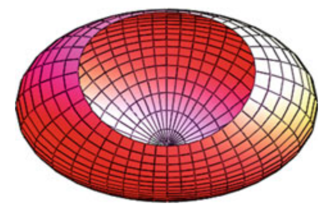


Fig. 2 The ellipsoid of revolution with the elliptical opening, $u_o \le u \le \pi$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \rho\sqrt{1 + \left(\frac{\omega}{2a^2}\rho^2 \sin 2u \cos^2 v\right)^2};\ F = 0;$$

$$B = \rho \sin u \sqrt{1 + \left(\frac{\omega}{2a^2}\rho^2 \sin 2v \sin^2 u\right)^2};$$

$$k_1 = \frac{ab}{\left[b^2 + \omega(\rho \sin u \cos v)^2\right]^{3/2}};$$

$$k_2 = \frac{1}{\rho\sqrt{1 - (1 - a^4/b^4)\sin^2 u \cos^2 v}}.$$

Coordinate lines *u, v* form *the geographic system of coordinates* but they are not lines of principal curvatures.

Additional Literature

Krivoshapko SN. Research on general and axisymmetric ellipsoidal shells used as domes, pressure vessels, and tanks. Applied Mechanics Reviews (ASME). 2007; vol. 60, No. 6, p. 336-355.

"Ellipsoid" by *Jeff Bryant*, Wolfram Demonstrations Project, 2007.

■ Ding–Dong Surface

A surface of revolution "*Ding–Dong Surface*" is like a surface of revolution "*Kiss surface.*"

Forms of definition of the surface

(1) Implicit equation: $x^2 + y^2 = (1 - z)z^2$

So, the studied surface of revolution is an algebraic surface of the third order. It is obtained by rotating curve

$$x = x(z) = z(1-z)^{1/2}$$

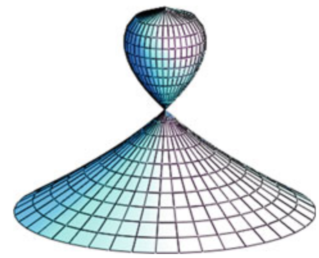
about an axis Oz.

(2) Parametrical equation (Fig. 1):

$x = x(u, v) = r(v)\cos u,\ y = y(u, v) = r(v)\sin u,$

$z = z(v) = v,$

where $r(v) = v\sqrt{1 - v}$; $-\infty \le v \le 1$; $0 \le u \le 2\pi$.



$(-1 \le v \le 1)$

Fig. 1

Additional Literature

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■ "Eight Surface"

A surface of revolution "*Eight Surface*" is generated by rotation of a curve

$$x = x(z) = 2z\left(1 - z^2\right)^{1/2}$$

about the axis Oz. The surface pictured in Fig. 1 is called an eight surface because it is a surface of revolution of a figure eight.

Forms of definition of the surface

(1) Implicit equation:

$$x^2 + y^2 = 4\left(1 - z^2\right)z^2.$$

Hence, the studied surface is *an algebraic surface of the fourth order*.

(2) Parametrical equations (Fig. 1):

$x = x(u, v) = \cos u \sin 2v, \quad y = y(u, v) = \sin u \sin 2v,$

$z = z(v) = \sin v,$

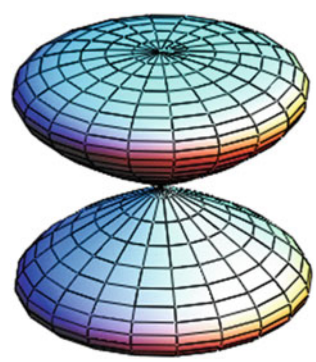


Fig. 1

where $-\pi/2 \le v \le \pi/2$; $0 \le u \le 2\pi$. The surface comes to a point at its very center.

Reference

The Eight Surface: http://www.math.hmc.edu/~gu/math142/mellon/curves_and_surfaces/surfaces/eightsurf.html

■ Surface of Revolution "Egg" of the Fourth Order

Eggshell is one of the perfect natural forms. Having researched *closed two-focus curves* of the fourth order, one can obtain an equation of mathematical model of the meridian cross section of an eggshell. G.V. Brandt considered that an egg form can be described by an implicit equation of the fourth order:

$$z^2 + y^2 = 3x(2a - x)\left[1 - c^2/(x + a)^2\right]/4,$$

where $2a$ is a length of major axis (an axis of rotation); c is the interfocus distance; $(a - c)/2$ is the distance the origin of a Cartesian coordinates from the first focus of meridional curve.

Parametrical equations of a surface of revolution "Egg" can be written in the form:

$$x = x, \quad y = y(x, \varphi) = r(x)\cos\varphi,$$
$$z = z(x, \varphi) = r(x)\sin\varphi,$$

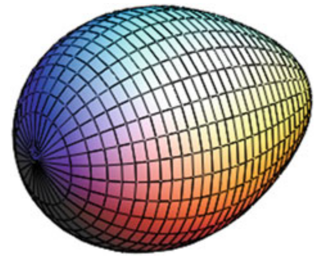


Fig. 1

where $r(x) = \sqrt{\frac{3}{4}x(2a - x)\left[1 - \frac{a^2\beta^2}{(x+a)^2}\right]}$, $\beta = c/a$ is a coefficient characterized a form of the meridian. A surface "*Quail Egg*" with $\beta = 0.75$ is presented in Fig. 1.

Reference

Brandt GV. The research of an equation of a shell formed by the two-focus curve. Sb. tr. VZPI: "Stroitelstvo i Arhitektura". Moscow: VZPI. 1973; p. 76-86.

■ Surface of Revolution "Egg" of the Third Order

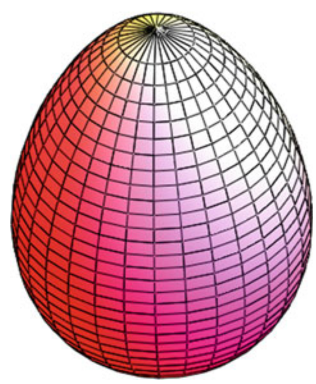
It is known also a surface of revolution "Egg" which is given by an implicit equation of the third order:

$$x^2 + y^2 = c^2 z(z - a)(z - b),$$

where a, b, c are constant parameters determining the form of a surface. Parametrical equations of the third-order surface of revolution "Egg" (Fig. 1) can be given as

$$x = x(u, v) = c\sqrt{u(u - a)(u - b)}\sin v,$$
$$y = y(u, v) = c\sqrt{u(u - a)(u - b)}\cos v,$$
$$z = z(u) = u,$$

where $a <= b$, then $0 \le v \le 2\pi$, $0 \le u \le a$.



$a = 1$ cm; $b = 1.5$ cm;
$c^2 = 0.85^2$ cm^{-1}

Fig. 1

■ Piriform Surface

This surface of revolution resembles a coming to the surface soft capacity with load. In English language literature, this surface is called "Piriform Surface".

Parametrical equations are

$$x = x(u, v) = b[\cos v(r + \sin v)]\cos u,$$
$$y = y(v) = a(r + \sin v),$$
$$z = z(u, v) = b[\cos v(r + \sin v)]\sin u,$$

where $0 \le u \le 2\pi$, $-\pi/2 \le v \le \pi/2$; a, b, and r are constant coefficients defining the form of the surface (Fig. 1).

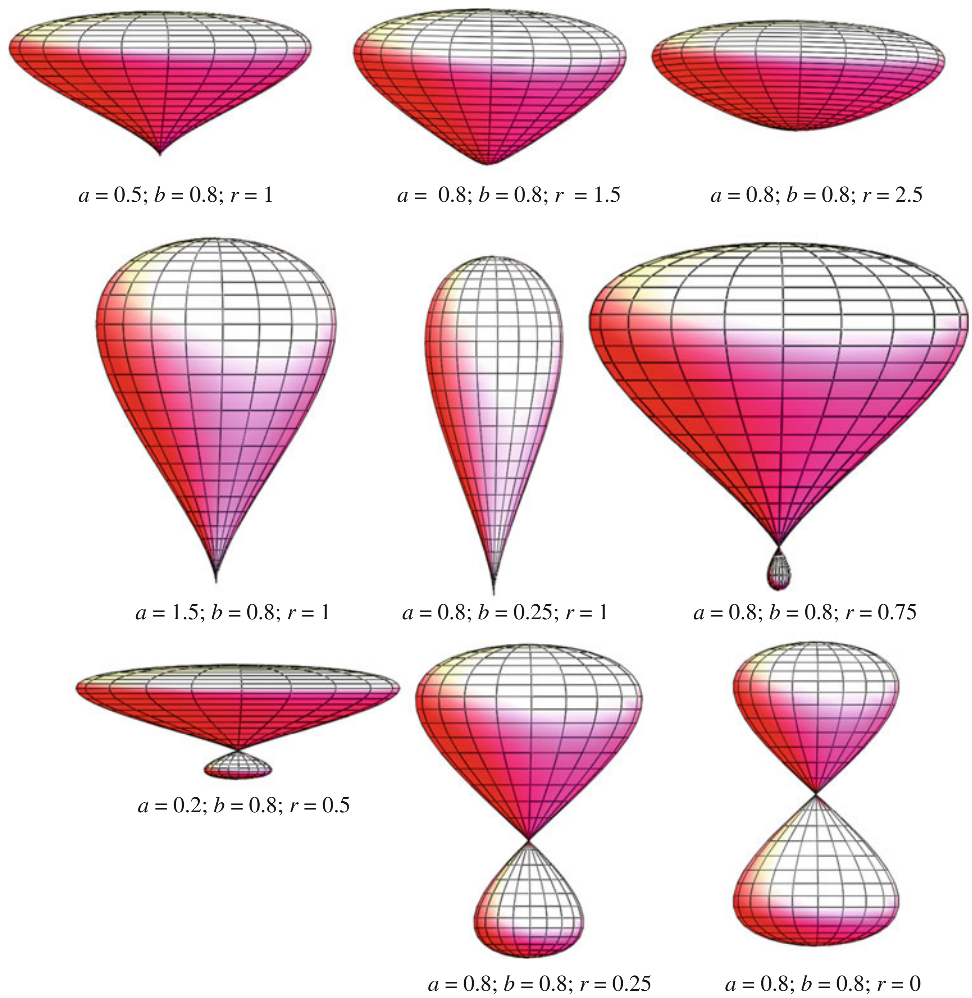


Fig. 1

■ "Drop"

Assuming certain values of constant parameters entering into parametrical equations of a surface of revolution "Drop," one can obtain the form of a drop in the process of falling.

Parametrical equations of the surface can be given as (Figs. 1 and 2):

$$x = x(u, v) = a(b - \cos u) \sin u \cos v,$$

$$y = y(u, v) = a(b - \cos u) \sin u \sin v,$$
$$z = z(u) = \cos u,$$

where $0 \le u \le \pi$, $0 \le v \le 2\pi$; a and b are constant coefficients defining the form of the surface.

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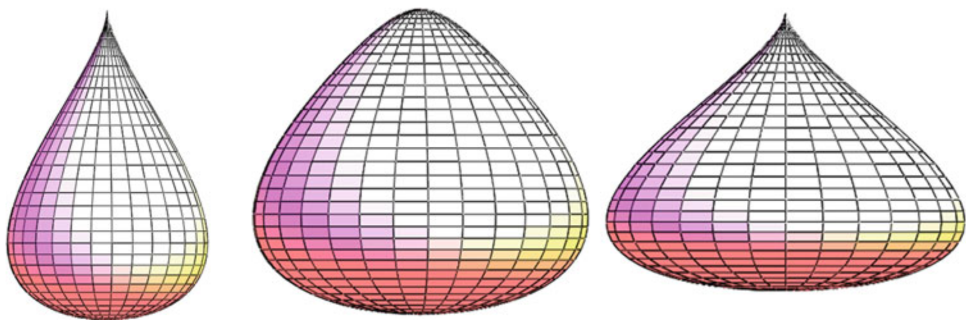
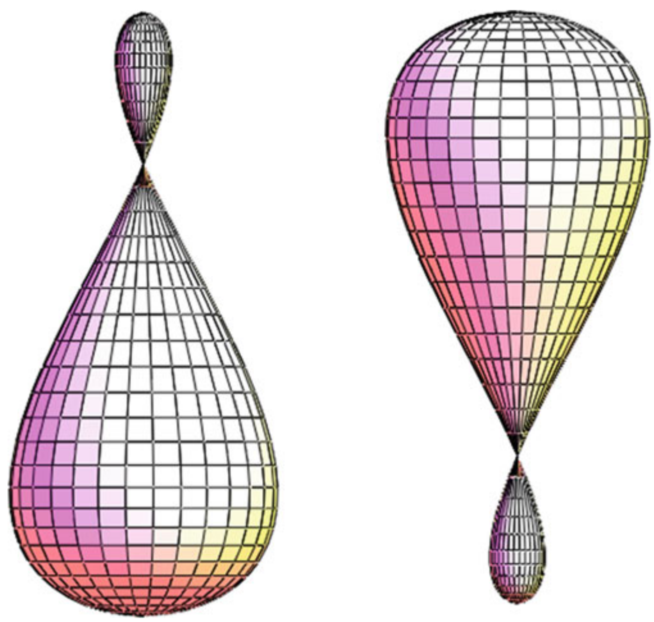
Parametrische Flächen und Körper. - http://www.3d-meir.de/tut3/Seite44.html



$a = 0.5$; $b = 1$ $a = 0.5$; $b = 2$ $a = 1$; $b = 1$

Fig. 1



$a = 0.5$; $b = 0.5$ $a = 0.5$; $b = –0.5$

Fig. 2

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Additional Literature

P.S.: Additional literature is given at the corresponding pages of the Chap. “2. Surfaces of Revolution”.

2.1 Middle Surfaces of Bottoms of Shells of Revolution Made by Winding of One Family of Threads Along the Lines of Limit Deviation

Shells of revolution made by winding of one family of threads along the lines of limit deviation are used in pressure vessels from composite materials. They consist of a cylindrical fragment and two bottoms that are jointed smoothly just between themselves along the edges. The bottoms end by the pole openings with metal flange for the fixing of the cover. A pressure vessel from composed materials made by a method of winding of high-strength threads is more adaptable to streamlined production and gives a reduction of 30–50 % in weight in comparison with metal analogies.

Inner forces appearing in the bottom under inner pressure must be oriented along the threads in its every point.

An equation of *a middle surface of bottoms of shells of revolution made by winding of one family of threads along the lines of limit deviation* is derived from the decision of a nonlinear ordinary differential equation:

$$\frac{y''}{y'(1+y'^2)} = \frac{2r}{r^2-t^2} - \frac{\mathrm{tg}^2\varphi}{r}$$

obtained on the base of a momentless theory of analysis of shells made of threads. The following conventions are used in the formula: $y = f_1(r)$ is an equation of a meridian of the middle surface of the bottom of revolution; r is a radial coordinate of a generatrix line of the bottom (meridian); the primes mean the differentiation with respect to a coordinate r; φ is an angle of the thread with a meridian of the surface of the bottom. In every point of the shell surface, a tread with an angle $+\varphi$ corresponds the thread with the angle $-\varphi$; a parameter t is equal to zero for the pole opening closed by the cover or to the radius r_p of the opening in the cover.

Trajectories of the threads of the shell must satisfy a condition of *technological realizably*, i.e., absolute value of tangent of the angle between the normal to the trajectory of a thread and the normal to the surface must not go over the coefficient of friction k of the thread on the surface in the process of winding. It can be written as

$$\left| \frac{r\varphi' \cos\varphi + \sin\varphi}{\frac{ry'' \cos^2\varphi}{1+y'^2} + y' \sin^2\varphi} \right| \le k.$$

For shell of revolution made by winding of one family of threads along the lines of limit deviation, an equation of generatrix surface $y = f_1(r)$ and an equation of the trajectories of the threads $\varphi = f_2(r)$ are calculated numerically from the solution of Augustin Louis Cauchy problem for a system of two differential equations that are the equation of generatrix curve of the surface of revolution and the equation of technological realizably with a sign of an equality in the right part and with a meaning $k_0 \le k$. An angle φ of a thread at the pole must be equal to 90° due to a condition of continuity of automatized winding.

The given differential equations give an opportunity to find a form of generatrixes of a surface of bottoms and the trajectory of threads of pressure vessels with maximally differing radiuses of pole openings.

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2.2 Middle Surfaces of Bottoms of Shells of Revolution Made by Plane Winding of Threads

Shells of revolution made by plane winding are used in pressure vessels from composite materials. They consist of a cylindrical fragment and two bottoms that are jointed smoothly just between themselves along the edges. The bottoms end by the pole openings with metal flange for the fixing of the cover. A pressure vessel from composed materials made by a method of winding of high-strength threads is more adaptable to streamlined production and gives a reduction of 30–50 % in weight in comparison with metal analogies.

Inner forces appearing in the bottom of the shell under action of inner pressure must be oriented along the threads in its every point. An equation of the generatrix of *the middle surface of bottoms of shells of revolution made by plane winding of threads* is derived from the decision of a nonlinear ordinary differential equation:

$$\frac{y''}{y'(1+y'^2)} = \frac{2r}{r^2-t^2} - \frac{\mathrm{tg}^2\varphi}{r}$$

obtained on the base of a momentless theory of analysis of shells made of threads. The following conventions are used in the formula: $y = y(r)$ is an equation of a meridian of the middle surface of the bottom of revolution; r is a radial coordinate of a generatrix curve of the surface of revolution of bottom. The primes mean the differentiation with respect to a coordinate r; φ is an angle of the thread with a meridian of the surface of revolution of the bottom. In every point of the shell surface, a tread with an angle $+\varphi$ corresponds the thread with the angle $-\varphi$; a parameter t is equal to zero for the pole opening closed by the cover or to the radius r_p of the opening in the cover.

The threads of plane winding are placed on the surface of revolution in the planes tangent to the pole openings of the both bottoms in conformity with an equation

$$\mathrm{tg}\varphi = \frac{ry' - y}{\sqrt{1+y'^2}\sqrt{r^2\mathrm{ctg}^2\gamma - y^2}},$$

where γ is the angle of the plane with a thread with the axis of rotation of a surface of the bottom. An angle φ of a thread at the pole must be equal to 90° due to a condition of continuity of winding.

An equation of a meridian of the middle surface $y = y(r)$ for a shell of revolution made by plane winding is turn up from the solution of A.L. Cauchy problem for a nonlinear ordinary differential equation

$$\frac{y''}{y'(1+y'^2)} = \frac{2r}{r^2 - t^2} - \frac{(ry' - y)^2}{r(1+y'^2)(r^2\mathrm{ctg}^2\gamma - y^2)},$$

which is obtained by equating corresponding parts of two given above differential equations. The given differential equations give an opportunity to find a form of generatrix curves of middle surfaces of bottoms and the trajectory of threads of pressure vessel both with equal and different radiuses of pole openings of two bottoms.

The calculated trajectory of laying of the thread in the process of winding must satisfy a condition of *technological realizably*, i.e., absolute value of tangent of the angle between the normal to the trajectory of a thread and the normal to the surface must not go over the coefficient of friction k of the thread on the surface in the process of winding. This condition is presented in the previous section.

Reference

Vasil'ev VV, Protasov VD, Bolotin VV et al. Composite Materials. Reference book. Moscow: "Mashinostroenie", 1990; 512 p.

2.3 Middle Surface of Bottoms of Shell of Revolution Made by Winding of Threads Along Geodesic Lines

Pressure vessels from composed materials made by a method of winding of high-strength threads along geodesic lines are more adaptable to streamlined production and give a reduction of 30–50 % in weight in comparison with metal analogies.

The laying of threads on a surface along geodesic lines maintains a stable position of threads in the process of their winding in conformity with A. Clairaut equation: $r \sin \varphi = r_{0,}$ where φ is the angle of the thread with the generatrix curve of a surface of revolution. In every point of the middle surface of a shell of revolution, a tread with an angle $+\varphi$ corresponds the thread with the angle $-\varphi$; r_0 is the radius of the pole opening. The form of a generatrix curve $y = y(r)$ of the middle surface of revolution of the bottom ensures the direction of inner forces, appearing in the shell of the bottom under action of inner pressure, along the threads. A generatrix of the surface of bottom with a flange is computed as a result of consistent solution of two differential equations:

$$\frac{\mathrm{d}y_1}{\mathrm{d}r} = -\frac{r(r^2 - t^2)\sqrt{a^2 - r^2}}{\sqrt{a^2(r^2 - r_0^2)(a^2 - t^2)^2 - r^2(a^2 - r_0^2)(r^2 - t^2)^2}}$$

where $b \le r < a$,

$$\frac{\mathrm{d}y_2}{\mathrm{d}r} = -\frac{r(b^2 - r_0^2)\sqrt{(a^2 - r_0^2)(r - r_0^2)}}{\sqrt{a^2(b^2 - r_0^2)^2(a^2 - r_0^2)^2 - r^2(r^2 - r_0^2)(b^2 - r_0^2)^2(a - r_0^2)}}$$

$r_0 \le r \le b$, $y = y(r)$ is a axial coordinate of a generatrix curve of the bottom; a is the radius of the cylindrical segment of the shell of revolution; b is the maximal radius of the flange; a parameter t is equal to zero for the pole opening closed by the cover or to the radius r_p of the opening in the cover.

A.L. Cauchy problem for the first differential equation is solved with a initial condition that is $y_1 = 0$ if $r = a$. For the second differential equation, an initial condition is $y_2 = y_1$ if $r = b$. The first and the second equation can be solved in *elliptical integrals*. Maximal radius of the flange for the convex surface of the bottom must satisfy a condition:

$$b \ge \frac{\sqrt{3}}{2} r_0 \sqrt{1 + \sqrt{1 - \frac{8t^2}{9r_0^2}}}.$$

The form of the studied middle surface is shown in Fig. 1. An equation of the meridian $y = y(r)$ was derived numerically with the help of presented differential equations. A problem was solved for a surface of revolution with the following parameters: $a = 3$ m; $b = 1.3$ m; $r_0 = 1$ m, $t = 0$. The surface of revolution runs smoothly into the cylindrical segment of the pressure vessel.

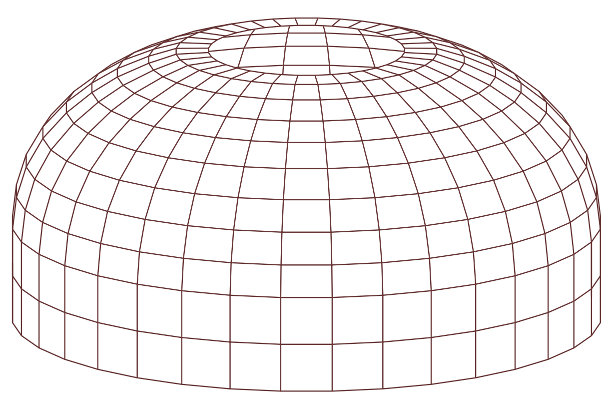

Fig. 1

References
Vasil'ev VV, Protasov VD, Bolotin VV et al. Composite Materials. Reference book. Moscow: "Mashinostroenie", 1990; 512 p.

Obraztzov IF, Vasil'ev VV, Bunakov VA. Optimal Design of Shells of Revolution from Composite Materials. Moscow: "Mashinostroenie", 1977; 144 p.

2.4 Middle Surfaces of Shells of Revolution with Given Properties

Many scientific works devote to discovering form of a meridian of the middle surface of thin-walled shell of revolution with given properties in advance. It is known the following criterions of selection of optimal form of shell of revolution: a cost of a shell, minimal weight [1], the absence of bending moments and tensile normal forces [2], the given stress state for acting external load [3], the given bearing capacity for optimal slope [4], maximal external load; minimal weight under limitation for value of the natural frequency and maximal displacements [5]; the absence of bending moments with taking into account inner pressure, dead weight and centrifugal forces [6]; maximal critical load [7, 8] or the selection of a form with taking into consideration another set of presented demands.

A condition of equi-strength of thin-walled shell of reservoir is assumed as a basis of analysis of *drop-shaped reservoir* for the liquid products [9]. Geometry of the middle surface of a shell is chosen on condition that tensile meridional and circular forces will be equal to each other and constant ($N_1 = N_2 = N =$ const) under an action of designed load. It means that a condition

$$1/R_1 + 1/R_2 = \gamma(h + y)/N = pN,$$

must be satisfied. This equation follows from the condition of equilibrium of a shell element (Laplace formula). Here R_1 and R_2 are radiuses of principle curvatures correspondingly in meridional and circular directions. The key designed load (inner pressure)

$$p = \gamma(h + y)$$

is a sum of hydrostatical pressure of liquid and uniform redundant pressure; y is the distance the peak from a considered point of the shell in the vertical direction; γ is a density of the product; h is a height of designed column of liquid.

In a paper [10], problems of existence of optimal forms of thin-walled shells possessing minimal mass and satisfying to corresponding geometrical limitations and satisfying to restrictions on acceptable number of cycles of external cyclical load were studied. In this paper, an equilibrium stress state of a membrane shell of revolution loaded by axisymmetric loads q_n, q_θ was described by the following equations:

$$\mathrm{d}(r_0 N_\alpha)/\mathrm{d}\alpha - N_\theta R_1 \cos\alpha + r_0 R_1 q_\alpha = 0,$$
$$N_\alpha/R_1 + N_\theta/R_2 = q_n,$$

$r_0 = R_2 \sin\alpha$. The symbolism is shown in Fig. 1 at Page 100.

E. Annaberdyev [11] offers a method of selection of the single surface of revolution passing through given parallels and having the given magnitudes of coefficients of the first fundamental form in the theory of surfaces

$$\mathrm{d}s^2 = E\mathrm{d}u^2 + G\mathrm{d}v^2.$$

We cannot design a surface of revolution when a finite number of its parallels is taken. A meridian of surface of revolution can be formed if we shall give the common tangents at the joints of the parallels for maintaining smoothness of the meridian.

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■ Surfaces of Revolution with Geometrically Optimal Rise

In applied geometry of surfaces, interest to methods of optimization of geometrical form of surfaces of revolution with given properties in advance arose time and again. It was considered that the most actual problem is the following: it is necessary to obtain a form of the surface with minimal area S covering the maximal volume V. It gives the lesser expenditure of materials and the lesser weight of the shell. The special criterion

$$n = V/S$$

was introduced into practice (Fig. 1).

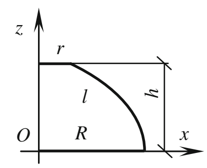

Fig. 1

An area S of the second-order surface and a volume covered by this surface can be defined with the help of the general formulas:

$$S = 2\pi \int_0^h x(z)\sqrt{1 + x(z)'^2}\mathrm{d}z, \quad V = \pi \int_0^h x(z)^2 \mathrm{d}z,$$

where $x = x(z)$ is an equation of a meridian; h is the rise of a surface, i.e., maximal rise of a surface over the plane xOy. A meridian is rotated about the axis Oz.

For concrete surfaces of revolution, these formulas give:

(1) **a truncated sphere**:

$$x = x(z) = \sqrt{a^2 - (z + \sqrt{a^2 - R^2})^2};$$
$$a = \sqrt{(R^2 - h^2 - r^2)^2/(4h^2) + R^2};$$
$$S = 2\pi ah; V = \frac{\pi h}{2}\left(R^2 + \frac{h^2}{3} + r^2\right);$$

$$n_{\text{sph.segm.}} = \frac{h(R^2 + h^2/3 + r^2)}{2\sqrt{(R^2 - h^2 - r^2)^2 + 4h^2R^2}};$$
$$n_{\text{sphere}} = \frac{R}{3};$$

(2) **a truncated cone**:

$$x = x(z) = R - (R - r)z/h;$$
$$S = \pi(R + r)\sqrt{(R - r)^2 + h^2};$$
$$V = \frac{\pi h}{3}\left(R^2 + r^2 + rR\right);$$
$$n_{\text{tr.c}} = \frac{h(R^2 + r^2 + rR)}{3(R + r)\sqrt{(R - r)^2 + h^2}},$$
$$n_{\text{c}} = \frac{hR}{3\sqrt{R^2 + h^2}};$$

(3) **a circular cylinder**:

$$S = 2\pi Rh; \quad V = \pi R^2 h; \quad n_{\text{cyl}}. = R/2.$$

(4) **a truncated paraboloid of revolution**:

$$x = x(z) = \sqrt{R^2 - z(R^2 - r^2)/h};$$

$$S = \frac{4\pi h}{3(R^2 - r^2)}\left\{\left[R^2 + \frac{(R^2 - r^2)^2}{4h^2}\right]^{3/2} - \left[r^2 + \frac{(R^2 - r^2)^2}{4h^2}\right]^{3/2}\right\};$$

$$V = \frac{\pi h}{2}(R^2 + r^2);\ n_{\text{par.}} = \frac{3Rh^3}{\left[(4h^2 + R^2)^{3/2} - R^3\right]};$$

$$n_{\text{tr.par.}} = \frac{3(R^4 - r^4)}{8\left[R^2 + (R^2 - r^2)^2/(4h^2)\right]^{3/2} - 8\left[r^2 + (R^2 - r^2)^2/(4h^2)\right]^{3/2}};$$

(5) **a truncated ellipsoid of revolution**:

$$x = x(z) = \sqrt{a^2 - \left(z/k + \sqrt{a^2 - R^2}\right)^2};$$

$$a \geq R;\ m^2 = a^2 - R^2;$$

$$\frac{c}{a} = k = \frac{h}{\sqrt{a^2 - r^2} - \sqrt{a^2 - R^2}};$$

$$V = \pi\left[a^2 h - \frac{(h + km)^3}{3k^2} + \frac{km^3}{3}\right];$$

$$n_{tr.el.} = \frac{V}{S},$$

where for **an oblate ellipsoid** with semi-axes $a > c$ ($k < 1$); $b^2 = 1 - k^2$, one has

$$S = \pi b\left[\frac{h + km}{k^2}\sqrt{\frac{k^4 a^2}{b^2} + (h + km)^2} - m\sqrt{\frac{a^2}{b^2} - R^2} + \frac{k^2 a^2}{b^2}\ln\frac{h + km + \sqrt{k^4 a^2/b^2 + (h + km)^2}}{k(m + \sqrt{a^2/b^2 - R^2}}\right];$$

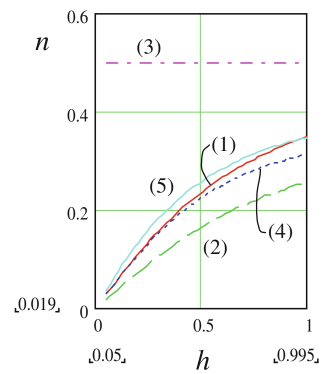

Fig. 2

for **a prolate ellipsoid** with semi-axes $a < c$ ($k > 1$); $t^2 = k^2 - 1 > 0$, one has

$$S = \pi t\left[\frac{h + km}{k^2}\sqrt{\frac{a^2 k^4}{t^2} - (h + km)^2} - m\sqrt{\frac{a^2 k^2}{t^2} - m^2} + \frac{a^2 k^2}{t^2}\left(\arcsin\frac{h + km}{ak^2}t - \arcsin\frac{mt}{ak}\right)\right].$$

Curves showing a change of the ratio $n = V/S$ with a change of a rise h give an opportunity to choose optimal parameters of the meridian for the given shell form (Fig. 2).

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Krivoshapko SN. Emel'yanova YuV. On a problem of surface of revolution with geometrically optimal rise. Montazh. i Spetz. Raboty v Stroit. 2006; 2, p. 11-14.

■ Middle Surface of Non-Bending Shell of Revolution Under Uniform Pressure

Under action of uniform pressure with corresponding boundary conditions, not only spherical and circular cylindrical shells deform without bending but also endless two-parametrical family of shells of revolution which includes a sphere and a cylinder as a particular case. In the process of axisymmetrical deformation, all normals to a middle surface do not turn, i.e., their angle of turn in the meridional plane is equal to zero. Besides, the angles of shearing between the meridians and parallels are equal to zero too and the angles between them remain equal to $\pi/2$.

Having assumed these propositions and using the first condition of Peterson-Codazzi

$$\frac{dR_2}{d\theta} = (R_1 - R_2)\frac{\cos\theta}{\sin\theta},$$

V.I. Gurevich and V.S. Kalinin derived a condition of absence of bending in shells of revolution in forces in the form:

$$\frac{R_2}{R_1}\frac{\mathrm{d}(N_2 - vN_1)}{\mathrm{d}\theta} + (1+v)(N_2 - N_1)\frac{\cos\,\theta}{\sin\,\theta} = 0$$

where R_1 and R_2 are the principal radiuses of curvatures of the meridian and the parallels accordingly; θ is the angle of a normal to the meridian with an axis of rotation; v is Poisson's ratio in theory of elasticity; N_1 and N_2 are the normal tensile or compressive forces reckoned per unit of curvilinear coordinates' length acting in the tangent plane of middle surface of the shell of revolution,

$$N_1 = \frac{pR_2}{2},\; N_2 = 0.5pR_2(2 - \frac{R_2}{R_1}).$$

A condition of absence of bending is correctly for shells of revolution subjected to any axisymmetrical loading. Substituting the values of normal forces in this condition, we can obtain its new interpretation:

$$\left(3 - \frac{R_2}{R_1}\right)\frac{\mathrm{d}R_2}{\mathrm{d}\theta} - R_2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{R_2}{R_1}\right) = 0$$

defining radiuses of principal curvatures of shell of revolution deforming without bending under action of uniform pressure.

It is obviously that not only radiuses of principal curvatures of sphere and cylinder satisfy this condition but shells with constant ratio $R_2/R_1 = 3$ too. In this case, $N_1 = N_2$. Assume that $z = f(x)$ is an equation of unknown meridian, then

$$R_1 = -\frac{(1+f'^2)^{3/2}}{f''},\quad R_2 = \frac{x}{\sin\theta} = \frac{x\sqrt{1+f'^2}}{f'}.$$

After substituting of values $R_1 = R_1(f)$ and $R_2 = R_2(f)$ into the differential equation of absence of bending, we can derive an equation of left branch of the meridian in the form of an integral:

$$z = f(x) = -\int\limits_{-\eta}^{x} \frac{2C_1C_2x^3\mathrm{d}x}{\sqrt{(C_1 - C_2x^2)^2 - 4C_1^2C_2^2x^6}},$$

which does not express itself in terms of elementary functions. Here, C_1 is constant.

In Fig. 1, taken from a paper of V.I. Gurevich and V.S. Kalinin, the meridians of non-bending shells of revolution having an angle $\theta = \pi/2$ when $x = \pm r_1$, i.e., $R_2 = r_1$, where r_1 is the radius of the support circle, are presented.

The surfaces represented in Fig. 1 divide by a sphere into closed and unclosed at the peak. Unclosed surfaces divide by a circular cylinder into the surfaces of negative and positive Gaussian curvatures near the support part.

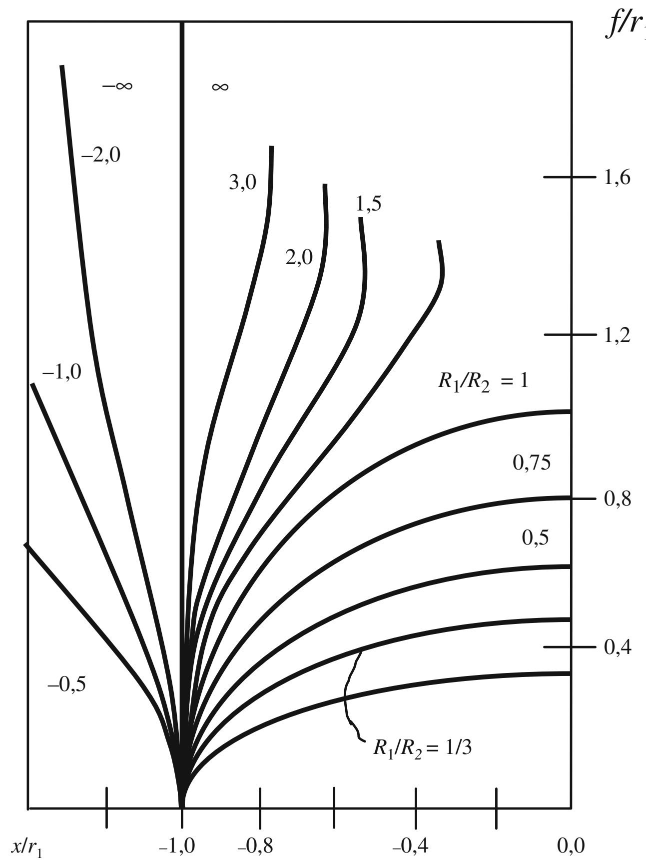

Fig. 1

Meridians were constructed under the condition that

$$C_1 = \frac{R_1}{r_1(r_1 - 3R_1)},\quad C_2 = \frac{R_1}{r_1^3(r_1 - R_1)}.$$

Dissertation of N.V. Cherdyntzev is devoted to seeking of forms of shells of revolution and differential equations of stress-strain state of non-bending shell of revolution under uniform external pressure are presented. An integral defining a form of the shell was reduced to a sum of two elliptical integrals and was presented also in the form of power series.

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2.5 Surfaces of Revolution with Extreme Properties

Let a plane curve $r = r(z)$ (Fig. 1) passing through the given points has the given length L and revolving about an axis Oz, forms a surface of revolution of the given area S. Besides this, the volume V bounded by this surface and by two planes that are perpendicular to the axis of revolution must have the greatest value. This is a classical variational problem about conditional extremum: if a curve $r = r(z)$ gives an extremum to an integral

$$V = \int_D \pi \cdot r^2 \mathrm{d}r$$

under conditions

$$L = \int_D \sqrt{1 + r'^2}\,\mathrm{d}z \quad \text{and} \quad S = \int_D 2\pi r \sqrt{1 + r'^2}\,\mathrm{d}z$$

then the constants λ_0, λ_1, and λ_2 (*Lagrange multipliers*) exist and the curve $r = r(z)$ gives the extremum to an integral

$$Q = \int_D H \mathrm{d}z$$

where

$$H = \lambda_0 \pi r^2 + \lambda_1 \sqrt{1 + r'^2} + 2\lambda_2 \pi r \sqrt{1 + r'^2}.$$

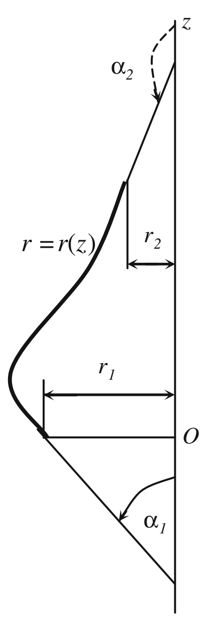

Fig. 1

Taking into consideration that this problem due to a reciprocity principle is equivalent to other two problems about conditional extremum:

(1) Obtain a plane curve $r = r(z)$ of a given length L which rotating about an axis Oz forms a surface of the minimal area bounding the given volume V.
(2) Obtain a plane curve $r = r(z)$ of the minimal length L which rotating about an axis Oz forms a surface of the given area S bounding the given volume V.

An Euler equation for the functional H is

$$H - \frac{\partial H}{\partial r'} = C,$$

because the function H does not depend explicitly on z, i.e., $H = H(r,r')$.

After transformation, we can derive an equation $z = z(r)$ in the integral form:

$$z = \int \frac{(C - \lambda_0 r^2)\mathrm{d}r}{\sqrt{4(\lambda_1 + \lambda_2 r)^2 - (C - \lambda_0 r^2)^2}} + \gamma.$$

In general case, this integral can be expressed with the help of elliptical integrals. But having specific values of λ_0, λ_1, λ_2, and C, it is possible to integrate in the elementary functions. In this case, we shall obtain *a sphere* and *a torus* when $\lambda_1^2 - C\lambda_2^2 = 0$.

So, a sphere and a torus satisfy to all extremal conditions.

The expressions for Gaussian and mean curvatures of extreme surfaces have the following form:

$$K = \frac{(C - \lambda_0 r^2)(\lambda_2 C + \lambda_0 \lambda_2 r^2 + 2\lambda_0 \lambda_1 r)}{4r(\lambda_1 + \lambda_2 r)^3},$$

$$2H = \frac{\lambda_1(C - 3\lambda_0 r^2) - 2\lambda_0 \lambda_2 r^3}{2r(\lambda_1 + \lambda_2 r)^2}$$

Giving different values to Lagrange constants, we can obtain different forms of surfaces possessing by extreme properties. There are well-known surfaces such as *cylindrical surface, sphere, torus*, *catenoid*, little known and insufficiently studied surfaces such as *nodoid* and *unduloid*, and recently presented surfaces such as "*Penka*" and *a surface of catenoidal type*, among them.

One paper is devoted to investigation of extremal surfaces of rotation for area-type functional. The solutions of differential Euler–Lagrange equation are obtained. Also, the symmetry property of this surface is proved; the examples of functionals are demonstrated and their corresponding solutions are given.

A theorem of existence for nonholonomic rotation surfaces of zero total curvature of the second kind was proved in a paper of O.V. Vasil'eva . An example of a nonholonomic surface of this class was constructed.

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■ Surface of Catenoidal Type

Substitute $\lambda_0^* = 0,\ \lambda_1^* \neq 0,\ \lambda_2^* \neq 0$ into general equation for generatrix curves

$$z = \int \frac{(C^* - \lambda_0^* r^2)\mathrm{d}r}{\sqrt{4(\lambda_1^* + \lambda_2^* r)^2 - (C^* - \lambda_0^* r^2)^2}} + \gamma$$

of surfaces of revolution possessing by extremal properties then we can formulate a problem in the following form: determine a surface formed by rotation of a curve $r = r(z)$ about an axis Oz limited by two planes, that are perpendicular to the axis of rotation, and having the least area of the surface with given length of a generatrix meridian $r = r(z)$.

Due to *reciprocity theorem*, such surface is equivalent to a surface of given area formed by rotation of a line $z = z(r)$ with the least length about an axis Oz. Then an expression for generatrix curves, represented before, will have the following form:

$$z = \int\limits_D \frac{C\mathrm{d}r}{\sqrt{4(\lambda_1 + r)^2 - C^2}},$$

where we introduced the following symbolisms:

$$\lambda_1 = \frac{\lambda_1^*}{\lambda_2^*},\ C = \frac{C^*}{2\lambda_2^*}.$$

Having fulfilled the specific manipulations, one can obtain an equation of the meridian $r = r(z)$ expressed in elementary functions:

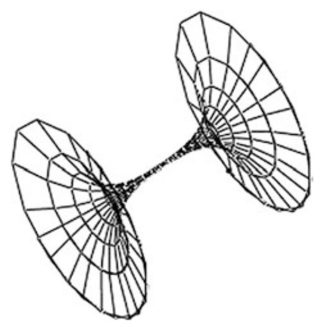

Fig. 1

$$r = C \cdot \cosh\frac{z - \gamma}{C} - \lambda_1.$$

The equation obtained is an equation of *a catenary* that is parallel transferred along an axis Oz at a distance of λ_1.

It should be noted that catenary is formed by *a focus of a parabola* in the process of rolling of this parabola along an axis Ox. The magnitude C is a parameter of the parabola. A value γ is defined by the initial position of the focus of the parabola.

A classical catenoid is formed by rotation of a catenary when this line is placed at the certain distance from the axis of rotation. A surface of revolution formed by rotation of a catenary displaced from this position will not be a minimal surface because the sum of principal curvatures of this surface is not equal to zero (Fig. 1).

Parametrical equations of *a surface of catenoidal type* can be written in the following form:

$$\begin{aligned} x = x(z,\beta)\ &= r(z)\cos\beta,\\ y = y(z,\beta)\ &= r(z)\sin\beta,\\ z = z. \end{aligned}$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \mathrm{ch}\frac{z-\gamma}{C},\ F = 0,\ B = r,$$
$$L = -\frac{1}{C},\ M = 0,\ N = \frac{B}{A},$$
$$k_z = k_1 = \frac{-1}{CA^2},\ k_\beta = k_2 = \frac{1}{AB},$$
$$K = \frac{-1}{CBA^3} = \frac{-C^2}{r(r+\lambda_1)^3} < 0,$$
$$2H = \frac{C}{r(r+\lambda_1)^2} \neq 0.$$

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Dao Chong Thi, *Fomenko AT*. Minimal surfaces and a problem of Plato. Moscow: "Nauka", 1987; 312 p

Pul'pinskiy YaS., *Cherevatskiy VB*. Modelling of extremum surfaces by soap films. Materialy Mezhdunarodnoy Nauchn. Konf. "Modelling as instrument of solving of technical and pertaining to the humanities problem". Part 1. Taganrog: TRTU, 2002; p. 62-65 (4 ref.).

■ "Penka"

Assuming $\lambda_0 \neq 0$; $\lambda_1 \neq 0$; $\lambda_2 = 0$ in the equation for generatrix curves of surfaces of revolution possessing extreme properties

$$z = \int \frac{(C^* - \lambda_0 r^2)\mathrm{d}r}{\sqrt{4(\lambda_1 + \lambda_2 r)^2 - (C^* - \lambda_0 r^2)^2}} + \gamma,$$

we can raise a problem in the following form: determine a curve

$$r = r(z)$$

of the given length in the process of rotation of which about an axis Oz, a surface of revolution is formed and together with two planes, that are perpendicular to the axis Oz, it envelops a maximal volume.

Assume $\lambda = \lambda_1/\lambda_0$, $C = C^*/\lambda_0$, then an integral expression for the generatrix meridian of a surface of revolution has the form:

$$z = \int \frac{(C - r^2)\mathrm{d}r}{\sqrt{4\lambda^2 - (C - r^2)^2}} + \gamma.$$

In this case, Gaussian and mean curvatures, radiuses of principal curvatures are

$$K = -\frac{(C - r^2)}{2\lambda^2}; \quad 2H = \frac{C - 3r^2}{2\lambda r};$$
$$R_1 = \frac{\lambda}{r}; \quad R_2 = \frac{2\lambda r}{(C - r^2)}.$$

Constants λ and C are determined due to the boundary conditions.

An equation of the generatrix meridian can be expressed with the help of elliptical integrals with taking into account the parameters λ, C and the conditions at the edges:

$$\left.\begin{array}{l} z = 2\sqrt{\pm\lambda}[\mathrm{E}(k,\varphi) - \mathrm{E}(k,\varphi_0)] + \sqrt{\pm\lambda}[\mathrm{F}(k,\varphi) - \mathrm{F}(k,\varphi_0)], \\ r = \sqrt{|2\lambda \pm C|}\cos\varphi, \end{array}\right\}$$

or

$$\left.\begin{array}{l} z = -\dfrac{C}{\sqrt{|2\lambda \pm C|}}[\mathrm{F}(k,\varphi) - \mathrm{F}(k,\varphi_0)] \\ \qquad + \sqrt{|2\lambda \pm C|}[\mathrm{E}(k,\varphi) - \mathrm{E}(k,\varphi_0)]; \\ r = \sqrt{|2\lambda \pm C|}\sqrt{1 - k^2\sin^2\varphi}, \end{array}\right\}$$

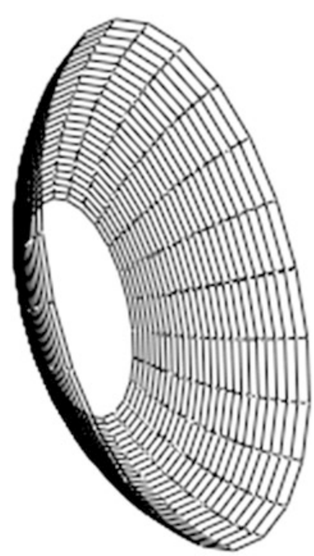

Fig. 1

where $F(k;\varphi)$ and $E(k;\varphi)$ are the elliptical integrals of the first and second orders, k is a module but φ is an amplitude of the elliptical integrals, φ_0 is an initial amplitude corresponding to $r = a$.

If $\lambda = \pm C/2$ and $C = 0$ then the integral expression for the generatrix curve is solved in quadrature: if $\lambda = \pm C/2$, then (Fig. 1).

$$z = \sqrt{\frac{C}{2}} \ln\left|\frac{\sqrt{2C} + \sqrt{2C - r_1^2}}{\sqrt{2C} + \sqrt{2C - r^2}} \cdot \frac{r}{r_1^2}\right| + \sqrt{2C - r^2} - \sqrt{2C - r_1^2};$$

if $C = 0$, then

$$z = \frac{1}{2}\left[r\sqrt{(2\lambda)^2 - r^2} - a\sqrt{(2\lambda)^2 - a^2}\right] - 2\lambda^2\left[\arcsin\frac{r}{2\lambda} - \arcsin\frac{a}{2\lambda}\right].$$

Having known the equation of a generatrix curve, it is easy to construct the surface of revolution with extremum properties with the help of parametrical equations:

$$x = x(r,\beta) = r\cos\beta,$$
$$y = y(r,\beta) = r\sin\beta,$$
$$z = z(r).$$

A surface with $\lambda = \pm C/2$ is called “PenKa” (Fig. 1).

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2.6 The Surfaces of Delaunay

In 1841, astronomer and mathematician C. Delaunay has picked out some surfaces of revolution described by him in his paper into an independent group.

In appendix of this paper, M. Sturm noted that the determination of equations of *Delaunay surfaces* is a variational problem on a conditional extremum.

For example, for *unduloid* and *nodoid*, the crux of the problem consists in the following: determine the functions $y(x)$, that are identified with meridians of surfaces of revolution, the volume of which can be calculated by a formula

$$V(y) = \pi\int_{x_0}^{x_1} y^2 \mathrm{d}x,$$

under condition of extremum of areas of their lateral surfaces

$$S(y) = 2\pi\int_{x_0}^{x_1} y\mathrm{d}s = 2\pi.$$

It is supposed that the edges of a surface of revolution are fixed.

This problem results in an equation of Euler–Lagrange:

$$y^2 + \frac{2ay}{\sqrt{1 + y'^2}} \mp b^2 = 0,$$

that is connected with an integral

$$F(y) = \pi\int_{x_0}^{x_1}(y^2\mathrm{d}x + 2ay\mathrm{d}s) = \pi\int_{x_0}^{x_1}\left(y^2 + 2ay\sqrt{1 + y'^2}\right)\mathrm{d}x.$$

Here, a is a corresponding real parameter; b is the second parameter.

It is recognized that the Delaunay surfaces are surfaces of revolution with *constant mean curvature*. With the exception of spheres, they are generated by *roulettes* in the process of their rotation about a curve along which the corresponding conics roll.

Roulettes are formed by focuses of parabola, ellipse, and hyperbola rolling without sliding along a straight line that is an axis of rotation.

Delaunay surfaces incorporate five surfaces of revolution that are *catenoids, unduloids, nodoids, spheres, and circular cylindrical surfaces*.

Let us present Euler–Lagrange equations for every type of surfaces of revolution:

$$\frac{y}{\sqrt{1 + y'^2}} - c = 0; \quad c > 0\,(\text{catenoid});$$

$$y^2 - \frac{1}{H}\frac{y}{\sqrt{1 + y'^2}} + b^2 = 0, \quad \frac{1}{2H} > b > 0\ (\text{unduloid});$$

$$y^2 - \frac{1}{H}\frac{y}{\sqrt{1 + y'^2}} - b^2 = 0, \quad b > 0\,(\text{nodoid});$$

$$y^2 - \frac{1}{H}\frac{y}{\sqrt{1 + y'^2}} = 0, \quad H > 0\ (\text{sphere});$$

$$y^2 - \frac{1}{H}\frac{y}{\sqrt{1 + y'^2}} + b^2 = 0, \quad H > 0,\, b > \frac{1}{2H}$$

(circular cylindrical surface).

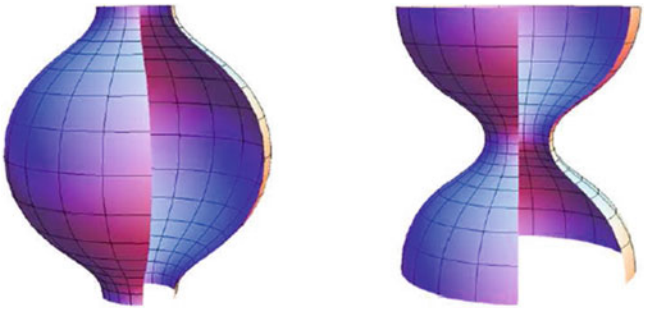

Fig. 1 Open parts of the bulb (*left*) and the neck (*right*) segments of the axially symmetric unduloid-like periodic surfaces of revolution obtained with the help of parametric equations by Djondjorov PA, et al

So, the Delaunay surfaces are included in a group of "Surfaces of Revolution with Extreme Properties" (p. 72). Axisymmetric surfaces of Delaunay's unduloids provide solutions of the shape equation in explicit parametric form. This class provides the analytical examples of surfaces with periodic curvatures studied by K. Kenmotsu and leads to some unexpected relationships among Jacobian elliptic functions and their integrals (Fig. 1).

Delaunay surfaces are used for description of processes in gas dynamics, for research of surfaces of soap films and bubbles.

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2.6.1 Nodoid and Unduloid Surfaces of Revolution

Substituting $\lambda_0 \neq 0,\ \lambda_1 = 0,\ \lambda_2 \neq 0$ into a general shape equation for generatrix curves of surfaces of revolution possessing extreme properties

$$z = \int \frac{(C - \lambda_0 r^2)\mathrm{d}r}{\sqrt{4(\lambda_1 + \lambda_2 r)^2 - (C - \lambda_0 r^2)^2}} + \gamma$$

we can obtain an integral equation of the generatrix:

$$z = \int \frac{(C - \lambda_0 r^2)}{\sqrt{4(\lambda_2 r)^2 - (C - \lambda_0 r^2)^2}} \mathrm{d}r + \gamma.$$

This integral equation describes a family of *curves of Shturm*. that are lines generated by a focus of a parabola or hyperbola in the process of rolling of corresponding curves along a straight.

In that case, we can state a problem in the following form: find a plane curve $r = r(z)$ that forms a body of rotation of the given volume V. This curve rotates about an axis Oz but the body must cover a minimal area S.

Due to the principle of mutuality, this problem is equivalent to the following problem: determine a plane curve $r = r(z)$ rotating about an axis Oz that forms a body of minimal volume V limited by the surface of the given area S (Fig. 1).

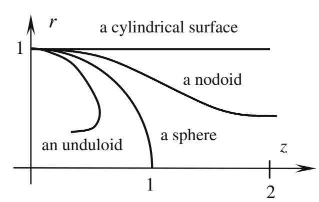

Fig. 1

Constant mean curvature is a remarkable property of nodoids and unduloids:

$$2H = -\frac{1}{\lambda_2} = \text{const},$$

but

$$K = \frac{(C^2 - r^4)}{4\lambda_2^2 r^4}.$$

So, an unduloid, or onduloid, is a surface with constant nonzero mean curvature obtained as a surface of revolution of an elliptic catenary: that is, by rolling an ellipse along a fixed line, tracing the focus, and revolving the resulting curve around the line. A nodoid is a surface of revolution with constant nonzero mean curvature obtained by rolling a hyperbola along a fixed line, tracing the focus, and revolving the resulting nodary curve about the line.

In 1828, Poisson has shown that a surface of separation of two mediums that are at balance is a surface of a constant mean curvature. But in this case, one neglects the dead weight. These surfaces can be modeled by soap films. A physical principle forming soap films, regulating their behavior, local and global properties is rather simple. A physical system keeps corresponding configuration only if the system cannot change easily the configuration having captured a position with less level of energy. An integral of general type is reduced into elliptical integrals of the first and second types:

$$x = -\frac{CF(k', \varphi)}{r} + rE(k', \varphi), \quad y = r\sqrt{1 - k'^2 \sin\varphi},$$

where

$$k = \frac{m}{r}; \quad k' = \sqrt{1 - k^2}$$

is an additional module of the integral. In ultimate cases, the integral for the studied surfaces can be reduced to an equation of sphere and circular cylindrical surface.

An analog of geometrical properties of shells of revolution under corresponding conditions is a condition of matching in strength (the same strength), i.e., an equality of circular and meridional forces in every cross section. A shell of revolution will be in equal strength state under action of inner pressure P and axial force P_0^z per unit length of the circular edge if

$$\frac{P_0^z}{2\pi r_1} = \lambda_2 P \sin\theta_0.$$

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■ Nodoid Surface Connecting Two Circular Cones

It is necessary to know Lagrange multipliers λ_0, λ_2; Euler constant C and a constant of integration γ for the unambiguous determination of a curve defined by an equation:

$$z = \int \frac{(C - \lambda_0 r^2)\mathrm{d}r}{\sqrt{4(\lambda_2 r)^2 - (C - \lambda_0 r^2)^2}} + \gamma,$$

These values can be obtained without using of integral conditions for areas and volumes of the surface.

Let us construct a conjugation of two circular cones with known radiuses r_1 and r_2 and with slopes α_1, α_2 of rectilinear generatrixes of the cones (Fig. 1). For this case, we shall use a nodoidal surface. The length of the surface along an axis Oz turns automatically.

The integral equation becomes

$$z = \int \frac{(C^* - r^2)}{\sqrt{4 \cdot (\lambda r)^2 - (C^* - r^2)^2}} \cdot \mathrm{d}r + \gamma,$$

where $C^* = \frac{C}{\lambda_0}$, $\lambda = \frac{\lambda_2}{\lambda_0}$.

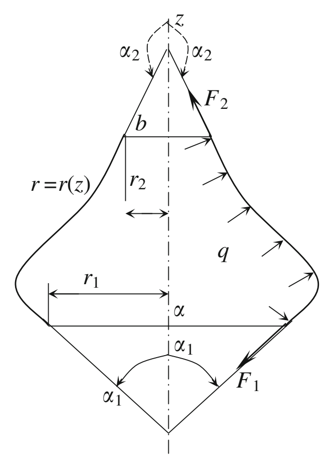

Fig. 1

The values r_1 and r_2, α_1 and α_2 must be connected between themselves.

Let us study a soap bubble subjected to inner pressure q. A contact of a soap film with the bases of the circular cones takes place in the sections a and b. In these sections, surface tension forces are directed along rectilinear generatrixes

These forces are

$$F_1 = \mu l_1 = 2\pi r_1 \mu \text{ and } F_2 = \mu l_2 = 2\pi r_2 \mu,$$

where μ is a coefficient of surface tension, l_i are the lengths of the contours of contact.

The conditions of equilibrium give

$$r_1 \sin\alpha_1 = r_2 \sin\alpha_2.$$

Using a Laplace formula for surface tension, we can get

$$\Delta p = 2H\mu.$$

So, we can design the surfaces both of positive and negative mean curvatures.

For the determination of coefficients λ and C, it is necessary to use an expression for derivative:

$$r'(z) = \frac{\sqrt{4\lambda^2 r^2 - (C^* - r^2)^2}}{(C^* - r^2)}$$

and boundary conditions: if $z = 0$ then $r = r_1$ and $r' = \tan\alpha_1$; but if $r = r_2$ then $r' = \tan\alpha_2$.

In addition, we have

$$\lambda = \frac{r_1^2 - r_2^2}{2(r_2\cos\alpha_1 - r_1\cos\alpha_2)},$$

$$C^* = \frac{r_1 r_2(r_1\cos\alpha_1 - r_2\cos\alpha_2)}{(r_2\cos\alpha_1 - r_1\cos\alpha_2)}.$$

In Fig. 2, copper nodoids are shown made by a method of galvanoplastics.

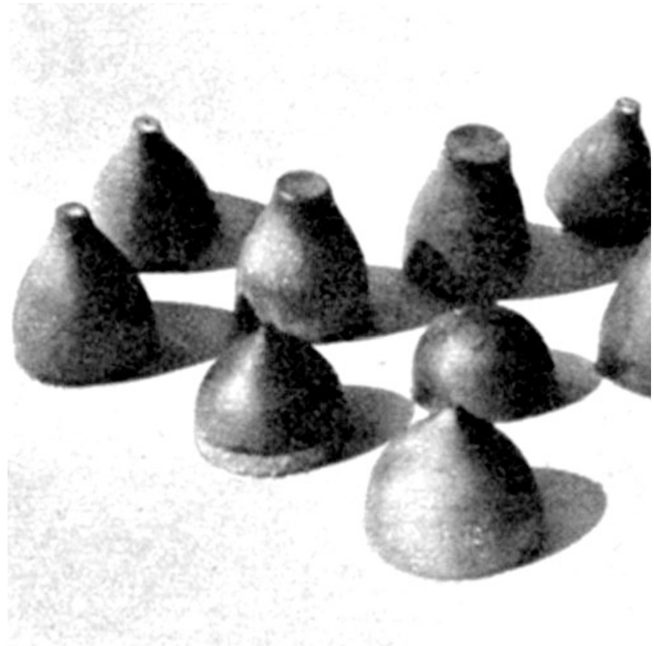

Fig. 2

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Translation Surfaces

3

A surface of translation (*a translation surface*) is a surface formed by parallel translation of a curve of some direction that is a generatrix curve L_1 along another curve that is a directrix curve L_2 (Fig. 1). So, a point M_0 of the curve L_1 slides along the curve L_2. The same surface can be obtained if we shall take a curve L_2 as a generatrix but a curve L_1 as a directrix.

Assume that $\boldsymbol{r}_1(u)$ and $\boldsymbol{r}_2(v)$ are the radius-vectors of the curves L_1 and L_2 accordingly, then a radius-vector of a translation surface is

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{r}_1(u) + \boldsymbol{r}_2(v) - \boldsymbol{r}_1(u_0),$$

where $\boldsymbol{r}_1(u_0) = \boldsymbol{r}_2(v_0)$ is the radius-vector of a point M_0 (Fig. 1). The lines u and v form *a transport net*. A transport net is *a conjugate Chebyshev net* on a two-dimensional surface in an Euclidean space.

S. Lie proved that a surface of translation is a geometrical locus of the middles of straight line segments that the ends of which lie on two support curves $\boldsymbol{\rho}(\alpha)$ and $\boldsymbol{t}(\beta)$. In this case, an equation of a translation surface can be written in the following form:

$$\boldsymbol{R} = [\boldsymbol{\rho}(\alpha) + \boldsymbol{t}(\beta)]/2.$$

Surfaces of translation are divided into *surfaces of right translation*, *surfaces of oblique translation*, and *velaroidal surfaces*.

A cylindrical surface is the simplest surface of translation. It can be formed by translation of any curve parallel to itself, lying on a cylindrical surface and intersecting all its generatrixes.

Directrix and generatrix curves of a surface of right translation lie at mutually perpendicular planes.

Surfaces of right translation can be defined by an explicit equation:

$$z = z(x, y) = z_1(x) + z_2(y),$$

where $z = z_1(x)$ is the equation of a plane generatrix curve L_1; $z = z_2(y)$ is the equation of a plane directrix curve L_2 (Fig. 1). Directrix and generatrix curves of a surface can be arbitrary curves but usually they are taken of the same type.

Only a cylindrical surface and a plane are *developable surface of translation*.

An existence of *conjugate Chebyshev's net* on the surface is an invariant criterion of a translation surface. So, a surface carrying a transport net is called a surface of translation. A translation surface is called *a special one* if its net of translation is *an isothermally conjugate net*.

Surfaces of oblique translation are formed by parallel translation of a plane curve and two of its symmetrical points touch the plane contour continuously.

These surfaces can be given by an explicit equation

$$z = g(u - v) + h(u + v),$$

where $z = g(u - v)$ is an equation of a plane generatrix curve L_1; $z = h(u + v)$ is an equation of a plane directrix curve L_2.

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_3, © Springer International Publishing Switzerland 2015

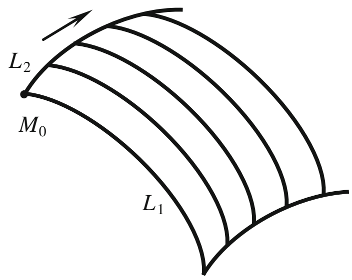

Fig. 1

A translation surface on the rectangular plane contour with generatrix curve that changes its curvature with movement is called a velaroidal surface (Fig. 2). Sometimes, a velaroidal surface is called *a funicular surface*.

For *the translation Weingarten surfaces* in Euclidean space, the following theorem holds (Dillen F et al.): A translation surface in $\boldsymbol{R}^3$ is a Weingarten surface if and only if it is either *a plane*, *a cylindrical surface*, *the minimal surface of Scherk*, or *an orthogonal elliptic paraboloid.*

In 1937, N.G. Chebotaryov investigated *a generalized surface of translation* possessing a family of curves which satisfy the following conditions: (a) only one curve of the family passes through every point of the surface; (b) each of these curves can be transferred in every other curve of the family by means of transformation of some group of Lie. A group of parallel translations ensures a formation of surfaces of right translation.

Fig. 2 A velaroid shallow reinforced concrete cover of Nekrasov Market in Sanct-Peterburg, 1960

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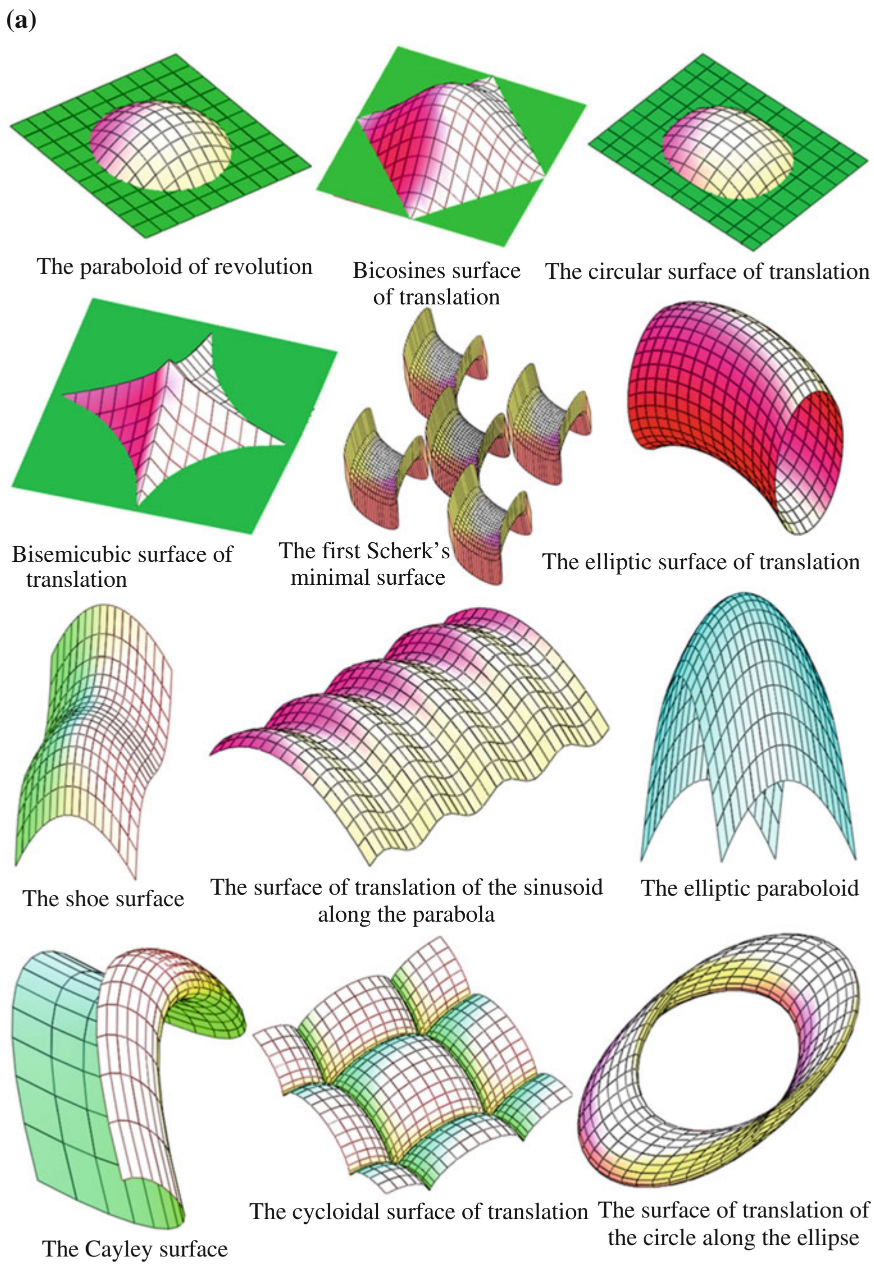
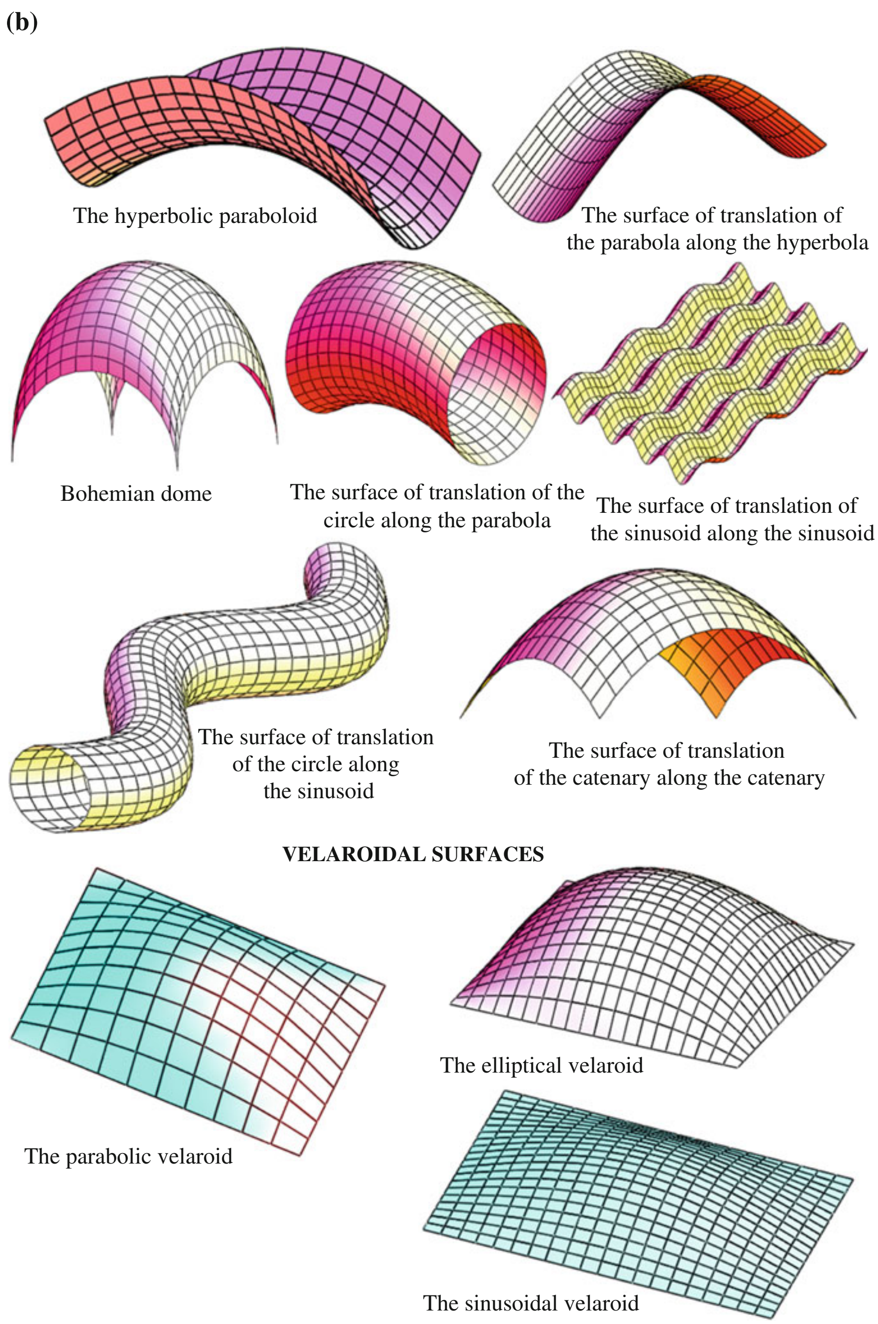
■ Translation Surfaces Presented in the Encyclopedia




(b)
The hyperbolic paraboloid
The surface of translation of the parabola along the hyperbola
Bohemian dome
The surface of translation of the circle along the parabola
The surface of translation of the sinusoid along the sinusoid
The surface of translation of the circle along the sinusoid
The surface of translation of the catenary along the catenary
VELAROIDAL SURFACES
The parabolic velaroid
The elliptical velaroid
The sinusoidal velaroid

3.1 Surfaces of Right Translation

A directrix L_2 and generatrix L_1 curves of *a surface of right* (*parallel*) *translation* lie in mutually perpendicular planes and have only one common point M_0 (Fig. 1).

Surfaces of right translation can be given by an explicit equation:

$$z = z(x, y) = z_1(x) + z_2(y),$$

where, for example, $z = z_1(x)$ is an equation of a plane generatrix curve L_1 but $z = z_2(y)$ is an equation of a plane directrix curve L_2. There are no translation surfaces of constant Gaussian curvature with $K = \text{const} \neq 0$.

Paraboloids are the most known surfaces of right translation.

These are *a paraboloid of revolution* with the same parabola as directrix and generatrix curves:

$$z = z(x, y) = \frac{x^2}{2p} + \frac{y^2}{2p},$$

an elliptical paraboloid having directrix and generatrix parabolas with different focus distances:

$$z = z(x, y) = \frac{x^2}{2p} + \frac{y^2}{2q}, \quad p > 0, \ \ q > 0,$$

a hyperbolic paraboloid having directrix and generatrix parabolas with the branches directed into different sides

$$z = z(x, y) = \frac{x^2}{2p} - \frac{y^2}{2q}, \quad p > 0, \ \ q > 0.$$

A cylindrical surface and *a plane* are the only developable surface of right translation. Assume a directrix curve in the form $z = z_2(y)$ and a generatrix curve in the form of a straight line $z = z_1 = a = \text{const}$, then we shall obtain a translation surface in the form of the right cylindrical surface (Fig. 2):

$$z = z(y) = a + z_2(y).$$

These surfaces were studied in Chap. 1 "Ruled Surfaces" in Subsect. "1.1.2. Cylindrical Surfaces."

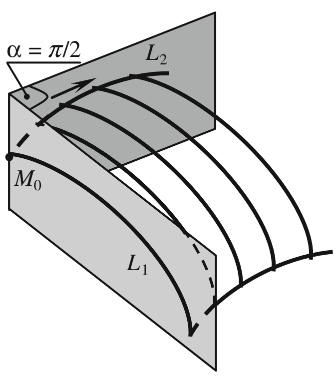


Fig. 1

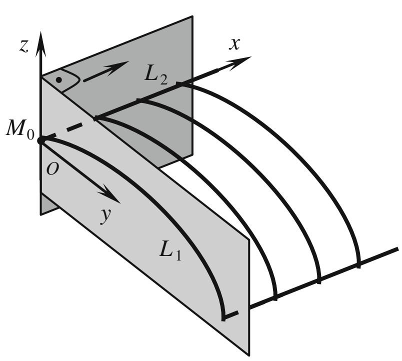


Fig. 2

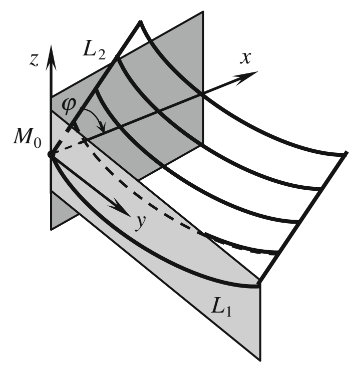


Fig. 3

Fig. 4 Cheryomushkinskiy Market in the form of the surface of parallel translation, Moscow, Russia

Assume a directrix curve in the form $z = z_2(y)$ and a generatrix curve in the form of a straight line $z = z_1(x) = x\tan\varphi$, then we shall obtain a translation surface in the form of the oblique cylindrical surface (Fig. 3):

$$z = z(x, y) = x\tan\varphi + z_2(y).$$

Oblique cylindrical surfaces were studied in Chap. "1. Ruled Surfaces" in Subsect. "1.1.2 Cylindrical Surfaces." Here, φ is a slope angle of the generatrix straight line with a positive direction of an axis x.

Surfaces of parallel translation possess an important technological property: poured-in-place shells designed in the form of surfaces of parallel translation can be made with the help of simple formwork (Fig. 4).

Additional Literature

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■ Circular Surface of Translation

A circular surface of translation is formed when a generating circle with a radius R_2 moves on a director circle with a radius R_1 (Fig. 1).

Forms of definition of the surface

(1) Explicit equation (Fig. 1):

$$z = z(x, y) = \sqrt{R_1^2 - \left(x - \frac{a}{2}\right)^2} - \sqrt{R_1^2 - \frac{a^2}{4}} + \sqrt{R_2^2 - \left(y - \frac{b}{2}\right)^2} - \sqrt{R_2^2 - \frac{b^2}{4}}, \quad 0 \le x \le a, \quad 0 \le y \le b.$$

Here

$$\left(x - \frac{a}{2}\right)^2 + \left(z + \sqrt{R_1^2 - \frac{a^4}{4}}\right)^2 = R_1^2,$$
$$\left(-\frac{b}{2}\right)^2 + \left(z + \sqrt{R_2^2 - \frac{b^4}{4}}\right)^2 = R_2^2$$

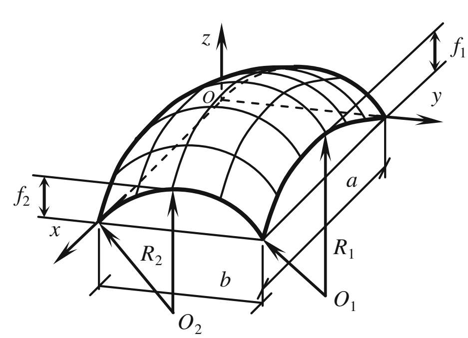

Fig. 1

are the equations of the directrix and generatrix circles lying at the mutually perpendicular places and furthermore $R_1 \ge a/2$; $R_2 \ge b/2$. The centers of the circles are at the same side of the surface.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = \frac{R_1^2}{R_1^2 - (x - a/2)^2},$$
$$F = \frac{(x - a/2)(y - b/2)}{\sqrt{R_1^2 - (x - a/2)^2}\sqrt{R_2^2 - (y - b/2)^2}},$$
$$B^2 = \frac{R_2^2}{R_2^2 - (y - b/2)^2},$$
$$M = 0,$$
$$L = \frac{-R_1^2}{R_1^2 - \left(x - \frac{a}{2}\right)^2}\sqrt{\frac{R_2^2 - (y - b/2)^2}{R_1^2R_2^2 - \left(x - \frac{a}{2}\right)^2\left(y - \frac{b}{2}\right)^2}},$$
$$N = \frac{-R_2^2}{R_2^2 - \left(y - \frac{b}{2}\right)^2}\sqrt{\frac{R_2^2 - (x - a/2)^2}{R_1^2R_2^2 - \left(x - \frac{a}{2}\right)^2\left(y - \frac{b}{2}\right)^2}},$$
$$k_x = -\sqrt{\frac{R_2^2 - (y - b/2)^2}{R_1^2R_2^2 - (x - a/2)^2(y - b/2)^2}},$$
$$k_y = -\sqrt{\frac{R_1^2 - (x - a/2)^2}{R_1^2R_2^2 - (x - a/2)^2(y - b/2)^2}}, \quad K > 0$$

(2) Explicit equation:

$$z = z(x, y) = R_1 - \sqrt{R_1^2 - x^2} \pm \left(R_2 - \sqrt{R_2^2 - y^2}\right).$$

The centers of the both circles lie at one side of the surface if we take the sign (+) but if we take the sign (–) then the centers of the both circles will lie at the different sides of the surface (Fig. 2).

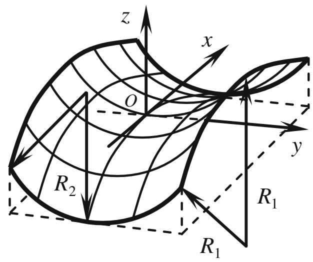

Fig. 2

(3) Vector equation (Fig. 3):

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{r}(u,v) = R_1[\sin\gamma - \sin(\gamma - u/R_1)]\boldsymbol{i} + R_2[\sin\varphi - \sin(\varphi - v/R_2)]\boldsymbol{j} \\ &\quad + \{R_1[\cos(\gamma - u/R_1) - \cos\gamma] + R_2[\cos(\varphi - v/R_2) - \cos\varphi]\}\boldsymbol{k};\end{aligned}$$

where u, v are the lengths of the arcs of directrix and generatrix circles;

$$\begin{aligned}&0 \le u \le 2R_1\gamma; \qquad 0 \le v \le 2\varphi R_2;\\ &f_1 = R_1(1-\cos\gamma), \quad f_2 = R_2(1-\cos\varphi)\end{aligned}$$

are the rises of these circles (Fig. 3) but 2φ and 2γ are their central angles. The rise of the surface in a point with coordinates $x = a/2$ and $y = b/2$ is

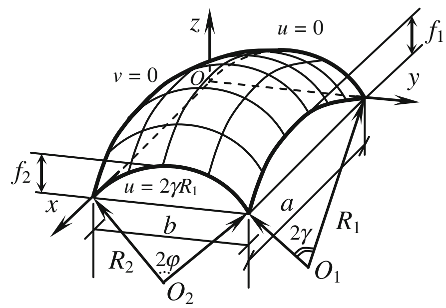

Fig. 3

$$z = f_1 + f_2.$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = B = 1; \quad F = \sin\left(\gamma - \frac{u}{R_1}\right)\sin\left(\varphi - \frac{v}{R_2}\right);$$

$$L = -\frac{\cos(\varphi - v/R_2)}{R_1\sqrt{1-F^2}} = k_u; \quad M = 0;$$

$$N = -\frac{\cos(\gamma - u/R_1)}{R_2\sqrt{1-F^2}} = k_v.$$

So, the chosen curvilinear coordinate net is not orthogonal but conjugate.

■ Cayley Surface

Cayley surface is an algebraic ruled surface of translation of negative Gaussian curvature. A. Cayley has studied this surface as geometrical illustration of his investigation devoted to a theory of pencils of binary quadratic forms. That is why, the surface was named after A. Cayley.

Forms of definition of the surface

(1) Explicit equation (Fig. 1):

$$z = xy - \frac{x^3}{6}.$$

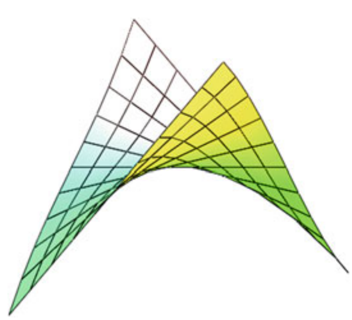

Fig. 1

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 1 + \left(y - \frac{x^2}{2}\right)^2, \quad F = x\left(y - \frac{x^2}{2}\right), \quad B^2 = 1 + x^2,$$

$$L = -\frac{x}{\sqrt{A^2B^2 - F^2}}, \quad M = \frac{1}{\sqrt{A^2B^2 - F^2}}, \quad N = 0,$$

$$k_x = \frac{L}{A^2}, \quad k_y = 0,$$

$$K = -\frac{1}{(A^2B^2 - F^2)^{3/2}} < 0, \quad H = -\frac{x(1+2y)}{2(A^2B^2 - F^2)^{3/2}}.$$

Curvilinear coordinates x, y on the surface are non-orthogonal and non-conjugate. Coordinate lines y (x = const) coincide with rectilinear generatrixes of the Cayley surface. The surface shown in Fig. 1 is given in the limits of $-1 \le x \le 1$ and $-1 \le y \le 1$. Consistent cross sections of the cubic Cayley surface by parallel planes in all three directions possess interesting two-dimensional interpretations. The planes that are orthogonal to the axis Oy demonstrate a transformation of the function with two extreme points into the function without extremum. A cross section of the surface by the plane $y = b$ gives a curve with two extreme points $x = \pm\sqrt{2b}$. There is a curve $z = -x^3/6$ in the cross section of the surface by the plane $y = 0$. The curve lying in the cross section of the surface by the plane $y = -b$ does not have extremum.

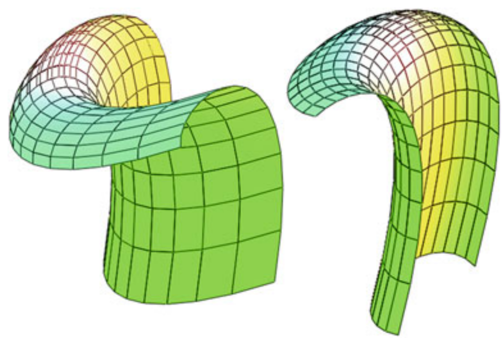

Fig. 2

The Cayley surface given as

$$z = xy - x^3$$

is called *a ruled improper affine sphere*.

(2) Parametrical form of definition (Fig. 2):

$$x = x(u, v) = u + v,$$
$$y = y(u, v) = \left(u^2 + v^2\right)/2 + a(v - u),$$
$$z = z(u, v) = \left(u^3 + v^3\right)/3 + a(v^2 - u^2).$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 1 + (u - a)^2 + u^2(u - 2a)^2,$$
$$F = 1 + (u - a)(v + a) + (u - 2a)(v + 2a)uv,$$
$$B^2 = 1 + (v + a)^2 + v^2(v + 2a)^2,$$
$$L = -\frac{(u - v)^2 + 4a(v + a - u)}{\sqrt{A^2B^2 - F^2}} = -N, \quad M = 0,$$
$$K = -\frac{\left[(u - v)^2 + 4a(v + a - u)\right]^2}{\left(A^2B^2 - F\right)^2} < 0.$$

Here, the surface referred to conjugate non-orthogonal system of curvilinear coordinates u, v. The surface shown in Fig. 2 is pictured within the limits $-2 \le u \le 2$ and $-2 \le v \le 2$ when $a = 1$.

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■ Surface of Translation of Catenary Along Catenary

A surface of translation of one catenary along other catenary is formed by a translation of a catenary that is remained parallel to itself along another catenary and a corresponding point of the first catenary slides on the second one. The directrix catenary and the generatrix catenary lie in mutually perpendicular planes.

Forms of definition of the surface

(1) Explicit form of the definition (Fig. 1):

$$z = z(x, y)$$
$$= -a\cosh\frac{x - b}{a} + a\cosh\frac{b}{a} - d\cosh\frac{y - c}{d} + d\cosh\frac{c}{d},$$

where $0 \le x \le 2b$; $0 \le y \le 2c$. This surface covers a rectangular plan $2b \times 2c$.

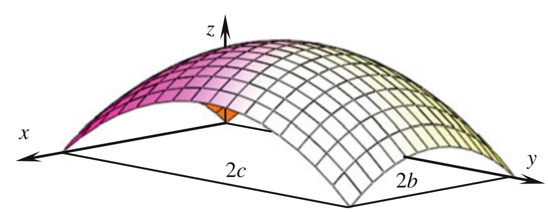

Fig. 1

A rise f_1 of the catenary $z = -a \cosh[(x - b)/a] + a \cosh(b/a)$ lying in the planes $y = 0$ and $y = 2c$ is determined from an equation:

$$f_1 = a\left[\cosh(b/a) - 1\right].$$

A rise f_2 of the catenary $z = -d \cosh[(y - c)/d] + d \cosh(c/d)$ lying in the planes $x = 0$ and $x = 2b$ is determined from an equation:

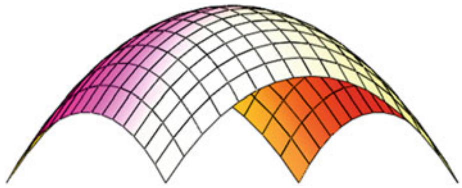

Fig. 2

$$f_2 = d\,[\cosh(c/d) - 1].$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = \cosh\frac{x-b}{a},\quad F = \sinh\frac{x-b}{a}\sinh\frac{y-c}{d},$$
$$B = \cosh\frac{y-c}{d},$$
$$A^2B^2 - F^2 = \cosh^2\frac{y-c}{d} + \sinh^2\frac{x-b}{a},$$
$$L = \frac{-\cosh\frac{x-b}{a}}{a\cdot\sqrt{\cosh^2\frac{y-c}{d} + \sinh^2\frac{x-b}{a}}},$$
$$N = \frac{-\cosh\frac{y-c}{d}}{d\cdot\sqrt{\cosh^2\frac{y-c}{d} + \sinh^2\frac{x-b}{a}}},$$
$$M = 0,\quad K = \frac{\cosh\frac{x-b}{a}\cosh\frac{y-c}{d}}{ad(A^2B^2 - F^2)^2} > 0,$$
$$k_x = \frac{-1}{a\cosh\frac{x-b}{a}\cdot\sqrt{\cosh^2\frac{y-c}{d} + \sinh^2\frac{x-b}{a}}},$$
$$k_y = \frac{-1}{d\cosh\frac{y-c}{d}\cdot\sqrt{\cosh^2\frac{y-c}{d} + \sinh^2\frac{x-b}{a}}}.$$

The surface if given in curvilinear non-orthogonal conjugated system of coordinates x, y.

(2) Explicit form of the definition:

$$z = z(x,y) = a\left(\cosh\frac{x}{a} - 1\right) + d\left(\cosh\frac{y}{d} - 1\right),$$
$$-\infty \le x \le \infty;\quad -\infty \le y \le \infty.$$

The surface plotted for x from $-b$ to b ($-b \le x \le b$) and y from $-c$ to c ($-c \le y \le c$) covers a rectangular plan $2b \times 2c$. The origin of a Cartesian coordinates is at the peak of the surface (Fig. 2).

The surface is generated by the parallel motion of the catenary $z = z(x) = a[\cosh(x/a) - 1]$ on the catenary $z = z(y) = d[\cosh(y/d) - 1]$. There is a closed curve $a\cosh(x/a) + d\cosh(y/d) = h + a + d$ in the cross section of the translation surface by the plane $z = h$.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = \cosh\frac{x}{a},\quad F = \sinh\frac{x}{a}\sinh\frac{y}{d},\quad B = \cosh\frac{y}{d},$$
$$L = \frac{\cosh(x/a)}{a\cdot\sqrt{\cosh^2\frac{y}{d} + \sinh^2\frac{x}{a}}},\quad M = 0,$$
$$N = \frac{\cosh(y/d)}{d\cdot\sqrt{\cosh^2\frac{y}{d} + \sinh^2\frac{x}{a}}},$$
$$K = \frac{\cosh(x/a)\cosh(y/d)}{ad(A^2B^2 - F^2)^2} > 0.$$

■ Surface of Translation of Circle Along Parabola

A surface of right translation of a circle along a parabola is formed by a generatrix circle of constant radius a in the process of its motion on a directrix parabola or on the contrary, in the process of the parallel translation of a parabola along a circle (Fig. 1).

Forms of definition of the surface

(1) Explicit equation (Fig. 1):

$$z = z(x,y) = \frac{4f}{c^2}y(y-c) + \sqrt{a^2 - \left(x - \frac{b}{2}\right)^2} - \sqrt{a^2 - \frac{b^2}{4}},$$

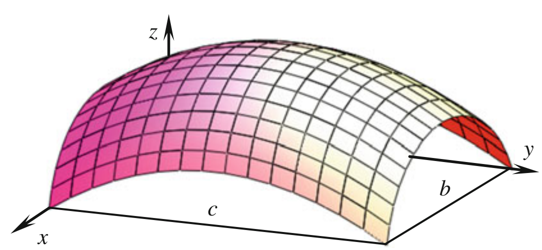

Fig. 1

where f is the rise of the parabola, i.e., it is the distance the peak of the parabola from the plane $z = 0$ at the edges $x = 0$ and $x = b$; $0 \le x \le b; 0 \le y \le c; b \le 2a$. The surface covers the rectangular plan $b \times c$.

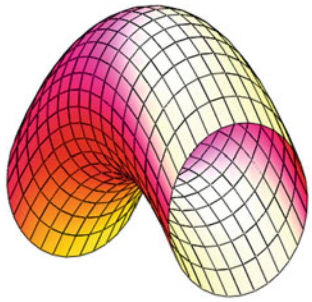

Fig. 2

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = \frac{a^2}{a^2 - (x - b/2)^2}, \quad F = -\frac{4f(x - b/2)(c - 2y)}{c^2\sqrt{a^2 - (x - b/2)^2}},$$

$$B^2 = \frac{c^4 + 16f^2(c - 2y)^2}{c^4},$$

$$L = \frac{-a^2}{\left[a^2 - (x - b/2)^2\right]^{3/2}\sqrt{A^2 + B^2 - 1}}, \quad M = 0,$$

$$N = \frac{-8f}{c^2\sqrt{A^2 + B^2 - 1}},$$

$$k_x = \frac{-1}{\sqrt{a^2 - (x - b/2)^2}\sqrt{A^2 + B^2 - 1}},$$

$$k_y = \frac{-8fc^2}{\left[c^4 + 16f^2(c - 2y)^2\right]\sqrt{A^2 + B^2 - 1}},$$

$$K = \frac{8fa^2}{\left[a^2 - (x - b/2)^2\right]^{3/2}(A^2 + B^2 - 1)^2}.$$

In Fig. 1, the surface of parallel translation of the circle along the parabola is presented. One can see the curvilinear non-orthogonal conjugated system of coordinate placed on the surface. The only coordinate lines $x = b/2$ and $y = c/2$ coincide with the lines of principal curvatures of the surface.

(2) Explicit equation:

$$z = z(x, y) = -\frac{y^2}{2p} \pm \sqrt{a^2 - x^2} - a, \quad -a \le x \le a.$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = \frac{a^2}{a^2 - x^2}, \quad F = \pm\frac{xy}{p\sqrt{a^2 - x^2}}, \quad B^2 = \frac{y^2 + p^2}{p^2},$$

$$L = \mp\frac{a^2}{(a^2 - x^2)^{3/2}\sqrt{A^2 + B^2 - 1}}, \quad M = 0,$$

$$N = \frac{-1}{p\sqrt{A^2 + B^2 - 1}},$$

$$k_x = \mp\frac{1}{\sqrt{A^2 + B^2 - 1}\sqrt{a^2 - x^2}},$$

$$k_y = \frac{-p}{\sqrt{A^2 + B^2 - 1}\,(y^2 + p^2)},$$

$$K = \pm\frac{a^2}{p(a^2 - x^2)^{3/2}(A^2 + B^2 - 1)^2}.$$

The translation surface shown in Fig. 2 contains the segments both of positive and negative Gaussian curvatures divided by the line with parabolic points.

The surface touches the xOy coordinate plane in the point with coordinates (0; 0; 0), i.e., in the peak of the directrix parabola.

A line of intersection of the studied translation surface with planes $z = b$ $(0 \le b \le -\infty)$ is given by explicit equation:

$$y = \pm\sqrt{2p(\pm\sqrt{a^2 - x^2} - a - b)}.$$

The lines of intersection of the surface with the planes $b = -a$; $b = -2a$ and $b = -2.2a$ are presented in Fig. 3.

The studied translation surface belongs simultaneously to a class of *cyclic surfaces* to a group of *cyclic surfaces with a plane of parallelism*. Any plane y = const can be taken as *a pane of parallelism*.

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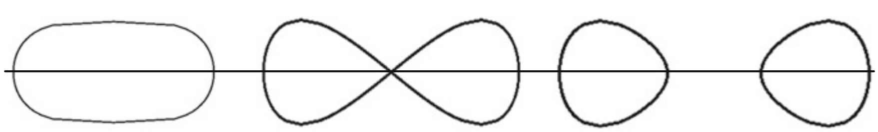

$p = 2a$; $b = -a$ $p = 2a$; $b = -2a$ $p = 2a$; $b = -2.2a$

Fig. 3

■ Surface of Translation of Sinusoid Along Parabola

Surface of translation of sinusoid along parabola is generated by parallel translation of a sinusoid

$$z = -c\sin\left(\varphi + \frac{n\pi y}{d}\right) + c\sin\varphi$$

along a plane parabola

$$z = -a(x-b)^2 + ab^2$$

where n is a number of whole semi-waves containing on the straight with a length of d.

If they take a sinusoid as a directrix but a parabola as a generatrix then the same surface will be formed.

A sinusoid and a parabola must be placed at the mutually perpendicular planes. A studied surface contains the segments both of negative and positive total curvatures.

Forms of definition of the surface

(1) Explicit equation (Fig. 1):

$$z = z(x,y) = -a(x-b)^2 - c\cos\frac{n\pi y}{d} + ab^2 + c.$$

Assume $0 \le x \le 2b$ and $0 \le y \le d$ then the translation surface will cover a rectangular plan $2b \times d$ but $z \le ab^2 + 2c$.

There are two curves

$$x = b \pm \sqrt{b^2 - \frac{h}{a} + \frac{c}{a}\left(1 - \cos\frac{n\pi y}{d}\right)}$$

in the cross section of the translation surface by a plane $z = h$.

Assume $z = h = ab^2$ then an equation of the line of the intersection of the surface with the taken plane are (Fig. 2)

$$x = b \pm \sqrt{2c/a}\,\sin\frac{n\pi y}{2d},$$

In Fig. 3, the lines of intersection of the surface with the plane $z = h = 0$ are shown.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + 4a^2(x-b)^2,$$
$$F = -2acn\pi\frac{x-b}{d}\sin\frac{n\pi y}{d},$$
$$B^2 = 1 + \frac{c^2n^2\pi^2}{d^2}\sin^2\frac{n\pi y}{d},$$
$$L = \frac{-2a}{\sqrt{A^2+B^2-1}}, \quad M = 0,$$
$$N = \frac{cn^2\pi^2}{d^2\sqrt{A^2+B^2-1}}\cos\frac{n\pi y}{d},$$
$$K = -\frac{2acn^2\pi^2}{d^2(A^2+B^2-1)^2}\cos\frac{n\pi y}{d}.$$

Hence, non-orthogonal conjugate system of curvilinear coordinates x, y defines the studied surface of translation.

The only coordinate lines $y = 0 + kd/n$ ($k = 0, 1, \ldots, n$) intersect the lines y at right angles. The coordinate line $x = b$ intersects the lines x at right angles too.

(2) Explicit equation (Figs. 4 and 5):

$$z = z(x,y) = -a(x-b)^2 + c\sin\frac{n\pi y}{d} + ab^2.$$

These surfaces (Figs. 1, 2, 3, 4 and 5) belong simultaneously to a class of *wave-shaped*, *waving*, *and corrugated surfaces*.

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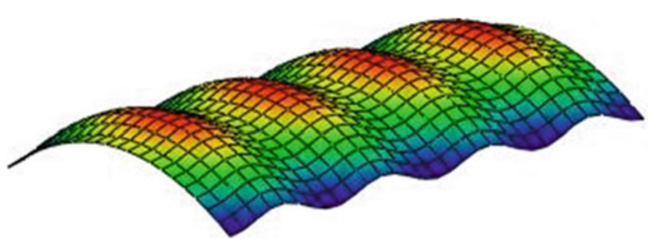

Fig. 1

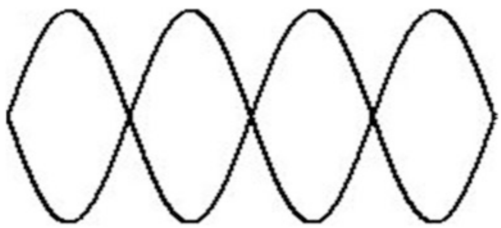

Fig. 2

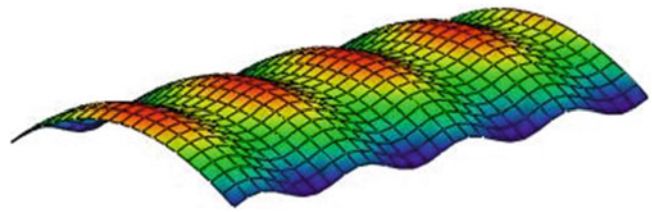

Fig. 4

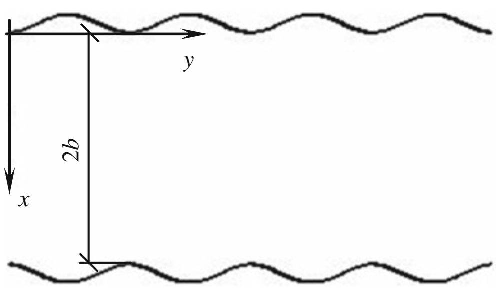

Fig. 3

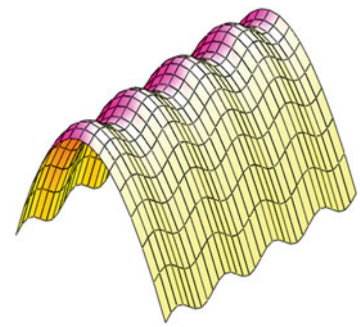

Fig. 5

■ Elliptic Surface of Translation

Elliptic surface of translation is formed by parallel translation of an ellipse so that a corresponding point of this ellipse moves on other ellipse. A directrix and generatrix ellipses are placed in mutually perpendicular planes.

Forms of the definition of an elliptic surface of translation

(1) Explicit equation (Fig. 1):

$$z = f_1 - b + \frac{b}{a}\sqrt{a^2 - \left(x - \frac{c}{2}\right)^2} + f_2 - m + \frac{m}{n}\sqrt{n^2 - \left(y - \frac{d}{2}\right)^2},$$

where

$$a = \frac{bc}{2\sqrt{f_1(2b - f_1)}}, \quad n = \frac{md}{2\sqrt{f_2(2m - f_2)}},$$

but f_1 is the rise of the ellipses lying in the planes $y = 0$ and $y = d$ ($f_1 < 2b$); f_2 is the rise of the ellipses placed in the planes $x = 0$ and $x = c$ ($f_2 < 2m$); $0 \le x \le c$; $0 \le y \le d$; a, b are the semi-axes of the ellipses lying in the cross sections of the translation surface by the planes $y = \text{const}$; n, m are the semi-axes of the ellipses lying in the cross sections of the translation surface by the planes $x = \text{const}$. The surface covers the rectangular plan $c \times d$, $c \le 2a; d \le 2n$.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 1 + \frac{b^2(c/2 - x)^2}{a^2\left[a^2 - (x - c/2)^2\right]},$$

$$F = \frac{mb(c/2 - x)(d/2 - y)}{an\sqrt{a^2 - (x - c/2)^2}\sqrt{n^2 - (y - d/2)^2}},$$

$$B^2 = 1 + \frac{m^2(d/2 - y)^2}{n^2\left[n^2 - (y - d/2)^2\right]},$$

$$A^2B^2 - F^2 = A^2 + B^2 - 1,$$

$$L = \frac{-ab}{\left[a^2 - (x - c/2)^2\right]^{3/2}\sqrt{A^2 + B^2 - 1}},$$

$$M = 0, \quad N = \frac{-nm}{\left[n^2 - (y - d/2)^2\right]^{3/2}\sqrt{A^2 + B^2 - 1}},$$

$$K = \frac{abnm}{\left[n^2 - (y - d/2)^2\right]^{3/2}\left[a^2 - (x - c/2)^2\right]^{3/2}(A^2 + B^2 - 1)^2} > 0.$$

Hence, non-orthogonal conjugate system of curvilinear coordinates x, y defines the studied surface of translation.

This variant of definition of the translation surface is used usually in strength analyses of thin shallow shells. In this case, it is assumed that $A \approx 1$, $B \approx 1$, $F \approx 0$.

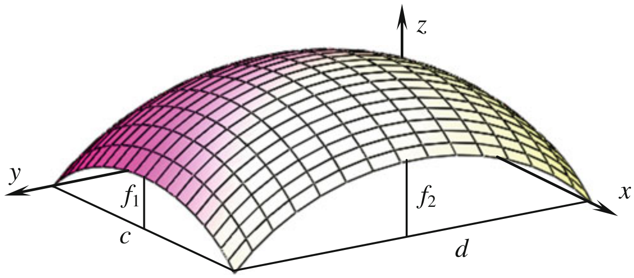

Fig. 1

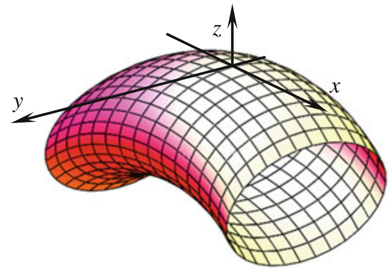

Fig. 2

(2) Explicit equation (Fig. 2):

$$z = \frac{b}{a}\sqrt{a^2 - x^2} + \frac{d}{c}\sqrt{c^2 - y^2} - b - d,$$

where a, b and c, d are the semi-axes of the directrix and generatrix ellipses.

The origin of a Cartesian coordinate system is placed at the peak of the surface: $-a \le x \le a$; $-c \le y \le c$.

■ Shoe Surface

"*Shoe Surface*" is a surface of parallel translation of a generatrix *parabola* along a directrix *cubic parabola*. The both parabolas lie in the mutually perpendicular planes.

Forms of definition of the surface

(1) Explicit equation (Fig. 1):

$$z = x^3/3 - y^2/2.$$

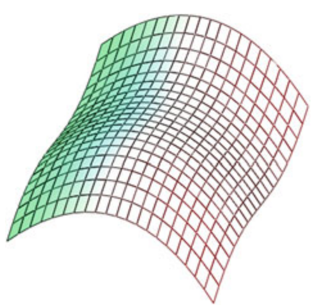

Fig. 1

In the cross section of the surface by planes x = const, parabolas lie but in the cross section of the surface by planes y = const, cubic parabolas are.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + x^4,\quad F = -x^2y,\quad B^2 = 1 + y^2,$$
$$A^2B^2 - F^2 = 1 + x^4 + y^2,$$
$$L = \frac{2x}{\sqrt{1 + x^4 + y^2}},\quad M = 0,$$
$$N = \frac{-1}{\sqrt{1 + x^4 + y^2}},$$
$$K = \frac{-2x}{(1 + x^4 + y^2)^2},\quad H = \frac{2x(1 + y^2) - x^4 - 1}{2(1 + x^4 + y^2)^{3/2}}.$$

Shoe surface is a surface of negative Gaussian curvature when $x > 0$ and a surface of positive Gaussian curvature if $x < 0$ but the curvilinear coordinate $x = 0$ contains only parabolic points with $K = 0$.

■ Surface of Translation of Parabola Along Hyperbola

Surface of translation of parabola along hyperbola is formed by the parallel translation of a parabola so that its corresponding point slides on a hyperbola. The same surface is generated if one moves the hyperbola along the parabola.

Forms of definition of the surface

(1) Explicit equation (Fig. 1):

$$z = \frac{2a - x}{a^2} f_1 x - \sqrt{c^2 + \frac{f_2(2c + f_2)}{d^2}(y - d)^2} + c + f_2,$$

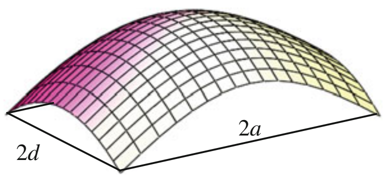

Fig. 1

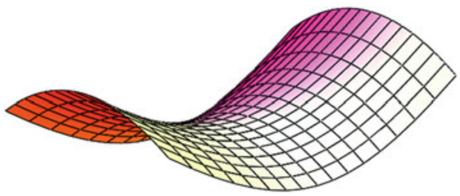

Fig. 2

where f_1 is the distance the coordinate plane xOy from the peak of a generatrix parabola lying in the plane xOz; f_2 is the same one but from the peak of directrix hyperbola lying in the plane yOz, $0 \le x \le 2a; 0 \le y \le 2d$. The studied surface of translation covers a rectangular plan $2a \times 2d$. The branches of the parabola and hyperbola are directed into the same side. Asymptotes of the hyperbola intersect a coordinate plane xOy at the angle of β:

$$\mathrm{tg}\beta = d \Big/ \sqrt{f_2(2c+f_2)}.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{4f_1^2}{a^4}(a-x)^2,$$
$$F = \frac{-2f_1 f_2(2c+f_2)(a-x)(y-d)}{a^2 d\sqrt{c^2d^2 + f_2(2c+f_2)(y-d)^2}},$$
$$B^2 = 1 + \frac{f_2^2(2c+f_2)^2(y-d)^2}{d^2\left[c^2d^2 + f_2(2c+f_2)(y-d)^2\right]},$$
$$L = \frac{-2f_1}{a^2\sqrt{A^2+B^2-1}},$$
$$N = \frac{-c^2 f_2(2c+f_2)d}{\left[d^2c^2 + f_2(2c+f_2)(y-d)^2\right]^{3/2}\sqrt{A^2+B^2-1}},$$
$$M = 0, \quad K = \frac{LN}{(A^2+B^2-1)^2} > 0.$$

(2) Explicit equation (Fig. 2):

$$z = -\frac{2a-x}{a^2}f_1 x - \sqrt{c^2 + \frac{f_2(2c+f_2)}{d^2}(y-d)^2} + c + f_2.$$

Now the branches of the parabola and hyperbola are directed into the opposite sides. This surface has $K < 0$.

■ Surface of Translation of Sinusoid Along Sinusoid

Surface of translation of sinusoid along sinusoid is formed by the parallel translation of a sinusoid $z_1 = c\,\sin(n\pi x/a)$ along another sinusoid $z_2 = d\,\sin(m\pi y/b)$, here n, m are the number of integer semi-waves containing on the straight line with the lengths equal to a and b correspondingly; c and d the amplitudes of the sinusoids. It is possible to hold that a sinusoid $z_1 = z_1(x)$ is a directrix curve but a sinusoid $z_2 = z_2(y)$ is a generatrix curve. The both sinusoids lie at mutually perpendicular planes.

This translation surface is defined by an explicit equation: (Figs. 1 and 2):

$$z = z(x,y) = c\sin(n\pi x/a) + d\sin(m\pi y/b).$$

Fig. 1

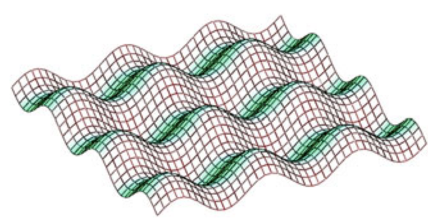

Fig. 2

Curvilinear coordinate net on the surface is non-orthogonal and conjugate. Only lines passing through the peaks of the surface intersect each other at right angle. The translation surface has the segments of both negative and positive total curvature. The surface shown in Fig. 1 has $c = d = 0.1$ m; $a = b = 3$ m; $n = m = 6$; in Fig. 2, the surface has $c = d = 0.3$ m; $a = b = 3$ m; $n = m = 6$.

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■ **Cycloidal Surface of Translation**

Cycloidal surface of translation is formed by the parallel translation of some cycloid curve

$$\boldsymbol{r}_1 = \boldsymbol{r}_1(t) = a(t + \sin t + \pi)\boldsymbol{i} + c(1 + \cos t)\boldsymbol{k}$$

along other one

$$\boldsymbol{r}_2 = \boldsymbol{r}_2(u) = b(u + \sin u + \pi)\boldsymbol{j} + d(1 + \cos u)\boldsymbol{k}.$$

The studied translation surface can be given by a vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, t) = \boldsymbol{r}_1(t) + \boldsymbol{r}_2(u).$$

A cycloidal surface of translation is a surface of the only positive total curvature (Fig. 1).

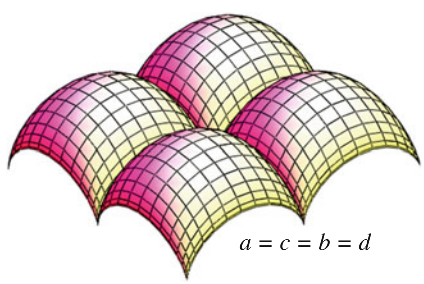

Fig. 1

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Additional Literature

P.S.: Additional literature is given on the corresponding pages of Chap. "3. Translation Surfaces" and in Chap. "5 in The Literature on Geometry, Application, and Analysis of Shells in the Form of Surfaces of Congruent Cross Section."

3.2 Translation Surfaces with Congruent Generatrix and Directrix Curves

Surfaces of translation with a plane supporting contour and having congruent curves as generatrixes and directrixes are called *translation surfaces with congruent generatrix and directrix curves*. It is recommended to add a prefix "bi" to names of these surfaces.

For example, a *bicircular translation surface* of the forth order (Fig. 1) is defined by an explicit equation:

$$z = 2r - \sqrt{r^2 - x^2} - \sqrt{r^2 - y^2}$$

or

$$x = r\sin u, \quad y = r\sin v, \quad z = r(\cos u + \cos v - 2),$$

where r is a radius of generator and generator circles; u, v are the central angles of the both circles. The beginning of Cartesian coordinates coincides with the highest point of the surface. A form of the horizontal cross sections approaches to a square with rounded off angles if the rise (h) of the surface approaches to a radius of the circles (r), i.e., if $h \to r$.

An explicit equation of horizontal cross sections is written in the form:

$$x = \pm\sqrt{4rh - 4r^2 - h^2 + y^2 + 2(2r - h)\sqrt{r^2 - y^2}},$$

where $h > 0$ is the distance between the origin of Cartesian coordinates and the horizontal secant plane.

A bi-elliptic translation surface of the forth order is defined by an equation:

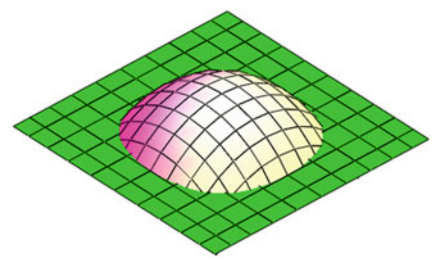

Fig. 1

$$z = 2b - \frac{b}{a}\left(\sqrt{a^2 - y^2} + \sqrt{a^2 - x^2}\right)$$

where a is a major semi-axis of the ellipses; b is a minor semi-axis of the ellipses. There is a curve

$$x = \pm\sqrt{y^2 - a^2 + 2a\sqrt{a^2 - y^2}}$$

in the cross section of the surface by a plane $z = b$.

A biparabolic translation surface (*paraboloid of revolution*) is given by an equation

$$z = \frac{y^2 + x^2}{2p}.$$

In the cross section of a biparabolic translation surface by a horizontal plane $z = h =$ const, a circumference

$$x = \pm\sqrt{2ph - y^2}$$

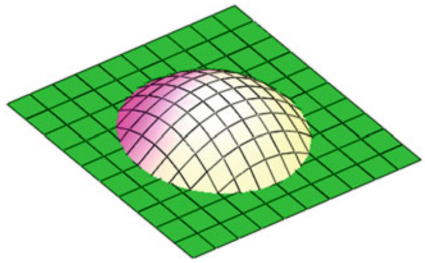

Fig. 2

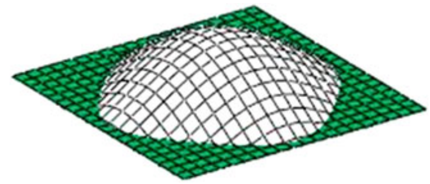

Fig. 3

lies where $h > 0$ is the distance between the origin of Cartesian coordinates and the horizontal secant plane.

A bihyperbolic translation surface is presented in Fig. 2. It can be given by the following equation:

$$z = -2a + \left(\sqrt{b^2 + y^2} + \sqrt{b^2 + x^2}\right)\frac{a}{b}$$

where a is a dimension of the real semi-axis but b is a dimension of the imaginary semi-axis of the hyperbola.

Having assumed $h = a$, we obtain an equation of the line if intersection of the surface and the plane $h = a$ in the form:

$$x = \pm\sqrt{9b^2 + y^2 - 6b\sqrt{b^2 + y^2}}.$$

Bi-agnésienne translation surface of the fifth order is shown in Fig. 3. Its explicit equation is written in the form:

$$z = 2a - \frac{a^3}{y^2 + a^2} - \frac{a^3}{x^2 + a^2},$$

where a is the diameter of the circumference taking part in the tracing of *the agnésienne* (*cubique d'Agnesi* or *witch of Agnesi*). A cross section of the surface by a horizontal plane $z = h$ gives a curve of the fourth order:

$$x = \pm a\sqrt{\frac{y^2h + a^2h - ay^2}{2ay^2 + a^3 - y^2h - a^2h}}.$$

The plane is limited by a curve having inflection points.

In the Encyclopedia, surfaces of this group are distributed over the other classes of surfaces. They are in class of "Surfaces of congruent cross sections," in "Surfaces of revolution," in "The second order surfaces," and so on.

Additional Literature

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3.3 Surfaces of Oblique Translation

Surfaces of oblique translation are created due to the movement of a plane curve on a directrix so that in the process of slipping of a generatrix curve on the plane unmovable contour, two symmetrical points of the generatrix touch the directrix continuously (Fig. 1).

Theoretically, such surface can be formed on a contour of an arbitrary outline but the surfaces created on a contour with symmetry, for example, circumferences, ellipses, squares, ovals, rectangles, are of the greatest interest.

If we want to study *a diagonal surface of translation* formed for a plane *rhombic contour* then the following relations

$$x = \frac{(u+v)c}{a} - c = \frac{(u+v)c}{\sqrt{c^2 + d^2}} - c,$$
$$y = \frac{(v-u)d}{a} = \frac{(v-u)d}{\sqrt{c^2 + d^2}}$$

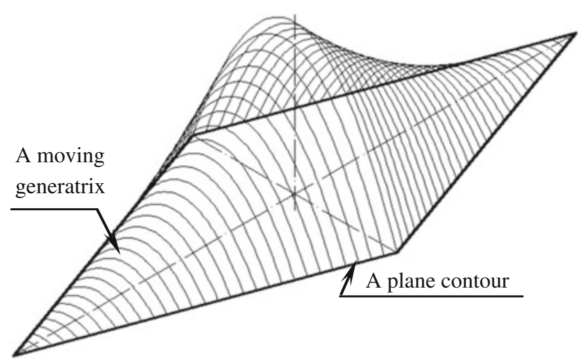

Fig. 1

between Cartesian x, y and oblique u, v coordinates would be useful (Fig. 2).

If they use oblique coordinates u, v as curvilinear coordinates on *the surface of diagonal translation* with a rhombic plane contour, then two sides of the plane contour will coincide with curvilinear coordinate lines $v = a$ and $u = a$.

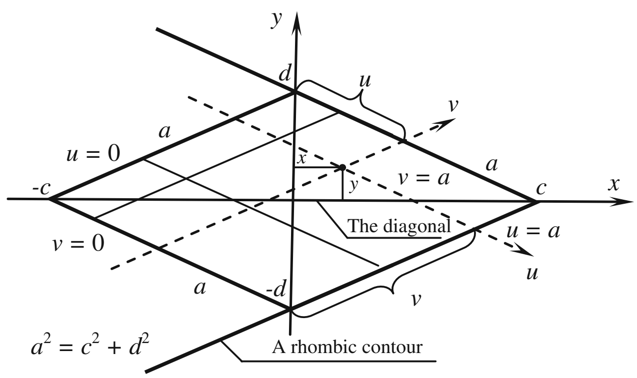

Fig. 2

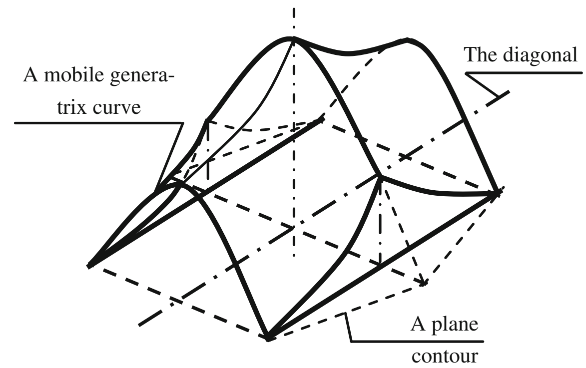

Fig. 4

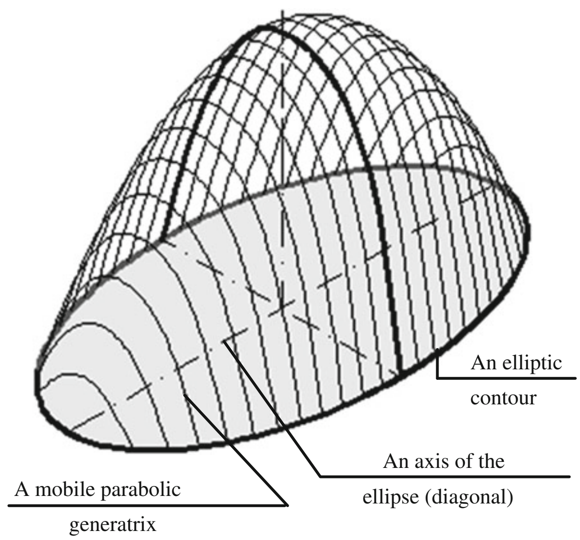

Fig. 3

Sometimes, using a method of design of surfaces of diagonal translation it is possible to obtain well known surfaces. For example, having taken an ellipse as a plane contour and any axis of the ellipse as a diagonal but a parabola as a movable generatrix curve (Fig. 3), we can form *an elliptic paraboloid* that is studied in Chap. "35. The Second Order Surfaces."

Surfaces of diagonal translation can be applied both with a plane boundary contour (Fig. 1) and with truncated borders (Fig. 4). In the last case, surpluses of bulk will be absent.

Additional Literature

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■ Volkov's Diagonal Circular Surface of Translation

A diagonal circular surface of translation of Volkov is formed by parallel movement of a plane circular curve along a diagonal of a square contour (Fig. 1). By the way, two symmetrical points of the circular generatrix slip on a plane unmovable square contour. Because two coordinate planes are the planes of symmetry of the surface, it is quite enough to research only its segment in the limit of changing of a variable x from 0 to c.

Forms of definition of the surface

(1) Implicit equation:

$$z^4 - 2z^2[2r^2 - y^2 - (c - x)^2] + [y^2 - (c - x)^2]^2 = 0,$$

where r is a constant radius of a generatrix circle; $2c$ is a length of diagonals of the square contour; $a = \sqrt{2}\, c$ is a length of a side of the square contour; $r > c$.

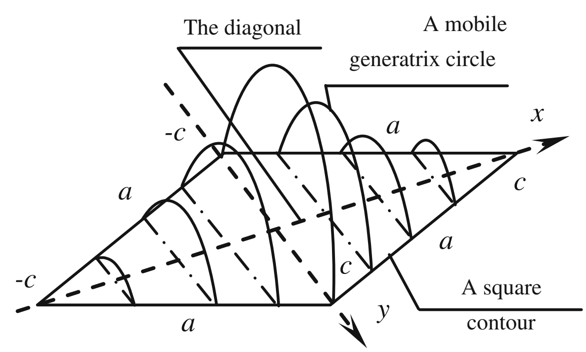

Fig. 1

The examined surface is *an algebraic surface of the fourth order*.

(2) Explicit equation:

$$z = \sqrt{r^2 - y^2} - \sqrt{r^2 - (c-x)^2}.$$

Volkov's diagonal circular surface of translation can be attributed to *surfaces of parallel translation of circle along circle*. A plane square contour $y = \pm (c - x)$ lies at the plane $z = 0$.

In the cross section of the surface by planes $x = x_0 = \text{const}$, generatrix circles with the radius r:

$$\left(z + \sqrt{r^2 - (c-x_0)^2}\right)^2 + y^2 = r^2$$

lie. Their centers are placed at the plane xOz but below the coordinate plane xOy at the quantity $m = \sqrt{r^2 - (c-x)}^2$.

In the cross section of the surface by planes $y = y_0 = \text{const}$, the circles with the radius r:

$$\left(z - \sqrt{r^2 - y_0^2}\right)^2 + (c-x)^2 = r^2$$

are placed. Their centers lie at the plane $x = c$ but they are over the coordinate plane xOy at the quantity $n = \sqrt{r^2 - y_0^2}$.

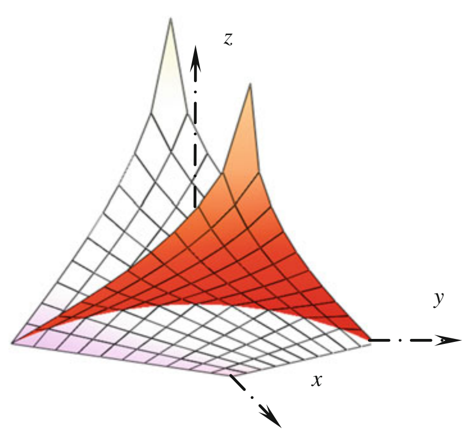

Fig. 2

The ordinate of the highest point of the surface is

$$z_{\max} = z(x=0, y=0) = r - \sqrt{r^2 - c^2}.$$

G.F. Volkov supposes that a variant of the surface with truncated boundary contour, that is square in plan with the sides equal to $t = c/2$ in the direction of the axis x and equal to $b = c$ in the direction of the axis y, is the most perspective one.

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{r^2}{r^2 - (c-x)^2}, \quad F = \frac{(c-x)y}{\sqrt{r^2 - (c-x)^2}\sqrt{r^2 - y^2}}, \quad B^2 = \frac{r^2}{r^2 - y^2},$$

$$A^2B^2 - F^2 = \frac{r^4 - y^2(c-x)^2}{[r^2 - (c-x)^2][r^2 - y^2]},$$

$$L = \frac{r^2[A^2B^2 - F^2]^{-1/2}}{[r^2 - (c-x)^2]^{3/2}}, \quad M = 0, \quad N = \frac{-r^2[A^2B^2 - F^2]^{-1/2}}{[r^2 - y^2]^{3/2}},$$

$$K = \frac{-r^4\sqrt{r^2 - y^2}\sqrt{r^2 - (c-x)^2}}{[r^4 - y^2(c-x)^2]^2} < 0.$$

The surface is *a surface of negative Gaussian curvature*.

(2) Parametrical equations (Fig. 2):

$$x = x(u,v) = \frac{u+v}{\sqrt{2}} - c,$$
$$y = y(u,v) = \frac{v-u}{\sqrt{2}},$$
$$z = z(u,v) = \sqrt{r^2 - \frac{(v-u)^2}{2}} - \sqrt{r^2 - \left(2c - \frac{u+v}{\sqrt{2}}\right)^2}$$

where $0 \le u \le a; 0 \le v \le a$ (see also Fig. 2 on p. 175 but with $d = c$). The second part of the surface will be symmetrical relatively the plane $x = 0$.

An equation of the line of division of two segments of the surface is

$$v = \sqrt{2}c - u.$$

Additional Literature

Volkov G.F. A shell of translation of negative curvature. Armotzement. Konstruk. v Stroitelstve. Leningrad, Gosstroyizdat, 1963; p. 48-58 (2 ref.).

■ Oblique Parabolic Surface of Translation

An oblique parabolic surface of translation or *a diagonal parabolic surface of translation* is formed by parallel translation of a plane parabola along the diagonal of a rhombic contour (see also Fig. 1 on the p. 175). Two symmetrical points of the parabolic generatrix slip on a plane unmovable contour.

Forms of definition of the surface

(1) Explicit equation:

$$z = \frac{d^2(c-x)^2}{2pc^2} - \frac{y^2}{2p}$$

where $2c$ and $2d$ are the length of the rhombic plane contour in the direction of the axes x and y relatively (see also Fig. 2 on the p. 177); p is a geometrical characteristic of the generatrix parabola. In the cross section of the surface by the plane $y = 0$, the parabola

$$z = d^2(c-x)^2/\left(2pc^2\right)$$

lies, but there is a generatrix parabola

$$z = \left(d^2 - y^2\right)/(2p)$$

in the cross section of the surface by the coordinate plane $x = 0$. The maximum rise of the surface over the plane $z = 0$ is

$$z_{\max} = d^2/(2p).$$

In Fig. 1, the truncated diagonal parabolic surface of translation is shown when $x = \pm c/2$ and $y = \pm d/2$.

The given formulas show that two space of a diagonal parabolic surface of translation are segments of segments of *a hyperbolic paraboloid* which is studied in "Ruled surfaces of negative Gaussian curvature."

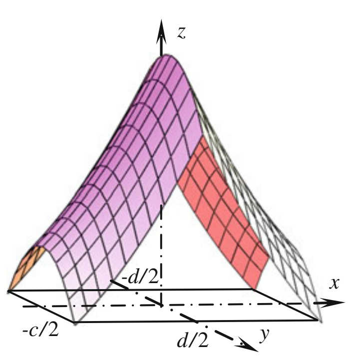

Fig. 1

Two coordinate planes $x = 0$ and $y = 0$ are the planes of symmetry of the surface and that is why it is quite enough to search only its segment in the limit from 0 to c.

(2) Parametrical form of definition (Fig. 2):

$$x = x(u,v) = \frac{(u+v)c}{a} - c,$$
$$y = y(u,v) = \frac{(v-u)d}{a},$$
$$z = z(u,v) = \frac{2d^2}{p}\left(1 - \frac{u+v}{a} + \frac{uv}{a^2}\right)$$

where $a = \sqrt{c^2 + d^2}$ is the length of the side of the rhombic contour.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 1 + \frac{4d^4}{a^2p^2}\left(\frac{v}{a} - 1\right)^2,$$
$$F = \frac{c^2 - d^2}{a^2} + \frac{4d^4}{a^2p^2}\left(\frac{v}{a} - 1\right)\left(\frac{u}{a} - 1\right),$$
$$B^2 = 1 + \frac{4d^4}{a^2p^2}\left(\frac{u}{a} - 1\right)^2,$$
$$L = N = 0, \quad M = \frac{4cd^3}{pa^4\sqrt{A^2B^2 - F^2}},$$
$$k_u = k_v = 0, \quad K = -\frac{M^2}{A^2B^2 - F^2} < 0.$$

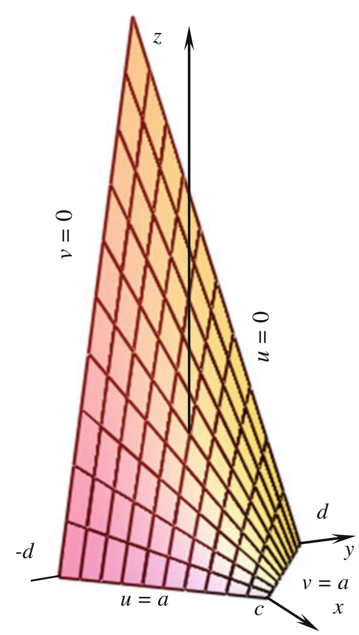

Fig. 2

Coordinate lines u, v are straight lines and coincide with two families of rectilinear generatrixes of a hyperbolic paraboloid.

In Fig. 2, the fragment of the diagonal parabolic surface of translation constructed in the limits

$$0 \le u \le a = \sqrt{c^2 + d^2},$$
$$0 \le v \le a = \sqrt{c^2 + d^2}$$

is presented.

Only the half of the presented surface cut off by a plane $x = 0$ will become one space of the diagonal parabolic surface of translation. The second half will be as mirror image of the first segment of the surface.

The both symmetrical spaces are placed upon the rhombic plane contour.

Coordinate lines $u = a$ and $v = a$ coincide with two sides of the rhombic plane contour.

A projection of a line $v = a - u$ that lies on the diagonal parabolic surface of translation, on a plane $z = 0$ coincides with a coordinate axis y.

3.4 Velaroidal Surfaces

Velaroidal surface is a surface placed on a plane rectangular plan with a generatrix curve of variable curvature. So, the surface is bounded by four mutually orthogonal contour straights ($k_x = k_y = 0$) lying at the same plan.

At present, three types of velaroidal surfaces are known that are parabolic velaroid (Fig. 1), sinusoidal velaroid (Fig. 2), and elliptical velaroid. Sometimes, a velaroidal surface is called *a funicular surface*.

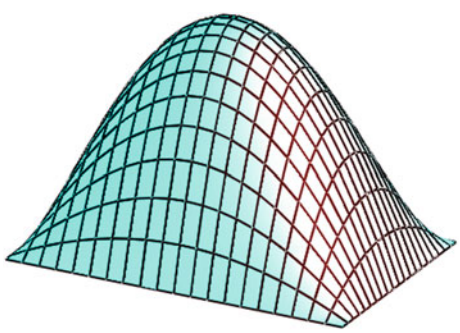

Fig. 1

Additional Literature

Mihailescu M, *Horvath I*. Velaroidal shells for covering universal industrial halls. Acta techn. Acad. sci. hung. 1977; 85 (1-2), p. 135-145.

Hadid Hassoun A, *Lynn Paul P*. Bending analysis of parabolic velaroidal shells. J. Struct. Div. Proc. Amer. Soc. Civ. Eng. 1980; 106, No 7, p. 1609-1621.

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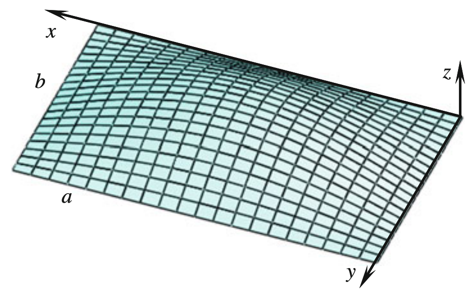

Fig. 2

■ Sinusoidal Velaroid

A sinusoidal velaroid is formed by two families of semi-waves of sinusoids lying in mutually perpendicular planes and with convexities directed in the same side (Fig. 2). Every family of the sinusoids has the same period. A sinusoidal velaroid as any other velaroid is bounded by a plane rectangular contour.

An explicit equation of a sinusoidal velaroid is written in the following form:

$$z = f \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

where a, b are the dimensions of a plane rectangular contour in the plan (Fig. 2). A maximum rise f if the surface over the plane $z = 0$ is at the point with coordinates $x = a/2$; $y = b/2$.

There are sinusoids with amplitudes

$$f \sin(\pi x_c / a)$$

in the cross sections of the surface by the planes $x = x_c = \text{const}$ and with amplitudes

$$f \sin(\pi y_c / b)$$

in the cross sections of the surface by the planes $y = y_c = \text{const}$. The sides of the plane rectangular contour coincide with coordinate lines $x = 0$, $x = a$ and $y = 0$, $y = b$.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 1 + f^2\frac{\pi^2}{a^2}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}, \quad F = f^2\frac{\pi^2}{4ab}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b},$$
$$B^2 = 1 + f^2\frac{\pi^2}{b^2}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b},$$
$$L = \frac{-f\pi^2}{a^2\sqrt{A^2+B^2-1}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \quad M = \frac{f\pi^2}{ab\sqrt{A^2+B^2-1}}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b},$$
$$N = \frac{a^2}{b^2}L,$$
$$k_x = \frac{-f\pi^2}{a^2A^2\sqrt{A^2+B^2-1}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \quad k_y = \frac{a^2A^2}{b^2B^2}k_x,$$
$$K = \frac{-f^2\pi^4}{2a^2b^2(A^2+B^2-1)^2}\left(\cos\frac{2\pi x}{a} + \cos\frac{2\pi y}{b}\right),$$
$$H = \frac{-f\pi^2}{2a^2b^2(A^2+B^2-1)^{3/2}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$
$$\left[a^2 + b^2 + f^2\pi^2\left(\cos^2\frac{\pi x}{a} + \cos^2\frac{\pi y}{b}\right)\right] \le 0.$$

Hence, the curvilinear coordinate net on the studied surface is non-orthogonal and non-conjugate. Only a coordinate line $x = a/2$ intersects a line $y = b/2$ at the right angle.

Mean curvature is equal to zero ($H = 0$) on the contour. Curvatures k_x, k_y of the coordinate lines are equal to zero on the whole contour.

Gaussian curvature of the surface can assume both negative and positive values. For example, a segment of the surface near its peak has $K > 0$ but at the angular points and at the middles of the sides of the contour, we have $K = 0$.

Additional Literature

Krivoshapko S, *Shambina S*. Forming of velaroidal surfaces on ring plan with two families of sinusoids. Abstracts. 16th Scientific-Professional Colloquium on Geometry and Graphics. Baška: Ministry of Science, Education and Sports of the Republic of Croatia, September 9-13, 2012; p. 19.

Structural Optimization. Vol. 2. Mathematical Programming. Ed. by M. Save and W. Prager. Plenum Press. NY and London. 1990, p. 205.

■ Surface of Velaroidal Type on Annulus Plan with Two Families of Sinusoids

Let us take two concentric circles with r_0 and R radiuses (Fig. 1) which we shall assume for contour curves of the surface. Let us assume that in any vertical section of *a–a* passing through the center *O*, a sinusoid lies, then

$$z = z(r) = -b\cos\frac{2\pi(r-c)}{R-r_0} + b,$$

where b is the variable height of a half of the wave of the sinusoid,

$$b = -0.5B(\cos(n\alpha) - 1);$$

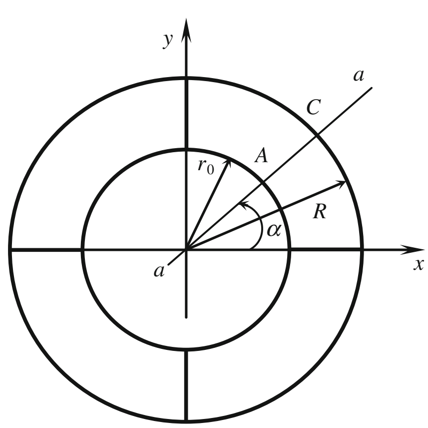

Fig. 1

B is the maximum height of a semi-wave; n is any number, $0 \le b \le B$. We consider that part of the sinusoid being between points A and C ($r_0 \le r \le R$) lies on a designed surface of velaroidal type (Fig. 2).

The set of curvilinear coordinate lines in the circular direction, we shall accept also in the form of sinusoids (Fig. 3)

$$z = z(\alpha) = -u(\cos(n\alpha) - 1),$$

where n is the number of identical fragments of a surface in the district direction, accepted necessarily, but $z = 0$ for $\alpha = 0$ always, i.e., line $\alpha = 0$ on the surface is a straight line;

$$u = -B\left(\cos\frac{2\pi(r-c)}{R-r_0} - 1\right).$$

One fragment is joined to the next similar fragment of the surface along the straight lines lying in the horizontal plane. For example, for Fig. 1, we have $n = 4$. Peripheral sinusoids on these straight lines have tangents which lie in the horizontal plane. Tangents to radial sinusoids in points of internal and external contours will lie in the horizontal plane also. Thus, we receive the parametrical equations of a studied surface as

$$x = x(r,\alpha) = r\cos\alpha,$$
$$y = y(r,\alpha) = r\sin\alpha,$$
$$z = z(r,\alpha) = \frac{B}{2}(1 - \cos(n\alpha))\left(1 - \cos\frac{2\pi(r-c)}{R-r_0}\right),$$

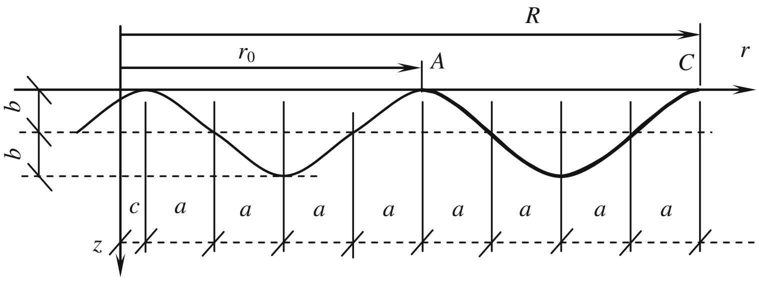

Fig. 2 *a–a*

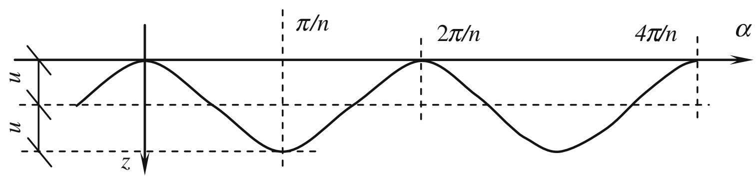

Fig. 3

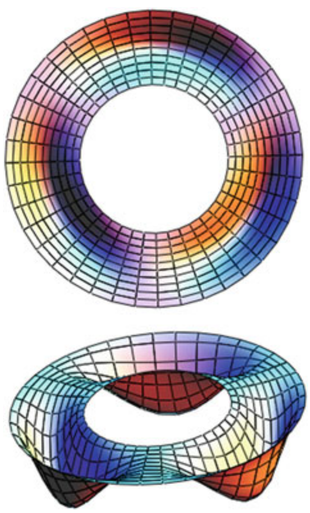

Fig. 4

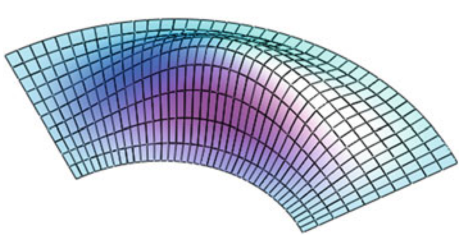

Fig. 5

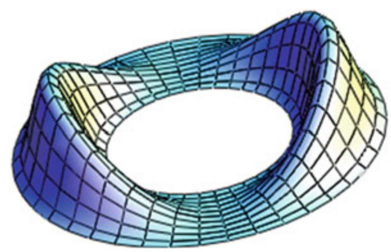

Fig. 6

$r_0 \le r \le R$, $0 \le \alpha \le 2\pi$, $a = (R - r_0)/4$, r = const are curvilinear coordinate lines on the surface projected on the horizontal plane as concentric circles.

According to the equations, we have contour lines with $z = 0$ when $\cos(n\alpha) = 1$, i.e., $n\alpha = 2m\pi$, or $\alpha_k = 2m\pi/n$, $0 \le \alpha \le \alpha_k$ is one cell of the surface. The surface with $n = 3$, $R = 4$ m, $r_0 = 2$ m; $B = 1$ m, $c = 0$ is shown in Fig. 4. In Fig. 5, one segment of the surface presented in Fig. 1 is shown. In Fig. 6, the surface has $n = 2$. These surfaces may be attributed to *surfaces of velaroidal type*.

Reference

Krivoshapko S.N., *Gil-Oulbe Mathieu*. Geometry and strength of a shell of velaroidal type on annulus plan with two families of sinusoids. Int. J. of Soft Computing and Engineering (IJSCE). 2013; Vol. 3, Iss. 3, p. 71-73.

■ Parabolic Velaroid

A parabolic velaroid belongs to surfaces of translation of positive Gaussian curvature on a plane rectangular plan with generatrix parabolas of variable curvature lying in parallel planes. So, a surface is bounded by four mutually orthogonal contour straights ($k_x = k_y = 0$). In some papers, parabolic velaroid is called *Shtaerman's surface with plane contour*.

Forms of definition of the surface

(1) Explicit equation (Fig. 1):

$$z = c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{x^2y^2}{a^2b^2}\right)$$

where a, b are half-spans in the direction of the coordinate axes x, y, relatively, c is a rise of the surface in its center.

A parabolic velaroid is limited by the contour straights coinciding with the coordinate lines $x = \pm a$; $y = \pm b$.

In the cross sections of the surface by planes x = const or y = const, parabolas lie with the rises that less than the rise c of the surface at the center.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{4c^2x^2}{a^4}\left(1 - \frac{y^2}{b^2}\right)^2,$$
$$F = \frac{4c^2xy}{a^2b^2}\left(1 - \frac{y^2}{b^2}\right)\left(1 - \frac{x^2}{a^2}\right),$$
$$B^2 = 1 + \frac{4c^2y^2}{b^4}\left(1 - \frac{x^2}{a^2}\right)^2,$$
$$L = \frac{2c}{a^2\sqrt{A^2 + B^2 - 1}}\left(1 - \frac{y^2}{b^2}\right),$$
$$M = -\frac{4cxy}{a^2b^2\sqrt{A^2 + B^2 - 1}},$$
$$N = \frac{2c}{b^2\sqrt{A^2 + B^2 - 1}}\left(1 - \frac{x^2}{a^2}\right),$$
$$K = \frac{4c^2}{a^2b^2(A^2 + B^2 - 1)^2}\left(1 - \frac{y^2}{b^2}\right)\left(1 - \frac{x^2}{a^2}\right) \geq 0.$$

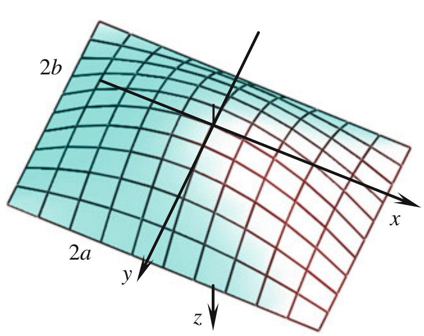

Fig. 1

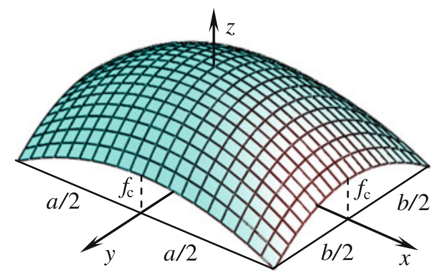

Fig. 2

(2) Explicit equation (Fig. 2):

$$z = 3f\left[\left(1 - \frac{4x^2}{3a^2}\right)\left(1 - \frac{4y^2}{3b^2}\right) - \frac{4}{9}\right].$$

Using this method of definition of a velaroidal surface, we must remember that a, b are spans of the surface. A surface with a rise of the surface at the center equal to $f_0 = 5f/3$ covers a rectangular plan $a \times b$. There are parabolas

$$z = \frac{4}{3}f\left(\frac{1}{2} - \frac{2y^2}{3b^2}\right) \text{ and } z = \frac{4}{3}f\left(\frac{1}{2} - \frac{2x^2}{3a^2}\right)$$

with the same rises $f_c = 2f/3$ on the edges $x = \pm(a/2)$ and $y = \pm(b/2)$ of the surface.

The studied velaroidal surface is called "Darbazi surface" in Georgia. This name has come from the name of the ancient Georgian stone cover. "Darbazi surface" is an inner segment of a parabolic velaroid.

Coordinate lines $x = \pm\sqrt{3}a/2$ and $y = \pm\sqrt{3}b/2$ have zero curvature, i.e., $k_x = k_y = 0$. A plane rectangular contour lies in the cross section of the surface by a plane $z = -4f/3$.

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■ Elliptical Velaroid

An elliptical velaroid belongs to surfaces of translation on a plane rectangular plan with generatrix ellipses of variable curvature lying in parallel planes. So, a surface is bounded by four mutually orthogonal contour straights.

An explicit equation of an elliptical velaroid can be written in the form:

$$z = z(x, y) = \sqrt{f^2 - \frac{f^2 - c^2}{a^2}(x^2 + y^2) + \frac{f^2 - c^2}{a^4}x^2y^2},$$

where a are the half-spans of the surface in the direction of the coordinate axes x, y; $(f - c)$ is the rise of the surface at its center (Fig. 1).

In the cross section of the surface by a plane $z = c$, a square $2a \times 2a$ lies. So, an elliptical velaroid covers a square plan $2a \times 2a$ and is bounded by straights lying in a plane $z = c$ and coinciding with the coordinate lines

$$x = \pm a; \quad y = \pm a.$$

In the cross sections of the surface by planes $x = 0$ and $y = 0$, the same ellipses with semi-axes f and $af\big/\sqrt{f^2 - c^2}$ lie. Assume $f^2 - c^2 = a^2$, then ellipses generate into circles with radius f (Fig. 2).

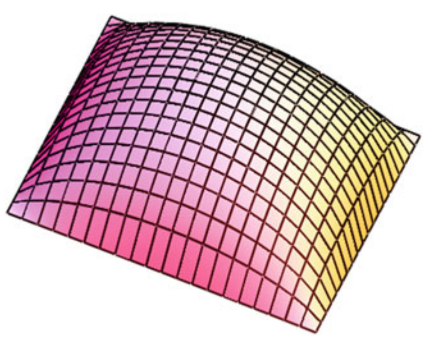

Fig. 1

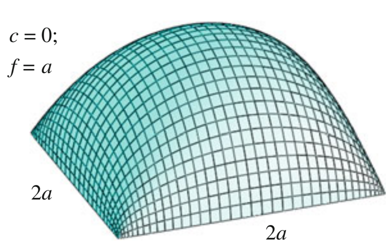

Fig. 2

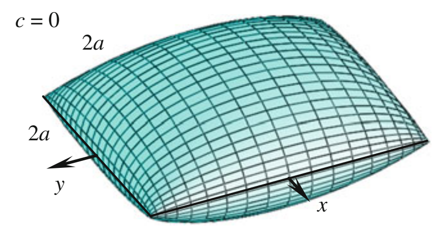

Fig. 3

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{x^2(f^2 - c^2)^2(y^2 - a^2)^2}{a^8\left[f^2 - \frac{f^2 - c^2}{a^2}\left(x^2 + y^2 - \frac{x^2y^2}{a^2}\right)\right]},$$

$$F = \frac{(f^2 - c^2)^2(x^2 - a^2)(y^2 - a^2)xy}{a^8\left[f^2 - \frac{f^2 - c^2}{a^2}\left(x^2 + y^2 - \frac{x^2y^2}{a^2}\right)\right]},$$

$$B^2 = 1 + \frac{y^2(f^2 - c^2)^2(x^2 - a^2)^2}{a^8\left[f^2 - \frac{f^2 - c^2}{a^2}\left(x^2 + y^2 - \frac{x^2y^2}{a^2}\right)\right]},$$

$$L = \frac{(f^2 - c^2)(y^2 - a^2)\left(f^2 - \frac{f^2 - c^2}{a^2}y^2\right)}{a^4\left[f^2 - \frac{f^2 - c^2}{a^2}\left(x^2 + y^2 - \frac{x^2y^2}{a^2}\right)\right]^{3/2}\sqrt{A^2 + B^2 - 1}},$$

$$M = \frac{(f^2 - c^2)xy\left[2f^2 - \frac{f^2 - c^2}{a^2}\left(x^2 + y^2 - \frac{x^2y^2}{a^2} + a^2\right)\right]}{a^4\left[f^2 - \frac{f^2 - c^2}{a^2}\left(x^2 + y^2 - \frac{x^2y^2}{a^2}\right)\right]^{3/2}\sqrt{A^2 + B^2 - 1}},$$

$$N = \frac{(f^2 - c^2)(x^2 - a^2)\left(f^2 - \frac{f^2 - c^2}{a^2}x^2\right)}{a^4\left[f^2 - \frac{f^2 - c^2}{a^2}\left(x^2 + y^2 - \frac{x^2y^2}{a^2}\right)\right]^{3/2}\sqrt{A^2 + B^2 - 1}}.$$

Elliptical velaroid has $K > 0$ at a central part of the surface but it has $K < 0$ at the angular parts of the surface.

Having assumed $c = 0$, we can obtain an explicit equation of an elliptical velaroid formed by semi-ellipses (Fig. 3) in the following form:

$$z = z(x, y) = \pm\frac{f}{a^2}\sqrt{a^4 - a^2(x^2 + y^2) + x^2y^2}.$$

A maximum rise of the surface at its center is equal to f. In the cross section of the surface by planes $x = 0$ or $y = 0$, ellipse with semi-axes a and f lie.

Additional Literature

Mihailescu M and Horvath I. Velaroidal shells for covering universal industrial halls. Acta techn. Acad. sci. hung. 1977; 85 (1–2), p. 135-145 (9 refs.)

4 Carved Surfaces

Carved surfaces are called surfaces with one family of plane lines of principal curvatures lying on planes that are orthogonal to the surface. The family of the plane lines of principal curvatures of a carved surface is *geodesic lines*. So the normals of these lines coincide with the normals of the surface. The carved surface may be characterized as the surface with *geodesic family of the lines of the principal curvatures*.

A carved surface, one of the evolutes of which is a developable surface, is called *a Monge surface*. This surface was named a Monge surface in honor of geometrician who first paid attention to such surfaces and investigated them in 1807. *An evolute surface* (an evolute) of the carved surface is called *a directrix surface*.

A Monge surface may be constructed with the help of *a kinematic method of rolling without slipping* of a plane with a plane curve (*meridian, profile*) over any developable surface. Having taken an arbitrary plane curve on a tangent plane P of the developable surface S and then having begun rolling of this plane P without slipping over the surface of the stationary directrix developable surface S (*stationary axoid*), we shall see the process of formation of a Monge surface by the generatrix curve. So, a Monge surface is formed by *the orthogonal trajectories* of the single-parametric system of planes. Orthogonal trajectories of the points of a meridian are called *parallels*. All meridians of a Monge surface are congruent. Meridians and parallels form two systems of the lines of principal curvatures of a Monge surface. The simplest example of a Monge surface is *any surface of revolution* that may be considered as a degenerated Monge surface. The single-parametric system of planes carrying a meridian degenerates into a pencil of planes passing through the axis of rotation of the surface. The cinematic method of forming of Monge surfaces gives an opportunity to divide them into three groups depending on the type of a directrix surface (stationary axoid), which are Monge surfaces with cylindrical, conic, and developable directrix surfaces.

Darboux treated them as surfaces traced out by a fixed curve on a plane that rolls on a developable surface. When the developable surface is a cylinder, he gives the general coordinates in his *Leçons sur la Theorie Generale des Surfaces* (Vol. I, p. 105).

The term Monge surface should not be confused with the term *Monge patch*, which is often used in differential geometry to denote a surface coordinate path of type (x, y, $f(x, y)$).

A. Naeve and J.-O. Eklundh hold that a Monge surface is *a Constant Section Twist Compensated Generalized Cylinder*.

Let us assume that

$$\alpha(u)x + \beta(u)y + \gamma(u)z = p(u)$$

is an equation of the single-parametric system of planes;

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$$x = x(u),\ y = y(u),\ z = z(u)$$

are parametrical equations of orthogonal trajectory referred to the same parameter u. A condition of orthogonality consists in execution of the following conditions along the curve:

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \lambda\alpha(u), \quad \frac{\mathrm{d}y}{\mathrm{d}u} = \lambda\beta(u), \quad \frac{\mathrm{d}z}{\mathrm{d}u} = \lambda\gamma(u) \quad \text{or}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\beta(u)}{\alpha(u)}, \quad \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\gamma(u)}{\alpha(u)}.$$

Based on the argument that a Monge surface is formed by movement of a generatrix plane curve along another directrix curve, so that the generatrix curve lies in the normal plane of the directrix curve and is connected rigidly with it, it is possible to write a vector equation of a carved surface as

$$\boldsymbol{r}(u,t) = \boldsymbol{\rho}(u) + x(t)\boldsymbol{e}(u) + y(t)\boldsymbol{g}(u),$$

where

$$\boldsymbol{e}(u) = \boldsymbol{v}\cos\theta + \boldsymbol{b}\sin\theta; \quad \boldsymbol{g}(u) = -\boldsymbol{v}\sin\theta + \boldsymbol{b}\cos\theta;$$

$\boldsymbol{e}(u)$, $\boldsymbol{g}(u)$ are the unit orthogonal vectors of mobile Cartesian system of coordinates placed in the normal plane of the directrix curve $\boldsymbol{\rho}(u)$; $x = x(t)$, $y = y(t)$ are the parametrical equations of the generatrix curve given for the same mobile Cartesian coordinate system; $\boldsymbol{v} = \boldsymbol{v}(u)$, $\boldsymbol{b} = \boldsymbol{b}(u)$ are the normal and binormal of the directrix curve;

$$\theta = -\int \kappa s\, \mathrm{d}u + \theta_o$$

is the angle of vector $\boldsymbol{v}$ with vector $\boldsymbol{e}$ or vector $\boldsymbol{b}$ with vector $\boldsymbol{g}$; κ is torsion of the directrix curve; θ_o is a constant of integration (an initial angle of vector $\boldsymbol{v}$ with vector $\boldsymbol{e}$); $s = |\rho'|$.

N.M. Onischuk and O.I. Svetlanova have studied *affine carved surfaces*.

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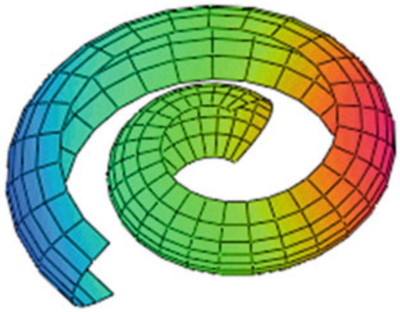
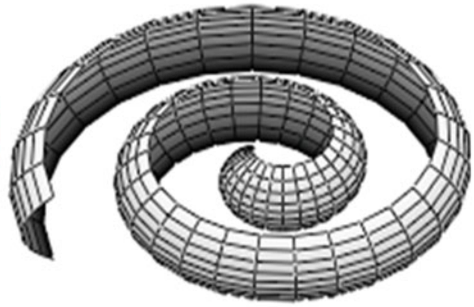
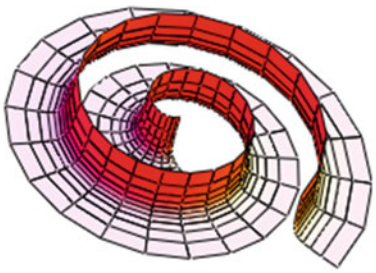
■ Carved Surfaces Presented in the Encyclopedia



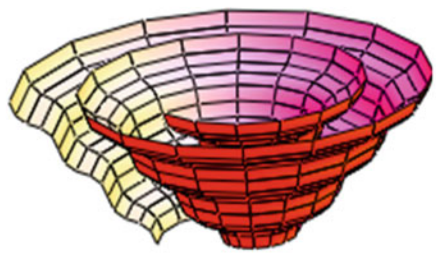
The Monge surface with the cylindrical directrix surface and meridian in the form of the catenary



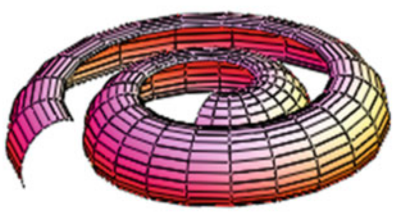
The Monge surface with the cylindrical directrix surface and the parabolic meridian



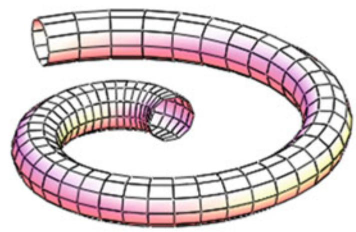
The Monge surface with the cylindrical directrix surface and the hyperbolic meridian



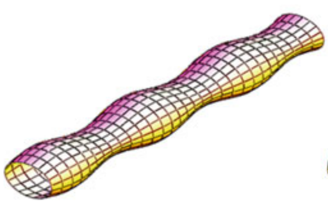
The Monge surface with the cylindrical directrix surface and with the sinusoid as meridian

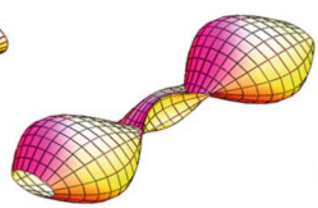


The Monge surface with the cylindrical directrix surface and meridian in the form of the cycloid

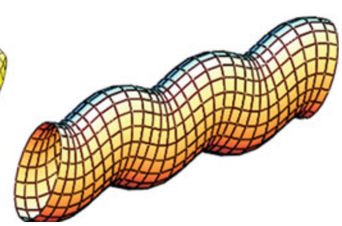


The tubular surface with a plane line of centers in the form of the evolvent of the circle

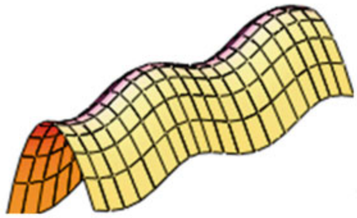




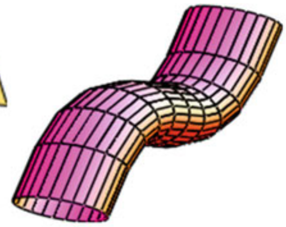
The carved surface with the directrix ellipse and the generatrix sinusoid



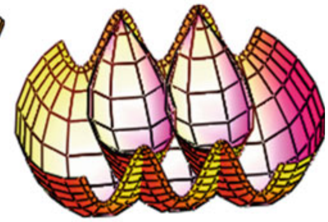
The carved surface with the directrix sinusoid and the generatrix ellipse



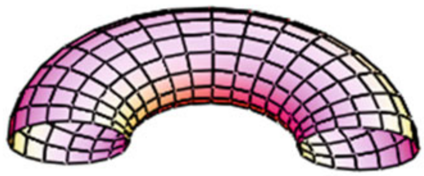
The carved surface with the directrix sinusoid and the generatrix parabola



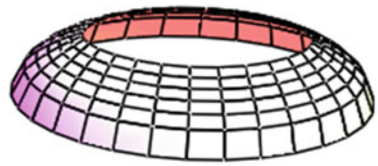
The carved surface with the directrix cubic parabola and the generating ellipse



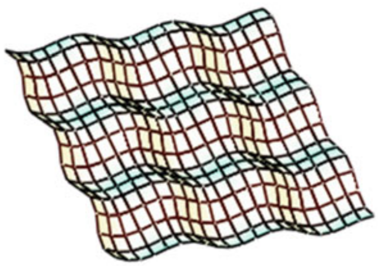
The carved surface with the directrix sinusoid and the generatrix cycloid



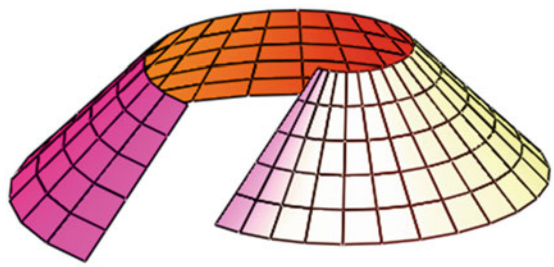
The carved surface with the directrix cycloid and the generatrix ellipse



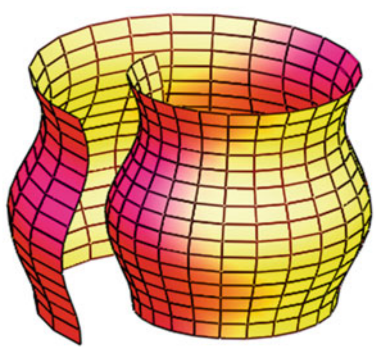
The carved surface with the directrix ellipse and the generatrix parabola



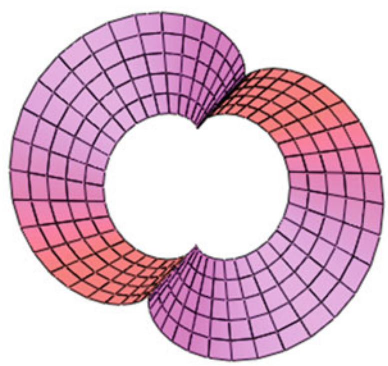
The carved sinusoidal surface



The Monge ruled surface with the circular cylindrical directrix surface



The Monge surface with the cylindrical directrix surface and with the sinusoid as meridian



The ruled conical limaçon of revolution

4.1 Monge Surfaces with a Circular Cylindrical Directrix Surface

If a directrix surface of a Monge surface is a right circular cylindrical surface, then all its parallels are the plane lines placed in parallel planes orthogonal to generatrix straight lines of the directrix cylinder. All meridians (generatrix curves) have equal curvatures at the points of the same parallel. Every two parallels cut off the equal arcs of the geodesic lines of principal curvatures (meridians). Assume points of one of the parallels for the beginning of set of the arc length for all meridians, then curvatures of the plane meridians will be of the same functions of their arc length. Taking into consideration that a directrix surface is a circular cylindrical surface, we can use generalized cylindrical coordinates *α, β* for the determination of an equation of a Monge surface. The Cartesian coordinates may be expressed in the following form:

$$x = x(\alpha, \beta) = r\cos\alpha - u\sin\alpha,$$
$$y = y(\alpha, \beta) = r\sin\alpha + u\cos\alpha,$$
$$z = \beta,$$

where r is the radius of the generatrix cylinder (Fig. 1);

$$u = u(\alpha, \beta) = f(\beta) - r\alpha.$$

A function $f = u(0, \beta)$ describes a form of the meridian. Using the presented parametric equations of the surface, we may determine the coefficients of the fundamental forms:

$$A = (f - r\alpha), \quad F = 0, \quad B^2 = 1 + f'^2,$$
$$L = \frac{A}{B}, \quad M = 0, \quad N = -\frac{f''}{B}$$

and obtain the principal radiuses of curvatures:

$$R_1 = AB, \quad R_2 = -\frac{B^3}{f''}.$$

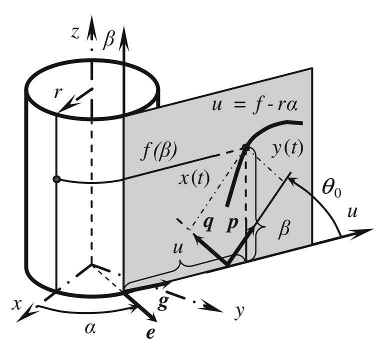


Fig. 1

Taking $r = 0$, we shall have *a surface of revolution*. Assume the meridian in the form of a straight line that is $f = c\beta + b$, then $u = c\beta + b - r\alpha$. In this case, parametrical equations of a Monge ruled surface with a circular cylindrical directrix surface (*stationary axoid*) can be written in the form:

$$x = x(\alpha, \beta) = r\cos\alpha - (c\beta + b - r\alpha)\sin\alpha,$$
$$y = y(\alpha, \beta) = r\sin\alpha + (c\beta + b - r\alpha)\cos\alpha,$$
$$z = \beta.$$

Having these equations of the surface, it is easy to derive the coefficients of the fundamental forms:

$$A = c\beta + b - r\alpha, \quad F = 0, \quad B^2 = 1 + c^2,$$
$$N = M = 0, \quad L = A/B, \quad aR_1 = AB, \quad R_2 = \infty.$$

Orthogonal trajectories (parallels) of the points of the plane generatrix curve, when we have a circular cylindrical directrix surface with a radius *r,* are the evolvents of the circle with the radius *r*.

A vector equation of a parallel may be written as

$$\boldsymbol{\rho}(\alpha) = r[\boldsymbol{e} + (\alpha_o - \alpha)\boldsymbol{g}],$$

where

$$\boldsymbol{e}(\alpha) = \boldsymbol{i}\cos\alpha + \boldsymbol{j}\sin\alpha; \ \boldsymbol{g}(\alpha) = -\boldsymbol{i}\sin\alpha + \boldsymbol{j}\cos\alpha$$

are the circular vector functions (Fig. 1).

Hence, an equation of the Monge surface with a circular cylindrical directrix surface is

$$\boldsymbol{r}(\alpha, t) = r[\boldsymbol{e} + (\alpha_o - \alpha)\boldsymbol{g}] + x(t)\boldsymbol{p}(\alpha) + y(t)\boldsymbol{q}(\alpha),$$

where

$$\boldsymbol{p}(\alpha) = \boldsymbol{g}(\alpha)\cos\theta_0 + \boldsymbol{k}\sin\theta_0;$$
$$\boldsymbol{q}(\alpha) = -\boldsymbol{g}(\alpha)\sin\theta_0 + \boldsymbol{k}\cos\theta_0.$$

For this case of definition of a Monge surface, formulas for the determination of coefficients of the fundamental forms can be written as

$$A = r(\alpha_0 - \alpha) + x(t)\cos\theta_0 - y(t)\sin\theta_0,$$
$$F = 0, \quad B = \sqrt{x'^2(t) + y'^2(t)},$$
$$L = -[x'(t)\sin\theta_0 + y'(t)\cos\theta_0]A/B, \quad M = 0$$
$$N = -[x'(t)y''(t) - x''(t)y'(t)]/B,$$
$$k_1 = -[x'(t)\sin\theta_0 + y'(t)\cos\theta_0]/(AB),$$
$$k_2 = -[x'(t)y''(t) - x''(t)y'(t)]/B^3.$$

A vector equation of a Monge ruled surface with a circular cylindrical directing surface is given in the Subsect. "1.1.1. Torse surfaces (torses)".

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■ Monge Surface with a Cylindrical Directrix Surface and a Parabolic Meridian

A Monge surface with a cylindrical directrix surface and a parabolic meridian can be defined as a surface formed by a moving parabola, the plane of which rolls without slipping above *a right circular cylinder*. Assuming a parabola as a generatrix curve and an evolvent of the circle as a directrix curve, it is possible to give another determination for a studied Monge surface: Monge surface with a cylindrical directrix surface and a parabolic meridian is generated by a plane curve in the form of a parabola moving along a directrix evolvent of the circle, so that a plane with the parabola lies all the time at the normal plane of the directrix evolvent of the circle and has rigid connection with it.

The forms of the definition of the Monge surface

(1) Parametrical form of the definition (Figs. 1, 2, 3 and 4):

$$x = x(\alpha, t) = r\cos\alpha - [r(\alpha_o - \alpha) + t\cos\theta_o - at^2\sin\theta_o]\sin\alpha,$$
$$y = y(\alpha, t) = r\sin\alpha + [r(\alpha_o - \alpha) + t\cos\theta_o - at^2\sin\theta_o]\cos\alpha,$$
$$z = z(t) = t\sin\theta_o + at^2\cos\theta_o,$$

where r is the radius of a directrix circular cylinder; θ_o is the slope angle of the axis of the parabola with the axis of the directrix cylinder; $0 \le \alpha \le \infty$; $-\infty \le t \le \infty$ (see also Fig. 1 on p. 189). For this form of definition, a vector equation of the directrix evolvent of the circle with a radius r is written as

$$\boldsymbol{\rho}(\alpha) = r[\boldsymbol{e} + (\alpha_o - \alpha)\boldsymbol{g}],$$

but parametrical equations of the same directrix evolvent are

$$x = x(\alpha) = r[\cos\alpha - (\alpha_o - \alpha)\sin\alpha],$$
$$y = y(\alpha) = r[\sin\alpha + (\alpha_o - \alpha)\cos\alpha].$$

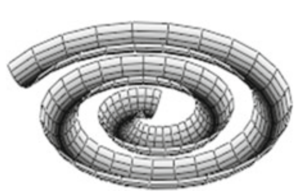


Fig. 1

Fig. 2

Parametrical equations of the generatrix parabola for the local coordinate system are presented in the form: $x = t$, $y = at^2$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = r(\alpha_o - \alpha) + t\cos\theta_o - at^2\sin\theta_o,$$
$$F = 0,\ B^2 = 1 + 4a^2t^2,$$
$$L = -\frac{\sin\theta_o + 2at\cos\theta_o}{B}A,$$
$$M = 0,\ N = -\frac{2a}{B},$$
$$k_1 = -\frac{\sin\theta_o + 2at\cos\theta_o}{AB},\ k_2 = -\frac{2a}{B^3}.$$

The surface is given in lines of the principal curvatures t, α. One family of the lines of principal curvatures denoted by t coincides with the generatrix parabolas, but the second family of the plane lines of principal curvatures α is the family of the evolvents of the circle with a radius r.

In Figs. 1, 2, 3, and 4, Monge surfaces with the cylindrical directrix surface and the parabolic meridian are shown. They have the following parameters:

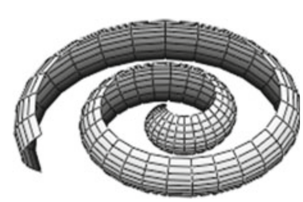
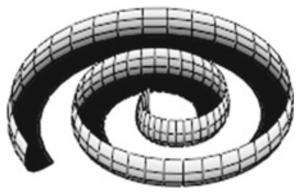


Fig. 3



$\theta = -\pi/4$

Fig. 4

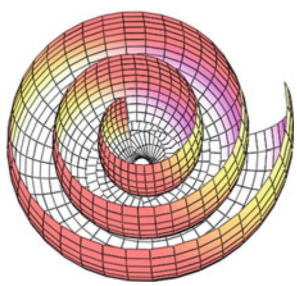


Fig. 5

$$r = 1 \text{ m}; \quad \alpha_o = 0; \quad a = 0.5 \text{ m}^{-1};$$
$$-2\,\text{m} \le t \le 2\,\text{m}; \quad \pi/2 \le \alpha \le 9\pi/2.$$

The value of parameter θ_o is shown under the corresponding figures.

In Fig. 5, the Monge surface is shown with parameters

$$r = 0.5 \text{ m}; \ \alpha_o = 0; \ a = 0.25 \text{ m}^{-1};$$
$$-4\,\text{m} \le t \le 4\,\text{m}; \quad \pi/2 \le \alpha \le 9\pi/2; \quad \theta_o = \pi/4.$$

(2) Parametrical equations in the cylindrical coordinates (Fig. 2):

$$x = x(\alpha, \beta) = r\cos\alpha - \left(\frac{\beta^2}{2p} - r\alpha + c\right)\sin\alpha,$$
$$y = y(\alpha, \beta) = r\sin\alpha + \left(\frac{\beta^2}{2p} - r\alpha + c\right)\cos\alpha,$$
$$z = \beta.$$

Coefficients of the fundamental forms of the surface:

$$A = \beta^2/(2p) + c - r\alpha, \quad F = 0, \quad B^2 = (p^2 + \beta^2)/p^2,$$
$$L = A/B, \quad M = 0, \quad N = -1/\sqrt{p^2 + \beta^2}.$$

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■ Monge Surface with a Cylindrical Directrix Surface and with a Sinusoid as Meridian

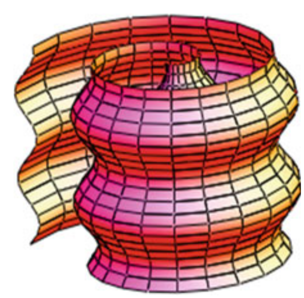
A Monge surface with a cylindrical directrix surface and a sinusoid curve as meridian may be defined as a surface formed by a moving sinusoid, the plane of which rolls without slipping above a right circular cylinder.

Assuming a sinusoid as a generatrix curve or a meridian and an evolvent of the circle as a directrix curve or a parallel, it is possible to give another determination for a studied Monge surface: Monge surface with a cylinder directrix surface and a sinusoidal generatrix is formed by a plane sinusoid moving along a directrix evolvent of the circle, so that a plane with the generatrix sinusoid lies all the time at the normal plane of directrix evolvent of the circle and is deadly linked with it.

The forms of the definition of the Monge surface

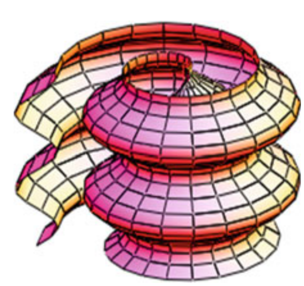
(1) Parametrical equations (Figs. 1, 2, and 3):

$$x = x(\alpha, t) = r\cos\alpha - [r(\alpha_o - \alpha) + t\cos\theta_o - c\sin(\mathrm{d}t)\sin\theta_o]\sin\alpha,$$
$$y = y(\alpha, t) = r\sin\alpha + [r(\alpha_o - \alpha) + t\cos\theta_o - c\sin(\mathrm{d}t)\sin\theta_o]\cos\alpha,$$
$$z = z(t) = t\sin\theta_o + c\sin(\mathrm{d}t)\cos\theta_o,$$



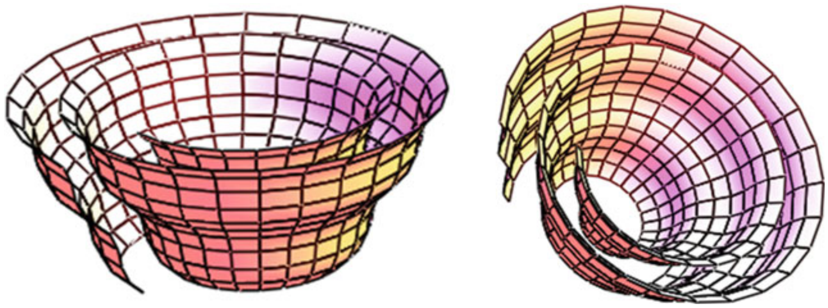
$r = 1$ m; $c = 1$ m; $d = 1$; $\theta_o = \pi/2$

Fig. 1



$r = 1$ m; $c = 2$ m; $d = 1$; $\theta_o = \pi/2$

Fig. 2



$r = 1$ m; $c = 1$ m; $d = 0.5$; $\theta_o = \pi/3$

Fig. 3

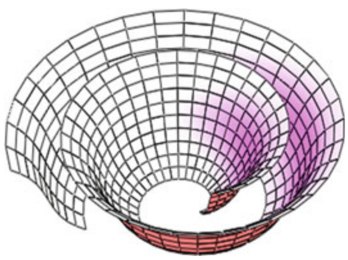


Fig. 4

where r is the radius of the directrix circular cylinder; θ_o is the slope angle of the axis of the sinusoid with the coordinate plane xOy; $0 \leq \alpha \leq \infty$; $-\infty \leq t \leq \infty$ (see also Fig. 1 on p. 189).

For this form of definition of the surface, a vector equation of the directrix evolvent of the circle with a radius r is written as

$$\boldsymbol{\rho}(\alpha) = r[\boldsymbol{e} + (\alpha_o - \alpha)\boldsymbol{g}],$$

but parametrical equations of the same directrix evolvent are

$$x = x(\alpha) = r[\cos\alpha - (\alpha_o - \alpha)\sin\alpha];$$
$$y = y(\alpha) = r[\sin\alpha + (\alpha_o - \alpha)\cos\alpha].$$

The parametrical equations of the generatrix sinusoid for local Cartesian coordinate system $\boldsymbol{q}$, $\boldsymbol{p}$ are used in the form:

$$x = x(t) = t,$$
$$y = y(t) = c\,\sin(\mathrm{d}t).$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = r(\alpha_o - \alpha) + t\cos\theta_o - c\sin(\mathrm{d}t)\sin\theta_o,$$
$$F = 0, \quad B^2 = 1 + c^2d^2\cos^2(\mathrm{d}t),$$
$$L = -\frac{\sin\theta_o + cd\cos(\mathrm{d}t)\cos\theta_o}{B}A,$$
$$M = 0, \quad N = \frac{cd^2\sin(\mathrm{d}t)}{B},$$
$$k_1 = -\frac{\sin\theta_o + cd\cos(\mathrm{d}t)\cos\theta_o}{AB},$$
$$k_2 = \frac{cd^2\sin(\mathrm{d}t)}{B^3}.$$

The surface is given in lines of the principal curvatures t, α. One family of the plane lines of principal curvatures denoted by t coincides with the generatrix sinusoids, but the second family of the plane lines of principal curvatures α is the family of the evolvents of the circle with a radius r.

In Figs. 1, 2 and 3, Monge surfaces with the cylindrical directrix surface and the sinusoidal meridian are shown. The values of the geometric parameters are presented under the corresponding figures.

The Monge surface shown in Fig. 4 has $r = 1$ m; $c = 1$ m; $d = 0.5$; $\theta_o = \pi/4$.

■ Monge Surface with a Cylindrical Directrix Surface and a Hyperbolic Meridian

A Monge surface with a cylindrical directrix surface and a hyperbolic meridian can be defined as a surface formed by a moving hyperbola, the plane of which rolls without slipping above a right circular cylinder.

It is possible to give another determination of a studied Monge surface: Monge surface with a cylinder generatrix surface and a hyperbolic meridian is generated by a plane hyperbola moving along a generatrix evolvent of the circle, so that a plane with the hyperbola lies all the time in the normal plane of the directrix and is deadly linked with it.

The forms of the definition of the Monge surface

(1) Parametrical equations (Figs. 1 and 2):

$$\begin{aligned} x = x(\alpha, t) &= r\cos\alpha - [r(\alpha_o - \alpha) + c\text{ch}t\cos\theta_o \\ &\qquad - d\text{sh}t\sin\theta_o]\sin\alpha, \\ y = y(\alpha, t) &= r\sin\alpha + [r(\alpha_o - \alpha) + c\text{ch}t\cos\theta_o \\ &\qquad - d\text{sh}t\sin\theta_o]\cos\alpha, \\ z = z(t) &= c\text{ch}t\sin\theta_o + d\text{sh}t\cos\theta_o, \end{aligned}$$

where r is the radius of a directrix circular cylinder; θ_o is the slope angle of the axis of the hyperbola with the coordinate plane xOy; $0 \le \alpha \le \infty$; (see also Fig. 1 on p. 189).

For this form of definition, a vector equation of the directrix evolvent of the circle with a radius r may be written as

$$\boldsymbol{\rho}(\alpha) = r[\boldsymbol{e} + (\alpha_o - \alpha)\boldsymbol{g}],$$

but parametric equations of the same directrix evolvent are

$$x = x(\alpha) = r[\cos\alpha - (\alpha_o - \alpha)\sin\alpha];$$
$$y = y(\alpha) = r[\sin\alpha + (\alpha_o - \alpha)\cos\alpha].$$

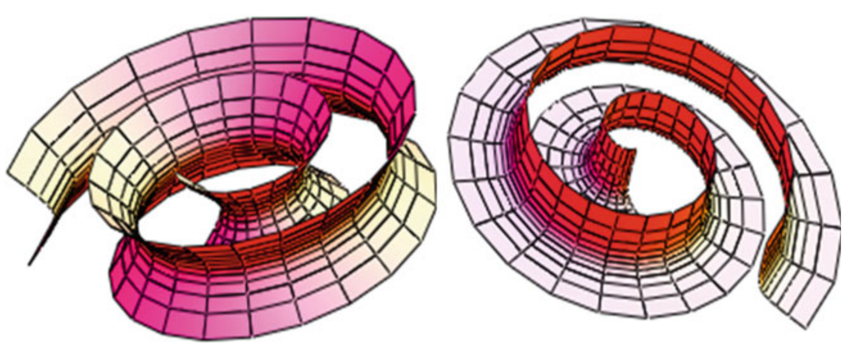
Parametrical equations of the generatrix hyperbola for the local Cartesian coordinate system $\boldsymbol{q}, \boldsymbol{p}$ are

$$x = x(t) = c\cosh t,$$
$$y = y(t) = d\sinh t.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

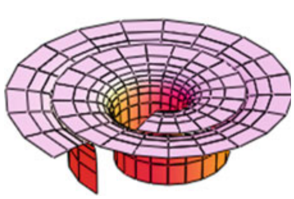
$$A = r(\alpha_o - \alpha) + c\cosh t\cos\theta_o - d\sinh t\sin\theta_o,$$
$$F = 0, \quad B^2 = c^2\sinh^2 t + d^2\cosh^2 t,$$
$$L = -\frac{c\sinh t\sin\theta_o + d\cosh t\cos\theta_o}{B}A,$$
$$M = 0, \; N = \frac{cd}{B},$$
$$k_1 = -\frac{c\sinh t\sin\theta_o + d\cosh t\cos\theta_o}{AB}, \quad k_2 = \frac{cd}{B^3}.$$

The surface is given in lines of the principal curvatures t, α.



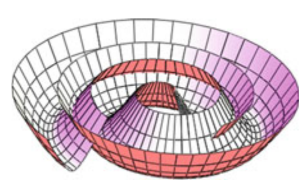
$\theta_o = \pi$; $r = 1$ m; $c = d = 1$ m; $\theta_o = \pi/4$

Fig. 1



$r = 6$ m; $c = d = 1$ m; $\theta_o = \pi/4$

Fig. 2



$r = 6$ m; $c = d = 1$ m; $\theta_o = \pi/2$

Fig. 3

One family of the plane lines of principal curvatures denoted by t coincides with the generatrix hyperbolas, but the second family of the plane lines of principal curvatures α is the family of the evolvents of the circle with a radius r.

In Figs. 1, 2 and 3, Monge surfaces with the cylindrical directrix surface and the hyperbolic meridian are shown. The values of the geometric parameters are presented under the corresponding figures.

Additional Literature

Ivanov VN, Rizwan Muhammad. Geometry of Monge surfaces and design of shells. Structural Mechanics of Engineering Constructions and Buildings. Moscow, Izd-vo ASV, 2002; Iss. 11, p. 27-36.

Ivanov VN. Some equations of a theory of surfaces with the system of the plane coordinate curves. Analysis of Shells of the Building Structures: Trudy UDN, Moscow: UDN, 1977; Vol. 83, Iss. 10, p. 37-48 (4 refs).

■ Monge Surface with a Cylindrical Directrix Surface and Meridian in the Form of a Cycloid

A Monge surface with a cylindrical directrix surface and a meridian in the form of a cycloid can be defined as a surface formed by a mobile cycloid, the plane of which rolls without slipping above a right circular cylinder.

Having assumed a cycloid as a generatrix curve (meridian) and an evolvent of the circle as a directrix curve (parallel), we can formulate another determination for a studied Monge surface: a Monge surface with a cylindrical directrix surface and a cycloid as generatrix is formed by a cycloid moving along a directrix evolvent of the circle, so that a plane with the cycloid coincides all the time with the normal plane of the directrix cycloid and is deadly linked with it.

The forms of the definition of the Monge surface

(1) Parametrical equations (Figs. 1 and 2):

$$\begin{aligned} x &= x(\alpha,t) = r\cos\alpha - [r(\alpha_o-\alpha) + c(t-\sin t)\cos\theta_o \\ &\qquad - c(1-\cos t)\sin\theta_o]\sin\alpha, \\ y &= y(\alpha,t) = r\sin\alpha + [r(\alpha_o-\alpha) + c(t-\sin t)\cos\theta_o \\ &\qquad - c(1-\cos t)\sin\theta_o]\cos\alpha, \\ z &= z(t) = c(t-\sin t)\sin\theta_o + c(1-\cos t)\cos\theta_o \end{aligned}$$

where r is the radius of a directrix circular cylinder; θ_o is the slope angle of the line of rolling of the circle with a radius c, the point on which traces the cycloid, with the coordinate plane xOy; $0 \le \alpha \le \infty$; $-\infty \le t \le \infty$ (see also Fig. 1 on p. 189).

For this form of definition of the surface, a vector equation of the directrix evolvent of the circle with the radius r may be written as

$$\boldsymbol{\rho}(\alpha) = r[\boldsymbol{e} + (\alpha_o - \alpha)\boldsymbol{g}],$$

but parametrical equations of the same directrix evolvent are

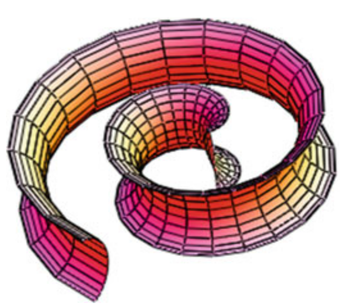
$$\begin{aligned} x &= x(\alpha) = r[\cos\alpha - (\alpha_o-\alpha)\sin\alpha]; \\ y &= y(\alpha) = r[\sin\alpha + (\alpha_o-\alpha)\cos\alpha]. \end{aligned}$$

Parametrical equations of the generatrix cycloid for the local Cartesian coordinate system $\boldsymbol{q}, \boldsymbol{p}$ are

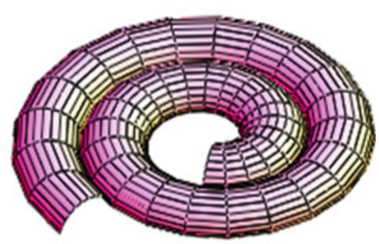
$$\begin{aligned} x &= x(t) = c(t-\sin t), \\ y &= y(t) = c(1-\cos t). \end{aligned}$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned} &A = r(\alpha_o-\alpha) + c(t-\sin t)\cos\theta_o - c(1-\cos t)\sin\theta_o, \\ &F = 0, \quad B^2 = 2c^2(1-\cos t), \\ &L = -\frac{(1-\cos t)\sin\theta_o + \sin t\cos\theta_o}{B}cA, \\ &M = 0, \quad N = \frac{c\sqrt{1-\cos t}}{\sqrt{2}}, \\ &k_1 = -c\frac{(1-\cos t)\sin\theta_o + \sin t\cos\theta_o}{AB}, \\ &k_2 = \frac{1}{2c\sqrt{2}\sqrt{1-\cos t}}. \end{aligned}$$

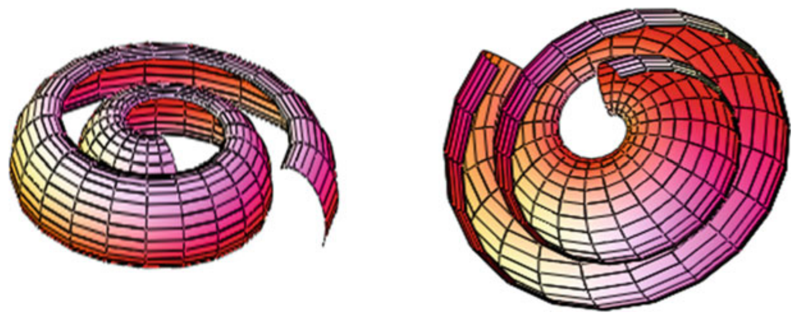


r = 1 m; c = 1 m; $\theta_o = \pi/2$;



r = 1 m; c = 1 m; $\theta_o = 0$

Fig. 1



$r = 1$ m; $c = 1$ m; $\theta_o = -\pi/4$;

Fig. 2

The surface is given in lines of principal curvatures t, α. One family of the plane lines of principal curvatures denoted by t coincides with the generatrix cycloids, but the second family of the plane lines of principal curvatures is the family of the evolvents of the circle with a radius r.

So, we have the surface of negative Gaussian curvature if $\theta_o = \pi/2$ (Fig. 1) and the surface consisting of two segments with different values of Gaussian curvature, i.e., $K < 0$ if $0 < t < \pi$ and $K > 0$ if $\pi < t < 2\pi$ (Fig. 2).

■ Monge Surface with a Cylindrical Directrix Surface and Meridian in the Form of Catenary

A Monge surface with a cylindrical directrix surface and a meridian in the form of a catenary can be determined as a surface formed by a mobile catenary

$$x(t) = t,$$
$$y(t) = a\cosh\frac{t}{a}.$$

the plane of which rolls without sliding above a right circular cylinder.

Having assumed a generatrix curve, i.e., the meridian and an evolvent of the circle

$$x(\alpha) = r[\cos\alpha - (\alpha_o - \alpha)\sin\alpha],$$
$$y(\alpha) = r[\sin\alpha + (\alpha_o - \alpha)\cos\alpha]$$

as a directrix curve, i.e., as parallel, we can give another determination for this surface: a Monge surface with a cylindrical directrix surface and with a meridian in the form of a catenary is formed by a moving generatrix catenary along a directrix evolvent of the circle, so that generatrix catenary lies in the normal plane of the directrix line. In the process of the movement, the catenary does not rotate itself about the tangent line to the directrix curve.

The forms of the definition of the Monge surface

(1) A parametrical form of the definition of the Monge surface with a cylindrical directrix surface and an arbitrary meridian:

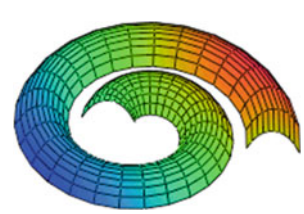
$$\begin{aligned} x = x(\alpha, t) &= r\cos\alpha - [r(\alpha_o - \alpha) + x(t)\cos\theta_o \\ &\quad - y(t)\sin\theta_o]\sin\alpha, \\ y = y(\alpha, t) &= r\sin\alpha + [r(\alpha_o - \alpha) + x(t)\cos\theta_o \\ &\quad - y(t)\sin\theta_o]\cos\alpha \\ z = z(t) &= x(t)\sin\theta_o + y(t)\cos\theta_o, \end{aligned}$$

where r is the radius of a directrix circular cylinder; θ_o is the slope angle of the generatrix curve with the coordinate plane xOy; $0 \le \alpha \le \infty$; $-\infty \le t \le \infty$ (see also Fig. 1 on p. 189).

Substituting the values of Cartesian coordinates of a generatrix catenary

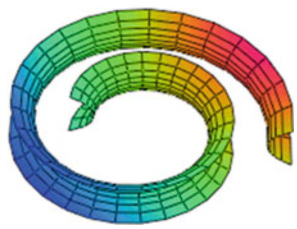
$$x(t) = t,\ y(t) = a\cosh(t/a)$$

into the common parametric equations given above, we determine the parametrical equations of the considered Monge surface:



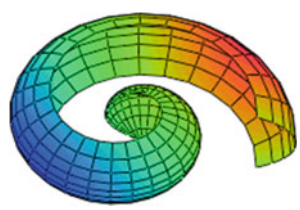
$\theta_o = 0$

Fig. 1



$\theta_o = -\pi/2;$

Fig. 2



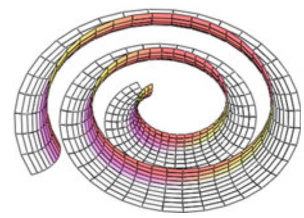
$\theta_o = -\pi/2;$

Fig. 3

$$x = x(\alpha, t) = r\cos\alpha - [r(\alpha_o - \alpha) + t\cos\theta_o - a\cosh\frac{t}{a}\sin\theta_o]\sin\alpha,$$
$$y = y(\alpha, t) = r\sin\alpha + [r(\alpha_o - \alpha) + t\cos\theta_o - a\cosh\frac{t}{a}\sin\theta_o]\cos\alpha,$$
$$z = z(t) = t\sin\theta_o + a\cosh\frac{t}{a}\cos\theta_o.$$

The formulas for the determination of coefficients of the fundamental forms of the surface are given in the Sect. "4.1. Monge surfaces with a circular cylindrical directrix surface" (p. 189). These formulas for the Monge surface considered here become simpler.

Coefficients of the fundamental forms of the surface and its principal curvatures:



$\theta_o = -\pi/4;$

Fig. 4

$$A = r(\alpha_o - \alpha) + t\cos\theta_o - a\cosh\frac{t}{a}\sin\theta_o,$$
$$F = 0,\ B = \cosh\frac{t}{a},$$
$$L = -\frac{\sin\theta_o + \sinh(t/a)\cos\theta_o}{B}A,$$
$$M = 0,\quad N = -\frac{1}{a},$$
$$k_1 = -\frac{\sin\theta_o + \sinh(t/a)\cos\theta_o}{AB},$$
$$k_2 = -\frac{1}{a\cosh^2(t/a)}.$$

The surface is given in the lines of principal curvatures t, α. One family of the plane lines of principal curvatures denoted by t coincides with the generatrix catenary, but the second family of the plane lines of principal curvatures α is the family of the evolvents of the circle with a radius r.

In Figs. 1, 2, 3 and 4, the Monge surfaces with the cylindrical directrix surface and the catenary as meridian are shown. The following geometrical parameters:

$$\alpha_o = 0;\ r = 0.5;\ a = 0.4;\ \pi \le \alpha \le 3\pi$$

are assumed.

The angle θ_o is presented in the corresponding figures.

4.2 Monge Surfaces with a Conic Directrix Surface

Monge surfaces with a conic directrix surface are formed by a plane curve lying in the plane rolling without sliding above *a circular cone*. So, the determinant of a surface considered here contains a right circular cone as *fixed axoid* and a plane as *loose axoid*. The method of the forming of surfaces gives an opportunity to draw a conclusion that these surfaces are *conic limaçons of revolution* (see also Sect. "34.1. *Rotational Surfaces*"), which belong to a class of *kinematic surfaces of general type*. These surfaces may be referred also to a class of *surfaces of the congruent cross sections* (see also Chap. 5).

I.A. Skidan proposes to use hyperbolic coordinates u, t, v for representation of Monge surfaces with a conic directrix surface (Fig. 1). He introduced into practice the following conventional signs: t is a parameter of mobile plane Ω^p, t = const is a plane Ω^p; u, v are orthogonal coordinates on the plane. An axes v coincides with a straight line of the tangency of the plane Ω^p and the cone Ω^c; v = const is a plane that is perpendicular to the axis of the cone Ω^c; an axis u is directed into the side of increasing of t; u = const is a one sheet hyperboloid of revolution and the cone Ω^c is an asymptotic cone for it.

Functions

$$x = \varphi(t, u, v),\ y = \psi(t, u, v),\ z = \xi(t, u, v)$$

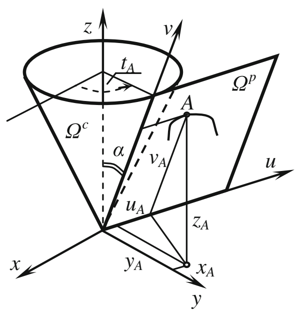


Fig. 1

give the Cartesian coordinates—hyperbolic coordinates relation. These ratios can be obtained if we relate the same point A of a Monge surface to the Cartesian coordinates and to the hyperbolic coordinates:

$$
\begin{aligned}
x &= x(u,t) = v\sin\alpha\cos t - u\sin t;\\
y &= y(u,t) = v\sin\alpha\sin t + u\cos t;\\
z &= v\cos\alpha.
\end{aligned}
$$

The inverse hyperbolic coordinates—Cartesian coordinates ratios can be written in the following form:

$$
\begin{aligned}
t &= 2\arctan\frac{y-\sqrt{x^2+y^2+z^2\mathrm{tg}^2\alpha}}{x+z\mathrm{tg}\alpha},\\
u &= \sqrt{x^2+y^2-z^2\mathrm{tg}^2\alpha},\\
v &= z\sec\alpha.
\end{aligned}
$$

The hyperbolic system of coordinates gives an opportunity to design a Monge surface with a fixed axoid in the form of a circular cone and with a loose axoid in the form of a plane with the help of the function

$$v = v(u,t).$$

Coefficients of the fundamental forms of the surfaces:

$$
\begin{aligned}
A^2 &= \left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2 = 1+\left(\frac{\partial v}{\partial u}\right)^2,\\
F &= (v-u)\sin\alpha+\frac{\partial v}{\partial u}\frac{\partial v}{\partial t},\\
B^2 &= \left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2\\
&= \left(\frac{\partial v}{\partial t}\right)^2-2u\sin\alpha\frac{\partial v}{\partial t}+u^2+v^2\sin^2\alpha,\\
L &= \frac{u\cos\alpha\frac{\partial^2 v}{\partial u^2}}{\sqrt{A^2B^2-F^2}},\\
M &= \cos\alpha\frac{u\frac{\partial^2 v}{\partial u\partial t}-u\sin\alpha\left(\frac{\partial v}{\partial u}\right)^2+v\sin\alpha\frac{\partial v}{\partial u}-\frac{\partial v}{\partial t}}{\sqrt{A^2B^2-F^2}},\\
N &= \cos\alpha\frac{u\frac{\partial^2 v}{\partial t^2}-2u\sin\alpha\frac{\partial v}{\partial t}\frac{\partial v}{\partial u}+\frac{\partial v}{\partial u}\left(v^2\sin^2\alpha+u^2\right)-v\sin\alpha\frac{\partial v}{\partial t}}{\sqrt{A^2B^2-F^2}}.
\end{aligned}
$$

Normal planes of the parallels of a Monge surface with conic directrix surface that are the tangent planes of the cone passing through its vertex. So, parallels are *spherical curves* placed at *the concentric spheres* with the centers in the vertex of the directrix cone.

The opposite is true also: a surface with a system of the lines of principal curvatures lying on concentric spheres is a Monge surface with a conic directrix surface.

Reference

Skidan IA. Kinematical surfaces in hyperbolic coordinates. Prikl. Geom. i Ingen. Grafika. Kiev, 1972; Iss. 14, p. 78-82.

■ Ruled Conic Limaçon of Revolution

Monge surfaces with a conic directrix surface are formed by a plane curve lying in the plane rolling without sliding above *a circular cone*. So, the determinant of a surface considered here contains a right circular cone as *fixed axoid* and a plane as *loose axoid*. The method of the forming of surfaces gives an opportunity to draw a conclusion that these surfaces are *conic limaçons of revolution* (see also Sect. "34.1. *Rotational Surfaces*") which belong to a class of *kinematic surfaces of general type*. These surfaces may be referred also to a class of *surfaces of the congruent cross sections* (see also Chap. 5).

Assuming a straight line as a generatrix curve on the mobile plane, we can obtain *a ruled conic limaçon of revolution*.

The forms of definition of an arbitrary conic limaçon of revolution

(1) Parametric equations:

$$
\begin{aligned}
x(u,v) &= [l+\varphi(v)](\sin\theta\cos u\cos w-\sin u\sin w)\\
&\quad -\psi(v)(\sin\theta\cos u\sin w+\sin u\cos w);\\
y(u,v) &= [l+\varphi(v)](\sin\theta\sin u\cos w+\cos u\sin w)\\
&\quad -\psi(v)(\sin\theta\sin u\sin w-\cos u\cos w);\\
z(u,v) &= [l+\varphi(v)]\cos\theta\cos w-\psi(v)\cos\theta\sin w,
\end{aligned}
$$

where l is the distance the vertex of the cone O from the beginning O_1 of the mobile system of the coordinates O_1XY in which the plane generatrix line

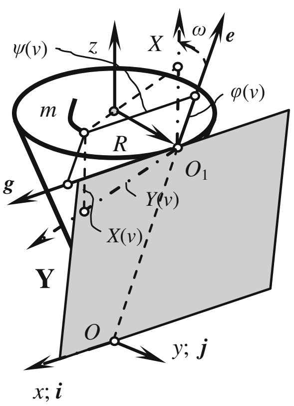


Fig. 1

$$X = X(v), \quad Y = Y(v)$$

is given. The distance l is taken along the line of the cone-plane tangency.

The mobile system of coordinates is placed in the rolling plane but its coordinate axes O_1X, O_1Y are turned at the angle ω relatively to mutually orthogonal unit vectors $\boldsymbol{e}(u)$ and $\boldsymbol{g}(u)$ (Fig. 1);

$$\begin{aligned}
\varphi(v) &= X(v)\cos\omega - Y(v)\sin\omega,\\
\psi(v) &= X(v)\sin\omega + Y(v)\cos\omega;\\
w(u) &= Ru/l = pu = \sin\theta\cdot u;\\
p &= R/l = \sin\theta;
\end{aligned}$$

where θ is the angle of the straight generatrix of the circular cone with its axis, u is the angle characterizing the rolling of the plane.

In Fig. 1, the initial position of the cone and the plane ($u = 0$) is shown.

Coordinate lines u form the spherical curves but coordinate lines v coincide with the generatrix plane curves m (Fig. 1).

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned}
A^2 &= [l\sin w + \varphi(v)\sin w + \psi(v)\cos w]^2\left(1-p^2\right)^2,\\
F &= 0, \quad B^2 = \dot\varphi^2 + \dot\psi^2 = \dot X^2 + \dot Y^2,\\
L &= -\frac{\dot\varphi\psi A}{B}, \quad M = 0,\ N = \frac{1}{B}\left(\dot X\,\ddot Y - \ddot X\,\dot Y\right),\\
k_1 &= \frac{L}{A^2} = -\frac{\dot\varphi\psi}{AB}, \quad k_2 = \frac{\left(\dot X\,\ddot Y - \ddot X\,\dot Y\right)}{B^3} = k_0,
\end{aligned}$$

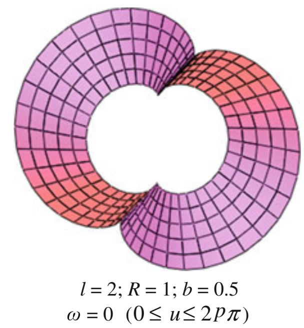


Fig. 2

where k_0 is the curvature of the plane generating curve m (Fig. 1). The points over the functions mean the differentiation with respect to the parameter v.

(2) Parametrical form of the definition of the ruled conic limaçon of revolution (Fig. 2):

$$\begin{aligned}
x(u,v) &= [l+\varphi(v)](\sin\theta\cos u\cos w - \sin u\sin w)\\
&\quad - \psi(v)(\sin\theta\cos u\sin w + \sin u\cos w),\\
y(u,v) &= [l+\varphi(v)](\sin\theta\sin u\cos w + \cos u\sin w)\\
&\quad - \psi(v)(\sin\theta\sin u\sin w - \cos u\cos w),\\
z(u,v) &= [l+\varphi(v)]\cos\theta\cos w - \psi(v)\cos\theta\sin w,
\end{aligned}$$

where

$$\varphi(v) = v\cos\omega - bv\sin\omega; \quad \psi(v) = v\sin\omega + bv\cos\omega.$$

Parametric equations of a ruled conic limaçon of revolution are derived from the general parametric equations for arbitrary conic limaçon of revolution after substitution of the values of

$$X = v, \quad Y = bv$$

into them.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned}
A &= [l\sin w + \varphi(v)\sin w + \psi(v)\cos w]\left(1-p^2\right)\\
F &= 0, \quad B^2 = 1 + b^2,\\
L &= -\dot\varphi\psi A/B, \quad M = 0,\ N = 0,\\
k_1 &= L/A^2 = -\dot\varphi\psi/(AB),\\
k_2 &= 0, \quad K = 0.
\end{aligned}$$

The ruled conic limacons of revolution are surfaces of *zero Gaussian curvature*, i.e., *torse surfaces*.

4.3 Carved Surfaces of General Type

Gaspard Monge introduced into practice surfaces that are formed by a plane generatrix curve lying in the plane which rolls without sliding over a cylinder or cone. Later, Darboux offered to add surfaces traced out by any fixed curve lying in a plane that rolls on *any torse surfaces* to Monge surfaces too. And at any time, every point of the generatrix curve lying in the plane executes a motion orthogonal to this plane.

Carved surfaces of general type are formed by a plane fixed generatrix curve, one point of which moves along any directrix curve and all the time, the generatrix curve must be in the normal plane of the directrix curve. In this case, all coordinate lines of one family are congruent curves but another family of coordinate lines is orthogonal to the plane lines of the first family. The net of curvilinear coordinates obtained by this method is conjugate one. Hence, carved surfaces of general type and Monge surfaces are the same surfaces. They differ only by a method of their forming.

The forms of definition of carved surfaces of general type

Assume that

$$\boldsymbol{\rho}(u) = x(u)\boldsymbol{i} + y(u)\boldsymbol{j} + z(u)\boldsymbol{k}$$

is a radius vector of a directrix curve. So, an arc length is

$$s = \sqrt{x'^2 + y'^2 + z'^2}.$$

The curvature of the directrix curve can be expressed as

$$k = \sqrt{\begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}^2 + \begin{vmatrix} x' & x'' \\ z' & z'' \end{vmatrix}^2 + \begin{vmatrix} y' & y'' \\ z' & z'' \end{vmatrix}^2} \Bigg/ \left(x'^2 + y'^2 + z'^2\right)^{3/2}.$$

The torsion of the directrix curve is given as

$$\kappa = \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} \Bigg/ \left[(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2\right].$$

It can also be shown that

$$\boldsymbol{\tau} = (x'\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k})/s$$

is the unit tangent vector to the directrix curve;

$$\begin{aligned}\boldsymbol{\beta} = [&(y'z'' - y''z')\boldsymbol{i} + (z'x'' - z''x')\boldsymbol{j} \\ &+ (x'y'' - x''y')\boldsymbol{k}]/(ks^3)\end{aligned}$$

is its unit vector of the binormal;

$$\begin{aligned}\boldsymbol{\nu} = \{&[y'(x'y'' - x''y') - z'(z'x'' - z''x')]\boldsymbol{i} \\ &+ [z'(y'z'' - y''z') - x'(x'y'' - x''y')]\boldsymbol{j} \\ &+ [x'(z'x'' - z''x') - y'(y'z'' - y''z')]\boldsymbol{k}\}/(ks^4)\end{aligned}$$

is the unit vector of the normal of the directrix curve.

The primed symbols mean the differentiation with respect to the parameter u:

$$\ldots' = d\ldots/\mathrm{d}u; \quad \ldots'' = d^2\ldots/\mathrm{d}u^2.$$

(1) Vector form of definition of the surface:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + X(v)\boldsymbol{e}_0(u) + Y(v)\boldsymbol{g}_0(u)$$

where

$$\begin{aligned}\boldsymbol{e}_0(u) &= \boldsymbol{\nu}\cos\theta + \boldsymbol{\beta}\sin\theta; \\ \boldsymbol{g}_0(u) &= -\boldsymbol{\nu}\sin\theta + \boldsymbol{\beta}\cos\theta, \\ \theta = \theta(u) &= -\int \kappa s\,\mathrm{d}u + \theta_0\end{aligned}$$

is the angle of the normal vector $\boldsymbol{\nu}(u)$ of the directrix curve with the vector $\boldsymbol{e}_0(u)$;

$$X = X(v), \quad Y = Y(v)$$

are parametric equations of the generatrix plane curve given in the mobile local system of coordinates XOY.

The angle $\theta(u)$ depends on the torsion of the directrix curve. For a plane directrix curve, we have $\theta = \theta_0 = \text{const}$, i.e., the generatrix plane curve moves in the normal plane of the directrix curve without rotation.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = s[1 - k(X\cos\theta - Y\sin\theta)]; \quad F = 0; \quad B = \sqrt{\dot{X}^2 + \dot{Y}^2};$$

$$L = \left(\dot{X}\sin\theta + \dot{Y}\cos\theta\right)sk\frac{A}{B}; \quad M = 0; \quad N = \frac{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}{B};$$

$$k_1 = k_u = \left(\dot{X}\sin\theta + \dot{Y}\cos\theta\right)\frac{sk}{AB};$$

$$k_2 = k_v = \frac{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}{B^3}.$$

where

$$\dot{\ldots} = \frac{d\ldots}{\mathrm{d}v}, \quad \ddot{\ldots} = \frac{d^2\ldots}{\mathrm{d}v^2},$$

Reference

Ivanov VN, Rizwan Muhammad. Geometry of Monge ruled surface and constructions of the shells. Structural Mechanics of Engineering Constructions and Buildings. 2002; Iss. 11, p. 27-36.

■ Carved Sinusoidal Surface

A carved sinusoidal surface is formed by a plane generatrix sinusoidal curve

$$X = X(v) = v, \quad Y = Y(v) = c\sin \mathrm{d}v,$$

in the process of motion of one of its point along a plane directrix sinusoidal curve

$$x = x(u) = u, \quad y = y(u) = a\sin bu.$$

The generating sinusoid all the time is in the normal plane of the directrix sinusoidal curve. A carved sinusoidal surface can be included into a group of the *wave-shaped surfaces.*

Forms of definition of the surface

(1) Vector form of definition:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + X(v)\boldsymbol{e}_0(u) + Y(v)\boldsymbol{g}_0(u)$$

where $\boldsymbol{\rho}(u) = u\boldsymbol{i} + a\sin bu\boldsymbol{j}$; $\boldsymbol{e}_0(u) = \boldsymbol{\beta} = \boldsymbol{k}$; $\boldsymbol{g}_0(u) = -\boldsymbol{\nu} = -(y'\boldsymbol{i} - x'\boldsymbol{j})/s$;

$$s = \sqrt{x'^2 + y'^2} = \sqrt{1 + a^2b^2\cos^2 bu}; \quad \theta = \pi/2.$$

Here, $\boldsymbol{\beta}$ is the unit vector of the binormal of the directrix curve; $\boldsymbol{\nu}$ is the unit vector of the principal normal of the directrix curve; $\ldots' = d\ldots/du$. Formulas are obtained from the general formulas given on p. 199.

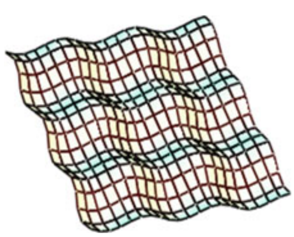

$a = c = 1$; $b = 0.5$; $d = 0.5$

Fig. 1

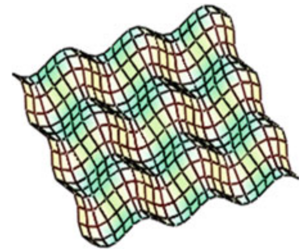

$a = 2$; $b = d = 0.5$; $c = 1$

Fig. 2

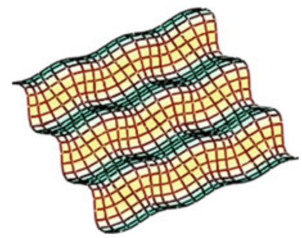

$a = 1$; $b = d = 0.5$; $c = 2$

Fig. 3

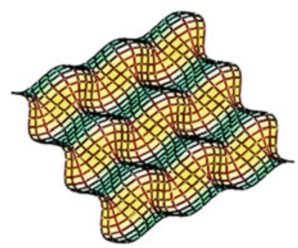

$a = c = 2$; $b = d = 0.5$

Fig. 4

(2) Parametric form of definition (Figs. 1, 2, 3, and 4):

$$x = x(u, v) = x(u) - \frac{y'(u)}{s}Y(v) = u - \frac{y'(u)}{s}Y(v),$$
$$y = y(u, v) = y(u) + \frac{x'(u)}{s}Y(v),$$
$$z = z(v) = X(v) = v.$$

■ Carved Surface with Directrix Cubic Parabola and Generating Ellipse

For the definition of a carved surface with a directrix cubic parabola $x = x(u) = u$, $y = y(u) = au^3$ and a generatrix ellipse $X = X(v) = c\cos v$, $Y = Y(v) = d\sin v$, one may use the general formulas given on page 199 but having assumed $\theta = 0$.

The generatrix ellipse all the time is in the normal plane of the directrix cubic parabola.

Forms of definition of the surface

(1) Parametric equations (Fig. 1):

$$x = x(u,v) = x(u) + \frac{y'(u)}{s}X(v) = u + \frac{y'(u)}{s}X(v),$$
$$y = y(u,v) = y(u) - \frac{x'(u)}{s}X(v), \quad z = Y(v).$$

(2) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + X(v)\mathbf{e_0}(u) + Y(v)\mathbf{g_0}(u),$$

where $\boldsymbol{\rho}(u) = u\boldsymbol{i} + au^3\boldsymbol{j}$, $\boldsymbol{e}_0(u) = \boldsymbol{\nu} = (y'\ \boldsymbol{i} - x'\ \boldsymbol{j})/s$; $\boldsymbol{g}_0(u) = \boldsymbol{\beta} = \boldsymbol{k}$;

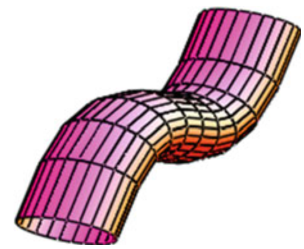


Fig. 1

$$s = \sqrt{x'^2 + y'^2} = \sqrt{1 + 9a^2u^4}.$$

Here, $\boldsymbol{\beta}$ is the unit vector of the binormal of the directrix curve; $\boldsymbol{\nu}$ is the unit vector of the principal normal of the cubic parabola; $\ldots' = d\ldots/\mathrm{d}u$.

■ Carved Surface with Directrix Ellipse and Generatrix Parabola

Carved surface is formed by a generatrix parabola $X = v$, $Y = Y(v) = cv^{1/2}$ when one of its point moves along a directrix ellipse

$$x = x(u) = a\cos u, \ y = y(u) = b\sin u.$$

Then (Fig. 1a, b)

$$x = x(u,v) = a\cos u - \frac{y'(u)}{s}c\sqrt{v},$$
$$y = y(u,v) = b\sin u + \frac{x'(u)}{s}c\sqrt{v},$$

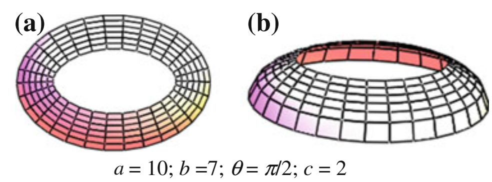


Fig. 1

$$z = z(v) = X(v) = v;$$
$$s = \sqrt{a^2\sin^2 u + b^2\cos^2 u}.$$

■ Carved Surface with Directrix Cycloid and Generatrix Ellipse

For definition of a carved surface with a directrix *cycloid*

$$x = x(u) = a(u - \sin u), \quad y = y(u) = a(1 - \cos u)$$

and a generatrix ellipse

$$X = X(v) = c\cos v, \quad Y = Y(v) = d\sin v$$

we may use general formulas given on page 199.

Taking into account that a cycloid is a plane curve, we can write the radius vector of a directrix curve in the form:

$$\boldsymbol{\rho}(u) = x(u)\boldsymbol{i} + y(u)\boldsymbol{j},$$

then $s = \sqrt{x'^2 + y'^2}$; $k = (x'y'' - y'x'')/s^3$ is the curvature of a directrix curve, $\boldsymbol{\kappa} = 0$, i.e., its torsion is equal to zero; $\boldsymbol{\tau} = (x'\boldsymbol{i} + y'\boldsymbol{j})/s$ is the unit tangent vector of the directrix curve; $\boldsymbol{\beta} = \boldsymbol{k}$ is the unit vector of its binormal; $\boldsymbol{\nu} = (y'\boldsymbol{i} - x'\boldsymbol{j})/s$ is the unit vector of the principal normal of the directrix curve;

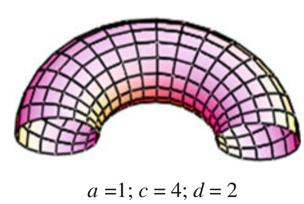
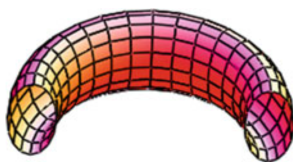


Fig. 1 $\theta = 0$

$a = 1;\ c = 4;\ d = 2$

Fig. 2 $\theta = \pi/4$



$a = 1;\ c = 4;\ d = 2$

Fig. 3 $\theta = \pi/2$

$$\ldots' = d\ldots/\mathrm{d}u;\ \ \ldots'' = d^2\ldots/\mathrm{d}u^2.$$

Forms of definition of the carved surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + X(v)\boldsymbol{e}_0(u) + Y(v)\boldsymbol{g}_0(u),$$

where

$$\boldsymbol{e}_0(u) = \boldsymbol{v}\cos\theta + \boldsymbol{\beta}\sin\theta = \frac{y\boldsymbol{i} - x'\boldsymbol{j}}{s}\cos\theta + \boldsymbol{k}\sin\theta;$$
$$\boldsymbol{g}_0(u) = -\boldsymbol{v}\sin\theta + \boldsymbol{\beta}\cos\theta = -\frac{y'\boldsymbol{i} - x'\boldsymbol{j}}{s}\sin\theta + \boldsymbol{k}\cos\theta;$$

θ = const is the angle of the normal $\boldsymbol{v}(u)$ of the directrix curve with the vector $\boldsymbol{e}_0(u)$; $X = X(v) = c\cos v$, $Y = Y(v) = d\sin v$ are parametrical equations of the generatrix ellipse written for the mobile local coordinate system XOY. For a plane directrix curve, we have θ = const, i.e., the generatrix ellipse moves in the normal plane of the directrix cycloid without rotation.

(2) Parametrical equations (Figs. 1, 2 and 3):

$$x = x(u, v) = x(u) + \frac{X(v)y'\cos\theta - Y(v)y'\sin\theta}{s}$$
$$= a(u - \sin u) + [c\cos v\cos\theta - d\sin v\sin\theta]\cos\frac{u}{2},$$
$$y = y(u, v) = y(u) - \frac{X(v)x'\cos\theta - Y(v)x'\sin\theta}{s}$$
$$= a(1 - \cos u) - [c\cos v\cos\theta - d\sin v\sin\theta]\sin\frac{u}{2},$$
$$z = z(v) = X(v)\sin\theta + Y(v)\cos\theta$$
$$= c\cos v\sin\theta + d\sin v\cos\theta,$$

where $s = 2a\sin(u/2)$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = s[1 + k(X\cos\theta - Y\sin\theta)];\ F = 0;\ B = \sqrt{\dot{X}^2 + \dot{Y}^2};$$
$$L = \left(\dot{X}\sin\theta + \dot{Y}\cos\theta\right)sk\frac{A}{B};\ M = 0; N = \frac{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}{B};$$

where

$$\dot{\ldots} = \frac{d\ldots}{\mathrm{d}v},\quad \ddot{\ldots} = \frac{d^2\ldots}{\mathrm{d}v^2},$$
$$k = -\frac{1}{4a\sin(u/2)} = -\frac{1}{2s},$$
$$k_1 = k_u = \left(\dot{X}\sin\theta + \dot{Y}\cos\theta\right)\frac{sk}{AB};$$
$$k_2 = k_v = \frac{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}{B^3}.$$

In the process of forming of the carved surface by kinematic method, the generatrix curve, for case in question this is an ellipse, all the time is placed in the normal plane of the directrix cycloid.

Additional Literature

Rizwan Muhammad. Design of the shells in form of Monge surface. Structural Mechanics of Engineering Constructions and Buildings. 2003; Iss. 12, Moscow: Izd-vo ASV, p. 63-68.

■ Carved Surface with Directrix Sinusoid and Generatrix Cycloid

For definition of a carved surface with a directrix sinusoid

$$x = x(u) = u,\quad y = y(u) = a\sin bu$$

and a generatrix cycloid

$$X = X(v) = c(v - \sin v),\quad Y = Y(v) = c(1 - \cos v)$$

it is necessary to use the general formulas given on page 199.

Taking into account that a directrix sinusoid is a plane curve, we can present the radius vector of a directrix curve in the form:

$$\boldsymbol{\rho}(u) = x(u)\boldsymbol{i} + y(u)\boldsymbol{j},$$

then

$$k = (x'y'' - y'x'')/s^3 = y''/s^3$$

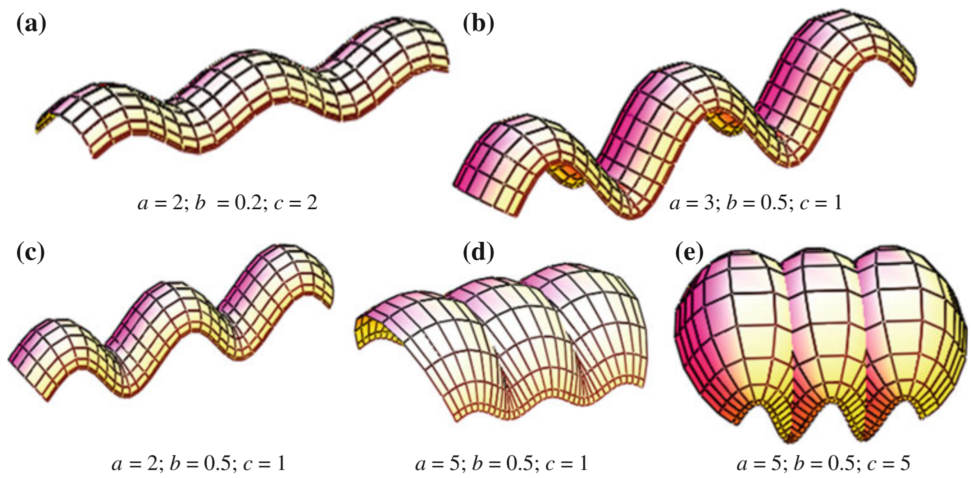


Fig. 1

is the curvature of a directrix sinusoid, $s = \sqrt{x'^2 + y'^2}$; $\kappa = 0$, i.e., the torsion of a sinusoid is equal to zero, $\boldsymbol{\tau} = (x'\boldsymbol{i} + y'\boldsymbol{j})/s$ is the unit tangent vector of a directrix curve; $\boldsymbol{\beta} = \boldsymbol{k}$ is its unit vector of the binormal; $\boldsymbol{v} = (y'\boldsymbol{i} - x'\boldsymbol{j})/s$ is the unit vector of the principal normal of the directrix curve.

$$\ldots' = d\ldots/\mathrm{d}u;\ \ldots'' = d^2\ldots/\mathrm{d}u^2.$$

The forms of definition of the carved surface

(1) Vector form of definition:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + X(v)\boldsymbol{e}_0(u) + Y(v)\boldsymbol{g}_0(u),$$

where

$$\boldsymbol{e}_0(u) = \boldsymbol{v}\cos\theta + \boldsymbol{\beta}\sin\theta = \frac{y'\boldsymbol{i} - x'\boldsymbol{j}}{s}\cos\theta + \boldsymbol{k}\sin\theta;$$
$$\boldsymbol{g}_0(u) = -\boldsymbol{v}\sin\theta + \boldsymbol{\beta}\cos\theta = -\frac{y'\boldsymbol{i} - x'\boldsymbol{j}}{s}\sin\theta + \boldsymbol{k}\cos\theta;$$

θ = const is the angle of the normal $\boldsymbol{v}(u)$ of the directrix curve with the vector $\boldsymbol{e}_0(u)$; $X = X(v) = c(v - \sin v)$, $Y = Y(v) = c(1 - \cos v)$ are parametrical equations of the generatrix cycloid given in the mobile local coordinate system XOY. For a plane directrix curve, we have θ = cons, i.e., the generatrix cycloid moves in the normal plane of the directrix sinusoid without rotation.

(2) Parametrical equations (Fig. 1):

$$x = x(u, v) = x(u) + \frac{X(v)y'\cos\theta - Y(v)y'\sin\theta}{s}$$
$$= u + \frac{(v - \sin v)\cos\theta - (1 - \cos v)\sin\theta}{s}abc\cos bu,$$
$$y = y(u, v) = y(u) - \frac{X(v)x'\cos\theta - Y(v)x'\sin\theta}{s}$$
$$= a\sin bu - c\frac{(v - \sin v)\cos\theta - (1 - \cos v)\sin\theta}{s},$$
$$z = z(v) = X(v)\sin\theta + Y(v)\cos\theta$$
$$= c[(v - \sin v)\sin\theta + (1 - \cos v)\cos\theta],$$

where $s = \sqrt{1 + a^2b^2\cos^2 bu}$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = s[1 + k(X\cos\theta - Y\sin\theta)];$$
$$F = 0;\ B = \sqrt{\dot{X}^2 + \dot{Y}^2} = 2c\sin(v/2);$$
$$L = (\dot{X}\sin\theta + \dot{Y}\cos\theta)sk\frac{A}{B};$$
$$M = 0;\ N = \frac{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}{B} = -c\sin\frac{v}{2};$$
$$k_1 = k_u = (\dot{X}\sin\theta + \dot{Y}\cos\theta)\frac{sk}{AB};$$
$$k_2 = k_v = \frac{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}{B^3} = -\frac{1}{4c\sin(v/2)},$$

by the way

$$k = -\frac{ab^2\sin bu}{s^3},\quad \dot{\ldots} = \frac{d\ldots}{\mathrm{d}v},\quad \ddot{\ldots} = \frac{d^2\ldots}{\mathrm{d}v^2}.$$

In the process of forming of the carved surface by kinematic method, the generatrix curve, for case in question this is a cycloid, all the time is placed in the normal plane of the directrix sinusoid. The carved surfaces shown in Fig. 1 have $\theta = \pi/2$.

Additional Literature

Barra Mario. The cycloid. Educ. Stud. Math. 1975; 6, No. 1, p. 93-98.

■ Carved Surface with Directrix Sinusoid and Generatrix Parabola

For definition of a carved surface with a directrix sinusoid

$$x = x(u) = u, \quad y = y(u) = a \sin bu$$

and a generatrix parabola $X = X(v) = v$, $Y = Y(v) = cv^2$ given in the mobile local system of coordinates XOY, we may use the general formulas given on page 199.

The forms of definition of the carved surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + X(v)\boldsymbol{e}_0(u) + Y(v)\boldsymbol{g}_0(u),$$

where

$$\boldsymbol{e}_0(u) = \boldsymbol{v}\cos\theta + \boldsymbol{\beta}\sin\theta = \frac{y'\boldsymbol{i} - x'\boldsymbol{j}}{s}\cos\theta + \boldsymbol{k}\sin\theta;$$
$$\boldsymbol{g}_0(u) = -\boldsymbol{v}\sin\theta + \boldsymbol{\beta}\cos\theta = -\frac{y'\boldsymbol{i} - x'\boldsymbol{j}}{s}\sin\theta + \boldsymbol{k}\cos\theta;$$

θ = const is the angle of the normal $\boldsymbol{v}(u)$ of a directrix curve with vector $\boldsymbol{e}_0(u)$. For a plane directrix curve, we have θ = cons, i.e., the generatrix parabola moves in the normal plane of the directrix sinusoid without rotation.

(2) Parametrical equations (Fig. 1):

$$x = x(u, v) = x(u) + \frac{X(v)y'\cos\theta - Y(v)y'\sin\theta}{s}$$
$$= u + \frac{\cos\theta - cv\sin\theta}{s}abv\cos bu,$$
$$= a\sin bu - v\frac{\cos\theta - cv\sin\theta}{s},$$
$$z = z(v) = X(v)\sin\theta + Y(v)\cos\theta = v\sin\theta + cv^2\cos\theta,$$

where $s = \sqrt{1 + a^2b^2\cos^2 bu}$.

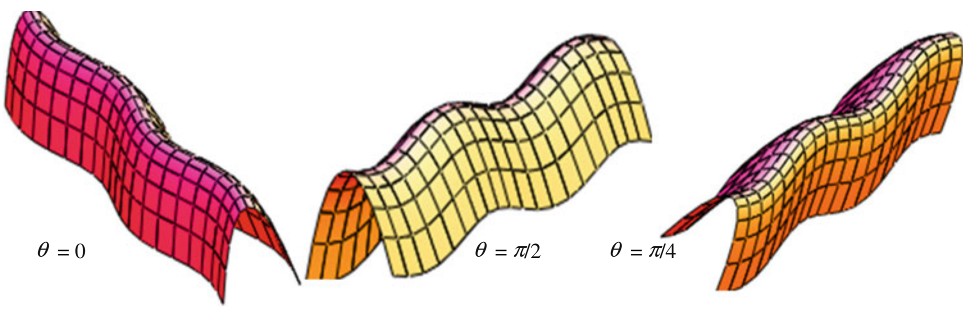


Fig. 1

■ Carved Surface with Directrix Sinusoid and Generatrix Ellipse

For definition of a carved surface with a directrix sinusoid $x = x(u) = u$, $y = y(u) = a \sin bu$ and a generatrix ellipse $X = X(v) = c \cos v$, $Y = Y(v) = d \sin v$, we can use general equations given on page 199 assuming $\theta = \pi/2$. Taking into account that a directrix sinusoid is a plane curve, we can write the radius vector of a directrix curve in the form: $\boldsymbol{\rho}(u) = u\boldsymbol{i} + y(u)\boldsymbol{j}$, then $k = y''/s^3$ is the curvature of the sinusoid, $s = \sqrt{1 + y'^2}$; $\boldsymbol{\kappa} = 0$, i.e., its torsion is equal to zero, $\boldsymbol{v} = (y'\boldsymbol{i} - \boldsymbol{j})/s$ is the unit vector of the principal normal of the directrix curve;

$$\ldots' = d\ldots/\mathrm{d}u; \ \ldots'' = d^2\ldots/\mathrm{d}u^2.$$

Forms of definition of the carved surface

(1) Vector form of definition:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + X(v)\boldsymbol{e}_0(u) + Y(v)\boldsymbol{g}_0(u)$$

where

$$\boldsymbol{e}_0(u) = \boldsymbol{k}; \quad \boldsymbol{g}_0(u) = -\boldsymbol{v}.$$

The generatrix ellipse moves in the normal plane of the directrix sinusoid without rotation.

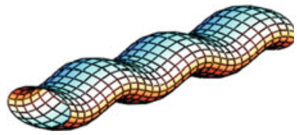


Fig. 1

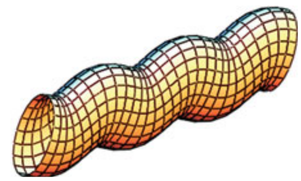


Fig. 2

(2) Parametric form of definition (Figs. 1 and 2):

$$x = x(u,v) = u - (abd\cos bu \sin v)/s,$$
$$y = y(u,v) = a\sin bu + (d\sin v)/s,$$
$$z = c\cos v.$$

The studied carved surfaces can be included into a group of the *wave-shaped surfaces.* If $c = d$, the surface degenerates into *a tubular surface with a plane sinusoidal line of centers* (see also a Subsect. "17.2.1. Normal cyclic surfaces").

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = s[1 - kY];\ F = 0;\ B = \sqrt{\dot{X}^2 + \dot{Y}^2};$$
$$L = sk\dot{X}\frac{A}{B};\ M = 0;\ N = \frac{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}{B} = \frac{cd}{B};$$

where

$$k_1 = k_u = \frac{sk}{AB}\dot{X};\ k_2 = k_v = \frac{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}{B^3} = \frac{cd}{B^3}.$$
$$\dot{\ldots} = \frac{d\ldots}{dv},\quad \ddot{\ldots} = \frac{d^2\ldots}{dv^2},$$

■ Carved Surface with Directrix Ellipse and Generatrix Sinusoid

For definition of a carved surface with a directrix ellipse $x = x(u) = a\cos u$, $y = y(u) = b\sin u$ and a generatrix sinusoid $X = X(v) = v$, $Y = Y(v) = c\sin dv$, we can use general equations given on the page 199 when $\theta = \pi/2$.

Taking into account that a directrix ellipse is a plane curve, we can write the radius vector of a directrix curve in the form:

$$\boldsymbol{\rho}(u) = x(u)\boldsymbol{i} + y(u)\boldsymbol{j},$$

then

$$k = (x'y'' - y'x'')/s^3 = ab/s^3,\ s = \sqrt{x'^2 + y'^2};$$

where k is the curvature of a directrix curve; $\kappa = 0$, i.e., its torsion is equal to zero; $\boldsymbol{v} = (y'\boldsymbol{i} - x'\boldsymbol{j})/s$ is the unit vector of the principal normal of the directrix curve, $\ldots' = d\ldots/du$; $\ldots'' = d^2\ldots/du^2$.

The forms of definition of the Monge's surface

(1) Vector form of definition:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + X(v)\boldsymbol{e}_0(u) + Y(v)\boldsymbol{g}_0(u),$$

where

$$\boldsymbol{e}_0(u) = \boldsymbol{k};\ \boldsymbol{g}_0(u) = -\boldsymbol{v}.$$

The generatrix sinusoid moves in the normal plane of the directrix ellipse without rotation.

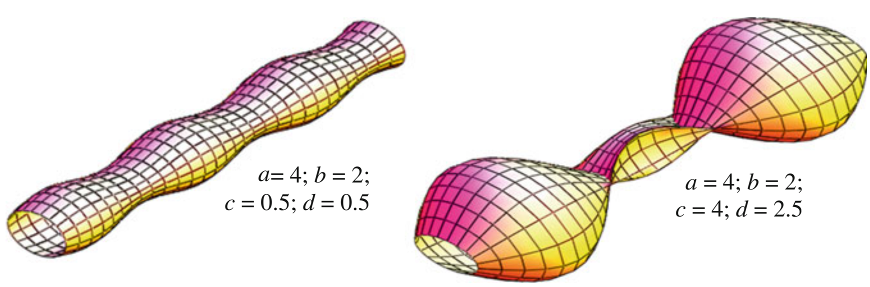


Fig. 1

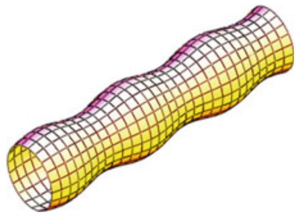


Fig. 2

(2) Parametrical equations (Figs. 1, 2):

$$x = x(u, v) = a\cos u - (cb\cos u\sin \mathrm{d}v)/s,$$
$$y = y(u, v) = b\sin u - (ac\sin u\sin \mathrm{d}v)/s,$$
$$z = v.$$

These carved surfaces may be included into a group of *waving surfaces.*

If $a = b$, then this surface degenerates into *a corrugated surface of revolution with general sinusoid,* see also Chap. "2. Surfaces of Revolution" (Fig. 4).

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Additional Literature

P.S.: Additional literature is given on corresponding pages of the Chap. "4. Carved Surfaces".

5 Surfaces of Congruent Sections

A surface of congruent sections is called a surface carrying on itself the continuous single-parametric family of plane lines. Such surface is formed by any *moving fixed plane line* (*generatrix*). A single-parametric family of the planes α that are carriers of these lines corresponds to single-parametric family of the cross sections of the surface. The segregation of the surfaces in question into a special class simplified the interpretation of methods of design of these surfaces by means of descriptive geometry.

Surfaces of plane-and-parallel translation relative to a projection plane π are the simplest types of surfaces of congruent sections. The plane-and-parallel translation of a figure relatively to the plane of the projections is the displacement of the figure in the space when its every point moves in its level plane. The horizontal projection of figure f changes its disposition but does not change its size when one carries out the plane-and-parallel translation relatively to the plane π. The frontal projections of the points of figure f move in this case along the straight lines that are parallel to an axis of the drawing.

Surface of right translation (see also "Translation surfaces") is a surface of congruent section too. The plane-parallel movement of a meridian f of *a surface of revolution* is the rotation of this meridian about a fixed straight line that is an axis of rotation. Hence, surfaces of revolution may be related to a class of surfaces of congruent sections.

The plane generatrixes f of *carved surface* are simultaneously geodesic lines and lines of the principle curvatures. The single-parametrical family of the carrier planes α with the generatrix lines f is orthogonal to the plane directrix curve (see also "Carver surfaces"). So, carved surfaces (Fig. 1) qualify as surfaces of congruent sections.

All *cyclic surfaces* with generatrix circles f of constant radius may be included into a class of surfaces of congruent section.

Rotational surfaces enter into one of the group of surfaces of congruent sections (see also "Kinematical surfaces of general type").

Helical surfaces (Fig. 2a) are formed by helical motion of arbitrary curve (see also "Helical surfaces"). So, they also may be included into a class of surfaces of congruent sections.

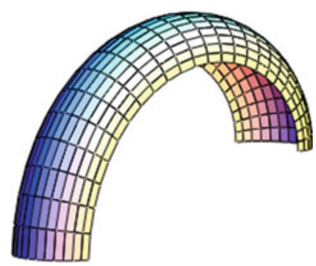

Fig. 1

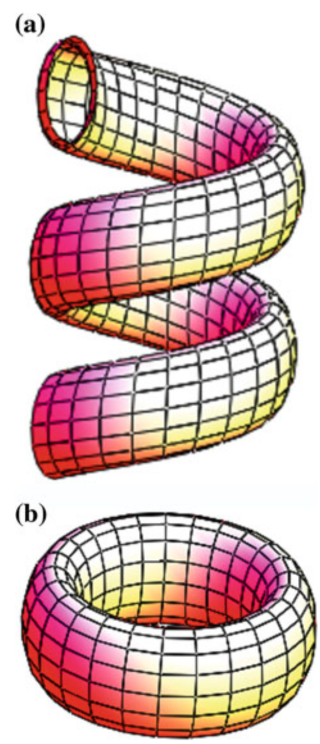

Fig. 2

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_5, © Springer International Publishing Switzerland 2015

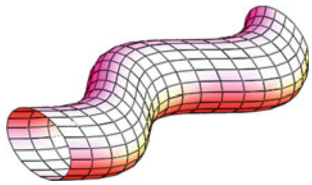

Fig. 3

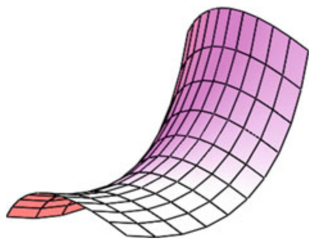

Fig. 4

Surfaces of congruent sections of general type may be obtained from surfaces of plane-and-parallel translation with the help of strains of different type. The simplest way of strain is a vertical continuous displacement of the generatrixes f. Under such strain, surfaces of right translation go over into the surfaces of right translation; surfaces of revolution (Fig. 2b) change to helical surfaces of a variable or constant pitch (Fig. 2a). Rotational surfaces deform into *the spiroidal surfaces*.

In Fig. 3, the surface of translation of the ellipse along the sinusoid is shown and, in Fig. 4, the surface of translation of a parabola over another parabola is given. The surface of translation of the ellipse over another ellipse is shown. Both the ellipses lie at the perpendicular planes. One can call all these surfaces (Figs. 3, 4 and 5) as surfaces of congruent sections.

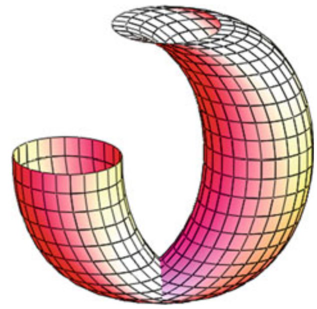

Fig. 5

It is possible to fulfill the deformation of a surface of plane-and-parallel translation into the surface of congruent section by continuous rotation of the planes α about their traces α_1. In this case, the value of the angle of the rotation must be given as a function of the same parameter that single-parametric family of the planes with curves f has.

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■ Bicosine Surface of Translation

A surface of right translation formed by a parallel movement of a generatrix cosine curve along a similar cosine curve is called *a bicosine surface of translation*. The directrix and generatrix cosine curves are placed at mutually perpendicular planes.

The known explicit equation of a bicosine surface of translation has the following form (Fig. 1):

$$z = z(x, y) = c\cos\frac{\pi x}{a} + c\cos\frac{\pi y}{a} - 2c,$$

where c is the amplitude of the cosine curves. A surface covers a square plan $a \times a$. At the cross section of a surface by a plane $z = -h$, a curve

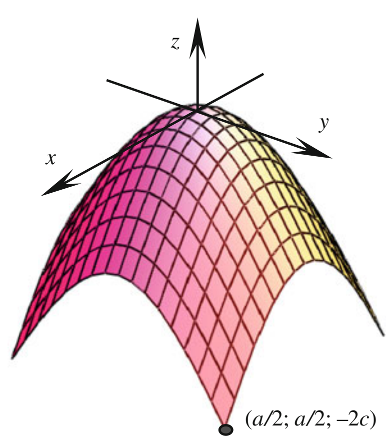

Fig. 1

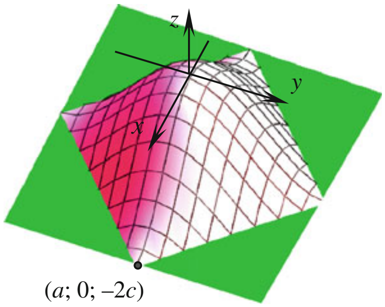

Fig. 2

$$y = \frac{a}{\pi}\arccos\left(-\frac{h}{c} + 2 - \cos\frac{\pi x}{a}\right)$$

lies. This curve disintegrates into four straight lines $y = \pm x \pm a$ when $h = 2c$ (Fig. 2). In this case, the bicosine surface of translation covers the square plan

$$\sqrt{2}a \times \sqrt{2}a; \quad -a \le x \le a; \; -a \le y \le a; .$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{c^2\pi^2}{a^2}\sin^2\frac{\pi x}{a}, \quad F = \frac{c^2\pi^2}{a^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a},$$
$$B^2 = 1 + \frac{c^2\pi^2}{a^2}\sin^2\frac{\pi y}{a},$$
$$A^2B^2 - F^2 = 1 + \frac{c^2\pi^2}{a^2}\left(\sin^2\frac{\pi x}{a} + \sin^2\frac{\pi y}{a}\right),$$
$$L = \frac{-1}{\sqrt{A^2B^2 - F^2}}\frac{c\pi^2}{a^2}\cos\frac{\pi x}{a},$$
$$M = 0, \quad N = \frac{-1}{\sqrt{A^2B^2 - F^2}}\frac{c\pi^2}{a^2}\cos\frac{\pi y}{a}.$$

The surface is given in curvilinear nonorthogonal conjugate coordinates x, y. A bicosine surface of translation is a special case of *a surface of translation of a cosine curve along a cosine curve* (see also “Translation surfaces”). The surface presented in Fig. 2 may be called a *diagonal cosine surface of translation* (see also “Surfaces of diagonal translation”).

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■ Bisemicubic Surface of Translation

A surface of translation formed by parallel motion of a bisemicubic parabola of Neil along a similar Neil parabola is called *a bisemicubic surface of translation.*

An explicit equation of a bisemicubic surface of translation is (Fig. 1).

$$z = \sqrt[3]{\frac{y^2}{c^2}} + \sqrt[3]{\frac{x^2}{c^2}}$$

Directrix and generatrix parabolas of Neil lie at the mutually perpendicular planes. The cross section of the surface by a plane $z = h$ gives a curve (Fig. 2).

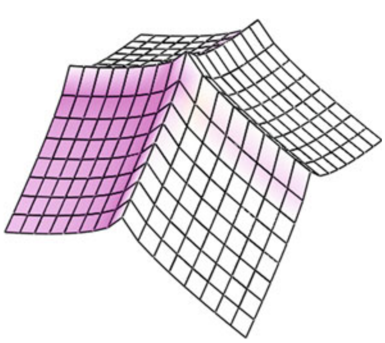

Fig. 1

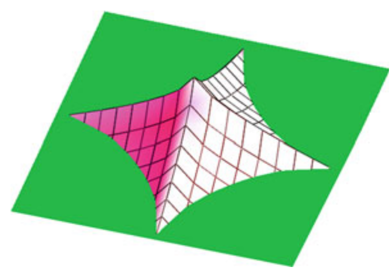

Fig. 2

$$x = \pm c\left(h - \sqrt[3]{\frac{y^2}{c^2}}\right)^{3/2}$$

In this case, the plane contour of the surface of translation with four vertical planes of symmetry has the obviously expressed concavities.

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■ Twisted Surface with Congruent Ellipses in Parallel Planes

A twisted surface with congruent curves in parallel planes is formed by the rotary-and-translational motion of a plane curve $X = X(v)$, $Y = Y(v)$ placed in a plane that is perpendicular to the axis of rotation. The origin of a mobile Cartesian coordinate system OXY is placed on a fixed coordinate axis Ox at the distance of a from an axis Oz.

Having assumed an ellipse

$$X = X(v) = b\cos v; \quad Y = Y(v) = c\sin v,$$

as a generatrix curve, we obtain *a twisted surface with congruent ellipses in parallel planes*.

Forms of definition of the surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = [a + X(v)]\boldsymbol{h}(u) + Y(v)\boldsymbol{n}(u) + tu\boldsymbol{k},$$

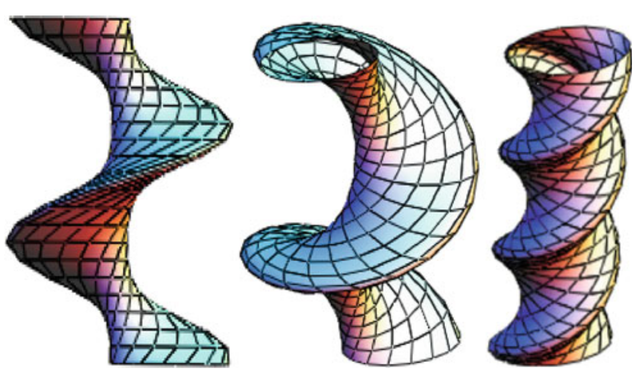
where u is the angle of the rotation counted off from an axis Ox in the direction of an axis Oy;

$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u, \quad \boldsymbol{n}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u$$

are the unit vectors placed on the coordinate plane xOy and their direction coincides with the direction of the mobile coordinate axes oX and oY; t is a parameter characterized by the speed of the translational motion of the generatrix curve along the axis of rotation (an Oz axis).

(2) Parametrical equations:

$$x = x(u, v) = (a + X)\cos u - Y\sin u,$$
$$y = y(u, v) = (a + X)\sin u + Y\cos u,$$
$$z = z(u) = ut.$$



$a = \varepsilon = (b^2 - c^2)^{1/2} = 1.73$

Fig. 1

Using this form of definition, we can take arbitrary plane curve as a generatrix curve including an ellipse (Figs. 1, 2, 3, 4 and 5).

$$X = X(v) = b\cos v; \quad Y = Y(v) = c\sin v$$

Assume a straight line $X = v$, $Y = 0$, and $a = 0$ as a generating curve, then parametric equations of the surface can be written as

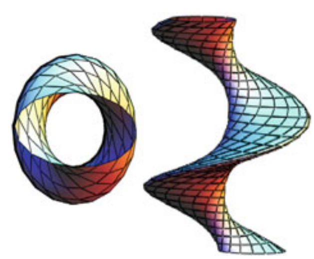
$$x = x(u, v) = v\cos u, \quad y = y(u, v) = v\sin u,$$
$$z = z(u) = ut.$$

These equations are parametrical equations of *a right helicoid*.

Assume that a generating straight line is given by equations $X = 0$ and $Y = v$, then parametric equations of a twisted surface are

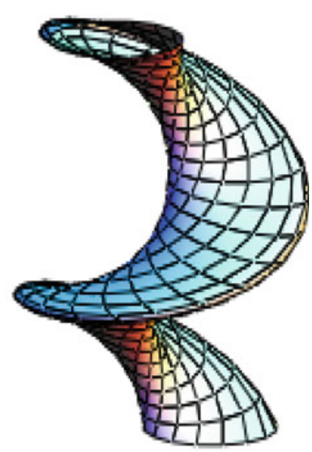
$$x = x(u, v) = a\cos u - v\sin u,$$
$$y = y(u, v) = a\sin u + v\cos u, \quad z = z(u) = ut.$$

These equations describe a surface of *a pseudodevelopable helicoid.*



$a = 0$

Fig. 2

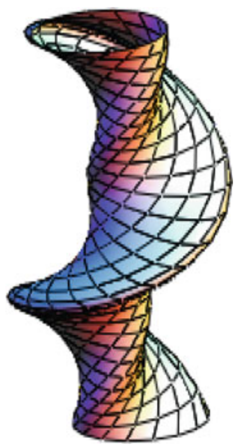


$a = b$

Fig. 3

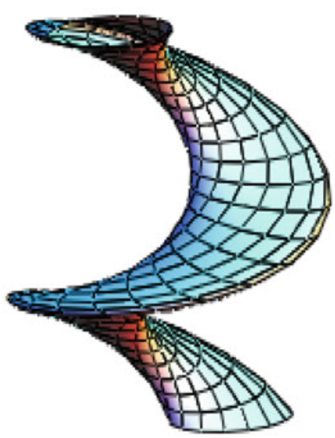
Coefficients of the fundamental forms of the surface:

$$A^2 = (a+X)^2 + Y^2 + t^2,$$
$$F = (a+X)Y' - X'Y,$$
$$B^2 = X'^2 + Y'^2,$$
$$A^2B^2 - F^2 = [(a+X)X' + YY']^2 + t^2B^2,$$
$$L = \frac{tF}{\sqrt{A^2B^2 - F^2}},$$
$$M = \frac{tB^2}{\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{t(X'Y'' - X''Y')}{\sqrt{A^2B^2 - F^2}}$$



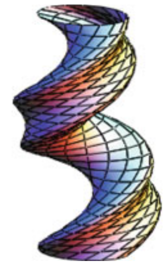
$a = -1$

Fig. 4



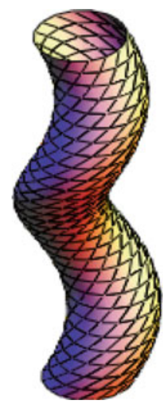
$a = 2.5$

Fig. 5



$a = 1$

Fig. 6



$a = 0.5$

Fig. 7

All twisted surfaces with the congruent ellipses in the parallel planes shown in Figs. 1, 2, 3, 4, 5 and 6 have b = 2 m; c = 1 m; t = 1 m; $0 \le v \le 2\pi$; $0 \le u \le 3\pi$. The parameter a is given under the figures.

In Fig. 6, the generatrix ellipse is given by the equations

$$X = X(v) = c \sin v, \ Y = Y(v) = b \cos v.$$

Assuming b = c, one can obtain a *right circular helical surface* (see also "Circular helical surfaces"). The surface shown in Fig. 7 has b = c = t = 1.

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Krivoshapko SN, Shambina SL. Surfaces of the congruent cross-sections of pendulum type on a circular cylinder. Geometrical Modelling and Computer Technologies: Theory, Practice, Education: Proc. VI Int. Science-and-Practical Conf., Ukraine, Kharkov, April 21-24, 2009. Kharkov: KhGUPT, 2009; p. 34-39.

Additional Literature

P.S.: Additional literature is given on the corresponding pages of the Chaps. "5. Surfaces of Congruent Sections", "2. Surfaces of Revolution", "4. Carved Surfaces", a Subsect. "17.2.1. Tubular Surfaces", a Sect. "3.1. Surfaces of Right Translation", of the Chaps. "7. Helical Surfaces" and "8. Spiral Surfaces".

6 Continuous Topographic and Topographic Surfaces

Topographic surfaces are called surfaces given by the discrete set of their horizontals. Such definition of a topographical surface is used mainly in mining art, building and in topography.

A continuous topographic surface is a topographic surface given by continuous set of *the level lines* (Fig. 1). Continuous topographic surfaces are used mainly in aircraft manufacture and shipbuilding. In these cases, a surface is generated by three families of lines: (1) sections by the horizontal planes (*lines of latitudes*), (2) sections by the frontal surfaces (*buttock lines*) and (3) sections by the profile planes (*lines of the side frames*). Assume some understandings: *surfaces of one series* are the continuous topographic surfaces formed by the distribution of the same family of level lines and *surfaces of different series* are the continuous topographic surfaces formed by the distribution of different families of level lines. If two continuous topographic surfaces of one series are intersecting then they are intersecting along the horizontals (frontal lines, profile lines). The line of intersection of two continuous topographic surfaces of different series is formed by *a method of ancillary planes of level* or due to points with the application of an analytical form of the definition of the line of intersection.

Surfaces of revolution may be regarded as continuous topographic surfaces with the generating family of lines that are a set of the concentric circles (Fig. 2). *Coaxial surfaces of revolution* are surfaces of one series because a line of intersection of coaxial surfaces of revolution is a parallel. Surfaces of revolution with the parallel axes are continuous topographic surfaces of different series. If such surfaces are intersecting then their lines of intersection are determined analytically or with the help of a graphical method of ancillary planes of level.

Algebraic surfaces given in an explicit form $z = z(x, y)$, *surfaces of right translation* (Fig. 3), *cyclic surfaces with a plane of parallelism* (Fig. 3), *Catalan surfaces* (see also Subsect. "1.2.1 Catalan Surfaces"), *surfaces of constant slope* (see also Subsect. "1.1.1 Torse Surfaces (Torses)"),

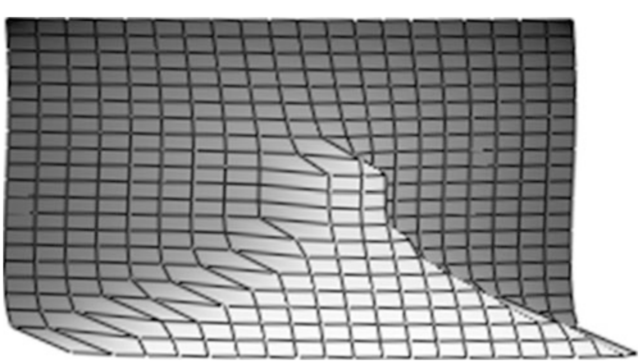

Fig. 1

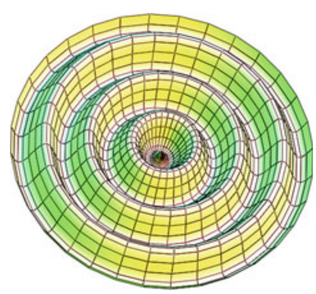

Fig. 2

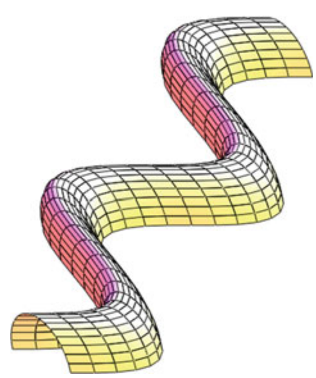

Fig. 3

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_6, © Springer International Publishing Switzerland 2015

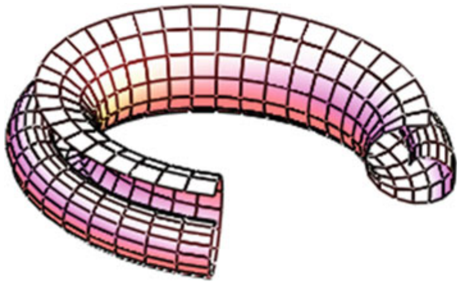

Fig. 4

Monge surfaces with a circular cylindrical directrix surface (see also Sect. 4.1) (Fig. 4) may be classified among continuous topographic surfaces.

The known geometrical properties of the surface of constant slope give the opportunity to use it for approximation of topographic surface. A.G. Varvaritza offered to use the formula

$$\alpha = Lh/(0.175\, s),$$

for the determination of a mean angle of slope of the terrain on a given territory. Here L is the length of all horizontals on the given territory; h is the height of the cross-section of the terrain, m; s is the area of the given territory, hectares.

Analytical topographic surfaces may be regarded as abstract models of real topographic surfaces of the given area. Analytical topographic surfaces can help to solve the practical problem of locating the minimum number of viewpoints to see the entire surface. The viewshed of a point on an irregular topographic surface is defined as the area visible from the point.

Additional Literature

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Wolf GW. Metric Surface Networks. In: Proc. of the Fourth International Symposium on Spatial Data Handling. 1990; p. 844-856.

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Goodchild MF, Lee Jay. Coverage problems and visibility regions on topographic surface. Annals of Operations Research. 1989; 18, p. 175-186.

■ Topographic Surface with the Given Elliptical Cross Sections

At some cases, it is better to design technical forms and building shells when we have the net frame of the future surface. For construction of a surface they give a single-parametric family of the plane u lines using some requirements given in advance and after that, a net of v-lines are constructed.

For construction of *a topographic surface with the given elliptic cross-sections,* it is necessary to give a single-parametric family of the plane coaxial ellipses lying at the parallel planes with the elevations given in advance. For example, a single-parametric family of ellipses may be given by equations:

$$x = v\cos u,$$
$$y = b\sin u,$$

where u are the lines of the family of ellipses with an equal dimension of the half-axis b; v are the lines of the family of curves, the projections of which on the plane xOy are parallel to the axis Ox.

Assume that ellipses $v = v_k$ lie in the plans $z = z_k$ correspondingly; $1 \le k \le n$ where n is an integer of planes with ellipses given in advance. A coordinate z of the level lines depends on a parameter v only, i.e.

$$z = z(v)$$

because the lines of the frame (ellipses) given in advance lie at the parallel planes.

The function $z = z(v)$ may be assumed in the form of *Lagrange polynomial* where number of terms of a series n must be equal to the number of the planes given in advance with the given ellipses:

$$z = z(v) = \sum_{k=1}^{n} \frac{\omega(v)}{(v - v_k)\omega'(v_k)} z_k$$
$$= \sum_{k=1}^{n} \frac{(v - v_1)(v - v_2)(v - v_3)\cdots(v - v_n)}{(v - v_k)\omega'(v_k)} z_k,$$

$$\omega'(v_k) = \left.\frac{d\omega(v)}{dv}\right|_{v=v_k}.$$

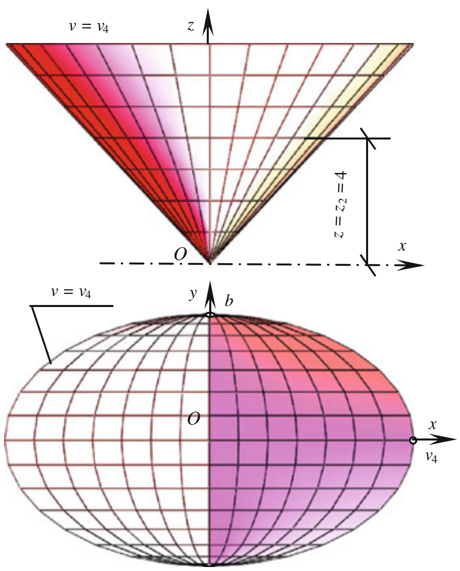


Fig. 1

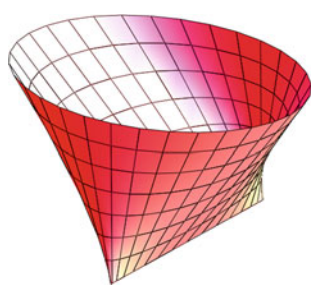


Fig. 2

Assume the ellipses with $v_1 = 1$, $v_2 = 3$, $v_3 = 4$, $v_4 = 6$ lying in the planes

$$z = z_1 = 2, \quad z = z_2 = 4, \quad z = z_3 = 5, \quad z = z_4 = 7$$

correspondingly, that is $n = 4$.

In this case, a topographic surface with the given elliptic cross-sections (Figs. 1 and 2) has the following parametrical equations:

$$\begin{aligned}
&x = x(u, v) = v\cos u,\\
&y = y(u) = b\sin u,\\
&z = z(v) = -\frac{(v-3)(v-4)(v-6)}{15} + \frac{2(v-1)(v-4)(v-6)}{3}\\
&\qquad\quad - \frac{5(v-1)(v-3)(v-6)}{6} + \frac{7(v-1)(v-3)(v-4)}{30}
\end{aligned}$$

Coefficients of the fundamental forms of the surface:

$$\begin{aligned}
A^2 &= v^2\sin^2 u + b^2\cos^2 u,\\
F &= -v\sin u\cos u,\\
B^2 &= \cos^2 u + z'(v),\\
A^2B^2 - F^2 &= b^2\cos^4 u + z'^2(v)A^2,\\
z'(v) &= \mathrm{d}z/\mathrm{d}v,\\
L &= -\frac{bvz'(v)}{\sqrt{A^2B^2 - F^2}}, \quad M = -\frac{bz'(v)\cos u}{\sqrt{A^2B^2 - F^2}},\\
N &= -\frac{z''(v)b\cos^2 u}{\sqrt{A^2B^2 - F^2}},\\
K &= \frac{b^2z'(v)\cos^2 u[vz''(v) - z'(v)\sin^2 u]}{(A^2B^2 - F^2)^2}
\end{aligned}$$

where

$$z'(v) = \frac{\mathrm{d}z}{\mathrm{d}v}, \quad z''(v) = \frac{\mathrm{d}^2z}{\mathrm{d}v^2}.$$

The surface is given in the curvilinear non-orthogonal non-conjugate coordinates u, v.

The surfaces shown Figs. 1 and 2, has

$$b = 4\ \mathrm{m}; \quad 0 \le v \le 7\ \mathrm{m}.$$

Additional Literature

Amirov M, Mihaylenko VE. On question of the construction of the net frame. Prikl. Geom. i Ingen. Graphika, Kiev. 1974; Iss. 18, p. 10-16 (3 refs).

■ "Trash Can"

A continuous topographic surface "Trash Can" contains ellipses as the level lines and that is why this surface and *a topographic surface with the given elliptic cross-sections* may be called *surfaces of one series*.

Parametric equations of the presented surface (Fig. 1) are written in the form:

$$x = x(u, v) = (b + v)\cos u,$$
$$y = y(u, v) = v \sin u,$$
$$z = z(v) = av^2.$$

where a, b are constants, $0 \le u \le 2\pi$. The lines of level $v = v_c$ = const are ellipses with half-axes

$$\sqrt{h/a}; \quad b + \sqrt{h/a},$$

and besides $h = av_c^2$ is a size (height) of the surface in the direction of the axis z. The line $v = 0$ is a straight line segment coinciding with the axis x, $-b \le x \le b$. The coordinate plane $x = 0$ intersects the surface along the parabola

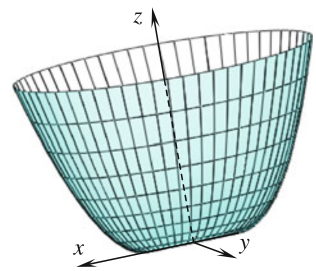


Fig. 1

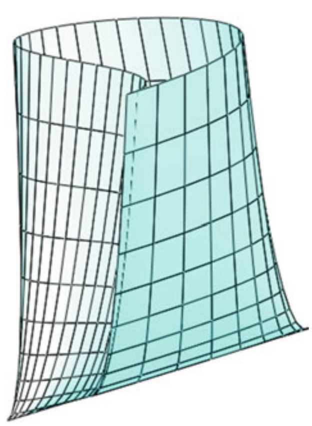
$z = ay^2$. A cross-section of the surface by the plane $y = 0$ contains a straight line segment $-b \le x \le b$ and two half-branches of the parabola:

$$z = a(x - b)^2 \quad \text{and} \quad z = a(x + b)^2.$$

The surface is symmetrical relative to the planes $y = 0$ and $x = 0$. The curvilinear system of coordinate u, v is non-orthogonal and non-conjugate but the surface itself has positive Gaussian curvature ($K \ge 0$) and besides $K = 0$ only along the straight line segment $v = 0$.

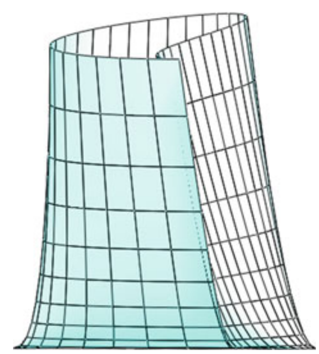
■ "Paper Bag"

This surface has received this name because of its some resemblance with a standard paper bag (Figs. 1 and 2). A paper bag is a preformed container made of paper, usually with an opening at one end. Conditionally, surface "Paper Bag" may be related to a class of *continuous topographic surfaces*.



$a = -2; \quad b = 1.2$
$0 \le v \le 2; -2\pi \le u \le 0$

Fig. 1



$a = 2; \quad b = 1.2$
$0 \le v \le 2; -2\pi \le u \le 0$

Fig. 2

This surface has the following parametrical form of definition:

$$x = x(u, v) = v \cos u,$$
$$y = y(u, v) = (v + bu)\sin u,$$
$$z = z(v) = av^2,$$

where a, b are arbitrary constants, $-2\pi \le u \le 0$, $0 \le v \le 2$.

■ Kappa Surface

Kappa Surface is formed by the rotation of a curve

$$x = x(t) = a\cos t\cot t, \quad y = y(t) = a\cos t$$

about an axis y (Fig. 1). A curve also known as Gutschoven's curve which was first studied by G. van Gutschoven around 1662. It was also studied by Newton and, some years later, by Johann Bernoulli. It is given by the Cartesian equation

$$(x^2 + y^2)y^2 = a^2x^2,$$

by the polar equation

$$r = a\cot\theta.$$

Parametrical form of definition of "Kappa surface":

$$x = x(u, v) = a\cos u\cos v,$$
$$y = y(v) = -a\cos v/tgv,$$
$$z = z(u, v) = a\sin u\cos v$$

where a is any constant; $0 \le u \le 2\pi$, $0 < v \le \pi/2$.

In Fig. 1, a parameter v changes within the limits: $0.25 < v \le \pi/2$. As it was shown before, surfaces of revolution are *continuous topographic surfaces*.

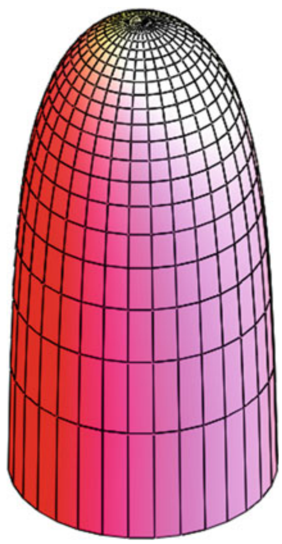


Fig. 1

Reference

Weisstein, Eric W. "Kappa Curve". From MathWorld - A Wolfram Web Resource. http://mathworld.wolfram.com/KappaCurve.html

■ Continuous Topographic Surface of Cassini

A continuous topographic surface of Cassini contains *ovals of Cassini* as level lines

$$\left(x^2 + y^2\right)^2 - 2c^2(x^2 - y^2) = a^4 - c^4,$$

or that is the same

$$\left(x^2 + y^2 + c^2\right)^2 - 4c^2x^2 = a^4,$$

where a is a parameter of changing of the form of Cassini ovals; c is a parameter of changing of an axis of a continuous topographic surface.

Ovals of Cassini are a geometric locus and the product of the distances from the points of the oval till the points ($-c$, 0) and (c, 0) is equal to a^2. When $c\sqrt{2} < a$ then an oval of Cassini is an oval line; if $c < a < c\sqrt{2}$, then an oval of Cassini is a curve with *a point of self-intersection* (singular point); but when $c > a$, a Cassini oval turns into two closed ovals with the centers at the points ($-c$, 0) and (c, 0).

Having a constant value of a but changing the parameter c, one can obtain a single-parametrical family of Cassini ovals. Distributing this family in the space along an arbitrary directrix curve, we can have surfaces of different form.

In some scientific works, the continuous topographic surfaces given by the implicit equations

$$[(x - a)^2 + y^2][(x + a)^2 + y^2] = z^4,$$

or that is the same

$$(x^2 - a^2)^2 + 2y^2(x^2 + a^2) + y^4 = z^4$$

are called *the Cassini surfaces* (Fig. 1).

This surface is symmetrical one relatively to all three coordinate planes (Fig. 1d). Two cavities of this surface touch each other at the points

$$x = \pm a; \quad y = z = 0$$

Forms of definition of the surface

(1) Implicit form of assignment [Kirillov]:

$$\left(x^2 + y^2\right)^2 - 2f_1^2(z)(x^2 - y^2) = a^4 - f_1^4(z),$$

where $c = f_1(z)$ is the function describing the form of an axis of the surface.

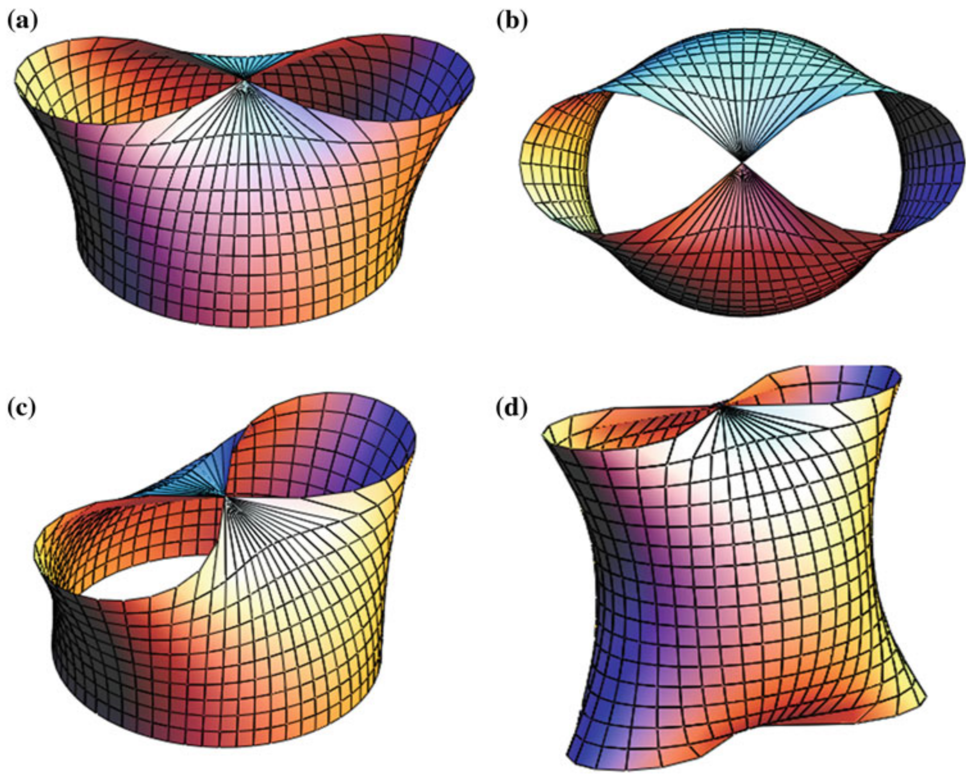


Fig. 1

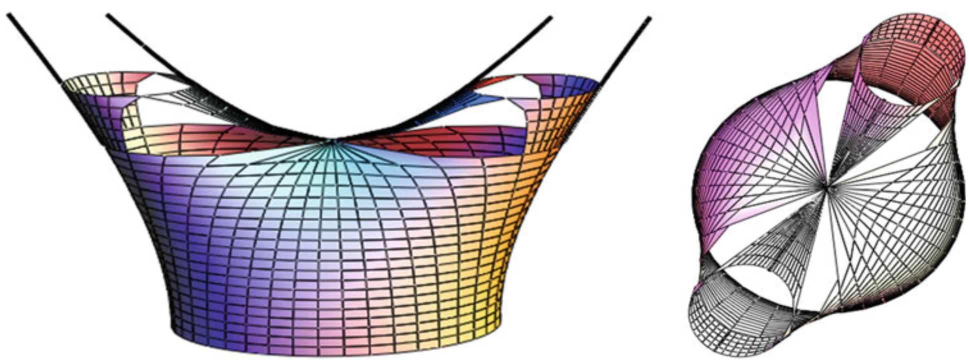


Fig. 2

(2) Parametrical equations (Fig. 1):

$$x = x(u, z) = r\cos u, \quad y = y(u, z) = r\cos u, \quad z = z,$$

where

$$r = r(u, z) = \sqrt{c^2 \cos 2u \pm \sqrt{a^4 - c^4 \sin^2 2u}}, \; c = f_1(z).$$

In Fig. 1, the studied surface with $c = f_1(z) = 3z$; $a = 8$ is shown. In this case, the equation of the surface is

$$\left(x^2 + y^2\right)^2 - 18z^2(x^2 - y^2) = 8^4 - 81z^4.$$

There is a circle with a radius $R = 8$ cm in the cross-section of the surface by the plane $z = 0$. Assume $z = 8/3$ cm then we obtain a *lemniscate of Bernoulli* (Fig. 1a–c). The surface represented in Fig. 1d has a parameter z changing between the limits $z = -8/3$ cm and $z = 8/3$ cm.

The surface contains two ramified canals (Fig. 2) when a parameter z goes over a limit equal to 8/3 cm.

Additional Literature

Kirillov SV. On construction of continuous topographical surfaces with complex cross-sections. KIbernetika Grafiki i Prikl. Geom. Poverhnostey. Moscow: MAI, 1972; Vol. IX, Iss. 243, p. 118-124.

6.1 Aerodynamic Surfaces Given by Algebraic Plane Curves

Aerodynamic surfaces are the surfaces of the swimming bodies created by the nature or man. These surfaces are formed by their main cross-sections lying in coordinate planes. The forms of the lines in the main cross-sections are chosen from conditions given in advance to a future surface.

Additional Literature

Ablisimov AV, *Besyadovskiy AR*. A practical method of the design of aerodynamic focuses. Trudy of LKI. Sredstva i Metody Povysheniya Morehodnyh kachestv Sudov. 1989; p. 123-127.

■ Aerodynamic Surface Given by a Continuous Framework of Elliptical Ribs

The design waterline of a studied surface has the form of *a generalized Agnesi curl*

$$y = \frac{L^2 B}{4x^2 + L^2} - \frac{B}{2}$$

at the cross-section of the surface by a plane xOy. *The midship section* at the cross-section of the surface by a plane yOz is made in the form of an ellipse

$$4y^2/B^2 + z^2/T^2 = 1$$

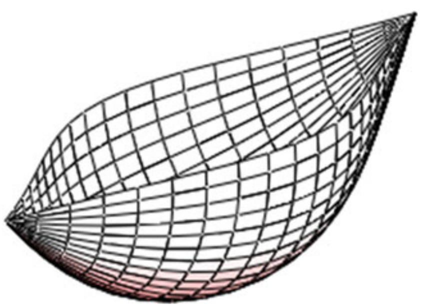


Fig. 1

but at the cross-section of the surface by a plane xOz, *the main buttock line* has a parabolic form

$$z = T - 4Tx^2/L^2.$$

Here T is a draught of the surface, B is its maximum width along an axis Oy, L is its length along an axis Ox. An implicit equation of the studied aerodynamic surface (Fig. 1) was obtained in the form:

$$\frac{y^2}{\left(\frac{L^2 B}{4x^2 + L^2} - \frac{B}{2}\right)^2} + \frac{z^2}{\left(T - \frac{4Tx^2}{L^2}\right)^2} = 1.$$

A surface of the 10th order with an Agnesi curl, an ellipse and a parabola lying in three principle coordinate planes has three planes of the symmetry coinciding with the coordinate planes; $-L/2 \le x \le L/2$; $-B/2 \le y \le B/2$; $-T \le z \le T$.

Additional Literature

Avdon'ev EA, *Protod'yakonov SM*. Research of geometry of some surfaces of the highest order. Prikl. Geom. i Ingen. Grafika, Kiev. 1975; Iss. 20, p. 138-142.

■ Aerodynamic Surfaces Given by a Continuous Framework of Water-Lines in the Form of Generalized Agnesi Curls

The regarded surface has the same curves in the main sections that an aerodynamic surface given by a continuous framework of elliptic transverse frames has.

An implicit equation of the studied aerodynamic surface (Fig. 1) was obtained in the form:

$$y = \frac{L^2(T - z)B\sqrt{T^2 - z^2}}{T[4x^2 T + L^2(T - z)]} - \frac{B}{2T}\sqrt{T^2 - z^2}.$$

The 6th order surface with *an Agnesi curl, an ellipse, a parabola* in three principle planes has one plane of the symmetry yOz;

$$0 \le y \le B/2; \quad -L/2 \le x \le L/2; \quad 0 \le z \le T.$$

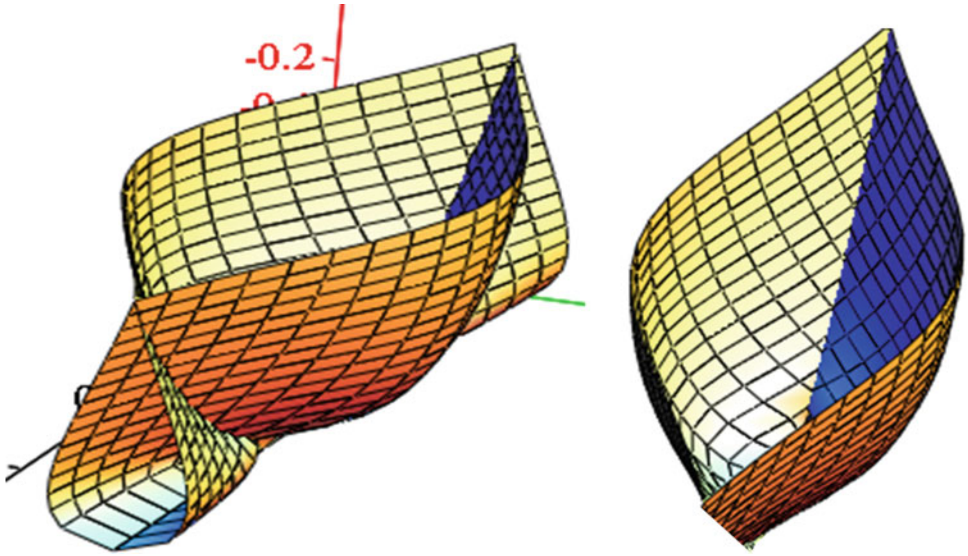


Fig. 1

Additional Literature

Avdon'ev EA, Protod'yakonov SM. Research of geometry of some surfaces of the highest order. Prikl. Geom. i Ingen. Grafika, Kiev. 1975; Iss. 20, p. 138-142.

Gotman ASh. Application of developable surfaces for designing well-streamlined ship shapes. Novosibirsk State Academy of Water Transport. http://shipdesign.ru

Gotman ASh. The design of hydro conic ship hull shapes with high hydrodynamic quality. *Proc. 14th Scientific and Methodological Seminar on Ship Hydrodynamics*.-Varna, 1985; 53 (I – II).

Corno M, Bottelli S, Panzani G, Tanelli M, Spelta C, Savaresi S.M. Improving high speed road-holding using actively controlled aerodynamic surfaces. 2013 European Control Conference (ECC), July 17-19, 2013; Zürich, Switzerland, p. 1493-1498.

■ The 6th Order Surface with Cartesian Folium, Ellipse, Cartesian Folium Lying in Three Principal Coordinate Sections

This surface has a Cartesian folium

$$y = \pm \frac{1.2713Bx}{L}\sqrt{\frac{3(L-x)}{L+3x}}$$

in the cross-section of the surface by a plane $z = 0$, an ellipse

$$4y^2/B^2 + z^2/T^2 = 1$$

in the cross-section of the surface by a plane $x = L/\sqrt{3}$ and a Cartesian folium

$$z = \pm \frac{2.5426Tx}{L}\sqrt{\frac{3(L-x)}{L+3x}}$$

in the cross-section of the surface by a plane $y = 0$. Here T is a draught of the surface along an axis Oz, B is its maximum width in the cross-section $x = L/\sqrt{3}$ along an axis Oy, L is its length along an axis Ox.

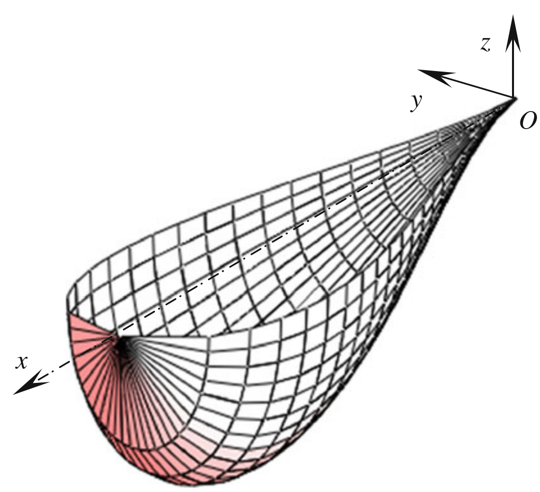


Fig. 1

An implicit equation of the studied aerodynamic surface (Fig. 1) was obtained in the form:

$$\frac{4L^2(L+3x)y^2}{19.395B^2x^2(L-x)} + \frac{L^2(L+3x)z^2}{19.395T^2x^2(L-x)} = 1.$$

The sixth order surface with a Cartesian folium, an ellipse, a Cartesian folium in three principal coordinate sections is formed by a family of ellipses lying in planes x = const.

Additional Literature

Avdon'ev EA, Protod'yakonov SM. Equations and characteristic of some algebraic surfaces of the highest orders. Prikl. Geom. i Ingen. Grafika, Kiev. 1976; Iss. 21, p. 108-20.

■ Surface of the 5th Order with Parabola, Ellipse, Parabola Lying in Three Principal Coordinate Sections

This surface with a parabola, an ellipse and a parabola lying in three principal coordinate sections has a parabola

$$y = \frac{B}{2} - \frac{Bx^2}{2L^2}$$

in the cross-sections of the surface by a plane xOy ($z = 0$); an ellipse (Fig. 1)

$$\frac{4y^2}{B^2} + \frac{z^2}{T^2} = 1$$

in the cross-section of the surface by a plane yOz and a parabola

$$z = -T + \frac{Tx^2}{L^2}$$

in the cross-section of the surface by a plane xOz. Here T is a draught of the surface; B is its maximum width along an axis Oy; $2L$ is its length along an axis Ox.

Availability of equations of the principal sections gives an opportunity to design surfaces with different conditions given in advance. Having the same principal sections, it is possible to design three surfaces substantially different from each other, see also "*Quartic surface with parabola, ellipse, parabola in three principal coordinate sections*". It is necessary for these cases to set up by a continuous framework of plane curves that are incident to a family of the planes parallel to one of three coordinate planes.

For practical aims, it is useful to count up *a coefficient of completeness* for every from obtained surfaces. A coefficient of completeness is the ratio of the area (or volume) bounded by a curve line (or by a surface) to the area (or volume) of a rectangle (or parallelepiped) having the same overall dimensions.

An explicit equation of the surface can be written as (Fig. 1):

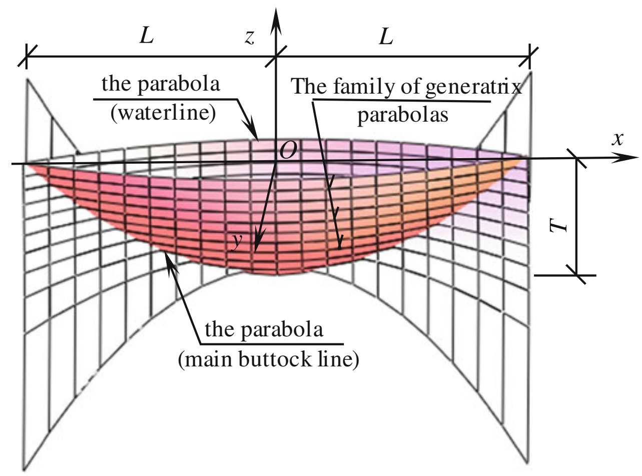


Fig. 1

$$y = \frac{B\sqrt{T^2 - z^2}}{2T} - \frac{B\sqrt{T^2 - z^2}}{2L^2(z+T)}x^2,$$

where $-L \le x \le L; -B/2 \le y \le B/2; -T \le z \le 0$. A designed surface is formed by a family of parabolas lying in planes $z = z_c$ = const:

$$y = \frac{B\sqrt{T^2 - z_c^2}}{2}\left[\frac{1}{T} - \frac{x^2}{L^2(z_c+T)}\right]$$

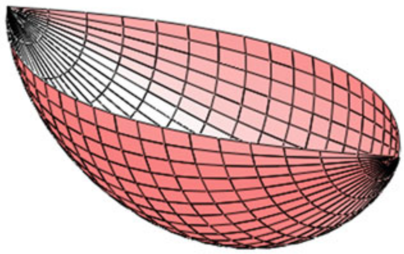
that are parallel to a coordinate plane xOy.

This surface can be also given by an implicit equation of the 5th order:

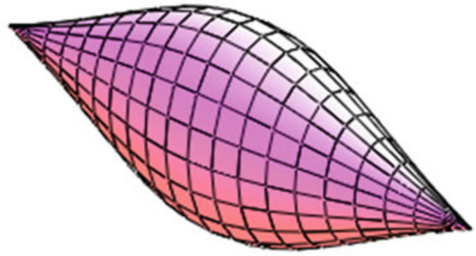
$$-\frac{x^4 z}{L^4} + \frac{2x^2z^2}{TL^2} + \frac{Tx^4}{L^4} - \frac{z^3}{T^2} - 4\frac{y^2 z}{B^2} - 4T\frac{y^2}{B^2} - \frac{z^2}{T} + z + T - 2T\frac{x^2}{L^2} = 0.$$

A coefficient of completeness for this surface is equal to 2/3.

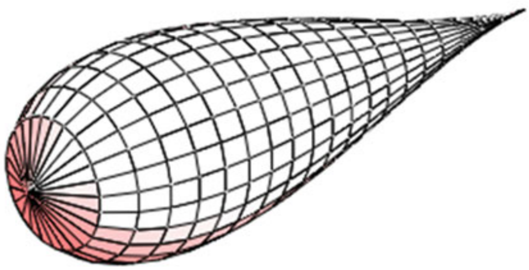
■ Aerodynamic Surfaces with a Continuous Framework of Plane Curves Presented in the Encyclopedia



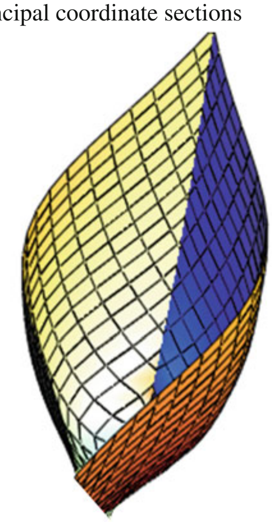
Quartic surface with parabola, ellipse, parabola in three principal coordinate sections



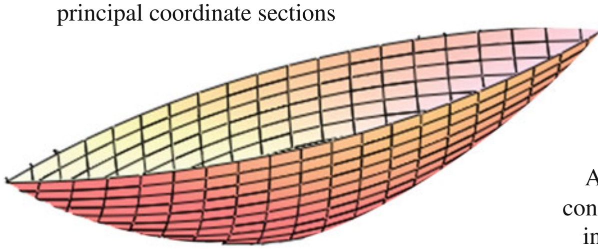
The 6^{th} order surface with Agnesi curl, ellipse, Agnesi curl lying in three principal coordinate sections



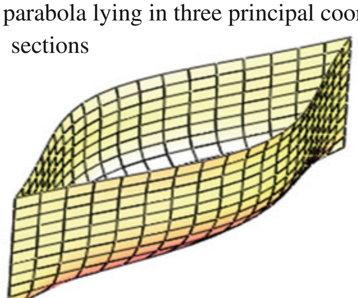
Quartic surface with the 4^{th} order curve, ellipse, the 4^{th} order curve in three principal coordinate sections



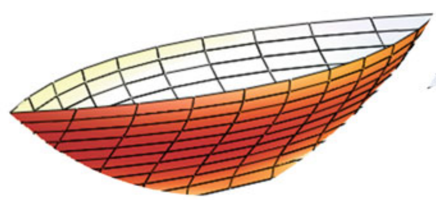
Aerodynamic surfaces given by a continuous framework of water-lines in the form of generalized Agnesi curls (the 6^{th} order surface with Agnesi curl, ellipse, parabola in three principal coordinate sections)



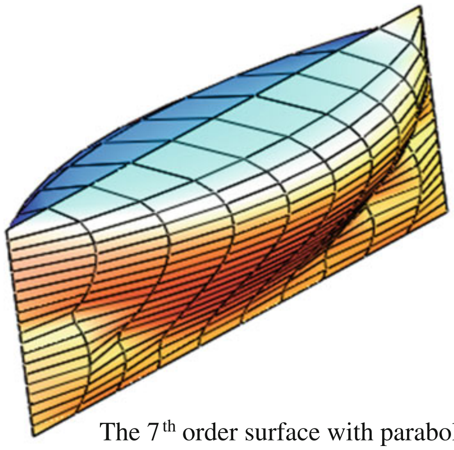
Surface of the 5^{th} order with parabola, ellipse, parabola lying in three principal coordinate sections



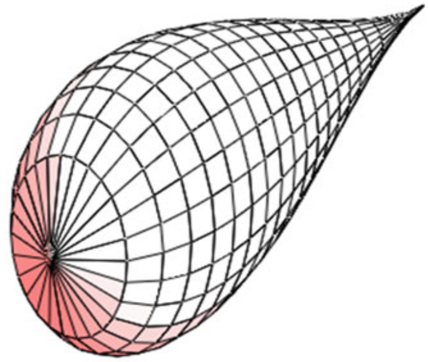
The 7^{th} order surface with Agnesi curl, Lame's curve of the third order, straight lines lying in three principal coordinate sections



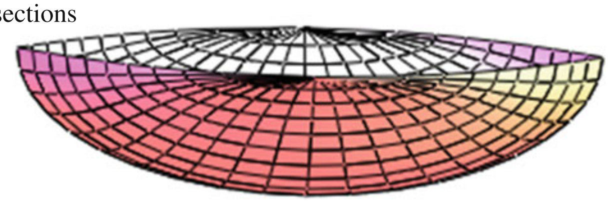
The 6^{th} order surface with parabola, the 4^{th} order curve, parabola lying in three principal coordinate sections



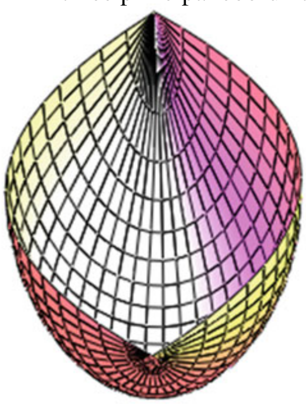
The 7^{th} order surface with parabola, the 4^{th} order curve, parabola lying in three principal coordinate sections



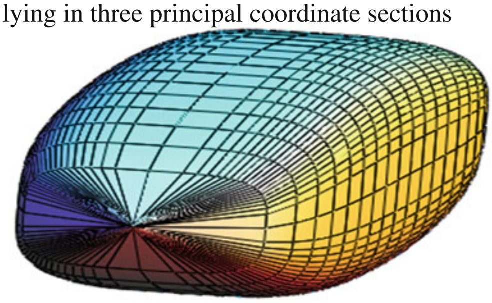
The 3^{rd} order surface with Cartesian folium, ellipse, Cartesian folium lying in three principal coordinate sections



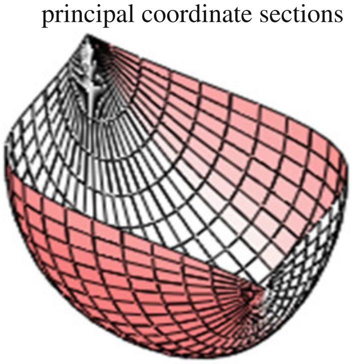
The 7^{th} order surface with parabola, Agnesi curl, ellipse lying in three principal coordinate sections



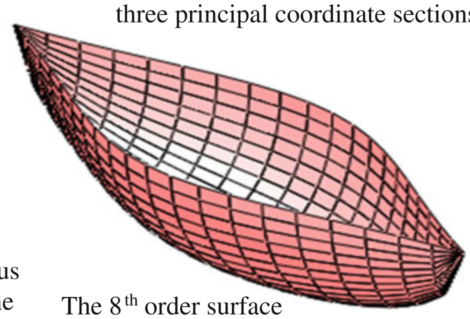
The 7^{th} order surface with parabola, ellipse, Cartesian folium lying in three principal coordinate sections



The 8^{th} order surface with Lame's curve of the 4^{th} order, Lame 's curve of the 4^{th} order, ellipse lying in three principal coordinate sections



Aerodynamic surface given by a continuous framework of elliptical ribs (Surface of the 10^{th} order with Agnesi curl, ellipse, parabola in three principal coordinate sections)



The 8^{th} order surface with Agnesi curl, ellipse, ellipse lying in three principal coordinate sections

7 Helical Surfaces

It is well known that apart from the trivial uniform motion, where nothing moves at all and all velocities are zero, there are the following three cases: (1) *uniform translations*, (2) *uniform rotations* with nonzero angular velocity about a fixed axis, and (3) *uniform helical motion* that are the superposition of a uniform rotation and a uniform translation parallel to the rotation's axis.

A helical surface is formed by a curve L in the process of its helical motion. The generatrix curve L uniformly revolves on an axis of revolution and at the same time executes a translational motion in the direction of the same axis that is called *a helical axis*. If the ratio of a value of the speed along the straight line to a value of the angular velocity is constant then a helical motion is called *an ordinary helical motion*. A surface formed by ordinary helical motion is called *an ordinary helical surface*. Trajectories of the points subjected to ordinary helical motion are *cylindrical helical lines of the constant lead* (*helixes*) lying on different coaxial circular cylinders. So, each point of the generatrix generates a helix and one helix and one position of the generatrix pass through each point of the surface. If we have for helical motion that a relation of the translational speed to the angular velocity is variable, then the trajectories of the points of the curve L are cylindrical helical lines of the variable lead but a helical surface itself is called *a helical surface of variable lead* (*pitch*).

Every line lying on a helical surface can be taken as a generating line L. Without any loss of generality, we ought to consider that a generating curve is placed in the plane of the axis of rotation. At least, for any helical surface formed by the helical motion of arbitrary line L, it is always possible to find a plane curve l lying in the plane of the axis of rotation and which will form the same helical surface under helical motion. If a generating curve L is assumed in the form of a straight line then the helical surface is called *a ruled helical surface*. If a fixed generating curve L is a circle then a helical surface is called *a circular helical surface with generatrix circles in the planes of pencil*.

So, taking into account the information written above, helical surfaces can be distributed with respect to the basic generatrix figure to three types: (a) ruled helical surfaces (when a straight line or its part is subdued to the helical movement), (b) circular helical surfaces (when a circle or its part is subdued to the helical movement), and (c) general helical surfaces (when an arbitrary curve or its part is subdued to the helical movement).

Assuming a coordinate axis Oz for an axis of rotation and giving an explicit equation to a plane generating curve l in the form of $z = f(r)$, it is possible to write a vector equation of the helical surface as

$$\begin{aligned}\boldsymbol{r} = \boldsymbol{r}(r,\varphi) &= r\cos\varphi\boldsymbol{i} + r\sin\varphi\boldsymbol{j} + [f(r) + a\varphi]\boldsymbol{k}\\ &= r\boldsymbol{e}(\varphi) + [f(r) + a\varphi]\boldsymbol{k}\end{aligned}$$

or in the form of an explicit equation as

$$z = f\left(\sqrt{x^2+y^2}\right) + a\,\mathrm{Arctan}(y/x).$$

A linear element ds can be described as

$$\mathrm{d}s^2 = (1+f'^2)\mathrm{d}r^2 + 2af'\mathrm{d}r\,\mathrm{d}\varphi + (r^2+a^2)\mathrm{d}\varphi^2.$$

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_7, © Springer International Publishing Switzerland 2015

Assume a vector equation of a helical surface in the form:

$$\boldsymbol{r} = \boldsymbol{r}(z, \varphi) = r(z)\cos\,\varphi\boldsymbol{i} + r(z)\sin\,\varphi\boldsymbol{j} + [z + a\varphi]\boldsymbol{k}.$$

then a linear element of the helical surface can be written as

$$\mathrm{d}s^2 = \left(r^{/2} + 1\right)\mathrm{d}z^2 + 2a\,\mathrm{d}z\,d\varphi + \left(r^2 + a^2\right)\mathrm{d}\varphi^2.$$

If $a = 0$, then a helical surface becomes *a surface of revolution*. Due to Bour's theorems, we can say that a metric of helical surface is a metric of rotation and a helical surface may be superimposed on a surface of revolution so that the helical lines, i.e., the trajectories of helical motion will superimpose on parallels. For example, a surface of a right helicoid (a ruled helical surface) can be developed on the surface of a catenoid (a surface of revolution). If a function $r(z)$ becomes a constant value, it means that the curve forming a helical surface lies on the circular cylinder with the axis coinciding with the axis of the motion. In this case, the surface obtained coincides with *a right cylindrical surface* (see also "Cylindrical helical strip" in Subsect. "1.1.2. Cylindrical Surfaces").

In the construction of planar intersections of helical surfaces generally only two types of intersection planes are used: (a) *normal intersection* by *normal plane* perpendicular to the surface axis, (b) *meridian intersection* by *meridian plane* passing through the surface axis.

Intersection curves are constructed as sets of separate points which are intersections of surface helices with *the intersection plane*.

Many papers devoted to analytical methods of forming helical surfaces are published. With the wide use of complex helical surfaces in screw components, it becomes more and more urgent to investigate efficient ways for the design and manufacture of components of this kind. For example, V. Ivanov and G. Nankov presented a generalized analytical method for profiling all types of rotation tools for forming helical surfaces.

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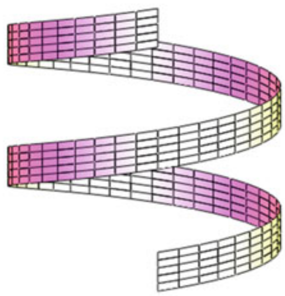
Pottmann H, Wallner J, *Leopoldseder S*. Kinematical methods for the classification, reconstruction and inspection of surfaces. SMAI 2001: Congrès national de mathém. appliquées et industrielles, Publ. de l'Équipe de Mathématiques Appliquées (Numéro special), Univ. de Technologie de Compiègne, 2001; p. 51-60.

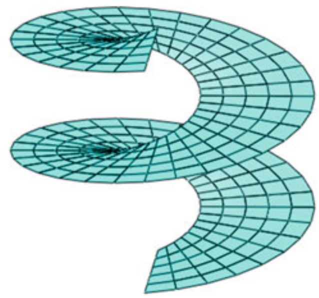
7.1 Ordinary Helical Surfaces

■ Ordinary Helical Surfaces Presented in the Encyclopedia

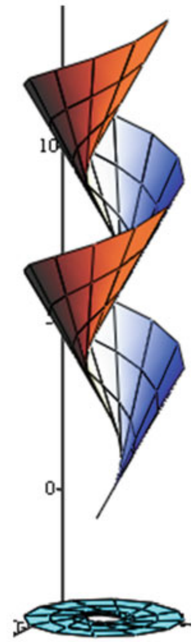
RULED HELICAL SURFACES

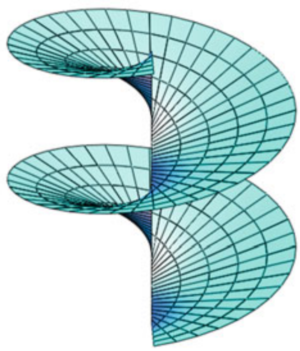


The cylindrical helical strip

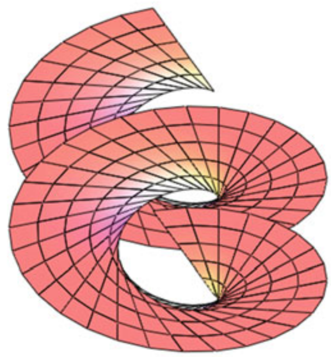


The right helicoid



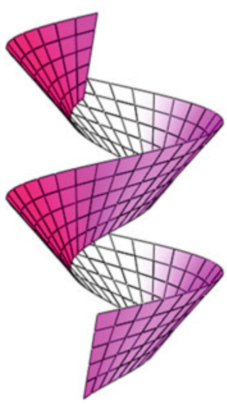


The oblique helicoid

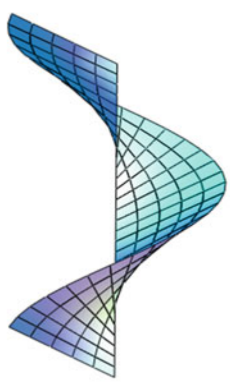


The pseudo-developable helicoid

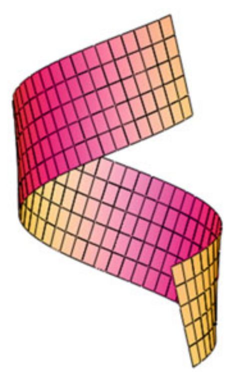
The evolvent helicoid



The convolute helicoid

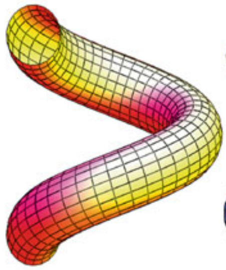


The degenerated convolute helicoid for a straight line

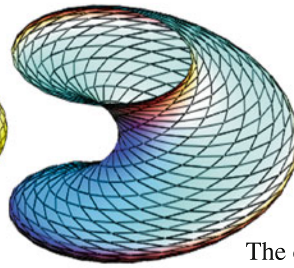


The helical surface generated by binormals of the cylindrical helix

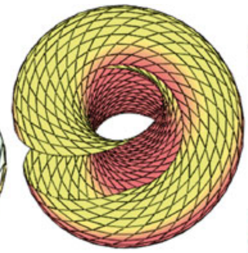
CIRCULAR HELICAL SURFACES



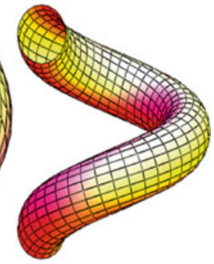
The tubular helical surface



The right circular helical surface

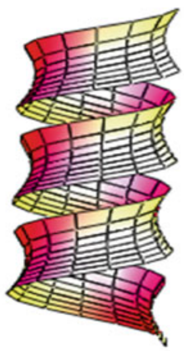


The circular helical surface with generating circles lying in the osculating plane of a helix of centers of the circles

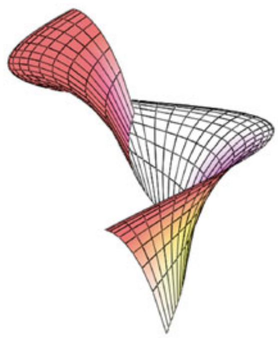


The circular helical surface with generatrix circles in the planes of pencil

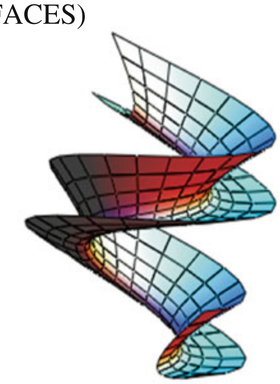
HELICAL SURFACES WITH ARBITRARY PLANE GENERATRIX CURVES
(GENERAL HELICAL SURFACES)



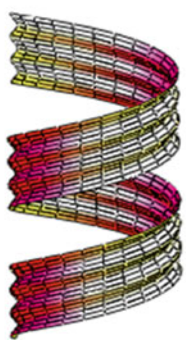
The helical surface with generatrix curve in the form of the hyperbola



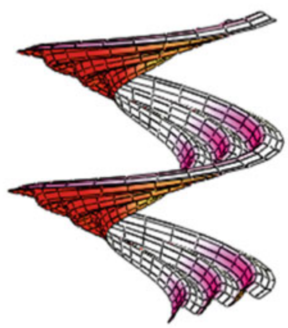
The Dini's surface



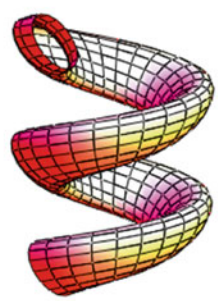
The helical surface with the parabolic generatrix of arbitrary position



The helical sinusoidal strip



The helical surface with the sinusoidal generatrix



The helical surface with the generatrix ellipse

7.1.1 Ruled Helical Surfaces

A ruled helical surface is formed by a straight generatrix disposed in arbitrary position under its ordinary helical motion (see also Chap. "7. Helical Surfaces").

All ruled surfaces including ruled helical surfaces may be divided into *torse surfaces* and *oblique ruled surfaces* or, which is the same, into *ruled surfaces of zero* and *negative Gaussian* curvature. *Evolvent helical surface* is the only one developable helical surface (see also "Open (evolvent) helicoid"). *Right helicoid* is the only *ruled helical minimal surface*. There are no ruled surfaces of positive Gaussian curvature.

In the industry, one more classification of ruled helical surfaces exists depending on the disposition of a straight generatrix L relative to the axis of a helix and on the normal intersection by normal plane perpendicular to the surface axis. If a straight line L intersects the helical axis then we have *a closed ruled helical surface*, but if a straight L does not intersect the helix axis then a ruled helical surface is called *an open ruled helical surface*. Closed ruled helical surfaces are always nondevelopable. Open ruled helical surfaces can be both developable and nondevelopable surfaces. Closed ruled helical surfaces are also called *surfaces of Archimedes*. Mario Hirz et al. gives the following definition: "If the generating straight line c is skew to the screw axis a and not perpendicular to a, the surface is called *generic helical ruled surface*." So, an open ruled helical surface can also be called a generic helical ruled surface.

In the cross section of a closed helical surface by a plane perpendicular to the helical axis *an Archimedes spiral* is placed, but in the cross section of an open ruled helical surface *an evolvent of the circle* lies.

Convolute and evolvent helicoids are related to a family of open ruled helical surfaces. Right and oblique helicoids originate from a family of closed ruled helical surfaces.

Equations of a ruled surface formed by ordinary helical motion of a straight line AB may be given as

$$\begin{aligned} x &= x(t,v) = a\cos v - t\sin\gamma\sin v,\\ y &= y(t,v) = a\sin v + t\sin\gamma\cos v,\\ z &= z(t,v) = pv + t\cos\gamma; \end{aligned}$$

or

$$\begin{aligned} x &= x(\rho,v) = a\cos v - [\rho^2 - a^2]^{1/2}\sin v,\\ y &= y(\rho,v) = a\sin v + [\rho^2 - a^2]^{1/2}\cos v,\\ z &= [\rho^2 - a^2]^{1/2}\cot\gamma + pv, \end{aligned}$$

where a is the least distance between the straight line AB and the axis Oz; γ is the angle of the generatrix straight line AB with the helical axis Oz; ρ is the radius of the projection of the trajectory of the motion of a point M of the surface on the plane xOy; a parameter t defines the position of the point M on the rectilinear generatrix. Having assumed $a = 0$, we obtain a closed ruled helical surface. According to the theorem of Schaal, the slope angle φ of the tangent plane at any point (t, v) of a helical surface with the central plane at the point $(t = 0;\ v)$ is determined by the formula

$$\tan\varphi = \frac{t}{p - a\cot\gamma}.$$

Sometimes they take values

$$n = \frac{\rho}{a} \quad \text{and} \quad z_0 = \frac{vH}{2\pi}$$

as curvilinear coordinates, where H is the pitch of the helical motion. In this case, parametric equations of the ruled helical surface can be written as

$$\begin{aligned} x &= x(n,z_0) = a\left[\cos\frac{2\pi z_0}{H} - \sqrt{n^2-1}\sin\frac{2\pi z_0}{H}\right],\\ y &= y(n,z_0) = a\left[\sin\frac{2\pi z_0}{H} + \sqrt{n^2-1}\cos\frac{2\pi z_0}{H}\right]\\ z &= z(n,z_0) = a\sqrt{n^2-1}\operatorname{ctg}\gamma + z_0. \end{aligned}$$

For all types of parametric equations of a helical surface, two families of coordinate lines on the surface consist of rectilinear generatrixes and helical lines.

An explicit equation of a ruled surface can be presented in the form:

$$\begin{aligned} z = &\pm\sqrt{x^2+y^2-a^2}\operatorname{ctg}\gamma + p\operatorname{Arctg}(y/x)\\ &\mp p\operatorname{Arctg}\left(\sqrt{x^2+y^2-a^2}/a\right). \end{aligned}$$

Different threads, helical staircases, screw propellers, screw conveyers, and spiral chutes have the form of ruled helical surfaces.

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■ Convolute Helicoid

A convolute helicoid is formed by a straight line *AB* that moves in the space all the time intersecting a helix and touching the side surface of a right circular cylinder with a radius *a*. The axis of the helix coincides with the axis of the cylinder but the angle between these skew lines is not equal to π/2 (Fig. 1).

In a cross section of a convolute helicoid by a plane that is perpendicular to the helical axis, *shortened or elongated evolvents* of a circle are placed in contrast to a cross section of *an evolvent helicoid* by a plane where *an ordinary evolvent* is placed.

Convolute helical surfaces are nondevelopable surfaces and the only one ruled open evolvent helical surface is a developable surface (see also "Open (evolvent) helicoid"). *The right convolute helicoid* traced by a helical motion of a straight line perpendicular to the helical axis is called *a pseudodevelopable helicoid* (see also "Ruled surfaces of negative Gaussian curvature").

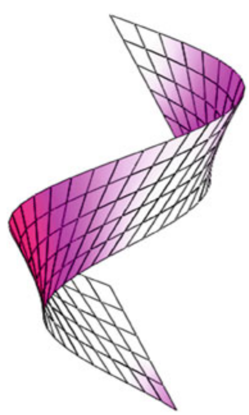

Fig. 1

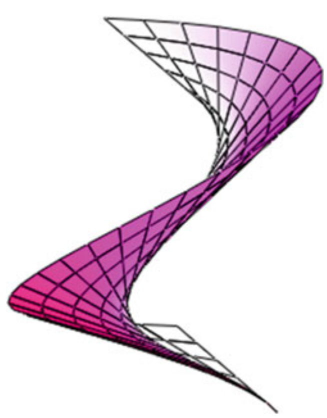

Fig. 2

Forms of definition of the surface

(1) Parametric equations (Figs. 1, 2, and 3):

$$x = x(t, v) = a\cos v - t\sin\gamma\sin v,$$
$$y = y(t, v) = a\sin v + t\sin\gamma\cos v,$$
$$z = z(t, v) = pv + t\cos\gamma;$$

where a is the shortest distance in a straight line AB from an axis Oz; γ is the angle of the generatrix straight line AB with the helical axis Oz; parameter t determines the disposition of a point M lying on the straight generatrix. The equations give both the spaces of a helicoid when $t > 0$ and $t < 0$.

Coefficients of the fundamental forms of the surface:

$$A = 1,\quad F = a\sin\gamma + p\cos\gamma,$$
$$B^2 = a^2 + p^2 + t^2\sin^2\gamma,$$
$$L = 0,\quad M = \frac{\sin\gamma(a\cot\gamma - p)}{\sqrt{(a\cot\gamma - p)^2 + t^2}},$$
$$N = \frac{[a(a\cot\gamma - p) + t^2\sin\gamma\cos\gamma]}{\sqrt{(a\cot\gamma - p)^2 + t^2}},$$
$$K = -\frac{(a\cot\gamma - p)^2}{[(a\cot\gamma - p)^2 + t^2]^2} < 0$$

The angle χ between the coordinate lines t and v is calculated with the help of the formula:

$$\cos\chi = (a\sin\gamma + p\cos\gamma)/\sqrt{a^2 + p^2 + t^2\sin^2\gamma}.$$

Rectilinear generatrixes are orthogonal to the helical lines v, if a constant angle γ is calculated by the formula:

$$\tan\gamma = -p/a.$$

Having assumed $\gamma = \pi/2$, we can obtain a right convolute helicoid (Fig. 3).

If the angle $\gamma = 0$, then a general convolute helicoid degenerates into cylindrical helical strip. If the directions of the generatrix straight lines coincide with the tangents to the directrix helical line then a developable helicoid is formed.

(2) Parametrical equations (Fig. 1):

$$x = (\rho, \varphi) = \rho\ \cos\ \varphi,$$
$$y = y(\rho, \varphi) = \rho\ \sin\ \varphi.$$
$$z = z(\rho, \varphi) = \pm\sqrt{\rho^2 - a^2}\,\mathrm{ctg}\,\gamma + p\varphi \mp p\arccos(a/\rho)$$
$$a \le \rho \le +\infty,\ \theta + v = \varphi.$$

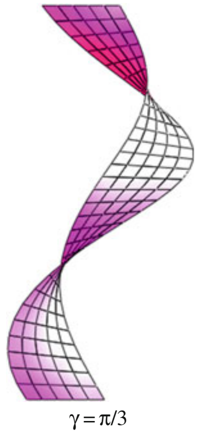

Fig. 3

Two signs correspond to values of θ from 0 to π/2 and from 0 to −π/2.

Coefficients of the first fundamental form of the surface:

$$A^2 = 1 + \frac{(\rho^2 \cot\,\gamma - ap)^2}{\rho^2(\rho^2 - a^2)},$$

$$F = \frac{p(\rho^2 \cot\gamma - ap)}{\rho\sqrt{\rho^2 - a^2}},\ B^2 = \rho^2 + p^2.$$

The angle χ between the coordinate lines ρ and φ can be calculated by the formula:

$$\cos\chi = \frac{p(\rho^2 \cot\gamma - ap)}{\sqrt{[\rho^2(\rho^2 - a^2) + (\rho^2 \cot\gamma - ap)^2][\rho^2 + p^2]}}.$$

Additional Literature

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■ Helical Surface Generated by Binormals of a Cylindrical Helix

A helical surface generated by binormals of a cylindrical helix

$$\boldsymbol{\rho} = \boldsymbol{\rho}(\varphi) = a\cos\,\varphi\boldsymbol{i} + a\sin\,\varphi\boldsymbol{j} + p\,\varphi\boldsymbol{k}$$

is *a convolute helical surface* given by a vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u,\ \varphi) = \boldsymbol{\rho}(\varphi) + u\boldsymbol{\beta},$$

where $\boldsymbol{\beta}$ is the unit vector of the binormal;

$$\boldsymbol{\beta} = \frac{p\sin\phi}{\sqrt{a^2+p^2}}\boldsymbol{i} - \frac{p\cos\phi}{\sqrt{a^2+p^2}}\boldsymbol{j} + \frac{a}{\sqrt{a^2+p^2}}\boldsymbol{k}$$

or

$$\boldsymbol{\beta} = \frac{a}{\sqrt{a^2+p^2}}\boldsymbol{k} - \frac{p}{\sqrt{a^2+p^2}}\boldsymbol{g};$$

$$\boldsymbol{g} = \boldsymbol{g}(\varphi) = -c\,\varphi\,\mathrm{in}\,\boldsymbol{i} + \cos\,\varphi\boldsymbol{j}$$

$\boldsymbol{g}$ is the unit vector oriented in the direction of a tangent to the projection of the helix on a plane xOy.

An equation of the binormal at a point $M(\boldsymbol{\rho})$ belonging to the helix may be written in the following form:

$$\frac{x - a\cos\varphi}{p\sin\varphi} = \frac{y - a\sin\varphi}{-p\cos\varphi} = \frac{z - p\varphi}{a}.$$

The binormal forms the constant angle γ with the axis of the helical surface:

$$\cos\gamma = \frac{a}{\sqrt{a^2+p^2}}.$$

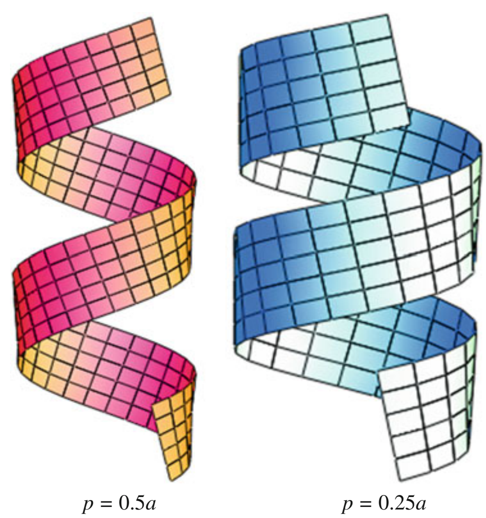

Fig. 1 $0 \le u \le 1$

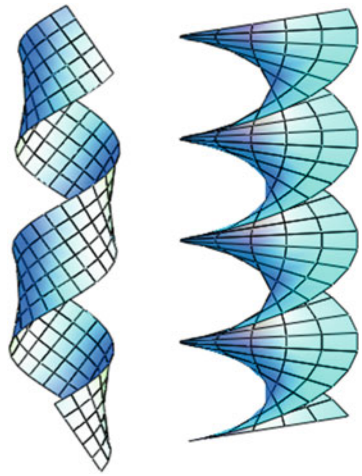

Fig. 2 $-1 \le u \le 1$

Forms of definition of the surface generated by binormals of the helix

(1) Parametrical equations (Figs. 1 and 2):

$$x = x(u, \varphi) = a \cos \varphi + u \frac{p \sin \varphi}{\sqrt{a^2 + p^2}},$$
$$y = y(u, \varphi) = a \sin \varphi - u \frac{p \cos \varphi}{\sqrt{a^2 + p^2}},$$
$$z = z(u, \varphi) = p\, \varphi + ua \Big/ \sqrt{a^2 + p^2}$$

where u is the parameter defining the disposition of the point on the binormal. The projection of the binormals to the plane xOy is placed along the tangents to the main circle which is in turn the projection of the cylindrical helical line $u = 0$ on the same plane xOy.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = 1, \quad F = 0,$$
$$B^2 = a^2 + p^2 + \frac{u^2 p^2}{a^2 + p^2},$$
$$L = 0, \quad M = -\frac{p}{B},$$
$$N = \frac{aB}{\sqrt{a^2 + p^2}}$$
$$k_\varphi = \frac{a}{B\sqrt{a^2 + p^2}}, \quad k_u = 0,$$
$$K = -p^2/B^4 < 0.$$

The obtained values of the fundamental forms confirm that a ruled surface of binormals of a helix is a surface of *negative Gaussian curvature* and the curvilinear coordinate net u, φ is nonconjugate orthogonal net. An equation of *the osculating plane* can be written as

$$x \sin \varphi - y \cos \varphi + az/p - a\varphi = 0.$$

A binormal of the helix is perpendicular to the osculating plane. On the osculating plane in a point M, the principal normal of a helix is parallel to a plane xOy and that is why it is the horizontal straight line. The tangent to the helix at the point M is a straight line of the maximum slope on the osculating plane.

Additional Literature

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7.1.2 Circular Helical Surfaces

A circular helical surface is generated by a generatrix circle of a constant radius r in the process of the ordinary helical motion relative to the helical axis Oz. Circular helical surfaces enter also in the class of *cyclic surfaces* (see also “Classification of Cyclic Surfaces” in Chap. “17. Cyclic Surfaces”).

According to the position of a directrix circle (*a basic circle*), circular helical surfaces S may be classified in the following form:

1. *Circular helical surface with a generatrix circle lying in the plane passing through the helical axis*;
2. *Tubular helical surface* formed by a constant radius circle lying in the normal plane of the directrix helix;
3. *Right circular helical surface* formed by a constant radius circle placed in the plane that is perpendicular to the helical axis;
4. *Circular helical surfaces with a generatrix circle lying in the osculating plane of a helix of the centers of the circles.*

It was found another classification of cyclical helical surfaces:

arch surface: when basic circle is located in the plane passing through the axis of the helical movement;

Archimedean serpentine: when basic circle is located in the plane perpendicular to the tangent line to the trajectory of the helical movement, i.e., to the helix of the circle center;

vinded column: when basic circle is located in the plane perpendicular to the axis of the helical movement.

So, we can see that both the classifications are identical.

Taking into attention the distance a from the center C of the generatrix circle until the helical axis, we can divide circular helical surfaces S into three groups:

1. The helical axis is out of the surface поверхности S; $a > r$;
2. The helical axis is on the surface S; $a = r$;
3. The helical axis is inside the surface S; $a < r$.

Every circle in the space is defined by *the basic vector of a circle,* the beginning of which coincides with the center of the circle, the direction coincides with the normal to the plane of the circle but the length is equal to a length of the radius of the circle. It is possible to form *a basis ruled surface* generated by the motion of a straight line carrying on itself basic vectors of the circular generatrixes of cyclic surface. The beginnings of the basic vectors of generatrix circles determine *a line of the centers on the basis surfaces*, but the ends of these vectors define *a line of the radius*.

A basis surface of a tubular helical surface is *a developable helicoid*. A circular helical surface with a generating circle lying in the plane passing through the helical axis has *a right convolute helicoid* (pseudodevelopable helicoid) as a basis surface. A basis surface of a right circular helical surface is *a cylindrical helical strip*. *A surface of binormals of a cylindrical helical line* is the basis surface for a circular helical surface with a generatrix circle lying in the osculating plane of a helix of the centers of the circles.

Parametric equations of *a circular helical surface S* can be given as

$$x = x(u, v) = r\cos(u + v)\sin\psi + x_0\cos v - y_0\sin v,$$
$$y = y(u, v) = r\sin(u + v)\sin\psi + x_0\sin v + y_0\cos v,$$
$$z = z(v) = r\cos\psi + pv + z_0$$

where x_o, y_o, and z_o are the coordinates of the centers C of the generatrix circle of radius r; a parameter ψ depends on u and can be found from the expression

$$n_1\cos u\sin\psi + n_2\sin u\sin\psi + n_3\cos\psi = 0;$$

n_1, n_2, and n_3 are the direction cosines of the basic vector of the generatrix circle given in a mobile system of coordinates executing the same helical motion. If v = const, we have a generatrix circle on the surface S and a line u = const is a heliacal line on the surface S.

A vector equation of *a circular helical surface* may be written as

$$\begin{aligned}\boldsymbol{r}(u, v) = r\boldsymbol{h}(u + v)\sin\psi + x_0\boldsymbol{h}(u) + y_0\boldsymbol{n}(u)\\ + (r\cos\psi + pv + z_0)\boldsymbol{k},\end{aligned}$$

where $\boldsymbol{r}(u, v)$ is the radius vector of the surface;

$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u,\ \boldsymbol{n}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u$$

are orthogonal unit radius vectors of the circle at the horizontal plane; $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors of Cartesian coordinates.

Additional Literature

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Krivoshapko SN, Christian A. Bock Hyeng. Static and dynamic analysis of thin-walled cyclic shells. International Journal of Modern Engineering Research. 2012; Vol. 2, Iss. 5, p. 3502-3508.

Krivoshapko SN, Christian A. Bock Hyeng. Classification of cyclic surfaces and geometrical research of canal surfaces. International Journal of Research and Reviews in Applied Sciences; 2012; Vol. 12, Iss. 3, p. 360-374.

P.S.: Additional literature is given also at the corresponding pages of the Subsect. "7.1.2. Circular Helical Surfaces".

■ **Circular Helical Surface with Generatrix Circles in the Planes of Pencil**

Assume that a circular helical surface S with a generatrix circle of a radius r lying at a plane passing through a helical axis Oz has a directrix helical line of centers:

$$x = x(v) = a\cos v, \quad y = y(v) = a\sin v, \quad z = z(v) = pv.$$

Forms of definition of the circular helical surface (*helical torus*)

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(v,\psi) = (a + r\sin\psi)\boldsymbol{e}(v) + (r\cos\psi + pv)\boldsymbol{k},$$

where $\boldsymbol{e}(v)$ is the unit circular vector function; ψ is a central angle of the generatrix circle read from the positive direction of the axis Oz; $0 \le \psi \le 2\pi$; r is the radius of a generatrix circle.

(2) Parametrical equations (Fig. 1):

$$x = x(v,\psi) = (a + r\ \sin\psi)\cos\ v,$$
$$y = y(v,\psi) = (a + r\ \sin\psi)\sin\ v,$$
$$z = z(v,\psi) = r\ \cos\psi + pv.$$

There is a generatrix circle of a radius r in the axial section of the surface S. The face cross section (*normal intersection* by *normal plane* that is perpendicular to the helical axis) is obtained if one takes $z = 0$, i.e., when

$$v = -\frac{r\cos\psi}{p}.$$

If $p = 0$, then studied helical surface S degenerates into *a circular torus* (see also Chap. "2. Surface of Revolution"). If $p = 0$ and $a = 0$, then studied surface degenerates into *a sphere* with a radius r.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = (a + r\sin\psi)^2 + p^2, \quad F = -rp\sin\psi, \quad B = r,$$

$$L = -\frac{(a + r\sin\psi)^2\sin\psi}{\sqrt{(a + r\sin\psi)^2 + p^2\cos^2\psi}},$$

$$M = \frac{rp\cos^2\psi}{\sqrt{(a + r\sin\psi)^2 + p^2\cos^2\psi}}$$

$$N = -\frac{r(a + r\sin\psi)}{\sqrt{(a + r\sin\psi)^2 + p^2\cos^2\psi}},$$

$$k_\psi = \frac{-(a + r\sin\psi)}{r\sqrt{(a + r\sin\psi)^2 + p^2\cos^2\psi}},$$

$$k_v = \frac{-\sin\psi(a + r\sin\psi)^2}{A^2\sqrt{(a + r\sin\psi)^2 + p^2\cos^2\psi}},$$

$$K = \frac{(a + r\sin\psi)^3\sin\psi - rp^2\cos^4\psi}{r[(a + r\sin\psi)^2 + p^2\cos^2\psi]^2}.$$

The angle χ between the nonorthogonal nonconjugate curvilinear coordinates v, ψ can be calculated by a formula

$$\cos\chi = \frac{p\ \sin\psi}{\sqrt{(a + r\ \sin\psi)^2 + p^2}}.$$

There are only two helical lines (when $\psi = 0$ and $\psi = \pi$) that are orthogonal to all generatrix circles. A studied helical surface has zones with both hyperbolic and elliptical points. These zones are divided by four lines with parabolic points. If $a = r$, then a helical axis is placed on the surface S (Fig. 2). A helical axis is placed inside the surface S if $a < r$ (Fig. 3). The intersection of the helical axis of a surface S with a generatrix curve takes place at the point for which

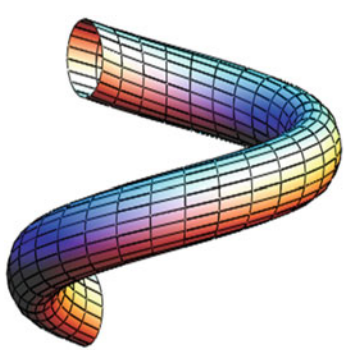

Fig. 1

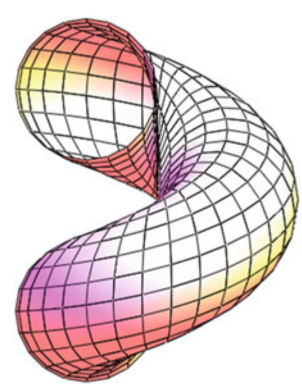

Fig. 2

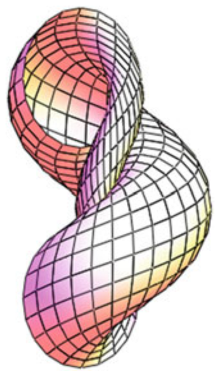

Fig. 3

$$a + r\sin\psi = 0 \text{ or when } \psi = \psi_0; \quad \sin\psi_0 = -a/r; \quad a < r.$$

Additional Literature

Lyukshin VC. Theory of Helical Surfaces for Design of the Cutting Tools. Moscow: Izd-vo “Mashinostroenie”, 1968; 372 p. (35 refs.).

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Pogrebezskaya MN. On the curvature of the helical surfaces. Izvestiya Vuzov, Mashinostroenie, 1965; No. 4, p. 5-15.

■ Tubular Helical Surface

A tubular helical surface (*normal helicoidal circular cylinder*) is formed by ordinary helical motion of a generatrix circle of a constant radius but a plane of the circle must coincide with the normal plane of the directrix helical line

$$x = x(v) = a\cos v, \quad y = y(v) = a\sin v, \quad z = z(v) = pv.$$

Tubular helical surfaces fall into a class of *cyclic surfaces* into a group of *canal surfaces*. The generatrix circles set up one family of lines of principal curvatures of the surface. A vector of a normal to the normal plane of the helix formed by the center of a generatrix circle is directed along the tangent to the helical line of the centers.

The helix angle β may be found from the formula:

$$\tan\beta = \frac{p}{a}.$$

Tubular helical surface is an envelope of a family of balls with a constant radius r; the centers of the balls are on the helical line.

Forms of definition of a tubular helical surface

(1) Vector equation:

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}(\vartheta, v) \\ &= (a + r\cos\vartheta)\boldsymbol{e}(v) + r\sin\vartheta\sin\beta\boldsymbol{g}(v) \\ &\quad + (pv - r\sin\vartheta\cos\beta)\boldsymbol{k}, \end{aligned}$$

where $\boldsymbol{e}(v)$, $\boldsymbol{g}(v)$ are the unit circular vector functions; ϑ is a central angle of a generatrix circle; $0 \le \vartheta \le 2\pi$; r is the radius of a generatrix circle; β is the helix angle, i.e., a slope angle with a plane $z = 0$,

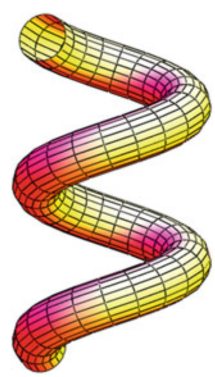

Fig. 1

$$\sin\beta = \frac{p}{\sqrt{p^2 + a^2}}, \quad \cos\beta = \frac{a}{\sqrt{p^2 + a^2}}.$$

(2) Parametrical equations (Fig. 1):

$$\begin{aligned} x &= x(\vartheta, v) = (a + r\cos\vartheta)\cos v - r\sin\vartheta\sin\beta\sin v, \\ y &= y(\vartheta, v) = (a + r\cos\vartheta)\sin v + r\sin\vartheta\sin\beta\cos v \\ z &= z(\vartheta, v) = pv - r\sin\vartheta\cos\beta. \end{aligned}$$

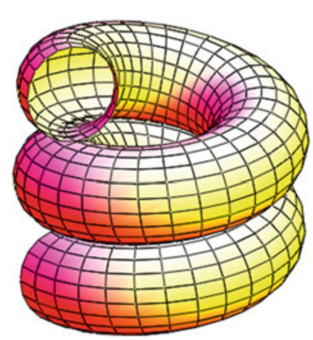

Fig. 2

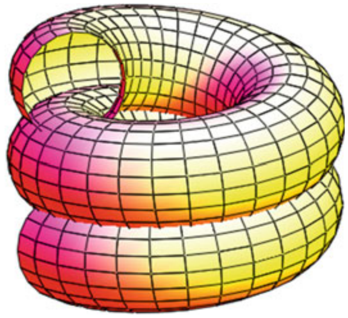

Fig. 3

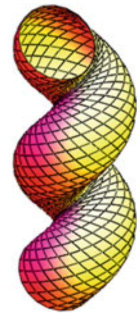

Fig. 4

Fig. 5 The helical cylindrical springs. **a** The helical cylindrical spring of compression. **b** The helical cylindrical spring of tension. **c** The helical cylindrical spring of torsion

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = r,\quad F = r^2 \sin\beta,$$
$$B^2 = r^2 \sin^2\vartheta \sin^2\beta + (a + r\cos\vartheta)^2 + p^2,$$
$$L = -r,\quad M = -\frac{rp}{\sqrt{a^2 + p^2}},$$
$$N = -\frac{rp^2 + a\cos\vartheta(a^2 + p^2 + ar\cos\vartheta)}{(a^2 + p^2)}$$
$$k_\vartheta = -\frac{1}{r},\quad k_v = \frac{N}{B^2},$$
$$K = \frac{a\cos\vartheta}{r(ar\cos\vartheta + a^2 + p^2)}.$$

An angle χ between nonorthogonal nonconjugate curvilinear coordinates v, ϑ is given by

$$\cos\chi = r\sin\beta \Big/ \sqrt{r^2\sin^2\vartheta\sin^2\beta + (a + r\cos\vartheta)^2 + p^2}.$$

If the pitch of a tubular helical surface is equal to r/π ($p = r/\pi$), then the turns of the tubular surface touch each other along the line $v = \pm\pi/2$ (Fig. 2). If $p < r/\pi$, then the turns of the surface intersect themselves (Fig. 3). In Fig. 4, the tubular helical surface with a radius of the basic cylinder equal to zero ($a = 0$) is shown.

Tubular helical surfaces are used for solid and hollow springs, in coil pipes of circular cross section (Fig. 5).

Additional Literature

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■ Right Circular Helical Surface

A right circular helical surface is formed by ordinary helical motion of a circle of constant radius which is placed on a plane perpendicular to the helical axis (Fig. 1). This plane with the normal directed along the helical axis is *a plane of parallelism* of the helical surface. The surface may be related to a class of *cyclic surfaces* to a group of *cyclic surfaces with a plane of parallelism*.

Parametric equations of a helix of the centers of generatrix circles:

$$x = x(v) = a\cos v,\quad y = y(v) = a\sin v,$$
$$z = z(v) = pv.$$

Forms of definition of a right circular helical surface

(1) Parametrical form of the definition (Figs. 1, 2, and 3):

$$x = x(\vartheta, v) = a\cos v + r\cos(\vartheta + v),$$
$$y = y(\vartheta, v) = a\sin v + r\sin(\vartheta + v),$$
$$z = z(v) = pv$$

where ϑ is a central angle of the generatrix circle; $0 \le \vartheta \le 2\pi$.

Coefficients of the fundamental forms of the surface:

$$A = r, \quad F = r(r + a\cos\vartheta),$$
$$B^2 = r^2 + a^2 + 2ra\cos\vartheta + p^2,$$
$$L = -\frac{rp}{\sqrt{p^2 + a^2\sin^2\vartheta}} = M,$$
$$N = -\frac{p(r + a\cos\vartheta)}{\sqrt{p^2 + a^2\sin^2\vartheta}},$$
$$K = \frac{ap^2\cos\vartheta}{r(p^2 + a^2\sin^2\vartheta)^2}.$$

In the points of parabolic lines $K = 0$, in the rest points of the surface we have $K > 0$ or $K < 0$. The coordinate lines ϑ, v are orthogonal only in points defined by condition:

$$r + a\cos\vartheta = 0.$$

In Fig. 1, the right circular helical surface is shown when $a > r$; in Fig. 2, the surface has $a = r$; the surface presented in Fig. 3 has $a < r$. The helical axis of the helical surface shown in Fig. 1 is out of the surface. The straight axis of the surface given in Fig. 2 lies on the surface, but the helical axis of the surface shown in Fig. 3 is within the right circular helical surface. The surfaces shown in Figs. 2 and 3 are also called *helical poles*.

An angle χ between coordinate lines v and ϑ is calculated by the formula:

$$\cos\chi = (r + a\cos\vartheta)/\sqrt{r^2 + a^2 + 2ra\cos\vartheta + p^2}.$$

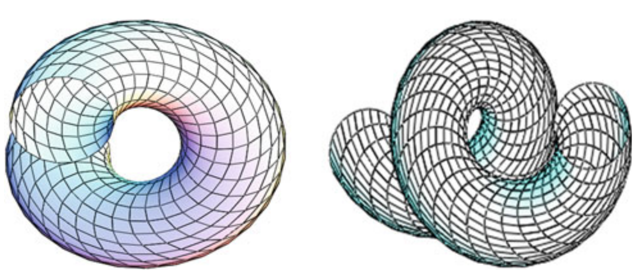


Fig. 1

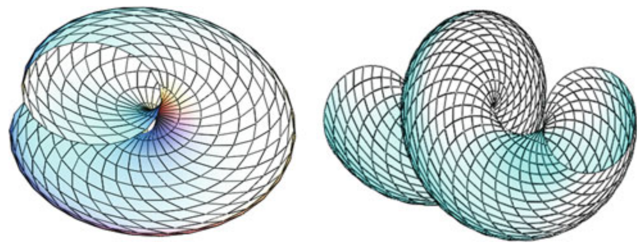


Fig. 2

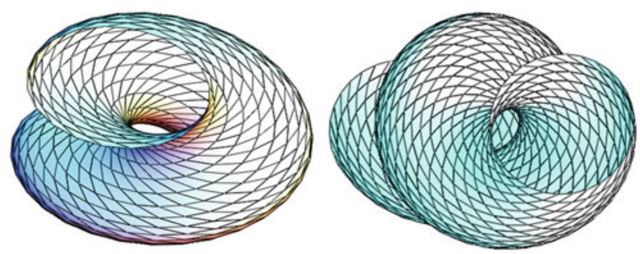


Fig. 3

(2) Parametrical form of the definition:

$$x = x(\rho, \varphi) = \rho\cos\varphi,$$
$$y = y(\rho, \varphi) = \rho\sin\varphi,$$
$$z = z(\rho, \varphi) = p(\varphi - \theta)$$

where $\cos\theta = (\rho^2 + a^2 - r^2)/(2ap)$.

The presented parametric equations of a right circular helical surface are used when a generatrix circle is given in polar coordinates ρ, θ:

$$\rho^2 - 2a\,\rho\cos\,\theta + a^2 - r^2 = 0 \text{ or}$$
$$\rho = a\,\cos\,\theta \pm \sqrt{r^2 - a^2\sin^2\theta}.$$

An implicit equation of the axial cross section of a right circular helical surface has the following form:

$$2a\,\cos(z/p) = x + (a^2 - r^2)/x.$$

(3) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(\vartheta, v) = (a + r\cos\vartheta)\boldsymbol{e}(v) + r\sin\vartheta\, g(v) + pv\boldsymbol{k},$$

where $\boldsymbol{e}(v)$, $\boldsymbol{g}(v)$ are the unit circular vector functions; ϑ is a central angle of the generatrix circle; $0 \le \vartheta \le 2\pi$.

■ Circular Helical Surface with Generating Circle Lying in the Osculating Plane of a Helix of Centers of the Circles

A circular helical surface with a generating circle lying in the osculating plane of a helix of centers of the circles (Figs. 1 and 2) falls also into a class of *cyclic surfaces*. *The basic vector* of a generatrix circle is directed along the binormal of the helical line of centers of directrix circles (see also Subsect. "7.1.2. Circular Helical Surfaces").

Parametrical equations of a helical line of the centers of generatrix circles:

$$x = x(v) = a\cos v,$$
$$y = y(v) = a\sin v,$$
$$z = z(v) = pv.$$

Parametrical equations of a studied helical surface are written in the form:

$$x = x(\vartheta, v) = (a + r\cos\vartheta)\cos v - r\sin\vartheta\sin\beta\sin v,$$
$$y = y(\vartheta, v) = (a + r\cos\vartheta)\sin v + r\sin\vartheta\sin\beta\cos v,$$
$$z = z(\vartheta, v) = pv - r\sin\vartheta\cos\beta$$

ϑ is a central angle of a generatrix circle; $0 \le \vartheta \le 2\pi$; r is the radius of a generatrix circle; β is the angle between the binormal of the helical line and the plane $z = 0$;

$$\tan\beta = -\frac{a}{p},\quad \sin\beta = \frac{a}{\sqrt{p^2 + a^2}},\quad \cos\beta = \frac{-p}{\sqrt{p^2 + a^2}},$$
$$\beta = \frac{\pi}{2} + \arctan\frac{p}{a}.$$

Coefficients of the fundamental forms of the surface:

$$A = r,\quad F = r\sqrt{a^2 + p^2}\left(\frac{ar}{a^2 + p^2} + \cos\vartheta\right),$$
$$B^2 = \frac{r^2a^2\sin^2\vartheta}{a^2 + p^2} + (a + r\cos\vartheta)^2 + p^2,$$
$$L = \frac{r^3p\cos\vartheta}{\Delta\sqrt{a^2 + p^2}},$$
$$M = -\frac{r^2p}{\Delta}\left(\sin^2\vartheta - \frac{ra\cos\vartheta}{a^2 + p^2}\right),$$
$$N = -\frac{r^2p}{\Delta\sqrt{a^2 + p^2}}\left[a - 2a\cos^2\vartheta.\right.$$
$$\left.-\frac{ra^2\cos\vartheta + rp^2\cos^3\vartheta}{a^2 + p^2}\right]$$

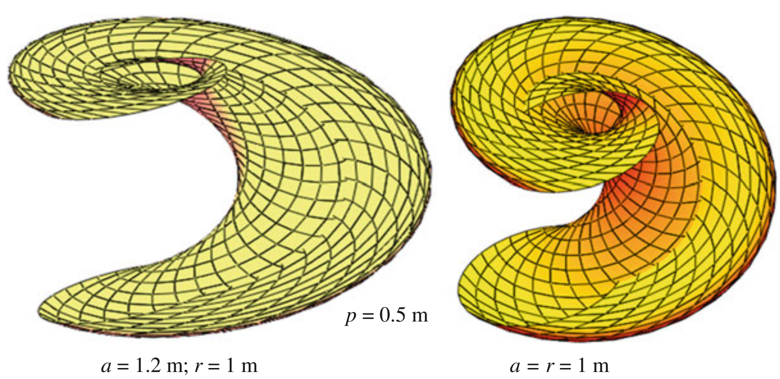


Fig. 1

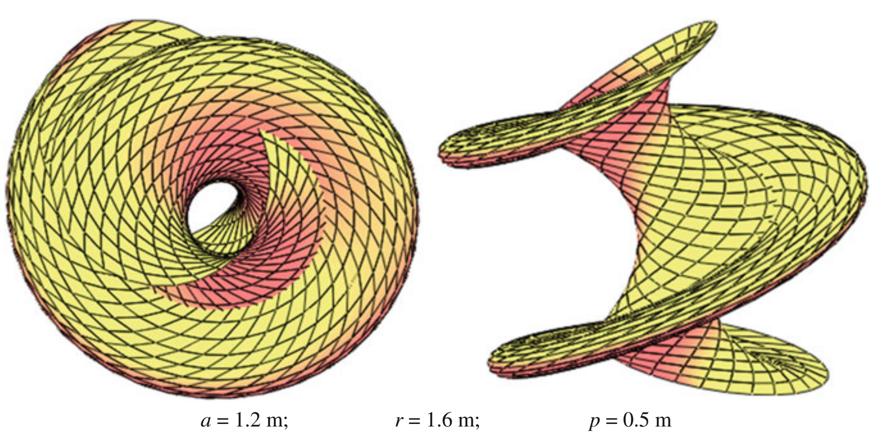


Fig. 2

where

$$\Delta^2 = A^2B^2 - F^2 = r^2\left[\frac{r^2p^2}{a^2+p^2}\cos^2\vartheta + (a^2+p^2)\sin^2\vartheta\right].$$

The coordinate lines ϑ coincide with the generatrix circles. An angle χ between nonorthogonal nonconjugate curvilinear coordinates v, ϑ is obtained by the formula:

$$\cos\chi = \frac{\sqrt{a^2+p^2}\left(\frac{ra}{a^2+p^2}+\cos\vartheta\right)}{\sqrt{\frac{r^2a^2\sin^2\vartheta}{a^2+p^2}+(a+r\cos\vartheta)^2+p^2}}.$$

Additional Literature

Lyukshin VC. Theory of Helical Surfaces for Design of the Cutting Tools. Moscow: Izd-vo "Mashinostroenie", 1968; 372 p. (35 refs).

7.1.3 Ordinary Helical Surfaces with Arbitrary Plane Generatrix Curves

Any surface formed by some curve (*profile*) rotating about an axis and at the same time executing a translation motion along the same axis is called *an ordinary helicoid of general type*. In addition, the translational speed of the profile is proportional to its angular velocity.

■ Dini's Helicoid

Dini's helicoid (*Dini's surface*) is formed by an ordinary helical motion of a tractrix (Fig. 1). If a translational speed and an angular velocity of the profile are arbitrary, then an obtained helical surface is called *a helical surface of a variable* pitch *with a plane generatrix curve in the form of a tractrix.*

If the ratio of a value of the speed of the profile along the surface axis to a value of the angular velocity is constant then the obtained surface is called *a Dini's surface* or *a helical pseudo spherical surface*. It is named after Ulisse Dini, an Italian mathematician of the late nineteenth and early twentieth centuries.

One family of the lines of principal curvatures of a Dini's surface consists of spherical lines. Tractrixes of the surface generate the second family of its lines of curvatures. The surface intersects the plane of the tractrix at the constant angle and in particular, this angle is equal to π/2 for a pseudosphere. *Pseudosphere* is a particular case of Dini's surface.

Parametrical equations of a Dini's surface (Fig. 1) have the following form:

$$x = x(u,v) = a\sin u\cos v,$$
$$y = y(u,v) = a\sin u\sin v,$$
$$z = z(u,v) = a\cos u + a\ln\left(\tan\frac{u}{2}\right) + bv,$$

where u is the angle of the helical axis with the tangent to the tractrix.

Coefficients of the fundamental forms of the surface and its curvatures:

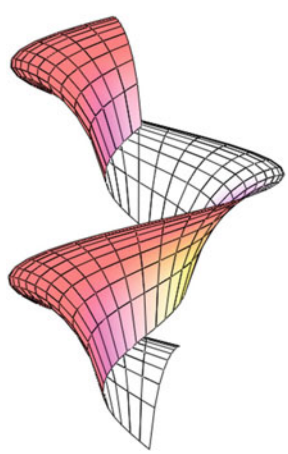


Fig. 1

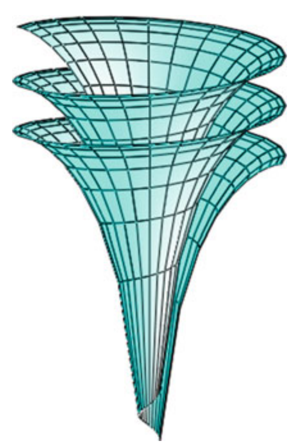


Fig. 2

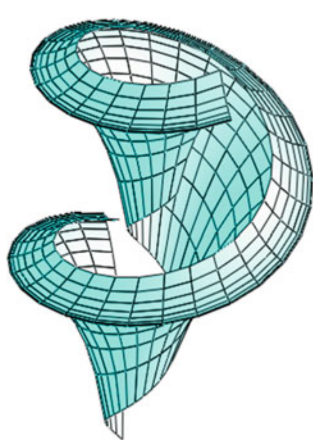


Fig. 3

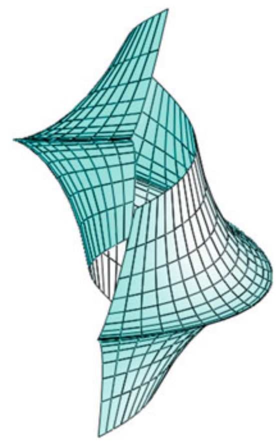


Fig. 4

$$A = a\cot u,\quad F = ab\frac{\cos^2 u}{\sin u},$$
$$B^2 = b^2 + a^2\sin^2 u,$$
$$A^2B^2 - F^2 = a^2(a^2+b^2)\cos^2 u,$$
$$L = -\frac{a^2\cot u}{\sqrt{a^2+b^2}},\quad M = -ab\frac{\cos u}{\sqrt{a^2+b^2}},$$
$$N = \frac{a^2\sin u\cos u}{\sqrt{a^2+b^2}},$$
$$k_u = k_1 = -\frac{\tan u}{\sqrt{a^2+b^2}},$$
$$k_v = \frac{a^2\sin u\cos u}{(b^2+a^2\sin^2 u)\sqrt{a^2+b^2}},$$
$$k_2 = \frac{\cot u}{\sqrt{a^2+b^2}},\quad K = -\frac{1}{a^2+b^2} = \text{const} < 0;$$
$$H = \frac{\cot 2u}{\sqrt{a^2+b^2}} \neq 0.$$

A Dini's surface is related to nonorthogonal nonconjugate curvilinear coordinates u, v. The coordinate lines u coincide with one family of the lines of principal curvatures of the surface. The area of an element of the surface bounded by the coordinate lines can be calculated by the formula:

$$\mathrm{d}s = a\left(a^2+b^2\right)^{1/2}\cos u\,\mathrm{d}u\mathrm{d}v.$$

Dini's surface comes also into a class of *surfaces of constant negative Gaussian curvature*. So, it can also be described as a surface of constant negative curvature created by twisting a pseudosphere.

Changing the geometrical parameters, it is possible to obtain different forms of the surface. Three variants of the Dini's surfaces are presented in Figs. 2, 3 and 4.

Additional Literature

Gray A. Modern Differential Geometry of Curves and Surfaces with Mathematica (2nd ed.). Boca Raton, FL: CRC Press. 1998; 1053 p.

Weisstein, Eric W. Dini's Surface". From MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com/DinisSurface.html

■ Helical Surface with Parabolic Generatrix of Arbitrary Position

A helical surface with a parabolic generatrix of arbitrary position is created by an ordinary helical motion of a parabola $Y(t) = ct^2$ with the Y-axis turned at an angle θ to the helical axis Oz. The vertex of the parabola is within a distance of the helical axis and moves along this axis in proportion to the angular velocity of the parabola (Fig. 1). The pitch of the cylindrical helical lines lying on the surface is equal to $2\pi b$, i.e., $H = 2\pi b$.

Form of the definition of the helical surface

(1) Parametrical equations (Fig. 1):

$$x = x(u,t) = \left(a + t\cos\theta + ct^2\sin\theta\right)\cos u,$$
$$y = y(u,t) = \left(a + t\cos\theta + ct^2\sin\theta\right)\sin u,$$
$$z = z(u,t) = bu - t\sin\theta + ct^2\cos\theta.$$

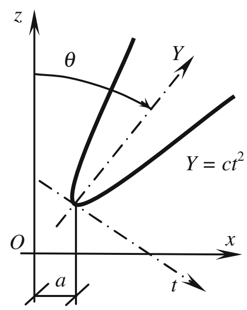


Fig. 1

Coefficients of the fundamental forms of the surface:

$$
\begin{aligned}
A^2 &= \left(a + t\cos\theta + ct^2\sin\theta\right)^2 + b^2;\\
F &= b(2ct\cos\theta - \sin\theta);\\
B^2 &= 1 + 4c^2t^2;\\
L &= \left(a + t\cos\theta + ct^2\sin\theta\right)^2 \frac{2ct\cos\theta - \sin\theta}{\Sigma};\\
M &= -\frac{b}{\Sigma}(\cos\theta + 2ct\sin\theta)^2;\\
N &= -\left(a + t\cos\theta + ct^2\sin\theta\right)\frac{2c}{\Sigma}
\end{aligned}
$$

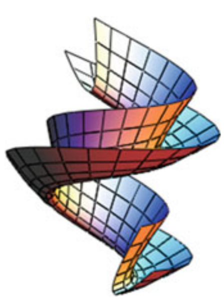
where

$$\Sigma^2 = B^2\left(a + t\cos\theta + ct^2\sin\theta\right)^2 + b^2(\cos\theta + 2ct\sin\theta)^2.$$

In Figs. 2, 3, 4, 5, and 6, five variants of the helical surfaces with the parabolic generatrix of the given position are shown. These surfaces can be of both positive (Fig. 4) and negative (Figs. 2, 3, 5, and 6) Gaussian curvature. Additional information can be taken from Figs. 7, and 8.

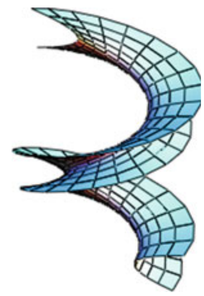
The studied helical surfaces are given in the curvilinear nonorthogonal nonconjugate coordinates u, t.

If $b = 0$, then the helical surfaces degenerate into *surfaces of revolution of parabola of arbitrary position* (see also "Surface of revolution of a parabola of arbitrary



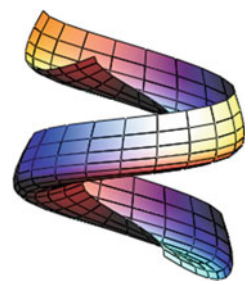
$a = 5$ m; $b = 2$ m;
$c = 1\ \mathrm{m}^{-1}$; $\theta = \pi/5$

Fig. 2



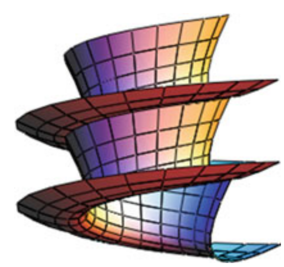
$a = 5$ m; $b = 3$ m;
$c = 1\ \mathrm{m}^{-1}$; $\theta = \pi/2$

Fig. 3



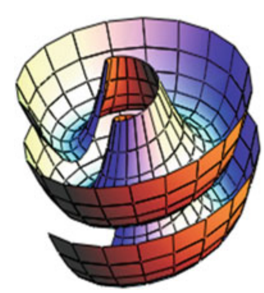
$a = 8$ m; $b = 1.5$ m;
$c = 0.6\ \mathrm{m}^{-1}$; $\theta = -\pi/4$

Fig. 4



$a = 4$ m; $b = 0.8$ m;
$c = 0.6\ \mathrm{m}^{-1}$; $\theta = \pi/4$

Fig. 5



$a = 4$ m; $b = 0.5$ m;
$c = 0.6\ \mathrm{m}^{-1}$; $\theta = 0$

Fig. 6

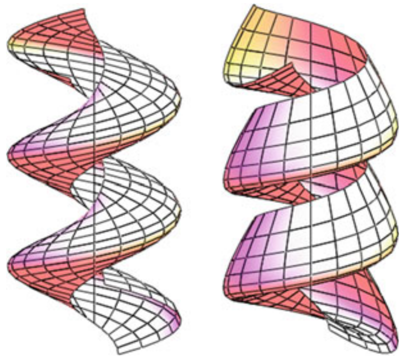


Fig. 7

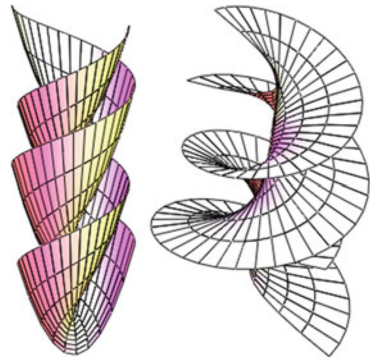


Fig. 8

position"). Having assumed $b = 0$, $\theta = 0$, and $a = 0$, we shall obtain *a paraboloid of revolution*.

Additional Literature

Ivanov VN. Geometry and design of shells on the basis of surfaces with a system of coordinate lines at the planes of pencil. Prostran. Konstruktzii zdaniy i Soor. Moscow: "Devyatka Print", 2004; Iss. 9, p. 26-35. (13 refs.).

■ Helical Surface with Sinusoidal Generatrix

A helical surface with a sinusoidal generatrix is formed by an ordinary helical motion of a sinusoid

$$Y(v) = c\sin(90^\circ + n\pi v/\mathrm{d}) = c\ \cos(n\pi v/\mathrm{d}),\ X(v) = v.$$

A local axis Y crosses the helical axis at an angle θ. The origin of the local system of Cartesian coordinates X, Y is on the helical directrix at the distance of a from the helical axis and moves along this axis proportionally to the angular velocity. The pitch of the helical lines lying on the surface is constant and equal to $2\pi b$, i.e., $H = 2\pi b$.

Form of the definition of the helical surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = \left(a + v\cos\ \theta + c\cos\frac{n\pi v}{d}\sin\theta\right)\cos u,$$
$$y = y(u, v) = \left(a + v\cos\ \theta + c\cos\frac{n\pi v}{d}\sin\theta\right)\sin u,$$
$$z = z(u, v) = bu - v\sin\theta + c\cos\frac{n\pi v}{d}\cos\theta,$$

where n is a number of integer half-waves of a sinusoid located on a length of d.

There is the helical surface with $a \neq 0$, $\theta = 0$ in Fig. 2; in Fig. 3, the surface having $\theta = 0$ and $a = 0$ is shown. In Fig. 4,

Fig. 1

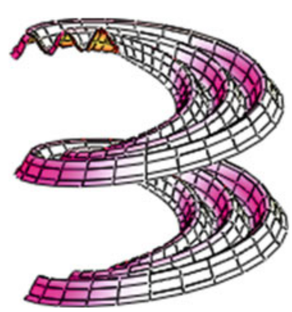


Fig. 2

Fig. 3

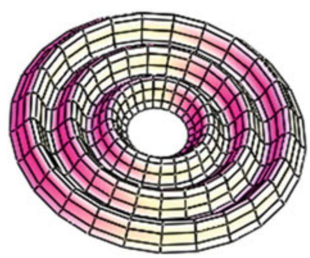


Fig. 4

the surface of revolution of the sinusoid with $\theta = 0$, $b = 0$, and $a \neq 0$ is shown.

A surface of revolution of an arbitrary sinusoid is shown in Fig. 5 (see also Chap. "2. Surfaces of Revolution"). For its forming, it is necessary to take $a = 0$, $\theta = 0$, and $b = 0$.

A surface of revolution of an inclined sinusoid is shown in Fig. 6. Here, $a \neq 0$, $b = 0$, and $\theta \neq 0$.

Having assumed $a \neq 0$, $b \neq 0$, and $\theta = \pi/2$, we can obtain *a helical sinusoidal strip*, Fig. 7 (see also "Helical Sinusoidal Strip" in Subsect. 7.1.3).

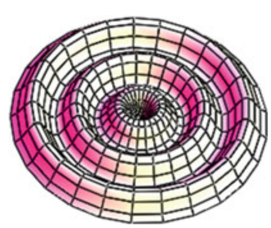


Fig. 5

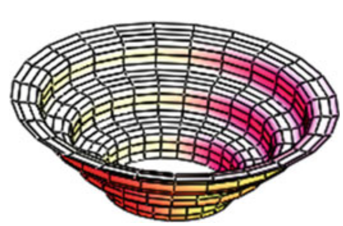


Fig. 6

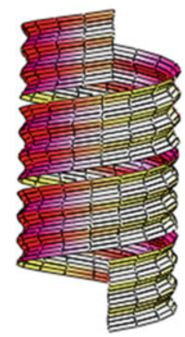


Fig. 7

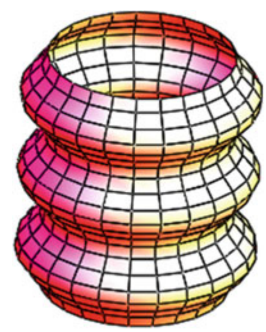


Fig. 8

Using the presented general formulas when $b = 0$, $\theta = \pi/2$, and $a \neq 0$, one may create *a corrugated surface of revolution of a general sinusoid* shown in Fig. 8 (see also Chap. "2. Surfaces of Revolution").

Coefficients of the fundamental forms of the surface:

$$A^2 = [a + v\cos\,\theta + c\cos(kv)\sin\theta]^2 + b^2,$$
$$F = -b[\sin\theta + ck\sin(kv)\cos\theta],$$
$$B^2 = 1 + c^2k^2\sin^2(kv),$$
$$A^2B^2 - F^2 = [a + v\cos\,\theta + c\cos(kv)\sin\theta]^2B^2 + b^2[\cos\theta - ck\sin(kv)\sin\theta]^2$$
$$L = [a + v\cos\theta + c\cos(kv)\sin\theta]^2[\sin\theta + ck\sin(kv)\cos\theta]\Big/\sqrt{A^2B^2 - F^2},$$
$$M = \frac{b}{\sqrt{A^2B^2 - F^2}}[\cos\,\theta - ck\sin(kv)\sin\theta]^2.$$
$$N = \frac{ck^2}{\sqrt{A^2B^2 - F^2}}[a + v\cos\theta + c\cos(kv)\sin\theta]\cos(kv)$$

where $k = n\pi/d$.

The surface is given at nonorthogonal nonconjugate curvilinear coordinates u, v.

Additional Literature

Ivanov VN. Geometry and design of shells on the basis of surfaces with a system of coordinate lines at the planes of pencil. Prostran. Konstruktzii zdaniy i Soor. Moscow: "Devyatka Print", 2004; Iss. 9, p. 26-35. (13 refs.).

■ Helical Surface with Generatrix Ellipse

A helical surface with a generatrix ellipse is formed by an ordinary helical movement of an ellipse

$$x_0(v) = c\cos v, \quad y_0(v) = d\sin v,$$

with the axis y_o intersecting the helical axis Oz of the surface at an angle θ (Fig. 1). The center of the ellipse is within a distance of the helical axis and moves along this axis in proportion to the angular velocity of the generatrix ellipse. The pitch of the helixes laying on the surface is equal to $2\pi b$, i.e.,

$$H = 2\pi b.$$

Parametrical equations of this surface are written in the following form:

$$x = x(u, v) = (a + c\ \cos v \cos\ \theta + d\ \sin v \sin\theta)\cos u,$$
$$y = y(u, v) = (a + c\ \cos v \cos\ \theta + d\ \sin v \sin\theta)\sin u,$$
$$z = z(u, v) = bu - c\ \cos v \sin\ \theta + d \sin v \cos\ \theta.$$

In Figs. 2, 3, and 4, the helical surfaces with the generatrix ellipse having the following geometric parameters $d > c$, $a \neq 0$, $b \neq 0$ are shown, but the surface in Fig. 2 has the angle θ between the local axis y_o and the helical axis Oz equal to $\pi/4$; in Fig. 3, $\theta = \pi/2$; in Fig. 4, $\theta = 0$.

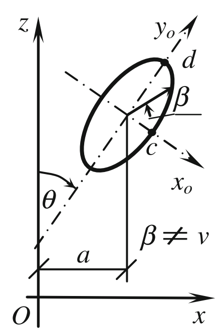


Fig. 1

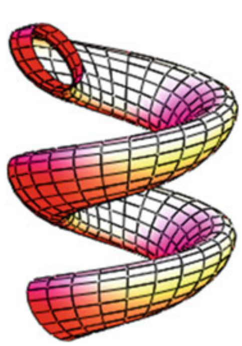


Fig. 2

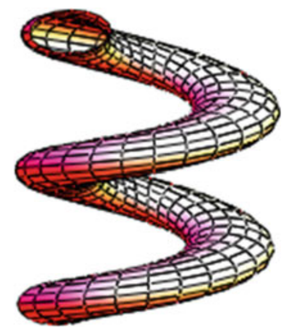


Fig. 3

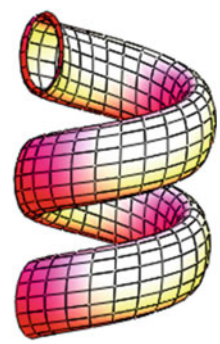


Fig. 4

The helical surface with a generatrix ellipse shown in Fig. 5 is formed when $d > c$, $a = d$, $b \neq 0$, and $\theta = \pi/2$. All generatrix ellipses in this case touch the helical axis of the surface. So, the helical axis lies on the surface.

The helical surface presented in Fig. 6 has $d > c$, $a < d$, $b \neq 0$, and $\theta = \pi/4$.

Assume $d > c$, $a \neq 0$, $b = 0$, and $\theta = 0$, then we shall create *an elliptical torus*, Fig. 7. If $d = c$, then *a helical surface with a generatrix ellipse* degenerates into *a circular helical surface with a generatrix circle lying in the plane passing through the helical axis* (see also Subsect. "7.1.2. Circular Helical Surfaces" and "Circular Helical Surface with Generatrix Circles in the Planes of Pencil").

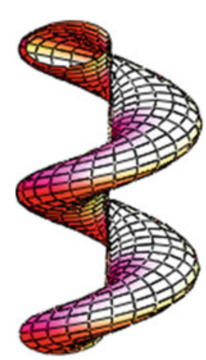


Fig. 5

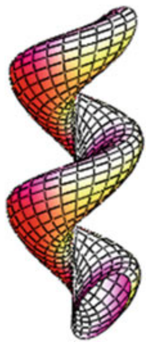


Fig. 6

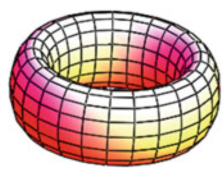


Fig. 7

A circular torus can be formed if $d = c$ and $b = 0$.

Coefficients of the fundamental forms of the surface:

$$A^2 = (a + c\cos v\cos\,\theta + \mathrm{d}\sin v\sin\theta)^2 + b^2,$$
$$F = b(c\sin v\sin\theta + \mathrm{d}\cos v\cos\,\theta),$$
$$B^2 = c^2\sin^2 v + \mathrm{d}^2\cos^2 v,$$
$$A^2B^2 - F^2 = (a + c\cos v\cos\,\theta + \mathrm{d}\sin v\sin\theta)^2B^2 + b^2(c\sin v\cos\,\theta - \mathrm{d}\cos v\sin\theta)^2,$$
$$L = -\frac{F}{b\sqrt{A^2B^2 - F^2}}(a + c\cos v\cos\theta + \mathrm{d}\sin v\sin\,\theta)^2,$$
$$M = \frac{b}{\sqrt{A^2B^2 - F^2}}(c\sin v\cos\theta - \mathrm{d}\cos v\sin\theta)^2,$$
$$N = -\frac{cd}{\sqrt{A^2B^2 - F^2}}(a + c\cos\theta + \mathrm{d}\sin v\sin\theta)$$
$$K = \Big[\mathrm{cd}(a + c\cos v\cos\theta + \mathrm{d}\sin v\sin\theta)^3(c\sin v\sin\theta + \mathrm{d}\cos v\cos\theta), - b^2(c\sin v\cos\theta - \mathrm{d}\sin\theta)^4\Big]/\left(A^2B^2 - F^2\right)^2.$$

The surface is given in nonorthogonal nonconjugate curvilinear coordinates u, v. It contains areas with positive and negative Gaussian curvature.

■ Helical Sinusoidal Strip

A helical sinusoidal strip is formed by an ordinary helical motion of a sinusoid

$$Y(v) = c\sin(90^\circ + n\pi v/d) = c\cos(n\pi v/d), \quad X(v) = v.$$

A local axis Y intersects the helical axis at an angle $\pi/2$ and an axis X is parallel to the helical axis. The origin of the local system of Cartesian coordinates X, Y is placed on a helical directrix at the distance of a from the helical axis and moves along this axis in proportion to the angular velocity of the generatrix sinusoid. The pitch of the helical lines lying on the surface is constant and equal to $2\pi b$, i.e.,

$$H = 2\pi b.$$

Parametrical equations of a helical sinusoidal strip (Fig. 1) are

$$x = x(u,v) = (a + c\cos\frac{n\pi v}{d})\cos u,$$
$$y = y(u,v) = (a + c\cos\frac{n\pi v}{d})\sin u,$$
$$z = z(u,v) = bu - v,$$

where n is a number of integer half-waves of a sinusoid located on a length of d; c is the amplitude of the sinusoid; $-\infty \le u \le \infty$; $0 \le v \le d$; $d \le 2\pi b$.

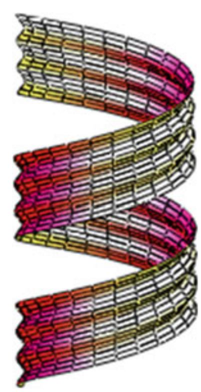


Fig. 1

Taking $d = 2\pi b$, it is possible to obtain the continuous helical surface without breaks between the strips with a d width. If $d > 2\pi b$, then the strips with a d width are placed overlapping each other. A helical sinusoidal strip degenerates into *a cylindrical helical strip* if $c = 0$ (see also Subsect. "1.1.2. Cylindrical Surfaces").

Coefficients of the fundamental forms of the surface:

$$A^2 = [a + c\cos(kv)]^2 + b^2, \quad F = -b,$$
$$B^2 = 1 + c^2k^2\sin^2(kv)$$
$$A^2B^2 - F^2 = [a + c\cos(kv)]^2\left[1 + c^2k^2\sin^2(kv)\right] + b^2c^2k^2\sin^2(kv)$$

$$L = \frac{[a + c\cos(kv)]^2}{\sqrt{A^2B^2 - F^2}}, \quad N = \frac{ck^2\cos(kv)}{\sqrt{A^2B^2 - F^2}}[a + c\cos(kv)], \quad \text{where} \quad k = n\pi/d,$$

$$M = \frac{bc^2k^2\sin^2(kv)}{\sqrt{A^2B^2 - F^2}}, \quad K = ck^2\frac{[a + c\cos(kv)]^3\cos(kv) - b^2c^3k^2\sin^4(kv)}{\left\{[a + c\cos(kv)]^2\left[1 + c^2k^2\sin^2(kv)\right] + b^2c^2k^2\sin^2(kv)\right\}^2}$$

■ Helical Surface with Generatrix Curve in the form of Evolvent of the Circle

A helical surface with generatrix curve in the form of an evolvent of the circle (Fig. 1) has the following parametrical equations:

$$x = x(u,v) = [a + x_0(v)\cos\vartheta + y_0(v)\sin\vartheta]\cos u,$$
$$y = y(u,v) = [a + x_0(v)\cos\vartheta + y_0(v)\sin\vartheta]\sin u,$$
$$z = z(u,v) = bu - x_0(v)\sin\vartheta + y_0(v)\cos\vartheta,$$

where $x_0(v) = c(\cos v + v\sin v), \quad y_0(v) = c(\sin v - v\cos v)$.

A local axis y_o intersects a helical axis at an angle θ. The origin of the local system of Cartesian coordinates x_o, y_o is placed on a helical directrix within a distance of the helical axis and moves along this axis in proportion to the angular velocity of the generatrix evolvent. The pitch of the helical lines lying on the surface is constant and equal to $2\pi b$. If $\theta = b = 0$, then a helical surface degenerates into a surface of revolution of the evolvent of the circle (Fig. 2).

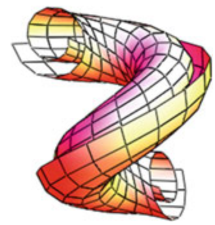


Fig. 1

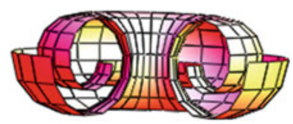


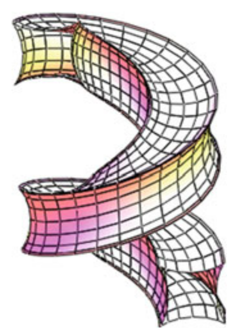
Fig. 2

■ Astroidal Helicoid

A helical surface with a generatrix curve in the form of an astroid (Fig. 1) is given by the following parametrical equations:

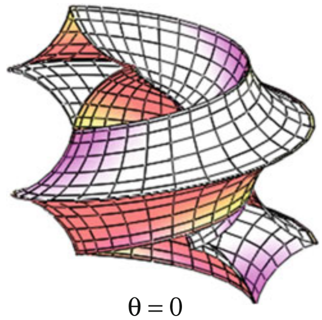
$$x = x(u,v) = [a + x_0(v)\cos\vartheta + y_0(v)\sin\vartheta]\cos u;$$
$$y = y(u,v) = [a + x_0(v)\cos\vartheta + y_0(v)\sin\vartheta]\sin u;$$
$$z = z(u,v) = bu - x_0(v)\sin\vartheta + y_0(v)\cos\vartheta,$$

where $x_0(v) = c\cos^3 v$, $y_0(v) = c\sin^3 v$, c is a radius of a rolling circle with a point forming the astroid.



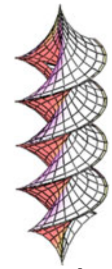
$\theta = \pi/4$

Fig. 1



$\theta = 0$

Fig. 2



$a = 0$

Fig. 3

A local axis y_o intersects a helical axis at an angle θ. The origin of the local system of Cartesian coordinates x_o, y_o is located on the helical directrix within the a distance of the helical axis and moves along this axis in proportion to the angular velocity. If $\theta = 0$ and $b = c/\pi$, then the coordinate lines $v = 0$ and $v = \pi$ touch each other (Fig. 2). An interesting surface shown in Fig. 3 is formed when $\theta = 0$ and $a = 0$.

■ Helical Surface with Generatrix Cycloid

A helical surface with a generatrix ordinary cycloid can be given by the following parametric equations:

$$x = x(u, v) = [a + x_0(v)\cos\vartheta + y_0(v)\sin\vartheta]\cos u;$$
$$y = y(u, v) = [a + x_0(v)\cos\vartheta + y_0(v)\sin\vartheta]\sin u;$$
$$z = z(u, v) = bu - x_0(v)\sin\vartheta + y_0(v)\cos\vartheta,$$

where $x_o(v) = c(v - \sin v)$, and $y_o(v) = c(1 - \cos v)$ are parametrical equations of a cycloid given in the local system of Cartesian coordinates x_o, y_o. The local axis y_o intersects the helical axis at an angle θ. The origin of the local system of Cartesian coordinates x_o, y_o is on the helical directrix at the distance of a from the helical axis and moves along this axis proportionally to the angular velocity. The pitch of the helixes lying on the surface is constant and equal to $2\pi b$ ($H = 2\pi b$).

In Fig. 1, the helical surface has $\theta = 0$. If $\theta = -\pi/2$, $b \neq 0$, $a \neq 0$, then we obtain the helical surface shown in Fig. 2. In

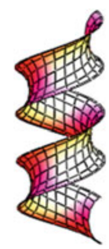


Fig. 2

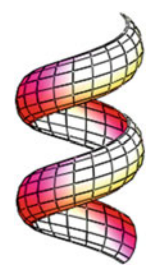


Fig. 3

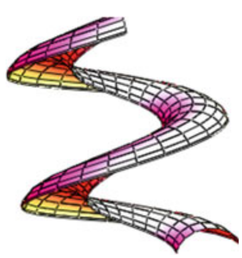


Fig. 1

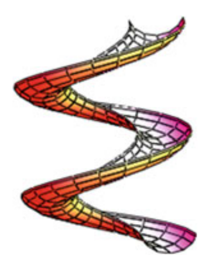


Fig. 4

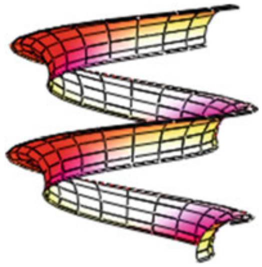


Fig. 5

Fig. 6

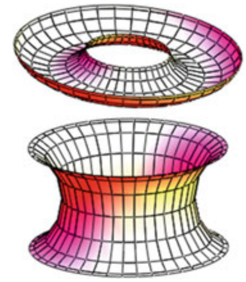


Fig. 7

Fig. 3, the helical surface is shown when $\theta = \pi/2$, $b \neq 0$, $a \neq 0$; but in Fig. 4, the surface has $\theta = \pi$. The helical surface with the inclined generatrix cycloid is given in Fig. 5, where it is assumed that $\theta = -\pi/4$, $b \neq 0$, $a \neq 0$. The helical surface with $a = 0$, $\theta = -\pi/4$, and $b \neq 0$ is represented in Fig. 6. Taking $b = 0$, we obtain a surface of revolution of cycloid. For example, in Fig. 7, the surfaces of revolution of the cycloid is shown when $\theta = \pi$ and when $\theta = -\pi/2$.

■ Helical Surface with Generatrix Curve in the Form of Hyperbola

A helical surface with generatrix curve in the form of a hyperbola is given by the following parametrical equations:

$$x = x(u,v) = [a + x_0(v)\cos\vartheta + y_0(v)\sin\vartheta]\cos u;$$
$$y = y(u,v) = [a + x_0(v)\cos\vartheta + y_0(v)\sin\vartheta]\sin u;$$
$$z = z(u,v) = bu - x_0(v)\sin\vartheta + y_0(v)\cos\vartheta,$$

where

$$x_0(v) = c\,\mathrm{ch}v, \quad y_0(v) = d\,\mathrm{sh}v$$

or

$$x_0(v) = v, \quad y_0(v) = e/v.$$

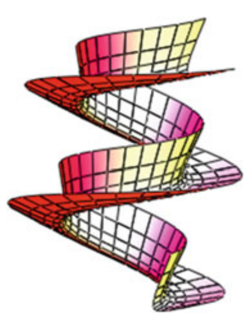


Fig. 1

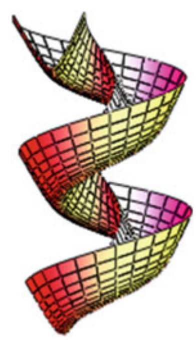


Fig. 2

The local axis y_o intersects the helical axis at an angle θ. The origin of the local system of Cartesian coordinates x_o, y_o is on the helical directrix at the distance of a from the helical axis and moves along this axis proportionally to the angular velocity of the hyperbola. The pitch of the helixes lying on the surface is constant and equal to $2\pi b$.

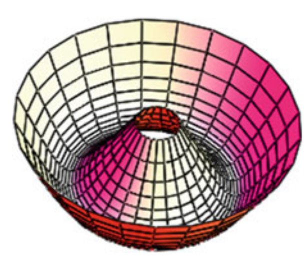


Fig. 3

Fig. 4

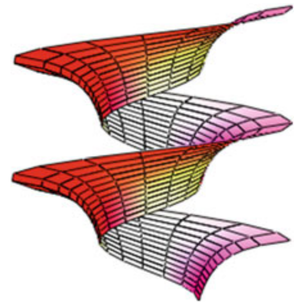


Fig. 5

The helical surface when $\theta = -\pi/4$ is shown in Fig. 1; in Fig. 2, it is presented when $\theta = -\pi/2$.

If $\theta = -\pi/2$ and $b = 0$, then the helical surface (Fig. 2) degenerates into *a surface of revolution of the hyperbola* (Fig. 3).

The helical surfaces given in Figs. 4 and 5 have $\theta = 0$ and $\theta = -\pi/2$ correspondingly; $v > 0$. Here, the generatrix hyperbola is taken as $y_o = e/x_o$.

The surface is given in nonorthogonal nonconjugate curvilinear coordinates u, v.

7.2 Helical Surfaces of Variable Pitch

A helical line l of variable pitch may be given by parametrical equations

$$x = x(v) = a\cos v, \quad y = y(v) = a\sin v, \quad z = z(v) = f(v).$$

A cylinder with a radius a on which the helical line l lies, is called *a basic cylinder*. When $f(v) = pv$, we have *an ordinary helical line*; if $f(v) = h \sin v$, it is the case of Mannheim–Darboux; if $f(v) = h \sin nv$ $(n \neq 1)$, it is the case of Kautny. An orthogonal projection of a helical line of variable pitch on a plane $z = 0$ is the circle of a radius a. The curvature k and the torsion κ of a helical line are calculated by formulas:

$$k = \frac{a\sqrt{a^2 + f'^2 + f''^2}}{(a^2 + f'^2)^{3/2}},$$
$$\kappa = \frac{(f' + f''')}{(a^2 + f'^2 + f''^2)}$$

A slope angle φ of a helical line l is defined as

$$\tan\varphi = f'(v)/a.$$

A helical motion of variable pitch of any curve forms *a helical surface of variable pitch*. Assume that a generatrix curve

$$X = f_1(t), \quad Y = f_2(t), \quad Z = f_3(t)$$

is given in mobile system of Cartesian coordinates $O'XYZ$. A point O' is located on fixed coordinate axis Oz and the axis Oz coincides with the mobile axis $O'Z$. In this case, parametric equations of a helical surface of variable pitch are written as:

$$\begin{aligned} x &= x(t, v) = f_1(t)\cos v - f_2(t)\sin v, \\ y &= y(t, v) = f_1(t)\sin v + f_2(t)\cos v \\ z &= z(t, v) = f(v) + f_3(t). \end{aligned}$$

The coordinate line t is a generatrix curve, the lines v are helical lines of variable pitch lying on a circular cylinder with radius equal to

$$\sqrt{f_1^2(t) + f_2^2(t)}.$$

The coefficients of the first fundamental form of the surface:

$$\begin{aligned} E &= A^2 = f_1'^2(t) + f_2'^2(t) + f_3'^2(t) \\ F &= -f_2(t)f_1'(t) + f_1(t)f_2'(t) + f'(v)f_3'(t) \\ G &= B^2 = f_1^2(t) + f_2^2(t) + f'^2(v), \end{aligned}$$

where …′ means the differentiation with respect to t.

The coefficients of the second fundamental form of the surface:

$$L = \frac{1}{\sqrt{A^2B^2 - F^2}} \begin{vmatrix} f_1'' & f_2'' & f_3'' \\ f_1' & f_2' & f_3' \\ -f_2 & f_1 & f' \end{vmatrix},$$

$$M = \frac{1}{\sqrt{A^2B^2 - F^2}} \begin{vmatrix} f_1'' & -f_2'' & 0 \\ f_1' & f_2' & f_3' \\ -f_2 & f_1 & f' \end{vmatrix},$$

$$N = \frac{1}{\sqrt{A^2B^2 - F^2}} \begin{vmatrix} -f_1 & -f_2 & f'' \\ f_1' & f_2' & f_3' \\ -f_2 & f_1 & f' \end{vmatrix}.$$

Ruled helical surfaces of variable pitch are created by helical motion of a straight line given by parametrical equations:

$$X = a, \quad Y = t\sin\gamma, \quad Z = t\cos\gamma.$$

Open ruled helical surfaces are determined by equations:

$$\begin{aligned} x &= x(t,v) = a\cos v - t\sin\gamma\sin v, \\ y &= y(t,v) = a\sin v + t\sin\gamma\cos v, \\ z &= z(t,v) = f(v) + t\cos\gamma \end{aligned}$$

or

$$\boldsymbol{r} = \boldsymbol{r}(t,v) = a\boldsymbol{e}(v) + t\sin\gamma\boldsymbol{g}(v) + \boldsymbol{k}[f(v) + t\cos\gamma].$$

Intersecting a ruled helical surface by a plane $z = 0$, one can obtain the end section of a ruled helical surface:

$$t = -f(v)/\cos\gamma.$$

Open right ruled helical surface of variable pitch is given by the same equations but it is necessary to take $\gamma = 90°$. *Closed ruled helical surface of variable pitch* can be constructed when $a = 0$. Assume the notation $t\sin\gamma = u$, then the parametrical equations of a closed ruled surface are simplified:

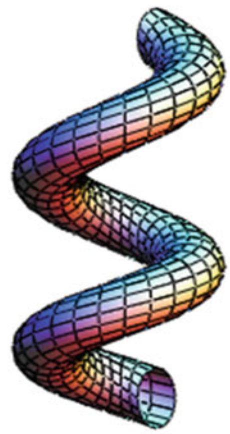
$$\begin{aligned} &x = x(u,v) = u\cos v, \quad y = y(u,v) = u\sin v, \\ &z = z(u,v) = u\operatorname{ctg}\gamma + f(v). \end{aligned}$$

The coefficients of the fundamental forms of the ruled helical surfaces are

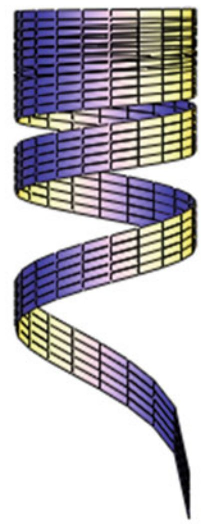
$$\begin{aligned} &A = 1, \quad F = a\sin\gamma + f'(v)\cos\gamma, \\ &B^2 = a^2 + f'^2(v) + t^2\sin^2\gamma, \\ &A^2B^2 - F^2 = \sigma^2\sin^2\gamma, \\ &\sigma^2 = [f'(v) - a\cot\gamma]^2 + t^2, \\ &L = 0, \quad M = \frac{\sin\gamma[a\cot\gamma - f'(v)]}{\sigma}, \\ &N = \left\{a[a\cot\gamma - f'(v)] + t^2\sin\gamma\cos\gamma + t\sin\gamma f''(v)\right\}/\sigma \end{aligned}$$

■ **Helical Surfaces of Variable Pitch Presented in the Encyclopedia**

The right helicoidal surface with the variable pitch

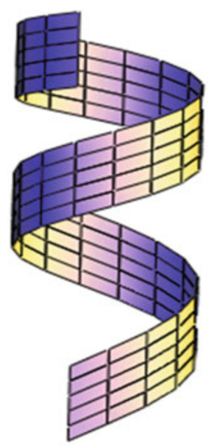


The cyclic helical surface with given slope angles of tangents at the beginning and at the end of the cylindrical helical line of centers of variable pitch

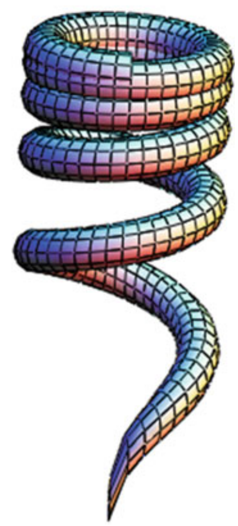


The cylindrical helical strip of the variable pitch

The helical surface of the variable pitch with arbitrary generatrix straight line



The cylindrical helical strip with the given slope angles of tangents at the beginning and at the end of the generatrix cylindrical helical line of centers of the variable pitch



The cyclic helical surface with the line of centers of the variable pitch

■ Cyclic Helical Surface with Given Slope Angles of Tangents at the Beginning and at the End of a Cylindrical Helical Line of Centers of Variable Pitch

A cylindrical helical line of variable pitch having parametrical equations

$$x(\varphi) = r\cos\varphi, \quad y(\varphi) = r\sin\varphi,$$
$$z = r\varphi\left(\tan\alpha_1 + \frac{\tan\alpha_2 - \tan\alpha_1}{2\varphi_k}\varphi\right)$$

is used for practical purposes as a line of centers of *a cyclic surface with circles in the planes of pencil*. Here, r is the radius of a cylinder on which the helical line of the centers is disposed; α_1 is the given slope angle of the tangent with the horizontal plane xOy at the beginning of the helical line of the centers, i.e., when $\varphi = 0$; α_2 is the given slope angle of the tangent at the end of the helical line of the centers, i.e., when $\varphi = \varphi_k$; $0 \le \varphi \le \varphi_k$. In Fig. 1, the development of the cylindrical surface with the radius r with three possible lines of centers is constructed: (1) the line 1 is formed when $\alpha_1 < \alpha_2$; (2) the line 2 is formed when $\alpha_1 > \alpha_2$; and (3) the line 3 is a line of constant slope with the slope angle equal to α_3, $\tan\alpha_3 = (\tan\alpha_1 + \tan\alpha_2)/2$.

Parametrical equations

$$x = x(\varphi,\beta) = (r + R\cos\beta)\cos\varphi,$$
$$y = y(\varphi,\beta) = (r + R\cos\beta)\sin\varphi,$$
$$z = z(\varphi,\beta) = r\varphi\left(\tan\alpha_1 + \frac{\tan\alpha_2 - \tan\alpha_1}{2\varphi_k}\varphi\right) + R\sin\beta$$

define *a cyclic surface with a given slope angles of tangents at the beginning* (α_1) *and at the end* (α_2) *of a cylindrical helical line of centers of variable pitch*. Here, φ is the angle counted off from the axis Ox in the direction of the axis Oy; $0 \le \phi \le \phi_k$; β is the angle in the plane of the generatrix circle counted off from the plane xOy in the direction of the axis Oz; R is the radius of the generatrix circles lying in the planes of pencil passing through the coordinate axis Oz. In Fig. 2, the cyclic helical surface having $\varphi_k = 4\pi$; $r = 1$ m; $\alpha_1 = 20°$; $\alpha_2 = 30°$; and $R = 0.5$ m is shown; $0 \le \beta \le 2\pi$. In this case, the cylindrical helical line of the variable pitch denoted by 1 (Fig. 1) is assumed as a directrix line.

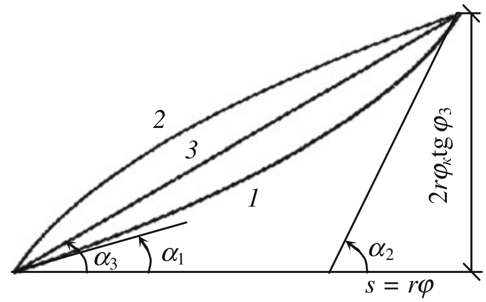


Fig. 1

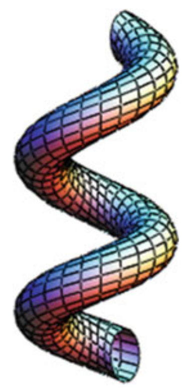


Fig. 2

■ Cylindrical Helical Strip of Variable Pitch

A cylindrical helical strip of variable pitch has a constant width along the straight generatrixes of a right circular cylinder where it is placed. The slope angles α of the tangents to a directrix line of the helical strip with the horizontal coordinate plane xOy change in the limits of $0 \le \alpha \le \pi/2$.

A directrix cylindrical line of a variable pitch

$$x(u) = a\cos(m\pi u),$$
$$y(u) = a\sin(m\pi u),$$
$$z(u) = b\sqrt{1 - (1-u)^2}$$

lies on the circular cylindrical surface of the radius a (Fig. 1). Parametric equations of the directrix line contain the constant geometrical parameter $m/2$, that is, a number of turns on the length of $0 \le z \le b$, when $0 \le u \le 1$.

Parametrical equations of a cylindrical helical strip of variable pitch (Fig. 2) are:

$$x = x(u) = a\cos m\pi u,$$
$$y = y(u) = a\sin m\pi u,$$
$$z = z(u,v) = b\sqrt{1-(1-u)^2} + v$$

where the line $v = 0$ coincides with the directrix of the cylindrical helical line of variable pitch.

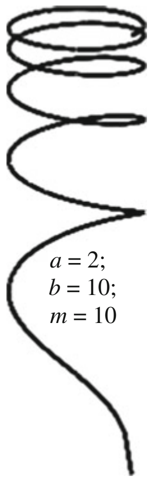


Fig. 1

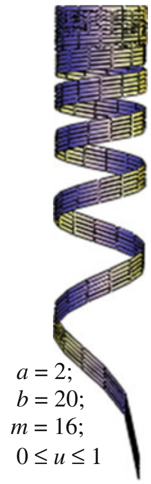


Fig. 2

The tangent lines to the coordinate lines v (Fig. 1) intersect the horizontal coordinate plane xOy at an angle

$$\alpha = 90^\circ - \varphi$$

where

$$\tan\,\varphi = \frac{am\pi}{b(1-u)}\sqrt{u(2-u)}.$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = m^2a^2\pi^2 + \frac{b^2(1-u)^2}{1-(1-u)^2},$$
$$F = \frac{b(1-u)}{\sqrt{1-(1-u)^2}},\quad B = 1,$$
$$A^2B^2 - F^2 = m^2a^2\pi^2,\quad L = -m^2a\pi^2,$$
$$M = N = K = 0,$$
$$k_1 = \frac{1}{a},\quad k_u = \frac{L}{A^2}\,k_2 = k_v = 0.$$

■ Right Helicoidal Surface with Variable Pitch

Rectilinear generatrixes of a *right helicoidal surface with a variable pitch* are parallel to a plane that is perpendicular to an axis Oz of the surface and pass through a cylindrical helical line of variable pitch

$$x(u) = a\,\cos m\pi u,$$
$$y(u) = a\,\sin m\pi u,$$
$$z(u) = b\sqrt{1-(1-u)^2}$$

and through the axis Oz of the surface. The angles α of the tangents to the directrix helical line with the horizontal coordinate plane xOy vary in the limits of $0 \le \alpha \le \pi/2$ (Fig. 1 in the previous section).

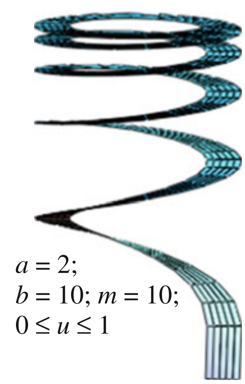


Fig. 1

Parametrical equations of a right helicoidal surface with the variable pitch (Figs. 1, 2, and 3) are:

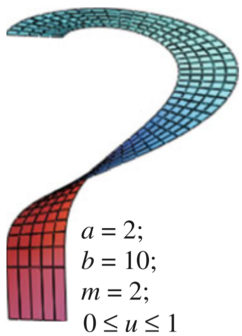


Fig. 2

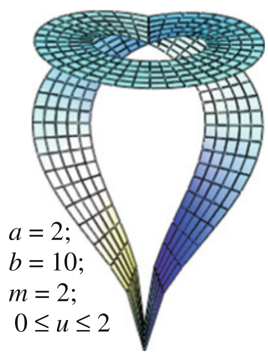


Fig. 3

$$x = x(u) = (a + v)\cos m\pi u,$$
$$y = y(u) = (a + v)\sin m\pi u$$
$$z = z(u, v) = b\sqrt{1 - (1 - u)^2}$$

where $m/2$ is a number of turns on the length of $0 \le z \le b$ when $0 \le u \le 1$, $-a \le v \le \infty$ (Figs. 1 and 2).

Coefficients of the fundamental forms of the surface:

$$A^2 = m^2(a + v)^2\pi^2 + \frac{b^2(1 - u)^2}{[1 - (1 - u)^2]},$$
$$F = 0, \quad B = 1,$$
$$L = \frac{bm\pi(a + v)}{A[1 - (1 - u)^2]^{3/2}},$$
$$M = \frac{mb\pi(1 - u)}{A[1 - (1 - u)^2]^{1/2}}, \quad N = 0$$
$$K = -\frac{m^2b^2\pi^2(1 - u)^2}{A^4[1 - (1 - u)^2]} \le 0,$$
$$H = \frac{bm\pi(a + v)}{2A^3[1 - (1 - u)^2]^{3/2}} \ne 0.$$

■ Cyclic Helical Surface with a Line of Centers of Variable Pitch

A cyclic helical surface with a line of centers of a variable pitch is formed by a circle of a constant radius, the center of which moves along the helical line of the centers L of the variable pitch (Fig. 1)

$$x(u) = a\cos m\pi u, \; y(u) = a\sin m\pi u,$$
$$z(u) = b\sqrt{1 - (1 - u)^2},$$

in the process of the helical motion.

A generatrix circle all the time is in the planes of pencil with the fixed straight line coinciding with the axis of the helical line Oz that is a coordinate axis Oz. The cylindrical helical line of the centers L lies at the circular cylindrical surface of the radius a; $m/2$ is a number of its turns on the length of $0 \le z \le b$, which means $0 \le u \le 1$.

The tangent straight lines to the cylindrical helical line L of the variable pitch cross the coordinate plane xOy at the angle

$$\alpha = 90° - \varphi$$

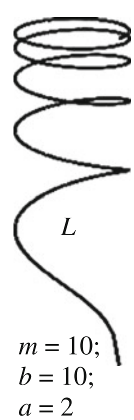


Fig. 1

where

$$\tan\varphi = \frac{am\pi\sqrt{u(2 - u)}}{b(1 - u)}.$$

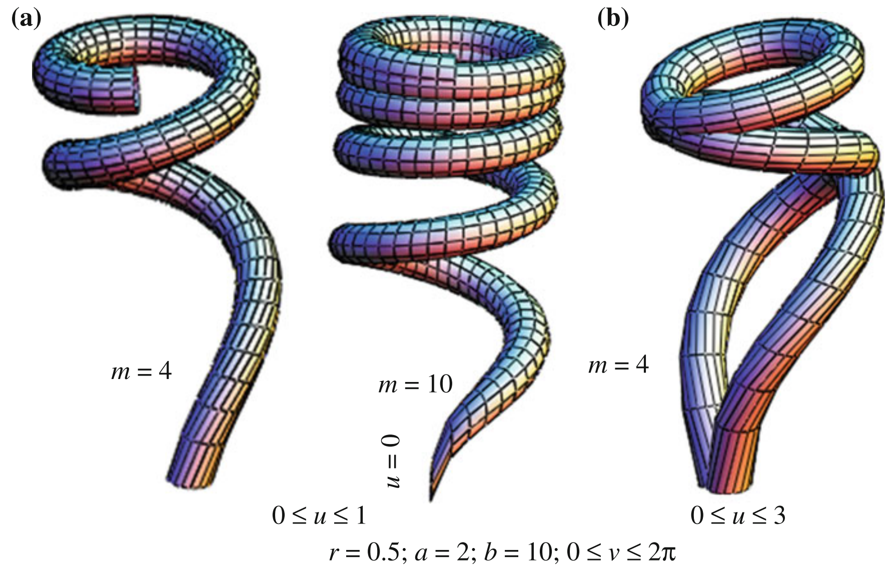


Fig. 2

The angle α changes from $\alpha = 0$ when $u = 1$ ($z = b$) till $\alpha = \pi/2$ when $u = 0$ ($z = 0$). The studied surface can be related both to a class of *helical surfaces* and to a class of *cyclic surfaces* to a group of *cyclic surfaces with circles at the planes of pencil*.

Parametrical equations of this type of helical surfaces (Fig. 2) can be written as

$$x = x(u, v) = (a + r\cos v)\cos m\pi u,$$
$$y = y(u, v) = (a + r\cos v)\sin m\pi u,$$
$$z = z(u, v) = b\sqrt{u(2-u)} + r\sin v,$$

where $r = \text{const}$ is a constant radius of a generatrix circle. In Fig. 2a, the cyclic surface with a geometrical parameter u changing in the limits of $0 \le u \le 1$ is shown. If one changes the geometrical parameter u within the limits $0 \le u \le 2$, then the surface will be closed (Fig. 2b). The remaining geometrical parameters of the surfaces presented in Fig. 2a, b are given for every figure personally.

Coefficients of the fundamental forms of the surface:

$$A^2 = m^2\pi^2(a + r\cos v)^2 + b^2\frac{(1-u)^2}{u(2-u)},$$
$$F = \frac{rb(1-u)\cos v}{\sqrt{u(2-u)}}, \quad B = r,$$
$$A^2B^2 - F^2 = m^2\pi^2r^2(a + r\cos v)^2 + \frac{b^2r^2(1-u)^2\sin^2 v}{u(2-u)};$$
$$L = \frac{-rm\pi(a + r\cos v)}{\sqrt{A^2B^2 - F^2}}\left\{m^2\pi^2(a + r\cos v)\cos v + \frac{b\sin v}{[u(2-u)]^{3/2}}\right\},$$
$$M = \frac{r^2m\pi b(1-u)\sin^2 v}{\sqrt{A^2B^2 - F^2}\sqrt{u(2-u)}},$$
$$N = \frac{-r^2m\pi(a + r\cos v)}{\sqrt{A^2B^2 - F^2}}.$$

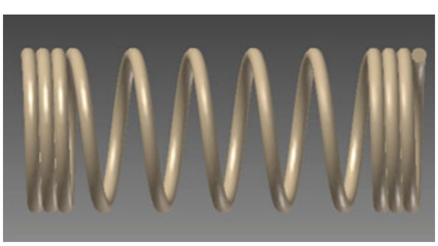


Fig. 3 The cyclic helical spring with the line of centers of the variable pitch [http://forums.autodesk.com/]

A cyclic helical surface with the line of centers of the variable pitch is given in curvilinear nonorthogonal non-conjugate coordinates u, v. The coordinate lines v coincide with the generatrix circles. The end circle $u = 1$ is a line of the principle curvature. The surface contains the segments of positive and negative Gaussian curvature.

Other forms of cyclic helical surfaces are known with the lines of centers of variable pitch (Fig. 3).

■ Cylindrical Helical Strip with Given Slope Angles of Tangents at the Beginning and at the End of the Directrix Cylindrical Helical Line of Centers of Variable Pitch

Helical surfaces of variable pitch may be seen in threads of different types, in the form of helical conveyers and worm presses where comparatively smooth changing of pitch is used.

For practical aims, a directrixcylindrical helical line of a variable pitch having parametrical equations in the form

$$x(\varphi) = r\cos\varphi, \quad y(\varphi) = r\sin\varphi,$$

$z = r\varphi\left(\tan\alpha_1 + \frac{\tan\alpha_2 - \tan\alpha_1}{2\phi_k}\varphi\right)$ is used. Here r is a radius of the cylinder with the helical line lying on it; α_1 is the given slope angle of the tangent with the horizontal plane xOy at the beginning of the helical line of the centers when $\varphi = 0$; α_2 is the given slope angle of the tangent at the end of the helical line of the centers when $\varphi = \varphi_k$; $0 \le \varphi \le \varphi_k$. In Fig. 1, the development of the cylindrical surface with the radius r with three possible lines of centers is constructed: (1) line 1 is formed when $\alpha_1 < \alpha_2$; (2) line 2 is formed when $\alpha_1 > \alpha_2$; and (3) line 3 is a line of constant slope with the slope angle equal to α_3,

$$\tan\alpha_3 = (\tan\alpha_1 + \tan\alpha_2)/2.$$

In Fig. 2, the same three types of the cylindrical helical lines 1′–3′ are shown. These helical lines are formed by the winding of the plane with tree lines 1–3 (Fig. 1) on the circular cylinder with radius r.

Parametrical equations of the cylindrical helical strip (Fig. 3) are

$$x = x(\varphi) = r\cos\varphi, \quad y = y(\varphi) = r\sin\varphi,$$
$$z = z(\varphi, u) = r\varphi\left(\tan\alpha_1 + \frac{\tan\alpha_2 - \tan\alpha_1}{2\varphi_k}\varphi\right) + u$$

where $0 \le \varphi \le \varphi_k$; and $-u_l \le u \le u_{\text{top}}$.

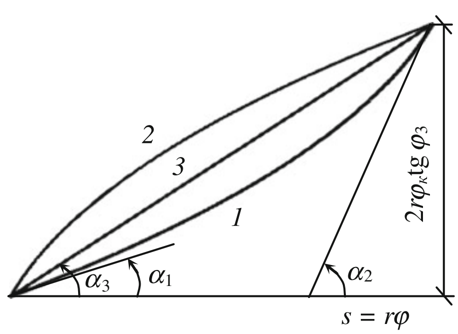


Fig. 1

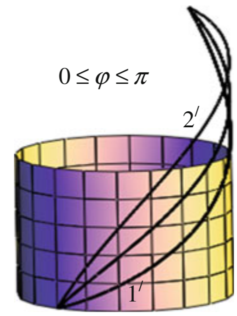


Fig. 2

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = r^2\left[1 + \left(\tan\alpha_1 + \frac{\tan\alpha_2 - \tan\alpha_1}{\varphi_k}\varphi\right)^2\right],$$
$$F = r\left(\tan\alpha_1 + \frac{\tan\alpha_2 - \tan\alpha_1}{\varphi_k}\varphi\right),$$
$$B = 1, \quad A^2B^2 - F^2 = r^2,$$
$$L = -r, \quad M = N = 0,$$
$$k_\varphi = -\frac{L}{A^2} = \frac{1}{r\left[1 + \left(\tan\alpha_1 + \frac{\tan\alpha_2 - \tan\alpha_1}{\varphi_k}\varphi\right)^2\right]}$$
$$k_u = k_2 = 0, \quad k_1 = \frac{1}{r}, \quad K = 0.$$

The studied cylindrical helical strip is a fragment of the right cylindrical surface of revolution with radius r. The curvilinear coordinates u, v are nonorthogonal and nonconjugate coordinates. The coordinate lines u coincide with the rectilinear generatrixes of the cylindrical surface.

In Fig. 3, the cylindrical helical strip of the variable pitch with $0 \le \varphi \le \varphi_k$; $\varphi_k = 4\pi$; $r = 1$ m; $\alpha_1 = 10°$; $\alpha_2 = 30°$; and $0 \le u \le 1$ m is shown. The cylindrical helical line of the variable pitch denoted as 1′, in Fig. 2 was taken as the

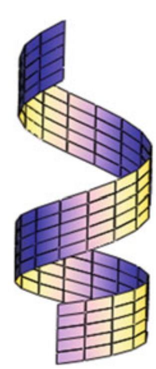


Fig. 3

Fig. 4 Long variable pitch screw, http://www.falconindustries.com/products-Complete-Screws.html

directrix helical line for design of the surface presented in Fig. 3.

Screws, helical conveyers, and threads with the lateral helical surfaces of variable pitch (Fig. 4) are seldom found.

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Additional Literature

P.S.: Additional literature is given at the corresponding pages of the Chap. "7. Helical Surfaces".

Spiral Surfaces

8

If a curve fulfilling helical motion is subjected simultaneously to similarity transformation with a coefficient of similarity which is proportional to the angle of rotation and with a constant center of similarity disposed on the axis of rotation, then it generates *a spiral surface*.

J.K. Whittemore has stated the following definition of spiral surface: "A spiral surface is the locus of the different positions of any curve which is rotated about an axis and at the same time subjected to a homothetic transformation with respect to a point of the axis in such a way that the tangent to the locus described by any point of the curve makes a constant angle with the axis".

The trajectories of points of the curve under this motion are disposed on the cones

$$x^2 + y^2 = z^2 \tan^2 \varphi$$

where φ is the angle of the axis of the right circular cone with its rectilinear generatrixes. Parametrical equations of the trajectories of the points of the generatrix curve (Fig. 1) are:

$$\begin{aligned} x &= x(t) = ce^{ht}\cos(b + \omega t),\\ y &= y(t) = ce^{ht}\sin(b + \omega t),\\ z &= de^{ht} \end{aligned}$$

where b, c, d are arbitrary constants; h = const. The projections of the points of the trajectories of the points of the curve on a plane xOy are *logarithmic spirals* (Fig. 1). The locus of each point of the curve generating a spiral surface lies on a right circular cone whose axis is the axis of the surface and is called simply a spiral or may be named "*a cylindro-conical helix*." Assume that b, c, d are arbitrary functions of a variable of u, then the parametrical equations presented above define an arbitrary spiral surface. The net set up from the trajectories of spiral motion and their orthogonal trajectories are called *a spiral net*. Darboux has proved that the square of the linear element of a spiral surface given in the parameters of a spiral net may be written in the following form

$$\mathrm{d}s^2 = U^2(u)e^{2v}\left(\mathrm{d}u^2 + \mathrm{d}v^2\right),$$

where U is a function of u alone.

Assume parametrical equations of *a conic spiral* as

$$\begin{aligned} x &= x(u) = ae^{mu}\cos u;\\ y &= y(u) = ae^{mu}\sin u;\\ z &= z(u) = a\lambda e^{mu}, \end{aligned}$$

then parametric equations of a spiral surface formed by the motion of any generatrix plane curve (Fig. 2)

$$\boldsymbol{\rho}(v) = x_o(v)\boldsymbol{p} + y_o(v)\boldsymbol{q}$$

lying in planes of pencil passing through a coordinate axis Oz may be written as

$$\begin{aligned} x &= x(u,v) = [ae^{mu} + x_o(v)\cos\theta + y_o(v)\sin\theta]\cos u;\\ y &= y(u,v) = [ae^{mu} + x_o(v)\cos\theta + y_o(v)\sin\theta]\sin u;\\ z &= z(u,v) = a\lambda e^{mu} - x_o(v)\sin\theta + y_o(v)\cos\theta. \end{aligned}$$

Here

$$x_o = x_o(v), \quad y_o = y_o(v)$$

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Fig. 1

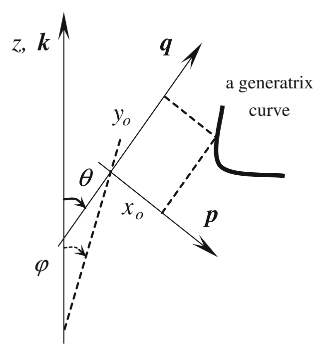

Fig. 2

are the parametrical equations of a generatrix curve given in a local coordinate system with the axis x_o, y_o coinciding with unit orthogonal vectors $\boldsymbol{p}$, $\boldsymbol{q}$ (Fig. 2). The beginning of the local coordinates lies on the directrix conic spiral; φ is the angle of the rectilinear generatrix of *the director cone* with the axis Oz;

$$\tan\varphi = 1/\lambda,\ \ \lambda = \cot\varphi;\ \ a = r_o \sin\varphi;\ \ r_o = \ \text{const.}$$

Coefficients of the first fundamental form of arbitrary spiral surface with a generatrix curve in the planes of pencil have the following form:

$$A^2 = a^2m^2\left(1+\lambda^2\right)e^{2mu} + \left(ae^{mu} + x_o\cos\theta + y_o\sin\theta\right)^2,$$
$$B^2 = x_o'^2 + y_o'^2,\ F = ame^{mu}\left[x_o'(\cos\theta - \lambda\sin\theta) + y_o'(\sin\theta + \lambda\cos\theta)\right].$$

A surface is given in a nonorthogonal nonconjugate system of curvilinear coordinates u, v, where the lines u coincide with the trajectories of the spiral motion but the lines v coincide with the generatrix plane curves. Let the parameter $\lambda = 0$, i.e., $\varphi = \pi/2$, then a conic spiral degenerates into the logarithmic spiral.

The spiral surfaces include *surfaces of revolution* too.

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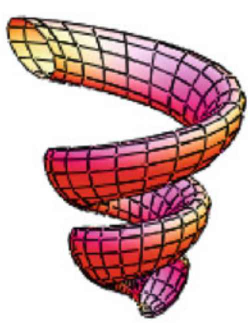
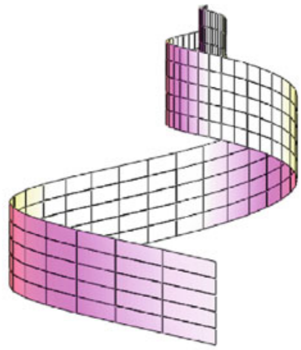
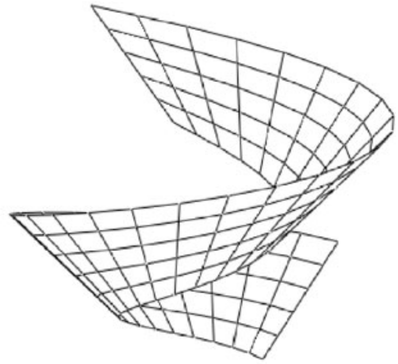
■ Spiral Surfaces Presented in the Encyclopedia



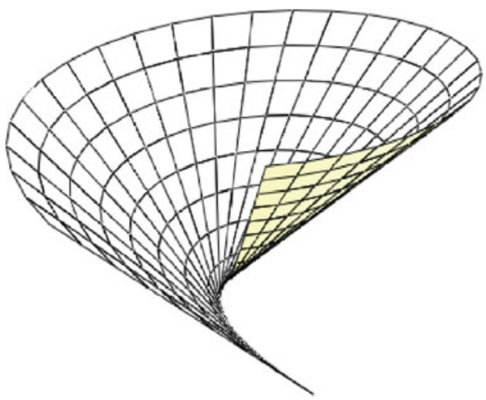
The spiral surface with
a generatrix ellipse



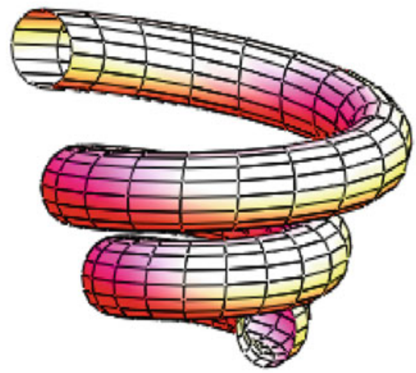
The cylindrical-and-conical
spiral strip



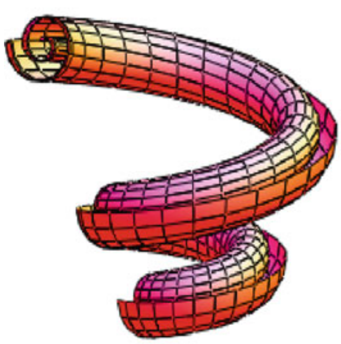
The developable conic
helicoid



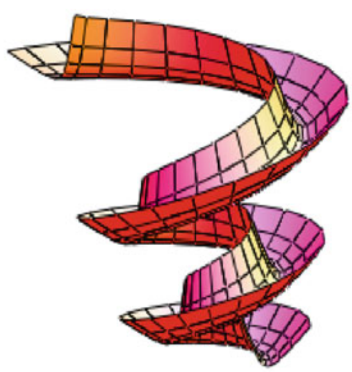
The torse with an edge of regression
given as $x = \bar{e}^t \cos t;\ y = \bar{e}^t \sin t,\ z = \bar{e}^t$



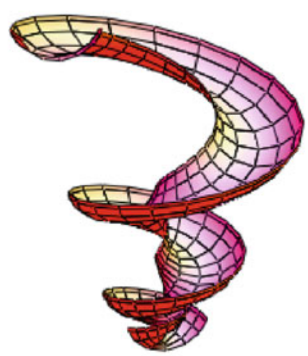
The circular spiral surface with
a generatrix circle of constant radius
lying in planes of the pencil



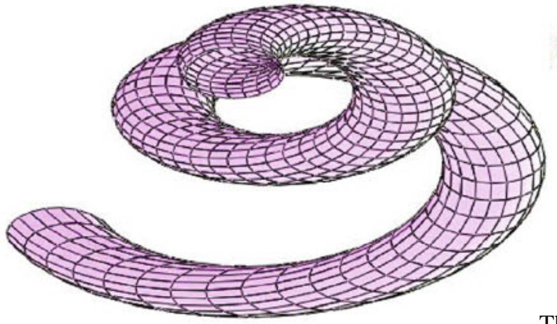
The spiral surface with
generatrix in the form of the
evolvent of a circle



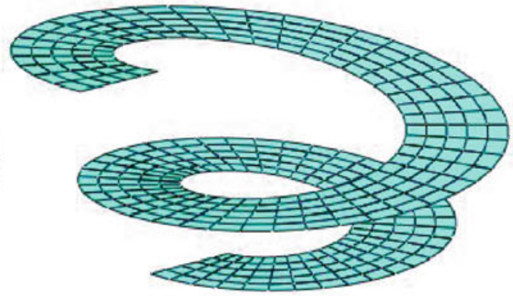
The spiral surface with
the hyperbolic generatrix



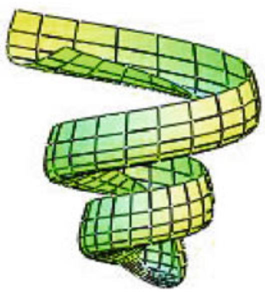
The spiral surface with
generatrix in the form
of the cycloid



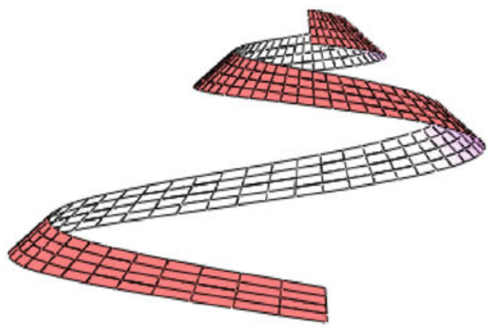
The right circular spiral surface



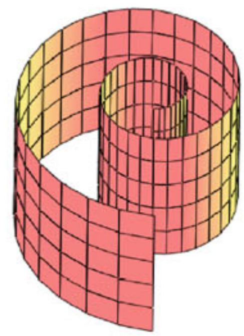
The spiral ruled surface with a straight generatrix perpendicular to an axis of a directrix conic spiral and to the tangent of the same spiral



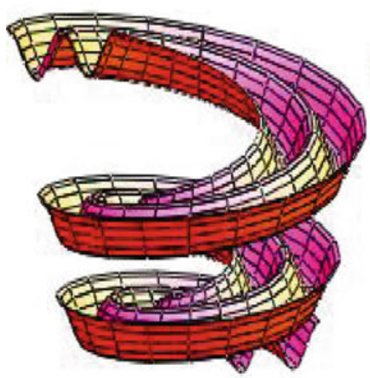
The spiral surface with the parabolic generatrix of arbitrary position



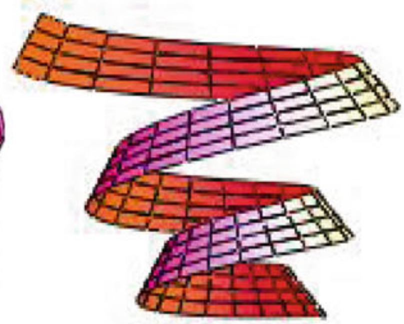
The spiral conical strip



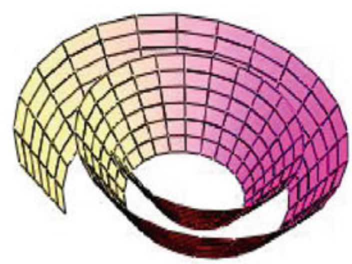
The right cylinder with the directrix logarithmic spiral



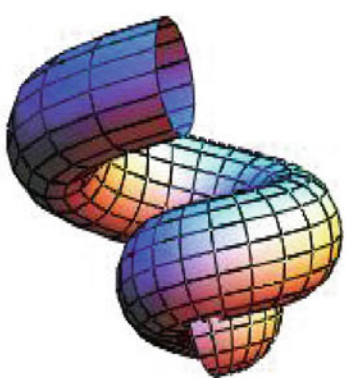
The spiral surface with the sinusoidal generatrix



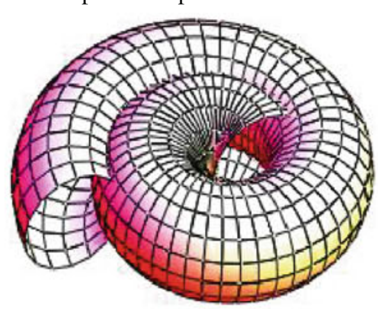
The spiral surface with straight generatrixes in the planes of pencil



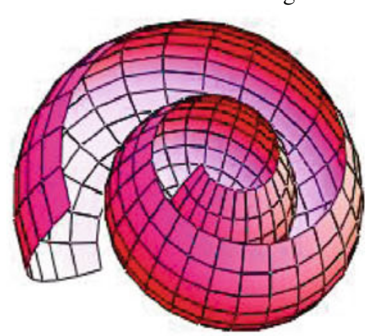
The ruled surface with straight generatrixes passing through the logarithmic spiral and intersecting the fixed axis under constant angle



The tubular spiral surface



The cyclic surface with generatrix circles in planes of a pencil and with a plane line of centers in the form of the logarithmic spiral



The spiral surface with the directrix logarithmic spiral and with the parabolic generatrix

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Additional Literature

P.S.: Additional literature is given at corresponding pages of the Chap. "8. Spiral Surfaces" and the Chap. "9. Spiral-Shaped Surfaces".

■ Spiral Ruled Surface with Straight Generatrix Perpendicular to an Axis of a Directrix Conic Spiral and to the Tangent of the Same Spiral

Assume a directrix conical spiral in the form:

$$x = x(\varphi) = r_0 \sin\lambda \cos\varphi \cdot e^{k\varphi},$$
$$y = y(\varphi) = r_0 \sin\lambda \sin\varphi \cdot e^{k\varphi},$$
$$z = z(\varphi) = r_0 \cos\lambda \cdot e^{k\varphi},$$

where λ is the angle of the axis Oz with a rectilinear generatrix of the cone; a longitude φ is the angle between the fixed plane xOz and a mobile plane of an axial cross-section; r_0, k are constants.

Let $\boldsymbol{c}(\varphi)$ is the unit vector which is perpendicular to the unit vectors $\boldsymbol{b}(\varphi)$ and $\boldsymbol{k}$, i.e.,

$$\boldsymbol{c}(\varphi) = \boldsymbol{b}(\varphi) \times \boldsymbol{k},$$
$$\boldsymbol{c}(\varphi) = \boldsymbol{i}\frac{y'}{\sqrt{x'^2 + y'^2}} - \boldsymbol{j}\frac{x'}{\sqrt{x'^2 + y'^2}},$$

where $\boldsymbol{b}(v)$ is the unit vector coinciding with the projection of the tangent vector to the conical spiral on the plane xOy; $\boldsymbol{k}$ is the unit vector coinciding with the axis Oz; x, y, z are the Cartesian coordinates of the directrix conical spiral; $\ldots' = \partial\ldots/\partial\varphi$.

A spiral ruled surface with a straight generatrix $u\boldsymbol{c}(v)$ which is perpendicular to an axis of the generatrix conical spiral and to a tangent of the same spiral is *a ruled surface of negative Gaussian curvature* (see also Sect. "1.2. Ruled Surfaces of Negative Total Curvature").

The surface belongs to a family of *Catalan's surfaces* (see also "Surfaces"). A plane z = const is *a plane of parallelism*.

Having known parametrical equations of a directrix conical spiral, we can write the equations of the spiral ruled surface (Fig. 1) in the form:

$$x = x(u,\varphi) = r_0 \sin\lambda \cos\varphi \cdot e^{k\varphi} + u\frac{\cos\varphi + k\sin\varphi}{\sqrt{1+k^2}},$$
$$y = y(u,\varphi) = r_0 \sin\lambda \sin\varphi \cdot e^{k\varphi} + u\frac{\sin\varphi - k\cos\varphi}{\sqrt{1+k^2}},$$
$$z = z(\varphi) = r_0 \cos\lambda \cdot e^{k\varphi},$$

where $|u|$ is the distance the conical spiral directrix from the corresponding point on the surface along a rectilinear generatrix.

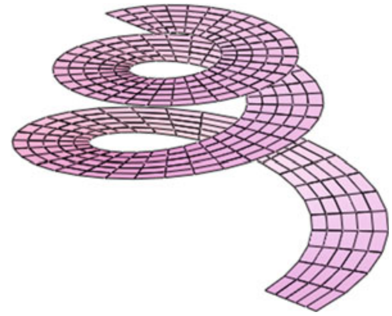

Fig. 1

Coefficients of the fundamental forms of the surface and its curvatures:

$$A = 1,\ F = 0,$$
$$B^2 = r_0^2 e^{2k\varphi}(\sin^2\lambda + k^2) + u^2 + 2ur_0 \sin\lambda e^{k\varphi}\sqrt{1+k^2},$$
$$L = 0,\ M = -r_0\cos\lambda e^{k\varphi}k/B,\ N = r_0 \cos\lambda e^{k\varphi}uk^2/B,$$
$$k_u = 0,\ k_\varphi = r_0\cos\lambda e^{k\varphi}uk^2/B^3,$$
$$K = -\frac{M^2}{B^2} = -\frac{r_0^2\cos^2\lambda e^{2k\varphi}}{B^4}k^2 < 0,\ H = \frac{N}{2B^2} = -\frac{k_\varphi}{2} \neq 0.$$

The surface is given in curvilinear orthogonal (F = 0) nonconjugate ($M \neq 0$) coordinates u, φ. The coordinate lines u coincide with the straight generatrixes of the spiral surface. The lines φ are spiral lines on the circular cones. The angle β of the tangent to the directrix conic line with the plane $z = 0$ may be found with the help of a formula:

$$\tan\beta = \frac{k\cot\lambda}{\sqrt{1+k^2}}.$$

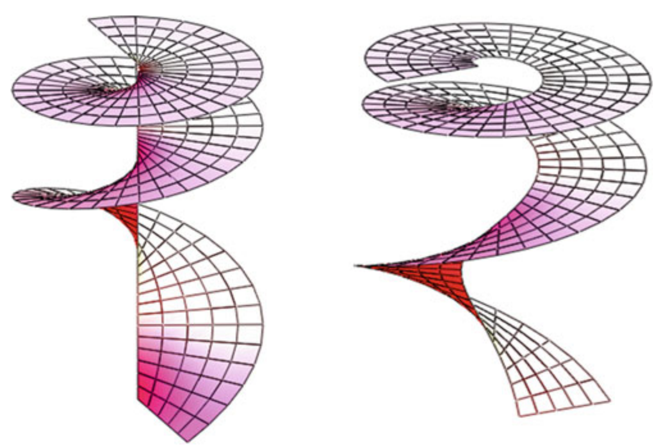

Fig. 2

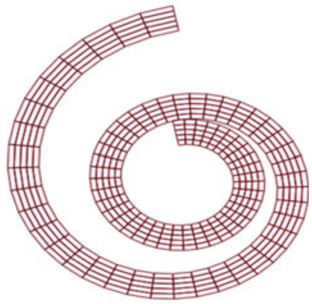

Fig. 3

If $\lambda = 0$ then a directrix cone degenerates into the straight line and the ruled spiral surface degenerates into a ruled helical circular surface with a variable pitch (Fig. 2). When $\lambda = \pi/2$, the spiral directrix line degenerates into a plane logarithmic spiral but the ruled spiral surface becomes a plane strip with the logarithmic directrix spiral (Fig. 3).

Additional Literature

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■ Spiral Surface with Straight Generatrixes in the Planes of Pencil

Assume a directrix conical spiral in the form:

$$x = x(u) = ae^{mu}\cos u,$$
$$y = y(u) = ae^{mu}\sin u,$$
$$z = z(u) = a\lambda e^{mu},$$

where $\lambda = \cot\varphi$; φ is the angle of the axis Oz with the straight generatrix of the cone on which the conic spiral lies; the longitude u is the angle of the plane xOz with the mobile plane of the axial cross-section; a, m are constants. *A spiral surface with straight generatrixes in the planes of pencil* is formed by a helical motion of a straight line crossing an axis of rotation at the angle θ (see also Chap. "8 Spiral Surfaces") and one point of the generatrix straight moves along the conic spiral (Fig. 1). All points of the straight generatrix trace *the conical spirals* that are *the slope lines*.

Having known parametrical equations of a directrix conical spiral, we can write the equations of the spiral ruled surface (Fig. 1) in the form:

$$x = x(u, v) = (ae^{mu} + v\sin\theta)\cos u,$$
$$y = y(u, v) = (ae^{mu} + v\sin\theta)\sin u,$$
$$z = z(u, v) = a\lambda e^{mu} + v\cos\theta.$$

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = a^2m^2e^{2mu}\left(1+\lambda^2\right) + (ae^{mu} + v\sin\theta)^2,$$
$$F = ame^{mu}(\sin\theta + \lambda\cos\theta), \quad B = 1,$$
$$A^2B^2 - F^2 = a^2m^2e^{2mu}(\cos\theta - \lambda\sin\theta)^2 + (ae^{mu} + v\sin\theta)^2,$$
$$L = \frac{-1}{\sqrt{A^2B^2 - F^2}}\Big\{(ae^{mu} + v\sin\theta)^2 + \cos\theta + (\cos\theta - \lambda\sin\theta)am^2e^{mu} \times (ae^{mu} - v\sin\theta)\Big\}$$

$$M = -\frac{ame^{mu}\sin\theta}{\sqrt{A^2B^2 - F^2}}(\cos\theta - \lambda\sin\theta), \quad N = 0,$$
$$k_u = \frac{L}{A^2}, \quad k_v = 0, \quad K = \frac{-M^2}{A^2B^2 - F^2} < 0.$$

The coefficients of the fundamental forms of the surface show that the spiral ruled surface is a surface of *negative Gaussian curvature* and it is given in the curvilinear non-orthogonal nonconjugate coordinates *u, v*.

The projections of the coordinate lines u on the plane xOy are *logarithmic spirals*. The coordinate lines v coincide with the rectilinear generatrixes of the spiral ruled surface.

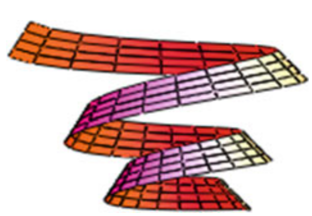

Fig. 1

Fig. 2

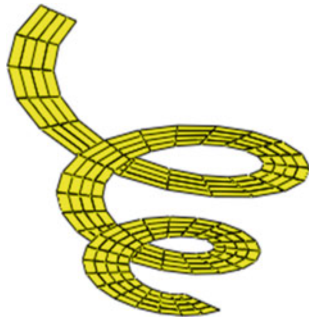

Fig. 3

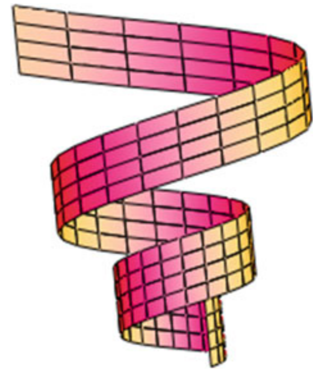

Fig. 4

If $\theta = \varphi$, then the considered spiral surface of negative Gaussian curvature degenerates into *a spiral conical strip* (Fig. 2) of zero Gaussian curvature (see also Subsect. "1.1.3. Conic Surfaces").

Having assumed $\theta = \pi/2$, one obtains *a spiral conical surface with the plane of parallelism xOy* (Fig. 3). This surface of negative Gaussian curvature belongs to a class of *Catalan's surfaces*.

If $\theta = 0$, then the considered spiral surface becomes *a cylindrical surface*, Fig. 4 (see also "*Cylindrical helical strip*" in Subsect. "1.1.2. Cylindrical Surface").

Putting $\lambda = 0$, one can obtain *a ruled surface with straight generatrixes passing through a logarithmic spiral and intersecting the fixed axis Oz at the constant angle* (see also Sect. "1.2. Ruled Surface of Negative Total Curvature").

All straight generatrixes lie in the planes of pencil.

■ Spiral Surface with Parabolic Generatrix of Arbitrary Position

Assume parametrical equations of a directrix conic spiral in the form:

$$x = x(u) = ae^{mu}\cos u,$$
$$y = y(u) = ae^{mu}\sin u,$$
$$z = z(u) = a\lambda e^{mu},$$

where $\lambda = \cot\varphi$; φ is the angle of the axis Oz with a straight generatrix of the cone on which the conic spiral lies; the longitude u is the angle of the plane xOz with a moving plane of the axial cross-section;

$$a = r_o \sin\varphi,$$

r_o, m are constants.

A spiral surface with a parabolic generatrix in the planes of pencil is formed by the helical motion of a parabola with the axis intersecting the axis of rotation at the angle θ (see also a Chap. "8. Spiral Surfaces"), but the origin of a local coordinate system in which the generating parabola is given in the form

$$y_o = bx_o^2$$

moves along the spiral line. All points of the generatrix parabolas trace *the conic spirals* that are *the slope lines*.

Parametrical equations of the surface are

$$x = x(u, v) = \left(ae^{mu} + v\cos\theta + bv^2\sin\theta\right)\cos u,$$
$$y = y(u, v) = \left(ae^{mu} + v\cos\theta + bv^2\sin\theta\right)\sin u,$$
$$z = z(u, v) = a\lambda e^{mu} - v\sin\theta + bv^2\cos\theta.$$

In Fig. 1, the spiral surface with $b < 0$, $\theta > \pi/2$ is shown or it is the same $b > 0$, but $\theta < 0$. In Fig. 2, the spiral surface having $b > 0$, $\theta = \pi/2$ is presented.

In Fig. 3, the axis of the generating parabola is parallel to the axis of rotation and that is why $b > 0$, $\theta = 0$.

The spiral surface shown in Fig. 4 has $b < 0$, $\theta = \pi/2$ or $b > 0$ but $\theta = -\pi/2$.

Fig. 1

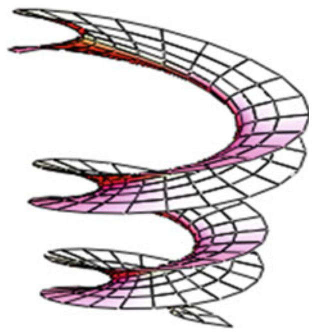

Fig. 2

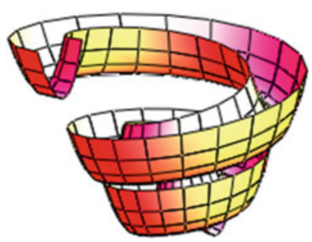

Fig. 3

Coefficients of the fundamental forms of the surface:

$$A^2 = r_o^2 m^2 e^{2mu} + D^2, \quad B^2 = 1 + 4b^2v^2,$$
$$F = ame^{mu}[\cos\theta - \lambda\sin\theta + 2bv(\sin\theta + \lambda\cos\theta)],$$
$$A^2B^2 - F^2 = D^2B^2 + d^2,$$

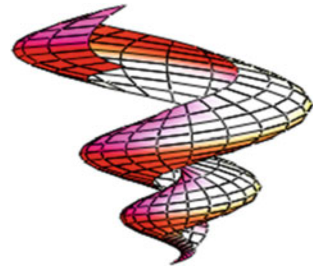

Fig. 4

where

$$\mathrm{d} = ame^{mu}[\sin\theta + \lambda\cos\theta - 2bv(\cos\theta - \lambda\sin\theta)],$$
$$D = ae^{mu} + v\cos\theta + bv^2\sin\theta,$$
$$L = \frac{1}{\sqrt{A^2B^2 - F^2}}\{D[D(\sin\theta - 2bv\cos\theta) - md] + 2ame^{mu}\mathrm{d}\},$$
$$M = \frac{d}{\sqrt{A^2B^2 - F^2}}(\cos\theta + 2bv\sin\theta),$$
$$N = -\frac{2bD}{\sqrt{A^2B^2 - F^2}}.$$

The spiral surface is given in curvilinear nonorthogonal nonconjugate coordinates *u, v*. The projections of coordinate lines *u* on the plane *xOy* are logarithmic spirals. The coordinate lines *v* coincide with the generating parabolas of the spiral surface.

Additional Literature

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■ Spiral Surface with a Generatrix Ellipse

A directrix conical spiral may be given by equations

$$x = x(u) = ae^{mu}\cos u, \quad y = y(u) = ae^{mu}\sin u,$$
$$z = z(u) = a\lambda e^{mu},$$

where $\lambda = \cot\varphi$; φ is the angle of the axis Oz with the straight generatrix of the cone on which the conical spiral lies; the longitude u is the angle of the plane xOz with the moving plane of the axial cross-section;

$$a = r_o\sin\varphi;$$

r_o, m are constants.

A spiral surface with the same ellipses lying in the planes of pencil is formed in the process of a helical motion of an ellipse, the center of which moves along *the conical spiral*. All points of the generating ellipses trace the conical spirals which are *the slope lines*.

Having known the parametrical equations of the directrix spiral, we can write parametrical equations of *a spiral surface with a generatrix ellipse* as

$$x = x(u,v) = (ae^{mu} + b\cos v\cos\theta + c\sin v\sin\theta)\cos u,$$
$$y = y(u,v) = (ae^{mu} + b\cos v\cos\theta + c\sin v\sin\theta)\sin u,$$
$$z = z(u,v) = a\lambda e^{mu} - b\cos v\sin\theta + c\sin v\cos\theta,$$

where θ is the angle of the axis of rotation z of the spiral surface with the axis y_o of the ellipse (see also Chap. "8. Spiral Surfaces"). The generating ellipse is given in a local system of Cartesian coordinates x_o, y_o by parametrical equations:

$$x_o = b\cos v, y_o = c\sin v.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = r_o^2 m^2 e^{2mu} + D^2,\ B^2 = b^2 \sin^2 v + c^2 \cos^2 v,$$
$$F = ame^{mu}[-b \sin v(\cos\theta - \lambda \sin\theta) + c \cos v(\sin\theta + \lambda\cos\theta)]$$
$$A^2B^2 - F^2 = D^2B^2 + d^2,$$

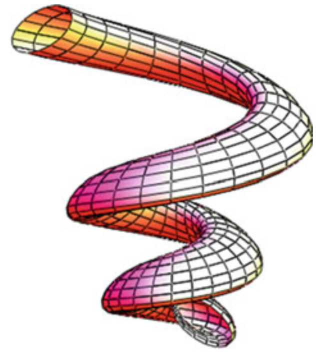
where

$$D = ae^{mu} + b\cos v\cos\theta + c\sin\theta,$$
$$d = -ame^{mu}[b\sin v(\sin\theta + \lambda\cos\theta) + c\cos v(\cos\theta - \lambda\sin\theta)],$$
$$L = \frac{1}{\sqrt{A^2B^2 - F^2}}\{D[D(-b\sin v\sin\theta - c\cos v\cos\theta) - md] + 2ame^{mu}d\},$$
$$M = \frac{d}{\sqrt{A^2B^2 - F^2}}(-b\sin v\cos\theta + c\cos v\sin\theta),$$
$$N = -\frac{bcD}{\sqrt{A^2B^2 - F^2}}.$$

A spiral surface is related to nonorthogonal nonconjugate system of the curvilinear coordinates u, v. The coordinate lines v coincide with the generating ellipses. The surface consists of segments of positive and negative Gaussian curvatures.

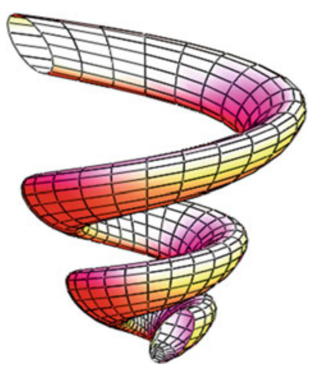
Assume $\varphi = \pi/2$ that is $\lambda = 0$, then a spiral surface with a generatrix ellipse (Fig. 1) degenerates into a spiral surface with the generatrix ellipse in the planes of pencil and with the plane line of the centers in the form of the logarithmic spiral. Changing the sign of the angle φ, one can change the form of the spiral surface (Figs. 1, 2, and 3). In Fig. 4, the spiral surface has $\theta = 0$ but in Fig. 5, the surface is with $\theta = \pi/2$.

If $b = c$ but $\theta = 0$ or $\theta = \pi/2$, then it is possible to design a *circular spiral surface with the generatrix circle of a constant radius lying in the planes of pencil* (see also



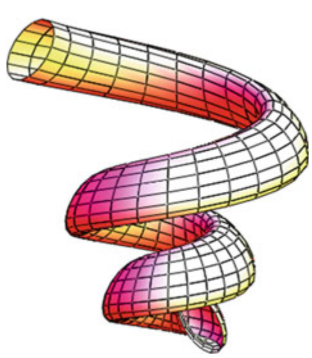
$\theta = -\pi/6$

Fig. 3



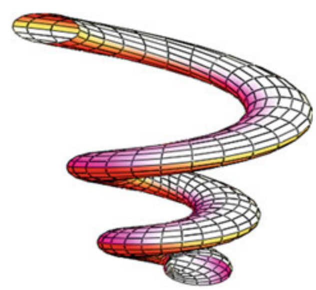
$\theta = \pi/4$

Fig. 1



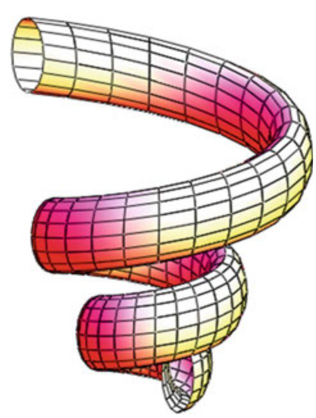
$\theta = -\pi/4$

Fig. 2



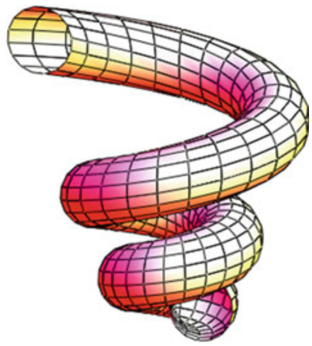
$\theta = 0$

Fig. 4



$\theta = \pi/2$

Fig. 5



$b = c$

Fig. 6

Sect. "17.4. Circular Surfaces with Circles in Planes of a Pencil"), Fig. 6.

Taking $b = c$ and $\varphi = \pi/2$ that is $\lambda = 0$ one may obtain a cyclic surface with a generatrix circle in the planes of pencil and with a plane line of the centers in the form of a logarithmic spiral (see also Subsect. "17.4.1. Cyclic Surfaces with Circles in the Planes of Pencil and with a Plane Center-to-Center Line").

■ Spiral Surface with a Hyperbolic Generatrix

Assume a directrix conical spiral in the form:

$$x = x(u) = ae^{mu}\cos u,$$
$$y = y(u) = ae^{mu}\sin u,$$
$$z = z(u) = a\lambda e^{mu},$$

where $\lambda = \cot\varphi$; φ is the angle of the axis Oz with a straight generatrix of the cone on which the conic spiral lies; the longitude u is the angle of the plane xOz with a moving plane of the axial cross-section; a, m are constants. *A spiral surface with a generatrix hyperbola in planes of a pencil* is formed by the helical motion of a hyperbola:

$$x_o = x_o(v) = b\cosh v, y_o = y_o(v) = c\sinh v$$

given in a local coordinate system x_o, y_o along the conical spiral. The axis y_o intersects the axis of rotation at an angle θ (see also Chap. "8. Spiral Surfaces"), and the origin of the local coordinate system moves along the conic spiral. All points of the generatrix hyperbolas trace *the conic spirals* which are *the slope lines*.

A spiral surface with a hyperbolic generatrix (Fig. 1) can be given by the following parametrical equations:

$$x = x(u,v) = (ae^{mu} + b\cosh v\cos\theta + c\sinh v\sin\theta)\cos u,$$
$$y = y(u,v) = (ae^{mu} + b\cosh v\cos\theta + c\sinh v\sin\theta)\sin u,$$
$$z = z(u,v) = a\lambda e^{mu} - b\cosh v\sin\theta + c\sinh v\cos\theta.$$

The spiral surface is given in the curvilinear nonorthogonal nonconjugate coordinates u, v. The projections of the coordinate lines u on the plane xOy are *logarithmic spirals*. The coordinate lines v coincide with the generating hyperbolas of the spiral surface.

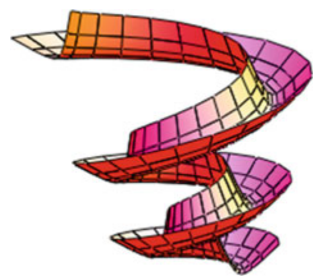


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2\left(1+\lambda^2\right)m^2e^{2mu} + D^2,\ B^2 = b^2\sinh^2 v + c^2\cosh^2 v,$$
$$F = ame^{mu}[b\sinh v(\cos\theta - \lambda\sin\theta) + c\cosh v(\sin\theta + \lambda\cos\theta)],$$

where

$$A^2B^2 - F^2 = D^2B^2 + d^2,$$
$$D = ae^{mu} + b\cosh v\cos\theta + c\sinh v\sin\theta,$$
$$d = ame^{mu}[b\sinh v(\sin\theta + \lambda\cos\theta) - c\cosh v(\cos\theta - \lambda\sin\theta)],$$
$$L = \frac{1}{\sqrt{A^2B^2 - F^2}}\{D[D(b\sinh v\sin\theta - c\cos\theta) - md] + 2ame^{mu}d\},$$
$$M = \frac{d}{\sqrt{A^2B^2 - F^2}}(b\cos\theta + c\cosh v\sin\theta),$$
$$N = -\frac{bcD}{\sqrt{A^2B^2 - F^2}}.$$

■ Spiral Surface with Generatrix in the Form of Cycloid

A spiral surface with generatrix in the form of a cycloid in the planes of a pencil is formed by helical motion of a cycloid

$$x_o = x_o(v) = b(v - \sin v), y_o = y_o(v) = b(1 - \cos v),$$

along the conical spiral. The cycloid is given in a local coordinate system x_o, y_o. *A cycloid* is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line. The axis y_o crosses the axis of rotation at an angle θ (see also Chap. "8. Spiral Surfaces") and the origin of the local coordinate system moves along the given spiral line

$$x = x(u) = ae^{mu}\cos u,$$
$$y = y(u) = ae^{mu}\sin u,$$
$$z = z(u) = a\lambda e^{mu},$$

where $\lambda = \cot\varphi$; φ is the angle of the axis Oz with the straight generatrix of the cone, on which the conic spiral lies; the longitude u is the angle of the plane xOz with a mobile plane of an axial cross-section; a, m are constants. All points of the generatrix cycloid trace the conical spirals which are the lines of slope.

The spiral surface with the generatrix in the form of a cycloid (Fig. 1) is defined by the following parametrical equations:

$$x = x(u, v) = [ae^{mu} + b(v - \sin v)\cos\theta + b(1 - \cos v)\sin\theta]\cos u,$$
$$y = y(u, v) = [ae^{mu} + b(v - \sin v)\cos\theta + b(1 - \cos v)\sin\theta]\sin u,$$
$$z = z(u, v) = a\lambda e^{mu} - b(v - \sin v)\sin\theta + b(1 - \cos v)\cos\theta.$$

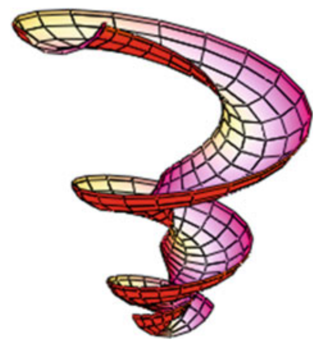


Fig. 1

The spiral surface is given in the curvilinear nonorthogonal nonconjugate coordinates u, v. The net set up from the trajectories of spiral motion and their orthogonal trajectories are called *a spiral net*. The projections of the coordinate lines u on the plane xOy are *logarithmic spirals*. The coordinate lines v (u = const) coincide with the generatrix cycloids of the spiral surface.

Taking $\varphi = \pi/2$ that is $\lambda = 0$ one may obtain a spiral surface with a generatrix cycloid in the planes of the pencil but with a plane directrix line in the form of a logarithmic spiral.

Additional Literature

Ivanov VN. Geometry and design of shells on the base of surfaces with a system of coordinate lines in the planes of pencil. Sb. Trudov: Prostran. Konstruk. Zdaniy i Soor. MOO «Prostranstvennye Konstruktzii». Moscow: «Devyatka Print», 2004; p. 26-35.

■ Spiral Surface with a Sinusoidal Generatrix

A spiral surface with a sinusoidal generatrix in the planes of a pencil (Figs. 1, 2, and 3) is generated by a helical motion of a sinusoid:

$$x_o = x_o(v) = v, y_o = y_o(v) = b\sin(\pi/2 - v) = b\cos v,$$

along a conical spiral. The sinusoid is given in a local coordinate system x_o, y_o. The axis y_o crosses the axis of rotation at an angle θ (see also Chap. "8. Spiral Surfaces"), and the origin of the local coordinate system moves along the given conical spiral:

$$x = x(u) = ae^{mu}\cos u,$$
$$y = y(u) = ae^{mu}\sin u,$$
$$z = z(u) = a\lambda e^{mu},$$

where $\lambda = \cot\varphi$; φ is the angle of the axis Oz with the straight generatrix of the cone, on which the conical spiral lies; a longitude u is the angle of the plane xOz with a mobile plane of an axial cross-section; a, m are constants. All points of the generatrix sinusoid trace the conic spirals which are *the slope lines*.

A studied spiral surface with a generatrix sinusoid of arbitrary location can be defined by the following parametrical equations:

$$x = x(u, v) = (ae^{mu} + v\cos\theta + b\cos v\sin\theta)\cos u,$$
$$y = y(u, v) = (ae^{mu} + v\cos\theta + b\cos v\sin\theta)\sin u,$$
$$z = z(u, v) = a\lambda e^{mu} - v\sin\theta + b\cos v\cos\theta,$$

where b is the amplitude of the generatrix sinusoid, $0 \leq \theta \leq 2\pi$.

In Fig. 1, the spiral surface with $\theta = 0$ is shown. The spiral surface presented in Fig. 2 has $\theta = \pi/3$ but the spiral surface given in Fig. 3 has $\theta = \pi/2$ i.e. the axis of the sinusoid is placed parallel to the axis of spiral i.e. to the coordinate axis Oz.

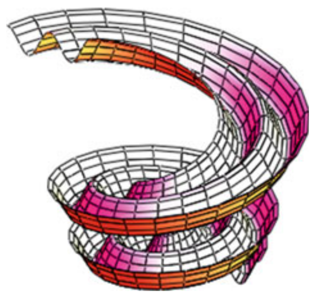


Fig. 1

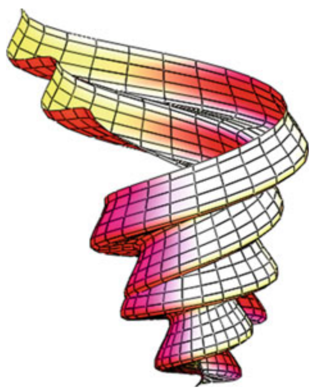


Fig. 2

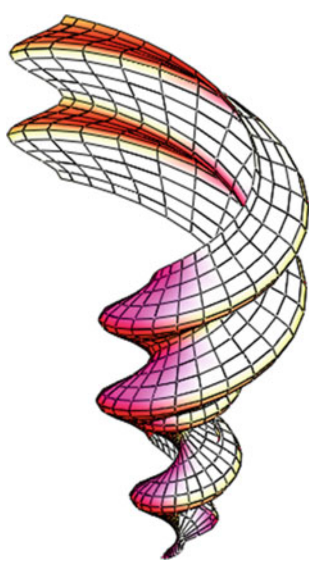


Fig. 3

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2m^2\left(1+\lambda^2\right)e^{2mu} + D^2,\ B^2 = 1 + b^2\sin^2 v,$$
$$F = ame^{mu}[\cos\theta - \lambda\sin\theta - b\sin v(\sin\theta + \lambda\cos\theta)],$$
$$A^2B^2 - F^2 = D^2B^2 + d^2,$$

where

$$D = ae^{mu} + v\cos\theta + b\cos v\sin\theta,$$
$$d = ame^{mu}[\sin\theta + \lambda\cos\theta + b\sin v(\cos\theta - \lambda\sin\theta)],$$
$$L = \frac{1}{\sqrt{A^2B^2 - F^2}}\{D[D(\sin\theta + b\sin v\cos\theta) - md] + 2ame^{mu}d\},$$
$$M = \frac{d(\cos\theta - b\sin v\sin\theta)}{\sqrt{A^2B^2 - F^2}},\ N = \frac{bD\cos v}{\sqrt{A^2B^2 - F^2}}.$$

The spiral surface is given in the curvilinear nonorthogonal nonconjugate coordinates u, v. The projections of the coordinate lines u on the plane xOy are *logarithmic spirals*. The coordinate lines v coincide with the sinusoidal generatrixes of the spiral surfaces.

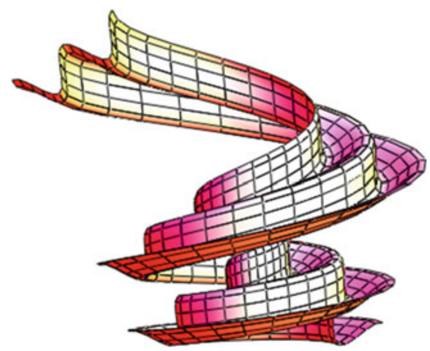


Fig. 4

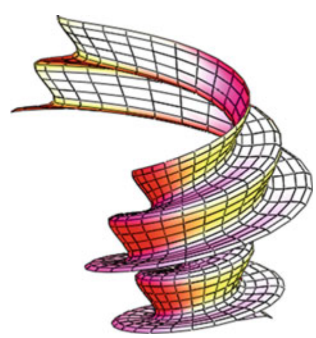


Fig. 5

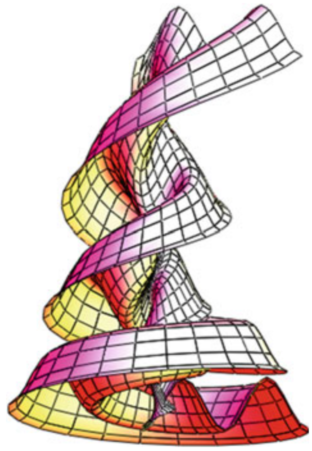


Fig. 6

If $b = 0$, then *a spiral surface with the sinusoidal generatrix* degenerates into *a spiral surface with the straight generatrixes in the planes of the pencil*.

Assume $\theta = \varphi$, that is $\lambda = \cot\theta$, then it means that the local axis x_o is directed along the normal to the conic surface of revolution, on which the given conical spiral lies and in this case, the axis y_o coincides with the straight generatrix of the cone. When $a = 0$ and $\theta = 0$, the spiral surface degenerates into *a surface of revolution of a general sinusoid* presented in Chap. "2. Surface of Revolution".

In Figs. 4, 5, and 6, the spiral surfaces with the generatrix sinusoid of the different position having $\theta = -\pi/6$ (Fig. 4), $\theta = -\pi/3$ (Fig. 5), and $\theta = -\pi/2$ (Fig. 6) are shown.

■ Spiral Surface with Generatrix in the Form of the Evolvent of a Circle

A spiral surface with a generatrix in the form of the evolvent of a circle in the plane of a pencil is formed by a helical motion of the evolvent of a circle

$$x_o = x_o(v) = b[\cos v + (v_o + v)\sin v],$$
$$y_o = y_o(v) = b[\sin v - (v_o + v)\cos v],$$

along the conic spiral; x_o, y_o are the local Cartesian coordinates, v is the angle taken from the axis x_o in the direction of the axis y_o; v_o = const. The axis y_o intersects the axis of rotation at an angle θ (see also Chap. "8. Spiral Surfaces") and the beginning of the local Cartesian coordinates moves along the given conical spiral:

$$x = x(u) = ae^{mu}\cos u,$$
$$y = y(u) = ae^{mu}\sin u,$$
$$z = z(u) = a\lambda e^{mu},$$

where $\lambda = \cot\varphi$; φ is the angle of the axe Oz with a straight generatrix of the cone, on which the conic spiral lies; a longitude u is the angle of plane xOz with a moving plane of the axial cross-section; a, m are constants. All points of the generatrix evolvent of a circle trace *the conic spirals* which are *the lines of slope*.

In the practical purposes they use the following definition of the spiral surface with the generatrix in the form of the evolvent of a circle with a radius b (Fig. 1):

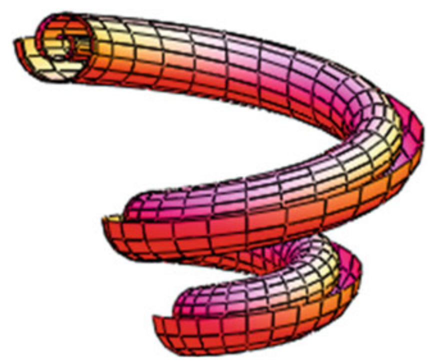


Fig. 1

$$x = x(u, v) = \{ae^{mu} + b[\cos v + (v_o + v)\sin v]\cos\theta$$
$$+ b[\sin v - (v_o + v)\cos v]\sin\theta\}\cos u,$$
$$y = y(u, v) = \{ae^{mu} + b[+(v_o + v)\sin v]\cos\theta$$
$$+ b[\sin v - (v_o + v)\cos v]\sin\theta\}\sin u,$$
$$z = z(u, v) = a\lambda e^{mu} - b[\cos v + (v_o + v)\sin v]\sin\theta$$
$$+ b[\sin v - (v_o + v)\cos v]\cos\theta.$$

This spiral surface is given in the curvilinear nonorthogonal nonconjugate coordinates u, v. The projections of the coordinate lines u on the plane xOy are *logarithmic spirals*. The coordinate lines v coincide with the generatrixes in the form of the evolvent of a circle of the spiral surface (Fig. 1).

■ Spiral Surface with Directrix Logarithmic spiral and with Parabolic Generatrix

A spiral surface with a directrix plane logarithmic spiral given in polar coordinates by an equation $\rho = ae^{mu}$ and with a parabolic generatrix given in a local system of coordinates by a parametrical equations $x_o(v) = v$; $y_o(v) = bv^2$ may be defined by the following parametric equations:

$$x = x(u,v) = \left[ae^{mu} + v\cos\theta + bv^2 sin\,\theta\right]\cos u;$$
$$y = y(u,v) = \left[ae^{mu} + v\cos\theta + bv^2\sin\theta\right]\sin u;$$
$$z = z(v) = -v\sin\theta + bv^2\cos\theta.$$

The axis y_o crosses the axis of rotation at an angle θ (see also Chap. "8. Spiral Surfaces") and an origin of the local coordinate system moves along the given plane logarithmic spiral; *a, b, m* are constants. Having put $\lambda = 0$, we can obtain the considered spiral surface as a particular case of *a spiral surface with a parabolic generatrix of an arbitrary position*.

In Fig. 1, the spiral surface with the directrix plane logarithmic spiral is shown when $\theta = 0$; in Fig. 2, it is given when $\theta = -\pi/2$; in Fig. 3, the surface has $\theta = \pi/2$; and in Fig. 4, it is presented when $\theta = \pi/4$.

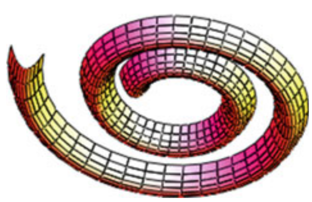


Fig. 1

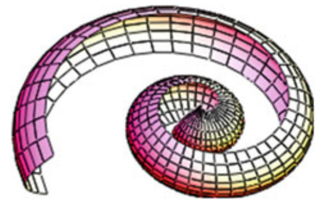


Fig. 2

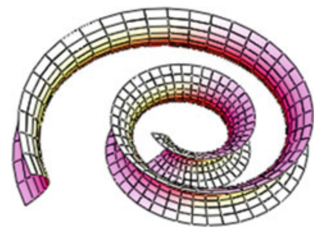


Fig. 3

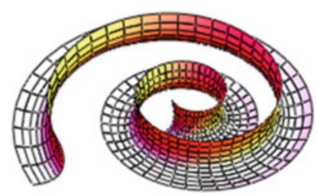


Fig. 4

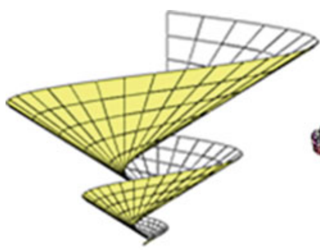
Additional Literature

Ivanov VN. Geometry and design of shells on the base of surfaces with a system of coordinate lines in the planes of pencil. Sb. Trudov: Prostran. Konstruk. Zdaniy i Soor. MOO «Prostranstvennye Konstruktzii». Moscow: «Devyatka Print», 2004; p. 26-35.

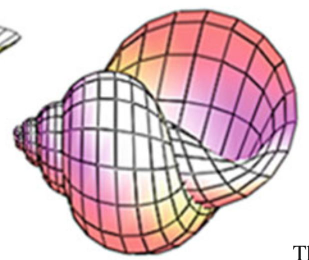
9 Spiral-Shaped Surfaces

Spiral-shaped surfaces bear a resemblance to *spiral surfaces* but these surfaces cannot be related to the same class because the spiral surface has the directrix curve only in the form of a spiral on a right circular cone and the generatrix curve does not change its form in the process of the motion along the conical spiral directrix line. But for a directrix curve of any spiral-shaped surface, one may take arbitrary spiral curve laying on any surface.

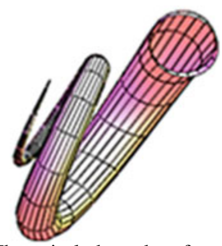
■ Spiral-Shaped Surfaces Presented in the Encyclopedia



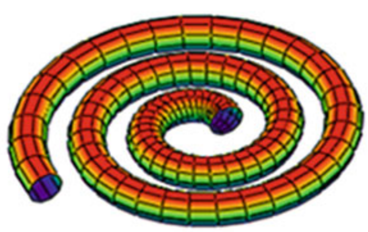
The developable helicoid with the cuspidal edge on the paraboloid of revolution



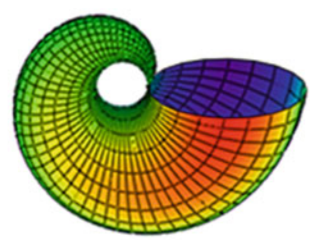
The spiral-shaped surface "Shell without vertex"



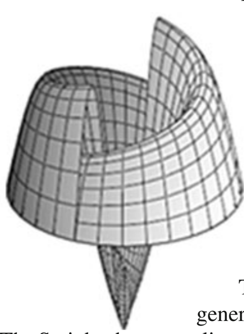
The spiral-shaped surface "Shell with vertex"



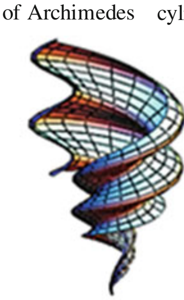
The cyclic surface with the generatrix circle in the planes of pencil and with the plane line of centers in the form of the spiral of Archimedes



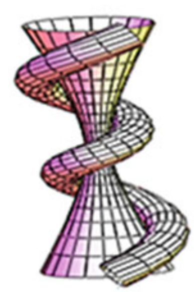
The cyclic surface with generatrix circles of variable radius and with a plane line of centers constructed about a circular cylinder



The Steinbach screw

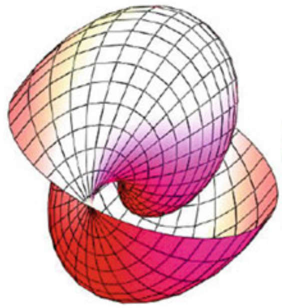


The spiral-shaped surface with generatrix sinusoids and with a directrix line of constant pitch on a circular cone

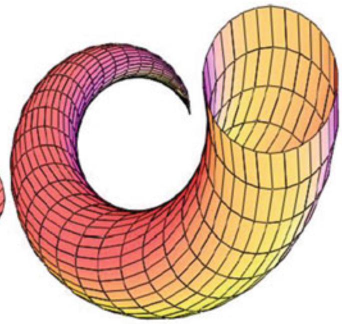


The tubular surface with a line of centers on the one-sheet hyperboloid of revolution

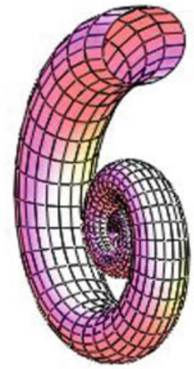
S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_9, © Springer International Publishing Switzerland 2015



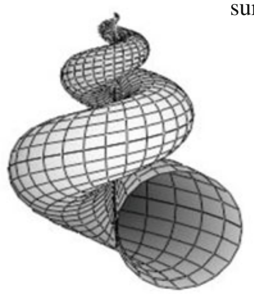
The snail surface



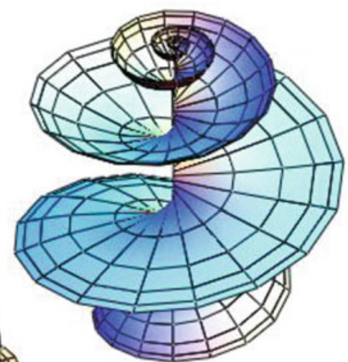
The right circular spiral-shaped surface with a generatrix circle of variable radius



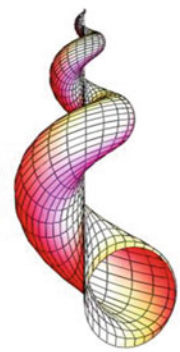
"Cornucopia"



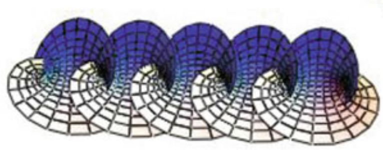
"Seashell"



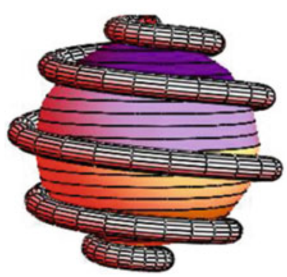
The hyperbolic helicoid



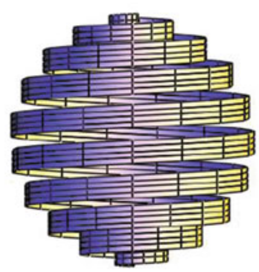
The spiral screw



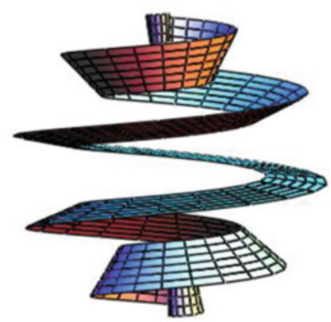
The transcendental surface with two families of plane isotropic coordinate lines



The tubular surface on the sphere

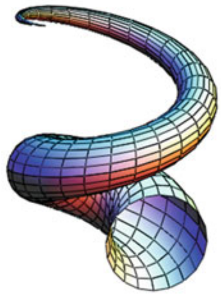


The cylindrical-and-spherical spiral-shaped strip

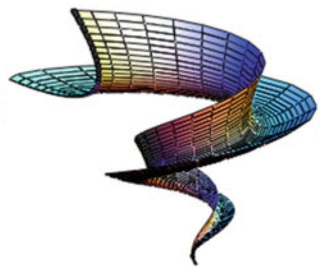


The conical surface with a directrix curve on the sphere when $p = 1/n$, where n is an integer, here $n = 10$

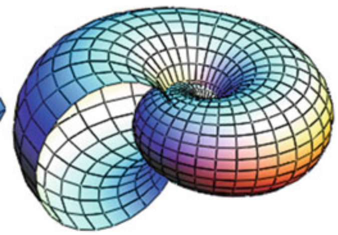
■ Spiral-Shaped Surfaces Presented in the Encyclopedia (sequel)



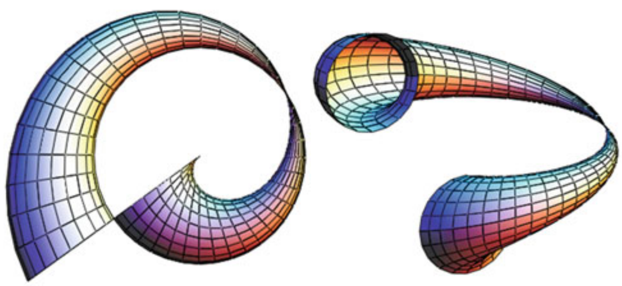
The cyclic surface in a cylinder



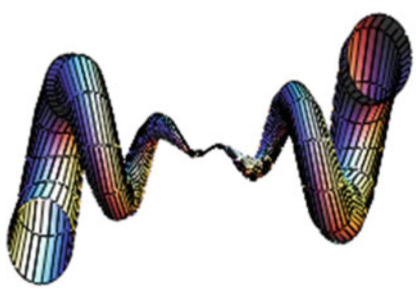
The spiral-shaped surface with the parabolic generatrixes and the directrix line of constant pitch on the circular cone



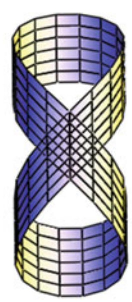
Normal cyclic surface with generatrix circles of variable radius and with a plane center-to-center line in the form of the logarithmic spiral [$R(u) = au$]



"Cyclic surface in the cylinder", transforming into "Cyclic surface about the cylinder"



The spiral-shaped surface with the elliptical generatrixes and with a line of centers of constant pitch on the circular cone



The cylindrical-and-spherical spiral-shaped strip

The ruled surface of the trajectory of movement of the straight generatrix of the evolvent helicoid in the process of its parabolic bending

■ Tubular Surface with a Line of Centers on One-Sheet Hyperboloid of Revolution

A one-sheet hyperboloid of revolution is given by an implicit equation:

$$x^2 + y^2 - a^2z^2 = c^2,$$

where c is the waist radius; $a = \tan\varphi$; φ is the angle of the axis of the hyperboloid with its straight generatrixes. Depending on an angle ω between the tangents to the slope line and the axis of the hyperboloid (slope angle ω), three types of slope lines can be disposed on the hyperboloid: (1) the straight generatrixes of the hyperboloid, when $\varphi = \omega$; (2) when $\varphi > \omega$; and (3) when $\varphi < \omega$.

Fig. 1

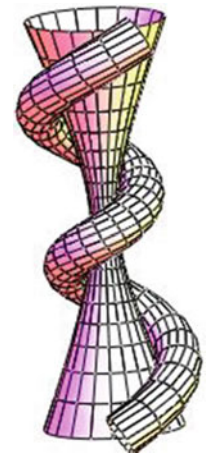


Fig. 2

Parametric equations of the line of slope on the one-sheet hyperboloid of revolution with $\varphi < \omega$ ($\tan\omega > \tan\varphi$) have the following form:

$$x(u) = c(m\sinh mu \cos u + \cosh mu \sin u),$$
$$y(u) = c(m\sinh mu \sin u - \cosh mu \cos u),$$
$$z(u) = (c/a)\sqrt{1+m^2}\sinh mu,$$

where $m = a/\sqrt{\tan^2\omega - a^2}$ (Fig. 1).

Taking this line of slope as a line of centers of a tubular surface, it is possible to form *a tubular surface with a line of centers on a one-sheet hyperboloid of revolution.*

Parametrical equations of this tubular surface are

$$x(u,v) = c[m\sinh mu \cos u + \cosh mu \sin u] - r((m/s)\sin v \cos u + \cos v \sin u),$$
$$y(u,v) = c[m\sinh mu \sin u - \cosh mu \cos u] + r(-(m/s)\sin v \sin u + \cos v \cos u),$$
$$z(u,v) = \sqrt{1+m^2}[(c/a)\sinh mu + (a/s)\sin v],$$

where

$$s = \sqrt{a^2(1+m^2)+m^2},$$

u, v are curvilinear coordinates. The tubular surface shown in Fig. 2 has

$$c = 5\ \text{m};\quad \omega = \pi/3;\quad \varphi = \pi/8;\quad r = 5\ \text{m};\quad -2\pi < u < 2\pi.$$

Additional Literature

Kirischiev RI. Lines of slope on surfaces of revolution of the second order and their projections. Matematika, Nekotorye eyo Prilozheniya i Metodika Prepodavaniya. Rostov-na-Donu. 1972; p. 80-94 (2 refs).

■ Ruled Surface of the Trajectory of Movement of Straight Generatrix of an Evolvent Helicoid in the Process of its Parabolic Bending

Let us take an annulus with inside radius a_0 and cut it along a straight line passing through the point with coordinates $x = a_0$, $y = 0$, and parallel to the y axis. This straight line is the tangent line to the inside contour.

Parabolic bending transforms the annulus into an open evolvent helicoid with a cuspidal edge in the form of a helix lying on the cylinder. Having known this propositions, it is possible to design the whole spectrum of developable (evolvent) helicoids with the increasing slope angles of their rectilinear generatrixes in the limit from 0 to $\pi/2$, i.e. $0 \le \varphi \le \pi/2$, using the parabolic bending of initial workpiece.

Using parabolic bending, we can make developable helicoids without any lap fold or rupture (break). For guaranteeing of parabolic bending of any developable helicoid into another developable helicoid, it is necessary to secure the motion of the straight generatrixes strictly along the given trajectories that are *ruled spiral-shaped surfaces of negative total curvature*. Hence, taking an annulus cut along a tangent to the inside contour, we can write the equations of all class of developable helicoids:

$$x = x(u,\varphi) = a_0\cos^2\varphi[\cos(ms) - um\sin(ms)],$$
$$y = x(u,\varphi) = a_0\cos^2\varphi[\sin(ms) + um\cos(ms)],$$
$$z = z(u,\varphi) = (s+u)\sin\varphi;\ m = 1/(a_0\cos\varphi),$$

where s = const is the length of the arch of the helical cuspidal edge; u is a curvilinear coordinate line coinciding with a straight generatrixes of the surface of the trajectories.

Assume $u = 0$, then it is possible to form the trajectory of the movement of any point (s = const) of the internal circular

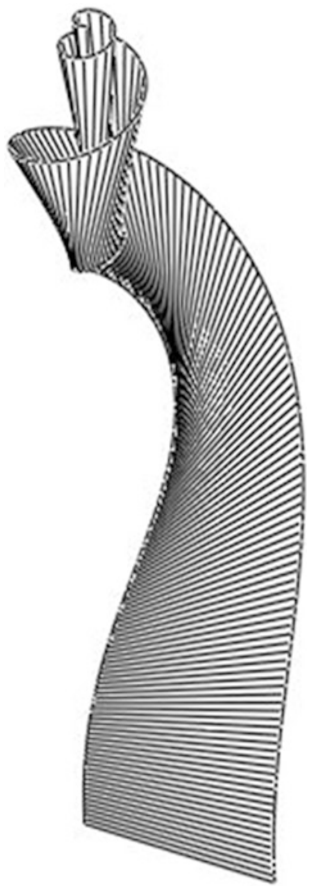


Fig. 1

Fig. 2

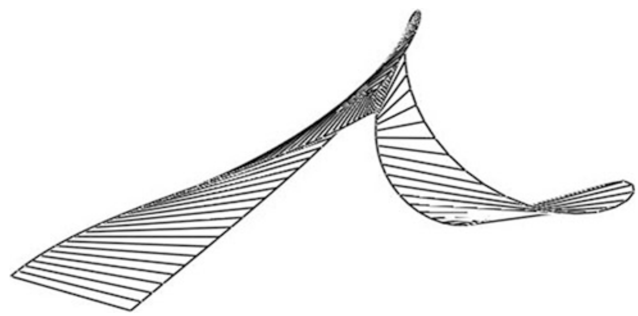


Fig. 3

contour of the annulus in the process of its bending into a developable helicoid. The parameter s changes for a whole annulus at the limits: $0 \le s \le 2\pi a_0$. In Fig. 1, the ruled surface of the trajectories of the straight generatrix $s = 2\pi a_0$ is shown which coincides with the line of cutting of the annulus. The given annulus is bent into the developable helicoid with $\varphi = 75^{\circ}$; $0 \le u \le 4m$.

Let us take the first straight generatrix $s = 0$, then the trajectory will be a ruled surface (Fig. 2):

$$x = x(u, \varphi) = a_0 \cos^2\varphi, y = y(u, \varphi) = u \cos \varphi,$$
$$z = z(u, \varphi) = u \sin \varphi.$$

Coefficients of the fundamental form of the surface:

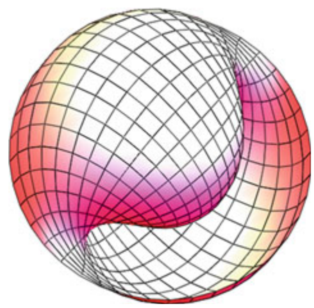
$$A = 1, \quad F = s \sin 2\varphi,$$
$$B^2 = a_0^2 \sin^2 2\varphi + \tan^2 \varphi u^2 s^2 / a_0^2 + (s + u)^2, \quad L = 0.$$

It is possible to write parametrical equations of the surface of the trajectories with taking into account that the first generatrix straight $s = 0$ is unmovable:

$$\begin{aligned} x = x(u, \varphi) &= a_0 \cos^2 \varphi[\cos(ms) - um \sin(ms)] \\ &\quad + a_0(1 - \cos^2 \varphi), \\ y = x(u, \varphi) &= a_0 \cos^3 \varphi[\sin(ms) + um \cos(ms)] \\ &\quad + \sin^2 \varphi(s + u), \\ z = z(u, \varphi) &= a_0 \cos^2 \varphi \sin \varphi(\sin(ms) + um \cos(ms)) \\ &\quad + \sin \varphi \cos \varphi(s + u). \end{aligned}$$

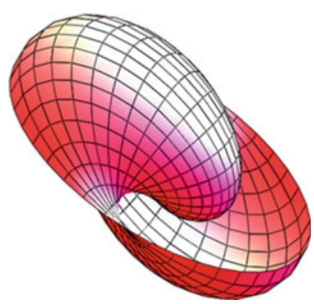
In Fig. 3, we have $s = 2\pi a_0$, $0 \le \varphi < 65^{\circ}$.

■ Snail surface



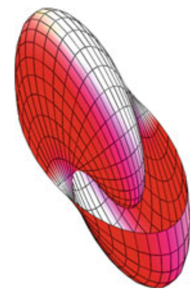
$a = b = c = 1$

Fig. 1



$a = b = 1;\ c = 0.5$

Fig. 2



$a = 0.5;\ b = 1;\ c = 0.25$

Fig. 3

Snail surface is given by parametrical equations

$$x = x(u, v) = au \sin u \cos v, \quad y = y(u, v) = bu \cos u \cos v,$$
$$z = z(u, v) = -cu \sin v,$$

$0 \le u \le 2\pi$, $-\pi \le v \le \pi$; a, b, c [m] are constants (Figs. 1, 2 and 3).

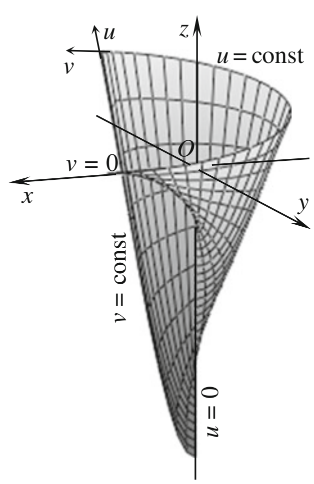
Reference

Parametrische Flächen und Körper.—http://www.3d-meier.de/tut3/Seite38.html

Paul Bourke. Surfaces and curves. University of Western Australia.—http://local.wasp.uwa.edu.au/~pbourke

■ Steinbach Screw

A surface called *Steinbach screw* contains a straight line about which it is disposed. Steinbach screw may be related to a class of spiral-shaped surfaces (Figs. 1–3).



$0 \le v \le 2\pi;\ 0 \le v \le 2$

Fig. 1

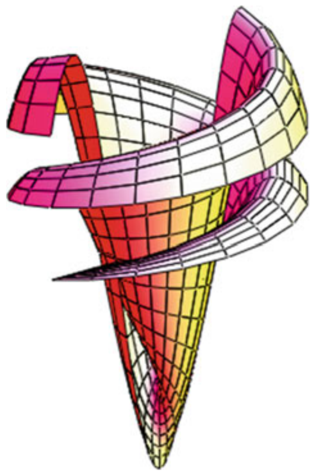
It is known a parametrical form of definition of this surface (Fig. 1):

$$x = x(u, v) = u \cos v,$$
$$y = y(u, v) = u \sin v$$
$$z = z(u, v) = v \cos u$$

The curvilinear coordinate line $u = 0$ ($z = v$) coincides with the coordinate line Oz. The curvilinear coordinate line $v = 0$ ($x = u$) coincides with the Ox axis (Fig. 1).

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + v^2 \sin^2 u, \quad F = -v \sin u \cos u,$$
$$B^2 = u^2 + \cos^2 u$$
$$A^2B^2 - F^2 = u^2 + \cos^2 u + u^2 v^2 \sin^2 u,$$
$$L = \frac{-uv \cos u}{\sqrt{u^2 + \cos^2 u + u^2 v^2 \sin^2 u}},$$
$$M = \frac{-\cos u - u \sin u}{\sqrt{u^2 + \cos^2 u + u^2 v^2 \sin^2 u}},$$
$$N = \frac{-u^2 v \sin u}{\sqrt{u^2 + \cos^2 u + u^2 v^2 \sin^2 u}}$$



$-4 \le u \le 4$

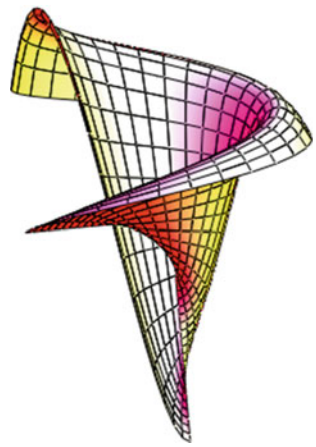
Fig. 2 $0 \le u \le \pi 2$

$$K = \frac{u^3 v^2 \sin u \cos u - (\cos u + u \sin u)^2}{(u^2 + \cos^2 u + u^2 v^2 \sin^2 u)^2},$$

$$H = -v \frac{u(1 + u^2 + \sin^2 u) \cos u + (u^2 + 2\cos^2 u + u^2 v^2 \sin^2 u) \sin u}{2(u^2 + \cos^2 u + u^2 v^2 \sin^2 u)^{3/2}}.$$

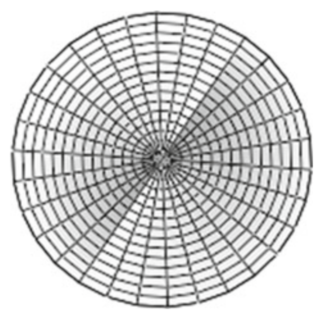
Mean curvature of the surface is equal to zero ($H = 0$) along the coordinate lines $u = 0$ and $v = 0$. The coordinate lines v (u = const) are imaged on the plane xOy in the forms of circles with radiuses u = const (Fig. 4). Equations of the projections of the coordinate lines u ($v = v_o$ = const) on the planes xOz and yOz can be written in the form:

$$z = v_o \cos \frac{x}{\cos v_o} \quad \text{and} \quad z = v_o \cos \frac{y}{\cos v_o}.$$



$0 \le u \le 4$

Fig. 3 $0 \le u \le \pi 2$



$0 \le v \le 2\pi$
$-\pi/2 \le u \le \pi/2$

Fig. 4

■ Spiral-Shaped Surface with Parabolic Generatrixes and Directrix Line of Constant Pitch on a Circular Cone

A surface is formed by parabolas lying at the planes of pencil with a fixed straight line passing through the axis of a cone. The vertexes of parabolas lay on a directrix line that is projected on a plane perpendicular to the axis of the cone as *spiral of Archimedes*.

Forms of definition of the surface

(1) Vector equation: $\boldsymbol{r}(u,v) = [au + \varphi(v)]\boldsymbol{h}(u) + [a\lambda u + \psi(v)]\boldsymbol{k}$,
where $\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u$ is the unit vector on the plane xOy; $\varepsilon(u)$ is a coefficient of similitude of the generatrix parabolas;

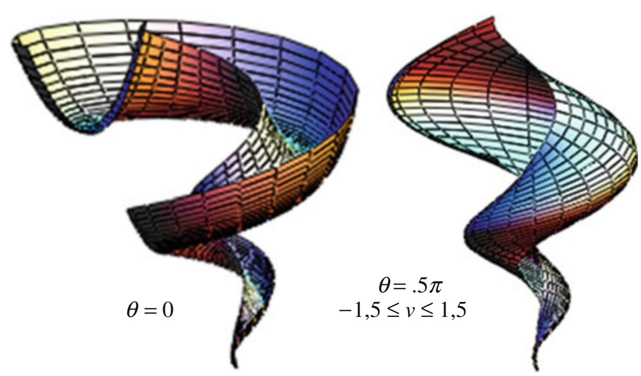


Fig. 1

$$\varphi(v) = X(v)\cos\theta - Y(v)\sin\theta,$$
$$\psi(v) = X(v)\sin\theta + Y(v)\cos\theta,$$

$X = X(v)$, $Y = Y(v)$ are parametric equations of the generatrix curve given in local system of the Cartesian coordinates, θ is the angle of the turn of the local axis oZ relatively to the axis Oz. For a surface with parabolic generatrixes:

$$X = v, Y = Y(v) = bv^2.$$

(2) Parametrical equations (Fig. 1, where $\varepsilon = tu$):

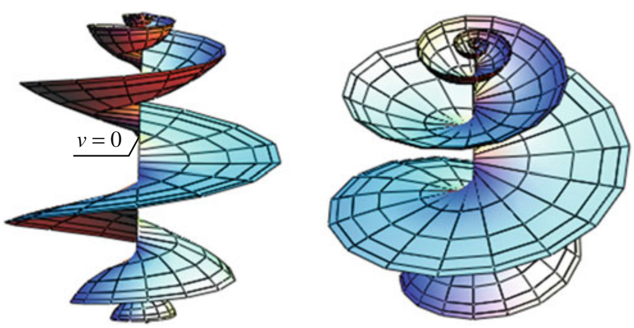
$$x = x(u, v) = [au + \varepsilon(u)\varphi(v)] \cos u,$$
$$y = y(u, v) = [au + \varepsilon(u)\varphi(v)] \sin u;$$
$$z = z(u, v) = [a\lambda u + \varepsilon(u)\psi(v)].$$

■ Hyperbolic Helicoid

Hyperbolic helicoid may be related to a class of spiral-shaped or to a class of helical-shaped surfaces.

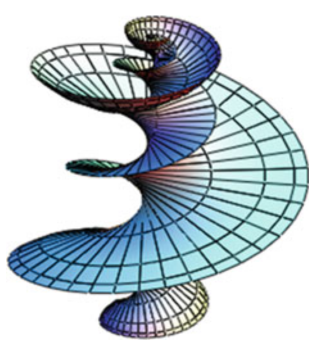
Forms of definition of surface of hyperbolic helicoid

(1) Parametrical equations (Figs. 1, 2, 3 and 4):



$t = 6;\ -\pi \le u \le 2\pi;\ 0 \le v \le 3.5$

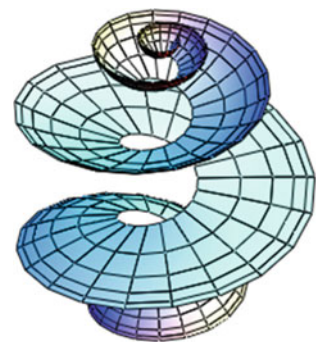
Fig. 1



$t = 0;\ -\pi \le u \le \pi;\ -1 \le v \le 3.5$

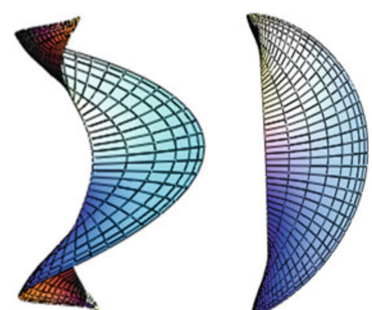
Fig. 2

$$x = x(u, v) = \frac{\sinh v}{1 + \cosh u \cosh v} \cos tu,$$
$$y = y(u, v) = \frac{\sinh v}{1 + \cosh u \cosh v},$$
$$z = z(u, v) = \frac{\sinh u \cosh v}{1 + \cosh u \cosh v},$$



$t = 4;\ -\pi \le u \le \pi;\ 1 \le v \le 3.5$

Fig. 3



$-\pi \le u \le \pi;$

$0.25 \le v \le 3.5;\quad 0 \le v \le 3.5;$

$t = 2;\quad t = 1$

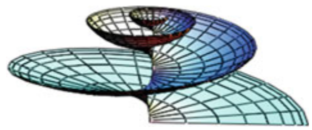
Fig. 4

where t is a constant parameter (*the torsion*); tu is the angle taken from the coordinate axis Ox in the direction of the axis Oy; u, v are the curvilinear coordinates. The coordinate lines v (u = const) are *plane lines lying in the planes of pencil* with the fixed straight line passing through the axis of a hyperbolic helicoid. The axis of a helicoid is a straight line coinciding with the curvilinear coordinate line $v = 0$ lying on the coordinate axis Oz.

The family of the coordinate lines v is projected on the coordinate plane xOy as a pencil of straight lines

$$y = x \tan(tu),$$

passing through a point $x = y = 0$ (Figs. 1, 2 and 3).



$u = 0$
$t = 6;\ 0 \le u \le \pi$

Fig. 5

The coordinate line $u = 0$ lays in the cross-section of the surface of a hyperbolic helicoid by a plane $y = 0$ and is a straight line coinciding with the coordinate axis Ox (Fig. 5).

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{t^2 \sinh^2 v + \cosh^2 v}{(1 + \cosh u \cosh v)^2};\quad F = 0;$$
$$B^2 = \frac{1}{(1 + \cosh u \cosh v)^2},$$
$$L = -\frac{t \sinh v \sinh u}{(1 + \cosh u \cosh v)} A;\quad M = \frac{tB^2}{A},$$
$$N = -\frac{t \sinh u \sinh v}{A(1 + \cosh u \cosh v)} B^2,$$
$$K = \frac{t^2 B^2}{A^2}\left(\sinh^2 v \sinh^2 u - \frac{1}{A^2}\right).$$

A surface of a hyperbolic helicoid is given in curvilinear orthogonal non-conjugate coordinates u, v.

In Figs. 1, 2, 3, 4 and 5, the hyperbolic helicoids with geometrical parameters shown in the corresponding figures are presented.

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Weisstein, Eric W. “Hyperbolic Helicoid.” From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/HyperbolicHelicoid.html

Weisstein, Eric W. CRC Concise Encyclopedia of Mathematics. Second edition. Chapman & Hall/CRC. 2003; 3226 p.

■ Cornucopia

A surface *“Cornucopia”* is a special case of a spiral-shaped surface “*Shell without vertex*” (see also “9.1. *Spiral-shaped cyclic surfaces with circles of variable radius at the planes of pencil*”) . Assume $a = b = 1$; $\lambda = 0$ and substitute these values into parametrical equations of a surface “*Shell without vertex*” and then we shall obtain the equations of a surface called *“Cornucopia”*.

So, a surface called “*Cornucopia”* is a *cyclic surface* with generatrix circles of variable radius

$$R(u) = e^{pu}$$

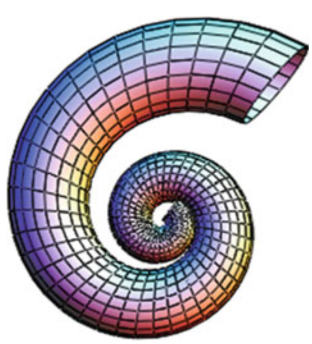
in the planes of pencil. A fixed straight line of a pencil of the planes passes over the coordinate axis Oz. A line of the centers of this cyclic surface is a plane logarithmic spiral

$$\rho = e^{mu}.$$

Forms of the definition of the surface

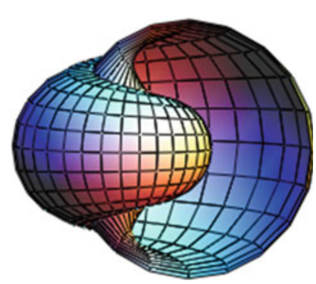
(1) Parametrical equations (Figs. 1 and 2):

$$x = x(u, v) = [e^{mu} + e^{pu} \cos v] \cos u,$$
$$y = y(u, v) = [e^{mu} + e^{pu} \cos v] \sin u$$
$$z = z(u, v) = e^{pu} \sin v.$$



$p = 0.1;\ m = 0.2$
$0 \le u \le 4\pi;\ 0 \le v \le 2\pi$

Fig. 1



$p = 0.15;\ m = 0.1;$
$0 \le u \le 5\pi;\ 0 \le v \le 2\pi$

Fig. 2

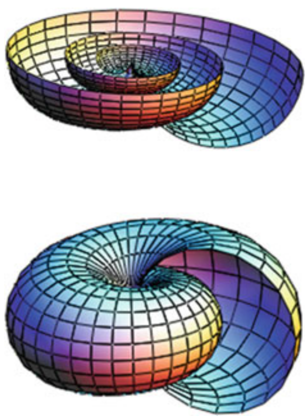


Fig. 3 $p = m = 0.1$

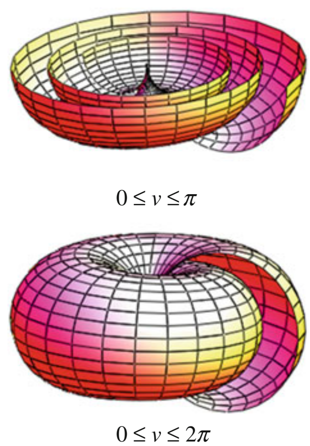


Fig. 4 $p = m = 0.05$

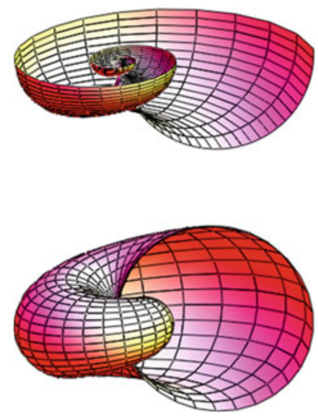


Fig. 5 $p = m = 0.2$

If we assume $p = 0$ in the parametrical equations of the studied surface, then a cyclic surface with a plane line of the centers in the form of a logarithmic spiral and with a generatrix circle of the constant radius $r = 1$ m in the planes of a pencil (see also "Cyclic surface with generatrix circle in the planes of pencil and the directrix in the form of a logarithmic spiral") will be obtained.

Coefficients of the fundamental forms of the surface:

$$A^2 = (me^{mu} + pe^{pu}\cos v)^2 + (e^{mu} + e^{pu}\cos v)^2 + p^2 e^{2pu}\sin^2 v,$$
$$F = -me^{pu}e^{mu}\sin v, \quad B = e^{pu},$$
$$A^2B^2 - F^2 = e^{2pu}\left[(me^{mu}\cos v + pe^{pu})^2 + (e^{mu} + e^{pu}\cos v)^2\right],$$
$$L = \frac{e^{pu}}{\sqrt{A^2B^2 - F^2}}\Big[2(me^{mu} + pe^{pu}\cos v)(pe^{pu} + me^{mu}\cos v) - (e^{mu} + e^{pu}\cos v)\left(m^2e^{mu}\cos v + p^2e^{pu}\right) + (e^{mu} + e^{pu}\cos v)^2\cos v\Big],$$
$$M = \frac{-e^{2pu}\sin v}{\sqrt{A^2B^2 - F^2}}(pe^{pu} + me^{mu}\cos v),$$
$$N = \frac{e^{2pu}}{\sqrt{A^2B^2 - F^2}}(e^{mu} + e^{pu}\cos v).$$

The surface is given in curvilinear non-orthogonal non-conjugate coordinates u, v. The coordinate lines $v = 0$ and $v = \pi$ are the lines of principal curvatures.

Coefficients of the fundamental forms of "Cornucopia" may be derived from the general formulas given in a Sect. "9.1. Spiral-shaped cyclic surfaces with circles of variable radius in the planes of pencil" when $a = b = 1$; $\lambda = 0$.

In Figs. 3, 4 and 5, the considered surfaces are presented when $p = m$. In this case, surface "*Cornucopia*" becomes a *canal surface of Joachimsthal* (see also Chap. "17. *Cyclic Surfaces*").

Additional Literature

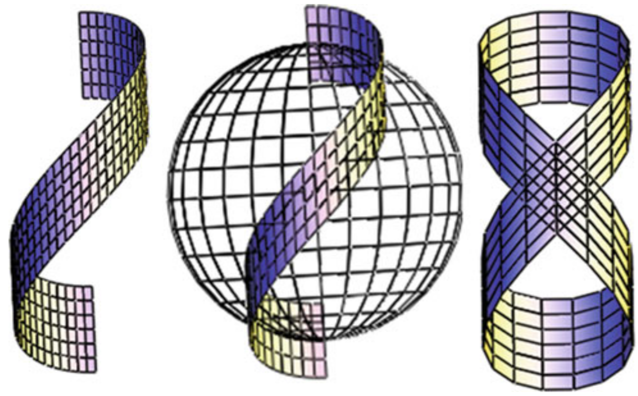
von Seggern D. CRC Standard Curves and Surfaces. Boca Raton. FL: CRC Press, 1993; p. 304.

■ Cylindrical-and-Spherical Spiral-Shaped Strip

A cylindrical surface with *a directrix spherical line*

$$\boldsymbol{E}_0(u) = a\boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\sin\omega,$$

where $\omega = pu$; $p = \text{const}$, disposed on a spherical surface with a radius a, may be formed by generatrix straight lines that are parallel to the chosen axis of the sphere. The surface designed by the method described above is called a *cylindrical-and-spherical spiral-shaped strip*.



$-\pi/2 \le u \le \pi/2$; $k_1 = 2/a$; $-\pi \le u \le \pi$

Fig. 1 $a = 20$; $p = 1$; $-6 \le u \le 6$

Forms of definition of a cylindrical-and-spherical spiral-shaped strip

(1) Vector form of the definition:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = a\boldsymbol{e}_0(u) + v\boldsymbol{k}.$$

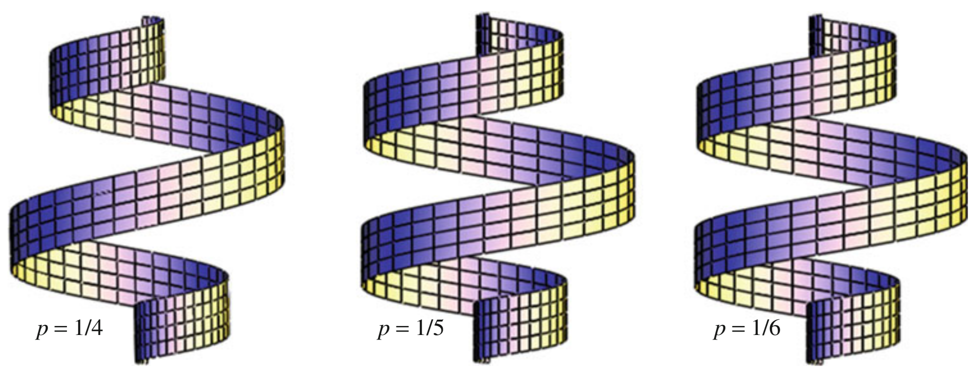
The unit vector $\boldsymbol{e}_0(u)$ is the normal of the sphere with a directrix curve.

(2) Parametrical form of the definition (Figs. 1, 2, 3 and 4):

$$x = x(u) = a\cos\omega\cos u,$$
$$y = y(u) = a\cos\omega\sin u,$$
$$z = z(u, v) = a\sin\omega + v,$$

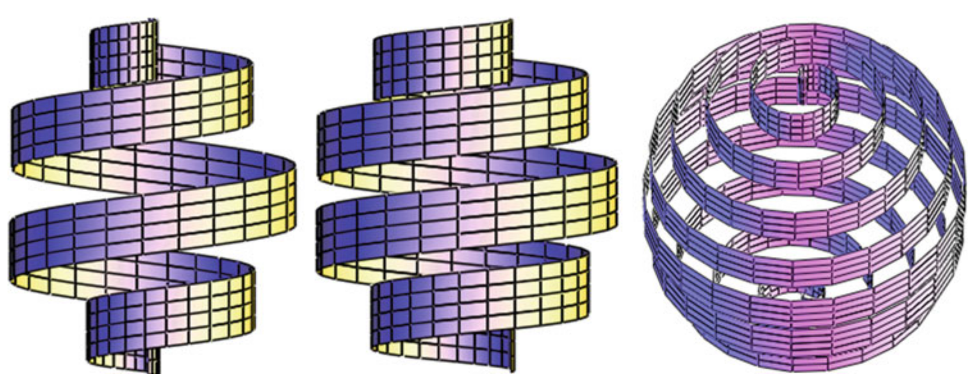
Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = a^2(p^2 + \cos^2\omega),\quad F = ap\cos\omega,\quad B = 1,$$
$$A^2B^2 - F^2 = a^2(p^2\sin^2\omega + \cos^2\omega),$$
$$L = -\frac{a\left[(1 + \sin^2\omega)p^2 + \cos^2\omega\right]}{\sqrt{p^2\sin^2\omega + \cos^2\omega}},\quad M = N = 0$$
$$k_u = -\frac{a\left[(1 + \sin^2\omega)p^2 + \cos^2\omega\right]}{A^2\sqrt{p^2\sin^2\omega + \cos^2\omega}},\quad k_v = k_2 = 0$$
$$k_1 = L/(A^2 - F^2),\quad K = 0$$



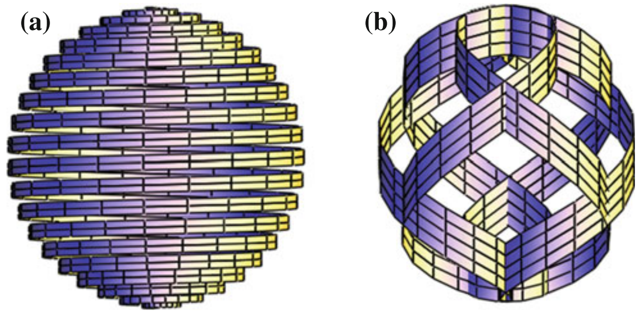
$-2\pi \le u \le 2\pi$; $-2.5\pi \le u \le 2.5\pi$; $-3\pi \le u \le 3\pi$;

Fig. 2 $a = 20$; $-4 \le u \le 4$



$-4\pi \le u \le 4\pi$; $p = 1/8$; $-4 \le v \le 4$; $-5\pi \le u \le 5\pi$; $p = 1/10$; $-10\pi \le u \le 10\pi$; $p = 1/20$; $-2 \le v \le 2$

Fig. 3 $a = 20$



$p = 1/40; -1 \le v \le 1;$
$-20\pi \le u \le 20\pi;$

$p = 2/3; -4 \le v \le 4;$
$-2\pi \le u \le 2\pi;$

Fig. 4 a = 20

For the construction of cylindrical-and-spherical spiral-shaped strips shown in Figs. 2, 3 and 4a, it is necessary to take a parameter $p = 1/n$, where n is an integer. In opposite case, we shall obtain the surfaces of the more complex form (Fig. 4b).

The projection of a cylindrical-and-spherical spiral-shaped strip on a plane z = const is a spiral curve: $\rho = a \cos \omega = a \cos (pu)$.

■ Transcendental Surface with Two Families of Plane Isotropic Coordinate Lines

Isotropic spherical representation of asymptotic lines of *a transcendental surface with two families of plane isotropic coordinate lines* forms two orthogonal pencils of the isotropic circles. In conjugated isotropic space, isotropic minimal surfaces Φ^* with the plane lines of principal curvature correspond to the surfaces Φ that are mentioned above. In spite of this, the surfaces Φ^* prove to be the transcendental isotropic minimal surfaces.

Parametrical equations of the studied surface (Figs. 1, 2, 3, 4 and 5) are

$$\begin{aligned} x &= x(u, v) = u + \cosh v \sin u, \\ y &= y(u, v) = v + \sinh v \cos u, \\ z &= z(u, v) = \sinh v \sin u, \end{aligned}$$

where u, v are isotropic asymptotical parameters.

Coefficient of the fundamental forms of the surface and its curvatures:

$$\begin{aligned} A^2 &= \left(2\cos u + \cosh v + \cos^2 u\right)\cosh v, \\ F &= \sinh v \cosh v \sin u \cos u, \\ B^2 &= -\sinh^2 v \cos^2 u + 2\cosh v(\cos u + \cosh v) \\ A^2B^2 - F^2 &= 2\cosh v\left[\left(\cos^2 u + 3\cosh^2 v\right)\cos u\right. \\ &\qquad \left. + \left(\cosh^2 v + 3\cos^2 u\right)\cosh v\right] \end{aligned}$$

$$L = 0, \quad M = \frac{(\cos u + \cosh v)^2}{\sqrt{A^2B^2 - F^2}}, \quad N = 0$$

$$k_u = k_v = 0, \quad k_1 = \frac{M}{AB + F}, \quad k_2 = \frac{-M}{AB - F}$$

$$K = -\frac{M^2}{A^2B^2 - F^2} < 0, \quad H = -\frac{MF}{A^2B^2 - F^2}$$

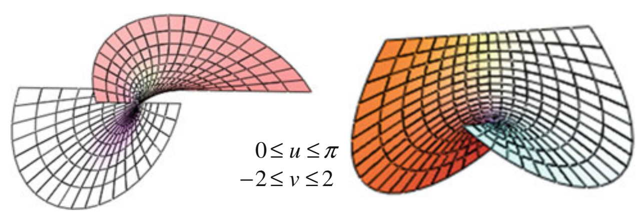


Fig. 1

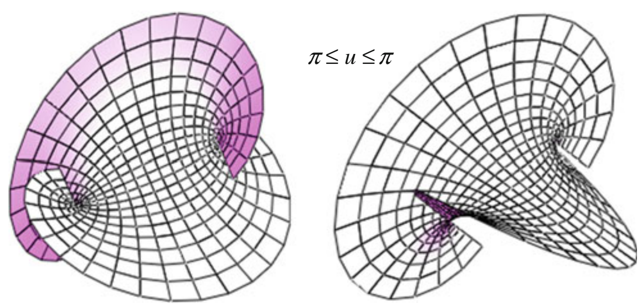


Fig. 2

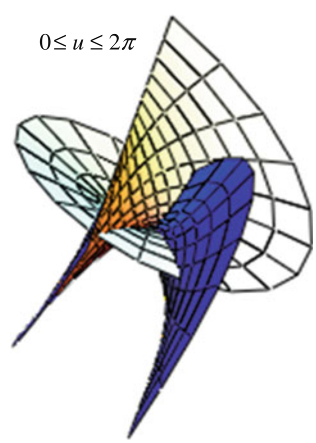


Fig. 3

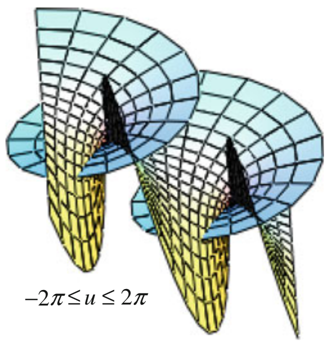


Fig. 4

The surface is given in the curvilinear non-orthogonal non-conjugate system of coordinates. The coordinates line u and v are plane lines.

The coordinates lines u ($v = v_c$ = const) are projected on the plane yOz in the form of the following curves:

$$z^2 = (y - v_c)^2 - \sinh^2 v_c.$$

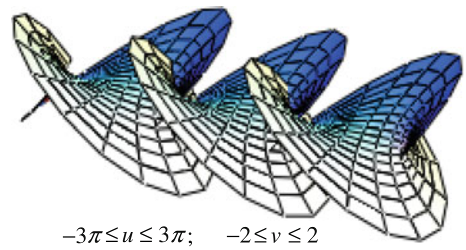


Fig. 5

In Figs. 1, 2, 3, 4 and 5, the examined surfaces bounded by the coordinate lines u and v are shown. In all figures, the surfaces are bounded by the contour lines $v = -2$ and $v = 2$, but the limits of the changing coordinate u are given in corresponding figures.

Additional Literature

Strubecker Karl. Über die Minimalflächen des isotropen Raumes, welche zugleich Affinminimalflächen sind. Monatsh. Math. 1977; 84, No. 4, s. 303-339.

■ Spiral-Shaped Surface with Elliptical Generatrixes and with a Line of Centers of Constant Pitch on a Circular Cone

A surface is formed by ellipses lying in the planes of pencil with a fixed straight line passing through the axis of a cone. The line of centers of a constant pitch on the circular cone is projected on a plane, which is perpendicular to the conical axis, as *spiral of Archimedes*.

Forms of definition of the surface

(1) Vector equation:

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{r}(u, v)\\ &= [au + \varphi(v)]\boldsymbol{h}(u) + [a\lambda u + \psi(v)]\boldsymbol{k}, \text{ where } \boldsymbol{h}(u)\\ &= \boldsymbol{i}\cos u + \boldsymbol{j}\sin u\end{aligned}$$

is the unit vector in a plane xOy; $\varepsilon(u)$ is a similarity factor of the generatrix ellipses taken according to given conditions of the design;

$$\varphi(v) = X(v)\cos\theta - Y(v)\sin\theta,$$
$$\psi(v) = X(v)\sin\theta + Y(v)\cos\theta.$$

$X = X(v)$, $Y = Y(v)$ are parametrical equations of the generatrix curves given in a local system of Cartesian coordinates. The origin of the coordinates is placed on a directrix curve; θ is the angle of turn of the local axis oZ relatively to the axis Oz. For a surface with elliptical generatrixes, we must assume:

$$X = X(v) = b\cos v, \quad Y = Y(v) = c\sin v.$$

(2) Parametrical form of the definition (Fig. 1):

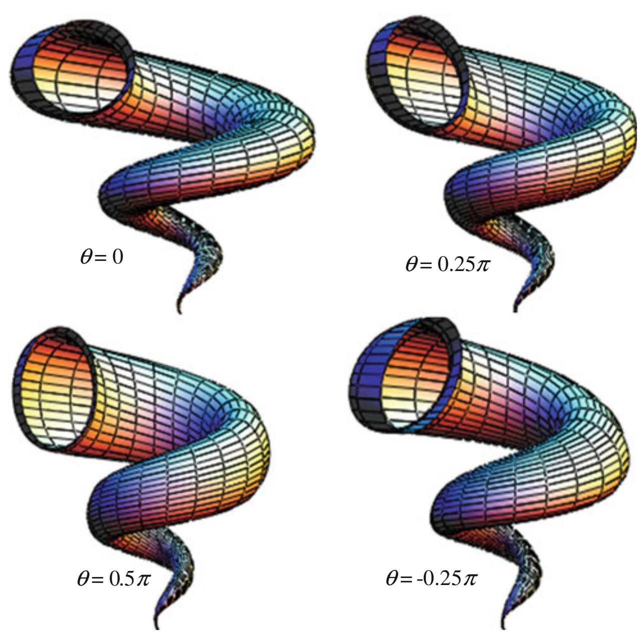


Fig. 1 $a = 0.8$; $\lambda = 0.2$; $b = 4$; $c = 3$; $t = 1$; $\varepsilon = tu$; $0 \le u \le 4\pi$

$$x = x(u,v) = [au + \varepsilon(u)\varphi(v)]\cos u,$$
$$y = y(u,v) = [au + \varepsilon(u)\varphi(v)]\sin u;$$
$$z = z(u,v) = [a\lambda u + \varepsilon(u)\psi(v)].$$

Additional Literature

Tevlin AM, Sulyukmanov FC. Some differential-and-geometrical characteristics of nonlinear quasi-helical surface. Voprosy Prikl. Geometrii. Moscow: MAI, 1972; Iss. 246, p. 110-114.

■ Seashell

A surface called "*Seashell"* is related to spiral- shaped cyclic surfaces with circles of variable radius lying at the planes of pencil. A fixed straight of the pencil of planes coincides with an axis of *the directrix conic spiral*.

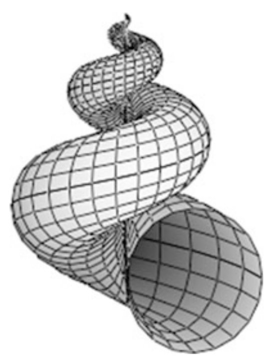


Fig. 1

Forms of the definition of the surface "Seashell"

(1) Parametrical equations (Fig. 1):

$$x = x(u,v) = 2[1 - e^{u/(6\pi)}]\cos u \cos^2(v/2),$$
$$y = y(u,v) = 2[-1 + e^{u/(6\pi)}]\sin u \cos^2(v/2),$$
$$z = z(u,v) = 1 - e^{u/(3\pi)} - \sin v + e^{u/(6\pi)}\sin v.$$

Fig. 2

(2) Parametrical equations, given by T. Nordstrand (Fig. 2):

$$x = x(u,v) = \left[\left(1 - \frac{v}{2\pi}\right)(1 + \cos u) + c\right]\cos nv,$$
$$y = y(u,v) = \left[\left(1 - \frac{v}{2\pi}\right)(1 + \cos u) + c\right]\sin nv$$
$$z = z(u,v) = \frac{bv}{2\pi} + a\sin u\left(1 - \frac{v}{2\pi}\right).$$

Additional Literature

Nordstrand T. Conic Spiral or Seashell. http://www.uib.no/people/nfyth/shelltxt.htm.

9.1 Spiral-Shaped Cyclic Surfaces with Circles of Variable Radius in the Planes of Pencil

Spiral-shaped cyclic surfaces with circles of variable radius in the planes of pencil may be related both to *a class of cyclic surfaces* and to *a class of spiral-shaped surfaces.* They have a conic helical curve $\rho = \rho(u) = ae^{mu}[h(u) + \lambda k]$ as a directrix curve. Here, $\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u$ is the circle of the unit radius at the plane *xOy*. The generating circles of variable radius $R(u)$ lay in the planes of pencil. The fixed straight line of the pencil coincides with the axis *Oz*.

Forms of definition of the surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + R(u)e(u,v),$$

Where $\boldsymbol{e}(u,v) = \boldsymbol{h}(u)\cos v + \boldsymbol{k}\sin v$.

(2) Parametrical equations:

$$x = x(u,v) = [ae^{mu} + R(u)\cos v]\cos u,$$
$$y = y(u,v) = [ae^{mu} + R(u)\cos v]\sin u,$$
$$z = z(u,v) = a\lambda e^{mu} + R(u)\sin v.$$

Coefficients of the fundamental forms of the surface:

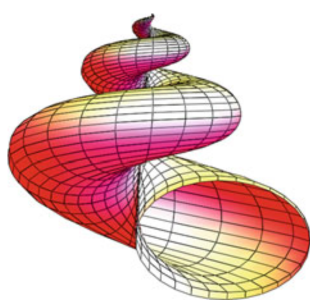
$$A^2 = a^2m^2(1+\lambda^2)e^{2mu} + R'^2 + [ae^{mu} + R(u)\cos v]^2 + 2ame^{mu}R'(\cos v + \lambda \sin v)$$
$$F = -ame^{mu}R(u)(\sin v - \lambda \cos v, \quad B = R(u),$$
$$A^2B^2 - F^2 = R^2\Big\{[ame^{mu}(\cos v + \lambda \sin v) + R']^2 + (ae^{mu} + R(u)\cos v)^2\Big\},$$
$$L = \frac{-R(u)}{\sqrt{A^2B^2 - F^2}}\big\{\big[a(1-m^2)e^{mu}\cos v + R(u)\cos^2 v - am^2\lambda e^{mu}\sin v - R''\big](ae^{mu} + R(u)\cos v) + 2(ame^{mu} + R'\cos v) \times [ame^{mu}(\cos v + \lambda \sin v) + R']\big\},$$
$$M = \frac{R(u)^2 \sin v}{\sqrt{A^2B^2 - F^2}}[R' + ame^{mu}(\cos v + \lambda \sin v)],$$
$$N = \frac{-R(u)^2(ae^{mu} + R(u)\cos v)}{\sqrt{A^2B^2 - F^2}}$$

where R' and R'' are differentiation with respect to a parameter u. Two surfaces "*Seashell*" and "*Snail surface*" of this subclass of surfaces were presented above at p. 288 and at p. 280.

■ Spiral Screw

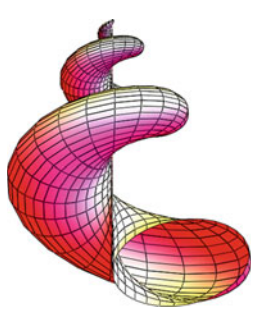
A surface *Spiral screw* (*Die Schnecke* in German) given by the following parametric equations

$$x = x(u,v) = a[1 - e^{u/(6\pi)}]\cos u \cos^2(v/2),$$
$$y = y(u,v) = a[-1 + e^{u/(6\pi)}]\sin u \cos^2(v/2),$$
$$z = z(u,v) = 1 - e^{u/(b\pi)} - \sin v + e^{u/(6\pi)}\sin v,$$



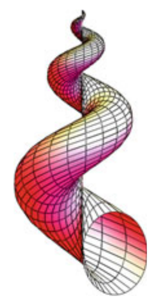
$a = 3; \quad b = 3$

Fig. 1



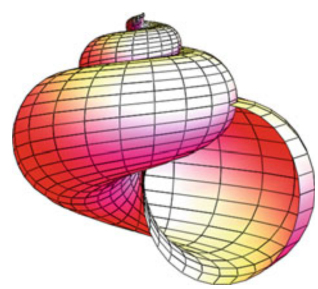
$a = 3; \quad b = 2.5$

Fig. 2



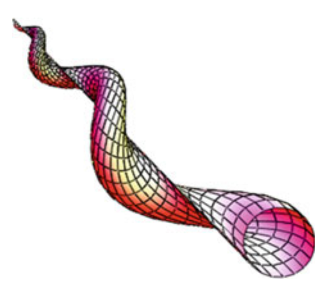
$a = 2; \quad b = 2.5$

Fig. 3



$a = 2; \quad b = 4$

Fig. 4



$a = 2; \quad b = 2.5$

Fig. 5

is *a cyclic surface with circles of variable radius laying at the planes of pencil*. The fixed straight line of the pencil coincides with an axis of *the directrix conic spiral*.

A cyclic surface “Spiral screw” presents the more wide version of a cyclic surface “Seashell” due to presence of additional coefficients *a, b*. If we shall take $a = 2$ m, $b = 3$, then both surfaces will coincide identically.

At Figs. 1, 2, 3, 4 and 5, different modifications of the surface “Spiral screw” are shown.

Reference

1. Parametrische Flächen und Körper.—http://www.3d-meier.de/tut3/Seite17.html

■ Spiral-Shaped Surface “Shell Without Vertex”

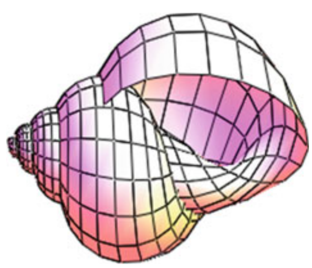
Spiral-shaped surface “Shell without vertex” is obtained as a special case of *spiral-shaped cyclic surfaces with generating circles of variable radius in the planes of pencil* when

$$R(u) = be^{pu}.$$

Forms of definition of the surface

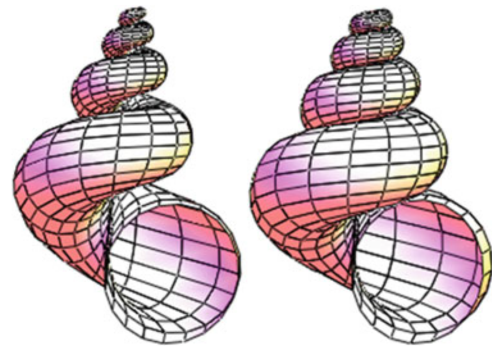
(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = [ae^{mu} + be^{pu} \cos v] \cos u,$$
$$y = y(u, v) = [ae^{mu} + be^{pu} \cos v] \sin u$$
$$z = z(u, v) = a\lambda e^{mu} + be^{pu} \sin v.$$



$a = 1; \lambda = 4; b = 1.5;$
$m = p = 0.1;\ 0 \leq u \leq 8\pi$

Fig. 1



$m = p = 0.1;$ $m = p = 0.08;$
$a = 1; \lambda = 4;\ b = 1;\ 0 \leq u \leq 8\pi$

Fig. 2

A surface defined by these parametrical equations is related to the curvilinear non-orthogonal non-conjugate coordinates u, v.

Coefficients of the fundamental forms of the surface may be derived by the formulas given above in the section “Spiral-shaped cyclic surfaces with circles of variable radius in the planes of pencil”. In Figs. 1 and 2, three spiral-shaped surfaces given by the same parametric equations are shown but they have different geometrical parameters. The vertexes at three surfaces shown in figures are absent and $R(u) = 0$, $0 \leq v \leq 2\pi$.

■ Spiral-Shaped Surface “Shell With Vertex”

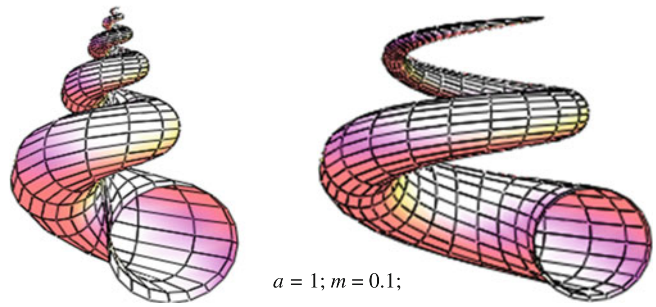
Spiral-shaped surface “Shell with vertex” is obtained as a special case of *spiral-shaped cyclic surfaces with generating circles of variable radius in the planes of pencil* when

$$R(u) = b(e^{pu} - 1).$$

Forms of definition of the surface

(1) Parametrical equations (Figs. 1, 2 and 3):

$$x = x(u, v) = [ae^{mu} + b(e^{pu} - 1) \cos v] \cos u,$$
$$y = y(u, v) = [ae^{mu} + b(e^{pu} - 1) \cos v] \sin u,$$
$$z = z(u, v) = a\lambda e^{mu} + b(e^{pu} - 1) \sin v.$$

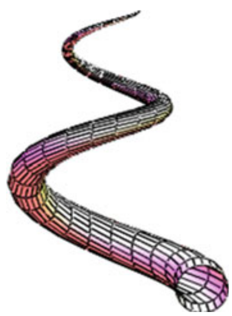


$a = 1; m = 0.1;$

$\lambda = 4; b = 1; p = 0.1;$
$0 \le u \le 8\pi$

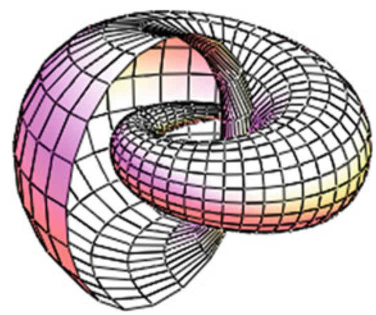
$\lambda = 1; b = 0.5; p = 0.08;$
$0 \le u \le 4\pi$

Fig. 1



$a = 1; m = 0.1; \lambda = 4;$
$b = 1; p = 0.05; \ 0 \le u \le 4\pi$

Fig. 2



$a = 1; m = 0; \lambda = 4; b = 1;$
$p = 0.1; \ 0 \le u \le 2\pi$

Fig. 3

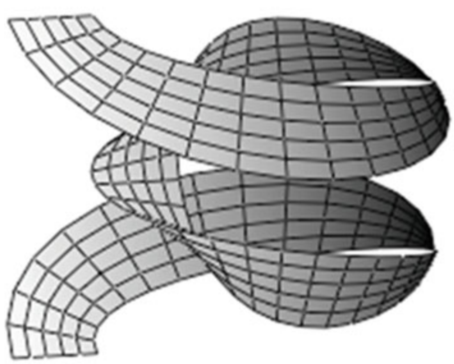
A surface defined by these parametrical equations is related to the curvilinear non-orthogonal non-conjugate coordinates u, v. Coefficients of the fundamental forms of the surface may be derived by the formulas given above in the section "Spiral-shaped cyclic surfaces with circles of variable radius in the planes of pencil" substituting a value of $R(u) = b(e^{pu} - 1)$ in them. In Figs. 1, 2 and 3, four spiral-shaped surfaces given by the same parametric equations are shown but they have different geometrical parameters. All four surfaces shown in figures have the vertexes and $R(u) = 0$, $0 \le v \le 2\pi$.

10 Helix-Shaped Surface

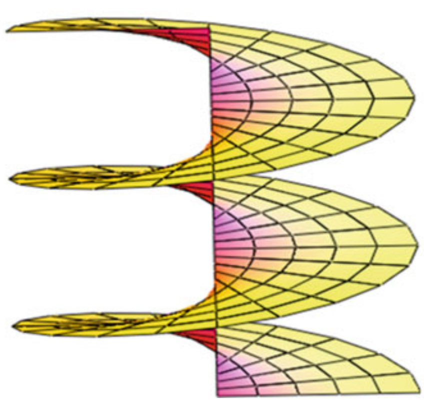
A helical surface is formed by a rigid curve which rotates uniformly about the helical axis lying in the same plane with the generatrix curve and, at the same time, executes a translational motion along the same axis. The trajectories of the points under their helical motion are cylindrical helical curves lying at the coaxial circular cylinders (see also Chap. "7. Helical Surfaces").

Surfaces, formed by generatrix curves executing besides ordinary helical motion relatively the helical axis any additional motion or deforming at certain law, are attributed to *helical-shaped surfaces*. At this case, the trajectories of the points of the generatrix curves will not be cylindrical helical lines. Helical-shaped surfaces can degenerate into helical surfaces under special selection of geometrical parameters.

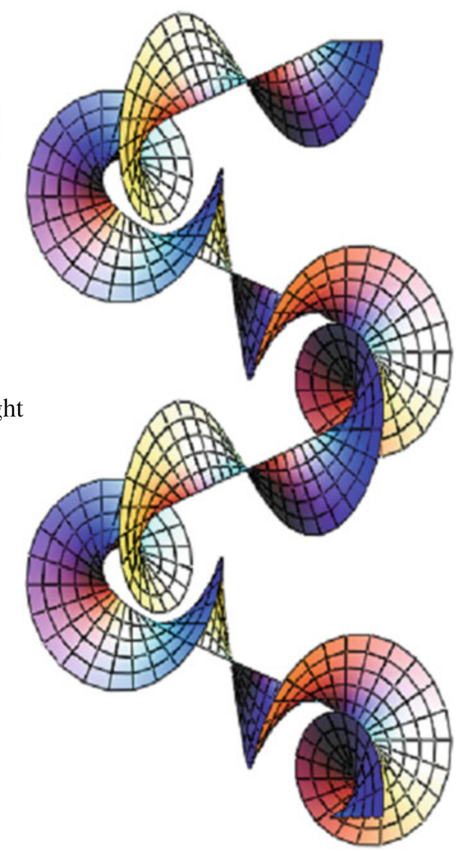
■ Helix-Shaped Surfaces Presented in the Encyclopedia



The helix-shaped twisted strip with straight generatrixes in the planes of pencil

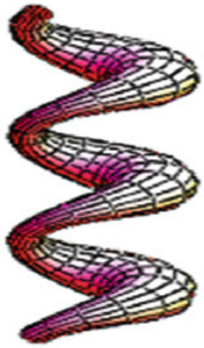


The elliptic helicoid

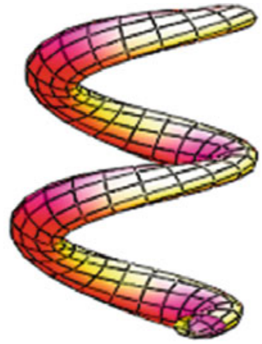


The helix-shaped preliminarily twisted strip

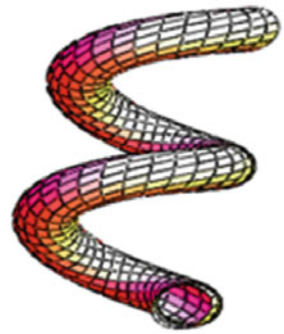
S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_10, © Springer International Publishing Switzerland 2015



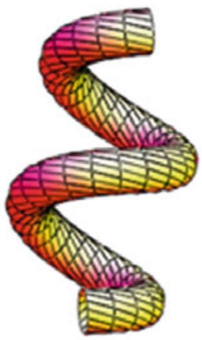
The normal cyclic helix-shaped surface consisting of the identical elements



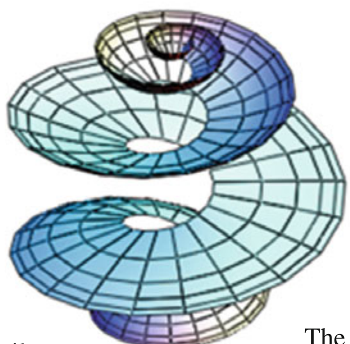
The helix-shaped preliminarily twisted surface of the elliptical cross section



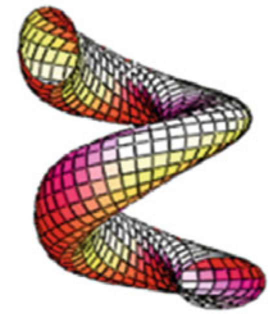
The helical twisted surface with the circles in the planes of pencil



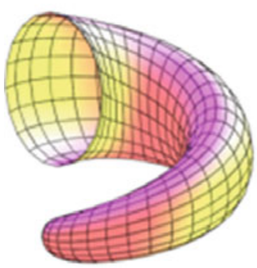
The helix-shaped preliminarily twisted surface of the circular cross section



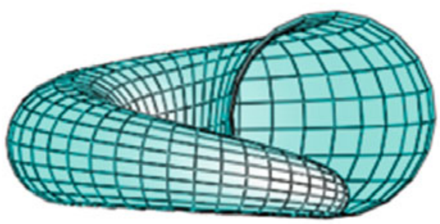
The hyperbolic helicoid



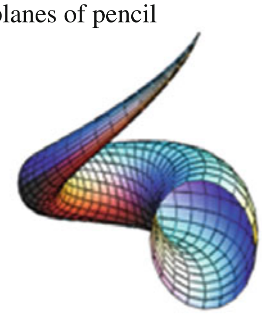
The helix-shaped twisted surface of the elliptical cross section in the planes of pencil



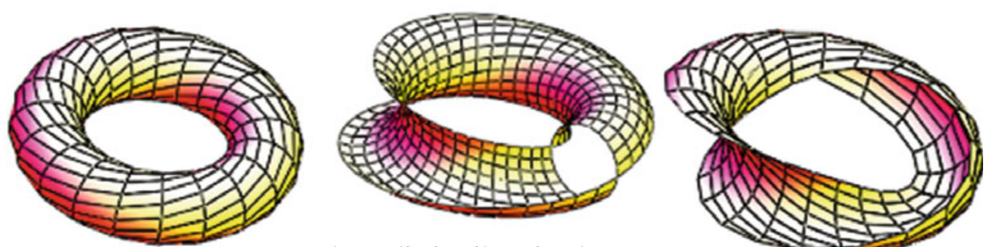
The helix-shaped surface with the variable elliptical cross section



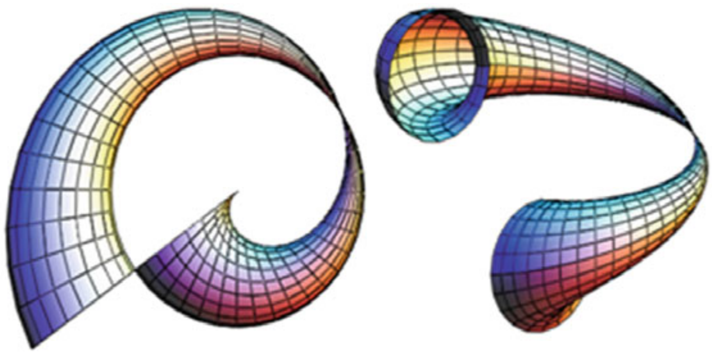
The helix-shaped surface with the generatrix circle of variable radius lying on a plane



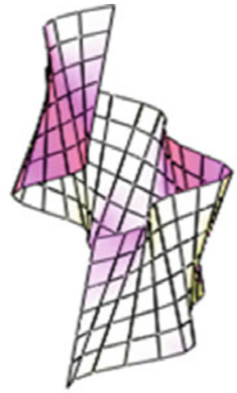
The cyclic surface in the cylinder



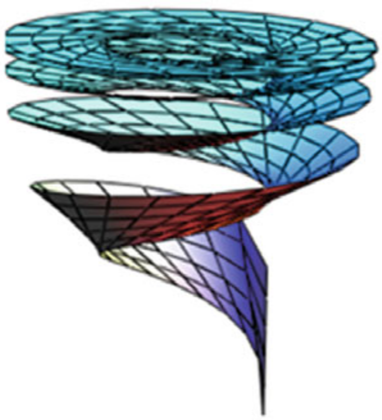
The preliminarily twisted torus



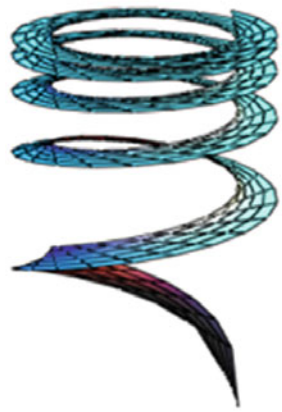
“Cyclic surface in a cylinder” transforming into “Cyclic surface about a cylinder”



The spiroidal ruled surface with the axoids “cylinder – cylinder”

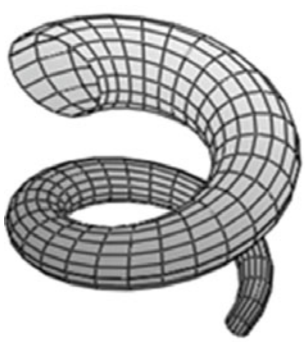


The developable helix-shaped surface with slope angles of straight generators changing from 0° till 90°



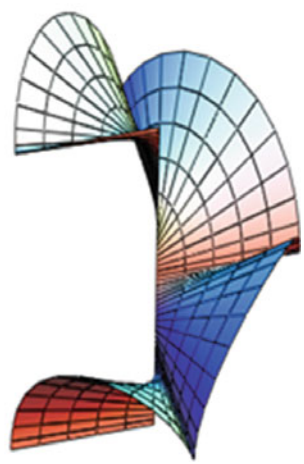
The pseudo-developable helix-shaped surface with variable pitch

The tubular helix-shaped surface with the line of centers of variable pitch



Quasi-helical cyclic surface with the boundary circles given in advance

The right waving helicoid



The rotational oblique helicoid

■ Elliptic Helicoid

An elliptic helicoid is a ruled surface of zero total curvature. It may be related both to a class of helix-shaped surfaces and to a class of ruled surfaces to a subclass of *ruled surfaces of negative Gaussian curvatures*.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = av\cos u, \quad y = y(u, v) = bv\sin u,$$
$$z = z(u) = cu,$$

where a, b are constants.

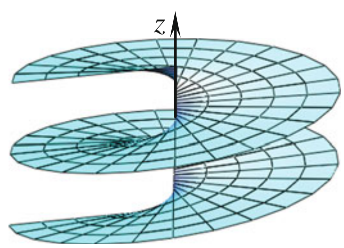


Fig. 1

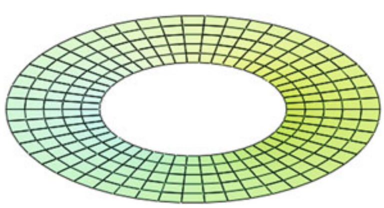


Fig. 2

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = a^2v^2\sin^2 u + b^2v^2\cos^2 u + c^2,$$
$$F = v(b^2 - a^2)\sin u\cos u,$$
$$B^2 = a^2\cos^2 u + b^2\sin^2 u,$$
$$L = N = 0,$$
$$M = \frac{abc}{\sqrt{a^2b^2v^2 + c^2(a^2\cos^2 u + b^2\sin^2 u)}},$$
$$k_u = k_v = 0,$$
$$K = \frac{-a^2b^2c^2}{(A^2B^2 - F^2)^2} < 0.$$

The straight generatrixes of the surface lay at the planes that are parallel to the coordinate plane xOy and besides this they are placed on the planes of pencil. The fixed straight line of the pencil of planes coincides with the coordinate axis Oz.

In the cross sections of an elliptic helicoid by the planes $z = z_o$ = const, the straight generatrixes of the surface lay:

$$y = y(x) = \frac{bx}{a}\tan\frac{z_o}{c}.$$

In Fig. 1, the elliptic helicoid is designed when $0 \le u \le 4\pi$ and $0 \le v \le 2$. An angle of slope of the tangents to the coordinate lines u may be found by a formula: $\cos(uz) = c/A$, so the coordinate lines u are not principal sloping lines. The projections of the lines u on the plane xOy are ellipses (Fig. 2):

$$\frac{x^2}{a^2v_o^2} + \frac{y^2}{b^2v_o^2} = 1.$$

(2) Explicit equation: $z = c\arctan(ay/bx)$.

If $a = b$, then an elliptic helicoid degenerates into *a right helicoid*.

■ Helix-Shaped Twisted Strip with Straight Generatrixes in the Planes of Pencil

Assume that a helical line l and a straight line passing through two points S and P belong to a surface and the point S is placed on the helix l but the point P is disposed on the helical axis.

A ruled surface in the form of helix-shaped twisted strip with straight generatrixes in the planes of pencil is created by motion of the point S together with the straight generatrix along the helical line l and with simultaneous rotation of this straight generatrix line in the planes of pencil passing through the helical axis of the helix l (Fig. 1).

Forms of definition of the surface

(1) Parametrical form of definition (Fig. 1):

$$x = x(u, v) = [a + u\cos(nv)]\cos v,$$
$$y = y(u, v) = [a + u\cos(nv)]\sin v,$$
$$z = z(u, v) = bv + u\sin(nv),$$

where u is the distance the point S, disposed at the helical directrix line, measured along the straight generatrix from a chosen point; v is the angle of the coordinate axis Ox with the axis Oy; n = const.

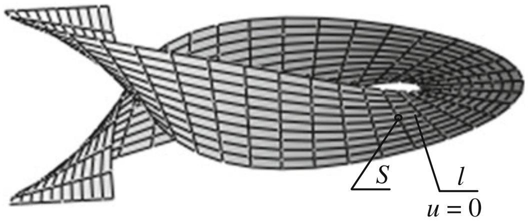


Fig. 1

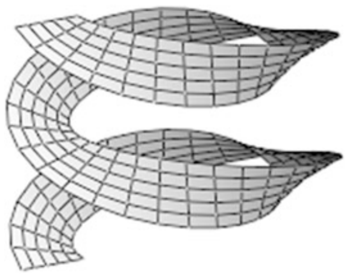


Fig. 2

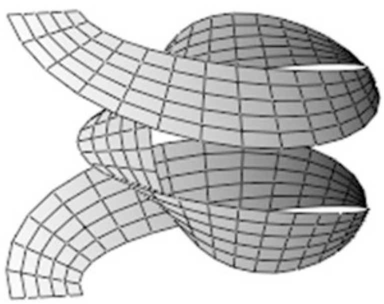


Fig. 3

The coordinate lines u coincide with the straight generatrixes of the surface. If $a = 0$, $n = 0$, then a helix-shaped twisted strip degenerates into *a right helicoid* (see also "Right Helicoid" in Chap. "19. Minimal Surfaces"). The coordinate line $u = 0$ coincides with the directrix helical line.

Coefficients of the fundamental forms of the surface:

$$A = 1, \quad F = b \sin nv,$$

$$B^2 = (un + b\cos nv)^2 + (a + u\cos nv)^2 + b^2 \sin^2 nv,$$

$$A^2B^2 - F^2 = (un + b\cos nv)^2 + (a + u\cos nv)^2$$

$$L = 0, \quad M = \frac{an - b\cos^2 nv}{\sqrt{A^2B^2 - F^2}},$$

$$N = \frac{\sin nv}{\sqrt{A^2B^2 - F^2}}\left[2un(un + b\cos nv) + (a + u\cos nv)^2\right],$$

$$K = -\frac{(an - b\cos^2 nv)^2}{(A^2B^2 - F^2)^2} < 0.$$

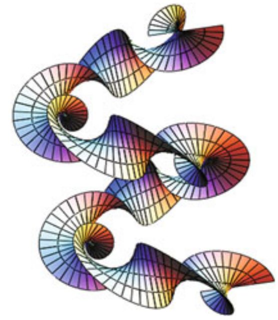


Fig. 4

Fig. 5

The ruled surface of the negative Gaussian curvature is related to curvilinear non-orthogonal non-conjugate coordinates u, v.

In Fig. 2, a considered surface with $n = 1$; $b = 1/\pi$ m; $a = 2$ m; -0.5 m $\le u \le 0.5$ m, $0 \le v \le 4\pi$ is shown. A helix-shaped twisted strip with $n = 0.5$; $a = 2$ m; $0 \le u \le 1$ m, $0 \le v \le 4\pi$; $b = 1/\pi$ m is shown in Fig. 3.

In Fig. 4, a ruled surface has $n = 3$; $a = 2$ m; -1 m $\le u \le 1$ m; $0 \le v \le 4\pi$; $b = 2/\pi$ m but in Fig. 5, they used the following geometrical parameters:

$$n = 5; \quad a = 2 \text{ m}; \quad -1 \text{ m} \le u \le 1 \text{ m};$$
$$0 \le v \le 4\pi; \quad b = 2/\pi\text{m}.$$

Additional Literature

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Ivanov VN. Geometry and design of shells on the base of surfaces with a system of coordinate lines in the planes of pencil. Prostranstv. Konstruktzii Zdaniy i Soor.: Sb. Statey. Moscow: OOO «Devyatka Print». 2004; Iss. 9, p. 26-35 (13 refs.).

■ Helix-Shaped Preliminarily Twisted Strip

A helix-shaped preliminarily twisted strip has a helical line of equal slope at a cylinder as a directrix curve and a mobile straight generatrix which is in the normal plane of the helical line in the process of its motion all the time. The rectilinear generatrix intersects the helical line in a point S, which moves along the helical directrix with the simultaneous rotation of this straight line about the point S in the normal plane of the helix.

The studied helix-shaped preliminarily twisted strip (Figs. 1, 2 and 3) can be defined by the following parametrical equations:

$$x = x(u,v) = a\cos u + v\left(\cos u\cos\theta + \frac{b}{s}\sin u\sin\theta\right),$$
$$y = y(u,v) = a\sin u + v\left(\sin u\cos\theta - \frac{b}{s}\cos u\sin\theta\right),$$
$$z = z(u,v) = bu + \frac{a}{s}v\sin\theta,$$

where $s = \sqrt{a^2+b^2}$, $\theta = \theta(u) = pu + \theta_o$; θ is a function determining the law of rotation of the straight generatrix about its point of intersection S with the directrix helix; p = const; a is the radius of the cylinder on which the directrix helical line is placed; $h = 2\pi b$ is the pitch of the directrix helical line.

Coefficients of the fundamental forms of the surface:

$$A^2 = \left(s + \frac{av}{s}\cos\theta\right)^2 + v^2\left(\frac{b}{s} - p\right)^2,$$
$$F = 0, \quad B^2 = 1, \quad A^2B^2 - F^2 = A^2$$
$$L = \frac{a}{sA}\left[v^2\left(\frac{b}{s} - p\right)\left(\frac{b}{s} - 2p\right) + \left(s + \frac{av}{s}\cos\theta\right)^2\right]\sin\theta,$$
$$M = -\frac{1}{A}\left(\frac{b}{s} - p\right)\left(s + 2v\frac{a}{s}\cos\theta\right), \quad N = 0,$$
$$K = -\frac{1}{A^4}\left(\frac{b}{s} - p\right)^2\left(s + 2v\frac{a}{s}\cos\theta\right)^2 \le 0, \quad H = \frac{L}{2A^2}$$

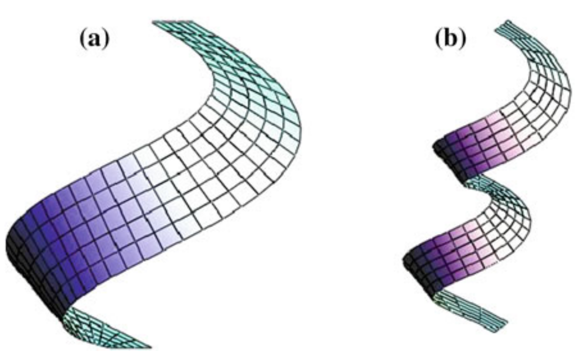

Fig. 1

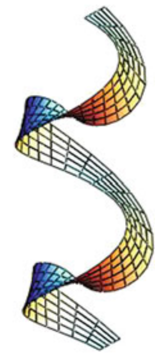
Fig. 2

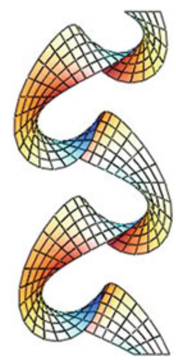
Fig. 3


Fig. 4

The studied helix-shaped surface is related to orthogonal but non-conjugate system of curvilinear coordinates u, v. The coordinate lines v coincide with the straight generatrixes of the surface. The curvilinear coordinate line v = 0 coincides with the directrix cylindrical helical line.

In Fig. 1, the helix-shaped preliminarily twisted strip with a = 2 m; b = 1 m; p = 0.5; $0 \le v \le 2$ m is shown, but in Fig. 1a, we have $0 \le u \le 2\pi$, and in Fig. 1b, we have $0 \le u \le 4\pi$.

The surface, shown in Fig. 2, has a = 2 m; b = 1 m; p = 1; $0 \le v \le 2$ m, $0 \le u \le 4\pi$. The surface, shown in Fig. 3, has the same values of the geometrical parameters but only

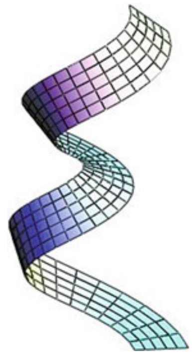

Fig. 5

p = 2. If a = 0 (Fig. 4), then a helix-shaped preliminarily twisted strip degenerates into *a right helicoid* (*helix- shaped preliminarily twisted strip*). If p = b/s, then the studied surface degenerates into a surface of zero total curvature (Fig. 5).

In Fig. 6, it is shown the surface having the geometrical parameters: $a = 4$ m; $b = 1$ m; $p = 3$; $0 \le v \le 2$ m, $0 \le u \le 4\pi$.

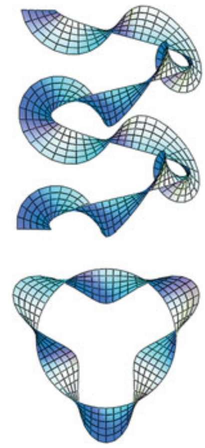

Fig. 6

In Fig. 6, the projection of the surface with the given above parameters on the horizontal plane xOy is presented too.

■ Helix-Shaped Surface with Variable Elliptical Cross Section

A line of the centers of elliptical cross sections *of a helix-shaped surface with variable elliptical cross section* lies on a circular cylinder. This line of the centers is *the sloping line* on the cylinder (Fig. 1).

Parametrical equations of the surface (Figs. 1 and 2) can be written in the following form:

$$x = x(u,v) = [R + c(u)\cos v]\cos u,$$
$$y = y(u,v) = [R + c(u)\cos v]\sin u,$$
$$z = z(u,v) = d(u)\sin v + pu,$$

where

$$c(u) = r + 2(C - r)\frac{u}{\pi},\ d(u) = r + 2(D - r)\frac{u}{\pi},$$
$$c' = \frac{\mathrm{d}c(u)}{\mathrm{d}u} = \frac{2}{\pi}(C - r),$$
$$d' = \frac{\mathrm{d}(u)}{\mathrm{d}u} = \frac{2}{\pi}(D - r),$$

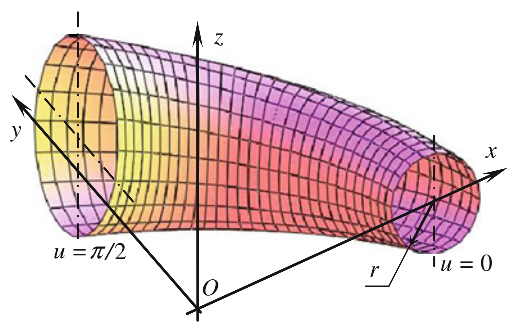

Fig. 1

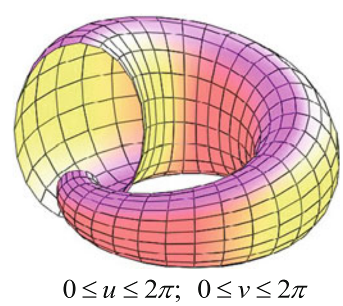

Fig. 2

R is the radius of the circular cylinder on which the line of the centers of elliptic cross sections lies; r is the radius of the circular cross section placed in the coordinate plane xOz ($u = 0$); u is the angle taken from the axis Ox in the direction of the axis Oy; $0 \le u \le \infty$; v is the angle taken from the coordinate plane xOy in the direction of the axis Oz; $0 \le v \le 2\pi$; C, D are the semi-axes of the elliptic cross section lying in the plane yOz ($u = \pi/2$); $2p\pi$ is the pitch of the helical line of the centers of the elliptic cross sections of the surface (Fig. 2).

The elliptical cross sections of the helix-shaped surface lay at the planes of pencil and the fixed straight line of the pencil passes through the coordinate axis Oz.

There is a circle

$$(x-R)^2 + z^2 = r^2$$

in the cross section of the surface by the plane $y = 0$ but the cross section of the surface by the plane $x = 0$ contains an ellipse:

$$y = R + C\cos v; \quad z = D\sin v,$$

or

$$\frac{(y-R)^2}{C^2} + \frac{z^2}{D^2} = 1.$$

Assume $p = 0$, then a *torus-shaped surface with variable elliptical cross section and with a circular line of the centers* (Fig. 3) will be obtained.

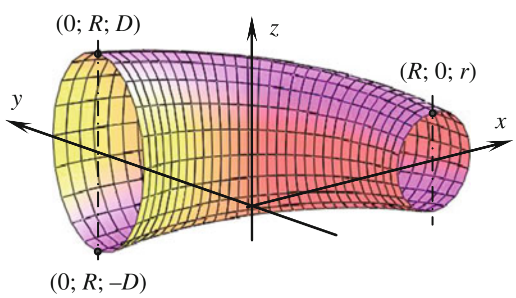

Fig. 3

Coefficients of the fundamental forms of the surface:

$$A^2 = c'^2\cos^2 v + [R + c(u)\cos v]^2 + [d'\sin v + p]^2,$$
$$F = -c(u)c'\sin v\cos v + d(u)(d'\sin v + p)\cos v,$$
$$B^2 = c^2(u)\sin^2 v + d^2(u)\cos^2 v$$
$$A^2B^2 - F^2 = d(u)c'\cos^2 v[d(u)c'\cos^2 v - 2c(u)d'\sin^2 v - 2c(u)p\sin v] + [R + c(u)]^2 \times [c^2(u)\sin^2 v + d^2(u)\cos^2 v] + (d'\sin v + p)c^2(u)\sin^2 v,$$
$$L = -\frac{\cos v}{\sqrt{A^2B^2 - F^2}}\left\{d(u)[2c'^2\cos^2 v + (R + c(u)\cos v)^2] + 2c(u)c'(d'\sin v + p)\right\},$$
$$M = \frac{2\sin v}{\pi\sqrt{A^2B^2 - F^2}} \times \left\{rR(D - C)\cos v + c^2(u)\left(D - r + \frac{\pi p\sin v}{2}\right)\right\},$$
$$N = \frac{-c(u)d(u)}{\sqrt{A^2B^2 - F^2}}[R + c(u)\cos v].$$

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■ Helix-Shaped Surface with Generatrix Circle of Variable Radius Lying on a Plane

A line of the centers of a helix-shaped surface with generatrix circle of variable radius lying on a plane is disposed on a circular cylinder of a radius R (Fig. 1). All generatrix circles of the regarded helix-shaped surface are disposed in the planes of pencil and touch the base plane along the circle of the radius R.

These *helix-shaped surfaces* may be regarded also as *cyclic surfaces with circles in the planes of pencil.*

Parametrical equations of the surface (Figs. 1 and 2) can be written in the following form:

$$x = x(u,v) = (R + r\sin v)\sin u,$$
$$y = y(u,v) = (R + r\sin v)\cos u,$$
$$z = z(u,v) = r(1 + \cos v),$$

where $r = r(u)$ is a function of the changing of the radius of the generatrix circle; R = const; v is an angle in a plane of a generatrix circle taken from the positive direction of the Oz

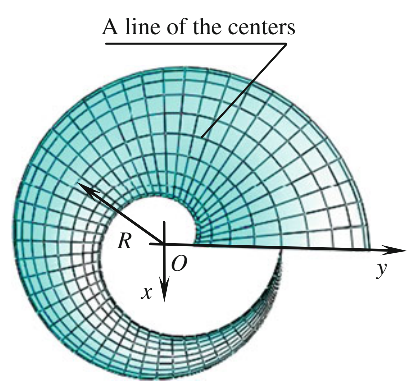

Fig. 1

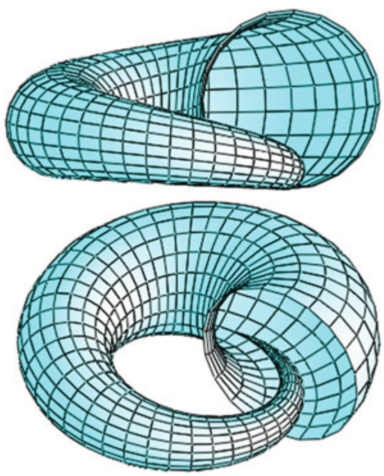

Fig. 2

axis in the direction of the coordinate plane $z = 0$; $0 \le v \le 2\pi$; u is the angle at the coordinate plane $z = 0$ taken from the coordinate axis Oy in the direction of the axis Ox.

The line of centers

$$x = R\sin u, \quad y = R\cos u, \quad z = r(u) = pu$$

is *a line of the constant slope*. The coordinate lines $u = \text{const}$ coincide with the generatrix circles. The coordinate line $v = \pi$ is a circle of the radius R lying in the coordinate plane $z = 0$ (Fig. 1).

Coefficients of the fundamental forms of the surface and its curvatures:

$$\begin{aligned}
&A^2 = (R + r\sin v)^2 + 2r'^2(1+\cos v),\\
&F = -rr'\sin v,\\
&B = r(u)\\
&A^2B^2 - F^2 = r^2[(R + r\sin v)^2 + 2r'^2(1+\cos v) - r'^2\sin^2 v],\\
&k_u = r\frac{r''(1+\cos v)(R + r\sin v) - A^2\sin v}{A^2\sqrt{A^2B^2 - F^2}},\\
&k_v = -\frac{R + r\sin v}{\sqrt{A^2B^2 - F^2}},
\end{aligned}$$

where

$$r' = \frac{\mathrm{d}r}{\mathrm{d}u}, \quad r'' = \frac{\mathrm{d}^2 r}{\mathrm{d}u^2}.$$

So, the surface is given in non-orthogonal non-conjugate system of the curvilinear coordinates u, v. It has segments both of positive and negative Gaussian curvatures. Assume $r = \text{const}$, then the studied helix-shaped surface degenerates into *a circular torus* (see also “Circular Torus” in Chap. “2. Surfaces of Revolution”).

Assume $r = r(u) = pu$ where $p = \text{const}$, then the values of the coefficients of the fundamental forms of the surface will be the following:

$$\begin{aligned}
&A^2 = (R + r\sin v)^2 + 2p^2(1+\cos v),\\
&F = -pr\sin v, \quad B = r(u) = pu,\\
&L = -\frac{rA^2\sin v}{\sqrt{A^2B^2 - F^2}},\\
&M = -\frac{pr^2(1+\cos v)\cos v}{\sqrt{A^2B^2 - F^2}},\\
&N = -\frac{r^2(R + r\sin v)}{\sqrt{A^2B^2 - F^2}},
\end{aligned}$$

hence,

$$\begin{aligned}
&k_u = -\frac{r\sin v}{\sqrt{A^2B^2 - F^2}},\\
&k_v = -\frac{R + r\sin v}{\sqrt{A^2B^2 - F^2}}.
\end{aligned}$$

In Fig. 1, the helix-shaped surface with $r = r(u) = pu = u/2$; $R = 4$ m; $0 \le u \le 2\pi$; $0 \le v \le 2\pi$ is given (Fig. 3). The surface, represented in Fig. 2, has

$$R = 10p, \quad 0 \le v \le 2\pi, \quad 0 \le u \le 5\pi/2.$$

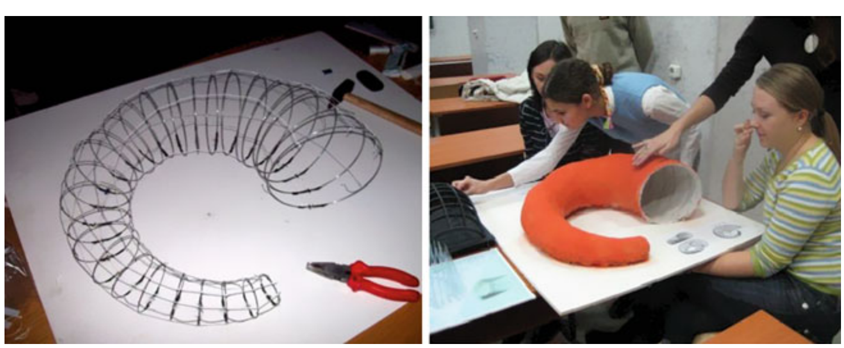

Fig. 3 Hand-making of the gypsum model of the helix-shaped surface shown in Fig. 2

■ Cyclic Surface in a Cylinder

Cyclic surface in a cylinder is formed by circles of variable radius $r = r(u)$ lying in the planes of pencil. The fixed straight of the pencil is an axis of a directrix helix disposed on the circular cylinder of a radius R (Fig. 1).

A point of the generatrix circle, which is at the most long distance from the axis of the directrix helix, moves along this helix

$$x = x(u) = R\cos u; \quad y = y(u) = R\sin u; \quad z = z(u) = bu.$$

Parametrical equations of cyclic surface in a cylinder are

$$\begin{aligned} x &= x(u,v) = [R - r(1-\cos v)]\cos u, \\ y &= y(u,v) = [R - r(1-\cos v)]\sin u, \\ z &= z(u,v) = bu + r\sin v, \end{aligned}$$

where u is an angle read in the plane xOy; v is an angle read in the plane of the generatrix circle. The coordinate line $v = 0$ coincides with the directrix helix on the cylinder of the radius R = const.

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} A^2 &= r'^2(1-\cos v)^2 + (b + r'\sin v)^2 + [R - r(1-\cos v)]^2, \\ B &= r, \quad F = r(r'\sin v + b\cos v) \\ A^2B^2 - F^2 &= r^2\left\{[r'(1-\cos v) + b\sin v]^2 + [R - r(1-\cos v)]^2\right\}, \\ L &= \frac{r}{\sqrt{A^2B^2-F^2}}\Big\{2r'(1-\cos v)[r'(1-\cos v) + b\sin v] \\ &\qquad - [R - r(1-\cos v)]^2\cos v \\ &\qquad + r''(1-\cos v)[R - r(1-\cos v)]\Big\}, \\ M &= \frac{r^2[r'(1-\cos v) + b\sin v]\sin v}{\sqrt{A^2B^2-F^2}}, \\ N &= -\frac{r^2[R - r(1-\cos v)]}{\sqrt{A^2B^2-F^2}}, \\ k_u &= \frac{L}{A^2}, \quad k_v = -\frac{[R - r(1-\cos v)]}{\sqrt{A^2B^2-F^2}}. \end{aligned}$$

If $r = r(u) = a - pu$, where a and p are constants, then formulas for the calculation of the coefficients of the fundamental forms of the surface may be written in the form:

$$\begin{aligned} A^2 &= p^2(1-\cos v)^2 + (b - p\sin v)^2 + [R - r(1-\cos v)]^2, \\ B &= r, \quad F = r(-p\sin v + b\cos v), \\ A^2B^2 - F^2 &= r^2\left\{[-p(1-\cos v) + b\sin v]^2 + [R - r(1-\cos v)]^2\right\}, \\ L &= \frac{r\left\{2p(1-\cos v)[p(1-\cos v) - b\sin v] - [R - r(1-\cos v)]^2\cos v\right\}}{\sqrt{A^2B^2-F^2}}, \\ M &= \frac{r^2[-p(1-\cos v) + b\sin v]\sin v}{\sqrt{A^2B^2-F^2}}, \\ N &= -\frac{r^2[R - r(1-\cos v)]}{\sqrt{A^2B^2-F^2}}. \end{aligned}$$

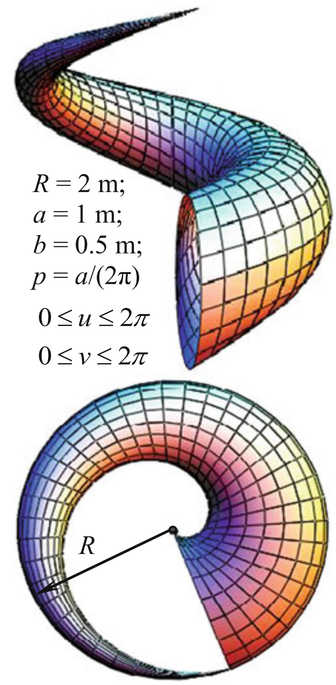

Fig. 1

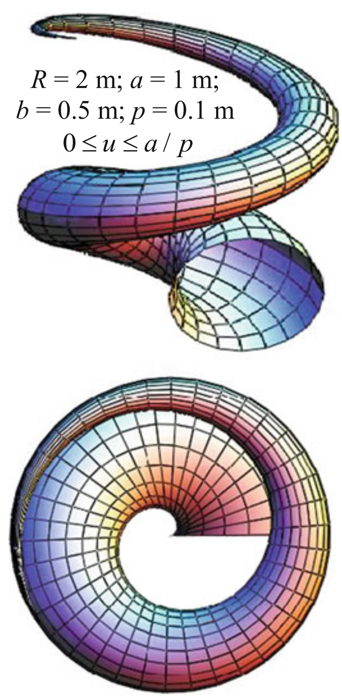

Fig. 2

A radius of a generatrix circle will be equal to zero when $u = a/p$ (Figs. 1 and 2). The parametric equations of the line of the centers of the studied cyclic surface may be written as:

$$x = x(u) = (R - r)\cos u$$
$$y = y(u) = (R - r)\sin u,$$
$$z = z(u) = bu.$$

Assume a law of changing of a radius of the generatrix circle in the form

$$r = r(u) = a - pu,$$

where a and p are constants, then the projection of the line of centers on the horizontal plane xOy is a *neoide*. So, in this case, a line of the centers lies on a right cylinder with a directrix neoide.

References

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Krivoshapko SN, Ivanov VN. Geometry, analysis, and design of the structures in the form of cyclic surfaces. Obzornaya Informatziyya. Ser. "Stroit. Materialy i Konstruktzii". Moscow: OAO VNIINTPI, 2010; Iss. 2, 61 p.

■ Quasi-helical Cyclic Surface with Boundary Circles Given in Advance

A cyclic surface with generatrix circles of variable radius in the planes of pencil and with a helical directrix line of centers

$$x = x(v) = r\cos v, \quad y = y(v) = r\sin v, \quad z = z(v) = bv$$

has values of the radiuses of initial (R_1) and last (R_2) generatrix circles an angle given in advance between the initial and final position of the planes of pencil. The fixed straight line of the pencil of the planes coincides with the fixed coordinate axis Oz. Such a cyclic surface is called *a quasi-helical cyclic surface with boundary circles given in advance*.

This quasi-helical surface has the following form of definition (Figs. 1, 2 and 3):

$$\begin{aligned} x &= x(u,v) = r\cos v + R(v)\cos u\cos v,\\ y &= y(u,v) = r\sin v + R(v)\cos u\sin v,\\ z &= z(u,v) = R(v)\sin u + bv,\\ R(v) &= \frac{v - v_2}{v_1 - v_2}R_1 + \frac{v - v_1}{v_2 - v_1}R_2,\\ R'(v) &= \frac{\mathrm{d}R(v)}{\mathrm{d}v} = \frac{R_2 - R_1}{v_2 - v_1}, \end{aligned}$$

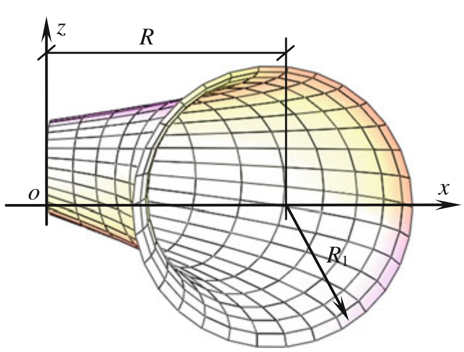


Fig. 1

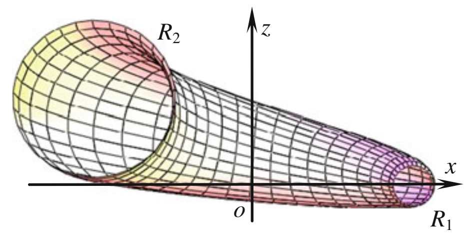


Fig. 2

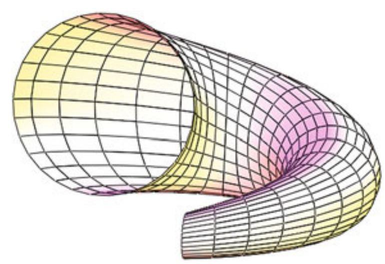


Fig. 3

where u is a central angle in the planes of the generatrix circles; $0 \le u \le 2\pi$; v is an angle taken from the axis Ox in the direction of the axis Oy; v_1 is an angle defining the position of the plane of the pencil with the initial circle of the radius R_1; v_2 is an angle defining the position of the plane of the pencil with the last circle of the radius R_2; r is the radius of the circular cylinder with the helical line of the centers lying on it.

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} &A = R(v),\\ &F = bR(v)\cos u,\\ &B^2 = [r + R(v)\cos u]^2 + R'^2(v) + 2bR'(v)\sin u + b^2,\\ &A^2B^2 - F^2 = R^2(v)\left\{[r + R(v)\cos u]^2 + [R'(v) + b\sin u]^2\right\},\\ &L = \frac{R^2(v)[r + R(v)\cos u]}{\sqrt{A^2B^2 - F^2}}, \quad M = -\frac{R^2(v)[R'(u) + b\sin u]}{\sqrt{A^2B^2 - F^2}}\sin u,\\ &N = \frac{R(v)\cos u}{\sqrt{A^2B^2 - F^2}}\left\{2R'(v)[R'(v) + b\sin u] + [r + R(v)\cos u]^2\right\}. \end{aligned}$$

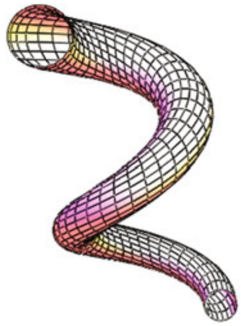


Fig. 4

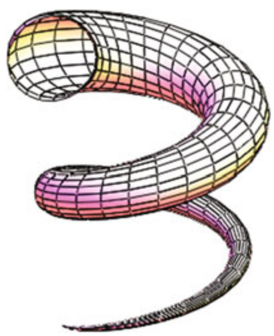


Fig. 5

The studied cyclic surface is given in non-orthogonal non-conjugate system of curvilinear coordinates u, v.

In Fig. 1, the surface with R_1 = 60 cm; $v_2 = \pi/2$; R_2 = 20 cm; v_1 = 0; r = 80 cm; $0 \le u \le 2\pi$; $0 \le v \le \pi/2$; b = 10 cm is shown.

The surface presented in Fig. 2 has R_1 = 10 cm; $v_2 = \pi$; R_2 = 40 cm; v_1 = 0; $0 \le u \le 2\pi$; r = 80 cm; b = 10 cm.

In Fig. 3, the surface with R_1 = 10 cm; R_2 = 40 cm; r = 80 cm; $0 \le u \le 2\pi$; $0 \le v \le 3\pi/2$; b = 10 cm is shown.

If $R_1 = R_2$, then we shall design *a circular helical surface with the generatrix circle lying in the plane passing through the helical axis* (see also Subsect. "7.1.2. Circular Helical Surfaces").

In Fig. 4, the surface with the following geometrical parameters:

$$R_1 = 10\ \text{cm};\quad R_2 = 20\ \text{cm};\quad r = 40\ \text{cm};\quad b = 10\ \text{cm};\quad 0 \le v \le 3\pi;\quad 0 \le u \le 2\pi$$

is given. But in Fig. 5, the geometrical parameters have the values of

$$R_1 = 0,\quad R_2 = 20\ \text{cm};\quad r = 40\ \text{cm};$$
$$b = 10\ \text{cm};\quad 0 \le v \le 4\pi;\quad 0 \le u \le 2\pi.$$

Additional Literature

Narzullayev SA. The frame method of design of cyclic surfaces with using of two systems of the coordinates. Prikl. Geom. i Ingenern. Grafika. Kiev, 1978; Iss. 25, p. 65-67 (3 refs.).

■ Developable Helix-Shaped Surface with Slope Angles of Straight Generators Changing from 0° till 90°

A developable helical-shaped surface with slope angles of straight generators from 0° *till* 90° is formed by the tangents to the helical line of a variable pitch (Fig. 1):

$$x(u) = a\cos m\pi u,$$
$$y(u) = a\sin m\pi u,$$
$$z(u) = b\sqrt{1-(1-u)^2},$$

lying on a circular cylindrical surface with a radius a; $m/2$ is a number of turns at the section of $0 \le z \le b$, i.e. when $0 \le u \le 1$.

The tangent straights to the cylindrical helical line of a variable pitch L, i.e. to the cuspidal edge *of the developable helical-shaped surface,* intersect the coordinate plane xOy at an angle $\alpha = 90° - \varphi$, where

$$\tan\varphi = \frac{am\pi}{b(1-u)}\sqrt{u(2-u)}.$$

Fig. 1

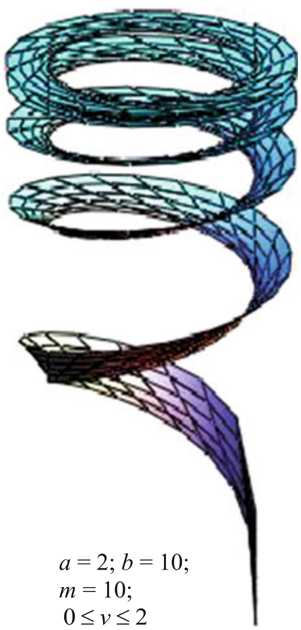


Fig. 2

The angle α changes from $\alpha = 0$, when $u = 1$ ($z = b$), till $\alpha = \pi/2$, when $u = 0$ ($z = 0$).

The studied surface may be related both to a class of *helical-shaped surfaces* and to *ruled surfaces* to the subclass of "*Torse Surface* (*Torses*)." If the cuspidal edge of a torse surface is known, then the definition of the surface and the determination of its coefficients of the fundamental forms offer no difficulty. For this, it is necessary to use the common formulas given on p. xxv at a Subsect. "1.1.1. Torse Surfaces (torses)."

Hence, parametrical equations of this developable surface (Figs. 2 and 3) may be presented as

$$x = x(u,v) = x(u) - v\frac{am\pi\sqrt{1-(1-u)^2}\sin m\pi u}{\sqrt{a^2m^2\pi^2[1-(1-u)^2]+b^2(1-u)^2}},$$

$$y = y(u,v) = y(u) + v\frac{am\pi\sqrt{1-(1-u)^2}\cos m\pi u}{\sqrt{a^2m^2\pi^2[1-(1-u)^2]+b^2(1-u)^2}},$$

$$z = z(u,v) = z(u) + v\frac{b(1-u)}{\sqrt{a^2m^2\pi^2[1-(1-u)^2]+b^2(1-u)^2}},$$

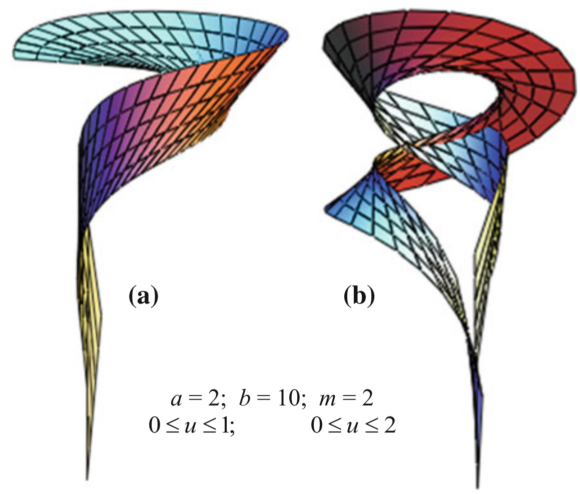


Fig. 3

where $x(u)$, $y(u)$, $z(u)$ are coordinates of the cuspidal edge of the studied torse surface given above.

Coefficients of the fundamental forms of the surface:

$$A^2 = F^2 + \frac{a^2m^2\pi^2v^2}{F^4}\left\{m^2\pi^2F^2 + \frac{b^2}{[1-(1-u)^2]^3}\right\},$$

$$F = \frac{\sqrt{a^2m^2\pi^2[1-(1-u)^2]+b^2(1-u)^2}}{\sqrt{1-(1-u)^2}}, \quad B = 1,$$

$$L = \frac{a^2m^3\pi^3bv^2(1-u)}{F^3\sqrt{A^2-F^2}[1-(1-u)^2]^{5/2}} \times \left\{3 + [1-(1-u)^2]^2m^2\pi^2\right\},$$

$$M = N = K = 0.$$

The surface is given in curvilinear non-orthogonal conjugate coordinates u, v. The geometrical parameters, used for the construction of the surfaces, are presented in corresponding drawings.

In Fig. 3a, the developable surface is shown with the geometrical parameter u changing in the limits of $0 \le u \le 1$.

In Fig. 3b, the surface has the following boundaries: $0 \le u \le 2$. If $0 \le u \le 2$, then the surface has the self-crossing and becomes closed.

■ Pseudo-developable Helix-Shaped Surface with Variable Pitch

A pseudo-developable helix-shaped surface with variable pitch is formed by the straight lines that are parallel to a coordinate plane xOy and pass through a helical line of a variable pitch L (Fig. 1)

$$x(u) = a\cos m\pi u, \quad y(u) = a\sin m\pi u,$$
$$z(u) = b\sqrt{1-(1-u)^2}$$

and remaining all the time tangent to the projection of this helical line on a plane perpendicular to the axis of the studied helix-shaped surface or, what is the same, on the coordinate plane xOy. The directrix helical line of a variable pitch L lays on the circular cylindrical surface of a radius a; $m/2$ is a number of its turns in the section $0 \le z \le b$, that is when $0 \le u \le 1$.

The tangent straight lines to the cylindrical helical line of a variable pitch L intersect the coordinate plane xOy at an angle $\alpha = 90° - \varphi$, where

$$\tan\varphi = \frac{am\pi\sqrt{u(2-u)}}{b(1-u)}.$$

The angle α changes from $\alpha = 0$, when $u = 1$ ($z = b$), till $\alpha = \pi/2$, when $u = 0$ ($z = 0$).

The studied surface may be related both to a class of *helix-shaped surfaces* and to a class of *ruled surfaces* to a subclass of *ruled surfaces of negative Gaussian curvature*, and at the more narrow meaning to a group of *Catalan's surfaces*.

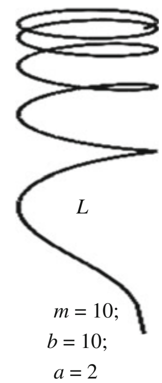


Fig. 1

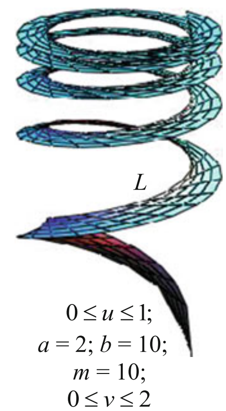


Fig. 2

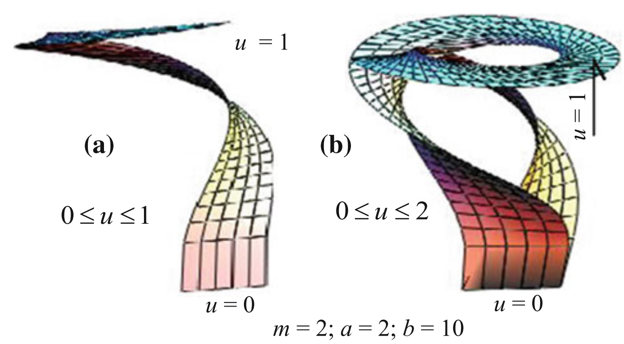


Fig. 3

Forms of definition of the surface

(1) Vector form of definition:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = a\boldsymbol{h}(m\pi u) + v\boldsymbol{n}(m\pi u) + z(u)\boldsymbol{k},$$

where

$$\boldsymbol{h}(m\pi u) = \boldsymbol{i}\cos(m\pi u) + \boldsymbol{j}\sin(m\pi u);$$
$$\boldsymbol{n}(m\pi u) = -\boldsymbol{i}\sin(m\pi u) + \boldsymbol{j}\cos(m\pi u);$$

v is a coordinate line coinciding with a straight generatrix of the surface.

(2) Parametrical form of definition (Figs. 2 and 3):

$$x = x(u, v) = x(u) - v\sin m\pi u,$$
$$y = y(u, v) = y(u) + v\cos m\pi u, \quad z = z(u),$$

where $x(u)$, $y(u)$, $z(u)$ are the coordinates of the directrix cylindrical helical line of a variable pitch of the ruled surface given above. In Fig. 3a, the *pseudo-developable surface is shown* when $0 \le u \le 1$, and in Fig. 3b, when $0 \le u \le 2$. If $0 \le u \le 2$, the surface becomes closed.

Coefficients of the fundamental forms of the surface:

$$A^2 = m^2\pi^2(a^2 + v^2) + b^2(1-u)^2/(2u - u^2),$$
$$F = am\pi, \quad B = 1,$$
$$L = m\pi b\frac{v + am\pi u(1-u)(2-u)}{\sqrt{A^2B^2 - F^2}[u(2-u)]^{3/2}},$$
$$M = \frac{m\pi b(1-u)}{\sqrt{A^2B^2 - F^2}\sqrt{u(2-u)}},$$
$$N = 0, \quad K \le 1.$$

■ Rotational Oblique Helicoid

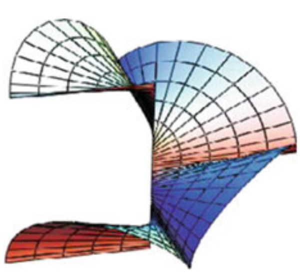
A rotational oblique helicoid is formed by a straight generatrix of a cone Ω, which rolls without sliding on a plane but the plane performs translation motion along an axis of the helicoid.

It is known the following form of the definition of a rotational oblique helicoid (Fig. 1):

$$\begin{aligned} x = x(u,v) &= v\cos\alpha\cos u \\ &\quad - v(\cos\varphi\sin\alpha\cos u + \sin\varphi\sin u)\tan\alpha, \\ y = y(u,v) &= v\cos\alpha\sin u \\ &\quad - v(\cos\varphi\sin\alpha\sin u - \sin\varphi\cos u)\tan\alpha, \\ z = z(u,v) &= v(1+\cos\varphi)\sin\alpha + \lambda u, \end{aligned}$$

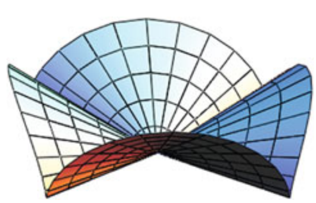
where $\varphi = nu$; $n = 1/\sin\alpha$; α is the angle of the axis of the mobile circular cone Ω with its straight generatrixes; $R = v\tan\alpha$; R is the radius of the base of the rolling cone Ω; v = const is a height of the cone Ω.

The angle of the axis Ox with the projection of the mobile axis of the cone Ω on the plane xOy appearing, when rolling the cone, is denoted as u; λ is a parameter characterizing the speed of the translational movement of the vertex of the cone Ω along the axis of the helicoid. When $\lambda = 0$, the surface degenerates into a *rotational surface with the axoids* "*plane-cone*", Fig. 2.



$n = 4;\ \lambda = 1;\ \sin\alpha = 1/n;$
$0 \le v \le R/\tan\alpha;$
$0 \le u \le 2\pi$

Fig. 1



$\lambda = 0$

Fig. 2

10.1 Helix-Shaped Preliminarily Twisted Surfaces with Plane Generatrix Curve

Helix-shaped preliminarily twisted surfaces with plane generatrix curve bearing on a class of *helical-shaped surfaces* have a directrix curve in the form of a helical line of constant slope on the cylinder. A plane generatrix curve is disposed in the normal plane of the helical line and besides the motion along this line, it rotates in the normal plane of the helix. The axis of rotation of a generatrix curve coincides with the tangent to the helix.

Forms of definition of a helix-shaped preliminarily twisted surface with a plane generatrix curve

(1) Vector form of the definition:

$$\begin{aligned} \boldsymbol{r}(u,v) &= \boldsymbol{\rho}(u) + [X(v)\cos\theta - Y(v)\sin\theta]\boldsymbol{v} \\ &\quad + [X(v)\sin\theta + Y(v)\cos\theta]\boldsymbol{\beta} \\ &= \{b - [X(v)\cos\theta - Y(v)\sin\theta]\}\boldsymbol{h}(u) \\ &\quad + [X(v)\sin\theta + Y(v)\cos\theta] \\ &\quad \times [-c\boldsymbol{n}(u) + b\boldsymbol{k}]/\sqrt{b^2+c^2} + cu\boldsymbol{k}, \end{aligned}$$

where

$$\boldsymbol{\rho}(u) = b\boldsymbol{h}(u) + cu\boldsymbol{k}$$

is a radius vector of a directrix helical line laying on the cylinder of a radius b;

$$X = X(v), \quad Y = Y(v)$$

are the parametric equations of the generatrix curve relatively to the local coordinate axes X, Y;

$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u$$

is the unit vector disposed at the plane xOy; $\boldsymbol{v} = -\boldsymbol{h}(u)$;

$$\boldsymbol{\beta} = [-c\boldsymbol{n}(u) + b\boldsymbol{k}]/\sqrt{b^2+c^2}$$

is the unit vector of the binormal of the directrix helical line;

$$\boldsymbol{n}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u.$$

Assume a new constant: $t = \sqrt{b^2+c^2}$.

(2) Parametrical form of the definition:

$$\begin{aligned} x = x(u,v) &= \{b - [X(v)\cos\theta - Y(v)\sin\theta]\}\cos u \\ &\quad + c\sin u[X(v)\sin\theta + Y(v)\cos\theta]/t, \\ y = y(u,v) &= \{b - [X(v)\cos\theta - Y(v)\sin\theta]\}\sin u \\ &\quad - c\cos u[X(v)\sin\theta + Y(v)\cos\theta]/t, \\ z = z(u,v) &= cu + b[X(v)\sin\theta + Y(v)\cos\theta]/t, \end{aligned}$$

where u is the angle in the plane xOy taken from the axis Ox in the direction of the axis Oy; $\theta = \theta(u)$ is an angle of twisting of the generatrix curve in the normal plane of the directrix helical line.

Coefficients of the fundamental forms of the surface:

$$A^2 = \left\{t - \frac{b}{t}[X(v)\cos\theta - Y(v)\sin\theta]\right\}^2 + \left(\frac{d\theta}{du} + \frac{c}{t}\right)^2 [X^2(v) + Y^2(v)],$$
$$F = \left(\frac{d\theta}{du} + \frac{c}{t}\right)\left[X(v)\frac{dY}{dv} - Y(v)\frac{dX}{dv}\right],$$
$$B^2 = \left(\frac{dX}{dv}\right)^2 + \left(\frac{dY}{dv}\right)^2,$$
$$\Sigma^2 = \left[t^2B^2\left\{1 - \frac{b}{t^2}[X\cos\theta - Y\sin\theta]\right\}^2 + \left(\frac{d\theta}{du} + \frac{c}{t}\right)^2\left(X(v)\frac{dX}{dv} + Y(v)\frac{dY}{dv}\right)^2\right],$$

$$L = \frac{-1}{\Sigma}\left\{\left(\frac{d\theta}{du} + \frac{c}{t}\right)\left(X\frac{dX}{dv} + Y\frac{dY}{dv}\right)\left[\frac{b}{t}\left(2\frac{d\theta}{du} + \frac{c}{t}\right)(X\sin\theta + Y\cos\theta) - \left(\frac{d\theta}{du} + \frac{c}{t}\right)\left(t - \frac{b}{t}(X\cos\theta - Y\sin\theta)\right)\right] + \left(t - \frac{b}{t}(X\cos\theta - Y\sin\theta)\right)\left[\frac{b}{t}\left(t - \frac{b}{t}(X\cos\theta - Y\sin\theta)\right) - \frac{d^2\theta}{du^2}(X\sin\theta + Y\cos\theta)\right]\left(\frac{dX}{dv}\sin\theta + \frac{dY}{dv}\cos\theta\right)\right\},$$
$$M = \frac{1}{\Sigma}\left\{B^2\left[t - \frac{b}{t}(X\cos\theta - Y\sin\theta)\right] + \frac{b}{t}\left(X\frac{dX}{dv} + Y\frac{dY}{dv}\right)\left(\frac{dX}{dv}\cos\theta - \frac{dY}{dv}\sin\theta\right)\right\}\left(\frac{d\theta}{du} + \frac{c}{t}\right),$$
$$N = \frac{1}{\Sigma}\left(\frac{dX}{dv}\frac{d^2Y}{dv^2} - \frac{dY}{dv}\frac{d^2X}{dv^2}\right)\left[t - \frac{b}{t}(X\cos\theta - Y\sin\theta)\right].$$

The surfaces in general case are given in curvilinear non-orthogonal non-conjugate coordinates u, v;

$$\Sigma^2 = A^2B^2 - F^2.$$

■ Helix-Shaped Preliminarily Twisted Surface of Elliptical Cross Section

A helix-shaped preliminarily twisted surface of elliptical cross section bearing on a class of *helical-shaped surfaces* has a directrix curve in the form of a helical line of constant slope on the cylinder.

A generatrix ellipse is disposed in the normal plane of the helical line and besides the motion along this line, it rotates at the normal plane of the helix. The axis of rotation of a generatrix ellipse coincides with the tangent of the helical line.

Forms of definition of the surface

(1) Vector form of the definition (Figs. 1a–e and 2):

$$\begin{aligned}\mathbf{r} = \mathbf{r}(u,v) &= \boldsymbol{\rho}(u) + [a\cos v\cos\theta - d\sin v\sin\theta]\mathbf{v} + [a\cos v\sin\theta + d\sin v\cos\theta]\boldsymbol{\beta}\\ &= \{b - [a\cos v\cos\theta - d\sin v\sin\theta]\}\mathbf{h}(u) + [a\cos v\sin\theta + d\sin v\cos\theta] \times [-c\mathbf{n}(u) + b\mathbf{k}]/\sqrt{b^2 + c^2} + cu\mathbf{k},\end{aligned}$$

where $\boldsymbol{\rho}(u) = b\mathbf{h}(u) + cu\mathbf{k}$ is the radius vector of the directrix helical line laying at the cylinder of a radius b;

$$X = a\cos v, \quad Y = d\sin v$$

are the parametrical equations of the generatrix ellipse relatively to the local coordinate axes X, Y; $\mathbf{h}(u) = \mathbf{i}\cos u + \mathbf{j}\sin u$ is the unit vector disposed in the plane xOy; $\mathbf{v} = -\mathbf{h}(u)$;

$$\boldsymbol{\beta} = [-c\mathbf{n}(u) + b\mathbf{k}]/\sqrt{b^2 + c^2}$$

is the unit vector of the binormal of the directrix helical line;

$$\mathbf{n}(u) = -\mathbf{i}\sin u + \mathbf{j}\cos u.$$

Assume a new constant: $t = \sqrt{b^2 + c^2}$.

(2) Parametrical form of the definition (Figs. 1a–e and 2):

$$\begin{aligned}x = x(u,v) &= \{b - [a\cos v\cos\theta - d\sin v\sin\theta]\}\cos u + c\sin u[a\cos v\sin\theta + d\sin v\cos\theta]/t,\\ y = y(u,v) &= \{b - [a\cos v\cos\theta - d\sin v\sin\theta]\}\sin u - c\cos u[a\cos v\sin\theta + d\sin v\cos\theta]/t\\ z = z(u,v) &= cu + b[a\cos v\sin\theta + d\sin v\cos\theta]/t,\end{aligned}$$

where u is the angle in the plane xOy taken from the axis Ox in the direction of the axis Oy; $\theta = \theta(u)$ is an angle of twisting of the generatrix curve in the normal plane of the directrix helical line.

Coefficients of the fundamental forms of the surface when $\theta(u) = pu$:

$$A^2 = \left\{t - \frac{b}{t}[a\cos v\cos\theta - d\sin v\sin\theta]\right\}^2 + \left(p + \frac{c}{t}\right)^2[a^2\cos^2 v + d^2\sin^2 v],$$

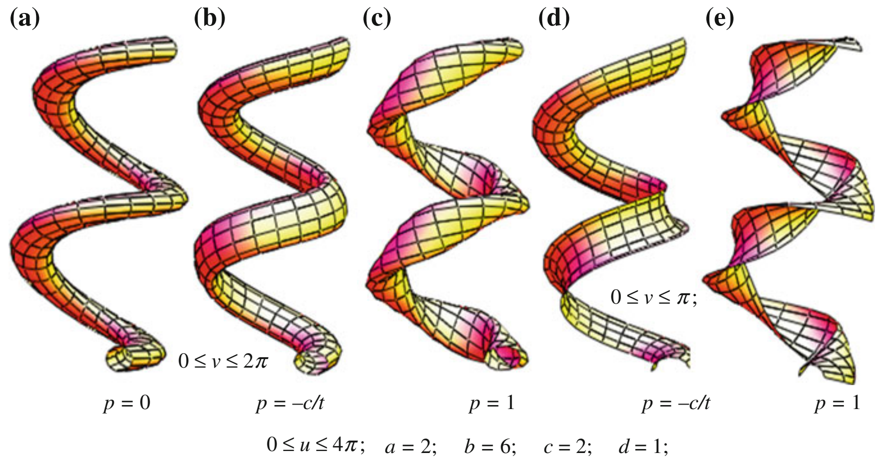


Fig. 1

$$F = ad\left(p+\frac{c}{t}\right),\quad B^2 = a^2\sin^2 v + d^2\cos^2 v,$$

$$\begin{aligned}\Sigma^2 &= A^2B^2 - F^2\\ &= t^2B^2\left\{1-\frac{b}{t^2}[a\cos v\cos\theta - d\sin v\sin\theta]\right\}^2\\ &\quad + \left(p+\frac{c}{t}\right)^2(d^2-a^2)^2\frac{\sin^2 2v}{4},\end{aligned}$$

$$\begin{aligned}M = \frac{p+c/t}{\Sigma}\Bigg\{&B^2\left[t-\frac{b}{t}(a\cos v\cos\theta - d\sin v\sin\theta)\right]\\ &-\frac{b}{t}(d^2-a^2)\frac{\sin 2v}{2}(a\sin v\cos\theta + d\cos v\sin\theta)\Bigg\},\end{aligned}$$

$$N = ad[t - b(a\cos v\cos\theta - d\sin v\sin\theta)/t]/\Sigma.$$

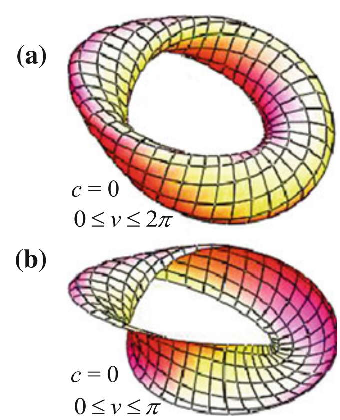


Fig. 2

A coefficient L of the second fundamental form of the surface is determined from the formula given at p. 308. In Fig. 1b, d, the surfaces are shown with the lines of the principle curvatures.

■ Helix-Shaped Preliminarily Twisted Surface of Circular Cross Section

A helix-shaped preliminarily twisted surface of circular cross section bearing on a class of *helical-shaped surfaces* has a directrix curve in the form of a helical line of constant slope on the cylinder and it is a special case of *a helix-shaped preliminarily twisted surface of an elliptical cross section.*

The generatrix circle is disposed at the normal plane of the helical line and besides the motion along this line, it rotates in the normal plane of the helix. The axis of rotation of a generatrix ellipse coincides with the tangent of the helical line.

The considered surface may be also regarded as *tubular helical surface* (see also Subsect. "7.1.2. Circular Helical Surfaces").

Forms of definition of the surface

(1) Vector form of the definition (Figs. 1a–e and 2):

$$\begin{aligned}\boldsymbol{r} = \boldsymbol{r}(u,v) &= \boldsymbol{\rho}(u) + a\cos(v+\theta)\boldsymbol{v} + a\sin(v+\theta)\boldsymbol{\beta}\\ &= [b - a\cos(v+\theta)]\boldsymbol{h}(u)\\ &\quad + a\sin(v+\theta)[-c\boldsymbol{n}(u) + b\boldsymbol{k}]/\sqrt{b^2+c^2} + cu\boldsymbol{k},\end{aligned}$$

where $\boldsymbol{\rho}(u) = b\boldsymbol{h}(u) + cu\boldsymbol{k}$ is the radius vector of the directrix helical line laying at the cylinder of a radius b; $X = a\cos v$, $Y = a\sin v$ are the parametrical equations of a generatrix ellipse relatively to the local coordinate axes X, Y;

$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u$$

is the unit vector disposed at the plane xOy; $\boldsymbol{v} = -\boldsymbol{h}(u)$;

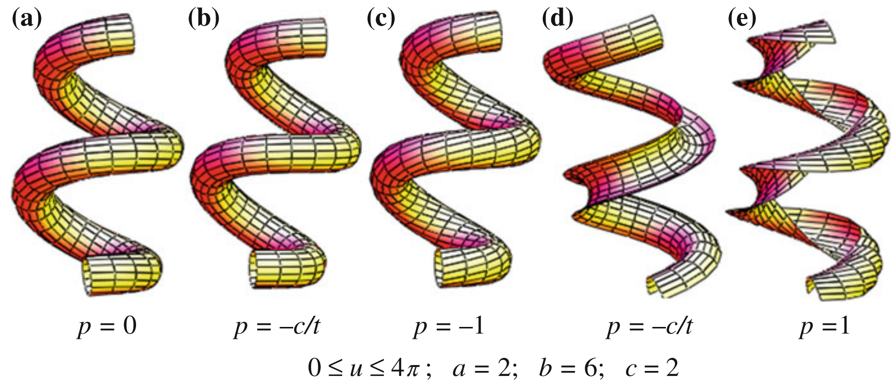


Fig. 1

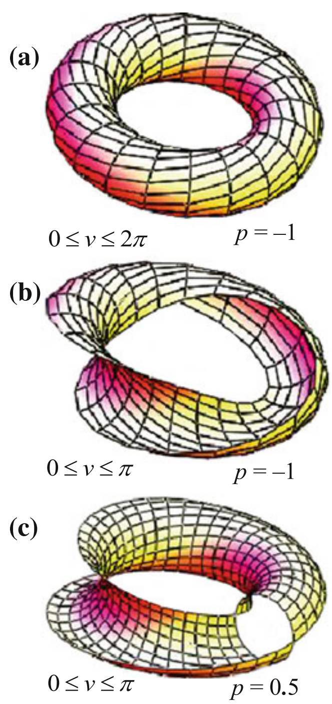


Fig. 2

$$\boldsymbol{\beta} = [-c\boldsymbol{n}(u) + b\boldsymbol{k}]/t$$

is the unit vector of the binormal of the directrix helical line;

$$\boldsymbol{n}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u; \quad t = \left[b^2 + c^2\right]^{1/2} = \text{const}.$$

(2) Parametrical form of the definition (Figs. 1a–e and 2):

$$\begin{aligned} x = x(u,v) &= [b - a\cos(v+\theta)]\cos u \\ &\quad + ca\sin u\sin(v+\theta)/t; \\ y = y(u,v) &= [b - a\cos(v+\theta)]\sin u \\ &\quad - ca\cos u\sin(v+\theta)/t; \\ z = z(u,v) &= cu + ba\sin(v+\theta)/t, \end{aligned}$$

where u is the angle in the plane xOy taken from the axis Ox in the direction of the axis Oy; $\theta = \theta(u)$ is an angle of twisting of the generatrix circle in the normal plane of the directrix helical line.

The generatrix circles are the lines of the principle curvatures.

Coefficients of the fundamental forms of the surface when $\theta(u) = pu$:

$$\begin{aligned} &A^2 = \left[t - \frac{ab}{t}\cos(v+\theta)\right]^2 + a^2\left(p + \frac{c}{t}\right)^2, \\ &F = a^2\left(p + \frac{c}{t}\right), \quad B^2 = a^2, \\ &\Sigma = \sqrt{A^2B^2 - F^2} = a\left[t - \frac{ab}{t}\cos(v+\theta)\right], \\ &L = -\frac{b}{t}\left[t - \frac{ab}{t}\cos(v+\theta)\right]\cos(v+\theta), \\ &M = a\left(p + \frac{c}{t}\right), \quad N = a, \\ &k_u = \frac{L}{A^2}, \quad k_v = \frac{1}{a}, \\ &K = -\frac{(b/t)[t - (ab/t)\cos(v+\theta)]\cos(v+\theta) + a(p + c/t)^2}{a[t - (ab/t)\cos(v+\theta)]^2}. \end{aligned}$$

The surface is given in non-orthogonal non-conjugate system of the curvilinear coordinates u, v. In Fig. 1b, d, the surfaces are shown with the lines of the principle curvatures. In Fig. 2, the surfaces have $c = 0$.

Additional Literature

Ivanov VN. Geometry and design of shells on the base of surfaces with a system of coordinate lines in the planes of pencil. Prostranstv. Konstruktzii Zdaniy i Soor.: Sb. Statey. Moscow: OOO «Devyatka Print». 2004; Iss. 9, p. 26-35 (13 refs.).

10.2 Helix-Shaped Twisted Surfaces with Plane Generating Curves in the Planes of Pencil

Helix-shaped twisted surfaces with plane generating curves in the planes of pencil are related to a class of *helix-shaped surfaces* because they have a directrix curve in the form of a helical line of the constant slope on the cylinder. A plane generatrix is disposed in the planes of a pencil with the fixed straight line of the pencil coinciding with the axis of the helix. Besides the ordinary helical motion along this axis, a generating curve rotates at the planes of the pencil. The axis of rotation of the generating curve coincides with the tangent to the projection of the helix on the coordinate plane that is perpendicular to the axis of the directrix helix.

Form of the definition of a helix-shaped twisted surface with a plane generating curve in the planes of pencil

(1) Parametric form of the definition:

$$x = x(u,v) = \{b + [X(v)\cos\theta - Y(v)\sin\theta]\}\cos u,$$
$$y = y(u,v) = \{b + [X(v)\cos\theta - Y(v)\sin\theta]\}\sin u,$$
$$z = z(u,v) = cu + [X(v)\sin\theta + Y(v)\cos\theta],$$

where u is the angle in the coordinate plane xOy taken from the coordinate axis Ox in the direction of the axis Oy; $\theta = \theta(u)$ is the angle of twisting of a plane generating curve in the plane of the pencil with the fixed straight coinciding with the axis Oz;

$$X = X(v), \quad Y = Y(v)$$

are the parametrical equations of a plane generating curve relatively to the local coordinate axes X, Y disposed in the planes of the pencil. The origin of the local coordinates is disposed on the directrix helical curve.

Coefficients of the fundamental forms of the surface:

$$A^2 = \{b + [X(v)\cos\theta - Y(v)\sin\theta]\}^2 + c^2 + 2c\frac{\theta}{du}[X(v)\cos\theta - Y(v)\sin\theta] + \left(\frac{d\theta}{du}\right)^2\left[X(v)^2 + Y(v)^2\right],$$

$$F = c\left[\frac{dX(v)}{dv}\sin\theta + \frac{dY(v)}{dv}\cos\theta\right] + \frac{d\theta}{du}\left[X(v)\frac{dY(v)}{dv} - Y(v)\frac{dX(v)}{dv}\right],$$

$$B^2 = \left(\frac{dX(v)}{dv}\right)^2 + \left(\frac{dY(v)}{dv}\right)^2,$$

$$\Sigma^2 = A^2B^2 - F^2 = \{b + [X(v)\cos\theta - Y(v)\sin\theta]\}^2 B^2 + \left\{\frac{d\theta}{du}\left[X(v)\frac{dX(v)}{dv} + Y(v)\frac{dY(v)}{dv}\right] + c\left[\frac{dX(v)}{dv}\cos\theta - \frac{dY(v)}{dv}\sin\theta\right]\right\}^2,$$

$$L = -\frac{1}{\Sigma}\Bigg\langle \{b + [X(v)\cos\theta - Y(v)\sin\theta]\} \Bigg\{[b + X(v)\cos\theta - Y(v)\sin\theta] \times \left[\frac{dX(v)}{dv}\sin\theta + \frac{dY(v)}{dv}\cos\theta\right] + \frac{d^2\theta}{du^2}\left[X(v)\frac{dX(v)}{dv} + Y(v)\frac{dY(v)}{dv}\right] + \left(\frac{d\theta}{du}\right)^2\left[X(v)\frac{dY(v)}{dv} - Y(v)\frac{dX(v)}{dv}\right]\Bigg\}$$
$$+ 2\frac{d\theta}{du}[X(v)\sin\theta + Y(v)\cos\theta] \times \left\{\frac{d\theta}{du}\left[X(v)\frac{dX(v)}{dv} + Y(v)\frac{dY(v)}{dv}\right] + c\left[\frac{dX(v)}{dv}\cos\theta - \frac{dY(v)}{dv}\sin\theta\right]\right\}\Bigg\rangle,$$

$$M = \frac{1}{\Sigma}\left\{\left[\frac{d\theta}{du}\left(X(v)\frac{dX(v)}{dv} + Y(v)\frac{dY(v)}{dv}\right) + c\left(\frac{dX(v)}{dv}\cos\theta - \frac{dY(v)}{dv}\sin\theta\right)\right] \times \left[\frac{dX(v)}{dv}\cos\theta - \frac{dY(v)}{dv}\sin\theta\right] - \left[b + (X(v)\cos\theta - Y(v)\sin\theta)\frac{d\theta}{du}B^2\right]\right\},$$

$$N = -\frac{b + [X(v)\cos\theta - Y(v)\sin\theta]}{\Sigma} \times \left[\frac{dX(v)}{dv}\frac{d^2Y}{dv^2} - \frac{d^2X}{dv^2}\frac{dY(v)}{dv}\right].$$

The surfaces of general type are given in non-orthogonal non-conjugate system of curvilinear coordinates u, v.

Additional Literature

Grigorenko YaM, Timonin AM. On one way to the numerical solving of the boundary problems of shells of complex geometry given in non-orthogonal curvilinear coordinate systems. Doklady UkrSSR. 1991; No. 4, p. 41-44 (4 refs.).

■ Helix-Shaped Twisted Surface of Elliptical Cross Section in the Planes of Pencil

A helix-shaped twisted surface of an elliptical cross section in the planes of a pencil belongs to a class of *helix- shaped surfaces* because it has a directrix curve in the form of a helical line of constant slope on the cylinder. The generatrix ellipses are disposed at the planes of the pencil with the fixed straight line of the pencil coinciding with the axis of the directrix helical line. Besides the ordinary helical motion along this axis, a generatrix ellipse rotates in the planes of the pencil. The axis of rotation of the generatrix ellipse coincides with the tangent to the circular projection of the helical line on the coordinate plane that is perpendicular to the axis of the helical line.

Form of definition of a helix-shaped twisted surface of elliptical cross section in the planes of pencil

(1) Parametrical equations (Fig. 1a–e):

$$x = x(u, v) = \{b + [a\cos v\cos\theta - d\sin v\sin\theta]\}\cos u,$$
$$y = y(u, v) = \{b + [a\cos v\cos\theta - d\sin v\sin\theta]\}\sin u,$$
$$z = z(u, v) = cu + [a\cos v\sin\theta + d\sin v\cos\theta],$$

where u is the angle in the coordinate plane xOy taken from the coordinate axis Ox in the direction of the axis Oy; $\theta = \theta(u)$ is the angle of twisting of a plane generating ellipse in the plane of the pencil with the fixed straight coinciding with the axis Oz;

$$X = X(v) = a\cos v, \quad Y = Y(v) = d\sin v$$

are the parametrical equations of a plane generating ellipse relatively to the local coordinate axes X, Y disposed in the planes of the pencil. The origin of the local coordinates is disposed on the directrix helical curve.

Coefficients of the fundamental forms of the surface when $\theta = pu$:

$$A^2 = (b + a\cos v\cos\theta - d\sin v\sin\theta)^2 + c^2 + 2cp(a\cos v\cos\theta - d\sin v\sin\theta) + p^2\left(a^2\cos^2 v + d^2\sin^2 v\right),$$
$$F = c(-a\sin v\sin\theta + d\cos v\cos\theta) + adp,$$
$$B^2 = a^2\sin^2 v + d^2\cos^2 v,$$
$$\Sigma^2 = A^2B^2 - F^2 = (b + a\cos v\cos\theta - d\sin v\sin\theta)^2B^2 + \left[\frac{p(d^2 - a^2)}{2}\sin 2v - c(a\sin v\cos\theta + d\cos v\sin\theta)\right]^2,$$
$$L = -\frac{1}{\Sigma}\Big\{(b + a\cos v\cos\theta - d\sin v\sin\theta)^2 \times (-a\sin v\sin\theta + d\cos v\cos\theta) + p^2ad(b + a\cos v\cos\theta - d\sin v\sin\theta) + 2p(a\cos v\sin\theta + d\sin v\cos\theta) \times \left[\frac{p(d^2 - a^2)}{2}\sin 2v - c(a\sin v\cos\theta + d\cos v\sin\theta)\right]\Big\},$$
$$M = -\frac{1}{\Sigma}\left\{\left[\frac{p(d^2 - a^2)}{2}\sin 2v - c(a\sin v\cos\theta + d\cos v\sin\theta)\right] \times (a\sin v\cos\theta + d\cos v\sin\theta) + b + pB^2(a\cos v\cos\theta - d\sin v\sin\theta)\right\},$$
$$N = -ad\frac{b + a\cos v\cos\theta - d\sin v\sin\theta}{\Sigma}\left(\sin^2 v - \cos^2 v\right).$$

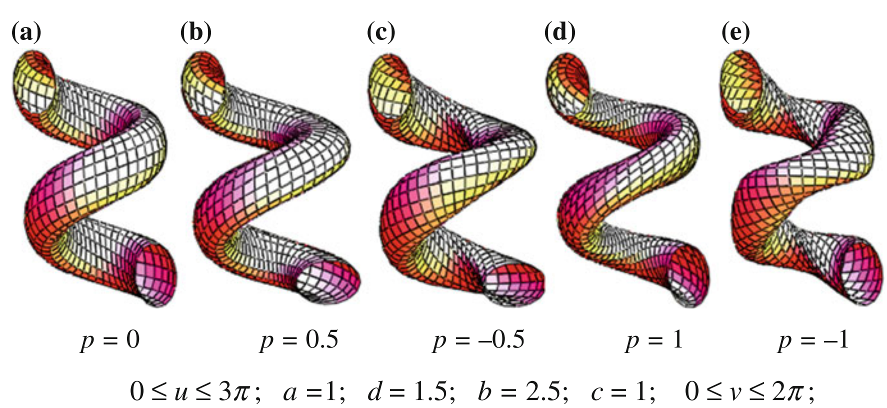


Fig. 1

Assume $p = 0$, then we shall obtain a *helical surface with a generatrix ellipse* (see also "Helical Surface with Generatrix Ellipse" in Subsect. "7.1.3. Ordinary Helical Surfaces with Arbitrary Plane Generatrix Curves"), Fig. 1a.

The helix-shaped twisted surface surfaces with different values of the parameter p are shown in Fig. 1b–e.

Reference

Ivanov VN. Geometry and design of shells on the base of surfaces with a system of coordinate lines in the planes of pencil. Prostranstv. Konstruktzii Zdaniy i Soor.: Sb. Statey. Moscow: OOO «Devyatka Print». 2004; Iss. 9, p. 26-35.

■ Helical Twisted Surface with Circles in the Planes of Pencil

A helical twisted surface with circles in the planes of a pencil is a special case of *a helix-shaped twisted surface of an elliptical cross section in the planes of a pencil* when major and minor semi-axes of a generatrix ellipse are equal to each other, i.e. $a = d$, and so the studied surface can be related to a class of *helix-shaped surfaces.* The generatrix circles are disposed at the planes of a pencil with a fixed

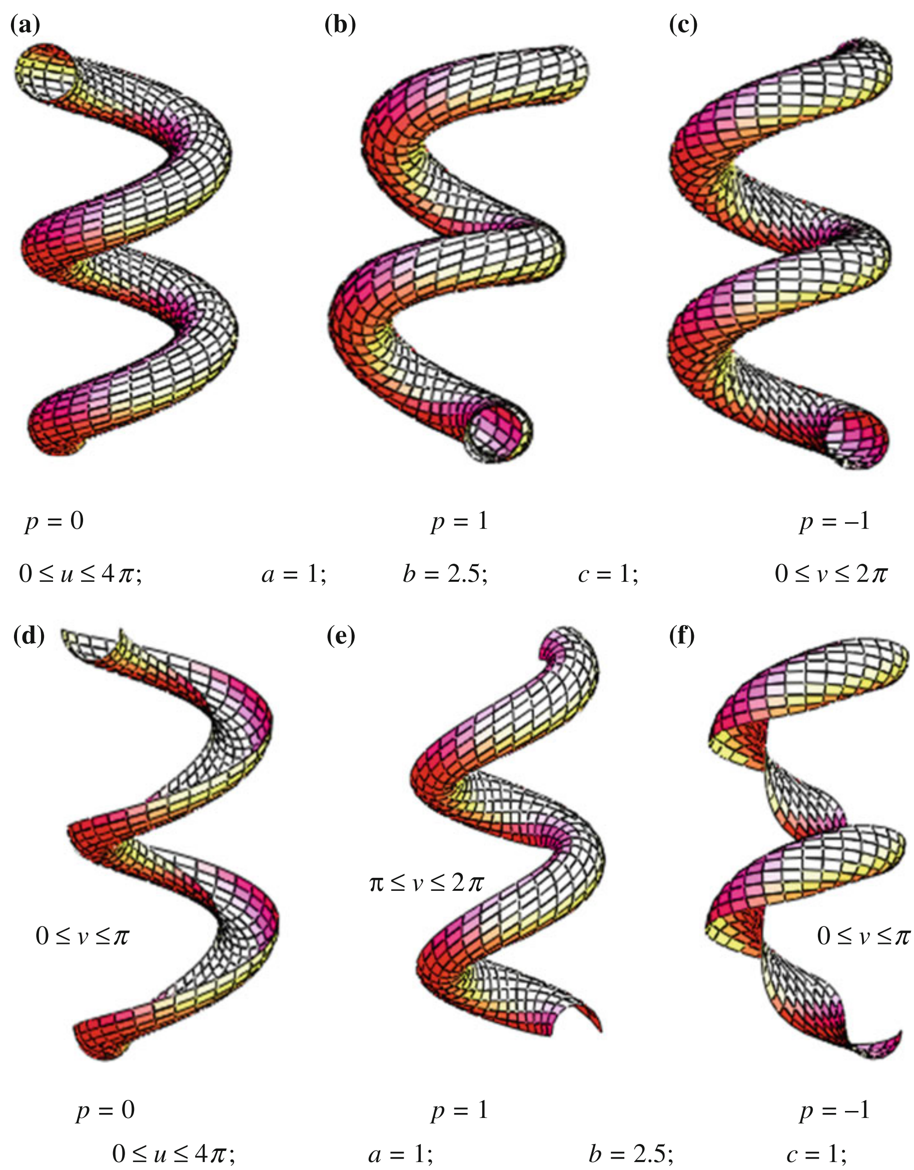


Fig. 1

straight line of the pencil coinciding with the axis of the directrix helical line. Besides the ordinary helical motion along this axis, the generatrix circle rotates in the planes of the pencil. The direction of the axis of rotation of a generatrix circle coincides with the direction of the tangent to the projection of the helical line on the coordinate plane perpendicular to the axis of the helical line.

Forms of definition of a helical twisted surface with circles in the planes of pencil

(1) Parametrical form of the definition (Fig. 1a–f):

$$x = x(u,v) = [b + a\cos(v+\theta)]\cos u,$$
$$y = y(u,v) = [b + a\cos(v+\theta)]\sin u,$$
$$z = z(u,v) = cu + a\sin(v+\theta),$$

where u is the angle in the coordinate plane xOy taken from the coordinate axis Ox in the direction of the axis Oy; $\theta = \theta(u)$ is the angle of twisting of a generatrix circle in the plane of the pencil with the fixed straight coinciding with the axis Oz;

$$X = X(v) = a\cos v, Y = Y(v) = a\sin v$$

are the parametrical equations of a generatrix circle relatively to the local coordinate axes X, Y disposed in the planes of the pencil. The origin of the local coordinates is disposed on the directrix helical curve.

Coefficients of the fundamental forms of the surface when $\theta = pu$:

$$A^2 = [b + a\cos(v+\theta)]^2 + c^2 + 2cpa\cos(v+\theta) + p^2a^2,$$
$$F = ca\cos(v+\theta) + a^2p,\ B^2 = a^2,$$
$$\Sigma^2 = A^2B^2 - F^2 = a^2\left\{[b + a\cos(v+\theta)]^2 + c^2\sin^2(v+\theta)\right\},$$
$$L = -\frac{a}{\Sigma}\langle[b + a\cos(v+\theta)]\{[b + a\cos(v+\theta]\cos(v+\theta) + ap^2\} - 2apc\sin^2(v+\theta)\rangle,$$
$$M = -\frac{-ca^2\sin^2(v+\theta) + b + pa^3\cos(v+\theta)}{\Sigma}$$
$$N = -a^2\frac{b + a\cos(v+\theta)}{\Sigma}\left(\sin^2 v - \cos^2 v\right).$$

Assume $p = 0$, then we shall obtain *a circle helical surface with a generatrix circle lying in a plane passing through the helical axis* (see also Subsect. "7.1.2. Circular Helical Surfaces"), Fig. 1a. In Fig. 1b–f, the total surfaces or their fragments with different values of the geometrical parameters are shown. When the parameter p changes, then only the form of the curvilinear coordinate lines u lying on the studied surface changes but the surface remains *a circular helical surface with a generatrix circle lying in a plane passing through the helical axis.*

(2) Vector form of the definition:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = [b + a\cos(v+\theta)]\boldsymbol{h}(u) + [cu + a\sin(v+\theta)]\boldsymbol{k},$$

where $\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u$; $\boldsymbol{\rho}(u) = b\boldsymbol{h}(u) + cu\boldsymbol{k}$ is a vector equation of a directrix helix on the cylinder of a radius b; $H = 2\pi c$ is a pitch of the directrix helix.

11 Blutel Surfaces

Blutel surface is formed by a single-parametric family of the conics and simultaneously envelopes a two-parametric family of second-order cones.

If the curves forming a conjugate net together with a family of the generatrix conics are also conics, then the planes of the generatrix conics pass through the fixed line *g*. And by the way, all generatrix conics pass through two points belong to the line *g* and these two points may coincide or be conjugated complexly. Analogously, all planes of the conjugate conics pass through a line *h* but all conjugate conics pass through two points at the line *h*.

Blutel surfaces include into themselves *quadrics* and *Dupin's cyclides*. Dupin cyclides are special instances of *double Blutel conic surfaces*, also known as *supercyclides*. These surfaces carry two families of conics being contained in two pencils of planes where tangent planes along the conics form quadratic cones.

The total classification of Blutel surfaces is given in a paper of W. Degen (1986). All of them are algebraic surfaces of the fourth order, as a rule. Besides, they represent a group of surfaces that are projectively equivalent to the Dupin cyclides.

Additional Literature

Degen W. Die Blutelschen Kegelschnittflächen, deren konjugierte Kurven ebanfaiis Kegelschnitte sind. Ber. math.-statist. Sek. Forschungszent. Graz. 1984; No. 1-2, p. 215-226.
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Bochillo GP. The fields of the quadratic cones and disposition on the variety of all hyperplane elements P_n. Differenzsialnaya Geometriya Mnogoobraziy Figur, Kaliningrad. 1988; No. 19, p. 15-19.
Degen W. Die zweifachen Blutelschen Kegelschnittflächen. Manuscr. math. 1986; 55, No. 1, p. 9-38.

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_11, © Springer International Publishing Switzerland 2015

12 Veronese Surfaces

Let M be a two-dimensional manifold; σ: $M \to S^4(1)$ an immersion into the four-dimensional unit sphere of the real Euclidean space R^5.

To each point $m_0 \in M$, let us associate a mobile orthogonal frame $\{m; v_1, \ldots, v_5\}$ of the space R^5 such that $m = \sigma(m_0)$; $v_1, v_2 \in T_m(\sigma(M))$.

The fundamental equations of the moving frames are:

$$\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2.$$

Then $\sigma(M) \subset S^4(1) \subset S^N(1)$ is a *Veronese surface* when $K = 1/3$; K is Gaussian curvature.

The Veronese surface may be defined as follows: In the Euclidean 3-space R^3, let us consider orthogonal coordinates (x, y, z) and the mapping $S^2(\sqrt{3}) \to S^4(1)$ given by

$$u_1 = \frac{1}{3}\sqrt{3}\cdot yz,\ u_2 = \frac{1}{3}\sqrt{3}\cdot xz,\ u_3 = \frac{1}{3}\sqrt{3}\cdot xy,$$
$$u_4 = \frac{1}{6}\sqrt{3}\cdot(x^2 - y^2),\ u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2)$$

$(u_1, \ldots, u_5)$ are orthogonal coordinates in R^5. To each point of the Veronese surface, we may associate the orthogonal frames such as $\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2$ with

$$\omega_1^3 = -\omega_2^4 = \sqrt{3}\omega^2/3;\ \omega_1^4 = \omega_2^3 = \sqrt{3}\omega/3^1;$$
$$\omega_3^4 = -2\omega_1^2.$$

For this Veronese surface, we get $K = 1/3$ and $k = -2/3$, where k is a curvature of the normal bundle. The Gaussian curvature K and the curvature of the normal bundle k are defined by

$$\mathrm{d}\omega_1^2 = -K\omega^1 \wedge \omega^2,\ \mathrm{d}\omega_3^4 = -k\omega^1 \wedge \omega^2.$$

The projections of the Veronese surfaces into the three-dimensional manifold are called Steiner surfaces. In a paper of Švec Alois (1988), some properties of the Veronese surfaces are ascertained and the conditions of separation of a Veronese surface from $\sigma(M)$ are proved.

For example, we can present:

Theorem 3 *Let σ: $M \to S^4(1)$, dim $M = 2$, be a minimal immersion; let M be compact. Suppose, on M, $K > 0$ and*

$$-\frac{7}{3}\min_M K < \min_M k \le \max_M k \le -2\max_M K.$$

Then $\sigma(M)$ is the Veronese surface.

Takehiro Iton (1988) proved the following theorem:

Let M be a compact connected-oriented surface minimally immersed in a unit sphere through the m-regular immersion. If its Gaussian curvature K satisfies

$$\frac{2}{(m+2)(m+3)} \le K \le \frac{2}{(m+1)(m+2)},\ 0 \le m,$$

then M is *a generalized Veronese surface*.

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13 Tzitzéica Surfaces

The centroaffine invariant

$$I = \frac{K}{d^4},$$

where K is the Gaussian curvature of a surface Σ and d is the distance from an arbitrary point of the surface Σ to the tangent plane to this surface at an arbitrary point of Σ, was introduced by G. Tzitzéica (1907). Gheorghe Tzitzéica (1873–1939) is the founder of the Romania school of differential geometry. A surface Σ for which the ratio K/d^4 is constant, is called *Tzitzéica's surface*.

As an example of a surface with constant ratio K/d^4, G. Tzitzéica has regarded a surface given by the equation

$$z(x^2 + y^2) = 1.$$

Let us consider the case in which Σ is a simple surface, given by the explicit Cartesian equation:

$$z = z(x, y).$$

In this case, the Gaussian curvature of the surface Σ is

$$K = \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2},$$

and the distance d from the origin to the tangent plane at an arbitrary point of the surface Σ is

$$d = \frac{|xz_x + yz_y - z|}{\sqrt{1 + z_x^2 + z_y^2}}.$$

Having nonzero centroaffine invariant I may result in the condition

$$I = \frac{K}{d^4}$$

in the nonlinear differential equation in partial derivatives:

$$z_{xx}z_{yy} - z_{xy}^2 = I(xz_z + yz_y - z)^4.$$

Moreover, the conditions d, $K \neq 0$ are equivalent to

$$z_{xx}z_{yy} - z_{xy}^2 \neq 0; \; xz_z + yz_y - z \neq 0.$$

The obtained differential equation is *a* Monge–Ampère equation:

$$z_{xx}z_{yy} - z_{xy}^2 = H(x, y, z, z_x, z_y).$$

Therefore, the differential equation $z_{xx}z_{yy} - z_{xy}^2 = I(xz_z + yz_y - z)^4$ must be called *Monge–Ampère*–Tzitzéica *equation*.

Popa Emil M. has derived a differential equation defining Tzitzéica surfaces of the type: $z = f\,[g(x) + h(y)]$.

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S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
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■ Tzitzéica Surface of the Second-Order with the Centroaffine Invariant $I = -a^2$

Three types of Tzitzéica surfaces of second-order with the centroaffine invariant $I = -a^2$ are known.

Forms of definition of the Tzitzéica's surface with $I = -a^2$

(1) Explicit form of the definition (Fig. 1):

$$z = z(x, y) = \sqrt{1 + axy}.$$

This surface of the negative Gaussian curvature ($K < 0$) is a *hyperbolic paraboloid*.

In Fig. 1, the surface with $c = a^{-1/2}$ is presented.

(2) Explicit form of the definition:

$$z = z(x, y) = \sqrt{1 + axy} + \varepsilon x.$$

(3) Explicit form of the definition:

$$z = z(x, y) = \sqrt{1 + a(x - \varepsilon_1 y)y} + \varepsilon_2 x.$$

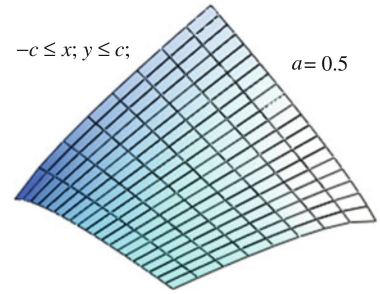

Fig. 1

Additional Literature

Udrişte C, Bîlă N. Symmetry group of Tzitzéica surfaces PDE. Balkan J. of Geometry and Its Applications. 1999; Vol. 4, No. 2, p. 123-140.

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■ Tzitzéica Surface of the Third-Order with the Centroaffine Invariant $I = 1/27$

A differential equation

$$z_{xx}z_{yy} - z_{xy}^2 = I(xz_z + yz_y - z)^4,$$

where centroaffine invariant $I = 1/27$, has a decision $z = 1/(xy)$. Using the theorem, proved by Udrişte C. and Bîlă N., it is possible to show that explicit equations

$$z = \frac{1}{xy} + \varepsilon x,\ z = \frac{1}{xy} + \varepsilon y,$$
$$z = \frac{1}{(x - \varepsilon_1 y)y} + \varepsilon_2 x,$$

where ε, ε_1, and ε_2 are constants, also define the Tzitzéica surfaces with

$$I = 1/27.$$

Forms of the definition of the Tzitzéica surface with $I = 1/27$

(1) Explicit equation (Fig. 1):

$$z = z(x, y) = 1/(xy).$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + 1/(x^4y^2),\ F = 1/(xy)^3,$$
$$B^2 = 1 + 1/(x^2y^4)$$
$$L = \frac{2y}{x\sqrt{x^4y^4 + x^2 + y^2}},$$
$$N = \frac{2x}{y\sqrt{x^4y^4 + x^2 + y^2}},$$

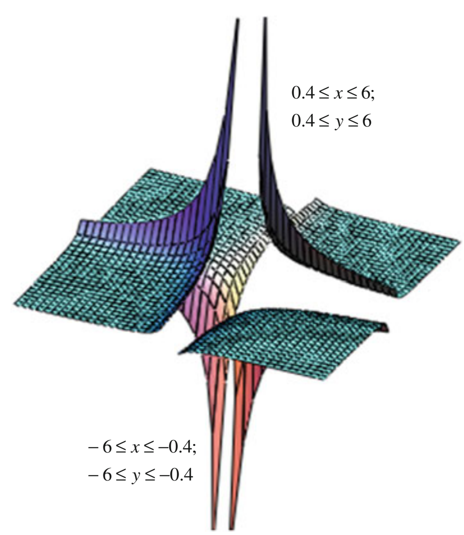

Fig. 1

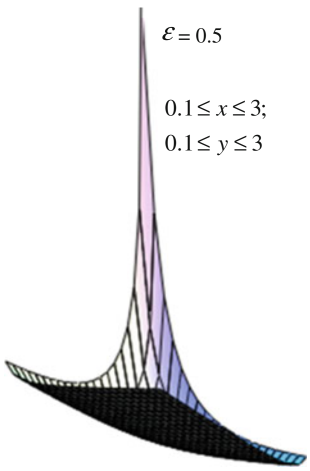

Fig. 2

$$A^2B^2 - F^2 = \left(x^4y^4 + x^2 + y^2\right)/\left(x^4y^4\right),$$
$$N = \frac{2x}{y\sqrt{x^4y^4 + x^2 + y^2}},$$
$$K = \frac{3x^4y^4}{(x^4y^4 + x^2 + y^2)^2} < 0.$$

The studied surface is *an algebraic surface of the third-order* of the positive Gaussian curvature. This surface is also called *an affine sphere of the elliptical type*.

(2) Explicit equation (Fig. 2):

$$z = 1/(xy) + \varepsilon x.$$

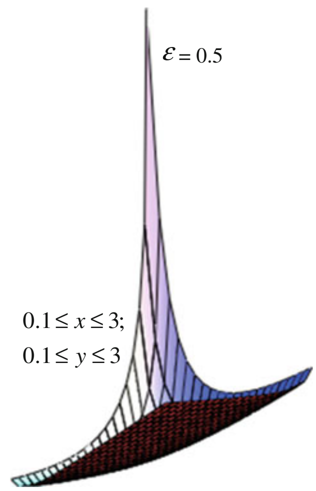

Fig. 3

(3) Explicit equation (Fig. 3):

$$z = \frac{1}{xy} + \varepsilon y.$$

(4) Explicit equation:

$$z = \frac{1}{(x - \varepsilon_1 y)y} + \varepsilon_2 x.$$

Reference

Udrişte C, Bîlă N. Symmetry group of Tzitzéica surfaces PDE. Balkan J. of Geometry and Its Applications. 1999; Vol. 4, No. 2, p. 123-140.

■ Tzitzéica Surface of the Second-Order with the Centroaffine Invariant $I = a^2$

A differential equation $z_{xx}z_{yy} - z_{xy}^2 = I(xz_z + yz_y - z)^4$, where centroaffine invariant $I = a^2$ = const has a decision $z = +\sqrt{1 - a(x^2 + y^2)}$, which defines the Tzitzéica surface of the second order. C. Udrişte and N. Bîlă have proved that the equations

$$z = +\sqrt{1 - a(x - \varepsilon y)^2 + y^2},$$
$$z = +\sqrt{1 - a(x^2 + y^2)} + \varepsilon x.$$

where ε is a constant, define also the Tzitzéica surface with $I = a^2$.

Forms of the definition of the surface

(1) Explicit equation (Fig. 1):

$$z = +\sqrt{1 - a(x^2 + y^2)}, \text{ where } z > 0.$$

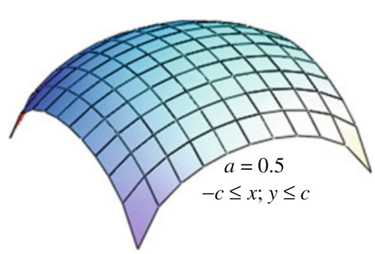

Fig. 1

This surface of the positive Gaussian curvature is a segment of *an ellipsoid of revolution*, which is limited by the sides

$$-\sqrt{1/a}<x<\sqrt{1/a},\ -\sqrt{1/a}<y<\sqrt{1/a},\ 0<z\leq 1.$$

In Fig. 1, the surface has $c = (2a)^{-1/2}$.

(2) Explicit equation (Fig. 2):

$$z=+\sqrt{1-a(x-\varepsilon y)^2+y^2};\ z>0.$$

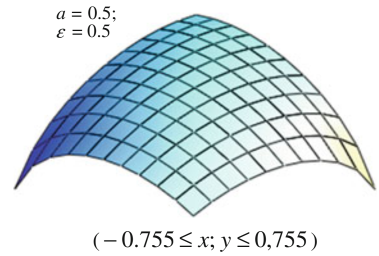

Fig. 2

(3) Explicit equation:

$$z=+\sqrt{1-a(x^2+y^2)}+\varepsilon x;\ z-\varepsilon x>0.$$

■ Tzitzéica Surface of the Third-Order with the Centroaffine Invariant I = –4/27

A Tzitzéica surface of the third order with centroaffine invariant I = –4/27 may be of three types.

Forms of the definition of the surface

(1) Explicit equation (Fig. 1):

$$z=z(x,y)=1/(x^2+y^2).$$

This surface of negative Gaussian curvature is a *surface of revolution of a hyperbola* $z = 1/x$ about *the axis* Oz. First, G. Tzitzéica proved the existence of a constant centroaffine invariant of this surface of revolution.

(2) Explicit equation (Fig. 2):

$$z=z(x,y)=\frac{1}{x^2+y^2}+\varepsilon x.$$

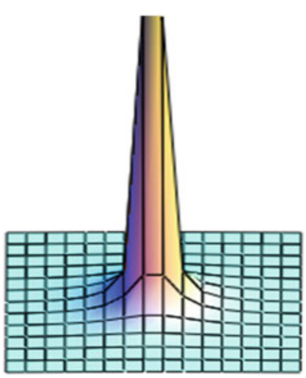

Fig. 1

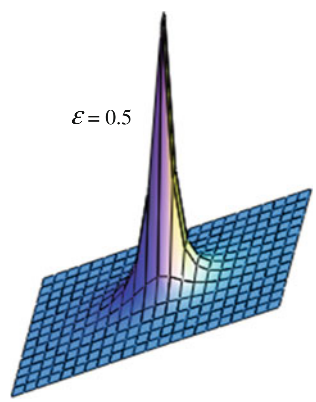

Fig. 2

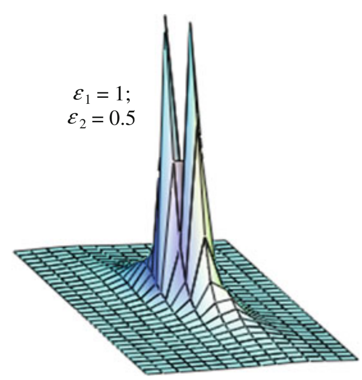

Fig. 3

(3) Explicit equation (Figs. 3 and 4):

$$z=z(x,y)=\frac{1}{(x-\varepsilon_1 y)^2+y^2}+\varepsilon_2 x$$

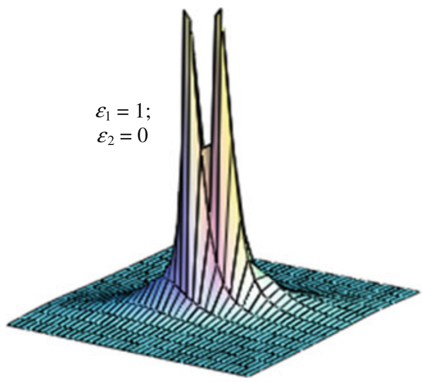

Fig. 4

All surfaces represented at Figs. 1, 2, 3 and 4 are designed with the boundaries:

$$-4 \leq x \leq 4;\ -4 \leq y \leq 4.$$

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14 Peterson Surfaces

Peterson surface is a surface having a conjugate net of conical or cylindrical lines which are the main base of the bending. For example, *Monge surfaces with a circular cylindrical directrix surface*, the corresponding *translation surfaces* and *surfaces of revolution* are Peterson surfaces. The indicatrix of rotations of Peterson surfaces is *right conoid*. In particular, *right helicoid* is the indicatrix for *carved surface*; *equilateral hyperbolic paraboloid* is the indicatrix for translation surface. First, this class of the surfaces was studied by K.M. Peterson as an example of surfaces assuming bending at the main base. Peterson (1866) has pointed at a class of surfaces capable to bend so that two appointed families of lines remain conjugated during all process of bending. Using his terminology, one may say that these lines are main base of bending for the considered surfaces.

Bending at the main base is a bending F_t of a surface F, when the directions of the extreme bending remain unchangeable. The net formed by the lines having the direction of the extreme bending is conjugated on each surface of F_t and is called a main *base of the bending*. In 1892, E. Goursat has shown that the family of the plane parallel cross sections at the Peterson surface remains plane parallel one after the bending as well. L. Bianshi has proved that to every infinitesimal bending of the surface, another surface is in keeping "linked" with the first one so that the normals at the corresponding points of the both surfaces are parallel and the asymptotic lines of the second surface correspond to the conjugate lines (base of the bending) on the first surface (B.K. Mlodzievskiy 1900).

Peterson surface carries on itself a conic conjugate net formed by the lines of touching of the cones circumscribed about the surface. The vertexes of these cones lay on two space curves. If both of these curves are plane and one of them lies in an improper plane then the net becomes a cylindrical–conical one. It has been established, catenoid is the only minimal surface among this type of Peterson surfaces (E.A. Korolev 1987).

One of a group of quasi-helical surfaces contains surfaces given by a vector equation depending on four functions of a single argument

$$\boldsymbol{R}(u,v) = [f(v) + u\psi(v)]\boldsymbol{e}(v) + [F(u) + \Phi(v)]\boldsymbol{k},$$
$$\text{where } \boldsymbol{e}(v) = \boldsymbol{i}\cos v + \boldsymbol{j}\sin v,$$

$F(u)$ is an equation of a meridian of the surface at the initial moment of the time, i.e., when $v = 0$. Some quasi-helical surfaces with special values of functions $f(v)$ and $\Phi(v)$ may be related to Peterson surfaces. For example, when $f(v) = 0$, *generalized Peterson surfaces* are obtained, produced from ordinary helical surfaces.

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DOI 10.1007/978-3-319-11773-7_14,

■ Bendings of a Three-Axial Ellipsoid

The last Peterson work "On bending of surfaces of the second-order" has appeared in 1882 after the death of its author. According to Peterson, if we have a triaxial ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

then its bendings are defined by equations

$$x=x(u,v)=r\cos\varphi\cos u,\ \ y=y(u,v)=r\sin\varphi\cos u,$$
$$z=z(u,v)=\int\sqrt{a^2(1-k^2)\sin^2 u+c^2\cos^2 u}du,$$

where

$$r=r(v)=\sqrt{a^2k^2-(a^2-b^2)\sin^2 v},$$
$$\varphi=\varphi(v)=\int\frac{a\sqrt{k^2(a^2\sin^2 v+b^2\cos^2 v)-(a^2-b^2)\sin^2 v}}{a^2k^2-(a^2-b^2)\sin^2 v}\mathrm{d}v.$$

Here, k is a parameter of bending, u and v are curvilinear coordinates conjugated on all surfaces of the given family, the lines u = const are conical lines, v = const are cylindrical lines. A parameter k = 1 corresponds to the initial surface.

Obviously, changing roles of the axes, it is possible to obtain three families of similar bendings and changing the signs before a^2, b^2, and c^2, one may find the bendings of other *central surfaces of the second other*.

Peterson has studied the bending of the second-order surfaces, the lengths of axes of which obtain the ultimate values. These values may be of three types: an axis may turn into infinity, or into zero, or become equal to another axis.

The curvature of the points moving away into infinity will tend to zero when the axes of the second-order surface turn into infinity. In this case, the surface becomes a plane. But we have three cases when the curvature tends not to zero but to the finite limit. It happens when (1) values b/a^2 and c/a^2 have the finite limits; (2) when value $a^3/(bc)$ has the finite limit, and (3) when values b/a and c/a have the finite limits. At the first case, we obtain the ordinary paraboloids the bendings of which have been found by Peterson. In two other cases, the surface of the second-order becomes a plane, but due to the choice of the orders of the infinities of the axes, the curvature of the part of the surface moving off does not tend to zero. If the points of an ellipsoid are related to elliptical coordinates on the surfaces then the linear element of the surface may be written in the form:

$$\mathrm{d}s^2=\frac{u-v}{4}\left[\frac{u\mathrm{d}u^2}{(a^2-u)(b^2-u)(c^2-u)}-\frac{v\mathrm{d}v^2}{(a^2-u)(b^2-u)(c^2-u)}\right].$$

All are real points of the surface when $a^2\geq u\geq b^2$, $b^2\geq v\geq c^2$.

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15 Surfaces of Bézier

An elementary surface of Bézier is defined with the help of a vector equation:

$$\boldsymbol{R}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}B_i^m(u)B_j^n(v)P_{ij},\quad 0\le u\le 1;\ \ 0\le v\le 1,$$

where $\boldsymbol{P}=\{\boldsymbol{P}_{ij}\},\quad i=0,1,\ldots,m;\quad j=0,1,\ldots,n$ is a given data;

$$B_i^m(u)=\binom{m}{i}u^i(1-u)^{m-i},\quad B_j^n(v)=\binom{n}{j}v^j(1-v)^{n-j}$$

are polynomials of S.N. Bernstein.

Parametrical equations of a surface of Bézier may be represented into a matrix form:

$$\begin{pmatrix}x(u,v)\\ y(u,v)\\ z(u,v)\end{pmatrix}=\left(u^0\ldots u^m\right)\boldsymbol{M}^T\begin{pmatrix}P_{00} & \ldots & P_{0n}\\ \ldots & \ldots & \ldots\\ P_{m0} & \ldots & P_{mn}\end{pmatrix}\boldsymbol{N}\begin{pmatrix}v^0\\ \ldots\\ v^n\end{pmatrix},$$

where $\boldsymbol{M}=(u_{ij})$ and $\boldsymbol{N}=(v_{ij})$ are square matrixes of the m and n orders:

$$u_{ij}=(-1)^{j-i}\binom{m}{j}\binom{j}{i},\quad v_{ij}=(-1)^{j-i}\binom{n}{i}\binom{j}{i}.$$

Surface of Bézier may consist of the rectangular or triangular segments. When the domain of changing of the parameters u and v are arbitrary rectangles of the view $\boldsymbol{R}=\{(u,v)\}$, where $a\le u\le b$, $c\le v\le d$, a vector equation of surfaces of Bézier has the following form:

$$\boldsymbol{R}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}B_i^m\left(\frac{u-a}{b-a}\right)B_j^n\left(\frac{v-c}{d-c}\right)P_{ij}.$$

Elementary surfaces of Bézier possess the interesting properties:

(1) they are smooth surfaces, (2) the boundary curves of elementary surfaces of Bézier are *curves of Bézier* of the corresponding degrees, their contour broken lines form the boundary of a polyhedral surface (*reference graph*), (3) an elementary surface of Bézier is an affine invariant, (4) it "repeats" the reference polyhedral surface, (5) if all vertexes of the file $\boldsymbol{P}$ lay at one plane, then a surface of Bézier determined by this file is a plane curvilinear polygon lying at this plane, (6) the changing of the only one vertex in the file (data) causes the conspicuous changing of the whole surface of Bézier, (7) a priory information about a surface of Bézier is sufficiently rough.

Additional Literature

http://www.rus-lib.ru/book/28/gr/55/165-176.html

■ Bicubic Surface of Bézier

Assume $m=n=3$ in a vector equation of a surface of Bézier

$$\boldsymbol{R}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}B_i^m(u)B_j^n(v)P_{ij},$$

then we shall obtain an elementary bicubic surface of Bézier that is defined by 16 vertexes:

$$\begin{pmatrix}P_{00}, P_{01}, P_{02}, P_{03}\\ P_{10}, P_{11}, P_{12}, P_{13}\\ P_{20}, P_{21}, P_{22}, P_{23}\\ P_{30}, P_{31}, P_{32}, P_{33}\end{pmatrix}$$

Using Bernstein polynomials, we can describe this surface by the following equation (Fig. 1):

$$\boldsymbol{R}(u,v)=\sum_{i=0}^{3}B_i^3(u)\left(\sum_{j=0}^{n}B_j^3(v)P_{ij}\right),\ 0\le u\le 1;\ 0\le v\le 1,$$

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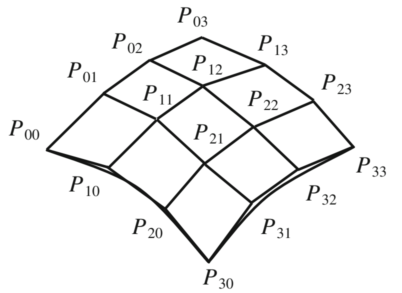

Fig. 1

or with the help of a matrix form:

$$\boldsymbol{R}(u,v) = \begin{pmatrix} B_0^3(u) B_1^3(u) B_2^3(u) B_3^3(u) \end{pmatrix} \times \begin{pmatrix} P_{00}, P_{01}, P_{02}, P_{03} \\ P_{10}, P_{11}, P_{12}, P_{13} \\ P_{20}, P_{21}, P_{22}, P_{23} \\ P_{30}, P_{31}, P_{32}, P_{33} \end{pmatrix} \begin{pmatrix} B_0^3(v) \\ B_1^3(v) \\ B_2^3(v) \\ B_3^3(v) \end{pmatrix} = \boldsymbol{U}^T \boldsymbol{M}^T \boldsymbol{PMV},$$

where

$$\boldsymbol{R}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix},$$

$$\boldsymbol{P} = \begin{pmatrix} P_{00}, P_{01}, P_{02}, P_{03} \\ P_{10}, P_{11}, P_{12}, P_{13} \\ P_{20}, P_{21}, P_{22}, P_{23} \\ P_{30}, P_{31}, P_{32}, P_{33} \end{pmatrix},$$

$$\boldsymbol{M} = \begin{pmatrix} 1 & -3 & 3 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix $\boldsymbol{M}$ is called *a basic matrix of a bicubic surface of Bézier*.

The combined bicubic surface of Bézier is a "C"-smooth (continuous) surface which is a composition of the elementary bicubic surfaces of Bézier.

An elementary rational bicubic surface of Bézier is given by the file (data) from 16 vertexes and is described by a vector equation:

$$\boldsymbol{R}(u,v) = \frac{1}{w(u,v)} \sum_{i=0}^{3} \sum_{j=0}^{3} w_{ij} B_i^3(u) B_j^3(v) P_{ij}, \ 0 \le u \le 1; \ 0 \le v \le 1,$$

where

$$w(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} w_{ij} B_i^3(u) B_j^3(v);$$

w_{ij} are nonnegative numbers, the sum of which is positive, are called *a scale* (*balance*). If the scales w_{ij} are equal between themselves then we obtain *a standard elementary surface of Bézier*.

The properties of a rational bicubic surface of Bézier arise from the properties of an elementary surface of Bézier. Besides, the behavior of a projective-invariant rational bicubic surface of Bézier is determined not only by the file of the vertexes but also by the set of the free parameters that are the scale factors w_{ij} under the given set of the vertex. Changing scale coefficients, it is possible to have an impact on a form of a rational bicubic surface of Bézier.

Bézier surfaces can be of any degree, but bicubic Bézier surfaces generally provide enough degrees of freedom for most applications.

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Quasi-ellipsoidal Surfaces

16

The forming of *quasi-ellipsoidal surfaces* is based on mathematical transformations applied to a canonic equation of *ellipsoid*. V.A. Nikityuk picked up three groups of quasi-ellipsoidal surfaces.

(1) *Quasi-ellipsoidal surfaces with three values of the semi-axes* are given by an equation:

$$\left(\frac{|x|}{a}\right)^n + \left(\frac{|y|}{b}\right)^m + \left(\frac{|z|}{c}\right)^k = 1,$$

where n, m, k are positive numbers.

A quasi-ellipsoid of this type has a closed surface with maximal dimensions along the axes x, y, z equal to $2a$, $2b$, $2c$, accordingly.

(2) *Quasi-ellipsoidal surfaces with six values of the semi-axes* are given by an equation:

$$\left(\frac{|x|}{a_1\theta(-x) + a_2\theta(x)}\right)^n + \left(\frac{|y|}{b_1\theta(-y) + b_2\theta(y)}\right)^m + \left(\frac{|z|}{c_1\theta(-z) + c_2\theta(z)}\right)^k = 1,$$

where n, m, k are positive numbers, $\theta(\xi)$ is *a Heaviside function*; $\theta(\xi) = 0$ if $\xi < 0$ and $\theta(\xi) = 1$ if $\xi \geq 0$. Application of Heaviside function gives an opportunity to introduce six different values of semi-axes of the quasi-ellipsoidal surface: a_1 when $x < 0$ and a_2 when $x \geq 0$; b_1 when $y < 0$ and b_2 when $y \geq 0$; c_1 when $z < 0$ and c_2 when $z \geq 0$. The quasi-ellipsoid has a closed surface with maximum dimensions along the axes x, y, and z equal to sum of the semi-axes: $a_1 + a_2$, $b_1 + b_2$ and $c_1 + c_2$, accordingly. If a quasi-ellipsoidal surface has different values of the semi-axes a_i, b_i, c_i, then it will not be symmetrical relatively to the coordinate planes yOz, xOz and xOy. The values of the exponents of n, m, k define the sign of the curvature of the segments of the surface and the existence of ribs.

(3) *Quasi-ellipsoidal surfaces with cylindrical insertions* along the axis z may be given by an equation:

$$\left(\frac{|x|}{a_1\theta(-x) + a_2\theta(x)}\right)^n + \left(\frac{|y|}{b_1\theta(-y) + b_2\theta(y)}\right)^m + \left(\frac{z\theta(z) + |(z + c_0)\theta(-z - c_0)|}{c_1\theta(-z) + c_2\theta(z)}\right)^k = 1,$$

where n, m, k are positive numbers, $\theta(\xi)$ is a Heaviside function; $\theta(\xi) = 0$ if $\xi < 0$ and $\theta(\xi) = 1$ if $\xi \geq 0$. Application of Heaviside function gives an opportunity to introduce six different values of semi-axes of the quasi-ellipsoidal surface: a_1 when $x < 0$ and a_2 when $x \geq 0$; b_1 when $y < 0$ and b_2 when $y \geq 0$; c_1 when z < 0 and c_2 when $z \geq 0$. The quasi-ellipsoid has a closed surface with maximum dimensions along the axes x, y and z equal to sum of the semi-axes: $a_1 + a_2$, $b_1 + b_2$, and $c_1 + c_2 + c_0$, accordingly.

A quasi-ellipsoid of this type may contain a cylindrical insertion by the length c_0 oriented along the axis z. A director line of the cylindrical part oriented along the axis z coincides with the line of the quasi-ellipsoid—the plane xOy intersection.

■ Quasi-ellipsoidal Surface with Three Values of Semi-axes

A quasi-ellipsoidal surface with three values of semi-axes is a closed surface given by an implicit equation:

$$\left(\frac{|x|}{a}\right)^n + \left(\frac{|y|}{b}\right)^m + \left(\frac{|z|}{c}\right)^k = 1,$$

where n, m, k are positive numbers.

A quasi-ellipsoid of this type has maximum dimensions along the axes x, y, z equal to $2a$, $2b$, $2c$, accordingly, where a, b, c are three semi-axes of a quasi-ellipsoid.

In Fig. 1, the quasi-ellipsoidal surface with semi-axes $a = 2$ m, $b = 1$ m, $c = 3$ m and with the values of the degrees $n = m = 2.5$; $k = 0.5$ is shown. The net on the surface is formed by parallels obtained by crossings of the surface by the planes that are perpendicular to the axis Oz, and by

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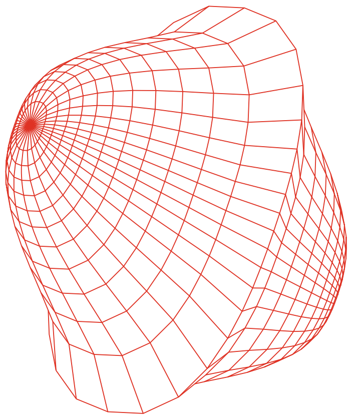

Fig. 1

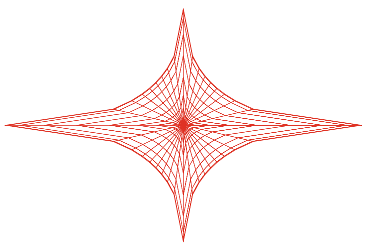

Fig. 2 $n = m = 0.5$

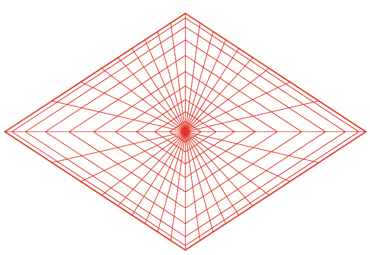

Fig. 3 $n = m = 1.0$

meridians obtained by crossings of the surface by the pencil of the planes passing through the axis *Oz*.

In Figs. 2, 3, 4, 5 and 6, the projections of the quasi-ellipsoids with the semi-axes $a = 2$ m, $b = 1$ m, $c = 3$ m on the coordinate plane *yOx* are presented. The value of the degree is $k = 0.5$.

When the values of exponents are less than unity, then a quasi-ellipsoid has concave segments of the surface (Fig. 2; $n = m = 0.5$).

When the values of exponents are equal to unity, then a quasi-ellipsoid takes a form of *an octahedron* (Fig. 3; $n = m = 1$).

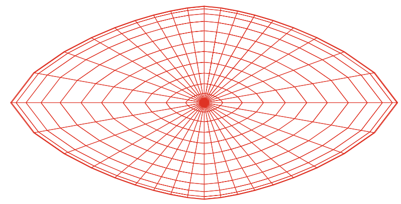

Fig. 4 $n = m = 1.5$

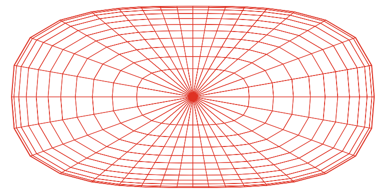

Fig. 5 $n = m = 3.0$

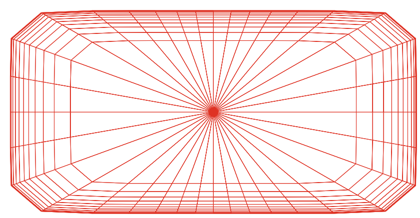

Fig. 6 $n = m = 10.0$

If the values of exponents are more than unity, then a quasi-ellipsoid has convex segments of the surface (Figs. 4, 5 and 6; $n = m = 1.5$; 3.0; 10.0, correspondingly).

If the values of exponents are more than two or they are equal to two, then a quasi-ellipsoid has *a smooth surface*.

When the values of the exponents tend to infinity, the surface of a quasi-ellipsoid tends to a surface of *a rectangular parallelepiped* (Fig. 6).

A quasi-ellipsoidal surface with $n = m = k$ is called *a superellipsoid*. Superellipsoids as computer graphics primitives were popularized by Alan H. Barr who used the name “superquadrics” to refer to both superellipsoids and supertoroids. The superellipsoid can be seen in the form of a dome that was built in April 1998 for Ruth Sussman of Umina, New South Wales, Australia.

The parallels of the quasi-ellipsoidal surfaces, shown in Figs. 1, 2, 3, 4, 5 and 6, are so-called *superellipses*. First, in 1818, Gabriel Lamé (1795–1870) has begun to study these closed curves given in Cartesian coordinates with the help of an equation

$$\left|\frac{x}{t}\right|^m + \left|\frac{y}{p}\right|^m = 1.$$

Superellipses with $t = p$ are known also as *curves of Lame* or *ovals of Lame*.

The Danish poet and scientist Piet Hein (1905–1996) has popularized the use of the superellipse in architecture, urban planning, and furniture making.

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16.1 Quasi-ellipsoidal Surfaces with Six Values of Semi-axes

■ Quasi-ellipsoidal Surface with Concave Segments Between Ribs

Quasi-ellipsoidal surfaces with six values of the semi-axes and with concave segments of the surface between ribs are given by an implicit equation:

$$\left(\frac{|x|}{a_1\theta(-x)+a_2\theta(x)}\right)^n+\left(\frac{|y|}{b_1\theta(-y)+b_2\theta(y)}\right)^n+\left(\frac{|z|}{c_1\theta(-z)+c_2\theta(z)}\right)^k=1,$$

where $0 < n < 1$; k is a positive number, $\theta(\xi)$ is the Heaviside function; see also Chap. “16. Quasi-Ellipsoidal Surfaces”, p. 331.

In Figs. 1 and 2, the projections of the quasi-ellipsoid on the coordinate planes xOy and yOz are shown. The quasi-ellipsoidal surfaces with six values of the semi-axes and with values of the exponents $0 < n < 1$ and $0 < k < 2$ have the ribs at the places of their intersection with the coordinate planes and the concave smooth segments of the surfaces between the ribs (Fig. 1). The quasi-ellipsoidal surfaces with the values of the exponents $0 < n < 1$ and $k \geq 2$ have the ribs at the places of their intersection with the coordinate planes, the smooth surface with the concave parallels and the meridians changing sign of curvature (Fig. 2).

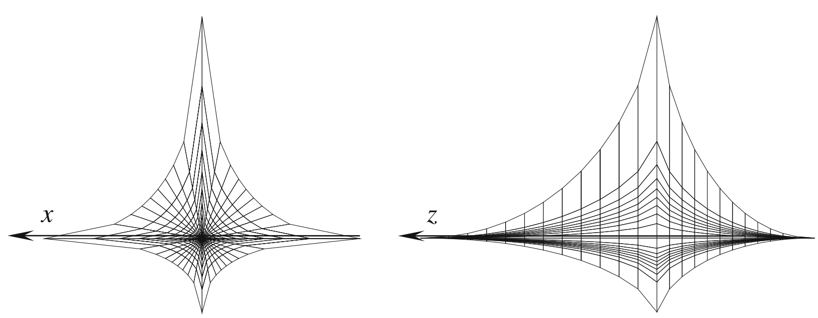

Fig. 1 $a_1 = a_2 = 1$; $b_1 = 1.5$; $b_2 = 0.5$; $c_1 = 1$; $c_2 = 1.5$; $n = 0.5$; $k = 0.7$

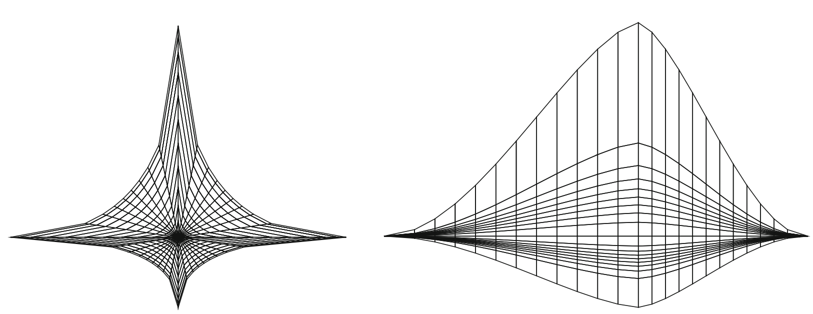

Fig. 2 $a_1 = a_2 = 1$; $b_1 = 1.5$; $b_2 = 0.5$; $c_1 = 1$; $c_2 = 1.5$; $n = 0.5$; $k = 3.5$

■ Ruled Quasi-ellipsoidal Surface

Ruled quasi-ellipsoidal surfaces are given by an implicit equation

$$\left(\frac{|x|}{a_1\theta(-x)+a_2\theta(x)}\right)+\left(\frac{|y|}{b_1\theta(-y)+b_2\theta(y)}\right)+\left(\frac{|z|}{c_1\theta(-z)+c_2\theta(z)}\right)^k=1,$$

where k is a positive number, $\theta(\xi)$ is the Heaviside function; see also Chap. "16. Quasi-Ellipsoidal Surfaces," p. 331.

In Figs. 1, 2 and 3, the projections of the quasi-ellipsoidal surfaces on the coordinate planes xOy and yOz are shown. If $k = 1$, then a ruled quasi-ellipsoidal surfaces degenerates into *an octahedron* (Fig. 2). The parallels between the longitudinal ribs are rectilinear but the meridians are convex (Fig. 3).

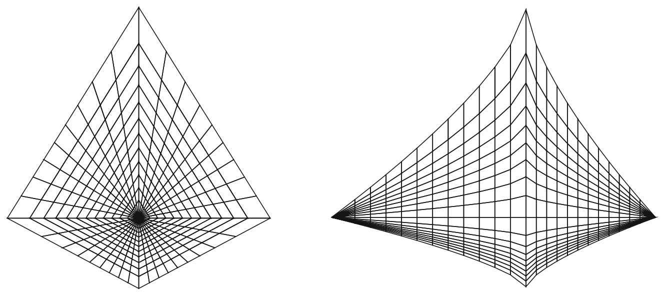

Fig. 1 $a_1 = a_2 = 1$; $b_1 = 1.5$; $b_2 = 0.5$; $c_1 = 1$; $c_2 = 1.5$; $n = 0.5$; $k = 0.7$

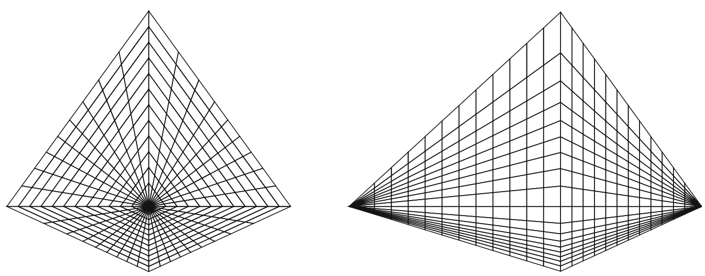

Fig. 2 $a_1 = a_2 = 1$; $b_1 = 1.5$; $b_2 = 0.5$; $c_1 = 1$; $c_2 = 1.5$; $k = 1$

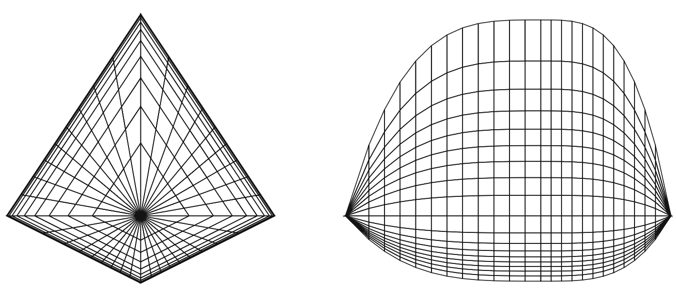

Fig. 3 $a_1 = a_2 = 1$; $b_1 = 1.5$; $b_2 = 0.5$; $c_1 = 1$; $c_2 = 1.5$; $k = 3.5$

■ Quasi-ellipsoidal Surface with Convex Segments Between Ribs

Quasi-ellipsoidal surfaces with six values of the semi-axes and with convex segments between the ribs are given by an equation:

$$\left(\frac{|x|}{a_1\theta(-x)+a_2\theta(x)}\right)^n+\left(\frac{|y|}{b_1\theta(-y)+b_2\theta(y)}\right)^n$$
$$+\left(\frac{|z|}{c_1\theta(-z)+c_2\theta(z)}\right)^k=1,$$

where $n > 1$, k is a positive numbers, $\theta(\xi)$ is a Heaviside function; $\theta(\xi) = 0$, $\theta(\xi) = 0$ if $\xi < 0$ and $\theta(\xi) = 1$ if $\xi \geq 0$.

Application of Heaviside function gives an opportunity to introduce six different values of semi-axes of the quasi-ellipsoidal surface: a_1 when $x < 0$ and a_2 when $x \geq 0$; b_1 when $y < 0$ and b_2 when $y \geq 0$; c_1 when $z < 0$ and c_2 when $z \geq 0$.

The quasi-ellipsoid has a closed surface with maximum dimensions along the axes *x, y,* and *z* equal to sum of the semi-axes: $a_1 + a_2$, $b_1 + b_2$, and $c_1 + c_2$, accordingly.

If a quasi-ellipsoidal surface has different values of the semi-axes a_i, b_i, c_i, then it will not be symmetrical relatively to the coordinate planes *yOz, xOz*, and *xOy*.

In Figs. 1, 2 and 3, the projections of the quasi-ellipsoidal surfaces with the convex segments on the coordinate planes *xOy* and *yOz* are presented.

The quasi-ellipsoidal surfaces of this type with the values of the exponents equal to $1 < n < 2$ and $k < 1$ have ribs at the places of intersection with the coordinate planes and have smooth surfaces between the ribs.

The parallels are convex and the curvature of the meridians changes its sign (Fig. 1).

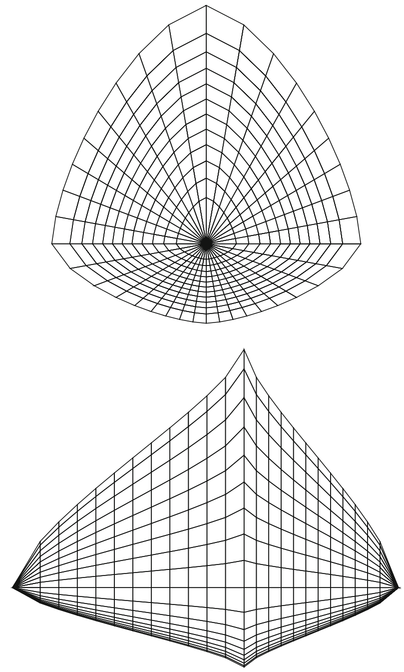

Fig. 1 $a_1 = a_2 = 1$; $b_1 = 1.5$; $b_2 = 0.5$; $c_1 = 1$; $c_2 = 1.5$; $n = 1.5$; $k = 0.7$

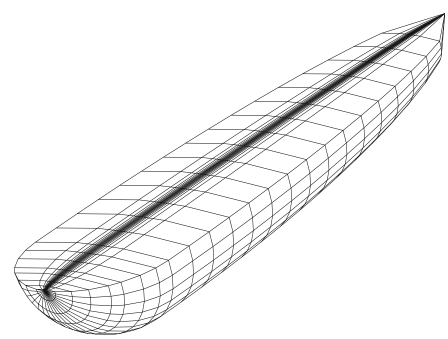

Fig. 2 $a_1 = 2.5$; $a_2 = 0.2$; $b_1 = b_2 = 3$; $c_1 = 2.5$; $c_2 = 15$; $n = k = 2$

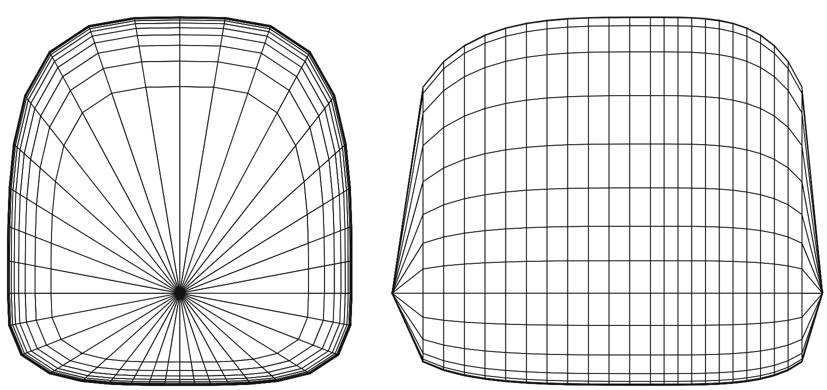

Fig. 3 $a_1 = a_2 = 1$; $b_1 = 1.5$; $b_2 = 0.5$; $c_1 = 1$; $c_2 = 1.5$; $n = k = 3.5$

If $k > 1$ and $1 < n < 2$, then the quasi-ellipsoid has the form shown in Fig. 2. Having assumed $n \geq 2$ and $k \geq 2$, one may construct a quasi-ellipsoidal surface with convex segments shown in Fig. 3.

When the values of the exponents tend to infinity, the surface of a quasi-ellipsoid tends to a surface of *a rectangular parallelepiped*.

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Nikityuk VA. Pressure Vessel. Patent of Russia No. 2109203, April 20, 1998.

16.2 Quasi-ellipsoidal Surfaces with Cylindrical Insertions

■ Quasi-ellipsoidal Surface with Convex Segments and a Cylindrical Insertion

Quasi-ellipsoidal surfaces with convex segments and cylindrical insertions along the axis z may be given by a single equation:

$$\left(\frac{|x|}{a_1\theta(-x)+a_2\theta(x)}\right)^n+\left(\frac{|y|}{b_1\theta(-y)+b_2\theta(y)}\right)^n + \left(\frac{z\theta(z)+|(z+c_0)\theta(-z-c_0)|}{c_1\theta(-z)+c_2\theta(z)}\right)^k=1,$$

where $n > 1$, k is a positive number, $\theta(\xi)$ is a Heaviside function; $\theta(\xi) = 0$ if $\xi < 0$ and $\theta(\xi) = 1$ if $\xi \geq 0$.

Application of Heaviside function gives an opportunity to introduce six different values of semi-axes of the quasi-ellipsoidal surface: a_1 when $x < 0$ and a_2 when $x \geq 0$; b_1 when $y < 0$ and b_2 when $y \geq 0$; c_1 when $z < 0$ and c_2 when $z \geq 0$.

The quasi-ellipsoid has a closed surface with maximum dimensions along the axes x, y, and z equal to sum of the semi-axes: $a_1 + a_2$, $b_1 + b_2$, and $c_1 + c_2 + c_0$, accordingly.

A quasi-ellipsoid of this type may contain a cylindrical insertion by the length of a_0 oriented along the axis x; by the length of b_0 oriented along the axis y, and by the length of c_0 oriented along the axis z.

A director line of the cylindrical part oriented along the axis x coincides with the line of the quasi-ellipsoid—the plane yOz intersection.

By analogy, director lines of cylindrical parts oriented along the axes y and z coincide with the lines of the quasi-ellipsoid—the planes xOz and xOz intersection, correspondingly.

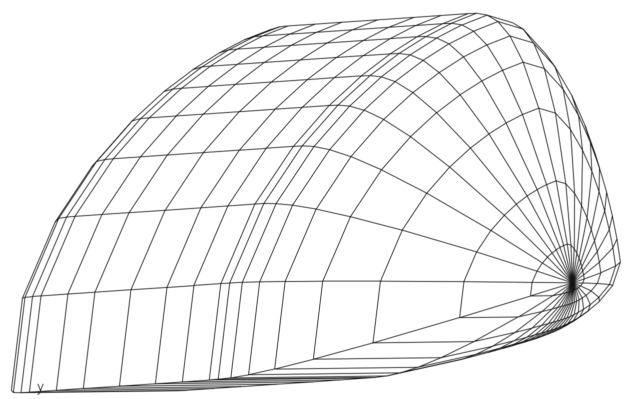

Fig. 1

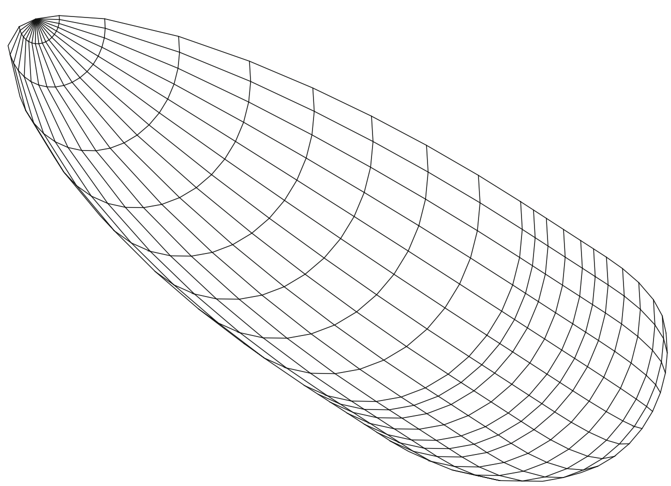

Fig. 2

In Fig. 1, the quasi-ellipsoidal surface with the cylindrical insertion having c_0 = 5 m along the coordinate axis z is shown. It has

n = 1.8; k = 3; a_1 = 10 m; a_2 = 3 m; b_1 = 5 m;
b_2 = 1 m; c_1 = c_2 = 2 m.

In Fig. 2, the quasi-ellipsoidal surface with the convex segments (n = k = 2) and with the cylindrical insertion having the length c_0 = 30 cm along the coordinate axis z is presented. The dimensions of the semi-axes are:

$$a_1 = 1.5\,\text{cm};\ a_2 = 20\,\text{cm};\ b_1 = b_2 = 20\,\text{cm};$$
$$c_1 = 120\,\text{cm};\ c_2 = 50\,\text{cm}.$$

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17 Cyclic Surfaces

A cyclic surface is formed by motion of a circle of variable or constant radius according to some law in the space (Fig. 1). An equation of a cyclic surface in a vector form is

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + R(u)\boldsymbol{e}(u, v),$$

where $\boldsymbol{r}(u, v)$ is the radius vector of a cyclic surface; $\boldsymbol{\rho}(u)$ is the radius vector of a directrix curve (*a line of the centers of the generatrix circles*); $R(u)$ is the law of changing of a radius of the generatrix circles; $\boldsymbol{e}(u, v)$ is a vector function of the circle of the unit radius in the plane of the generatrix circle with the normal $\boldsymbol{n}(u)$ (Fig. 2); $\boldsymbol{e}_0(u)$, $\boldsymbol{g}_0(u)$ is the unit vectors of the orthogonal system of coordinates lying in the plane of the generatrix circle; v is a polar angle in the plane of the generatrix circle.

$$\begin{aligned} \boldsymbol{e}(u, v) &= \cos \boldsymbol{e}_0(u) + \sin v\boldsymbol{g}_0(u); \\ \boldsymbol{g}(u, v) &= -\sin v\boldsymbol{e}_0(u) + \cos v\boldsymbol{g}_0(u); \\ \boldsymbol{e}_0(u) \times \boldsymbol{g}_0(u) &= \boldsymbol{e}(u, v) \times \boldsymbol{g}(u, v) = \boldsymbol{n}(u) \end{aligned}$$

In 1869, Alfred Enneper wrote that a surface in Euclidean space R^3 is called *cyclic* if is foliated by circles or by their arcs. First examples of cyclic surfaces are the surfaces of revolution. Enneper also proved that the planes of the foliation of a minimal surface must be parallel and catenoid is the only *cyclic minimal surfaces.*

Coefficients of fundamental forms of the surface:

$$\begin{aligned} E &= s^2 + 2\,s[(\boldsymbol{te})R' + (\boldsymbol{tg})R(\boldsymbol{e}'_0\boldsymbol{g}_0) - R(\boldsymbol{tn})(\boldsymbol{en}')] \\ &\quad + R^{/2} + R^2\left[(e'_0\boldsymbol{g}_0)^2 + (\boldsymbol{en}')^2\right]; \\ G &= R^2, \quad F = R[s(\boldsymbol{tg}) + R(\boldsymbol{e}'\boldsymbol{g}_0)]; \\ \sigma &= \sqrt{[s(\boldsymbol{te}) + R']^2 + [s(\boldsymbol{tn}) - R(\boldsymbol{en}')]^2}; \quad s = |\boldsymbol{\rho}'| \ s' = \partial s/\partial u; \\ L &= \{[(\boldsymbol{te})s + R']T_1 - [s(\boldsymbol{tn}) - R(\boldsymbol{en}')]T_2\}/\sigma; \\ M &= R\{[(\boldsymbol{tn})s - (\boldsymbol{en}')R][(e'_0g_0) - [(\boldsymbol{te})s + R'](\boldsymbol{gn}')\}/\sigma, \\ T_1 &= s'(\boldsymbol{tn}) + s^2k(\boldsymbol{n\nu}) + 2R'(\boldsymbol{en}') - R[(\boldsymbol{en}'') + 2(\boldsymbol{e}'_0\boldsymbol{g}_0)(\boldsymbol{gn}')] \\ T_2 &= s'(\boldsymbol{te}) + s^2k(\boldsymbol{e\nu}) - R[(\boldsymbol{e}'_0\boldsymbol{g}_0)^2 + (\boldsymbol{en}')^2] + R''; \end{aligned}$$

where $\boldsymbol{t} = \boldsymbol{\rho}'s$; $\boldsymbol{t}$, $\boldsymbol{\nu}$ are the unit vectors of the tangent and normal to the line of centers.

Surfaces of revolution, circular helical surfaces, and *tubular surfaces* with an arbitrary line of centers are the most used cyclic surfaces.

Canal surfaces, *normal cyclic surfaces, cyclic surfaces with circles in planes of a pencil, and cyclic surfaces with a plane of parallelism* form a class of cyclic surfaces. The more detailed classification of the cyclic surfaces is given in the next page.

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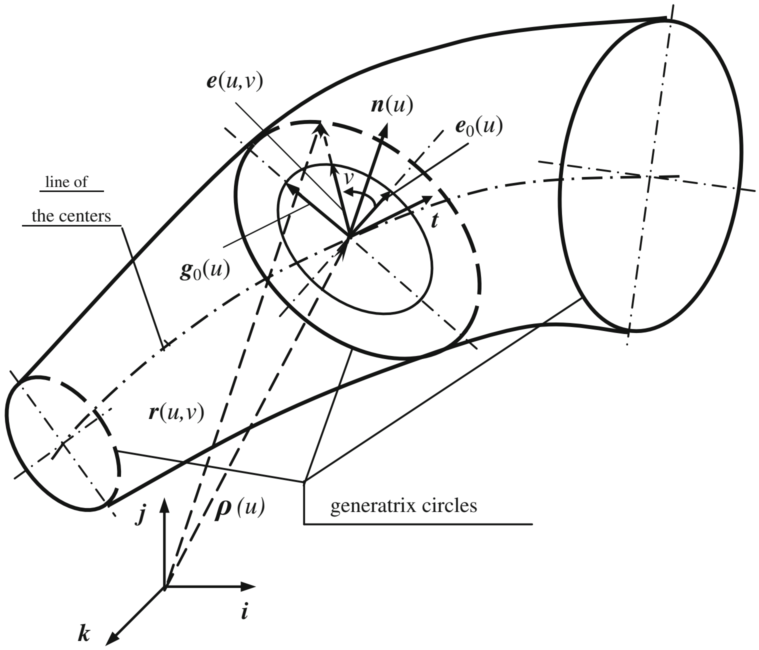

Fig. 1

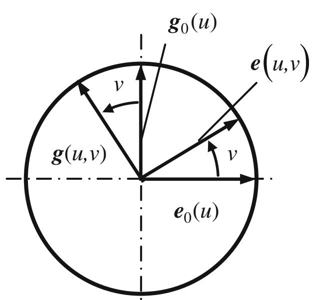

Fig. 2

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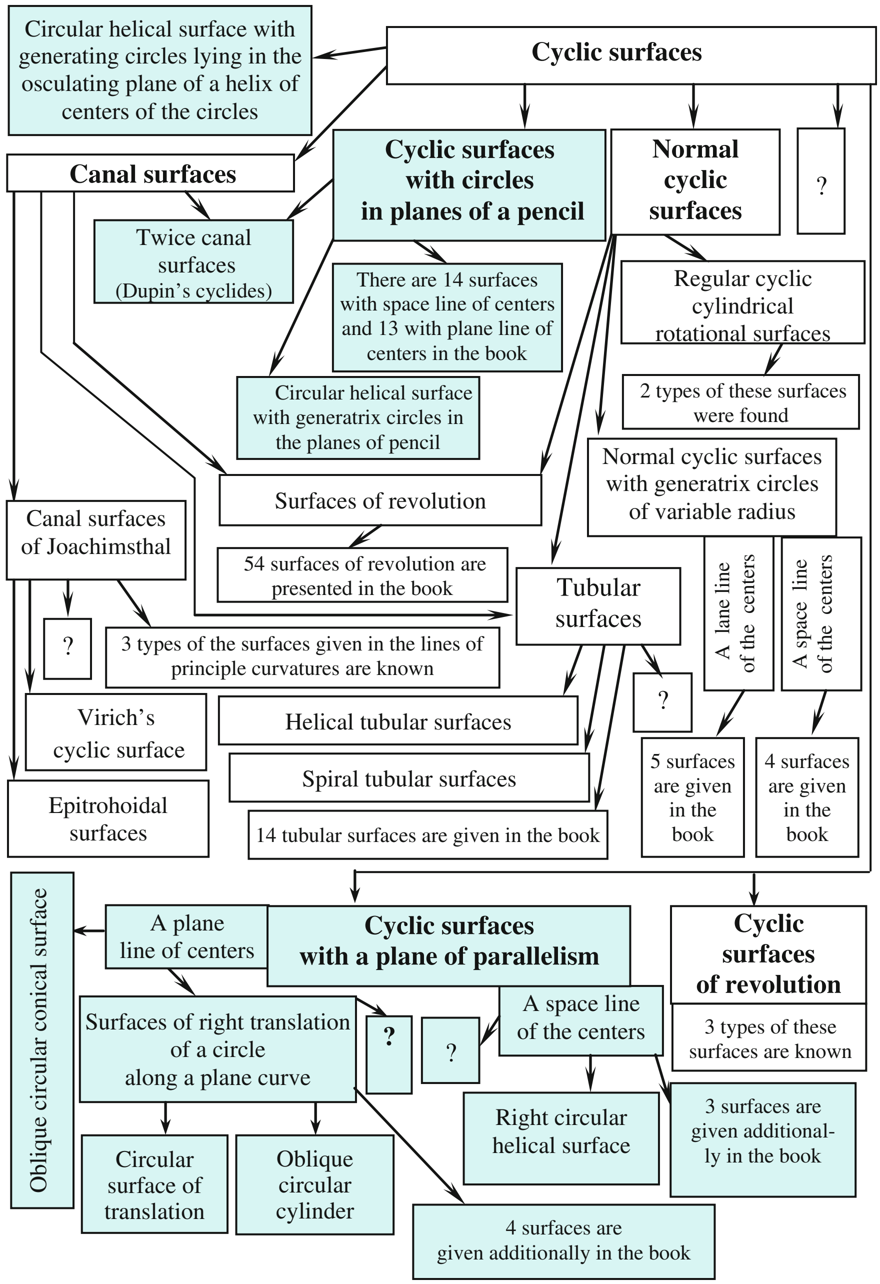
■ Classification of Cyclic Surfaces



The classification of the cyclic surfaces includes both the well-known groups of surfaces and cyclic surfaces which are known only to not great numbers of geometricians. It is meant that some of the cyclic surfaces not included in the classification must take their place in the vacant cells. The complete list of known cyclic surfaces is given in the table of contents in this chapter. Some cyclic surfaces appear simultaneously in another classes of surfaces. For example, a subgroup of the cyclic surfaces "Surfaces of Revolution" sets up an individual class of the same name.

Any circle at the space may be given by a vector, the beginning of which coincides with the center of the circle, but its direction coincides with the direction of the normal to a plane, in which the circle lies. The length of the normal is assumed equal to a radius of the circle and this vector is called *a defining vector of the circle*. So, arbitrary cyclic

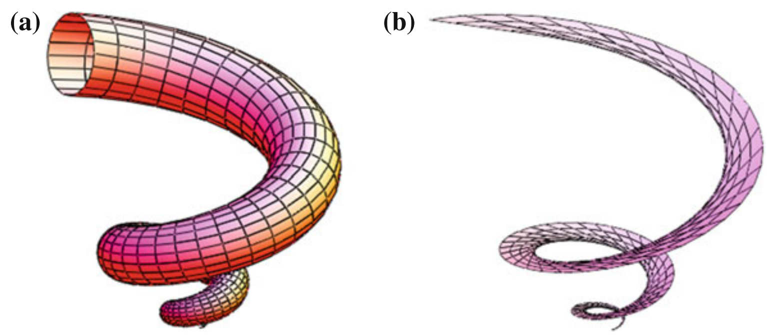


Fig. 1

surface may be related in space to the ruled surface formed by the motion of the defining vector of the circle. The ruled surface derived by this method is called *a basis surface* or *a base of the cyclic surface*. The beginnings of defining vectors trace *the line of centers* on the basis surface but their ends define *the line of radius*.

In Fig. 1a, the normal spiral cyclic surface is shown and in Fig. 1b, the basic surface of the cyclic surface, presented in Fig. 1a, is given.

The normals to the planes of the generatrix circles, i.e., *the defining vectors* coincide with the tangents to the directrix line of centers of normal cyclic surfaces.

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P.S.: Additional literature is given at corresponding pages of this chapter.

17.1 Canal Surfaces

A canal surface or *channel surface* is called a surface, one family of lines of principle curvature of which consists of circles. The plane of every circle crosses the surface under a constant angle. This assertion follows the second theorem of Forsythe: "If a plane crosses the surface under a constant angle, then the line of their intersection is the line of principle curvature of the surface."

Normalie of every circular line of the principle curvature is the cone and one sheet of the evolute of a canal surface degenerates into a curve G, which is the geometric locus of the vertexes of these cones. *Normalie* is a ruled surface formed by the normals of a surface drawn in all points of a curve lying on this surface.

Canal surfaces are also *Peterson surfaces* possessing conjugate set of conic or cylindrical lines. Canal surface is the envelope surface of the one-parametric family of the spheres with centers in points of the curve Peternell and Pottmann (1997) have written: "A canal surface Φ is defined as envelope of a one-parameter set of spheres $\Sigma(t)$, centered at *a spine curve* $m(t)$." So, the curve G is called a spine curve $m(t)$.

A vector equation of the family of the spheres is:

$$(\boldsymbol{r}-\boldsymbol{\rho})^2 = R^2,$$

where $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ is a radius vector of the curve G, $R = R(s)$ is a radius of the corresponding sphere; s is the length of an arc of the curve G, that is the geometric locus of the centers of the spheres of the family; $\boldsymbol{r} = \boldsymbol{r}(s)$ is a radius vector of any point lying on a corresponding spherical surface. If an inequality $|R'| \le 1$ is realized when the envelope canal surface is a real surface and this is the necessary and sufficient condition.

Theorem A real canal surface determined by a rational spine curve and a rational radius function possesses real rational parameterizations. M. Peternell and H. Pottmann (1997) proved that this theorem is not a characterization of rational canal surfaces, but a sufficient condition. Additionally, it admits a generalization on envelopes of rational one-parameter sets of cones of revolution.

The particular types of canal surfaces are:

1. *Tubular surfaces* are envelope surfaces of single-parametric family of spheres of constant radius. Any space or plane curve may be taken as a curve G (see also a Subsect. "17.2.1. Tubular Surfaces"). The circular lines of the principle curvatures lie in the normal planes of the curve G.
2. *Dupin cyclides*. Both families of lines of principal curvatures consist of circles, i.e., the cyclides are twice canal surfaces (see also a Subsect. "17.1.2. Dupin Cyclides"). They can be obtained as the image of the torus under a conformal transformation of the ambient space.
3. *Joachimsthal's canal surfaces*. A line of centers G of a Joachimsthal's canal surface is a plane curve. The circular lines of the principle curvatures lie in the planes of the pencil (see also a Subsect. "17.1.1. Canal Surfaces of Joachimsthal").
4. *Surfaces of revolution*. For this case, geometric locus of centers of spheres, that is the curve G, degenerates into the straight line, i.e., into the rotation axis (see also a Chap. "2. Surfaces of Revolution").
5. Canal surfaces formed by a generating circle of constant radius moving in the tangent plane of the line of centers of the generating circles and rotating about the tangents of the line of centers at the angle

$$\theta(s) = -\int \kappa \, \mathrm{d}s$$

where κ is the torsion of the line of centers of the generating circles. If the line of centers of generating circles is a plane curve then the generating circle of constant radius moves along the line of centers without rotation.

An arbitrary family of the circles of the unite sphere is the spherical mapping of the family of lines of principle curvatures of the infinite number of canal surfaces. Let $\boldsymbol{t} = \boldsymbol{t}(s_1)$ is the radius vector of a curve G_1 that is the spherical indicatrix of an arbitrary family of the circles of the unite sphere; s_1 is the length of an arc of the curve G_1, then the radius vector $\boldsymbol{r}$ of the curve G is determined by quadratures

$$\boldsymbol{r} = \int \boldsymbol{t} f(s_1)\mathrm{d}s_1,$$

where $f(s_1)$ is an arbitrary function. Assume $\cos\varphi = \mathrm{d}R/\mathrm{d}s$, where φ is the angle of the intersection of a plane of the circle with an envelope canal surfaces;

$$\mathrm{d}s = f(s_1)\mathrm{d}s_1;$$

then

$$R = \int \cos\varphi f(s_1)\mathrm{d}s_1.$$

If a family of the circles of the spherical mapping is given, then taking arbitrary function $f(s_1)$, it is possible to find the curve G, radiuses of the spheres R and the desired canal surface.

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17.1.1 Canal Surfaces of Joachimsthal

A surface of Joachimsthal is called a surface with a set of the plane lines of principle curvatures lying in the planes of a pencil. *A canal surface* is called a cyclic surface with a set of circles, which are the plane lines of principle curvatures.

If the circles of a canal surface lie in the planes of a pencil, then this surface is *a canal surface of Joachimsthal*. Canal surface of Joachimsthal is related also to a subclass of *cyclic surfaces with circles in the planes of a pencil*. A line of the centers of canal surfaces of Joachimsthal is a plane curve and so, canal surfaces of Joachimsthal are included in a group of *cyclic surfaces with circles in the planes of pencil and with a plane line of centers* (see also Subsect. "17.4.1. Cyclic Surfaces with Circles in the Planes of Pencil and with a Plane Center-to-Center Line"). A cyclic surface with circles in the planes of pencil and with a plane line of centers will be a canal surface, when a condition

$$\left[r^2(u) - R^2(u)\right]' = 0,$$

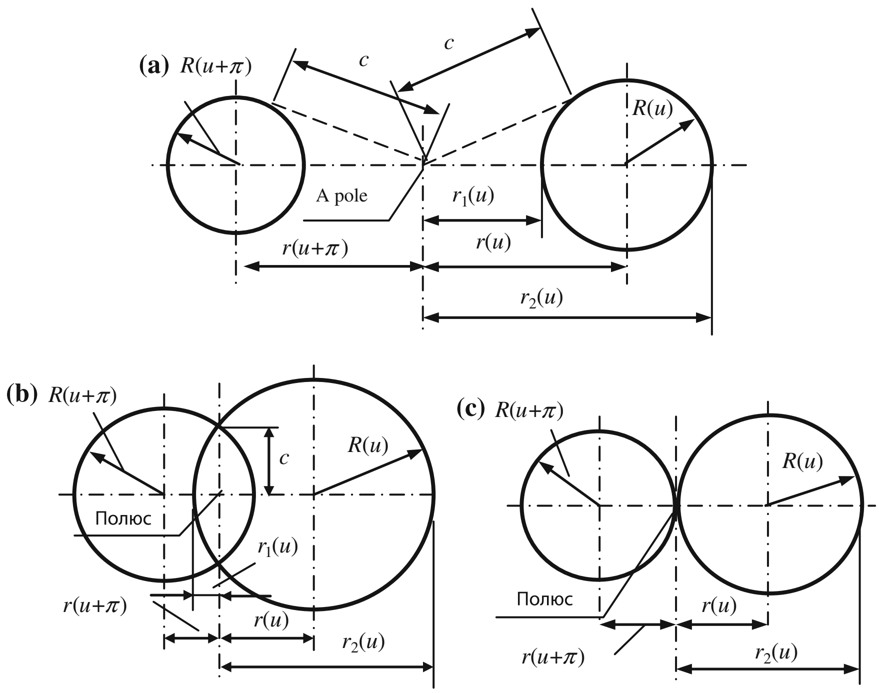

Fig. 1

or

$$r^2(u) - R^2(u) = [r(u) - R(u)] \cdot [r(u) + R(u)]$$
$$= r_1(u) \cdot r_2(u) = \pm c^2,$$

is satisfied. Here, c = const.

Due to this condition, we may obtain three methods of formation of a canal surface of Joachimsthal.

(I) A surface is formed by the rotation of a circle with a changing radius, so that the distance a pole of the surface from the point of tangency with a generatrix circle is constant and equal to c, that is (Fig. 1a)

$$r(u) > R(u), \quad r^2(u) - R^2(u) = +c^2.$$

(II) A surface is formed by the rotation of a circle with a changing radius about a common chord with a length equal to $2c$ (Fig. 1b), then (Fig. 1b)

$$r(u) < R(u), \quad r^2(u) - R^2(u) = -c^2.$$

(III) A surface is formed by the rotation of a circle with a changing radius about a common tangent (Fig. 1c). At this case (Fig. 1c)

$$r(u) = R(u), \quad r_1(u) = 0,$$
$$r_2(u) = 2 \cdot R(u), \ c = 0.$$

Coefficients of the fundamental forms of the surface are determined by the formulas given in the Subsect. "17.4.1. Cyclic Surfaces with Circles in the Planes of Pencil and with a Plane Center-to-Center Line" in view of the connection between the function of the radius of the line of centers $r(u)$ and the function of the radius of the generatrix circles $R(u)$. The curvilinear system of coordinates will not be orthogonal because the coordinate lines v = const are not the lines of principle curvatures.

The equations of canal surfaces of Joachimsthal in the lines of principal curvatures and the formulas for calculation of the coefficients of the fundamental forms of the surface are given in "Canal surface of Joachimsthal in the lines of principle curvatures."

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■ Canal Surfaces of Joachimsthal in the Lines of Principle Curvature

A canal surface of Joachimsthal may be designed with a directrix curve in the form of

(a) a line of the centers of the generatrix circles, $r(u)$ is the distance a pole of the surface from the center of the generatrix circle. In this case, an equation of the surface may be written as:

$$\boldsymbol{r}(u,v) = \left[r(u) + R(u)\frac{D_{12}(u,v)}{D_{22}(u,v)}\right]\boldsymbol{h}(u) + 2R(u)\frac{r_2(u)f(v)}{aD_{22}(u,v)}\boldsymbol{k};$$

(b) a line traced by the internal point of the generatrix circle lying in the plane of the line of the centers, then

$$r_1(u) = r(u) - R(u)$$

(see also Fig. 1 in the Subsect. "17.1.1. Canal Surface of Joachimsthal"). In this case, a vector equation of the desired surface is written as:

$$\boldsymbol{r}(u,v) = \frac{1}{D_{12}(u,v)}[G_2(v)r_1(u)\boldsymbol{h}(u) + aG_{11}(u)f(v)\boldsymbol{k}];$$

(c) a line traced by the external point of the generatrix circle lying in the plane of the line of the centers, then

$$r_2(u) = r(u) + R(u)$$

(see also Fig. 1 in the Subsect. "17.1.1. Canal Surface of Joachimsthal"). In this case, a vector equation of the desired surface can be given as:

$$\boldsymbol{r}(u,v) = \frac{1}{D_{22}(u,v)}[G_2(v)r_2(u)\boldsymbol{h}(u) + aG_{12}(u)f(v)\boldsymbol{k}].$$

In the presented vector equations, the following notations were used:

$$D_{1i} = 1 - \frac{r_i^2(u)}{a^2}f^2(v),\ D_{2i} = 1 + \frac{r_i^2(u)}{a^2}f^2(v),$$
$$G_{1i} = \frac{r_i^2(u)}{a^2} - (\pm\frac{-2}{a^2}),\ G_2(v) = 1 + (\pm\frac{c^2}{a^2})f^2(v);\ i = 1;\ 2,$$

a is a constant that is the characteristic dimensional parameter of the surface; $f(v)$ is any inverse-symmetrical function changing in the interval $-\infty \le v \le \infty$.

For example, if we take $f(v) = v$, then $-\infty \le v \le \infty$. But if we take $f(v) = \tan v$, then $-\pi/2 \le v \le \pi/2$; c is a constant taken from a condition of the forming of a canal surface of Joachimsthal:

$$r^2(u) - R^2(u) = r_1(u)r_2(u) = r_0^2 - R_0^2 = r_{10}r_{20} = \pm c^2,$$

where

$$r_0 = r(0),\quad R_0 = R(0).$$

A sign ($\pm$) before c^2 determines a method of the formation of a canal surface of Joachimsthal (see also Fig. 1 in the Subsect. "17.1.1. Canal Surface of Joachimsthal").

Coefficients of the fundamental forms and the principle curvatures of surface are determined due to the corresponding formulas of differential geometry (see also Chap. "Surfaces").

For our case, they may be lead to the following view:

$$A = \frac{G_2}{D_{2i}}\sqrt{r_i^2(u) + r_i'^2(u)},$$
$$F = 0,\quad B = a\frac{G_{1i}}{D_{2i}}\frac{\mathrm{d}f(v)}{\mathrm{d}v},$$
$$L = -\frac{G_2}{D_{2i}^2\sqrt{r_i^2(u) + r_i'^2(u)}} \times \left\{2\left[r_i^2(u) + r_i'^2(u)\right] - \frac{r_i(u)}{a}\left[r_i(u) + r_i''(u)\right]D_{2i}\right\},$$
$$M = 0,\quad N = -\frac{G_{1i}r_i^2(\mathrm{d}f/\mathrm{d}v)^2}{D_{1i}^2\sqrt{r_i^2(u) + r_i'^2(u)}},$$
$$k_1 = \frac{2\left[r_i^2(u) + r_i'^2(u)\right] - r_i\left(r_i + r_i''\right)D_{2i}}{G_2\left[r_i^2(u) + r_i'^2(u)\right]^{3/2}},$$
$$k_2 = \frac{r_i^2}{G_{1i}\sqrt{r_i^2(u) + r_i'^2(u)}}.$$

Assume $i = 1$, then we shall derive the expressions of the coefficients of the fundamental forms and the principle curvatures of a surface formed in accordance with the item b; if $i = 2$, then the expressions will be obtained for a surface formed in accordance with the item c.

In the given formulas, $r_i(u)$ is an equation of any plane curve given in the polar system of coordinates.

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■ Epitrochoidal Surface

A point *M,* located in the plane of a circle of a radius *a* rolling without sliding along another stationary circle of a radius *b*, traces *an epitrochoid curve*. And the planes of these two circles form the constant angle γ. The distance the point *M* from the center of the mobile circle is equal to μa ($\mu = 1$, or $\mu < 1$, or $\mu > 1$).

Changing parameter γ from 0 to 2π, it is possible to receive a system of epitrochoid curves, which will form *an epitrochoidal surface* Φ. A surface Φ envelopes the system of the balls and touches with them along circles (Fig. 1).

According to the theorem of Joachimsthal, we may say that a family of the circles of an epitrochoidal surface is a family of the lines of principle curvatures, so a surface Φ is a special case of a canal surface of Joachimsthal.

Forms of definition of an epitrochoidal surface

(1) Implicit form of definition:

$$\left(x^2 + y^2 + z^2 - 2\,\mu a x\right)^2 = 4a^2\left(x^2 + y^2\right).$$

The presented implicit form of definition of an epitrochoidal surface has been derived by Steblyanko when $a = b$. A plane xOy crosses an epitrochoidal surface along an epitrochoid curve, which is called a limaçon of Pascal as well.

(2) Parametrical equations:

$$x = x(\alpha, v) = 2R(\alpha)\cos^2 v \cos\alpha,$$
$$y = y(\alpha, v) = 2R(\alpha)\cos^2 v \sin\alpha,$$
$$z = z(\alpha, v) = R(\alpha)\sin 2v,$$

where

$$R(\alpha) = a(1 + \mu\cos\alpha)$$

is a radius of a generatrix circle, α is the angle of the axis Ox with a plane of the generatrix circle; $0 \le \alpha \le 2\pi$; $v = \gamma/2$ is an angle of the radius vector of the surface with the plane of the stationary circle; $-\pi/2 \le v \le \pi/2$. Using this method of the representation of a surface, we must remember that the surface is generated by rotation of a mobile circle with a radius *a* about its tangent at the point of tangency with the

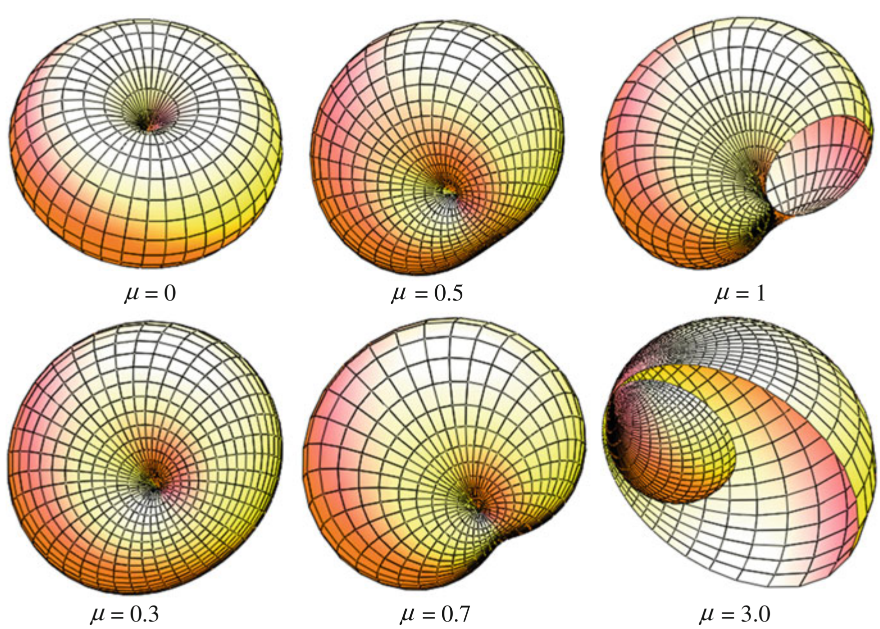

Fig. 1

stationary circle of a radius $b = a$. Generatrix circles of the surface lie in the planes of one pencil. The origin of the coordinates is placed in *the double conic point* of an epitrochoidal surface.

Coefficients of the fundamental forms of the surface:

$$A^2 = 4\left(R^2\cos^2 v + a^2\mu^2\sin^2\alpha\right)\cos^2 v,$$
$$F = 2a\mu R\sin 2v\sin\alpha,\quad B^2 = 4R^2,$$
$$L = -2\frac{a\mu R\cos\alpha + R^2\cos 2v + 2a^2\mu^2\sin^2\alpha}{\sqrt{R^2 + a^2\mu^2\sin^2\alpha}}\cos v,$$
$$M = -\frac{2a\mu R\sin 2v\sin\alpha}{\sqrt{R^2 + a^2\mu^2\sin^2\alpha}},$$
$$N = \frac{-4R^2}{\sqrt{R^2 + a^2\mu^2\sin^2\alpha}}.$$

The surface is given in a nonorthogonal and nonconjugate system of curvilinear coordinates.

(3) Vector form of definition of the surface given in the lines of principle curvatures (Fig. 1):

$$\mathbf{r} = \mathbf{r}(\alpha,\beta) = 2R(\alpha)[\cos\alpha i + \sin\alpha j + R(\alpha)f(\beta)k]/D(\alpha,\beta),$$

where

$$D(\alpha,\beta) = 1 + R^2(\alpha)f^2(\beta)/a^2,$$

$f(\beta)$ is any twice differentiated function, for example, $f(\beta) = \tan\beta$.

Additional Literature

Krivoshapko SN, Gil-oulbe Mathieu. Geometrical and strength analysis of thin pseudospherical, epitrochoidal, catenoidal shells, and shells in the form of Dupin cyclides. Shells in Architecture and Strength Analysis of Thin-Walled Civil-Engineering and Machine-Building Constructions of Complex Forms: Proc. Int. Conf., June 4-8, 2001, Moscow: Izd-vo RUDN, 2001, p. 183-192.

■ Joachimsthal Cosine Canal Surfaces of the 1st Type

Joachimsthal cosine canal surfaces of the 1*st type* are formed by the circles of changing radius lying in the planes of pencil. By the way, points of generatrix circles, the most distant from the pole, lie in the plane perpendicular to the axis of the pencil and trace *a plane circular sinusoid*. The distance *c*, taken from a point of intersection of the axis of the pencil with the plane of a sine curve until the point of the touching of a straight line passing through the pole with a generatrix circle, holds constant (see also Fig. 1a in the Subsect. "17.1.1. C*anal Surfaces of Joachimsthal*"). A circular sinusoid is a plane curve located around a circle (Fig. 1)

$$r_2(u) = a\{1 + \mu[1 + \cos(pu)]\},$$

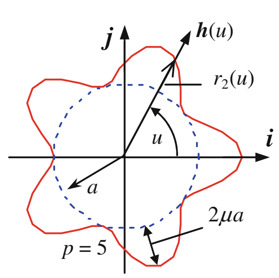

Fig. 1

μ is the ratio of the amplitude of the sine curve to the radius of the circle a; p is an integer of waves of the sine curve.

An equation of a Joachimsthal cosine canal surface of the 1st type in the lines of the principle curvatures (see also Sect. "Canal surfaces of Joachimsthal in the lines of principle curvature") is:

$$\boldsymbol{r}(u,v) = \frac{1}{D_{22}(u,v)}[G_2(v)r_2(u)\boldsymbol{h}(u) + aG_{12}f(v)\boldsymbol{k}],$$

where

$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u;$$
$$D_{22} = 1 + \frac{r_2^2(u)}{a^2}f^2(v),$$
$$G_{12} = \frac{r_2^2(u)}{a^2} - \frac{-^2}{a^2},$$
$$G_2(v) = 1 + \frac{c^2}{a^2}f^2(v);$$
$$i = 1;2;\ f(v) = \tan v.$$

The types of the cosine canal surfaces of Joachimsthal with different significances of the parameters μ, c, and p are shown in Fig. 2.

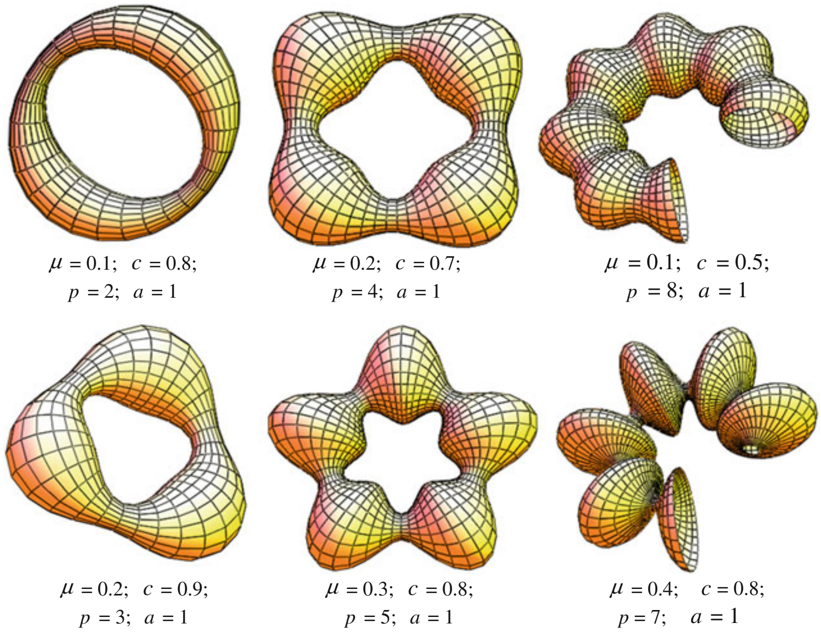

Fig. 2

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{G_2(v)}{D_{22}(u,v)}\sqrt{r_2^2(u) + r_2'^2(u)},$$

$$F = 0,\ B = \frac{aG_{12}(u)}{D_{22}(u,v)\cos^2 v},$$

$$L = -\frac{G_2}{D_{22}^2\sqrt{r_2^2(u) + r_2'^2(u)}}\left\{2\left[r_2^2(u) + r_2'^2(u)\right] - r_2(u)\left[r_2(u) + r_2''(u)\right]D_{22}\right\},$$

$$M = 0,\ N = \frac{-G_{12}r_2^2\cos^{-4}}{D_{22}^2\sqrt{r_2^2(u) + r_2^2(u)}},$$

$$k_1 = \frac{2\left[r_2^2(u) + r_2'^2(u)\right] - r_2(u)\left[r_2(u) + r_2''(u)\right]D_{22}}{G_2\left[r_2^2(u) + r_2'^2(u)\right]^{3/2}},$$

$$k_2 = \frac{r_2^2(u)}{a^2G_{12}\sqrt{r_2^2(u) + r_2'^2(u)}}$$

The derivatives of r_2 with respect to parameter u are shown with the help of the prime.

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Ivanov VN. Investigations of geometry of canal surfaces of Joachimsthal. The Problems of the Theory and Practice in the Engineering Investigations. XXXIII Scientific Conference. RUDN. Moscow: Isd-vo RUDN, 1977; p. 115-118.

Ivanov VN. Design of shells on the base of the canal surfaces of Joachimsthal. Bulletin of Peoples' Friendship University of Russia. Ser.: "Ingenern. Issledovaniya". Special issue: «Geometry and Analyses of Thin-Walled Space Structures», 2002; No. 1, p. 12-21 (4 refs.).

■ Joachimsthal Cosine Canal Surfaces of the 2nd Type

Joachimsthal cosine canal surfaces of the 2nd *type* are formed by the circles of changing radius lying in the planes of pencil. By the way, a set of planes with generatrix circles pass through a fixed straight line and this fixed straight coincides with a common chord of all generatrix circles. Points of generatrix circles, the most distant from the pole, lie in the plane perpendicular to the axis of the pencil and trace *a plane circular sinusoid* (see also Fig. 1b in the Subsect. "17.1.1. C*anal Surfaces of Joachimsthal*") (Fig. 1).

A circular sinusoid is a plane curve located around a circle

$$\boldsymbol{r}_2(\boldsymbol{u}) = \boldsymbol{a}\{1 + \boldsymbol{\mu}[1 + \mathbf{cos}\,(\boldsymbol{pu})]\},$$

μ is the ratio of the amplitude of the sine curve to the radius of the circle a; p is an integer of waves of the sine curve.

A vector equation of cosine canal surfaces of Joachimsthal of the 2nd type in the lines of the principle curvatures (see also Sect. "Canal surfaces of Joachimsthal in the lines of principle curvature") is written in the form:

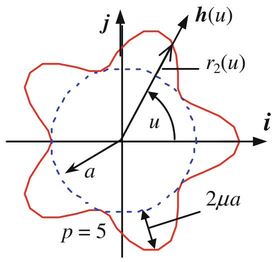

Fig. 1

$$\boldsymbol{r}(u,v) = \frac{1}{D_{22}(u,v)}[G_2(v)r_2(u)\boldsymbol{h}(u) + aG_{12}(u)f(v)\boldsymbol{k}];$$

where

$$\mathbf{h}(u) = i\,\mathbf{cos}\,u + j\,\mathbf{sin}\,u;$$

$$D_{22} = 1 + \frac{r_2^2(u)}{a^2}\tan^2(v),$$

$$G_{12} = \frac{r_2^2(u)}{a^2} + \left(\frac{c^2}{a^2}\right),$$

$$G_2(v) = 1 - \left(\frac{c^2}{a^2}\right)\tan^2(v); \quad i = 1; 2.$$

The types of the cosine canal surfaces of Joachimsthal with different significances of the parameters μ, c, and p are shown in Fig. 2.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{G_2}{D_{22}(u,v)}\sqrt{r_2^2(u) + r_2'^2(u)},$$

$$F = 0, \; B = \frac{aG_{12}(u)}{D_{22}(u,v)\cos^2 v},$$

$$L = -\frac{G_2}{D_{22}^2\sqrt{r_2^2(u) + r_2'^2(u)}}\left\{2\left[r_2^2(u) + r_2'^2(u)\right] - r_2(u)\left[r_2(u) + r_2''(u)\right]D_{22}\right\},$$

$$M = 0, \; N = \frac{-r_2^2(u)G_{12}\cos^{-4}}{D_{22}^2\sqrt{r_2^2(u) + r_2'^2(u)}},$$

$$k_1 = \frac{2\left[r_2^2(u) + r_2'^2(u)\right] - r_2(u)\left[r_2(u) + r_2''(u)\right]D_{22}}{G_2\left[r_2^2(u) + r_2'^2(u)\right]^{3/2}},$$

$$k_2 = \frac{r_2^2(u)}{a^2 G_{12}\sqrt{r_2^2(u) + r_2'^2(u)}},$$

where $r_2'(u) = -ap\mu\sin(up)$; $r_2''(u) = -ap^2\mu\cos(up)$.

References

Ivanov VN. Investigations of geometry of canal surfaces of Joachimsthal. The Problems of the Theory and Practice in the Engineering Investigations. XXXIII Scientific Conference. RUDN. Moscow: Isd-vo RUDN, 1977; p. 115-118.

Ivanov VN. Design of shells on the base of the canal surfaces of Joachimsthal. Bulletin of Peoples' Friendship University of Russia. Ser.: "Ingenern. Issledovaniya". Special issue: «Geometry and Analyses of Thin-Walled Space Structures», 2002; No. 1, p. 12-21 (4 refs.).

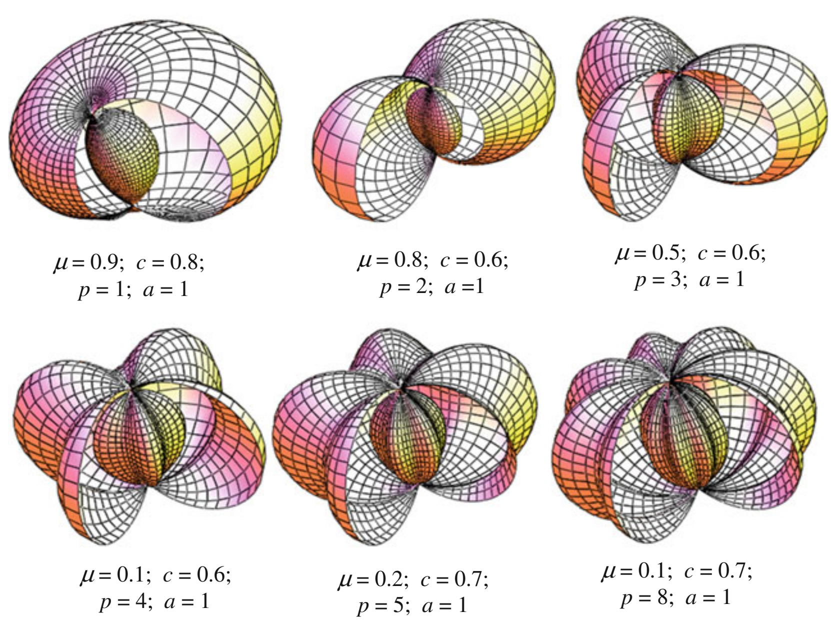

Fig. 2

■ Joachimsthal Cosine Canal Surfaces of the 3rd Type

Joachimsthal cosine canal surfaces of the 3*rd type* are formed by the circles of changing radius lying in the planes of pencil. By the way, a set of planes with generatrix circles pass through a fixed straight line and this fixed straight coincides with a common tangent line of all generatrix circles.

Points of generatrix circles, the most distant from the pole, lie in the plane perpendicular to the axis of the pencil and trace *a plane circular sinusoid* (see also Fig. 1c in the Subsect. "17.1.1. Canal Surfaces of Joachimsthal").

A circular sinusoid is a plane curve located around a circle (Fig. 1)

$$r_2(u) = a\{1 + \mu[1 + \cos(pu)]\}$$

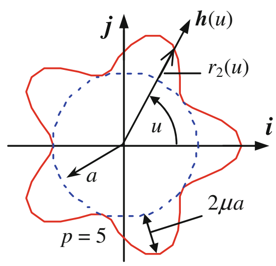

Fig. 1

μ is the ratio of the amplitude of the sine curve to the radius of the circle a; p is an integer of waves of the sine curve.

A vector equation of a cosine canal surface of Joachimsthal of the 3rd type in the lines of the principle curvatures (see also Sect. "Canal surfaces of Joachimsthal in the lines of principle curvature") is written as:

$$\boldsymbol{r}(u, v) = \frac{r_2(u)}{D_{22}(u, v)}\left[\boldsymbol{h}(u) + \frac{r_2(u)}{a}\operatorname{tg}(v)\boldsymbol{k}\right],$$

where

$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u;$$

$$D_{22} = 1 + \frac{r_2^2(u)}{a^2}\tan^2(v).$$

$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u;$$

$$D_{22} = 1 + \frac{r_2^2(u)}{a^2}\tan^2(v).$$

The types of the cosine canal surfaces of Joachimsthal with different significances of the parameters μ, p are shown at Fig. 2.

Coefficients of the fundamental forms of the surface and its principal curvatures:

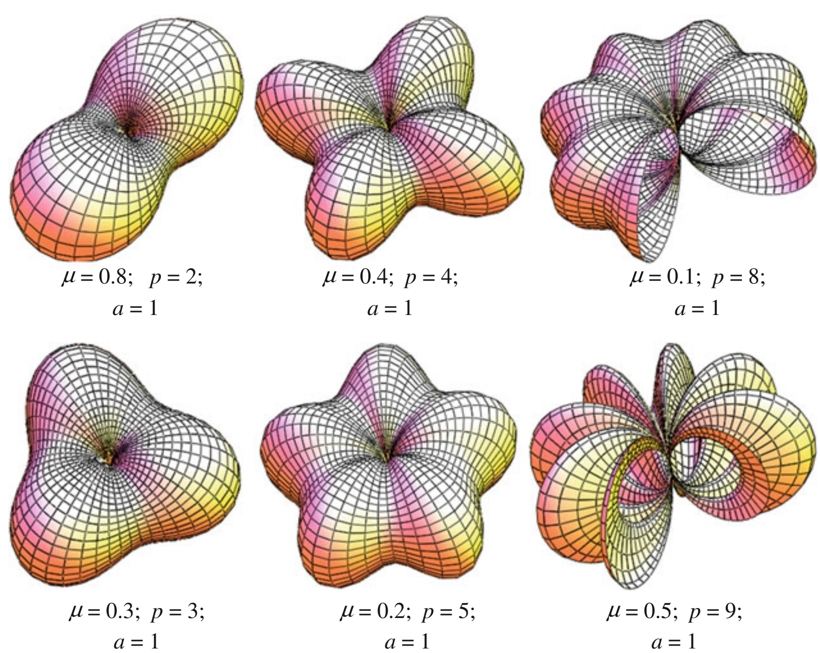

Fig. 2

$$A = \frac{1}{D_{22}(u,v)}\sqrt{r_2^2(u) + r_2'^2(u)},$$
$$F = 0,\ B = \frac{r_2^2(u)}{aD_{22}(u,v)\cos^2 v},$$
$$L = -\frac{1}{D_{22}^2\sqrt{r_2^2(u) + r_2'^2(u)}}\left\{2\left[r_2^2(u) + r_2'^2(u)\right]\right.$$
$$\left. - r_2(u)\left[r_2(u) + r_2''(u)\right]D_{22}\right\},$$
$$M = 0,\ N = \frac{-r_2^4(u)\cos^{-4} v}{a^2 D_{22}^2\sqrt{r_2^2(u) + r_2'^2(u)}},$$
$$k_1 = \frac{2\left[r_2^2(u) + r_2'^2(u)\right] - r_2(u)\left[r_2(u) + r_2''(u)\right]D_{22}}{\left[r_2^2(u) + r_2'^2(u)\right]^{3/2}},$$
$$k_2 = \frac{1}{\sqrt{r_2^2(u) + r_2'^2(u)}}$$

where

$$r_2'(u) = -ap\mu\sin(up);$$
$$r_2''(u) = -ap^2\mu\cos(up).$$

Additional Literature

Ivanov VN. Design of shells on the base of the canal surfaces of Joachimsthal. Bulletin of Peoples' Friendship University of Russia. Ser.: "Ingenern. Issledovaniya". Special issue: «Geometry and Analyses of Thin-Walled Space Structures», 2002; No. 1, p. 12-21 (4 refs.).

Ivanov VN., Gil-oulbe Mathieu. On question of geometry and design of shells in the form of canal surfaces of Joachimsthal. Structural Mechanics of Engineering Constructions and Buildings. 1994; No. 3, p. 68-75.

17.1.2 Dupin Cyclides

A surface with two families of lines of principle curvatures consisting of circles, when a radius of the circles of one family depends on one parameter only but a radius of the circles of other family depends on other variable parameter, is called *Dupin cyclide* or *cyclide of Dupin*. Dupin cyclides are often simply known as "*cyclides.*" There are several equivalent definitions of Dupin cyclides. Some of them will be presented below.

A cyclide is a special case of *a canal surface*. This property means that Dupin cyclides are natural objects in *Lie sphere geometry*. Both *evolute surfaces* of a cyclide degenerate into the plane curves G_1 and G_2, and so it may be designed as an envelope of a set of the spheres with centers in the points of the curve G_1, or as an envelope of a set of the spheres with centers at the points of the curve G_2. So, they appear as envelopes of one-parameter families of spheres in a twofold way.

Let $\boldsymbol{r}_1(u)$ and $\boldsymbol{r}_2(v)$ are radius vectors of the curves G_1 and G_2 accordingly, then equations of two families of the spheres may be written as

$$(\boldsymbol{r} - \boldsymbol{r}_1)^2 = A^2(u),$$
$$(\boldsymbol{r} - \boldsymbol{r}_2)^2 = B^2(u).$$

One sphere of the first family and one sphere of other family touch a cyclide at every point of the cyclide, therefore any two spheres of different families touch each other. A condition of touching of spheres of different families

$$(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2 = (A + B)^2$$

shows that the curves G_1 and G_2 are the focal curves of the second other.

A cyclide is an envelope of a family of the spheres that are tangent to three fixed spheres. Let us take three arbitrary spheres of the first family and then any sphere touching them will belong to other family.

Dupin cyclides are of three types. They were discovered by Charles Dupin in his 1803 dissertation under Gaspard Monge.

The evolutes of a *cyclide of the first type* are the focal ellipse and hyperbola:

$$y = b,\ z = 0,\ x = c\cos v \quad \text{and}$$
$$z = b\,\mathrm{sh}u,\ x = \pm a\,\mathrm{ch}u$$

correspondingly, and $b^2 = c^2 - a^2$. The equations show that an ellipse and a hyperbola lay at the mutually perpendicular planes. The focal ellipse and hyperbola may be given by their canonic equations:

$$\frac{x^2}{c^2} + \frac{y^2}{b^2} = 1,\ z = 0 \quad \text{and}$$
$$\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1,\ y = 0.$$

The equations of two families of the spheres can be written as:

$$(y - b\sin v)^2 + z^2 + (x - c\cos v)^2 = (a\cos v + d)^2.$$

and

$$y^2 + (z - b\,\mathrm{sh}u)^2 + (x \mp a\,\mathrm{ch}u)^2 = (\mp c\,\mathrm{ch}u - d)^2.$$

The evolutes of *a cyclide of the second type* are the focal parabolas:

$$x = u,\ y = 0,\ z = \frac{2u^2 - p^2}{4p} \quad \text{and}$$
$$x = 0,\ y = v,\ z = \frac{-2v^2 + p^2}{4p}.$$

The focal parabolas of a cyclide of the second type may be given also as:

$$x = 0,\ y^2 = 4\,l(z + l) \quad \text{and}$$
$$x^2 = -4lz,\ y = 0.$$

The evolutes of *a Dupin cyclide of the third type* are the focal circle and straight line:

$$x = a\cos u,\ y = a\sin u,\ z = 0 \quad \text{and}$$
$$x = 0,\ y = 0,\ z = v,$$

correspondingly.

In this case, parametrical equations of a cyclide of the third type may be presented in the following form:

$$x = \left[\left(\sqrt{a^2 + v^2} - b\right) a\cos u/\sqrt{a^2 + v^2}\right],$$
$$y = \left[\left(\sqrt{a^2 + v^2} - b\right) a\sin u/\sqrt{a^2 + v^2}\right],$$
$$z = bv/\sqrt{a^2 + v^2},$$

b = const. Eliminating a parameter v from these equations, it is possible to derive an implicit equation of a cyclide of the third type:

$$\left(x^2 + y^2 + z^2 + a^2 - b^2\right)^2 = 4a^2\left(x^2 + y^2\right),$$

showing that a Dupin cyclide of the third type is a circular torus formed by the rotation of the circle

$$(x - a)^2 + z^2 = b^2; \quad y = 0$$

about the axis Oz.

Dupin cyclides are *algebraic surfaces of order three and four* whose lines of curvature are circles. The set of circles which intersect Dupin cyclides twice orthogonally are called *ortho-circles*.

Dupin cyclides are known to be the images of cylinders, or cones of revolution, or tori under inversions. Depending on the location of the center of the inversion and of the choice of the input surface they obtain different types of Dupin cyclides.

Dupin cyclides can be obtained by certain projections from *supercyclides*.

Additional Literature

Degen W.L.F. On the origin of supercyclides. In R. Cripps (ed.). The Mathematics of Surfaces. VIII, Information Geometers, Winchester 1998, p. 297-312.

Schrott Michael, *Odehnal Boris*. Ortho-circles of Dupin cyclides. Journal for Geometry and Graphics. 2006; Vol. 10, No. 1, p. 73-98.

■ Dupin Cyclides of the First Type (of the Fourth Order)

The cuspidal edges *of the normalies* of two sets of the lines of principle curvatures of the cyclide generate two *evolute surfaces*, which for the Dupin cyclides of the first type degenerate into focal ellipse and hyperbola lying in two mutually perpendicular planes (see also a Subect. “17.1.2. Dupin Cyclides”).

The forms of definition of a Dupin cyclide of the first type

(1) Implicit equations:

$$(x^2 + y^2 + z^2 - \mu^2 + b^2)^2 = 4(cx - a\mu)^2 + 4b^2y^2,$$

where $a^2 = c^2 - b^2$.

Dupin cyclides of the first type are algebraic surfaces of the fourth order.

(2) Parametrical equations:

$$x = x(u, v) = \frac{c\cos v(-d \mp c\,\mathrm{ch}u) \pm a\,\mathrm{ch}u(a\cos v - d)}{-a\cos v \mp c\,\mathrm{ch}u},$$
$$y = y(u, v) = \frac{(\pm c\,\mathrm{ch}u + d)b\sin v}{\pm c\,\mathrm{ch}u + a\cos v},$$
$$z = z(u, v) = \frac{(a\cos v - d)b\,\mathrm{sh}u}{\mp a\cos v - c\,\mathrm{ch}u},$$

where $b^2 = c^2 - a^2$; $d = \text{const}$.

The surface of a cyclide is given in the lines of principle curvatures u, v. If a, b, c are constant values and $d = 0$, then a cyclide (Fig. 1) has two conic points with coordinates (0, b, 0) and (0, $-b$, 0). Assume $0 < d < a$, then cyclide will have the form shown in Fig. 2. The cyclide has two conic points with coordinates

$$\left(ad/c;\ \mp b\sqrt{c^4 - a^2d^2}/c^2;\ 0\right).$$

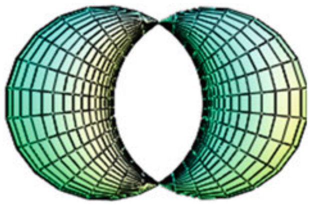

Fig. 1

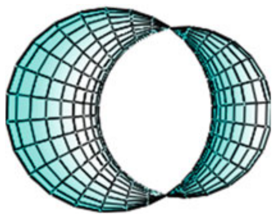

Fig. 2

If $d = a$, then the cycloid has one conic point with coordinates (c, 0; 0), Fig. 3. If $a < d < c$, then conic points are absent (Fig. 4). One degenerated point with coordinates (a, 0; 0) will appear, if one assumes $d = c$ (Fig. 5).

The cyclide has two conic points with coordinates

$$\left(ad/c;\ 0;\ \pm b\sqrt{d^2 - c^2}/c\right),$$

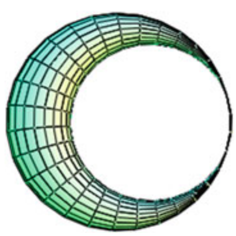

Fig. 3

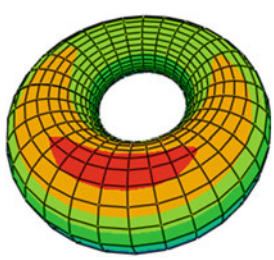

Fig. 4

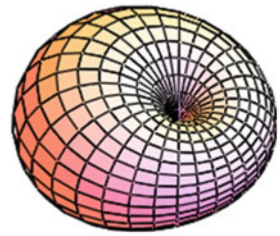

Fig. 5

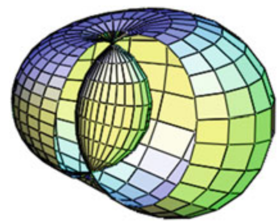

Fig. 6

when $d > c$ (Fig. 6). If $a = 0$; $b = c$; $d = c$, then we shall obtain a circular torus with one degenerated point, and having taken $d > c$, we can obtain a torus with two conic points.

(3) Parametrical equations:

$$x(\theta,\psi) = \frac{c\mu}{a} - \frac{b^2(\mu + a\cos\theta)}{a(c - a\cos\theta\cos\psi)},$$
$$y(\theta,\psi) = -\frac{b(c + \mu\cos\psi)}{c - a\cos\theta\cos\psi}\sin\theta,$$
$$z(\theta,\psi) = \frac{b(\mu + a\cos\theta)}{c - a\cos\theta\cos\psi}\sin\psi.$$

Coefficients of the fundamental forms and principle radii of curvatures of the surface:

$$A^2 = \frac{b^2(c + \mu\cos\psi)^2}{(c - a\cos\theta\cos\psi)^2},\ \text{and}$$
$$B^2 = \frac{b^2(\mu + a\cos\theta)^2}{(c - a\cos\theta\cos\psi)^2},\ F = 0,$$
$$R_2 = \mu + a\cos\theta,\ R_1 = \mu + c/\cos\psi.$$

Using this form of definition of a cyclide, we must remember that focal ellipse and hyperbola are given in the form:

$$(0, -b\sin\theta, -c\cos\theta), (-b\tan\psi,\ 0, -a\sec\psi),\ \text{correspondingly.}$$

The coordinate lines θ, ψ are the lines of the principle curvatures. Having taken $a = 0$, we shall obtain a torus but if $c = a = 0$, we shall have a sphere.

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Ivanov VN. On Dupin cyclides as Joachimsthal channel surfaces. The 10th International Conference on Geometry and Graphics, Ukraine, Kiev, 2002, July 28- August 2. Kiev. 2002; Vol. 2, p. 350-354 (14 refs.)

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■ Dupin Cyclide of the Second Type (of the Third Order)

The cuspidal edges *of the normalies* of two sets of the lines of principle curvatures of the cyclide generate two *evolute surfaces*, which for the Dupin cyclides of the second type degenerate into two focal parabolas lying in two mutually perpendicular planes (see also a Subect. “17.1.2. Dupin Cyclides”).

The lines of principle curvatures of a cyclide of the second type lie on two pencils of planes, i.e., the cyclides are Joachimsthal canal surfaces. Two circles lie in every plane. It is possible to show that cyclides of the second type are the limiting case of cyclides of the first type.

The forms of definition of Dupin cyclides of the second type

(1) Parametrical equations (Figs. 1 and 2):

$$x = x(u, v) = \frac{uD_2}{D_1 + D_2},$$
$$y = y(u, v) = \frac{vD_1}{D_1 + D_2},$$
$$z = z(u, v) = \frac{D_2(2u^2 - p^2) - D_1(2v^2 - p^2)}{4p(D_1 + D_2)},$$

where

$$D_1 = \frac{2u^2 + p^2 + q}{4p},$$
$$D_2 = \frac{2v^2 + p^2 - q}{4p}; \; q = \text{const};$$

p is a parameter of the focal parabolas (see also a Subsect. “17.1.2. Dupin Cyclides”). The surface of a cyclide is given in lines of principle curvatures u, v.

(2) Parametrical equations (Figs. 1 and 2): μ = const,

$$x = x(t, \theta) = \frac{2\,l\theta t^2 - 2\mu\theta}{1 + t^2 + \theta^2},$$
$$y = y(t, \theta) = \frac{2\,l(1 + \theta^2)t + 2\mu t}{1 + t^2 + \theta^2}$$
$$z = z(t, \theta) = \frac{\mu(t^2 + \theta^2 - 1) - l(1 - t^2 + \theta^2)}{1 + t^2 + \theta^2}.$$

In this case, the focal parabolas have coordinates:

$$\left[0, 2lt, l\left(t^2 - 1\right)\right] \text{ and } \left[2l\theta, 0, -l\theta^2\right].$$

Coefficients of the fundamental forms and principle radii of curvatures of the surface:

$$A = \frac{2(\mu + l + l\theta^2)}{1 + t^2 + \theta^2},$$
$$F = 0, \; B = \frac{2(\mu - lt^2)}{1 + t^2 + \theta^2},$$
$$R_1 = \mu + l(1 + \theta^2),$$
$$R_2 = \mu - l^2.$$

The surface of a cyclide is related to lines of principle curvatures t, θ.

A circle $y^2 + z^2 = (\mu + l)^2$ and a straight line $z = \mu - l$ lay at the cross section plane $x = 0$; a circle $x^2 + (z + l)^2 = \mu^2$ and a straight line $z = \mu + l$ lay at the plane $y = 0$.

If $\mu > 0$, the straight line crosses the circle and two conic points (Fig. 1) are formed. If $\mu < 0$, the straight line does not intersect the circle (Fig. 2). Both surfaces extend to infinity.

(3) Implicit equation:

$$z(x^2 + y^2 + z^2) + (y^2 + z^2)(l - \mu)$$
$$- x^2(l + \mu) - (z + l - \mu)(l + \mu)^2 = 0.$$

Dupin cyclides of the second type are *algebraic surfaces of the third order*.

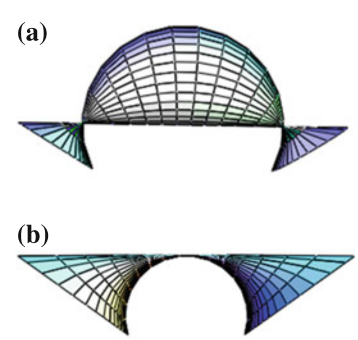

Fig. 1

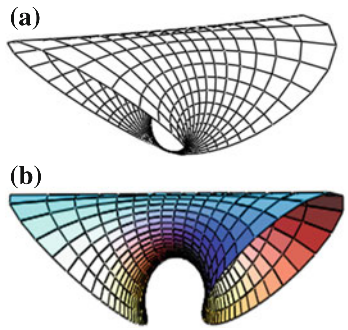

Fig. 2

(4) Parametrical equations (Figs. 1 and 2):

$$x = x(\alpha,\beta) = a\mu\frac{G_2(\beta)}{D(\alpha,\beta)},$$

$$y = y(\alpha,\beta) = a\mu\frac{G_2(\beta)}{D(\alpha,\beta)}\tan\alpha,$$

$$z = z(\alpha,\beta) = a\frac{G_1(\beta)}{D(\alpha,\beta)}f(\beta),$$

where

$$\begin{aligned} D(\alpha,\beta) &= 1 + r_2^2(\alpha)f^2(\beta);\\ G_1(\alpha) &= r_2(\alpha) - (\pm c^2),\\ r_2(\alpha) &= \frac{\mu}{\cos\alpha},\\ G_2(\beta) &= 1 \pm c^2 f^2(\beta), \end{aligned}$$

f(*β*) is any twice differentiated symmetric function. For example, one may take $f(\beta) = \tan\beta$, then $-\pi \le \beta \le \pi$.

In Fig. 1a, the cyclide having $c > \mu$ is shown. In Fig. 1b, the cyclide has $c = \mu$. In Fig. 2a, the cyclide having $c = 0$ is presented and in Fig. 2b, it has $c < \mu$.

17.2 Normal Cyclic Surfaces

A normal cyclic surface is formed by motion of a circle of changing $R(u)$ or constant R radius along any directrix curve $\boldsymbol{\rho}(u)$ that is called a *line of centers* but the generatrix circle must be at the normal plane of the line of centers of the circles all time. *Normal cyclic surfaces* are a subclass of cyclic surfaces (see also Subsect. "Classification of cyclic surfaces" in this chapter). A normal $\boldsymbol{n}(u)$ to the plane of a generatrix circle coincides with the tangent line $\boldsymbol{t}(u)$ to the directrix curve $\boldsymbol{\rho}(u)$, so $\boldsymbol{n}(u) = \boldsymbol{t}(u)$.

A vector equation of a normal cyclic surface may be written as:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + R(u)\boldsymbol{e}(u,v),$$

where $\boldsymbol{r}(u, v)$ is a radius vector of a cyclic surface; $\boldsymbol{\rho}(u)$ is the radius vector of the line of centers of generatrix circles; $R(u)$ is a low of changing of the radius of a generatrix circles;

$$\boldsymbol{e}(u,v) = \boldsymbol{e}_0(u)\cos v + \boldsymbol{g}_0(u)\sin v$$

is the vector function of a circle of unit radius lying in the plane of the generatrix circle with the normal $\boldsymbol{n}(u)$; $\boldsymbol{e}_0(u)$ and $\boldsymbol{g}_0(u)$ are the unit vectors of the orthogonal system of Cartesian coordinates lying in the normal plane of the line of centers of a generatrix circle. If a vector $\boldsymbol{e}_0(u)$ originates an angle θ with the normal of the line of centers of generatrix circles $\boldsymbol{\nu}(u)$, then the coordinate grid of the normal cyclic surface will be orthogonal and

$$\theta = -\int \kappa s\,\mathrm{d}u.$$

For a plane line of centers, we must take: $\theta = 0$, i.e., $\boldsymbol{e}_0(u) = \boldsymbol{\nu}(u)$.

For the determination of the coefficients of the fundamental forms of a normal cyclic surface, we may use the following formulas:

$$\begin{aligned} &E = A^2 = R'^2(u) + s^2[1 - kR(u)\cos\omega]^2,\\ &F = 0,\ G = B^2 = R^2(u), \end{aligned}$$

where

$$\begin{aligned} s &= |\boldsymbol{\rho}'| = |\partial\boldsymbol{\rho}/\partial u|;\ \omega = v + \theta;\\ L &= \Big\langle R'(u)\Big\{s' - \big[2skR'(u) + (sk)'R(u)\big]\cos\omega\\ &\qquad - s^2 k\kappa R(u)\sin\omega\Big\} - s[1 - R(u)k\cos\omega]\\ &\qquad \Big\{[1 - R(u)k\cos\omega]ks^2\cos\omega + R''(u)\Big\}\Big\rangle\frac{1}{A},\\ M &= \frac{R(u)R'(u)sk\sin\omega}{A},\ N = \frac{R(u)[s - R(u)sk\cos\omega]}{A}, \end{aligned}$$

where $s' = \partial s/\partial u$; k, κ are curvature and torsion of the line of centers.

In general case, generatrix circles of normal cyclic surfaces are not the lines of principle curvatures ($M \neq 0$), so normal cyclic surfaces are not canal surfaces (see also a Sect. "17.1. Canal Surfaces").

Cyclic surfaces may be by canal surfaces if

(1) $M = 0$ when $R'(u) = 0$, i.e., when R = const.

In this case, we obtain *a tubular surface* (see also a Subsect. "17.2.1. Tubular Surfaces").

For the determination of the coefficients of the fundamental forms of tubular surfaces, we may use the formulas presented above but considering $R'(u) = 0$:

$$A = s[1 - kR)\cos\omega],$$
$$F = 0, \ B = R,$$

where

$$s = |\boldsymbol{\rho}'| = |\partial\boldsymbol{\rho}/\partial u|;$$
$$\omega = v + \theta;$$
$$L = \langle -s[1 - R(u)k\cos\omega] \times [1 - R(u)k\cos\omega]ks^2\cos\omega)\rangle\frac{1}{A}$$
$$M = 0, \ N = \frac{R(u)[s - R(u)sk\cos\omega]}{A},$$

(2) $M = 0$ when $k = 0$, i.e., a line of the centers must be a straight line.

In this case, a normal cyclic surface degenerates into *a surface of revolution* with a rotation axis of z. A function $R(u) = R(z)$ determines a meridian of a surface of revolution (see also Chap. "2. Surface of Revolution").

The formulas for the determination of the coefficients of the fundamental forms of normal cyclic surfaces simplify for surfaces of revolution:

$$A^2 = 1 + R'^2(u),$$
$$F = 0, \ B = R(u),$$
$$L = -\frac{R''(u)}{\sqrt{1 + R'^2(u)}}, \ M = 0,$$
$$N = \frac{R(u)}{\sqrt{1 + R'^2(u)}},$$
$$k_1 = \frac{R''(u)}{[1 + R'^2(u)]^{3/2}},$$
$$k_2 = \frac{1}{R(u)\sqrt{1 + R'^2(u)}}.$$

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Ivanov VN, Gritzishen IV. Geometry of shells in the form of normal cyclic surfaces. The Problems of Nonlinear Analysis of the Large-Span Space Constructions, Science Session of MOO "Space Structures", April 20, 2010; p. 29-30.

17.2.1 Tubular Surfaces

Tubular surfaces are related to *normal cyclic surfaces* considered in the Sect. "17.2. Normal Cyclic Surfaces", but it is necessary to assume

$$R(u) = R = \text{const}, \quad \text{therefore}$$
$$\mathrm{d}R(u)/\mathrm{d}u = 0.$$

A vector equation of tubular surfaces may be written as:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + R\mathbf{e}(u, v),$$

where $\boldsymbol{r}(u, v)$ is a radius vector of a cyclic surface; $\boldsymbol{\rho}(u)$ is a radius vector of the line of centers of generatrix circles; $\boldsymbol{e}(u, v)$ is the vector function of the circle of the unit radius lying in the plane of a generatrix circle with the normal

$$\boldsymbol{t}(u) = \boldsymbol{\rho}'/s; \text{ where } s = |\boldsymbol{\rho}'| = |\partial\boldsymbol{\rho}/\partial u|.$$

It is possible to give another definition of tubular surfaces: Let $\boldsymbol{r}$: I $\rightarrow$ R^3 be a smooth, regular space curve. The tubular surface associated to $\boldsymbol{r}$, of radius a, is the envelope of the family of spheres of radius a, with the center on the curve. Clearly, if $\boldsymbol{r}$ is a straight line, then the tubular surface of radius a associated to it is just *the circular cylinder* of radius a, having $\boldsymbol{r}$ as symmetry axis. If, on the other hand, $\boldsymbol{r}$ is a circle, then the corresponding tubular surface is *a torus*.

The formulas for the determination of the coefficients of the fundamental forms of cyclic surfaces and their principal curvatures (see also Sect. "17.2. Normal Cyclic Surfaces") simplify for tubular surfaces and have the following form:

$$E = A^2 = s^2[1 - kR\cos(v+\theta)]^2,$$
$$F = 0,\ G = B^2 = R^2,$$

where

$$\omega = v + \theta;$$
$$\theta = \theta(u) = -\int \kappa s\,\mathrm{d}u + \theta_0;$$
$$L = -s^2k[1 - Rk\cos(v+\theta)]\cos(v+\theta),$$
$$M = 0,\ N = R,$$
$$k_1 = \frac{k\cos(\theta+v)}{1 - kR\cos(v+\theta)},\ k_2 = \frac{1}{R},$$

where k, κ are curvature and torsion of a line of centers. Hence, generatrix circles in the tubular surfaces coincide with one family of the lines of principle curvatures and the coordinate net u, v is the net of the lines of principle curvatures.

Tubular surfaces are a special case of *canal surfaces*. As an example of a tubular surface, we can give well-known *Escher surface* (Fig. 1).

A vector of the normal of the tubular surface lies in the plane of generatrix circles. The generatrix circles are *geodesic lines*. The surfaces with a family of the plane lines of principle curvatures, which are geodesic lines at the same time, are related simultaneously to a class of *carved surfaces* (see also Chap. "4. Carved Surfaces"). So, tubular surfaces may be related both to cyclic surfaces and to carved surfaces.

If a tubular surface has a plane line of centers, then $\theta = 0$ because torsion of plane curve is equal to zero ($\kappa = 0$) and coefficients of the fundamental forms of surface and its principal curvatures for a tubular surface may be obtained by the formulas:

Fig. 1

$$E = A^2 = s^2[1 - kR\cos v]^2,$$
$$F = 0,\ G = B^2 = R^2,$$
$$L = -s^2k[1 - Rk\cos v]\cos v,$$
$$M = 0,\ N = R,$$
$$k_1 = \frac{k\cos v}{1 - kR\cos v)},\ k_2 = \frac{1}{R},$$

where k is the curvature of a plane line of centers.

At the present time, *circular torus* (see also Chap. "2. Surfaces of Revolution"), which is tubular surface of revolution, and tubular helical surface (see also "Tubular Helical Surfaces" in Subsect. "7.1.2. Circular Helical Surfaces"), which may be related to a class of helical surfaces at the same time, are the most known tubular surfaces.

Tubular surfaces are among the surfaces which are easier to describe both analytically and "operationally." They are still under active investigation, both for finding best parameterizations or for application in different fields.

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Whatham JF. Thin shell analysis of circular pipe bends. Transl. Institution of Engineers, Australia. 1981; Vol. CE 23, No. 4, p. 234-245.
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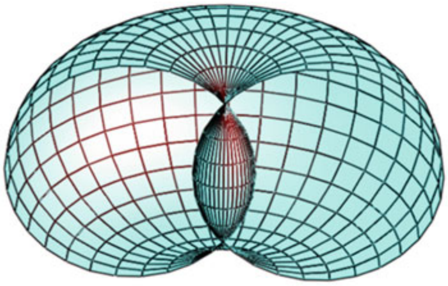
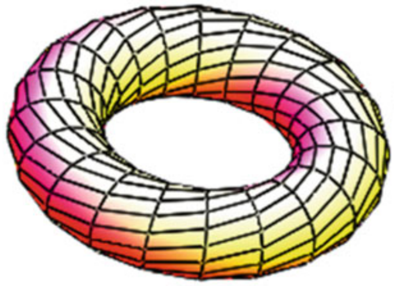
http://imp-world-r.narod.ru/articles/escher_math/escher_math.html (2013).

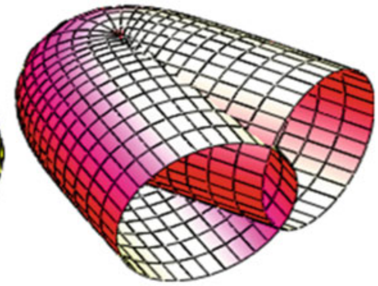
■ **Tubular Surfaces Presented in the Encyclopedia**



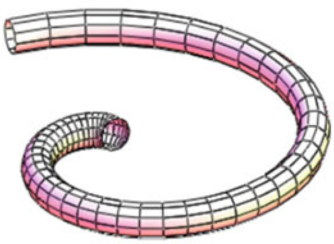
Closed circular torus



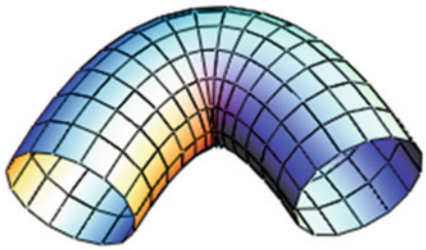
The preliminarily twisted circular torus



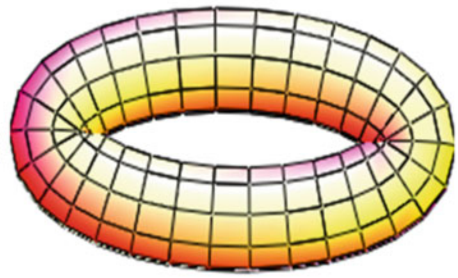
The tubular surface with the plane hyperbolic line of centers



The tubular surface with the plane line of centers in the form of the evolvent of the circle

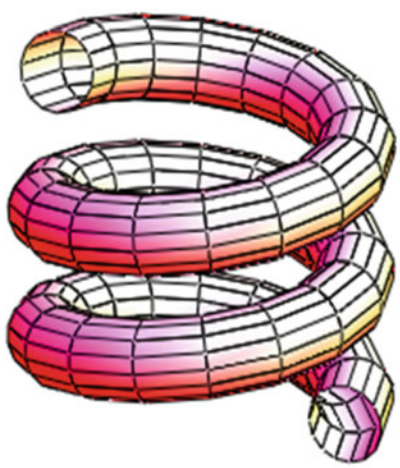


The tubular surface with the plane parabolic line of centers

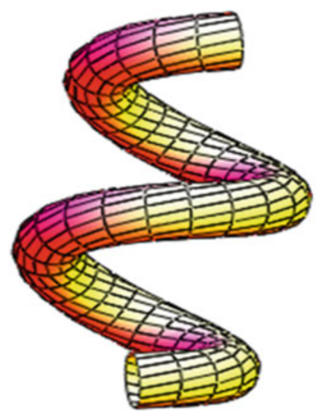


The tubular surface with the plane elliptical line of centers

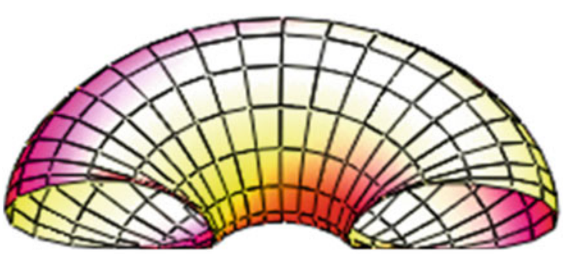
The tubular spiral surface



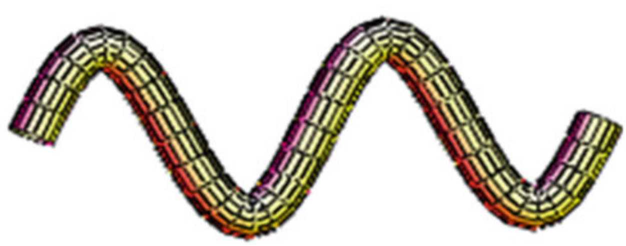
The tubular helical surface



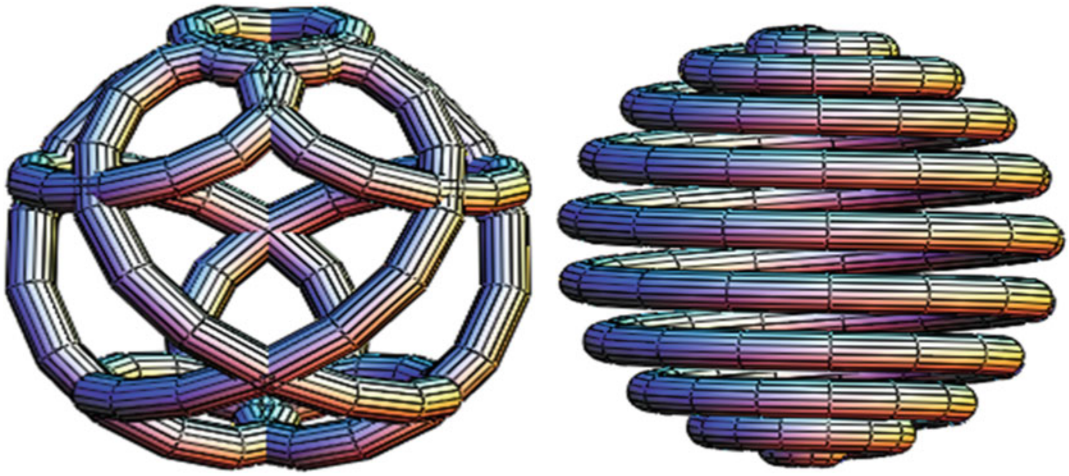
The helix-shaped preliminarily twisted surface of the circular cross section



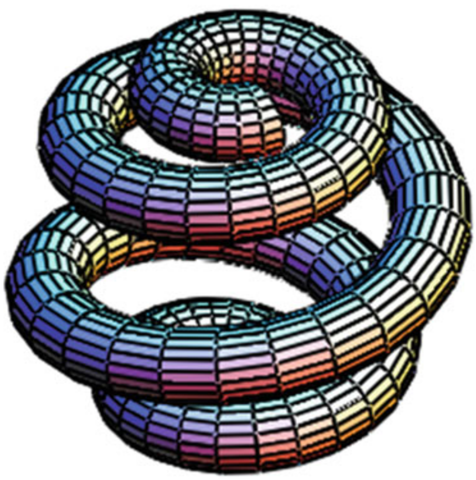
The tubular surface with theplane line of centers in the form of the cycloid



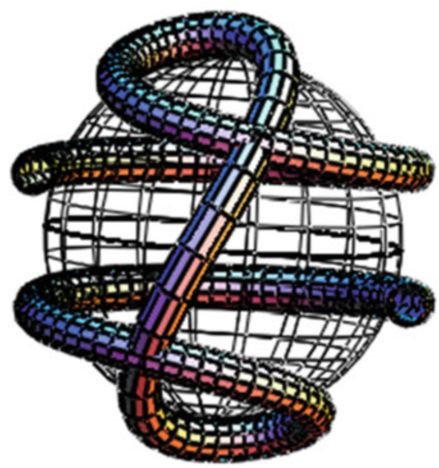
The tubular surface with the plane sinusoidal line of centers



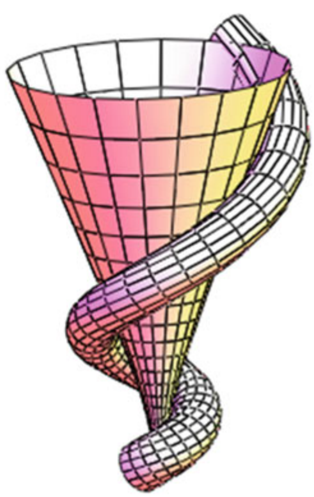
The tubular surface on the sphere



The tubular loxodrome

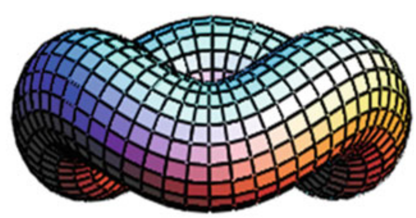


The tubular surface winding the sphere

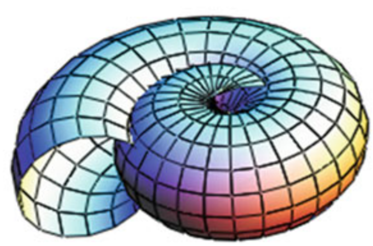


The tubular surface with a center-to-center line on the one-sheet hyperboloid of revolution

The tubular helix-shaped surface with a center-to-center line of the variable pitch



The wave-shaped torus on the sphere



The tubular surface with a plane line of centers in the form of the logarithmic spiral

■ Tubular Spiral Surface

Tubular spiral surface has the centerline in the form of a conic spiral and belong to normal cyclic surfaces with a generatrix circle of constant radius.

Forms of definition of a tubular surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + a\boldsymbol{e}(u, v),$$

where $\boldsymbol{r}(u, v)$ is a radius vector of a tubular spiral surface; $\boldsymbol{\rho}(u)$ is a radius vector of a conic spiral taken as a line of centers of generatrix circles with the radius a;

$$\begin{aligned}\boldsymbol{\rho}(u) &= be^{pu}(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u + \lambda\boldsymbol{k})\\ &= be^{pu}[\boldsymbol{h}(u) + \lambda\boldsymbol{k}],\end{aligned}$$

$\boldsymbol{e}(u, v)$ is a vector function of a circle of the unit radius at the normal plane of a generatrix circle with a normal

$$\begin{aligned}t(u) &= \boldsymbol{\rho}'/s;\\ \text{where } s &= |\boldsymbol{\rho}'| = |\partial\boldsymbol{\rho}/\partial u|;\\ e(u, v) &= \boldsymbol{v}\cos\omega + \boldsymbol{\beta}\sin\omega;\\ \omega &= v - \frac{p\lambda u}{\sqrt{1 + (1 + \lambda^2)p^2}} = v - cu,\end{aligned}$$

where c = const, $v = \frac{-\boldsymbol{h}+p\boldsymbol{n}}{\sqrt{1+p^2}}$;

$$\begin{aligned}\boldsymbol{\beta} &= \frac{-p\lambda(p\boldsymbol{h} + \boldsymbol{n}) + (1 + p^2)\boldsymbol{k}}{\sqrt{1 + (1 + \lambda^2)p^2}\sqrt{1 + p^2}},\\ \boldsymbol{n} &= -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u.\end{aligned}$$

(2) Parametrical equations:

$$\begin{aligned}x = x(u, v) &= be^{pu}\cos u - \frac{a}{\sqrt{1 + p^2}}\\ &\times \left[(p\sin u + \cos u)\cos\omega - \frac{p\lambda(\sin u - p\cos u)}{\sqrt{1 + (1 + \lambda^2)p^2}}\sin\omega\right],\\ y = y(u, v) &= be^{pu}\sin u - \frac{a}{\sqrt{1 + p^2}}\\ &\times \left[(\sin u - p\cos u)\cos\omega + \frac{p\lambda(p\sin u + \cos u)}{\sqrt{1 + (1 + \lambda^2)p^2}}\sin\omega\right],\\ z = z(u, v) &= b\lambda e^{pu} + \frac{a\sqrt{1 + p^2}\sin\omega}{\sqrt{1 + (1 + \lambda^2)p^2}}.\end{aligned}$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned}A &= be^{pu}\sqrt{1 + (1 + \lambda^2)p^2} - \frac{\sqrt{1 + p^2}a\cos\omega}{\sqrt{1 + (1 + \lambda^2)p^2}},\\ F &= 0,\ B = a,\\ L &= \frac{\sqrt{1 + p^2}A}{\sqrt{1 + (1 + \lambda^2)p^2}}\cos\omega,\\ M &= 0,\ N = a,\\ k_1 &= k_u = \frac{\sqrt{1 + p^2}\cos\omega}{A\sqrt{1 + (1 + \lambda^2)p^2}},\\ k_2 &= k_v = \frac{1}{a}.\end{aligned}$$

The tubular spiral surface is given in the lines of principle curvatures u, v. The tubular spiral surface with the conic spiral as the centerline of the generatrix circles is represented in Fig. 1. In Figs. 2 and 3, the tubular spiral surfaces with plane lines of centers in the form of *the logarithmic spirals* that are projection of the conic spirals on the plane perpendicular to the axis of spiral surface.

In Fig. 4, the tubular surface with the symmetrical conic spiral line of centers is shown.

An equation of the symmetrical conic spiral has the following form:

$$\begin{aligned}\boldsymbol{\rho}(u) &= be^{p|u|}\text{sign}(u)[\boldsymbol{i}\cos u + \boldsymbol{j}\sin u + \lambda\boldsymbol{k}]\\ &= be^{p|u|}\text{sign}(u)[\boldsymbol{h}|u| + \lambda\boldsymbol{k}].\end{aligned}$$

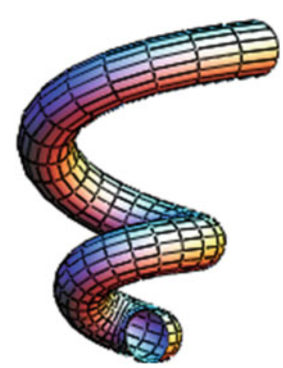
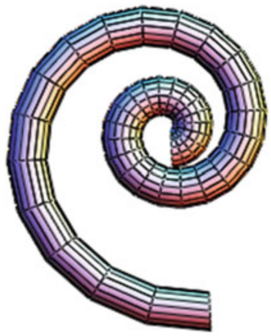
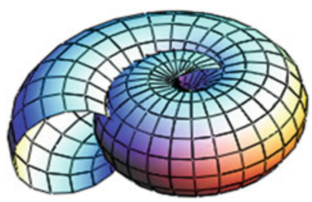


Fig. 1



$a = 1;\ b = 1;\ \lambda = 0;$
$p = 0.18;\ 2\pi \le u \le 6\pi$

Fig. 2



$a = 3;\ \ b = 1;\ \ \lambda = 0;$
$p\ = 0.2;\ \ 2\pi \le u \le 5\pi$

Fig. 3

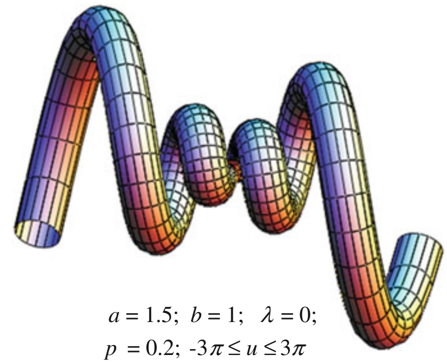


Fig. 4

Additional Literature

Krivoshapko SN, Christian A. Bock Hyeng. Geometrical research of rare types of cyclic surfaces. International Journal of Research and Reviews in Applied Sciences. 2012; Vol. 12, Iss. 3, p. 346-359.

■ Tubular Surface with a Plane Line of Centers in the Form of the Evolvent of a Circle

A tubular surface with a plane line of centers in the form of the evolvent of a circle

$$x = x(\alpha) = r\cos\alpha - r(\alpha_0 - \alpha)\sin\alpha,$$
$$y = y(\alpha) = r\sin\alpha + r(\alpha_0 - \alpha)\cos\alpha$$

is formed by motion of a circle of a constant radius lying in the plane (*loose axoid*) which rolls without sliding over a circular cylinder (*fixed axoid*) of the radius r.

The method of construction of the surface in question shows that it may be related to a class of *cyclic surfaces*, or to a class of *carved surfaces* (see also a Subsect. "Monge Surfaces with a Circular Cylindrical Directrix Surface"), or to a class of *kinematical surfaces of the common type* (see also a Sect. "34.1. Rotational Surface").

Forms of definition of the surface

(1) Parametrical equations (Figs. 1 and 2):

$$x = x(\alpha,\beta) = r\cos\alpha - [r(\alpha_0 - \alpha) + a\cos\beta]\sin\alpha,$$
$$y = y(\alpha,\beta) = r\sin\alpha - [r(\alpha_0 - \alpha) + a\cos\beta]\cos\alpha,$$
$$z = z(\beta) = a\sin\beta,$$

where a is a radius of a generatrix circle; α is the angle of the coordinate axis Ox with the axis Oy; β is the angle in the plane of a generatrix circle, taken from the plane xOy; $0 \le \beta \le 2\pi$.

In Fig. 1, the tubular surface with the plane centerline in the form of the evolvent of a circle of the radius a = 2 m is shown; $\pi/2 \le \alpha \le 3\pi$; $0 \le \beta \le 2\pi$. In Fig. 2, the tubular surface with r = 2 m; a = 2 m; α_0 = 0; $\pi/2 \le \beta \le 3\pi/2$; $\pi/2 \le \alpha \le 3\pi/2$ is presented.

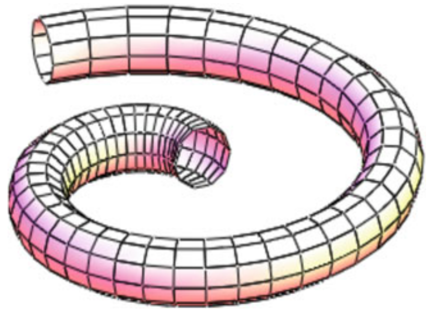


Fig. 1

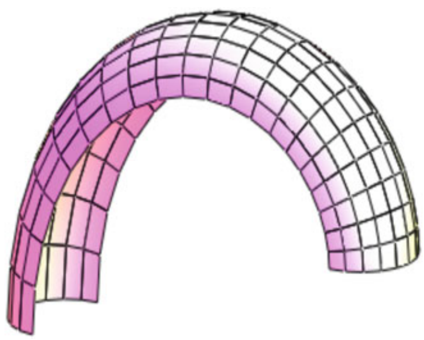


Fig. 2

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = (a\cos\beta - r\alpha)^2,$$
$$F = 0,\ B = a,$$
$$L = -(a\cos\beta - r\alpha)\cos\beta,$$
$$M = 0,\ N = -a,$$
$$k_1 = -\frac{\cos\beta}{(a\cos\beta - r\alpha)},$$
$$k_2 = -\frac{1}{a},$$
$$K = \frac{\cos\beta}{a(a\cos\beta - r\alpha)}.$$

A tubular surface with a plane line of the centers in the form of the evolvent of a circle (*a circle involute* or *the involute of the circle*) contains segments of positive (Fig. 2) and negative Gaussian curvature divided by the curvilinear coordinate lines $\beta = \pi/2$ and $\beta = 3\pi/2$ on which parabolic points with zero Gaussian curvature ($K = 0$) are disposed. The surface is given in the lines of principle curvatures β and α.

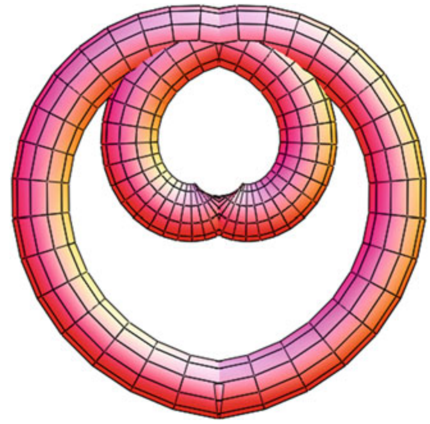


Fig. 3

(2) Parametrical form of the definition with the help of generalized cylindrical coordinates:

$$x = x(\alpha,\gamma) = r\cos\alpha - \left(\sqrt{a^2-\gamma^2} + c - r\alpha\right)\sin\alpha,$$
$$y = y(\alpha,\gamma) = r\sin\alpha + \left(\sqrt{a^2-\gamma^2} + c - r\alpha\right)\cos\alpha,$$
$$z = \gamma,$$

where c = const. If $r = 0$, then a considered tubular surface degenerates into *a circular torus*:

$$x = x(\alpha,\gamma) = -\left(\sqrt{a^2-\gamma^2} + c\right)\sin\alpha,$$
$$y = y(\alpha,\gamma) = \left(\sqrt{a^2-\gamma^2} + c\right)\cos\alpha,$$
$$z = \gamma.$$

Having assumed $c = 0$ in the equations of a circular torus given above, one can obtain parametric equations of a sphere.

In Fig. 3, the tubular surface with a plane line of centers in the form of symmetrical fragments of the evolvent of a circle is shown. The angular parameter α changes in the interval:

$$-2.45\pi \le \alpha \le 2.45\pi.$$

In this case, a tubular surface with the intersecting branches is obtained.

Additional Literature

Rekach VG, *Krivoshapko SN*. Analysis of Shells of Complex Form. Moscow: Izd-vo UDN, 1988; 176 p.

■ Tubular Surface with a Plane Line of Centers in the Form of a Cycloid

A tubular surface with a plane line of centers in the form of a cycloid

$$x = x(u) = b(\cos u + u \sin u),$$
$$y = y(u) = b(\sin u - u \cos u)$$

is formed by the motion of a circle of a constant radius remaining in the normal plane of the cycloid all the time and this plane with the cycloid rolls without sliding over a circular cylinder (stationary axoid) having a radius r.

Tubular surfaces belong to a subclass of *cyclic normal surfaces*.

The general information on *tubular surfaces* is given at a Subsect. "17.2.1. Tubular Surfaces," where it is set fourth tat a vector equation of *an arbitrary tubular surface* may be represented as

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + R\boldsymbol{e}(u, v),$$

where $\boldsymbol{r}(u, v)$ is a radius vector of a tubular surface; $\boldsymbol{\rho}(u)$ is the radius vector of the centerline of generatrix circles; $\boldsymbol{e}(u, v)$ is the vector function of a circle of the unit radius in the plane of generatrix circle with the normal

$$\boldsymbol{t}(u) = \boldsymbol{\rho}'/|\boldsymbol{\rho}'|.$$

Forms of definition of the surface in question

(1) Parametrical equations:

$$X = X(u, v) = x(u) + \frac{Ry'(u)\cos v}{\sqrt{x'^2(u) + y'^2(u)}},$$
$$Y = Y(u, v) = y(u) - \frac{Rx'(u)\cos v}{\sqrt{x'^2(u) + y'^2(u)}},$$
$$Z = Z(v) = R \sin v,$$

where R is a constant radius of a generatrix circle; v is the angle at the plane of a generatrix circle, read from the plane xOy in the direction of the axis Oz; $0 \le v \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \left\{1 + \frac{[x'(u)y''(u) - y'(u)x''(u)]R\cos v}{[x'^2(u) + y'^2(u)]^{3/2}}\right\}^2 \times [x'^2(u) + y'^2(u)],$$
$$F = 0,\ B = R,$$
$$L = -\frac{[x'(u)y''(u) - y'(u)x''(u)]A\cos v}{[x'^2(u) + y'^2(u)]},$$
$$M = 0,\ N = -R,$$
$$k_1 = k_u = -\frac{[x'(u)y''(u) - y'(u)x''(u)]\cos v}{[x'^2(u) + y'^2(u)]A},$$
$$k_2 = k_v = -\frac{1}{R},\ K = k_1k_2.$$

A tubular surface with a plane line of centers is given in the lines of principle curvatures u, v. The formulas presented above may be used for tubular surfaces with arbitrary plane line of centers given by parametrical equations

$$x = x(u),\ y = y(u).$$

(2) Parametrical equations:

$$X = X(u, v) = b(\cos u + u\sin u) + R\cos v \sin u,$$
$$Y = Y(u, v) = b(\sin u - u\cos u) - R\cos v \cos u,$$
$$Z = Z(v) = R\sin v.$$

They may be used if the centerline is given in the following form (Figs. 1 and 2):

$$x(u) = b(\cos u + u\sin u),$$
$$y(u) = b(\sin u - u\cos u)$$

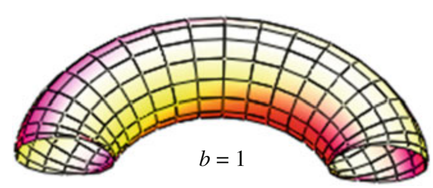


Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = bu + R\cos v,\quad F = 0,\quad B = R,$$
$$L = -A\cos v,\quad M = 0,\quad N = -R,$$
$$k_u = k_1 = -\frac{\cos v}{A},\quad k_v = k_2 = -\frac{1}{R},$$
$$K = \frac{\cos v}{AR}.$$

Tubular surface with a plane line of centers of generatrix circles in the form of the cycloid contains the segments of positive (Fig. 2) and negative Gaussian curvature divided by the curvilinear coordinate lines $v = \pi/2$ and $v = 3\pi/2$. On these lines, parabolic points with Gaussian curvature equal to zero ($K = 0$) are disposed.

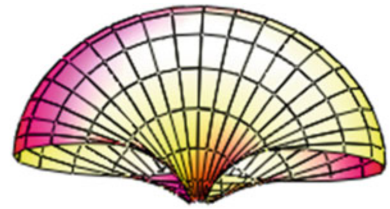


Fig. 2

Reference

Ivanov VN. Geometry and design of tubular shells. Bulletin of Peoples' Friendship University of Russia. Ser.: "Ingenern. Issledovaniya". 2005; No. 1 (11), p. 109-114.

■ Tubular Surface with a Plane Parabolic Line of Centers

A tubular surface with parabolic plane line of centers

$$y = bx^2$$

is formed by motion of a circle of a constant radius remaining all the time in the normal plane of the parabola. Tubular surfaces belong to a subclass of *normal cyclic surfaces*. The general information on *tubular surfaces* is given at a Subsect. "17.2.1. Tubular Surfaces," where it is set forth that a vector equation of *an arbitrary tubular surface* may be represented as

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + R\boldsymbol{e}(u, v),$$

where $\boldsymbol{r}(u,\ v)$ is a radius vector of a tubular surface; $\boldsymbol{\rho}(u)$ is the radius vector of the centerline of generatrix circles; $\boldsymbol{e}(u,\ v)$ is the vector function of a circle of the unit radius in the plane of generatrix circle with the normal

$$\boldsymbol{t}(u) = \boldsymbol{\rho}'/|\boldsymbol{\rho}'|.$$

Forms of definition of the tubular surface

(1) Parametrical equations:

$$X = X(u,v) = x(u) + \frac{Ry'(u)\cos v}{\sqrt{x'^2(u) + y'^2(u)}},$$
$$Y = Y(u,v) = y(u) - \frac{Rx'(u)\cos v}{\sqrt{x'^2(u) + y'^2(u)}},$$
$$Z = Z(v) = R\sin v,$$

where R is a constant radius of a generatrix circle; v is the angle at the plane of a generatrix circle, read from the plane xOy in the direction of the axis Oz; $0 \le v \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \left\{1 + \frac{[x'(u)y''(u) - y'(u)x''(u)]R\cos v}{[x'^2(u) + y'^2(u)]^{3/2}}\right\}^2 \times [x'^2(u) + y'^2(u)],$$
$$F = 0,\quad B = R;$$
$$L = -\frac{[x'(u)y''(u) - y'(u)x''(u)]A\cos v}{[x'^2(u) + y'^2(u)]},$$
$$N = -R,\quad M = 0,$$
$$k_1 = k_u = -\frac{[x'(u)y''(u) - y'(u)x''(u)]\cos v}{[x'^2(u) + y'^2(u)]A},$$
$$k_2 = k_v = -\frac{1}{R},\ K = k_1k_2.$$

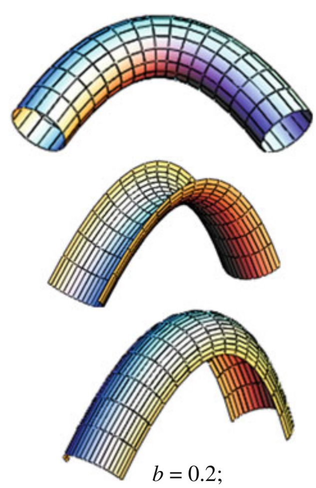


Fig. 1

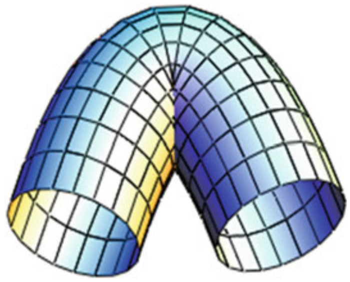


Fig. 2

A tubular surface with a plane line of centers is given in the lines of principle curvatures u, v. The formulas presented above may be used for tubular surfaces with arbitrary plane line of centers given by parametrical equations

$$x = x(u), \quad y = y(u).$$

(2) Parametrical equation (Figs. 1 and 2):

$$X = X(u,v) = u + \frac{2buR\cos v}{\sqrt{1+4b^2u^2}},$$
$$Y = Y(u,v) = bu^2 - \frac{R\cos v}{\sqrt{1+4b^2u^2}},$$
$$Z = Z(v) = R\sin v.$$

They may be used if the centerline is given in the following form:

$$x(u) = u, y(u) = bu^2.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \left[1 + \frac{2bR\cos v}{(1+4b^2u^2)^{3/2}}\right]^2 \times (1+4b^2u^2).$$
$$F = 0,\ B = R;$$
$$L = \frac{-2bA\cos v}{(1+4b^2u^2)},$$
$$M = 0,\ N = -R;$$
$$k_u = k_1 = \frac{-2b\cos v}{A(1+4b^2u^2)}$$
$$k_v = k_2 = -\frac{1}{R},$$
$$K = \frac{2b\cos v}{AR(1+4b^2u^2)}.$$

Tubular surface with a plane line of centers of generatrix circles in the form of the parabola contains the segments of positive and negative Gaussian curvature divided by the curvilinear coordinate lines $v = \pi/2$ and $v = 3\pi/2$. On these lines, parabolic points with Gaussian curvature equal to zero ($K = 0$) are disposed.

Additional Literature

Stasenko IV. The influence of the initial imperfections on the stress state of thin-walled curvilinear tubes. Tr. MVTU. M.: MVTU, 1980; No. 332, p. 146-160.

■ Tubular Surface with a Plane Hyperbolic Centerline

A tubular surface with a plane hyperbolic centerline is formed by motion of a circle of constant radius remaining all the time in the normal plane of the hyperbola

$$x = x(u) = b\,\mathrm{ch}\,u,$$
$$y = y(u) = c\,\mathrm{sh}\,u.$$

Tubular surfaces belong to a subclass of *cyclic normal surfaces*. The general information on *tubular surfaces* is given at a Subsect. "17.2.1. Tubular Surfaces", where it is set fourth that a vector equation of *an arbitrary tubular surface* may be represented as

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + R\boldsymbol{e}(u,v),$$

where $\boldsymbol{r}(u,\ v)$ is a radius vector of a tubular surface; $\boldsymbol{\rho}(u)$ is the radius vector of the centerline of generatrix circles; $\boldsymbol{e}(u,\ v)$ is the vector function of a circle of the unit radius in the plane of generatrix circle with the normal

$$\boldsymbol{t}(u) = \boldsymbol{\rho}'/|\boldsymbol{\rho}'|.$$

Forms of definition of the tubular surface

(1) Parametrical equations:

$$X = X(u,v) = x(u) + \frac{Ry'(u)\cos v}{\sqrt{x'^2(u)+y'^2(u)}},$$
$$Y = Y(u,v) = y(u) - \frac{Rx'(u)\cos v}{\sqrt{x'^2(u)+y'^2(u)}},$$
$$Z = Z(v) = R\sin v,$$

where R is a constant radius of a generatrix circle; v is the angle at the plane of a generatrix circle, read from the plane xOy in the direction of the axis Oz; $0 \le v \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \left\{1 + \frac{[x'(u)y''(u) - y'(u)x''(u)]R\cos v}{[x'^2(u) + y'^2(u)]^{3/2}}\right\}^2 \times [x'^2(u) + y'^2(u)],$$

$$F = 0,\ B = R,$$

$$L = -\frac{[x'(u)y''(u) - y'(u)x''(u)]A\cos v}{[x'^2(u) + y'^2(u)]},$$

$$M = 0,\ N = -R$$

$$k_1 = k_u = -\frac{[x'(u)y''(u) - y'(u)x''(u)]\cos v}{[x'^2(u) + y'^2(u)]A},$$

$$k_2 = k_v = -\frac{1}{R},\ K = k_1 k_2.$$

A tubular surface with a plane line of centers is given in the lines of principle curvatures u, v. The formulas presented above may be used for tubular surfaces with arbitrary plane line of centers given by parametrical equations

$$x = x(u), \quad y = y(u).$$

(2) Parametrical equations (Figs. 1 and 2):

$$X = X(u,v) = b\mathrm{ch}u + \frac{Rc\mathrm{ch}u\cos v}{\sqrt{b^2\mathrm{sh}^2u + c^2\mathrm{ch}^2u}},$$

$$Y = Y(u,v) = c\mathrm{sh}u - \frac{Rb\mathrm{sh}u\cos v}{\sqrt{b^2\mathrm{sh}^2u + c^2\mathrm{ch}^2u}},$$

$$Z = Z(v) = R\sin v.$$

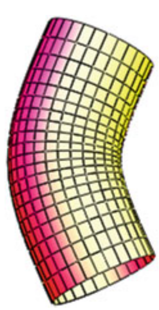


Fig. 1

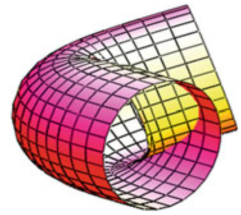


Fig. 2

They may be used if the centerline is given in the following form:

$$x(u) = d\mathrm{ch}u, \quad y(u) = c\mathrm{sh}u.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \left[1 - \frac{cbR\cos v}{\left(b^2\mathrm{sh}^2u + c^2\mathrm{ch}^2u\right)^{3/2}}\right]^2 \times \left(b^2\mathrm{sh}^2u + c^2\mathrm{ch}^2u\right),$$

$$F = 0,\ B = R;$$

$$L = \frac{cbA\cos v}{\left(b^2\mathrm{sh}^2u + c^2\mathrm{ch}^2u\right)},$$

$$M = 0,\ N = -R;$$

$$k_u = k_1 = \frac{cb\cos v}{A\left(b^2\mathrm{sh}^2u + c^2\mathrm{ch}^2u\right)},$$

$$k_v = k_2 = -\frac{1}{R},$$

$$K = \frac{-cb\cos v}{AR\left(b^2\mathrm{sh}^2u + c^2\mathrm{ch}^2u\right)}.$$

Tubular surface with a plane line of centers of generatrix circles in the form of the hyperbola contains the segments of positive and negative Gaussian curvature divided by the curvilinear coordinate lines $v = \pi/2$ and $v = 3\pi/2$. On these lines, parabolic points with Gaussian curvature equal to zero ($K = 0$) are disposed.

Additional Literature

Ivanov VN. On calculation of tubular shells on membrane theory. Dokl. VIII Nauchno-Tehn. Konf. Inzhenern. Fakulteta. Moscow: UDN, 1972, p. 26-28.

■ Tubular Surface with a Plane Elliptical Line of Centers

A tubular surface with an elliptical plane line of centers is formed by motion of a circle of constant radius remaining all the time in the normal plane of the ellipse

$$x(u) = b\cos u, \quad y(u) = c\sin u.$$

Tubular surfaces belong to a subclass of *cyclic normal surfaces*. The general information on *tubular surfaces* is given at a Subsect. "17.2.1. Tubular Surfaces," where it is set forth that a vector equation of *an arbitrary tubular surface* may be represented as

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + R\boldsymbol{e}(u, v),$$

where $\boldsymbol{r}(u, v)$ is a radius vector of a tubular surface; $\boldsymbol{\rho}(u)$ is the radius vector of the centerline of generatrix circles; $\boldsymbol{e}(u, v)$ is the vector function of a circle of the unit radius in the plane of generatrix circle with the normal

$$\boldsymbol{t}(u) = \boldsymbol{\rho}'/|\boldsymbol{\rho}'|.$$

Forms of definition of the tubular surface

(1) Parametrical equations:

$$X = X(u, v) = x(u) + \frac{Ry'(u)\cos v}{\sqrt{x'^2(u) + y'^2(u)}},$$
$$Y = Y(u, v) = y(u) - \frac{Rx'(u)\cos v}{\sqrt{x'^2(u) + y'^2(u)}},$$
$$Z = Z(v) = R\sin v,$$

where R is a constant radius of a generatrix circle; v is the angle at the plane of a generatrix circle, read from the plane xOy in the direction of the axis Oz; $0 \le v \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \left\{1 + \frac{[x'(u)y''(u) - y'(u)x''(u)]R\cos v}{[x'^2(u) + y'^2(u)]^{3/2}}\right\}^2 \times [x'^2(u) + y'^2(u)],$$
$$F = 0, \quad B = R;$$
$$L = -\frac{[x'(u)y''(u) - y'(u)x''(u)]A\cos v}{[x'^2(u) + y'^2(u)]},$$
$$M = 0, \quad N = -R;$$
$$k_1 = k_u = -\frac{[x'(u)y''(u) - y'(u)x''(u)]\cos v}{[x'^2(u) + y'^2(u)]A},$$
$$k_2 = k_v = -\frac{1}{R}, \quad K = k_1k_2.$$

A tubular surface with a plane line of centers is given in the lines of principle curvatures u, v. The formulas presented above may be used for tubular surfaces with arbitrary plane line of centers given by parametrical equations

$$x = x(u), \quad y = y(u).$$

(2) Parametrical equations:

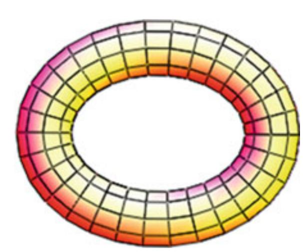
$$X = X(u, v) = b\cos u + \frac{cR\cos v\cos u}{\sqrt{b^2\sin^2 u + c^2\cos^2 u}},$$
$$Y = Y(u, v) = c\sin u + \frac{bR\cos v\sin u}{\sqrt{b^2\sin^2 u + c^2\cos^2 u}},$$
$$Z = Z(v) = R\sin v.$$

They may be used if the centerline is given in the following form: (Figs. 1 and 2)

$$x(u) = b\cos u, \quad y(u) = c\sin u$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \left[1 + \frac{bcR\cos v}{\left(b^2\sin^2 u + c^2\cos^2 u\right)^{3/2}}\right]^2 \times \left(b^2\sin^2 u + c^2\cos^2 u\right),$$
$$F = 0, \quad B = R;$$
$$L = \frac{-bcA\cos v}{\left(b^2\sin^2 u + c^2\cos^2 u\right)},$$
$$M = 0, \quad N = -R;$$
$$k_u = k_1 = \frac{-cb\cos v}{A\left(b^2\sin^2 u + c^2\cos^2 u\right)},$$
$$k_v = k_2 = -\frac{1}{R},$$
$$K = \frac{cb\cos v}{AR\left(b^2\sin^2 u + c^2\cos^2 u\right)}.$$



$b = 4$; $c = 3$

Fig. 1

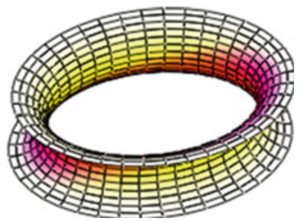


Fig. 2

Tubular surface with a plane line of centers of generatrix circles in the form of the ellipse contains the segments of positive and negative (Fig. 2) Gaussian curvature divided by the curvilinear coordinate lines $v = \pi/2$ and $v = 3\pi/2$. On these lines, parabolic points with $K = 0$ are disposed.

■ Tubular Surface with a Plane Sinusoidal Centerline

A tubular surface with a sinusoidal plane line of centers is formed by motion of a circle of constant radius, remaining all the time in the normal plane of the sinusoid

$$x(u) = u, \quad y(u) = b\sin cu.$$

Tubular surfaces belong to a subclass of *cyclic normal surfaces*. The general information on *tubular surfaces* is given at a Subsect. "17.2.1. Tubular Surfaces," where it is set forth that a vector equation of *an arbitrary tubular surface* may be represented as

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + R\boldsymbol{e}(u,v),$$

where $\boldsymbol{r}(u, v)$ is a radius vector of a tubular surface; $\boldsymbol{\rho}(u)$ is the radius vector of the centerline of generatrix circles; $\boldsymbol{e}(u, v)$ is the vector function of a circle of the unit radius in the plane of generatrix circle with the normal

$$\boldsymbol{t}(u) = \boldsymbol{\rho}'/|\boldsymbol{\rho}'|.$$

Forms of definition of the tubular surface

(1) Parametrical equations:

$$X = X(u,v) = x(u) + \frac{Ry'(u)\cos v}{\sqrt{x'^2(u) + y'^2(u)}},$$
$$Y = Y(u,v) = y(u) - \frac{Rx'(u)\cos v}{\sqrt{x'^2(u) + y'^2(u)}},$$
$$Z = Z(v) = R\sin v,$$

where R = const is a radius of a generatrix circle; v is the angle at the plane of a generatrix circle read from the plane xOy in the direction of the axis Oz; $0 \le v \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \left\{1 + \frac{[x'(u)y''(u) - y'(u)x''(u)]R\cos v}{[x'^2(u) + y'^2(u)]^{3/2}}\right\}^2 \times [x'^2(u) + y'^2(u)],$$
$$F = 0, \quad B = R;$$
$$L = -\frac{[x'(u)y''(u) - y'(u)x''(u)]A\cos v}{[x'^2(u) + y'^2(u)]},$$
$$M = 0, \quad N = -R;$$
$$k_1 = k_u = -\frac{[x'(u)y''(u) - y'(u)x''(u)]\cos v}{[x'^2(u) + y'^2(u)]A},$$
$$k_2 = k_v = -\frac{1}{R}, \quad K = k_1k_2.$$

A tubular surface with a plane line of centers is given in the lines of principle curvatures u, v. The formulas presented above may be used for tubular surfaces with arbitrary plane line of centers given by parametrical equations

$$x = x(u), \quad y = y(u).$$

(2) Parametrical equations (Fig. 1):

$$X = X(u,v) = u + \frac{cbR\cos v\cos cu}{\sqrt{1 + c^2b^2\cos^2 cu}},$$
$$Y = Y(u,v) = b\sin cu - \frac{R\cos v}{\sqrt{1 + c^2b^2\cos^2 cu}},$$
$$Z = Z(v) = R\sin v.$$

They may be used if the centerline is given in the following form:

$$x(u) = u, \quad y(u) = b\sin cu.$$

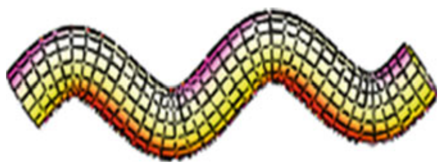


Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \left[1 - \frac{bc^2 R \cos v \sin cu}{(1 + c^2 b^2 \cos^2 cu)^{3/2}}\right]^2 \times \left(1 + c^2 b^2 \cos^2 cu\right),$$
$$F = 0,\ B = R;$$
$$L = \frac{bc^2 A \cos v \sin cu}{(1 + c^2 b^2 \cos^2 cu)},$$
$$M = 0, \quad N = -R;$$
$$k_u = k_1 = \frac{c^2 b \cos v \sin cu}{A(1 + c^2 b^2 \cos^2 cu)},$$
$$k_v = k_2 = -\frac{1}{R},$$
$$K = \frac{-c^2 b \cos v \sin cu}{AR(1 + c^2 b^2 \cos^2 cu)}.$$

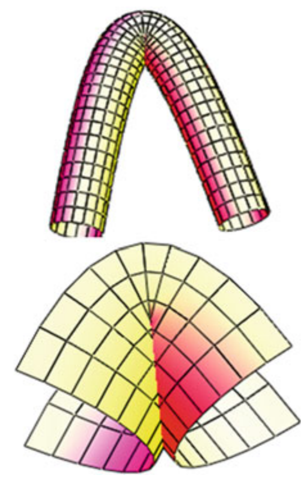


Fig. 2

Tubular surface with a plane line of centers of generatrix circles in the form of the sinusoid contains the segments of positive and negative Gaussian curvature divided by the curvilinear coordinate lines $v = \pi/2$ and $v = 3\pi/2$. On these lines, parabolic points with $K = 0$ are disposed.

In Fig. 2, the fragments of the tubular surface with the plane sinusoidal centerline is given when $b = 8$ cm; $c = 0.5\ \text{cm}^{-1}$; $R = 1$ cm.

■ Tubular Surface on the Sphere

A line of the centers of *a tubular surface on the sphere* is placed on the spherical surface. The unit normals $\boldsymbol{e}_0$ to the line of the centers coincide with the normals of the spherical surface.

A vector equation of the circle of the unit radius that is disposed in the normal plane of the centerline of the tubular surface is:

$$\boldsymbol{e}(u, v) = \cos v\boldsymbol{e}_0 + \sin v\boldsymbol{g}_0,$$

where

$$\boldsymbol{e}_0 = \boldsymbol{e}_0(u) = (\boldsymbol{i} \cos u + \boldsymbol{j} \sin u) \cos \omega + \boldsymbol{k} \sin \omega,$$

and

$$\omega = pu;\ p = \text{const};$$
$$\boldsymbol{g}_0 = \boldsymbol{g}_0(u) = \left[\boldsymbol{e}_0'/s \times \boldsymbol{e}_0\right];$$
$$s = \left[\omega^{/2} + \cos^2 \omega\right]^{1/2}.$$

Forms of definitions of a tubular surface on a sphere

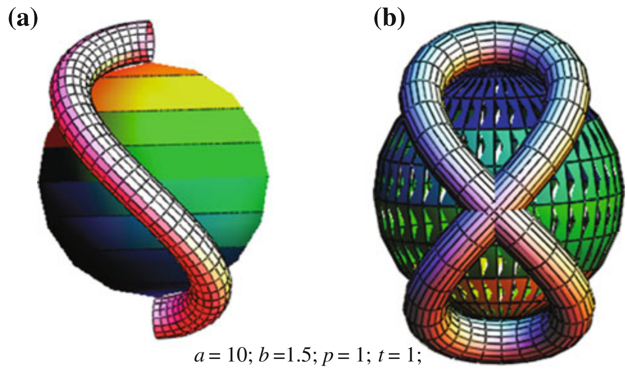
(1) Vector form of assignment (Figs. 1, 2, 3, 4 and 5):

$$\boldsymbol{r}(u, v) = a\boldsymbol{e}_0(u) + b\boldsymbol{e}(u, v),$$

where a is the radius of the base spherical surface with a centerline lying on it; b = const is a radius of a generatrix circle of the tubular surface.

(2) Parametrical equations (Figs. 1, 2, 3, 4 and 5):

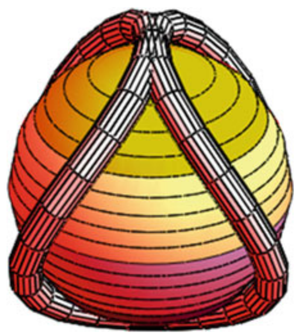
$$x = x(u, v) = (a + b \cos v) \cos \omega \cos u + \frac{b}{s} \sin v(\sin \omega \cos \omega \cos u - \omega' \sin u),$$
$$y = y(u, v) = (a + b \cos v) \cos \omega \sin u + \frac{b}{s} \sin v(\sin \omega \cos \omega \sin u + \omega' \cos u),$$
$$z = z(u, v) = (a + b \cos v) \sin \omega - \frac{b}{s} \sin v \cos^2 \omega,$$



$a = 10$; $b = 1.5$; $p = 1$; $t = 1$;
$-0.5\pi \le u \le 0.5\pi$

$t = 1$;
$-\pi \le u \le \pi$

Fig. 1



$a = 10$; $b = 1$; $p = 2$;
$t = 1$; $-\pi \le u \le \pi$

Fig. 2

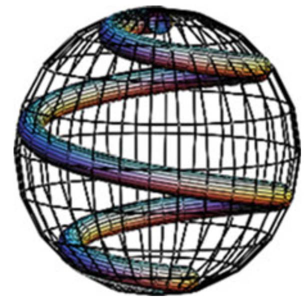
where

$$\omega = pu, \quad \omega' = p,$$

v is the angle, read in the normal plane of a line of centers of a tubular surface on a sphere.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = s(a + b\cos v) + b\left(1 + \frac{\omega'^2}{s^2}\right)\sin\omega\sin v,$$
$$F = 0, \quad B = b;$$
$$L = A\left[s\cos v + \left(1 + \frac{\omega'^2}{s^2}\right)\sin\omega\sin v\right],$$
$$M = 0, \quad N = b;$$
$$k_1 = k_u = L/A^2, \quad k_2 = k_v = 1/b.$$



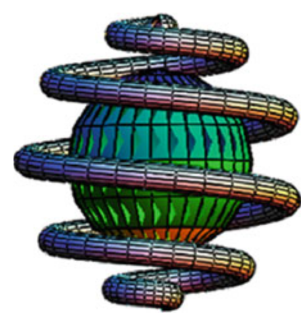
$a = 10$; $b = 1$; $p = 1/6$;
$t = -1$; $-3\pi \le u \le 3\pi$

Fig. 3

Assume, that

$$a = a_s + bt,$$

where a_s is a radius of the chosen sphere. If $t = 1$, then a tubular surface lies on the external surface of the chosen sphere



$a = 10$; $b = 1$; $p = 1/6$;
$t = 3$; $-5\pi \le u \le 5\pi$

Fig. 4

$a = 10$; $b = 1$; $p = 5/3$;
$-3\pi \le u \le 3\pi$

Fig. 5

(Figs. 1a, b and 2). If $t = -1$, then a tubular surface lies on the internal surface of the sphere with a radius a_s (Fig. 3).

Assume, that $t > 1$, then a tubular surface will be out of the chosen sphere (рис. 5), but if $t < -1$, then a tubular surface will be located inside of the sphere not touching it. If $t = 0$, the chosen sphere coincide with the base spheres (Figs. 1a, b and 2). All tubular surfaces shown in Figs. 1a, b, 3, 4, and 5 have the same base sphere of the radius a_s and $0 \le v \le 2\pi$. The tubular surfaces at Figs. 1, 4 and 5 may be related to the class of spiral-shaped surfaces (see Sect. "Spiral-Shaped Surfaces").

■ Tubular Helix-Shaped Surface with a Center-to-Center Line of Variable Lead

*A t*ubular helix-shaped surface with a center-to-center line of variable lead is formed by a circle of constant radius, the center of which moves along a helical line L of the variable lead (Fig. 1):

$$x(u) = a\cos m\pi u,$$
$$y(u) = a\sin m\pi u,$$
$$z(u) = b\sqrt{1-(1-u)^2} = b\varphi(u).$$

A generatrix circle remains at the normal planes of the helical line of the centers all the time. The cylindrical helical line of the centers L lays on the cylindrical surface of the radius a; $m/2$ is the number of its turns in the limit of a zone $0 \le z \le b$, i.e., when $0 \le u \le 1$.

The tangent straight lines to the cylindrical helical line L of the variable lead intersect the coordinate plane xOy at the angle $\alpha = 90° - \varphi$, where

$$\tan\varphi = \frac{am\pi\sqrt{u(2-u)}}{b(1-u)}.$$

The angle α changes from $\alpha = 0$ when $u = 1$ ($z = b$) until $\alpha = \pi/2$ when $u = 0$ ($z = 0$).

This surface in question may be related both to a class of *helix-shaped surfaces* and to a subclass of *normal cyclic surfaces* that are included in a class of *cyclic surfaces*.

Forms of definition of a tubular helix-shaped surface

(1) Vector equation (Fig. 2):

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + r\boldsymbol{t}(u,v),$$

where

$$\boldsymbol{\rho}(u) = a\boldsymbol{h}(m\pi u) + z(u)\boldsymbol{k}$$

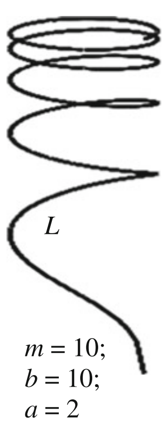


Fig. 1

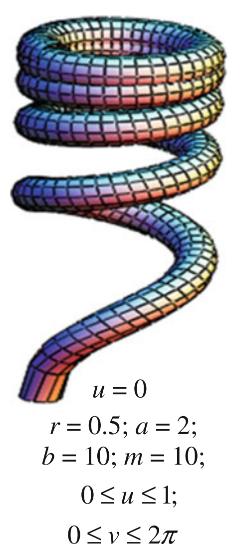


Fig. 2

is a radius vector of the helix-shaped line of centers of generatrix circles of a constant radius r;

$$\boldsymbol{h}(m\pi u) = \boldsymbol{i}\cos(m\pi u) + \boldsymbol{j}\sin(m\pi u);$$
$$\boldsymbol{n}(m\pi u) = -\boldsymbol{i}\sin(m\pi u) + \boldsymbol{j}\cos(m\pi u);$$
$$\boldsymbol{t}(u, v) = \boldsymbol{h}(m\pi u)\cos v + \boldsymbol{q}(m\pi u)\sin v$$

is the unit vector lying at the normal plane of the line of the centers; v is an angle read in the normal plane of the line of the centers,

$$\boldsymbol{q}(m\pi u) = (\boldsymbol{h} \times \boldsymbol{\tau}) = [-z'(u)\boldsymbol{n} + m\pi a\boldsymbol{k}]]/s;$$
$$s = \sqrt{a^2m^2\pi^2 + b^2(1-u)/\varphi},$$
$$\boldsymbol{\tau}(u) = \boldsymbol{\rho}'(u)/|\boldsymbol{\rho}'(u)| = \boldsymbol{\rho}'(u)/s = [m\pi a\boldsymbol{n} + z'(u)\boldsymbol{k}]/s$$

is the unit tangent vector of the line of the centers. The derivation on parameter u is shown by primes.

In Fig. 2, the cyclic surface having the geometric parameter u changing in the limits $0 \le u \le 1$ is shown.

If the geometrical parameter u changes in the limits of $0 \le u \le 2$, then the surface will be a closed surface. In Fig. 3, the surfaces with the parameters:

$a = 2$ m, $r = 0.5$ m, $0 \le u \le 2$, $m = 2$ and $m = 3$ are shown.

(2) Parametrical equations:

$$x = x(u, v) = (a + r\cos v)\cos m\pi u + \frac{rb}{s}\varphi'(u)\sin m\pi u \sin v,$$

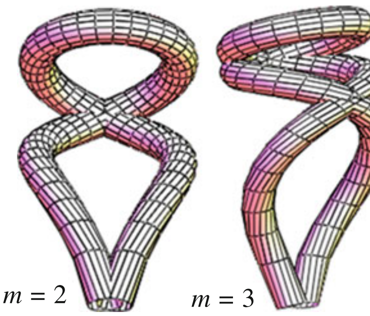


Fig. 3

$$y = y(u, v) = (a + r\cos v)\sin m\pi u - \frac{rb}{s}\varphi'(u)\cos m\pi u \sin v,$$
$$z = z(u, v) = b\varphi + \frac{am\pi r}{s}\sin v.$$

The tubular helix-shaped surface with a line of the variable lead is given in the curvilinear nonorthogonal nonconjugate coordinates u, v. The coordinate lines v coincide with the generatrix circles. The surface contains the parts of positive and negative Gaussian curvature.

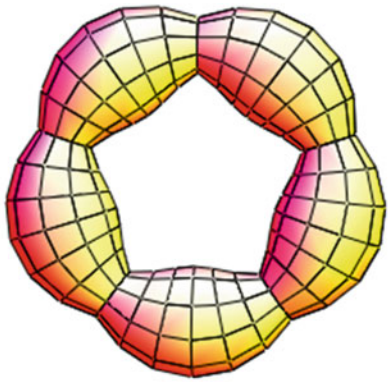
Additional Literature

Mamuzok GA, Miroshnichenko AV, Surkova GI. Algorithm of automatized design of tubular surfaces. Mat. Modeli i Sistemy Obrab. Inform. i Primeneniya Resheniy. Kharkov. 1988; p. 123-127.

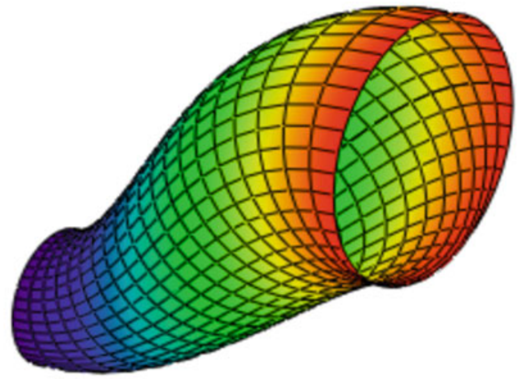
17.2.2 Normal Cyclic Surfaces with Generatrix Circle of Variable Radius

Tubular surfaces and *normal surfaces with generatrix circle of variable radius* belong to one subclass of cyclic surfaces, just to a Sect. 17.2 "Normal Cyclic Surfaces." But tubular surfaces have constant radius of generatrix circles and generatrix circles coincide with one set of lines of principal curvatures of surface. Generatrix circles of normal cyclic surfaces with generatrix circle of variable radius are not lines of principal curvatures.

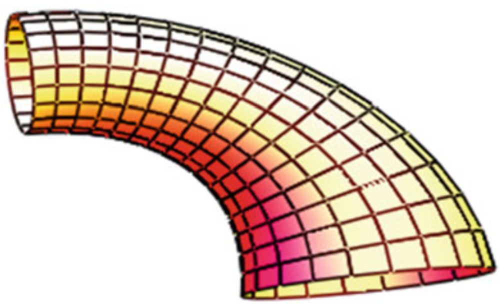
■ **Normal Cyclic Surfaces with Generatrix Circle of Variable Radius Presented in the Encyclopedia**



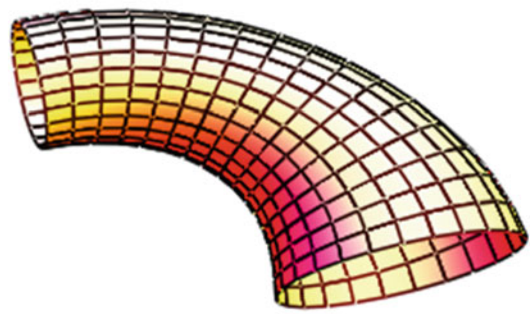
The normal cyclic surface with the plane circular line of centers and with a generator circle of the variable radius



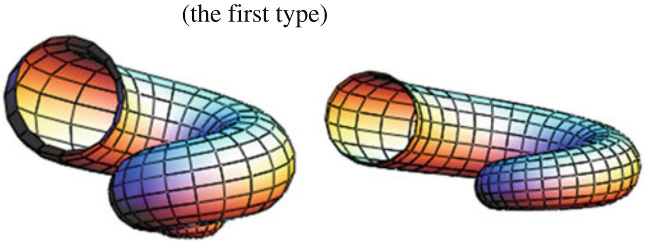
Connecting canal for two cylindrical surfaces with parallel axes



The normal cyclic surface with the elliptical line of centers and with a generatrix circle of the variable radius (the first type)

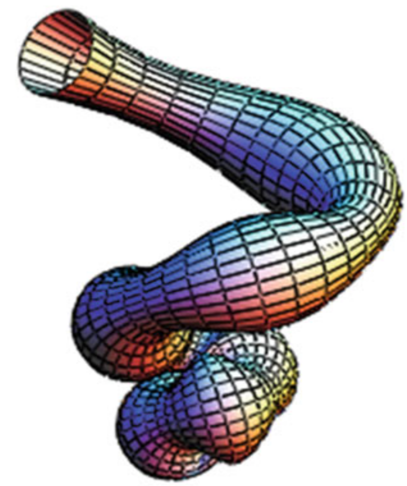


The normal cyclic surface with the elliptical line of centers and with a generatrix circle of the variable radius (the second type)

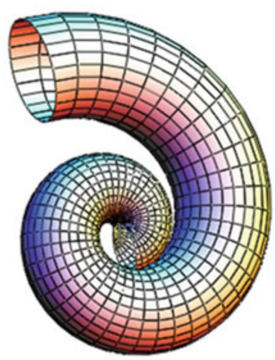


The normal cyclic surface with the generatrix circle of the variable radius and with the plane center-to-center line in the form of the conical spiral

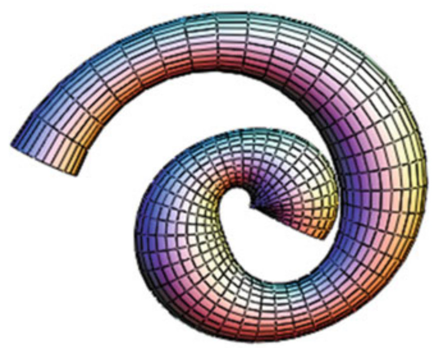
The normal cyclic helix-shaped surface consisting of the identical elements



The normal cyclic surface with a generatrix circle of the variable radius and with the plane center-to-center line in the form of the conical spiral [$R(u) = a(1 + c\sin tu)$]

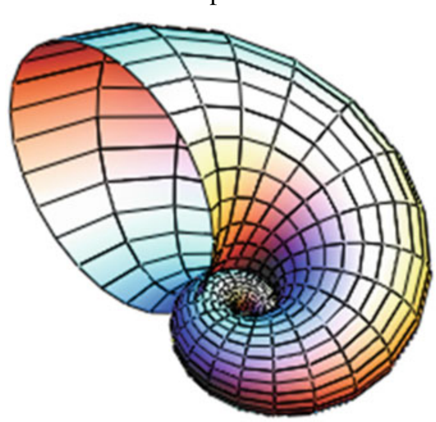


$R(u) = au$

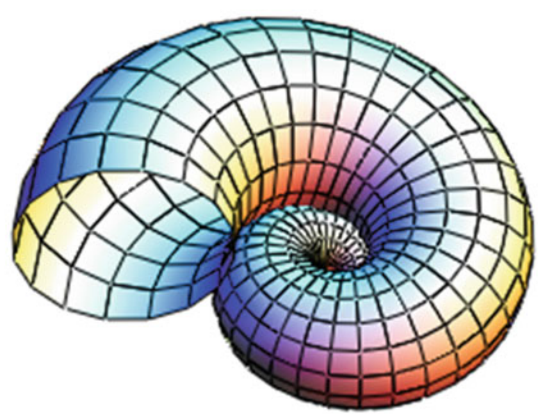


$R(u) = a(1 + c\sin tu)$

The normal cyclic surfaces with the generatrix circles of variable radius and with the plane center-to-center line in the form of the logarithmic spiral



The normal cyclic surface with a generatrix circle of the variable radius and with the plane center-to-center line in the form of the conical spiral [$R(u) = ae^{cu}$]



The normal cyclic surface with a generatrix circle of the variable radius and with the plane center-to-center line in the form of the conical spiral [$R(u) = au$]

■ Normal Cyclic Surface with Plane Circular Line of Centers and with a Generator Circle of Variable Radius

A normal cyclic surface with a plane circular line of centers and with a generator circle of a variable radius $R(u) = a(1 - d\cos pu)$ may be related both to a group of *cyclic surfaces with circles in planes of pencil and with a plane line of centers* and to a group of normal circular surfaces with a generator circle of variable radius. Normal planes of a plane circular line of centers degenerate into the planes of the pencil with the fixed straight of the pencil passing through the center of the circular line of the centers.

Forms of definition of the surface

(1) Parametrical equations (Figs. 1, 2 and 3):

$$x = x(u,v) = [b + R(u)\cos v]\cos u,$$
$$y = y(u,v) = [b + R(u)\cos v]\sin u,$$
$$z = z(u,v) = R(u)\sin v.$$

where b is a radius of the circular line of the centers (a directrix circle); v is the angle in the plane of a generatrix circle taken from the plane xOy; $0 \le v \le 2\pi$; u is the angle read from the axis Ox in the direction of the axis Oy; $0 \le u \le \infty$;

$$R(u) = a(1 - d\cos pu)$$

is the variable radius of a generatrix circle. Assume R = const, then the surface in question degenerates into a circular torus.

Coefficients of the fundamental forms of the surface:

$$A^2 = R'^2 + [b + R(u)\cos v]^2,$$
$$F = 0,\ B = R(u),$$
$$A^2B^2 - F^2 = A^2R^2(u),$$
$$L = \frac{-1}{A}\Big\{2R'^2\cos v - R''[b + R(u)\cos v]$$
$$\times + [b + R(u)\cos v]^2\cos v\Big\},$$
$$M = -\frac{R(u)R'}{A}\sin v,$$
$$N = -\frac{R(u)}{A}[b + R(u)\cos v]$$

The cyclic surface is given in curvilinear orthogonal non-conjugate coordinates u, v. It contains segments of both positive and negative total curvatures. Changing parameters in the parametrical equations of the surface in question, it is possible to obtain different types of the normal cyclic surfaces with a plane circular line of centers and with a variable radius of a generatrix circle (Figs. 1, 2 and 3). The coordinate lines v coincide with the circular generatrixes of the surface.

Reference

Ivanov VN. On one subclass of the normal cyclic surfaces. Structural Mechanics of Engineering Constructions and Buildings. Moscow: Izd-vo ASV, 2004; Iss. 13, p. 20-27.

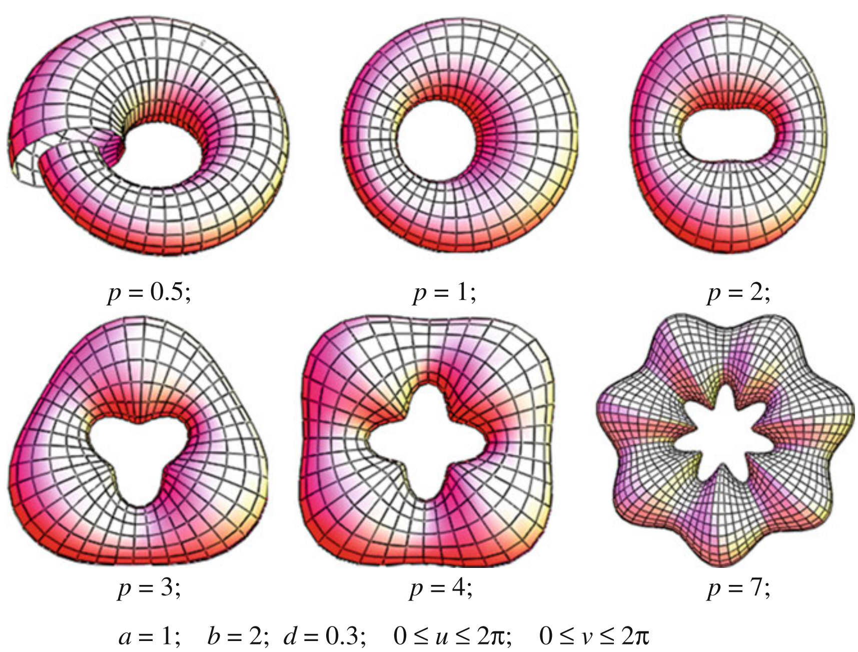
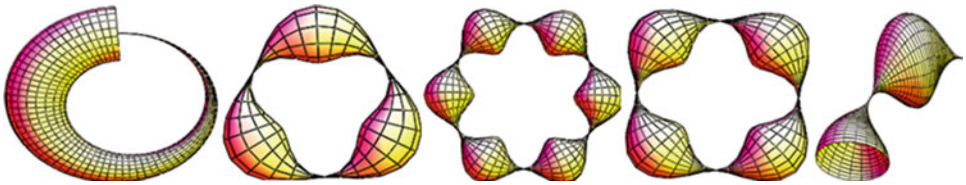


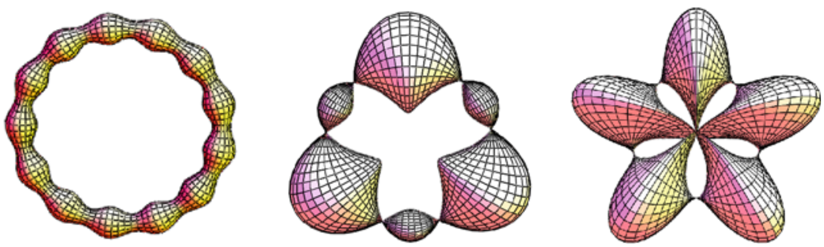
Fig. 1



$p = 0{,}5$ $p = 3$ $p = 6$ $p = 4$ $p = 6$

$a = 1;\quad b = 5;\quad d = 1;\quad 0 \le u \le 2\pi;\quad 0 \le v \le 2\pi;\quad 0 \le u \le \pi/2$

Fig. 2



$a = 1; b = 8; d = 0.3;$ $p = 7;\ 0 \le u \le 2\pi$ $a = 3; b = 5; d = 2;$ $p = 3;\ 0 \le u \le 2\pi$ $a = 2; b = 1; d = 1;$ $p = 5;\ 0 \le u \le 2\pi$

Fig. 3

■ Normal Cyclic Helix-Shaped Surface Consisting of Identical Elements

A normal cyclic helix-shaped surface consisting of identical elements is formed by motion of a circle of a variable radius along a helical line of constant slope. In this case, a generatrix circle is placed all the time at the normal plane of the helix of the centers and its radius changes under a sinusoidal law. It explains the appearance of the identical segments of the surface.

The forms of definition of the surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + R(u)\boldsymbol{e}(u, v),$$

where

$$\boldsymbol{\rho}(u) = b\boldsymbol{h}(u) + cu\boldsymbol{k};$$

r(u, v) is a radius vector of a cyclic surface; $\boldsymbol{\rho}$(u) is a radius vector of a helical line of the centers of the generatrix circles;

$$R(u) = a(1 - d\cos pu)$$

is a law of changing of a radius of the generatrix circle; u is the angle read from the axis Ox in the direction of the axis Oy; b is a radius of a cylinder with the helical line of centers lying on it; 2πc is the lead of the helical line of the centers;

$$\begin{aligned}\boldsymbol{h}(u) &= \boldsymbol{i}\cos u + \boldsymbol{j}\sin u;\\ \boldsymbol{e}(u, v) &= -\boldsymbol{h}\ \cos\omega + (-c\boldsymbol{n}/s + b\boldsymbol{k}/s)\sin\omega\\ &= \left(-\cos u\cos\omega + \frac{c}{s}\sin u\sin\omega\right)\boldsymbol{i}\\ &\quad - \left(\frac{c}{s}\cos u\sin\omega + \sin u\cos\omega\right)\boldsymbol{j} + \frac{b}{s}\boldsymbol{k}\end{aligned}$$

is the unit vector lying at the normal plane of the helical line of the centers;

$$\boldsymbol{n}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u; s^2 = b^2 + c^2;$$

k = b/s^2 is the curvature of the helical line of the centers;

$$\omega = v + \theta(u);\ \theta(u) = -cu/s.$$

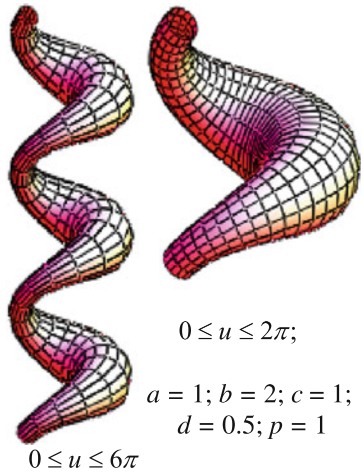


Fig. 1

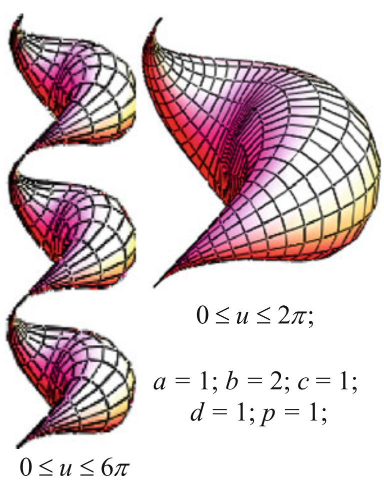


Fig. 2

(2) Parametrical equations (Figs. 1, 2 and 3):

$$x = x(u, v) = b\cos u + R(u)\left(-\cos u\cos\omega + \frac{c}{s}\sin u\sin\omega\right),$$
$$y = y(u, v) = b\sin u - R(u)\left(\sin u\cos\omega + \frac{c}{s}\cos u\sin\omega\right),$$
$$z = z(u, v) = cu + \frac{b}{s}R(u)\sin\omega.$$

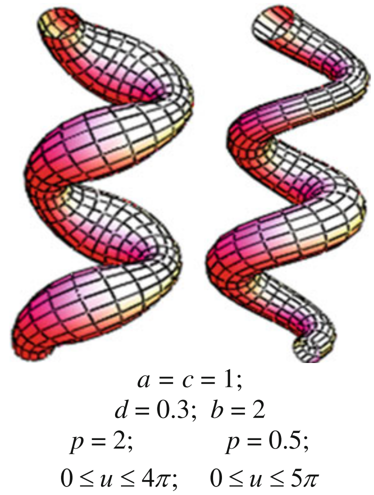


Fig. 3

Coefficients of the fundamental forms of the surface:

$$A^2 = (apd\sin pu)^2 + (b^2 + c^2) \times \left[1 - \frac{bR(u)}{b^2 + c^2}\cos\omega\right]^2,$$
$$F = 0,\ B = R(u),$$
$$L = -\frac{\sqrt{b^2 + c^2}}{A}\left\{\frac{abpd}{b^2 + c^2}\left[2apd\sin pu\cos\omega + \frac{cR(u)\sin\omega}{\sqrt{b^2 + c^2}}\right]\sin pu + \left[1 - \frac{bR(u)}{b^2 + c^2}\cos\omega\right]\left[b\cos\omega - \frac{b^2R(u)\cos^2\omega}{b^2 + c^2} + adp^2\cos pu\right]\right\},$$
$$M = \frac{apdbR(u)\sin pu\sin\omega}{A\sqrt{b^2 + c^2}},$$
$$N = \frac{R(u)\sqrt{b^2 + c^2}}{A}\left[1 - \frac{bR(u)}{b^2 + c^2}\cos\omega\right].$$

The cyclic surface is given in the curvilinear orthogonal nonconjugate coordinates u, v. The generatrix circles are not the lines of principle curvatures. Having taken $d = 0$, we may obtain *a tubular helical surface* and circular generatrixes will turn into lines of principal curvatures.

References

Ivanov VN. Shells with middle normal cyclic surfaces. Montazhn. i Spetz. Raboty v Stroitel'stve. 2006; No. 4, p. 37-39.
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■ Normal Cyclic Surface with Generatrix Circles of Variable Radius and with a Plane Center-to-Center Line in the Form of a Logarithmic Spiral

Logarithmic spiral is a horizontal projection of *conic spiral curve*. So, the normal surface in question is a special case of *a normal cyclic surface with a generatrix circle of variable radius and with a line of centers in the form of a conic spiral*. Logarithmic spiral intersects all rays emerging from the center under constant angle.

Forms of definition of the surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + R(u)\boldsymbol{e}(u,v),$$

where $\boldsymbol{\rho}$(u) is a radius vector of a logarithmic spiral taken as the line of centers of the generatrix circles with a law of changing of the radius assumed in the form R = R(u);

$$\boldsymbol{\rho}(u) = be^{pu}(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u) = be^{pu}\boldsymbol{h}(u),$$

$\boldsymbol{e}(u, v)$ is a vector function of a circle of the unit radius at the plane of a generatrix circle with the normal $\boldsymbol{t}(u) = \boldsymbol{\rho}'/s$; where $s = |\boldsymbol{\rho}'| = |\partial\boldsymbol{\rho}/\partial u|$;

$$\boldsymbol{e}(u,v) = \boldsymbol{v}\cos v + \boldsymbol{k}\sin v;$$
$$v = \frac{-\boldsymbol{h} + p\boldsymbol{n}}{\sqrt{1+p^2}};$$
$$\boldsymbol{n} = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u.$$

(2) Parametrical equations:

$$x = x(u,v) = be^{pu}\cos uu - \frac{R(u)}{\sqrt{1+p^2}} \times (p\sin u + \cos u)\cos v,$$
$$y = y(u,v) = be^{pu}\sin u - \frac{R(u)}{\sqrt{1+p^2}} \times (\sin u - p\cos u)\cos v,$$
$$z = z(u,v) = R(u)\sin v.$$

If we have a constant radius of a generatrix circle, i.e., R = const, then we shall obtain tubular surfaces with plane line of centers in the form of the logarithmic spiral (see also "Tubular Spiral Surface" in a Subsect. "17.2.1. Tubular Surfaces").

Coefficients of the fundamental forms of the surface:

$$A^2 = (1+p^2)\left(be^{pu} - \frac{R(u)\cos v}{\sqrt{1+p^2}}\right)^2 + R'^2(u),\ F = 0,\ B = R,$$

$$L = \frac{\sqrt{1+p^2}}{A}\left\{\left(bpe^{pu} - \frac{2R'(u)\cos v}{\sqrt{1+p^2}}\right)R'(u) - \left(be^{pu} - \frac{R(u)\cos v}{\sqrt{1+p^2}}\right) \times \left[\sqrt{1+p^2}\left(be^{pu} - \frac{R(u)\cos v}{\sqrt{1+p^2}}\right)\cos v + R''(u)\right]\right\}$$

$$M = \frac{R(u)R'(u)\sin v}{A},\ N = \frac{\sqrt{1+p^2}}{A}R(u)\left(be^{pu} - \frac{R(u)\cos v}{\sqrt{1+p^2}}\right),$$

where

$$R'(u) = \frac{\mathrm{d}R(u)}{\mathrm{d}u},\ R''(u) = \frac{\mathrm{d}^2R(u)}{\mathrm{d}u^2}$$

The generatrix circles coincide with coordinate lines v but they are not the lines of principle curvatures. In Fig. 1a, b, the surfaces with a law of changing of the radius of the generatrix circles written as $R = R(u) = au$ are shown. The surface shown in Fig. 2 has $R = R(u) = ae^{cu}$. In Fig. 3, the

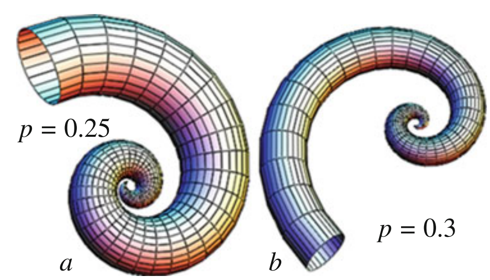
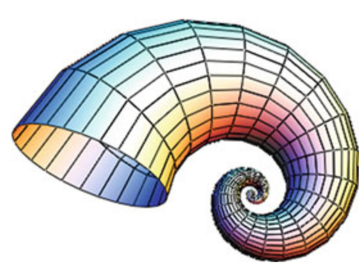
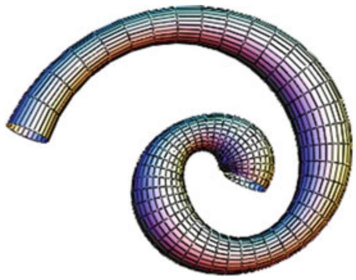


Fig. 1 $a = 0.5; b = 1;\ 0 \le u \le 4\pi$



$a = 0.5;\ b = 1;$
$0 \le u \le 4\pi;\ p = 0.3;\ c = 0.3$

Fig. 2



$a = 5$; $b = 1$; $3\pi \le u \le 6\pi$;
$p = 0.2$; $c = 0.25$

Fig. 3

surface with generatrix circles $R = R(u) = a(1 + c \sin tu)$ is presented. Assuming a function of changing of the radius of generatrix circles $R(u)$, one may construct the diversifying normal cyclic surfaces with the line of the centers in the form of a logarithmic spiral.

Reference

Ivanov VN. Shells with middle normal cyclic surfaces. Montazhn. i Spetz. Raboty v Stroitel'stve. 2006; No. 4, p. 37-39.

■ Normal Cyclic Surface with Generatrix Circle of Variable Radius and with a Plane Center-to-Center Line in the Form of a Conical Spiral

A line of the centers of a studied surface lies on a circular cone; it is a line of a constant slope and projected to a plane perpendicular to the axis of the cone as *a logarithmic spiral*. The surface relates to *normal cyclic surfaces*.

Form of definition of the surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \boldsymbol{\rho}(u) + R(u)\boldsymbol{e}(u, v),$$

where $\boldsymbol{\rho}$(u) is a radius vector of the conic spiral taken as the line of the centers of the generatrix circles with the law of changing of the radius as R = R(u);

$$\begin{aligned}\boldsymbol{\rho}(u) &= be^{pu}(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u + \lambda\boldsymbol{k})\\ &= be^{pu}[\boldsymbol{h}(u) + \lambda\boldsymbol{k}],\end{aligned}$$

$\boldsymbol{e}(u, v)$ is a vector function of the circle of the unit radius at the plane of a generatrix circle with the normal

$$\boldsymbol{t}(u) = \boldsymbol{\rho}'/s;$$

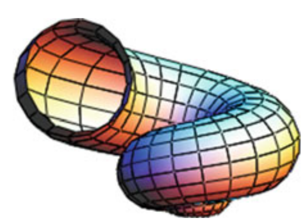
where

$$\begin{aligned}s &= |\boldsymbol{\rho}'| = |\partial\boldsymbol{\rho}/\partial u|;\\ \boldsymbol{e}(u, v) &= \boldsymbol{v}\cos\omega + \boldsymbol{\beta}\sin\omega;\\ \omega &= v - \frac{p\lambda u}{\sqrt{1 + (1 + \lambda^2)p^2}} = v - cu, \ c = \text{const},\\ \boldsymbol{v} &= \frac{-\boldsymbol{h} + p\boldsymbol{n}}{\sqrt{1 + p^2}} \ \boldsymbol{\beta} = \frac{-p\lambda(p\boldsymbol{h} + \boldsymbol{n}) + (1 + p^2)\boldsymbol{k}}{\sqrt{1 + (1 + \lambda^2)p^2}\sqrt{1 + p^2}},\\ \boldsymbol{n} &= -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u.\end{aligned}$$

(2) Parametrical equations:

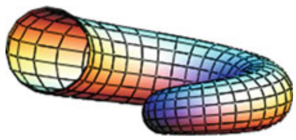
$$\begin{aligned}x = x(u, v) &= be^{pu}\cos u\\ &\quad - \frac{R(u)}{\sqrt{1 + p^2}}\left[(p\sin u + \cos u)\cos\omega - \frac{p\lambda(\sin u - p\cos u)}{\sqrt{1 + (1 + \lambda^2)p^2}}\sin\omega\right]\\ y = y(u, v) &= be^{pu}\sin u\\ &\quad - \frac{R(u)}{\sqrt{1 + p^2}}\left[(\sin u - p\cos u)\cos\omega + \frac{p\lambda(p\sin u + \cos u)}{\sqrt{1 + (1 + \lambda^2)p^2}}\sin\omega\right]\\ z &= z(u, v)\\ &= b\lambda e^{pu} + \frac{R(u)\sqrt{1 + p^2}\sin\omega}{\sqrt{1 + (1 + \lambda^2)p^2}}.\end{aligned}$$

In Figs. 1 and 2, the surfaces with the law of changing of a radius of the generatrix circles in the form $R = R(u) = au$ are shown. The surfaces shown in the Figs. 3 and 5 have



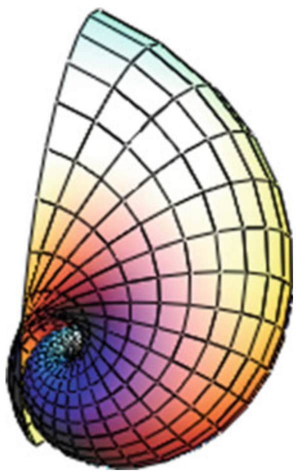
$a = 1$; $b = 1$; $p = 0.2$; $\lambda = 1$

Fig. 1 $2\pi \le u \le 6\pi$



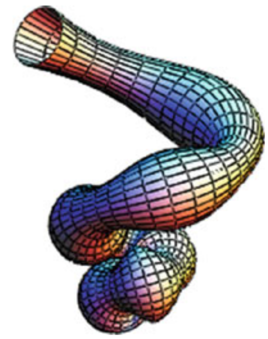
$a = 0.5;\ b = 1;$
$p = 0.25;\ \lambda = 0.5$

Fig. 2 $0 \le u \le 2\pi$



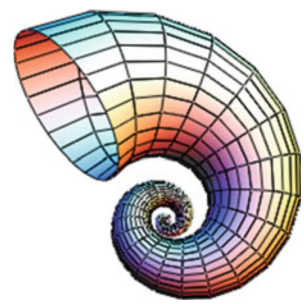
$a = 0.5;\ b = 1;\ \lambda = 0.5;$
$p = 0.3;\ c = 4;$

Fig. 3 $0 \le u \le 4\pi$



0.5; $a = 0.5;\ \lambda = 0.5;\ c = 0.28;$
$b = 1.5;\ p = 0.25$

Fig. 4 $0 \le u \le 6\pi$



$a = 10,\ \lambda = 2;\ \ b = 1{,}5;$
$p = 0.18;\ c = 0.3;\ t = 3$

Fig. 5 $2\pi \le u \le 6\pi$

$R = R(u) = ae^{cu}$. In Fig. 4, the surface with generatrix circles $R = R(u) = a(1 + c \sin tu)$ is presented. A surface of this type may be considered as *a waving surface* (Fig. 4).

Coefficients of the fundamental forms of surface and its curvatures may be obtained with the help of the formulas given in a Sect. "17.2. Normal Cyclic Surfaces".

Taking an arbitrary function of changing of a radius of a generatrix circle $R(u)$, it is possible to design various normal cyclic surfaces with a line of the centers of the generatrix circles in the form of a conic spiral.

If $\lambda = 0$, the surfaces degenerate into *normal cyclic surfaces with generatrix circles of variable radius and with a plane centerline in the form of a logarithmic spiral*. If a radius of the generatrix circle is constant (R = const), then we shall obtain a tubular surface.

Additional Literature

Ivanov VN. Shells with middle normal cyclic surfaces. Montazhn. i Spetz. Raboty v Stroitel'stve. 2006; No. 4, p. 37-39.

■ Connecting Canal for Two Cylindrical Surfaces with Parallel Axes

Connection of two pipelines of circular cross section with parallel axes is possible to fulfill with the help of a reducer in the form of *a connecting canal for two cylindrical surfaces with parallel axes*. In general case, a connecting canal is related to *normal cyclic surfaces with a generatrix circle of variable radius* and with a plane centerline in the form of a sinusoidal curve (Fig. 1):

$$z = \alpha, y = y(\alpha) = -d\left(\cos\frac{\pi\alpha}{2b} - 1\right)$$

Forms of the definition of surface of the connecting canal

(1) Parametrical equations (Figs. 1 and 2):

$$\begin{aligned} x &= x(\alpha,\beta) = r(\alpha)\cos\beta, \\ z &= \alpha - r(\alpha)\sin\beta\sin\varphi(\alpha), \\ y &= y(\alpha,\beta) = -d\left(\cos\frac{\pi\alpha}{2b} - 1\right) \\ &\quad + r(\alpha)\sin\beta\cos\varphi(\alpha), \end{aligned}$$

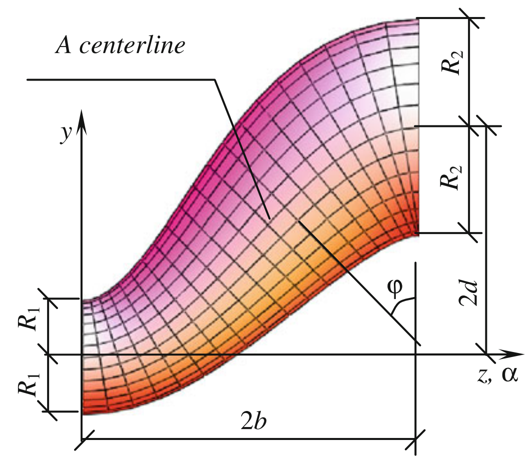


Fig. 1

where

$$r = r(\alpha) = \frac{(R_2 - R_1)}{2}\left(1 - \cos\frac{\pi\alpha}{2b}\right) + R_1$$

is the law of changing of the radius of the connecting canal,

$$\cos\phi(\alpha) = \frac{1}{\sqrt{1 + \frac{\pi^2 d^2}{4b^2}\sin^2\frac{\pi\alpha}{2b}}},$$
$$\mathrm{tg}\phi(\alpha) = \frac{\pi d}{2b}\sin\frac{\pi\alpha}{2b},$$

$\varphi(\alpha)$ is the angle of the tangent to the centerline of the cyclic surface with the coordinate axis Oz (Fig. 1); $2d$ is the distance between the parallel axes of the connected cylinders with radiuses R_1 and R_2; $R_2 \geq R_1$; $2b$ is the distance between the face planes of the connected cylinders; β is an angle in the plane of the generatrix circle read from the coordinate axis Ox; $0 \leq \alpha \leq 2b$; $0 \leq \beta \leq 2\pi$.

The coordinate lines β coincide with generatrix circles of the connecting canal of the variable radius.

In Fig. 2, the surface of the connecting canal is shown when

$$R_2 = 2R_1, d = 2R_1, b = 3R_1, 0 \leq \alpha \leq 2b;\ 0 \leq \beta \leq 2\pi.$$

Having assumed $R_1 = R_2$, we may construct a tubular surface with a plane sinusoidal line of centers (see also a Subsect. "17.2.1. Tubular Surfaces").

If $d = 0$ (Fig. 1), a surface of the connecting canal degenerates into *a surface* of *conjugation of two coaxial cylinders of different diameters* presented in a Chap. "2. Surfaces of Revolution."

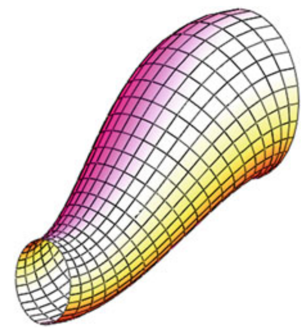
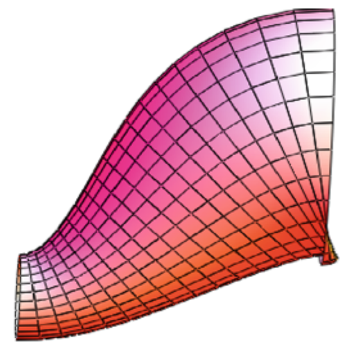
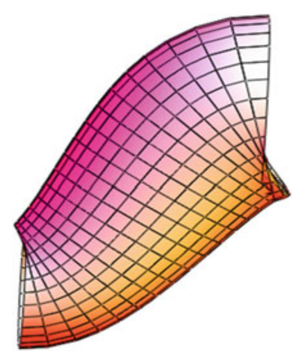


Fig. 2



$R_1 = 1$m; $R_2 = 3$ m;
$d = 2$ m; $b = 3$ m

Fig. 3



$R_1 = 2\ m$; $R_2 = 3\ m$;
$d = 3\ m$; $b = 3\ m$

Fig. 4

Designing a cyclic surface in question, we must pay attention to the ratios between geometrical parameters R_1, R_2, d, b, because folds can appear in face zones (Figs. 3 and 4).

Additional Literature

Gulyaev VI, Bazhenov VA, Gozulyak EA, Gaydaychuk VV. Analysis of shells of the complex form. Kiev: «Budivelnik», 1990; 190 p.

■ Normal Cyclic Surface with an Elliptical Line of Centers and with a Generatrix Circle of Variable Radius (the First Type)

A normal cyclic surface with an elliptical line of centers and with a generatrix circle of variable radius (*the first type*) may be used as a model surface of a connecting segment of two pipelines of different diameters with axes intersecting at right angle (Fig. 1). The considered cyclic surface has a plane elliptical line of centers given by parametrical equations

$$x = 0,\quad y = y(\alpha) = \rho(\alpha)\cos\alpha,\quad z = z(\alpha) = \rho(\alpha)\sin\alpha,$$

where

$$\rho(\alpha) = \frac{bd}{\sqrt{d^2\sin^2\alpha + b^2\cos^2\alpha}}$$

is a polar radius of an ellipse

$$\frac{y^2}{d^2} + \frac{z^2}{b^2} = 1.$$

Forms of definition of the cyclic surface

(1) Parametric form of assignment (Figs. 1 and 2):

$$X = R(\alpha)\sin\beta,$$
$$Y = y(\alpha) + R(\alpha)\cos\beta\cos\varphi(\alpha),$$
$$Z = z(\alpha) + R(\alpha)\cos\beta\sin\,\varphi(\alpha),$$

where an angle $\varphi(\alpha)$ is shown in Fig. 1;

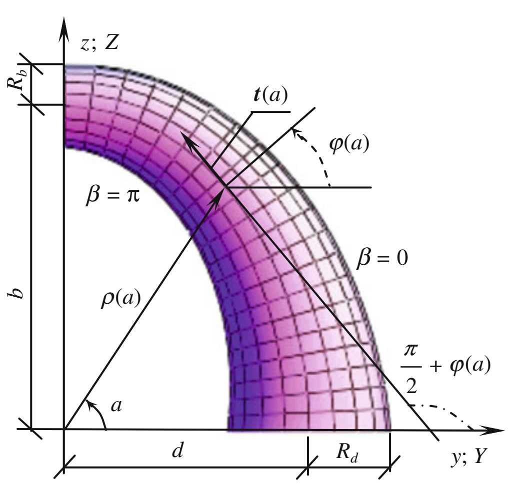


Fig. 1

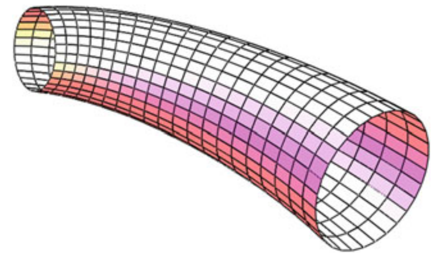


Fig. 2

$$R(\alpha) = 2\alpha(R_b - R_d)/\pi + R_d$$

is a linear law of changing of a radius of the generatrix circles; R_d, R_b are the radiuses of the generatrix circles at the initial ($\alpha = 0$) and finite ($\alpha = \pi/2$) position; $0 \le \beta \le 2\pi$; $0 \le \alpha \le \pi/2$.

A unit tangent vector $\boldsymbol{t}(\alpha)$ to the elliptical line of the canters

$$\boldsymbol{r} = \boldsymbol{r}(\alpha) = y(\alpha)\boldsymbol{j} + z(\alpha)\boldsymbol{k}$$

is given as

$$\boldsymbol{t}(\alpha) = \frac{\boldsymbol{r}'(\alpha)}{|\boldsymbol{r}'(\alpha)|} = \cos\left[\frac{\pi}{2} + \varphi(\alpha)\right]\boldsymbol{j} + \cos\varphi(\alpha)\boldsymbol{k}$$
$$= \frac{y'(\alpha)\boldsymbol{j}}{\sqrt{y'^2(\alpha) + z'^2(\alpha)}} + \frac{z'(\alpha)\boldsymbol{k}}{\sqrt{y'^2(\alpha) + z'^2(\alpha)}},$$
$$\ldots' = \frac{\partial\ldots}{\partial\alpha}$$

and parametrical equations of the cyclic surface may be written as:

$$X = X(\alpha,\beta) = R(\alpha)\sin\beta,$$
$$Y = Y(\alpha,\beta) = y(\alpha) + \frac{R(\alpha)\cos\beta\, z'(\alpha)}{\sqrt{y'^2(\alpha) + z'^2(\alpha)}},$$
$$Z = Z(\alpha,\beta) = z(\alpha) - \frac{R(\alpha)\cos\beta\, y'(\alpha)}{\sqrt{y'^2(\alpha) + z'^2(\alpha)}}.$$

Coefficients of the first fundamental forms of the surface:

$$A^2 = R'^2(\alpha) + \left[1 + R(\alpha)\cos\beta\frac{(y'z'' - z'y'')}{(y'^2 + z'^2)^{3/2}}\right]^2 \times \left(y'^2 + z'^2\right),$$
$$F = 0,\ B = R(\alpha).$$

The surface is given in the curvilinear orthogonal system of coordinates α, β. Coordinate lines β coincide with the generatrix circles.

We can obtain a *tubular surface with a plane elliptical line of centers of generatrix circles* taking $R(\alpha)$ = const.

The parametrical equations of the cyclic surface in question may be written at the detailed form:

$$X = X(\alpha, \beta) = R(\alpha)\sin\beta,$$
$$Y = Y(\alpha, \beta) = \frac{bd\cos\alpha}{\sqrt{d^2\sin^2\alpha + b^2\cos^2\alpha}} + \frac{b^2R(\alpha)\cos\beta\cos\alpha}{\sqrt{d^4\sin^2\alpha + b^4\cos^2\alpha}},$$
$$Z = Z(\alpha, \beta) = \frac{bd\sin\alpha}{\sqrt{d^2\sin^2\alpha + b^2\cos^2\alpha}} + \frac{d^2R(\alpha)\cos\beta\sin\alpha}{\sqrt{d^4\sin^2\alpha + b^4\cos^2\alpha}}.$$

Additional Literature

Gulyaev VI, Bazhenov VA, Gozulyak EA, Gaydaychuk VV. Analysis of shells of the complex form. Kiev: "Budivelnik", 1990; 190 p.

■ Normal Cyclic Surface with an Elliptical Line of Centers and with a Generatrix Circle of Variable Radius (the Second Type)

A normal cyclic surface with an elliptical line of centers and with a generatrix circle of variable radius (*the second type*) has a plane elliptical line of centers given by parametrical equations

$$x = 0, y = y(u) = d\cos u,$$
$$z = z(u) = b\sin u,$$

$0 \le u \le \pi/2$, where b, d are the semi-axes of ellipse

$$\frac{y^2}{d^2} + \frac{z^2}{b^2} = 1.$$

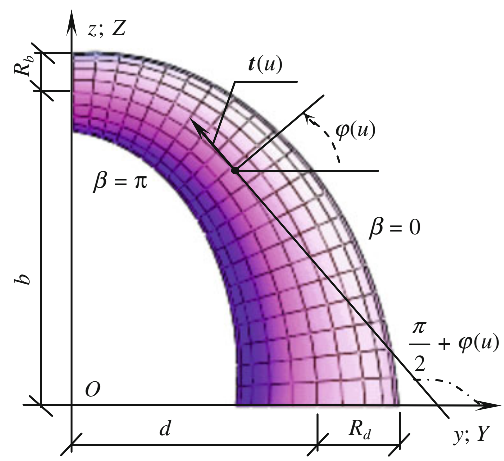


Fig. 1

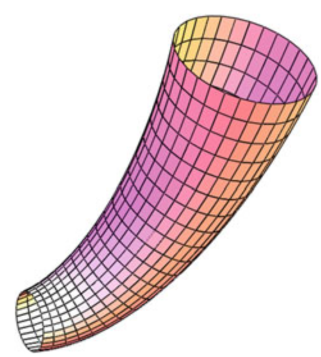


Fig. 2

Forms of definition of the cycle surface

(1) Parametric form of assignment (Figs. 1 and 2):

$$X = X(u, \beta) = R(u)\sin\beta,$$
$$Y = Y(u, \beta) = y(u) + R(u)\cos\beta\cos\varphi(u),$$
$$Z = Z(u, \beta) = z(u) + R(u)\cos\beta\sin\varphi(u),$$

where an angle φ(u) is shown in Fig. 1;

$$R(u) = 2u(R_b - R_d)/\pi + R_d$$

is a linear law of changing of a radius of the generatrix circles; R_d, R_b are the radiuses of the generatrix circles in the initial (u = 0) and finite (u = π/2) position (Fig. 1); $0 \le \beta \le 2\pi$. Considering that a unit tangent vector t(u) to the elliptical line

$$\boldsymbol{r} = \boldsymbol{r}(u) = y(u)\boldsymbol{j} + z(u)\boldsymbol{k}$$

is given by the equation

$$\boldsymbol{t}(u) = \frac{\boldsymbol{r}'(u)}{|\boldsymbol{r}'(u)|} = \cos\left[\frac{\pi}{2} + \varphi(u)\right]\boldsymbol{j} + \cos\varphi(u)\boldsymbol{k} = \frac{y'(u)\boldsymbol{j}}{\sqrt{y'^2(u) + z'^2(u)}} + \frac{z'(u)\boldsymbol{k}}{\sqrt{y'^2(u) + z'^2(u)}},$$
$$\ldots' = \frac{\partial\ldots}{\partial u}$$

parametric equations of the cyclic surface may be written as:

$$
\begin{aligned}
X &= X(u,\beta) = R(u)\sin\beta,\\
Y &= Y(u,\beta) = y(u) + \frac{R(u)\cos\beta\, z'(u)}{\sqrt{y'^2(u)+z'^2(u)}},\\
Z &= Z(u,\beta) = z(u) - \frac{R(u)\cos\beta\, y'(u)}{\sqrt{y'^2(u)+z'^2(u)}}.
\end{aligned}
$$

Coefficients of the first fundamental forms of the surface:

$$
\begin{aligned}
A^2 &= \left(\frac{\partial R(u)}{\partial u}\right)^2 + \left[1 + \frac{dbR(u)\cos\beta}{(y'^2+z'^2)^{3/2}}\right]^2 \left(y'^2 + z'^2\right),\\
F &= 0, \quad B = R(u).
\end{aligned}
$$

The surface is given in the curvilinear orthogonal system of coordinates α, β. Coordinate lines β coincide with the generatrix circles.

We can obtain a *tubular surface with a plane elliptical line of centers of generatrix circles* taking $R(u) = R = \text{const}$.

The parametrical equations of the cyclic surface in question presented above may be written at the detailed form:

$$
\begin{aligned}
X &= X(u,\beta) = R(u)\sin\beta,\\
Y &= Y(u,\beta) = d\cos u + \frac{bR(u)\cos\beta\cos u}{\sqrt{d^2\sin^2 u + b^2\cos^2 u}},\\
Z &= Z(u,\beta) = b\sin u + \frac{d\cdot R(u)\cos\beta\sin u}{\sqrt{d^2\sin^2 u + b^2\cos^2 u}}.
\end{aligned}
$$

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17.3 Cyclic Surfaces with a Plane of Parallelism

Cyclic surface with a plane of parallelism is created by motion of a generatrix circle of variable or constant radius and the planes of the circles are parallel to any plane that is called *a plane of parallelism*.

A vector equation of *a cyclic surface* is

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = \boldsymbol{\rho}(u) + R(u)\boldsymbol{e}(u,v),$$

where $\boldsymbol{r}(u, v)$ is a radius vector of a cycle surface; $\boldsymbol{\rho}(u)$ is a radius vector of a directrix curve that is called *a centerline of generatrix circles*; $R(u)$ is a law of changing of a radius of the generatrix circles; $\boldsymbol{e}(u, v)$ is a vector function of the unit radius at the plane of a generatrix circle with the normal $\boldsymbol{n}$ = const (see also Fig. 2, p. 370 in this chapter); $\boldsymbol{e}_0(u)$, $\boldsymbol{g}_0(u)$ are the unit vectors of the orthogonal system of Cartesian coordinates lying at the plane of a generatrix circle; v is a polar angle at the plane of a generatrix circle.

As n = const, then

$$\frac{d\boldsymbol{n}}{du} = \frac{d^2\boldsymbol{n}}{du^2} = 0,$$

hence, formulas for the determination of the coefficients of the fundamental forms of surface given in this chapter simplify.

Coefficients of the fundamental forms of surface for a cyclic surface with a plane of parallelism:

$$
\begin{aligned}
E &= A^2 = s^2 + 2s\left[(\boldsymbol{te})R' + (\boldsymbol{tg})R(u)(\boldsymbol{e}_0'\boldsymbol{g}_0)\right]\\
&\quad + R'^2 + R^2(u)(\boldsymbol{e}_0'\boldsymbol{g}_0)^2,\\
F &= R\left[s(\boldsymbol{tg}) + R(u)(\boldsymbol{e}_0'\boldsymbol{g}_0)\right],\ G = B^2 = R^2,\\
\sigma &= \sqrt{A^2B^2 - F^2}\Big/R(u) = \sqrt{\left[s(\boldsymbol{te}) + R'\right]^2 + s^2(\boldsymbol{tn})^2},\\
s &= |\boldsymbol{\rho}'|,\\
L &= \frac{1}{\sigma}\left\{\left[(\boldsymbol{te})s + R'(u)\right]\left[s'(\boldsymbol{tn}) + s^2k(\boldsymbol{nv})\right]\right.\\
&\quad \left. - s(\boldsymbol{tn})\left[s'(\boldsymbol{te}) + s^2k(\boldsymbol{ev}) - R(u)(\boldsymbol{e}_o'\boldsymbol{g}_o)^2 + R'(u)\right]\right\}
\end{aligned}
$$

where $\boldsymbol{t} = \boldsymbol{\rho}'/s$; $\boldsymbol{t}$, $\boldsymbol{v}$ are the unit vectors of the tangent and normal to the line of the centers.

The choice of the vector function $\boldsymbol{e}_0(u)$ does not have influence on the form of a cyclic surface in question, therefore it is possible to assume

$$(\boldsymbol{e}_o'\boldsymbol{g}_o) = 0.$$

Then $M = 0$ and we shall have a conjugate coordinate set u, v. Having

$$\boldsymbol{e}'_o = (\boldsymbol{e}'_o \boldsymbol{g}_o)\boldsymbol{g} - (\boldsymbol{e}_o \boldsymbol{n}')\boldsymbol{n} = \boldsymbol{0},$$

it is possible to prove that $\boldsymbol{e}_o = \text{const}$, $\boldsymbol{g}_o = \boldsymbol{n} \times \boldsymbol{e}_o = \text{const}$. So, the vectors $\boldsymbol{e}_o$, $\boldsymbol{g}_o$, $\boldsymbol{n}$ do not change when they move along the centerline and it is possible to assume that

$$\boldsymbol{g}_o = \boldsymbol{i}, \boldsymbol{n} = \boldsymbol{j}, \boldsymbol{e}_o = \boldsymbol{k},$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors of the Cartesian coordinate system. At this case, formulas for the determination of the coefficients of the fundamental forms of the surface will be:

$$\begin{aligned}
&E = A^2 = s^2 + 2\,s(\boldsymbol{te})R' + R'^2,\\
&F = Rs(\boldsymbol{tg}),\; G = B^2 = R^2,\\
&\sigma = \frac{\sqrt{A^2B^2 - F^2}}{R(u)} = \sqrt{[s(\boldsymbol{te}) + R']^2 + s^2(\boldsymbol{tj})^2},\\
&s = |\boldsymbol{\rho}'|,\\
&L = \frac{1}{\sigma}\{[(\boldsymbol{te})s + R'][s'(\boldsymbol{tj}) + s^2k(\boldsymbol{jv})]\\
&\qquad -s(\boldsymbol{tj})[s'(\boldsymbol{te}) + s^2k(\boldsymbol{ev}) + R']\},\\
&M = 0,\; N = \frac{R(u)s}{\sigma}(\boldsymbol{tj}).
\end{aligned}$$

Two special types of *cyclic surfaces with a plane of parallelism* are the most known. These are

(a) when a centerline is a straight line and the vector $\boldsymbol{n}$ is parallel to this straight line, then we obtain *a surface of revolution*;
(b) when a radius of a generatrix circle is constant, that is R = const, then we obtain *a cyclic surface of translation*.

Grigorenko et al. (1983) has offered for application a cyclic surface with a plane of parallelism xOz having a generatrix circle with a radius $R(u)$ and a plane sinusoidal centerline given in the form: $\boldsymbol{r}(u) = u\boldsymbol{i} + d[\cos(\pi u/2b) - 1]\,\boldsymbol{k}$. So, this surface may be defined by parametrical equations:

$$\begin{aligned}
x(u,v) = R(u)\sin v, y(u) = u, z(u,v)\\
= d[\cos(\pi u/2b) - 1] + R(u)\cos v,
\end{aligned}$$

where d is an amplitude of the plane sinusoidal centerline.

Additional Literature

Rekach VG, *Krivoshapko SN*. Analysis of Shells of Complex Geometry. Moscow: Izd-vo UDN, 1988; 176 p. (71 refs.).

Grigorenko YaM, *Gulyaev VI, Gozulyak EA*, *Ashuri K*. Stress-strain state of tubular shells subjected to action of uniform pressure. Applied Mechanics (Kiev). 1983; No. 8, Vol. 1, p. 11-18.

Shulikovskiy VI. Classical Differential Geometry. Moscow: Fizmatizdat, 1963; 540 p.

■ Right Circular Spiral Surface

A right circular spiral surface (Fig. 1) is formed by the motion of a circle of constant radius lying in a plane that is perpendicular to the axis of a spiral line along this spiral line

$$\begin{aligned}
x &= x(u) = e^{ku} r_o \sin\lambda \cos u,\\
y &= y(u) = e^{ku} r_o \sin\lambda \sin u,\\
z &= z(u) = e^{ku} r_o \cos\lambda,
\end{aligned}$$

remaining parallel to the plane of parallelism. It follows from these formulas that r_o is a constant value, k is some positive or negative constant number, λ is the angle of the axis Oz (the axis of the spiral) with the generatrix straight line of the cone with the conical spiral lying on it, u is the angel of the plane xOz with the moving plane of the axial cross section.

The conical spiral is the centerline of the generatrix circles of the constant radius (see also this chapter). Every plane z = const may be taken as plane of parallelism of the considered surface.

Right circular spiral surface may be put simultaneously into a class of *cyclic surfaces* (see also a Sect. “17.3. Cyclic Surfaces with a Plane of Parallelism”) and into a class of *spiral surfaces* (see also a Chap. “8. Spiral Surfaces”).

Form of definition of a right circular spiral surface

(1) Parametrical equations (Fig. 1a, b):

$$\begin{aligned}
x &= x(u,v) = e^{ku} r_o \sin\lambda \cos u + R\cos v,\\
y &= y(u,v) = e^{ku} r_o \sin\lambda \sin u + R\sin v\\
z &= z(u) = e^{ku} r_o \cos\lambda
\end{aligned}$$

where R is a constant radius of a generatrix circle; v is the angle read from the axis Ox in the direction of the axis Oy. This parameter defines the position of a point on the generatrix circle; $0 \le v \le 2\pi$.

In the cross sections of the surface by the planes z = const, the coordinate lines v lie and coincide with generatrix circles of the constant radius R.

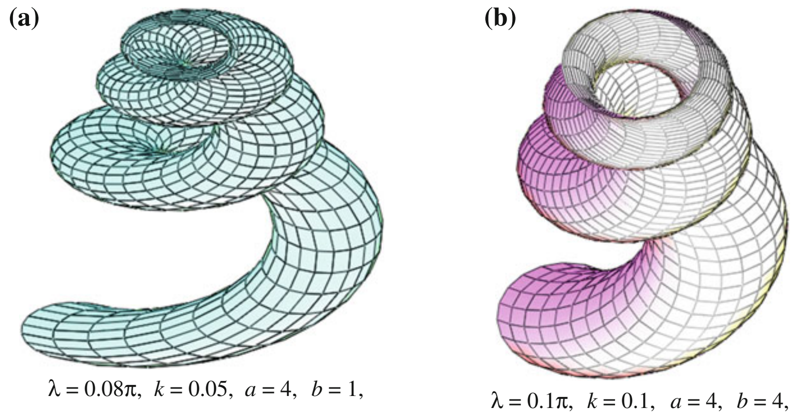

Fig. 1 $0 \le u \le 5.5\pi$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = r_o^2 e^{2ku}(k^2 + \sin^2\lambda),$$
$$F = r_o R \sin\lambda \cdot e^{ku}[\sin v(\sin u - k\cos u) + \cos v(\cos u + k\sin u)],$$
$$B = r;$$
$$L = \frac{kr_o\cos\lambda}{\sqrt{A^2B^2 - F^2}}e^{ku}F,$$
$$M = 0,\ N = \frac{kr_oR^2\cos\lambda}{\sqrt{A^2B^2 - F^2}}e^{ku};$$
$$k_u = \frac{kF\cos\lambda}{A\sqrt{A^2B^2 - F^2}\sqrt{k^2 + \sin^2\lambda}},$$
$$k_v = \frac{kr_o\cos\lambda}{\sqrt{A^2B^2 - F^2}}e^{ku},$$
$$K = \frac{k^2r_o^2FR^2\cos^2\lambda}{\left(A^2B^2 - F^2\right)^2}e^{2ku}.$$

The right circular spiral surface is given in nonorthogonal conjugate curvilinear coordinates u, v. The surface contains the segments of the positive and negative Gaussian curvature K.

The sign of Gaussian curvature of the surface depends on the sign of the coefficient F of the first fundamental form of surface. This coefficient F comes into the formula for the determination of Gaussian curvature of the surface.

The fragments of a spiral surface with different signs of Gaussian curvature are separated by the line

$$v = v(u) = \arctan\left(\frac{\cos u + k\sin u}{k\cos u - \sin u}\right),$$

along which Gaussian curvature is equal to zero ($K = 0$).

Additional Literature

Potapov YuS, Fominskiy LP, Potapov SYu. Energy of Rotation. Kishinev. 2001; 400 p.
Rekach VG, Krivoshapko SN. Analysis of Shells of Complex Geometry. Moscow: Izd-vo UDN, 1988; 176 p. (71 refs.).
Krivoshapko SN, Ivanov VN. Geometry, Analysis and Design of Structures in the form of cyclic surfaces: Review information, Ser. "Stroit. Materialy i Konstruktzii", Moscow. 2010; OAO VNIINTPI, Iss. 2, 61 p.

■ Surface of Translation of a Circle Along a Sinusoid

A surface of translation of a circle along a sinusoid is formed by parallel translation of a circle on condition that its definite point slides above a sinusoid. The directrix sinusoid and the generatrix circle lie at the mutual orthogonal planes. A surface of translation of a circle along a sinusoid may be also formed if one takes the sinusoid

$$x = x(z) = a\sin(n\pi z/b)$$

as the plane *line of the centers* of generatrix circles of the constant radius. The surface in question may be attached both to a class of *translation surfaces* (see also a Chap. "3. Translation Surfaces") and to a class of *cyclic surfaces* (see also a Sect. "17.3. Cyclic Surface with a Plane of Parallelism").

Form of definition of a surface of translation of a circle along a sinusoid

(1) Parametric form of definition (Fig. 1):

$$x = x(z, v) = a \sin \frac{n\pi z}{b} + r \cos v,$$
$$y = y(v) = r \sin v,$$
$$z = z,$$

where n is the number of the whole half-waves containing on the section with the length of b; r is a constant radius of the generatrix circles.

The coordinate lines v coincide with the generatrix circles, $0 \le v \le 2\pi$. The generatrix circles lie at the cross sections of the surface by the planes z = const.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{a^2 n^2 \pi^2}{b^2} \cos^2 \frac{n\pi z}{b},$$
$$F = -ar\frac{n\pi}{b} \sin v \cos \frac{n\pi z}{b},\ B = r,$$
$$A^2B^2 - F^2 = \frac{r^2}{b^2}\left[b^2 + a^2n^2\pi^2 \cos^2 v \cos^2 \frac{n\pi z}{b}\right],$$
$$L = \frac{an^2\pi^2 \cos v}{b\sqrt{b^2 + a^2n^2\pi^2 \cos^2 v \cos^2(n\pi z/b)}} \sin \frac{n\pi z}{b},$$
$$M = 0,\ N = \frac{rb}{\sqrt{b^2 + a^2n^2\pi^2 \cos^2 v \cos^2(n\pi z/b)}},$$
$$K = \frac{an^2\pi^2 b^2 \cos v}{r[b^2 + a^2n^2\pi^2 \cos^2 v \cos^2(n\pi z/b)]^2} \sin \frac{n\pi z}{b}.$$

A surface of translation of a circle over a sinusoid is related to a system of nonorthogonal conjugate curvilinear coordinates u, v. The circles lying at the cross sections of the surface by the planes $z = bi/n$ (i = 0; 1; .., n) are the lines of principle curvatures. The sinusoids $v = 0$ and $v = \pi$ also coincide with the lines of the principle curvatures. The surface contains the segments of positive and negative Gaussian curvature K, which are separated by the plane coordinate lines $v = \pi/2$ and $v = 3\pi/2$ containing parabolic points with $K = 0$.

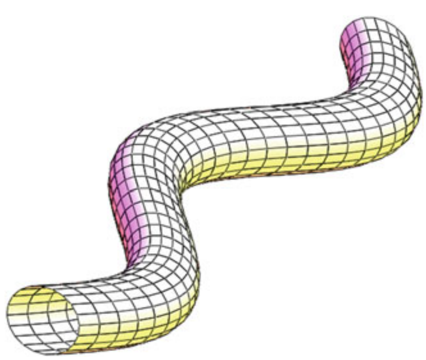

Fig. 1

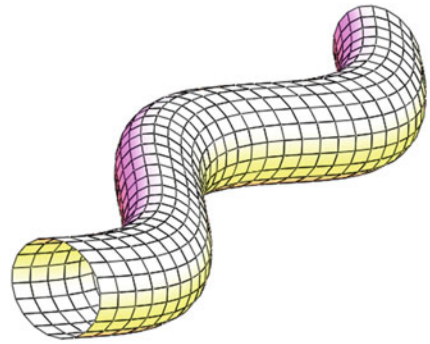

Fig. 2

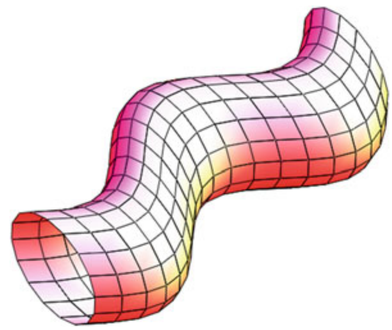

Fig. 3

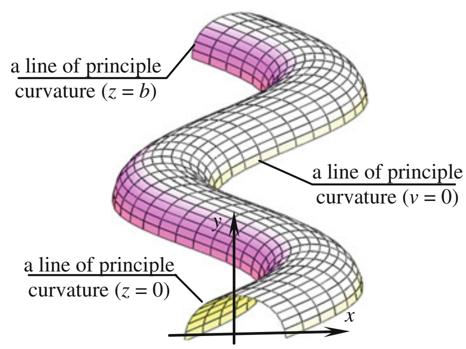

Fig. 4

If a = 0, then the surface degenerates into a *cylindrical surface of revolution* (see also a Subsect. "1.1.2. Cylindrical Surfaces").

In Fig. 1, the surface of translation with $n = 3$, $r < a$, $0 \le v \le 2\pi$, $0 \le z \le b$ is shown. The surface of translation with $n = 3$, $r = a$, $0 \le v \le 2\pi$, $0 \le z \le b$ is presented in Fig. 2 but with $n = 3$, $r > a$, $0 \le v \le 2\pi$, $0 \le z \le b$ is shown in Fig. 3. In Fig. 4, the surface of translation has $n = 3$, $r < a$, $0 \le v \le \pi$, $0 \le z \le b$.

Additional Literature

Rekach VG, *Krivoshapko SN*. Analysis of Shells of Complex Geometry. Moscow: Izd-vo UDN, 1988; 176 p. (71 refs.).

■ Surface of Translation of a Circle Along an Elliptical Centerline

A surface of translation of a circle along an elliptical centerline can be formed, if an ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is taken as *a plane line of the centers* of generatrix circles of a constant radius a and a coordinate plane xOy is assumed as a plane of parallelism (Fig. 1). This translation surface is a special case of *an elliptical surface of translation*. The considered surface may be related both to a class of *surfaces of translation* (see also a Sect. "3.1. Surfaces of Right Translation") and to a class of *cyclic surfaces* (see also a Sect. "17.3. Cyclic Surfaces with a Plane of Parallelism").

If $b > c$ and $a < b$, we obtain a surface shown in Fig. 1a. A surface of translation having $b = c$ and $a < b$ is presented in Fig. 1b, i.e., in this case, we have a circular centerline.

If $a = b = c$, then we have a surface of translation of a circle along the same circular centerline (Fig. 2), the fragment of this surface is called "Bohemian dome" (Fig. 3).

Forms of definition of the surface in question

(1) Parametrical equations (Figs. 1, 2 and 3):

$$x=x(u)=a\cos u,$$
$$y=y(u,v)=b\cos v+a\sin u,$$
$$z=z(v)=c\sin v,$$

where a is a constant radius of generatrix circles, b and c are the semi-axes of the ellipse (centerline of generatrix circles);

$0\le u\le 2\pi;\ 0\le v\le 2\pi;\ -a\le x\le a;\ -c\le z\le c.$

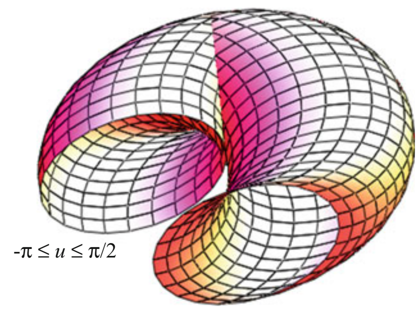

Fig. 2

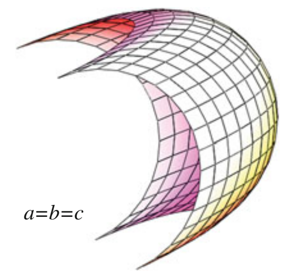

Fig. 3

Coefficients of the fundamental forms of the surface:

$$A=a,\quad F=-ab\sin v\cos u,$$
$$B^2=b^2\sin^2 v+c^2\cos^2 v,$$
$$A^2B^2-F^2=a^2\left(b^2\sin^2 v\sin^2 u+c^2\cos^2 v\right),$$
$$L=\frac{-ac\cos v}{\sqrt{b^2\sin^2 v\sin^2 u+c^2\cos^2 v}},$$
$$M=0,\ N=\frac{-bc\sin u}{\sqrt{b^2\sin^2 v\sin^2 u+c^2\cos^2 v}},$$
$$K=\frac{bc^2\sin u\cos v}{a\left(b^2\sin^2 v\sin^2 u+c^2\cos^2 v\right)^2}.$$

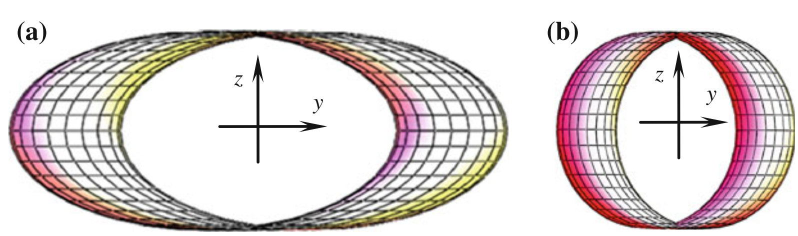

Fig. 1

The surface of translation contains segments of positive and negative Gaussian curvature, which are separated by the plane coordinate lines $u = 0$ and $u = \pi$ on which parabolic points are located. Bohemian dome, shown at Fig. 3, are designed with the following geometric parameters:

$$a = b = c, -\pi/2 \leq u \leq \pi/2;\ 0 \leq v \leq \pi.$$

(2) Explicit equation:

$$y = b\sqrt{c^2 - z^2}\Big/c + \sqrt{a^2 - x^2}.$$

An explicit form of definition shows that a considered cyclic surface belongs to surfaces of translation; $-a \leq x \leq a; -c \leq z \leq c$.

(3) Implicit equation:

$$\begin{aligned}&\left(x^2 + y^2 + b^2z^2/c^2 - b^2 - a^2\right)^2\\ &= 4b^2(c^2 - z^2)(a^2 - x^2)/c^2.\end{aligned}$$

So, the surface of a right translation of a circle along elliptical centerline is an algebraic surface or the fourth other (see also a Chap. "36. Algebraic Surface of the High Orders").

Additional Literature

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Kornienko AV. Sections of translation surfaces of one type by one-parametric family of vertical planes. Prikl. Geo mi Imgenern. Grafika. Kiev. 1972; Iss. 15, p. 49-50.

Shulikovskiy VI. Classical Differential Geometry. Moscow: Fizmatizdat, 1963; 540 p.

■ Surface of a Helical Pole

A surface of a helical pole is a special case of a *right circular helical surface* (see also a Subsect. "7.1.2. Circular Helical Surfaces"). This surface may be included both to a class of *cyclic surfaces* and to a class of *helical surfaces*. Parametrical equations of a helical line of centers of generatrix circles of a constant radius r are:

$$\begin{aligned}x &= x(v) = a\cos v,\\ y &= y(v) = a\sin v,\\ z &= z(v) = pv.\end{aligned}$$

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$\begin{aligned}x &= x(\vartheta, v) = a\cos v + r\cos(\vartheta + v),\\ y &= y(\vartheta, v) = a\sin v + r\sin(\vartheta + v),\\ z &= z(v) = pv,\end{aligned}$$

where ϑ is the central angle of a generatrix circle; $0 \leq \vartheta \leq 2\pi$. A surface of a helical pole is a right circular helical surface with a $\leq$ r. In Fig. 1, the surface of helical pole with a = r = p = 2 m is shown. The surface of helical pole having two generatrix circles is represented in Fig. 2.

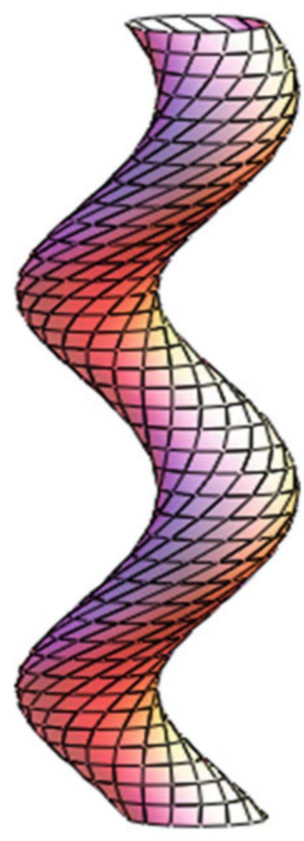


Fig. 1

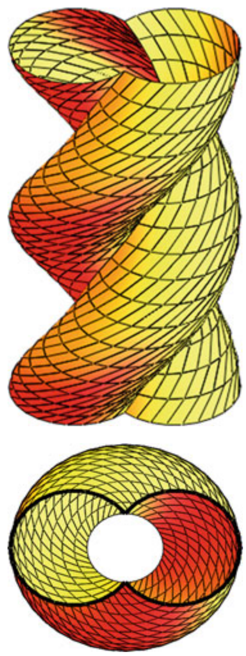


Fig. 2

Additional Literature

Bubennikov AV. Descriptive Geometry. Lectures 25-28. Educational television, Moscow: VZPI, 1966; 62 p.

■ Right Circular Spiral-Shaped Surface with a Generatrix Circle of Variable Radius

*A r*ight circular spiral-shaped surface with a generatrix circle of variable radius is formed by circles of variable radius lying in parallel planes. A line of centers of generatrix circles coincides with a conic spiral:

$$x = x(u) = e^{ku} r_o \sin\lambda \cos u,$$
$$y = y(u) = e^{ku} r_o \sin\lambda \sin u,$$
$$z = z(u) = e^{ku} r_o \cos\lambda$$

where r_o is a constant value, k is any positive or negative constant number; λ is the angle of the axis Oz (axis of a cone) with the generatrix straight line of the cone with the conic spiral disposed on it.

The surface may be included both into a class of *cyclic surfaces* and into a class of *spiral surfaces*.

Form of definition of the right circular spiral-shaped surface

(1) Parametrical equation:

$$x = x(u, v) = e^{ku} r_o \sin\lambda \cos u + R\cos v,$$
$$y = y(u, v) = e^{ku} r_o \sin\lambda \sin u + R\sin v$$
$$z = z(u) = e^{ku} r_o \cos\lambda$$

where R = R(u) is a radius of the generatrix circles; v is an angle read from the axis Ox in the direction of the axis Oy.

The parameter v defines the location of the point on the generatrix circle; $0 \le v \le 2\pi$. In the sections of the surface by the planes z = const, the coordinate lines v lie, which coincide with the generatrix circles.

The surface with

$$R(u) = u/2$$

is given in Fig. 1.

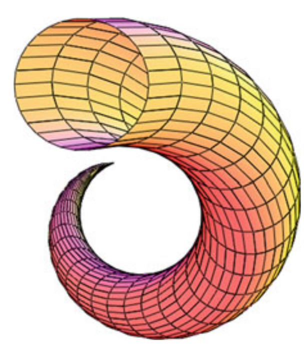
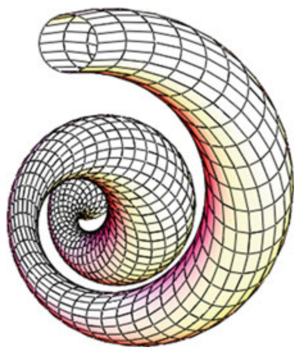


Fig. 1 .



$a = 2$ m, $b = 1$ m

Fig. 2 .

Coefficients of the fundamental forms of the surface:

$$A^2 = r_o^2 e^{2ku}(k^2 + \sin^2\lambda) + 2r_o \sin\lambda e^{ku} R'[(k\cos u - \sin u)\cos v + (k\sin u + \cos u)\sin v] + R'^2,$$
$$F = r_o \sin\lambda \cdot e^{ku} R[(k\sin u + \cos u)\cos v - (k\cos u - \sin u)\sin v],$$
$$L = \frac{r_o R k e^{ku}\cos\lambda}{\sqrt{A^2B^2 - F^2}}\left(\frac{F}{R} + kR' - R''\right),$$
$$M = 0,$$
$$N = \frac{r_o R^2 k e^{ku}\cos\lambda}{\sqrt{A^2B^2 - F^2}}.$$

In Fig. 2, the cyclic surface with the sinusoidal law of change of the radius of the generatrix circle

$$R(u) = a - b\sin u$$

is presented, $0 \le u \le 4\pi$.

■ Right Circular Surface on a Cylinder

Assume that the center of a circle of a constant radius r moves on the plane xOy along fixed circle of a constant radius R and at the same time, moves in the direction of the axis Oz (Fig. 1).

A cyclic surface with a plane of parallelism xOy formed so is called *a right circular surface on a cylinder.*

Form of definition of the surface

(1) Parametric form of definition:

$$x = x(t, \beta) = R\sin\alpha + r\sin\beta = R\sin(c + b\sin t) + r\sin\beta,$$
$$y = y(t, \beta) = R\cos\alpha + r\cos\beta = R\cos(c + b\sin t) + r\cos\beta,$$
$$z = z(t) = at,$$
$$\alpha = c + b\sin t,$$

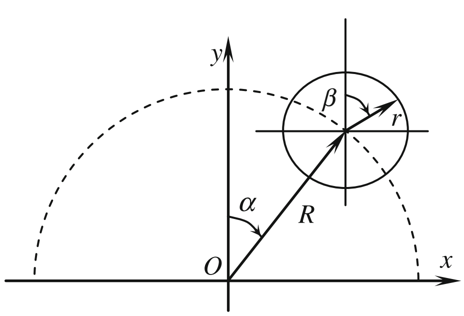


Fig. 1

where α is an angle changing under a given law (Fig. 1); β is the central angle of the generatrix circle, read from the axis Oy in the direction of the axis Ox, $0 \le \beta \le 2\pi$; t is a variable parameter; a is a constant determining a length of the cyclic surface in the direction of the axis Oz; b is an amplitude of the sinusoid due to the law of forming of which, the angle α changes (Fig. 2); c is a constant defining the position of the sinusoid $\alpha = \alpha(t) = c + b \sin t$ in the direction of the axis α.

In Fig. 3, the cyclic surface with the plane of parallelism xOy having

$$R = 3\text{ m};\ r = 1\text{ m};\ 0 \le \beta \le 2\pi;\ 0 \le t \le 3\pi$$

is shown.

Let the angle α changes at the interval $-\pi/4 \le \alpha \le \pi/4$, hence $c = 0$; $b = \pi/4$. If the length of the surface in the direction of the axis Oz equals 6 m, then

$$z = at = a3\pi = 6 \text{ or } a = 2/\pi.$$

In Fig. 4, the cyclic surface has the same geometric parameters $R = 3$ m; $r = 1$ m, but $-\pi/2 \le \beta \le \pi/2$; $0 \le \alpha \le \pi/2$, i.e., $c = \pi/4$; $b = \pi/4$. Assume $\Delta z = 6$ m, then taking into account the condition that $3\pi/2 \le t \le 9\pi/2$ one can obtain:

$$a3\pi = 6\text{ m or } a = 2/\pi.$$

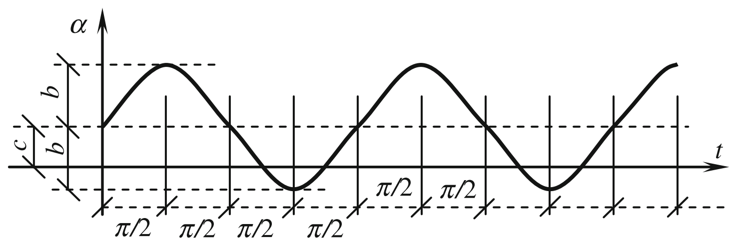


Fig. 2

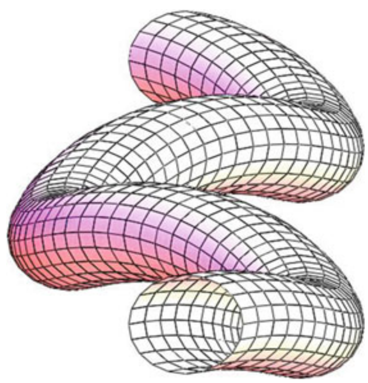


Fig. 3

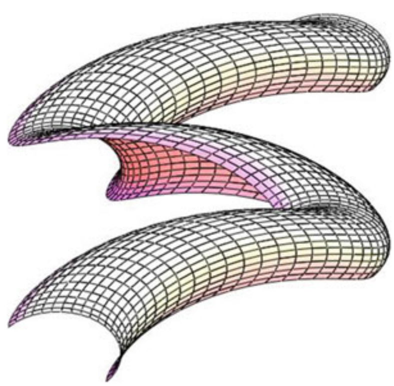


Fig. 4

Let us consider one more variant of the cyclic surface (Fig. 5) with

$$R = 3\,\mathrm{m}; r = 1\,\mathrm{m};$$
$$\pi/2 \le \beta \le 3\pi/2;\ -\pi/2 \le \alpha \le \pi/2,$$

therefore $c = 0$; $b = \pi/2$. Assume $3\pi/2 \le t \le 11\pi/2$, then having $\Delta z = 6$ m we obtain $a = 3/(2\pi)$.

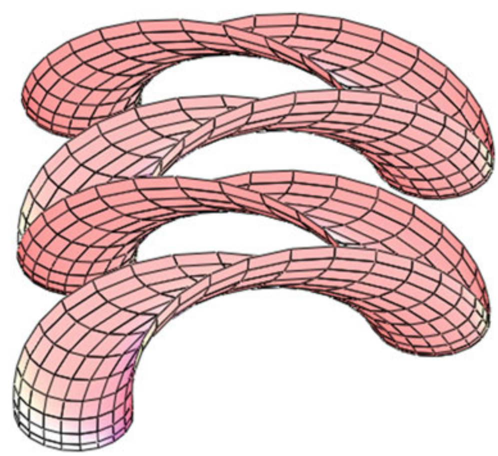


Fig. 5

Coefficients of the fundamental forms of the surface:

$$A^2 = b^2R^2\cos^2 t + a^2,$$
$$F = rbR\cos t\cos(c + b\sin t - \beta),\ B = r;$$
$$L = \frac{arbR\{\sin t\sin(c + b\sin t - \beta) - b\cos^2 t\cos(c + b\sin t - \beta)}{\sqrt{A^2B^2 - F^2}},$$
$$M = 0,\ N = -\frac{ar^2}{\sqrt{A^2B^2 - F^2}}.$$

Coordinate lines β coincide with the generatrix circles of constant radius r

$$(x - R\sin\alpha)^2 + (y - R\cos\alpha)^2 = r^2,$$

the centers of which lie on the circle of the radius R (Fig. 1).

17.4 Cyclic Surfaces with Circles in Planes of Pencil

Cyclic surfaces with circles in planes of pencil is formed by circles of constant of variable radius, the center of which moves along a given line of the centers of the circles and the generatrix circles lie at the planes of pencil. *Helical cyclic surfaces, canal surfaces of Joachimsthal*, *Dupin cyclides* belong to this class of surfaces. A vector equation of a cyclic surface with circles in the planes of pencil has the following form (Fig. 1):

$$\boldsymbol{r}(u,v) = \rho(u)\boldsymbol{h}(u) + z(u)\boldsymbol{k} + R(u)\boldsymbol{e}(u,v),$$

where $\boldsymbol{r}(u, v)$ is a radius vector of a cyclic surface; $\rho(u)\boldsymbol{h}(u)$ is the projection of the radius vector of the line of the centers of the generatrix circles on a plane perpendicular to *the fixed straight line of the pencil of the planes*; $z(u)$ is a law of the motion of the center of the generatrix circle parallel to the fixed straight line of the pencil of the planes; $R(u)$ is a radius of generatrix circles; $\boldsymbol{h}(u)$ is the vector function of the circle of the unit radius lying in a plane perpendicular to the fixed straight line of the pencil of the planes (Fig. 2a); $\boldsymbol{e}(u, v)$ is the vector function of the circle of the unit radius lying in the planes of the pencil (Fig. 2b).

Coefficients of the fundamental forms of surface of the cyclic surfaces with circles in planes of pencil: (see Fig. 2);

$$A^2 = \rho'^2 + z'^2 + R'^2 + R^2\theta'^2 + (\rho + R\cos\omega)^2 + 2\rho'(R\cos\omega)' + 2z'(R\sin\omega)',$$

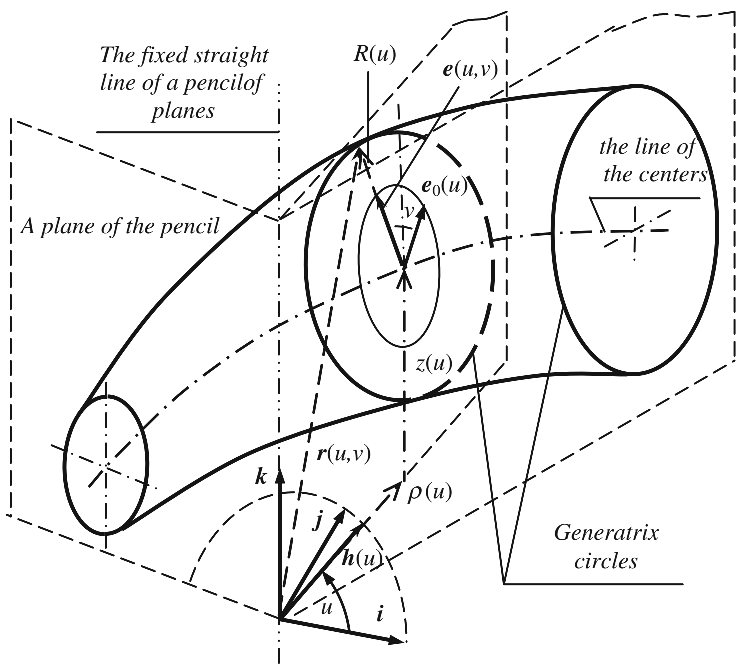


Fig. 1

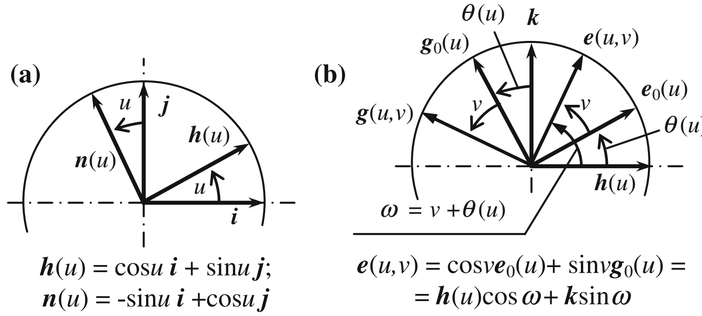


Fig. 2

$$F = R(R\theta' - \rho' \sin\omega + z' \cos\omega),\ B = R,$$
$$\omega = v + \theta(u)$$
$$L = \left\{\left[(\rho + R\cos\omega - \rho'')\cos\omega - z''\sin\omega + R\theta'^2 - R''\right](\rho + R\cos\omega) + 2\left[\rho' + (R\cos\omega)'\right](\rho'\cos\omega + z'\sin\omega + R')\right\}/\psi,$$
$$M = [R\theta'(\rho + R\cos\omega) - (\rho'\cos\omega + z'\sin\omega + R')R\sin\omega]/\psi,$$
$$N = R(\rho + R\cos\omega)/\psi;$$
$$(A^2B^2 - F^2)/R^2 = \psi^2 = \rho'^2 + z'^2 + R'^2 + (\rho + R\cos\omega)^2 + 2R'(\rho'\cos\omega + z'\sin\omega) - (\rho'\sin\omega - z'\cos\omega)^2,$$

where $\ldots\prime = \partial\ldots/\partial u$.

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Ivanovв VN. Cyclic surfaces (Geometry, classification and design of shells). Shell in Architecture and Strength Analysis of Thin-Walled Civil-Engineering and Machine-Building Constructions of Complex Form: Int. Scientific Conference, Moscow, June 4-8, 2001. Moscow: Izd-vo RUDN, 2001; p. 127-134 (18 refs.).

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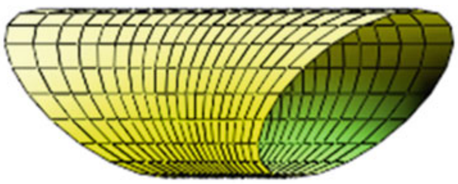
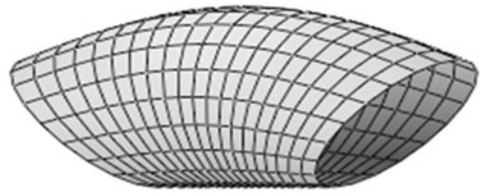
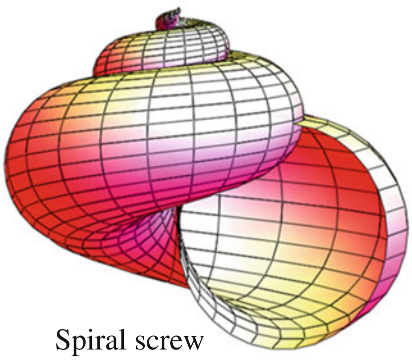
■ Cyclic Surfaces with Circles in Planes of Pencil Presented in the Encyclopedia



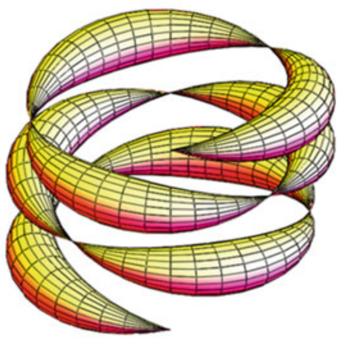
The cyclic surface with the circles of variable radius in the planes of the pencil andwith three straight parallel directrixes



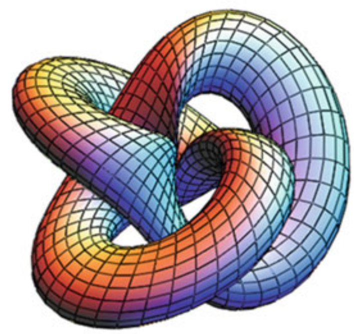
The cyclic surface with the circles in the planes of pencil with the straight directrix and the fixed straight of the pencil lying on the same side of the plane center-to-center line



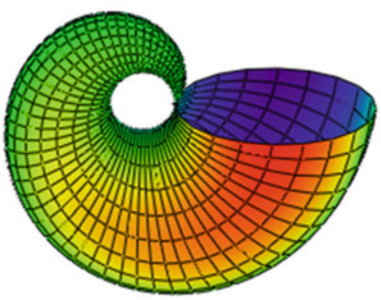
Spiral screw



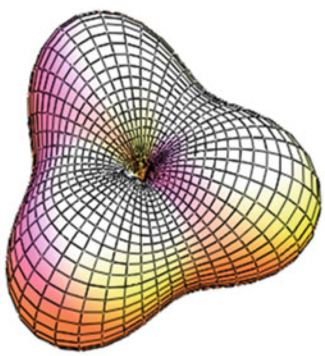
“Hornlet”



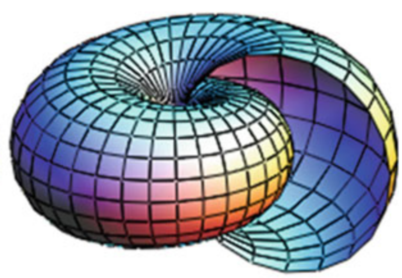
“Clover knot”



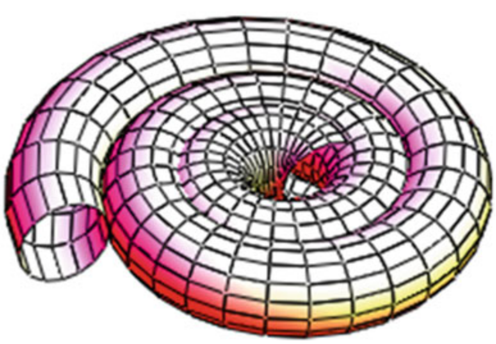
The cyclic surface with generatrix circles of variable radius and with the plane line of centers constructed about the circular cylinder



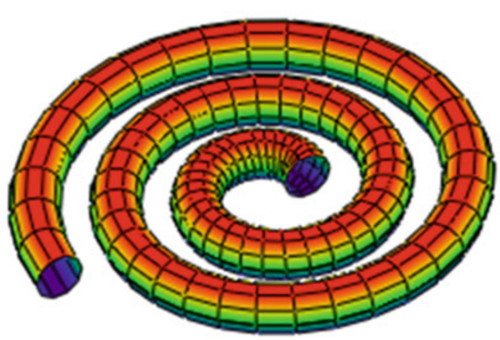
Joachimsthal cosine canal surfaces of the 3^{d} type



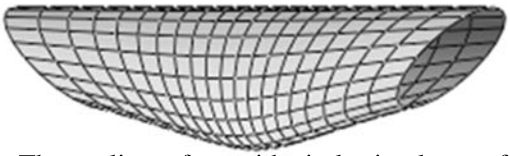
“ Cornucopia ”



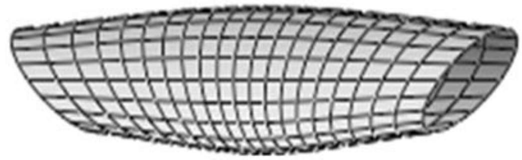
The cyclic surface with a generatrix circle in the planes of pencils and with the plane line of centers in the form of the logarithmic spiral



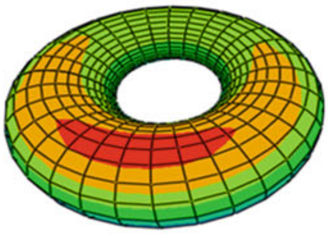
The cyclic surface with a generating circle in the planes of pencil and with the plane line of centers in the form of the spiral of Archimedes



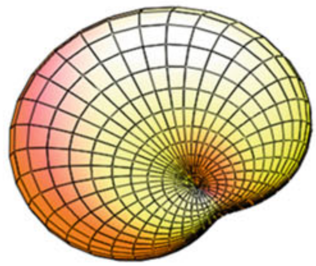
The cyclic surface with circles in planes of pencil with the straight directrix and the fixed straight of pencil lying on different sides of the plane center-to-center line



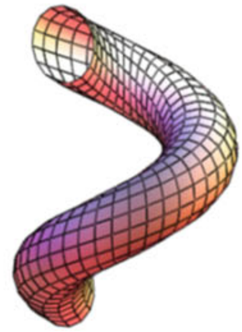
The cyclic surface with circles in planes of pencil and with the straight line of centers



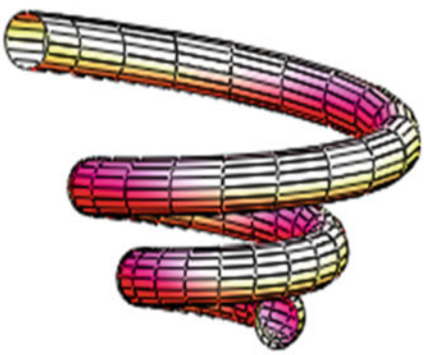
Dupin cyclide of the first type (of the forth order)



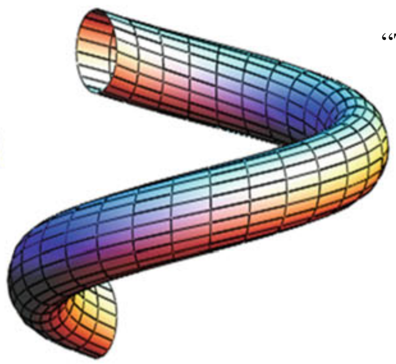
The epitrochoidal surface



“The Saint Elias surface”

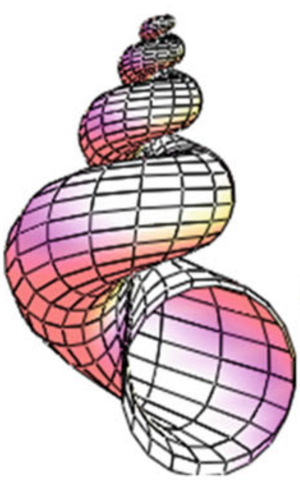


The circular spiral surface with the generatrix circle of constant radius lying in the planes of the pencil

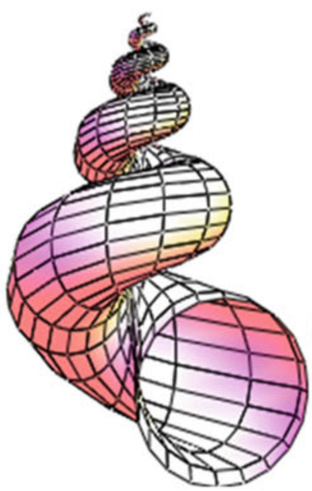


The circular helical surface with the generatrix circles in the planes of the pencil

The cyclic surface with circles in the planes of meridians of the sphere and with a center-to-center line on the same sphere

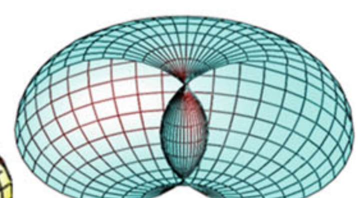


The spiral-shaped surface “Shell without vertex”

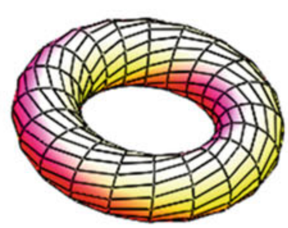


The spiral-shaped surface “Shell with vertex”

“Seashell”



The spindle circular



The preliminarily twisted circular torus

■ Circular Spiral Surface with a Generatrix Circle of Constant Radius Lying in Planes of a Pencil

Let us take a generatrix conic spiral:

$$x = x(u) = ae^{mu}\cos u,$$
$$y = y(u) = ae^{mu}\sin u,$$
$$z = z(u) = a\lambda e^{mu},$$

where

$$\lambda = \cot an\varphi;$$

φ is the angle of the axis Oz with a straight generatrix line of a cone on which the conic line lay; the longitude u is the angle of the plane xOz with the mobile plane of the axial cross section;

$$a = r_o \sin\varphi,$$

r_o, m are constants.

A circular spiral surface with a generatrix circle of constant radius lying in planes of a pencil is created by helical motion of a circle, the center of which moves along *the conical spiral line*.

All points of the generatrix circles of a constant radius R trace the *conical spirals,* which are *the slope lines*.

This surface may be included both into a class of *cyclic surfaces* and into a class of *spiral surfaces* (see also Chap. "8. Spiral Surfaces").

Form of definition of the circular spiral surface

(1) Parametrical equations (Fig. 1):

$$x = x(u,v) = (ae^{mu} + R\cos v)\cos u,$$
$$y = y(u,v) = (ae^{mu} + R\cos v)\sin u,$$
$$z = z(u,v) = a\lambda e^{mu} + R\sin v,$$

where R is a constant radius of the generatrix circle.

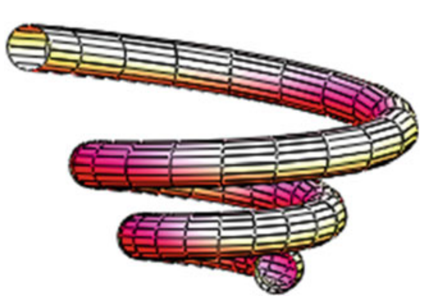


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = r_o^2 m^2 e^{2mu} + (ae^{mu} + R\cos v)^2,$$
$$F = -aRme^{mu}(\sin v - \lambda\cos v),$$
$$B = R;$$
$$A^2B^2 - F^2 = R^2\Big[(ae^{mu} + R\cos v)^2 + a^2m^2e^{2mu} \times (\cos v + \lambda\sin v)^2\Big],$$
$$L = \frac{-R}{\sqrt{A^2B^2 - F^2}}\Big[(ae^{mu} + R\cos v)^2\cos v + am^2e^{mu}(\cos v + \lambda\sin v)(ae^{mu} - R\cos v)\Big],$$
$$M = \frac{amR^2e^{mu}\sin v}{\sqrt{A^2B^2 - F^2}}(\cos v + \lambda\sin v),$$
$$N = -\frac{(ae^{mu} + R\cos v)}{\sqrt{A^2B^2 - F^2}}R^2.$$

The cyclic spiral surface is related to a nonorthogonal nonconjugate system of the curvilinear coordinates u, v. The coordinate lines v coincide with the generatrix circles. The surface contains the segments of positive and negative Gaussian curvatures.

If $\varphi = \pi/2$, i.e., $\lambda = 0$, then a circular spiral surface (Fig. 1) degenerates into *a circular spiral surface with a generatrix circle of a constant radius in the planes of the pencil and with the plane line of the centers in the form of the logarithmic spiral* (see also a Subsect. "17.4.1. Cyclic Surfaces with Circles in the Planes of Pencil and with a Plane Center-to-Center Line"), Figs. 2 and 3.

A surface of spiral spring is very much like a circular spiral surface in question but it is not a circular spiral surface in question.

Additional Literature

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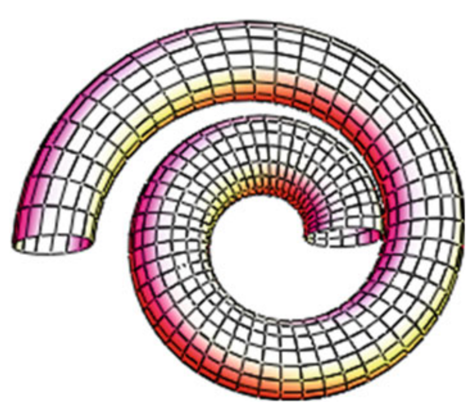
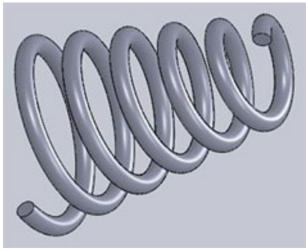


Fig. 2



Sedaghat Pisheh H, Shahabadi M, Komijany Y and Mohajerzadeh S. One-sided anchored structures for the realization of conical spiral antennas. The Second IASTED International Conference on Antennas, Radar, and Wave Propagation (ARP 2005), July 19-21, 2005, Banff, Alberta, Canada. 2005; p. 396-401.
http://grabcad.com/library/spiral-spring

Fig. 3

■ "The Saint Elias Surface"

The Saint Elias surface is a partial case of *a circular helical surface with the generatrix circle lying in the plane passing through the helical axis* (see also a Subsect. "7.1.2. Circular Helical Surfaces"). The surface may be included into a class of *cyclic surfaces* or into a class of *helical surfaces*. It is formed by the helical motion of a circle of a constant radius. The line of the centers of the Saint Elias surface is a helical circular line of a constant slope (helix):

$$x = x(v) = 2\cos v, y = y(v) = 2\sin v, z = z(v) = v.$$

Forms of definition of the Saint Elias surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(v, \psi) = (2 + \sin\psi)\boldsymbol{e}(v) + (\cos\psi + v)\boldsymbol{k},$$

where $\boldsymbol{e}(v)$ is the unit circular vector function; ψ is a central angle of a generatrix circle read from the direction of the axis Oz.

(2) Parametrical equations (Fig. 1):

$$x = x(v, \psi) = (2 + \sin\psi)\cos v.$$
$$y = y(v, \psi) = (2 + \sin\psi)\sin v$$
$$z = z(v, \psi) = \cos\psi + v$$

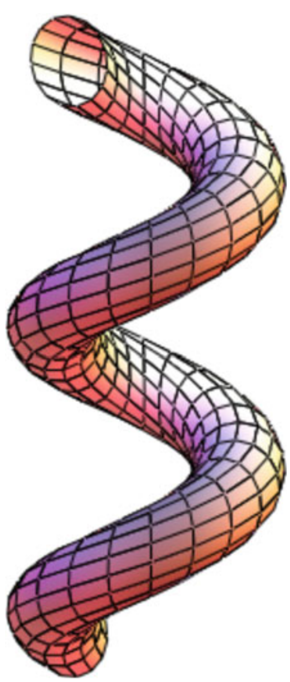


Fig. 1

The Saint Elias surface is a circular helical surface with the generatrix circulars in the planes of the pencil with $a = 2$ m; $r = p = 1$ m. The axial cross section of the surface is the generatrix circle of the radius $r = 1$ m. Assume $z = 0$, that is, $v = -\cos\psi$, then we have the end cross section.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = (2 + \sin\psi)^2 + 1,$$
$$F = -\sin\psi, \ B = 1,$$
$$L = -\frac{(2 + \sin\psi)^2 \sin\psi}{\sqrt{5 + 4\sin\psi}},$$
$$M = \frac{\cos^2\psi}{\sqrt{5 + 4\sin\psi}},$$
$$N = -\frac{(2 + \sin\psi)}{\sqrt{5 + 4\sin\psi}},$$
$$k_\psi = \frac{-(2 + \sin\psi)}{\sqrt{5 + 4\sin\psi}},$$
$$k_v = \frac{-\sin\psi(2 + \sin\psi)^2}{A^2\sqrt{5 + 4\sin\psi}}.$$
$$K = \frac{(2 + \sin\psi)^3 \sin\psi - \cos^4\psi}{(5 + 4\sin\psi)^2}$$

Additional Literature

The Saint Elias surface. http://www.geometrie.h10.ru/plocha/plocha3R.htm

■ Cyclic Surface with Circles in the Planes of Pencil and with a Waving Line of Centers on a Cylinder

Assume parametrical equations of a line of centers in the following form:

$$X = X(u) = R\cos u,$$
$$Y = Y(u) = R\sin u,$$
$$Z = Z(u) = bu + a\sin pu$$

and begin to move a circle of a constant radius r along it in the planes of the pencil passing through the Oz axis, then we may obtain the considered cyclic surface. The planes of the pencil pass through a fixed straight line coinciding with the axis of the centerline. This surface may be included into a class of *wave-shaped, waving, and corrugated surfaces*.

Form of definition of the surface

(1) Parametrical equation (Figs. 1 and 2):

$$x = x(u,v) = (R + r\cos v)\cos u,$$
$$y = y(u,v) = (R + r\cos v)\sin u,$$
$$z = z(u,v) = (bu + a\sin pu) + r\sin v,$$

where 2πb is a lead of the waving line of the centers, v is an angle read from the plane z = const in the direction of the axis Oz; $0 \le v \le 2\pi$ (Fig. 1), $0 \le v \le 3.5\pi$ (Fig. 2), u is an angle read from the axis x in the direction of the axis y.

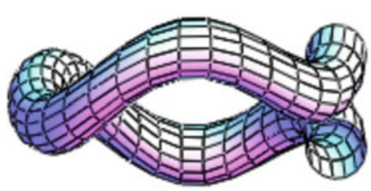


Fig. 1

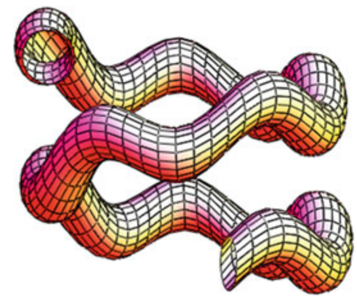


Fig. 2

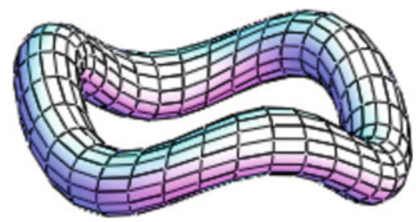


Fig. 3

Coefficients of the fundamental forms of the surface:

$$A^2 = (R + r\cos v)^2 + (b + ap\cos pu)^2,$$
$$F = r(b + ap\cos pu)\cos v,\ B = r;$$
$$L = \frac{-r(R + r\cos v)}{\sqrt{A^2B^2 - F^2}}\left[(R + r\cos v)\cos v + ap^2\sin pu\sin v\right].$$
$$M = \frac{r^2(b + ap\cos pu)}{\sqrt{A^2B^2 - F^2}}\sin^2 v,$$
$$N = \frac{-r^2(R + r\cos v)}{\sqrt{A^2B^2 - F^2}}$$

The coordinates lines $v = \pi/2$ and $v = 3\pi/2$ are orthogonal to the generatrix circles. If $b = 0$, then the surface degenerates into a wave-shaped torus (Fig. 3), that touches the plane $z = -a - r$ in the p points.

■ Cyclic Surface with Circles in the Planes of Meridians of the Sphere and with a Center-to-Center Line on the Same Sphere

The axis of a sphere coinciding with a coordinate axis Oz is a fixed straight line of the pencil of planes with circles of constant radius, which form *a cyclic surface with the circles in the planes of the meridians of the sphere and with the line of the centers on the same sphere*. So, the cyclic surface has a directrix *spherical line*

$$\boldsymbol{E}_0(u) = a\boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\sin\omega,$$

where

$$\omega = pu + c(p = \text{const},\ c = \text{const})$$

and this line is disposed on the surface of the sphere of a radius a. It is necessary to take $p = 1/n$, where n is an integer if one wants to obtain a spherical spiral (Fig. 1).

Fig. 1

The unit vector $\boldsymbol{e}_0(u)$ is a normal of the sphere on which the directrix line is placed.

The generatrix circles of a constant radius b are given in the local coordinate system

$$X(v) = b\cos v,\ Y(v) = b\sin v,$$

disposed in the planes of the meridians of the sphere. The origin of the coordinates is disposed on the directrix spherical line.

Forms of definition of the surface

(1) Vector equation:

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{r}(u,v)\\ &= (a\cos\omega + b\cos v)\boldsymbol{h}(u) + (a\sin\omega + b\ \sin v)\boldsymbol{k},\end{aligned}$$

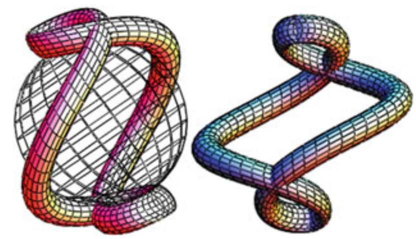


Fig. 3

where

$$\boldsymbol{h} = \boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u.$$

(2) Parametrical equations (Figs. 2, 3, 4, 5 and 6):

$$\begin{aligned}x &= x(u,v) = (a\cos\omega + b\cos v)\cos u,\\ y &= y(u,v) = (a\cos\omega + b\cos v)\sin u,\\ z &= z(u,v) = a\sin\omega + b\sin v.\end{aligned}$$

The geometrical parameters of the surfaces shown in Fig. 2 are given under the figures. In Fig. 3, the surface with the following geometrical parameters:
$a = 10$ cm; $b = 2$ cm; $p = 1/2$; $-2\pi \le u \le 2\pi$; $-\pi \le v \le \pi$ is shown

The surface presented in Fig. 4 is constructed for the parameters: $a = 10$ cm; $b = 2$ cm; $p = 2$. In Fig. 5, the surface has $a = 10$ cm; $b = 1$ cm; $p = 4$; $-\pi \le u \le \pi$, $-\pi \le v \le \pi$. In Fig. 6, the parameters of the surface are $a = 10$ cm; $b = 1$ cm; $p = 3/2$; $-2\pi \le u \le 2\pi$; $-\pi \le v \le \pi$.

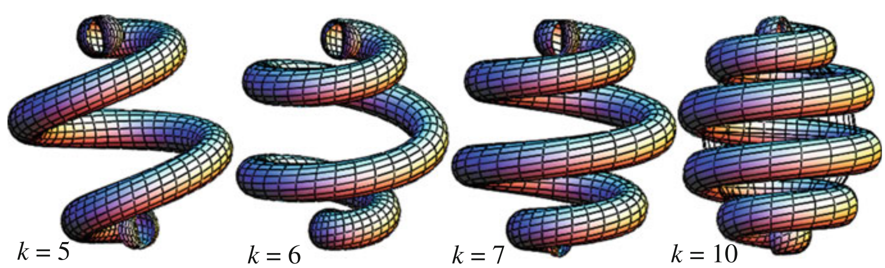


Fig. 2

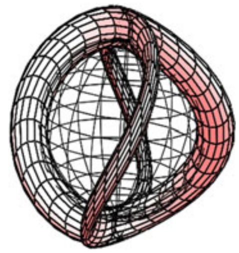


Fig. 4

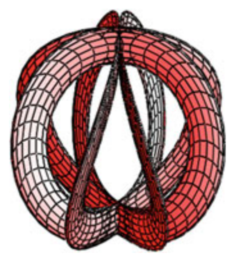


Fig. 5

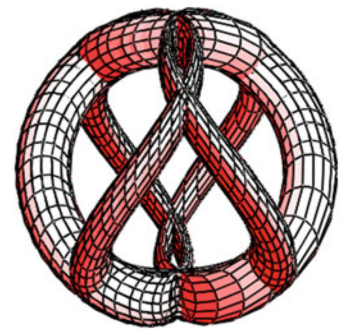


Fig. 6

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2p^2 + (a\cos\omega + b\cos v)^2,$$
$$F = abp\cos(v - \omega), \quad B = b; ,$$
$$A^2B^2 - F^2 = [a\cos\omega + b\cos v]^2 + a^2p^2\sin^2(v - \omega),$$
$$L = -\{2a^2p^2\sin\omega\cos(v - \omega) + (a\cos\omega + b\cos v)[ap^2\cos(v - \omega) + (a\cos\omega + b\cos v)\cos v]\}\Big/\sqrt{A^2b^2 - F^2}$$
$$M = -abp\sin v\cos v\Big/\sqrt{A^2b^2 - F^2}$$
$$N = -b(a\cos\omega + b\cos v)\Big/\sqrt{A^2b^2 - F^2}.$$

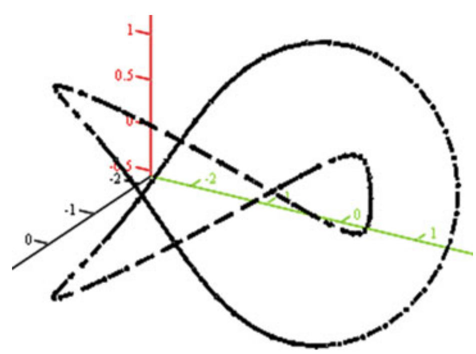
The surface is given in the curvilinear nonorthogonal nonconjugate coordinates u, v.

■ "Clover Knot"

Cyclic surface "*Clover knot*" (*Trefoil Knoten* in German) is formed by circles of a constant radius a lying in the planes of a pencil and moving along a centerline in the form of a closed space curve (Fig. 1)

$$x = x(u) = \left(R + a\cos\frac{u}{2}\right)\cos\frac{u}{3},$$
$$y = y(u) = \left(R + a\cos\frac{u}{2}\right)\sin\frac{u}{3},$$
$$z = z(u) = a + \sin\frac{u}{2},$$

where R and a are constants having an influence on the form of the centerline.



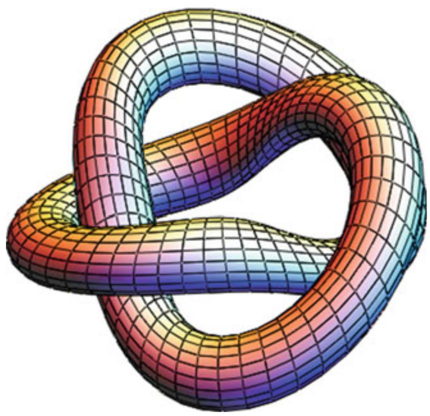
$R = 2$; $a = 0{,}5$;
$0 \le u \le 12\pi, \;\; 0 \le v \le 2\pi$

Fig. 1

Forms of definition of the surface

(1) Parametrical equations (Figs. 1 and 2):

$$x = x(u, v) = \left(R + a\cos\frac{u}{2}\right)\cos\frac{u}{3} + a\cos\frac{u}{3}\cos(v - \pi),$$
$$y = y(u, v) = \left(R + a\cos\frac{u}{2}\right)\sin\frac{u}{3} + a\sin\frac{u}{3}\cos(v - \pi),$$
$$z = z(u, v) = a + \sin\frac{u}{2} + a\sin(v - \pi),$$
$$0 \le u \le 12\pi, \; 0 \le v \le 2\pi.$$



$R = 2$; $a = 0{,}5$;
$0 \le u \le 12\pi$, $0 \le v \le 2\pi$

Fig. 2

(2) Vector equation:

$$\boldsymbol{\rho} = \boldsymbol{\rho}(u, v) = \boldsymbol{r}(u) + a[(\boldsymbol{i}\cos(u/3) + \boldsymbol{j}\sin(u/3))\cos(v-\pi) + \boldsymbol{k}\sin(u/3)],$$

where r(u) is a radius vector of the centerline, $0 \le u \le 12\pi$, $0 \le v \le 2\pi$.

Reference

Parametrische Flächen und Körper. – http://www.3d-meier.de/tut3/Seite5.html

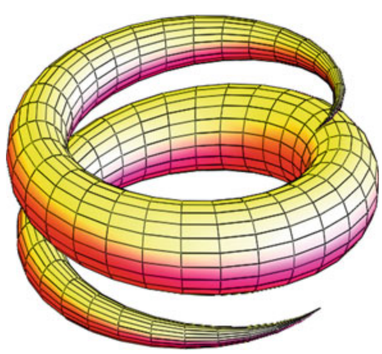
■ "Hornlet"

Hornlet is a cyclic surface with generatrix circles of a variable radius lying in the planes of a pencil with a helical centerline of the constant lead.

Form of definition of the surface

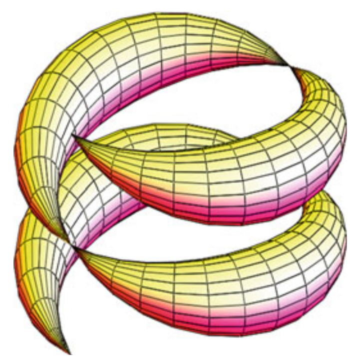
(1) Parametric form of the definition (Figs. 1, 2, 3 and 4):

$$x = x(u, v) = [a + \sin(b\pi u)\sin(b\pi v)]\sin(c\pi v),$$
$$y = y(u, v) = [a + \sin(b\pi u)\sin(b\pi v)]\cos(c\pi v),$$
$$z = z(u, v) = \cos(b\pi u)\sin(b\pi v) + hv,$$



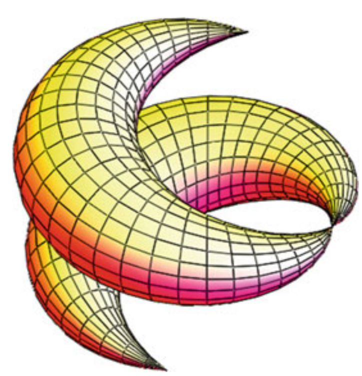
$a = 4$; $c = 4$; $b = 1$; $h = 4$

Fig. 1



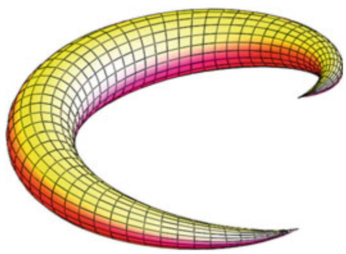
$a = 4$; $c = 4$; $b = 4$; $h = 6$

Fig. 2



$a = 2$; $c = 3$; $b = 2$; $h = 4$

Fig. 3



$a = 4; c = 2; b = 1; h = 4$

Fig. 4

where *a, b, c, h* are constants but *b* is a number of the hornlets at the given interval of changing of the *v* parameter; $0 \le v \le 1$.

17.4.1 Cyclic Surfaces with Circles in the Planes of Pencil and with a Plane Center-to-Center Line

For description of *cyclic surfaces with circles in the planes of pencil and with a plane center-to-center line,* one may use the formulas of a Sect. "17.4. Cyclic Surfaces with Circles in Planes of Pencil" substituting

$$z(u) = 0,\ \theta(u) = 0,\ \omega = v$$

into them. In this case, a vector equation of the cyclic surface with a plane center-to-center line is

$$\mathbf{r}(u, v) = \rho(u)\mathbf{h}(u) + R(u)\mathbf{e}(u, v),$$

where $\mathbf{r}(u,\ v)$ is a radius vector of the surface in question; $\rho(u)\mathbf{h}(u)$ is the projection of the radius vector of the center-to-center line of the generatrix circles at the plane perpendicular to a fixed straight line of the pencil of planes; $R(u)$ is a radius of generatrix circles; $\mathbf{h}(u)$ is the vector function of the circle of the unit radius lying at the plane perpendicular to the fixed straight line of the pencil (Fig. 1); $\mathbf{e}(u,\ v)$ is the vector function of the circle of the unit radius lying in the planes of the pencil.

Coefficients of the fundamental forms of the surface:

$$A^2 = \rho'^2 + R'^2 + (\rho + R\cos v)^2 + 2\rho' R' \cos v,$$
$$F = -R\rho' \sin v,\ B = R;$$
$$(A^2B^2 - F^2)/R^2 = \psi^2 = (\rho + R\cos v)^2 + (\rho' \cos v + R')^2,$$
$$L = \frac{1}{\psi}\{[(\rho + R\cos v - \rho'')\cos v - R''](\rho + R\cos v) + 2[\rho' + R'\cos v](\rho' \cos v + R')\},$$
$$M = -\frac{R(\rho' \cos v + R')\sin v}{\psi},$$
$$N = \frac{R(\rho + R\cos v)}{\psi},$$

where

$$\ldots' = \partial \ldots / \partial uv$$

The points 1 and 2 (Fig. 1) of the generatrix circles trace two curves

$$r_1(u) = \rho(u) - R(u) \text{ and } r_2(u) = \rho(u) + R(u)$$

at the plane of the centerline. Here, $\rho(u)$ is the distance the pole O from the centerline of the generatrix circles; $R(u)$ is a radius of the generatrix circles. The pole O is a point of intersection of the fixed straight line of the pencil of planes with the line of the centers of the generatrix circles. The contour curves with $r_1(u)$ and $r_2(u)$ may be taken as directrixes of cyclic surfaces with plane line of centers (Fig. 1).

A vector equation of a cyclic surface with the directrix curve $r_1(u)$ has the following form:

$$\mathbf{r}(u, v) = r_1(u)\mathbf{h}(u) + 2R(u)\cos v\ \mathbf{e}(u, v),$$

where $2 \cos v\mathbf{e}(u,\ v)$ is a polar equation of the circle of the unit radius with the origin of the coordinates at the point 1 of the generatrix circle (Fig. 1);

$$v = \gamma/2;\ -\pi/2 \le v \le \pi/2.$$

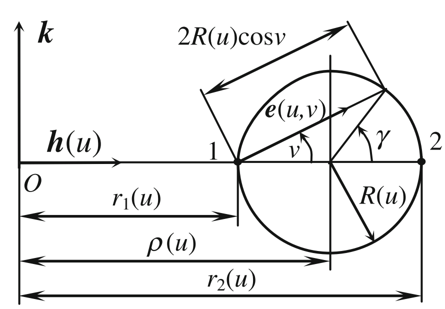


Fig. 1 .

Coefficients of the fundamental forms of the surface:

$$A^2 = r_1'^2 + 4R'(r_1' + R')\cos^2 v + (r_1 + 2R\cos^2 v)^2,$$
$$F = -2R(r_1' + R')\sin 2v, \quad B = 2R;$$
$$\psi^2 = (A^2B^2 - F^2)/R^2 = r_1'^2 - 4(r_1' + R')$$
$$\times (r_1' \sin^2 v - R'\cos^2 v)\cos^2 v + (r_1 + 2R\cos^2 v)^2;$$
$$L = \left\{(r_1 + 2R\cos^2 v)\left[(r_1 + 2R\cos^2 v - r_1'')\cos 2v - 2R''\cos^2 v\right]\right.$$
$$\left. +2(r_1' + 2R'\cos^2 v)(r_1'\cos 2v + 2R'\cos^2 v)\right\}/\psi$$

$$M = -2R\left(r_1'\cos 2v + 2R'\cos^2 v\right)\frac{\sin 2v}{\psi},$$
$$N = \frac{4R(r_1 + 2R\cos^2 v)}{\psi}.$$

In general case, generatrix circles of a cyclic surface with generatrix circles in the planes of pencil and with a plane center-to-center line are not the lines of principle curvatures. Sometimes, the generatrix circles of cyclic surfaces of this subsection may be the lines of principle curvatures but the radius $R(x)$ of the generatrix circles must change by a special law. The cyclic surfaces with the generatrix circles in planes of pencil which are the lines of principle curvatures are called Joachimsthal canal surfaces (see also a Subsect. "17.1.1. Canal Surfaces of Joachimsthal").

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Ivanov VN. Geometry and design of shells on the base of surfaces with a system of coordinates lines in the planes of pencil. Prostr. Konstruk. Zdaniy i Soor. Sb. Nauchn. Tr. MOO "Prostranst. Konstruktzii", Moscow: "DEVYATKA PRINT", 2004; Iss. 9, p. 26-35.

Ivanov V. Canal Joachimsthal surfaces with plane line of the centers. Issled. Prostranstv. Sistem. Moscow: Izd-vo RUDN, 1996; p. 32-36 (3 refs.).

■ Cyclic Surface with a Generatrix Circle in the Planes of Pencil and with a Plane Line of Centers in the Form of a Logarithmic Spiral

Cyclic surface with a generatrix circle in the planes of pencil and with a plane line of centers in the form of a logarithmic spiral

$$x = x(u) = ae^{mu}\cos u, y = y(u) = ae^{mu}\sin u$$

is formed by a circle of constant radius, the center of which moves along a logarithmic spiral (Figs. 1 and 2) but the circle all the time stays at the planes passing through the coordinate axis Oz taken as the fixed straight line of the pencil of planes.

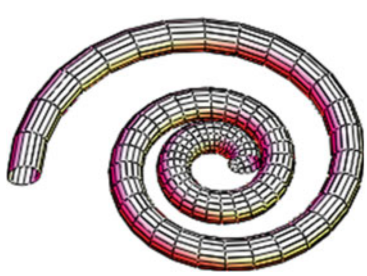


Fig. 1

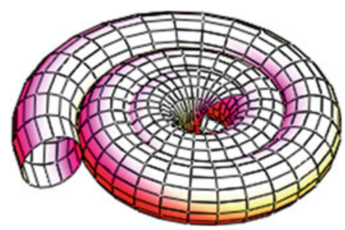


Fig. 2

The considered cyclic surface is a special case of *a circular spiral surface with the generatrix circle of constant radius lying in the planes of a pencil* (see also a Sect. "17.4. Cyclic Surfaces with Circles in Planes of Pencil"). The surface in question may be related both to a class of *cyclic surfaces* and to a class of *spiral surfaces* (see also Chap. "8. Spiral Surfaces").

Form of definition of the surface

(1) Parametrical equations:

$$x = x(u, v) = (ae^{mu} + r\cos v)\cos u,$$
$$y = y(u, v) = (ae^{mu} + r\cos v)\sin u,$$
$$z = z(v) = r\sin v,$$

where r is a radius of the generatrix circle, a and m are constants, $-\infty \le u \le \infty$; $0 \le v \le 2\pi$.

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2e^{2mu}m^2 + (ae^{mu} + r\cos v)^2,$$
$$F = -amre^{mu}\sin v,\ B = r;$$
$$A^2B^2 - F^2 = a^2m^2r^2e^{2mu}\cos^2 v + r^2(ae^{mu} + r\cos v)^2$$
$$L = \frac{-r\cos v}{\sqrt{A^2B^2 - F^2}}\Big[(ae^{mu} + r\cos v)^2 + am^2e^{mu}(ae^{mu} - r\cos v)\Big],$$
$$M = \frac{amr^2e^{mu}\sin v\cos v}{\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{-r^2(ae^{mu} + r\cos v)}{\sqrt{A^2B^2 - F^2}},$$
$$k_u = \frac{L}{A^2},$$
$$k_v = -\frac{(ae^{mu} + r\cos v)}{\sqrt{A^2B^2 - F^2}}.$$

The surface is given in nonorthogonal and nonconjugate curvilinear coordinates u, v. The plane coordinate lines $v = 0$ and $v = \pi$ lying in the cross section of the surface by a plane $z = 0$ are the lines of principle curvatures. The coordinate lines v = const are plane logarithmic spirals and the coordinate lines u = const are the generatrix circles of the radius r.

The cyclic surface with geometrical parameters $a = 0.57$ m; $m = 0.12$; $r = 1$ m; $3\pi \le u \le 8\pi$, shown in Fig. 1, does not have points of self-intersection but the cyclic surfaces with geometrical parameters $a = 0.5$ m; $m = 0.089$; $r = 1$ m; $3\pi \le u \le 8\pi$, shown in Fig. 2, intersects itself.

Additional Literature

Manselli P, Pucci C. Risultati di unicità per curve evolute ed evolventi di se stesse. Boll. Unione mat. ital. A. 1991; 5, No. 3, p. 373-379.

Lockwood EH. Book of Curves (Ch. 11 “The Equiangular Spiral”, p. 98-109). Cambridge: Cambridge University Press, 1961; 212 p.

■ Cyclic Surface with Generatrix Circles of Variable Radius and with a Plane Line of Centers Constructed About a Circular Cylinder

Cyclic surface with generating circles of variable radius and with a plane line of centers constructed about a circular cylinder may be defined by parametrical equations:

$$x = x(u, v) = [R + au(1 + \cos v)]\cos u,$$
$$y = y(u, v) = [R + au(1 + \cos v)]\sin u,$$
$$z = z(v) = au\sin v,$$

where $au = r(u)$ is the law of changing radius of the generatrix circles lying in the planes of pencil passing through a fixed straight line (coordinate axis Oz); R is a constant radius of the internal circular space (Fig. 1) traces by the point of the generatrix circle with the curvilinear coordinate $v = \pi$; $0 \le v \le 2\pi$; $0 \le u \le \infty$. The cyclic surface in question has the plane centerline:

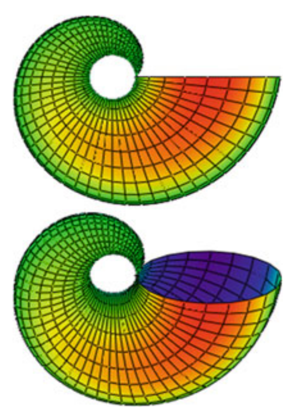


Fig. 1

$$x = x(u) = (R + au)\cos u,$$
$$y = y(u) = (R + au)\sin u,\ z = 0.$$

■ Cyclic Surface with a Generating Circle in the Planes of Pencil and with a Plane Line of Centers in the Form of Spiral of Archimedes

Cyclic surface with a generatrix circle in the planes of pencil and with a plane line of centers in the form of spiral of Archimedes

$$\rho = a\varphi \text{ or } x = x(\varphi) = a\varphi\cos\varphi, \quad y = y(\varphi) = a\varphi\sin\varphi$$

is formed by a circle of a constant radius, the center of which moves along the spiral of Archimedes (Fig. 1). The center of the circle slides along the directrix spiral of Archimedes but the circle itself remains all the time in planes passing through the coordinate axis Oz that is assumed as a fixed straight line of the pencil of planes.

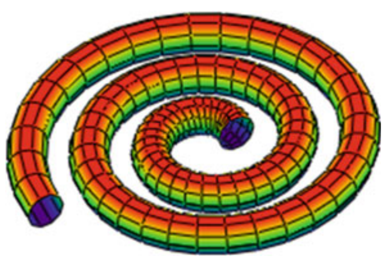


Fig. 1

Form of definition of the surface

(1) Parametrical equations:

$$x = x(\varphi, v) = (a\varphi + r\cos v)\cos\varphi$$
$$y = y(\varphi, v) = (a\varphi + r\cos v)\sin\varphi, \quad z = z(v) = r\sin v$$

where r is a constant radius of a generatrix circle, a is a constant, $0 \le \phi \le \infty$; $0 \le v \le 2\pi$.

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2 + (a\varphi + r\cos\varphi)^2,$$
$$F = -ar\sin v, \; B = r;$$
$$A^2B^2 - F^2 = r^2\left[a^2\cos^2 v + (a\varphi + r\cos v)^2\right];$$
$$L = \frac{-r\cos v}{\sqrt{A^2B^2 - F^2}}\left[2a^2 + (a\varphi + r\cos v)^2\right],$$
$$M = \frac{ar^2\sin v\cos v}{\sqrt{A^2B^2 - F^2}},$$
$$N = -\frac{r^2(a\varphi + r\cos v)}{\sqrt{A^2B^2 - F^2}}$$
$$k_\phi = \frac{L}{A^2}, \quad k_v = -\frac{(a\varphi + r\cos v)}{\sqrt{A^2B^2 - F^2}}$$

The surface is given in nonorthogonal and nonconjugate curvilinear coordinates φ, v. The plane coordinate lines $v = 0$ and $v = \pi$ lying in the cross section of the surface by a plane $z = 0$ are the lines of principle curvatures. The coordinate lines v = const are plane *neoids*, the coordinate lines u = const are the generatrix circles of the constant radius r.

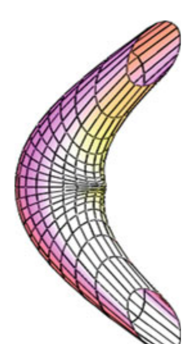
■ Cyclic Surface with a Generatrix Circle in the Planes of Pencil and with a Plane Centerline in the Form of the 2nd Order Curve

Cyclic surface with a generatrix circle in the planes of pencil and with a plane centerline in the form of the 2nd *other curve* is formed by motion of a circle of constant radius along a directrix curve.

It is rationally to use equations of directrix curves (centerlines) of cyclic surfaces with plane generatrix curves in the planes of pencil in *a polar coordinate system*. For the 2nd order curves, a polar equation can be written as

$$\rho(u) = \frac{a\varepsilon}{\eta(u)},$$

where $\rho(u)$ is a radius of a point of the curve; u is a polar angle; $\eta(u) = 1 + p\varepsilon\cos u; \varepsilon = \sqrt{1 + t(a/b)^2}$; a, b are parameters of the curves of the 2nd order (for parabola $b = 1$). If $t = p = 1$, we have a hyperbola; if $t = \pm1$, $p = -1$, we have an ellipse; for a parabola (Fig. 1), we must take $t = 0$, $p = 1$ and so $\varepsilon = 1$. The fixed straight line of the pencil



$b = 1$, $a = 1$, $p = 1$,
$-0.55\pi \le u \le 0.55\pi$

Fig. 1 $R = 0.4$

of planes goes through *the focus of the 2nd order curve* and coincides with the Oz axis (Fig. 1).

Forms of definition of the surface

(1) Vector equation:

$$\boldsymbol{r}(u,v) = \rho(u)\boldsymbol{h}(u) + R\boldsymbol{e}(u,v),$$

where $\boldsymbol{r}(u, v)$ is a radius vector of a cyclic surface; $\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u$ is a vector function of the circle of the unit radius in the plane Oxy; R is a radius of a generatrix circle; $\boldsymbol{e}(u,v) = \boldsymbol{h}(u)\cos v + \boldsymbol{k}\sin v$ is a vector function of the circle of the unit radius in a plane of the pencil; i, j, k are the orthogonal unit vectors.

(2) Parametrical equations:

$$x(u,v) = (\rho(u) + r\cos v)\cos u;$$
$$y(u,v) = (\rho(u) + r\cos v)\sin u;$$
$$z(u,v) = R\sin v.$$

Coefficients of the first fundamental form of the surface:

$$A^2 = a^2\varepsilon^4\frac{t^2\sin^2 v}{\eta^4} + \left(\frac{a\varepsilon}{\eta} + R\cos v\right)^2;$$
$$F = -Ra\varepsilon^2\frac{t\sin v}{\eta^2} \quad B = R^2,$$
$$A^2B^2 - F^2 = \left(\frac{a\varepsilon}{\eta} + R\cos v\right)^2 R^2.$$

■ Cyclic Surface with Circles in Planes of Pencil and with a Straight Line of Centers

Cyclic surface with circles in planes of pencil and with a straight line of centers is formed by a circle of a constant radius in the process of motion of its center along a straight line but the circles must remain all the time in the planes of a pencil and pass through the fixed straight line which is perpendicular to the straight line of the centers.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u,v) = \left(\frac{a}{\cos u} + r\cos v\right)\sin u;$$
$$y = y(u,v) = r\cos v\cos u,$$
$$z = z(v) = r\sin v,$$

where r is a constant radius of a generatrix circle, a is the shortest distance the straight line of the centers of the generatrix circles from the fixed straight line of the pencil that is parallel to the coordinate axis Oz; $-\pi/2 < u < \pi/2$, $0 \le v \le 2\pi$.

The fixed straight line passes over the point with coordinates (0; $-a$; 0).

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{a^2}{\cos^4 u} + \frac{2ar\cos v}{\cos u} + r^2\cos^2 v,$$
$$F = -\frac{ar\sin u\sin v}{\cos^2 u}, \; B = r;$$
$$A^2B^2 - F^2 = r^2\left[\left(\frac{a}{\cos u} + r\cos v\right)^2 + \frac{a^2\sin^2 u\cos^2 v}{\cos^4 u}\right];$$
$$L = \frac{r^2\cos^2 v}{\sqrt{A^2B^2 - F^2}}\left(r\cos v + \frac{a}{\cos u} - \frac{2a\sin^2 u}{\cos^3 u}\right),$$
$$M = -\frac{r^2 a\sin v\cos v\sin u}{\cos^2 u\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{r^2}{\sqrt{A^2B^2 - F^2}}\left(\frac{a}{\cos u} + r\cos v\right).$$

The surface is given in nonorthogonal, nonconjugate system of the curvilinear coordinates u, v.

In Fig. 2, the surface with the straight line of the centers having $r < |a|$; $0 \le v \le 2\pi$; $-\pi/4 < u < \pi/4$ is shown.

When $a = 0$, the surface becomes *a spherical surface*. The coordinate lines $u = 0$, $v = 0$ and $v = \pi$ are the lines of the principal curvatures.

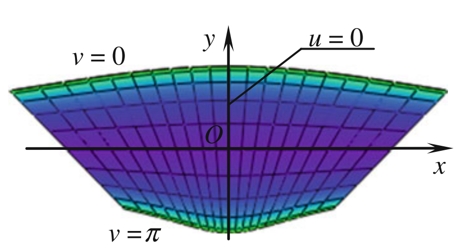


Fig. 1

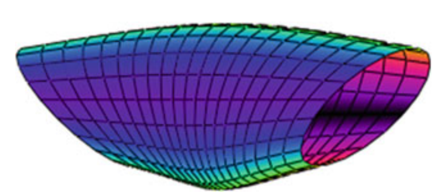


Fig. 2

The coordinate planes yOz and xOy are the planes the symmetry of the surface in question.

(1) Implicit equation:

$$x^2y^2 = (a + y)^2\left(r^2 - y^2 - z^2\right).$$

In the cross sections of the surface by planes $x = d = \text{const}$ (Fig. 1), the closed curves

$$z = \pm\frac{\sqrt{(a+y)^2(r^2 - y^2) - d^2y^2}}{(a+y)},$$

that are symmetrical relative to the coordinate axis Oy, are located.

In the cross sections of the surface by the planes $y = c = \text{const}$ (Fig. 1), ellipses

$$\frac{x^2 - y^2}{(a+c)^2(r^2 - c^2)} + \frac{z^2}{(r^2 - c^2)} = 1$$

lie. It should be noted that $c < r$.

Reference

Krivoshapko SN. Cyclic surfaces with the circles in the planes of a pencil and with the straight directrixes. Structural Mechanics of Engineering Constructions and Buildings. Moscow. 2004; Iss. 13, p. 8-13.

■ Cyclic Surface with Circles in Planes of Pencil with a Straight Directrix and a Fixed Straight of Pencil that are Lying on Different Sides of a Plane Center-to-Center Line

Cyclic surface with circles in planes of pencil with a straight directrix and a fixed straight of pencil lying on different sides of a plane center-to-center line is formed by a circle of a constant radius in the process of the motion of its point that is at the most distance from the fixed straight line of the pencil of planes along the straight line. And the circles must remain all the time at the plans of pencil passing through the fixed straight line which is perpendicular to the directrix straight line (Fig. 1).

Forms of definition of the surface

(1) Parametrical equations:

$$x = x(u,v) = \left[\frac{a}{\cos v} - r(1 - \cos v)\right]\sin u,$$
$$y = y(u,v) = r(1 - \cos v)\cos u,$$
$$z = z(v) = r\sin v,$$

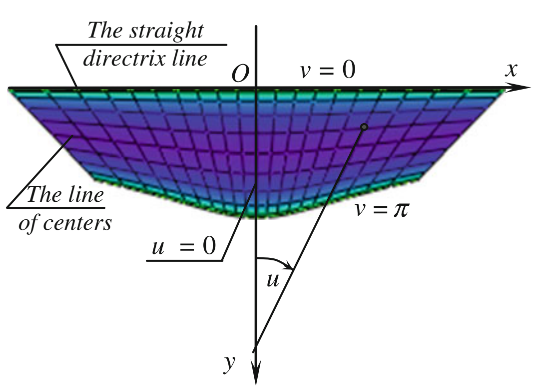


Fig. 1

where r is a constant radius of the generatrix circles; a is the shortest distance the directrix straight line of the generatrix circles from the fixed straight line of the pencil of planes that is parallel to the coordinate axis Oz (Fig. 1); $0 \le v \le 2\pi$; $-\pi/2 < u < \pi/2$. The directrix straight line coincides with the coordinate axis Ox (Fig. 1).

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{a^2}{\cos^4 u} - \frac{2ar(1 - \cos v)}{\cos u} + r^2(1 - \cos v)^2$$
$$= \left[\frac{a}{\cos u} - r(1 - \cos v)\right]^2 + \frac{a^2\sin^2 u}{\cos^4 u},$$
$$F = -\frac{ar\sin u\sin v}{\cos^2 u}, \quad B = r;$$
$$A^2B^2 - F^2 = r^2\left[\left(\frac{a}{\cos u} - r(1 - \cos v)\right)^2 + \frac{a^2\sin^2 u\cos^2 v}{\cos^4 u}\right];$$
$$L = \frac{r^2\cos v(1 - \cos v)}{\sqrt{A^2B^2 - F^2}}\left[\frac{a}{\cos u} - r(1 - \cos v) - \frac{2a\sin^2 u}{\cos^3 u}\right],$$
$$M = \frac{r^2 a\sin v\cos v\sin u}{\cos^2 u\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{-r^2}{\sqrt{A^2B^2 - F^2}}\left[\frac{a}{\cos u} - r(1 - \cos v)\right].$$

The surface is given in nonorthogonal, nonconjugate system of the curvilinear coordinates u, v.

In Fig. 2, the cyclic surface in question with $0 \le v \le 2\pi$; $-\pi/4 < u < \pi/4$ is shown.

The curvilinear coordinate $u = 0$ coincides with a generatrix circle and lies in the plane of symmetry ($x = 0$) of the

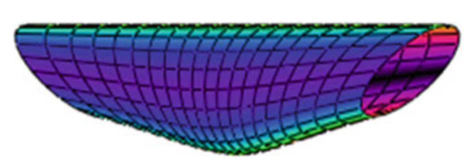


Fig. 2

cyclic surface. The curvilinear coordinate $v = 0$ coincides with the straight directrix line of the cyclic surface.

The coordinates lines $v = 0$ and $v = \pi$ lie in another plane of symmetry ($z = 0$) of the cyclic surface.

(1) Implicit equation:

$$x^2y^2\left[x^2y^2 + 2(a-y)^2(y^2+z^2-2r^2)\right] + (a-y)^4\left[(y^2+z^2) - 4r^2y^2\right] = 0$$

Reference

Krivoshapko SN. Cyclic surfaces with the circles in the planes of a pencil and with the straight directrixes. Structural Mechanics of Engineering Constructions and Buildings. Moscow. 2004; Iss. 13, p. 8-13.

■ Cyclic Surface with Circles in Planes of Pencil, with a Straight Directrix and a Fixed Straight of Pencil that are Lying on the Same Side of a Plane Center-to-Center Line

Cyclic surface with circles in planes of pencil, with a straight directrix and a fixed straight of pencil that are lying on the same side of a plane center-to-center line is formed by a circle of a constant radius in the process of the motion of its point that is the nearest to the fixed straight line of the pencil of planes along the straight line. And the circles must remain all the time at the planes of the pencil passing through the fixed straight line which is perpendicular to the directrix straight line and simultaneously to the plane in which this directrix straight and the plane line of the centers of the generatrix circles lie (Fig. 1).

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u,v) = \left[\frac{a}{\cos u} + r(1+\cos v)\right]\sin u,$$
$$y = y(u,v) = r(1+\cos v)\cos u$$
$$z = z(v) = r\sin v$$

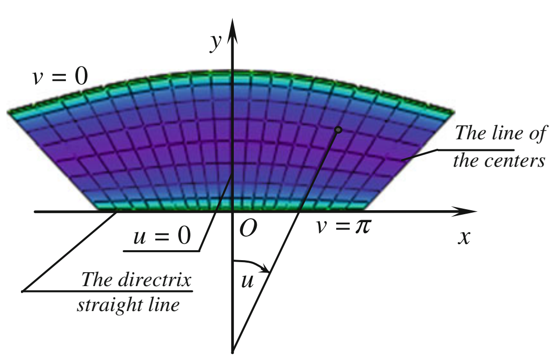


Fig. 1

where r is a constant radius of the generatrix circles; a is the shortest distance the directrix straight line of the generatrix circles from the fixed straight line of the pencil of planes that is parallel to the coordinate axis Oz; $0 \le v \le 2\pi$; $-\pi/2 < u < \pi/2$. The directrix straight line coincides with the coordinate axis Ox (Fig. 1).

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{a^2}{\cos^4 u} + \frac{2ar(1+\cos v)}{\cos u} + r^2(1+\cos v)^2,$$
$$F = -\frac{ar\sin u\sin v}{\cos^2 u},$$
$$B = r;$$
$$A^2B^2 - F^2 = r^2\left\{\left[\frac{a}{\cos u} + r(1+\cos v)\right]^2 + \frac{a^2\sin^2 u\cos^2 v}{\cos^4 u}\right\};$$
$$L = \frac{r^2\cos v(1+\cos v)}{\sqrt{A^2B^2-F^2}}\left[r(1+\cos v) + \frac{a}{\cos u} - \frac{2a\sin^2 u}{\cos^3 u}\right],$$
$$M = -\frac{r^2 a\sin v\cos v\sin u}{\cos^2 u\sqrt{A^2B^2-F^2}},$$
$$N = \frac{r^2}{\sqrt{A^2B^2-F^2}}\left[\frac{a}{\cos u} + r(1+\cos v)\right].$$

The surface is given in nonorthogonal, nonconjugate system of curvilinear coordinates u, v. In Fig. 2, the cyclic surface with the straight line of the centers with $0 \le v \le 2\pi$; $-\pi/4 < u < \pi/4$ is shown. If $a = 0$, the surface becomes *a closed circular torus* (see also Chap. "2. Surfaces of Revolution"). The coordinate lines $v = 0$, $v = \pi$ and $u = 0$ are the lines of principle curvatures.

In Fig. 3, the model of a cyclic surface in question is presented. The frame of this surface was made from the same steel rings.

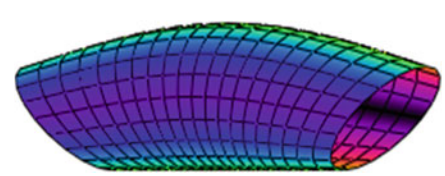


Fig. 2

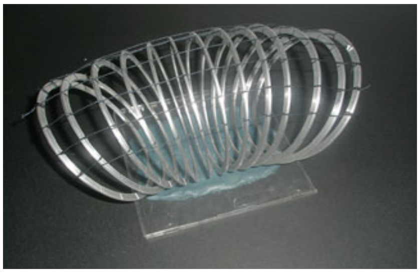


Fig. 3

(2) Implicit equation:

$$x^2y^2 + (a+y)^2(z^2+y^2-2r^2) = 2r(a+y)^2\sqrt{r^2-z^2}.$$

The coordinate planes $x = 0$ and $z = 0$ are planes of symmetry of the cyclic surface.

Reference

Krivoshapko SN. Cyclic surfaces with the circles in the planes of a pencil and with the straight directrixes. Structural Mechanics of Engineering Constructions and Buildings. Moscow. 2004; Iss. 13, p. 8-13.

■ Cyclic Surface with Circles of Variable Radius in Planes of Pencil and with Three Straight Parallel Directrixes

Cyclic surface with circles of variable radius in planes of pencil and with three straight parallel directrixes is formed by a circle of the variable radius, so that their centers lie on the same straight directrix. The straight lines connecting the points that are at the most distance from *the fixed straight line* and the nearest to it are parallel to the straight line of the centers. The circles must be all the time at the planes of the pencil passing through the fixed straight line which is perpendicular to the plane in which three directrix parallel straight lines are placed.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u,v) = (a + r\cos v)\tan u,$$
$$y = y(v) = r\cos v,$$
$$z = z(u,v) = r\frac{\sin v}{\cos u},$$

where r is a minimal radius of the generatrix circle lying at the plane $x = 0$; a is the shortest distance the straight line of the centers from the fixed straight line of the pencil of the planes which is parallel to the coordinate axis Oz; $0 \le v \le 2\pi$; $-\pi/2 < u < \pi/2$. The straight line of the centers of the generatrix circles coincides with the axis Ox (Fig. 1). The radiuses R of the generatrix circles are calculated with the help of formula:

$$R = R(u) = z = z(u,v) = \frac{r}{\cos u}.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{(a + r\cos v)^2 + r^2\sin^2 v\sin^2 u}{\cos^4 u},$$
$$F = -\frac{ar\sin u\sin v}{\cos^3 u},\ B = \frac{r}{\cos u};$$
$$A^2B^2 - F^2 = \frac{r^2}{\cos^6 u}\left[(a + r\cos v)^2 + (r^2 - a^2)\sin^2 v\sin^2 u\right];$$
$$L = -\frac{r^2\sin^2 v(a + r\cos v)}{\cos^3 u\sqrt{A^2B^2 - F^2}},$$
$$M = -\frac{r^2(r + a\cos v)\sin v\sin u}{\cos^4 u\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{r^2(a + r\cos v)}{\cos^3 u\sqrt{A^2B^2 - F^2}};$$
$$k_v = \frac{(a + r\cos v)}{\cos u\sqrt{A^2B^2 - F^2}},$$
$$K = \frac{-r^4\sin^2 v}{\cos^8 u(A^2B^2 - F^2)^2} \cdot \left[(a + r\cos v)^2\cos^2 u + (r + a\cos v)^2\sin^2 u\right] \le 0,$$

The surface is given in nonorthogonal, nonconjugate system of curvilinear coordinates u, v. In Fig. 2, the cyclic surface with three straight directrix lines is shown when

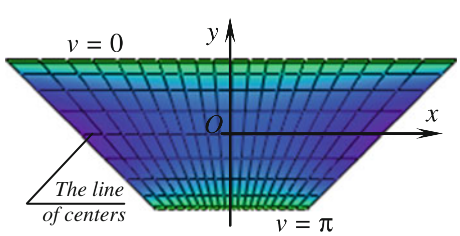


Fig. 1

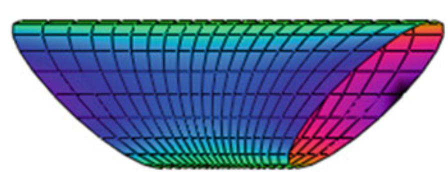


Fig. 2

$0 \le v \le 2\pi$; $-\pi/4 < u < \pi/4$. The coordinate lines $v = 0$, $v = \pi$ and $u = 0$ are the lines of principle curvatures. The cyclic surface in question is a surface of *negative Gaussian curvature* ($K < 0$). Only along the coordinate lines $v = 0$ and $v = \pi$, we have the parabolic points ($K = 0$).

(2) Implicit equation:

$$x^2(r^2 - y^2) - (a + y)^2(z^2 + y^2 - r^2) = 0.$$

The coordinate planes $x = 0$ and $z = 0$ are the planes of symmetry of the cyclic surface. There are two branches of a hyperbola

$$-\frac{x^2}{a^2} + \frac{z^2}{r^2} = 1$$

at the cross section of the surface by the plane $y = 0$.

Reference

Krivoshapko SN. Cyclic surfaces with the circles in the planes of a pencil and with the straight directrixes. Structural Mechanics of Engineering Constructions and Buildings. Moscow. 2004; Iss. 13, p. 8-13.

17.5 Cyclic Surfaces of Revolution

Cyclic surfaces of revolution are formed by rotation of any circle placed arbitrarily about an axis of revolution. The plane cross sections of these surfaces are bicircular elliptical and rational curves of the fourth order that are of great importance in technique. The plane cross sections with two mutually perpendicular axes of symmetry arouse the most interest. For example, there are *Perseus sections* on a torus. For every cyclic surface of revolution of general type, one may select any *paraboloid of revolution* which is called *a Perseus paraboloid*. Every plane tangent to a Perseus paraboloid intersects the cyclic surface of revolution along a curve having two mutually perpendicular axis of revolution. Such curves are called *Perseus sections*.

A vector equation of a line of the centers of the generatrix circles is

$$b\boldsymbol{h}(u) = b(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u),$$

where b is a radius of the centerline. The disposition of a generatrix circle is determinated with the help of Euler angles, i.e., θ is the angle of the vector $\boldsymbol{h}(u)$ with the trace of the intersection of the plane with a generatrix circle of a radius a and the coordinate plane xOy (angle of rotation about the axis Oz); ω is the angle of the plane with a generatrix circle with the axis of rotation.

Forms of definition of the cyclic surfaces of revolution

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = b\boldsymbol{h}(u) + a\boldsymbol{e}(u, v),$$

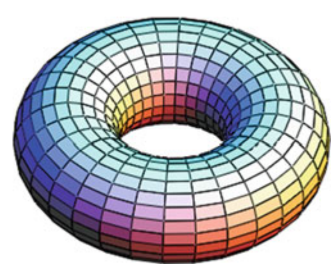
where

$$\boldsymbol{p}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u,$$

$$\begin{aligned}\boldsymbol{e}(u, v) = {} & \boldsymbol{h}(\cos\theta\cos v - \sin\theta\sin\omega\sin v) \\ & + \boldsymbol{p}(\sin\theta\cos v + \cos\theta\sin\omega\sin v) + \boldsymbol{k}\cos\omega\sin v.\end{aligned}$$

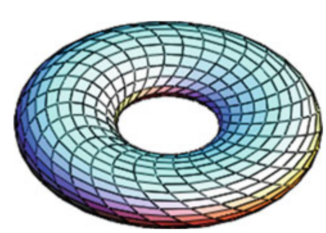
(2) Parametrical equations (Figs. 1, 2 and 3):

$$\begin{aligned}x(u, v) &= b\cos u + a[(\cos\theta\cos v - \sin\theta\sin\omega\sin v)\cos u \\ &\quad -(\sin\theta\cos v + \cos\theta\sin\omega\sin v)\sin u]; \\ y(u, v) &= b\sin u + a[(\cos\theta\cos v - \sin\theta\sin\omega\sin v)\sin u \\ &\quad +(\sin\theta\cos v + \cos\theta\sin\omega\sin v)\cos u], \\ z(u, v) &= a\cos\omega\sin v.\end{aligned}$$



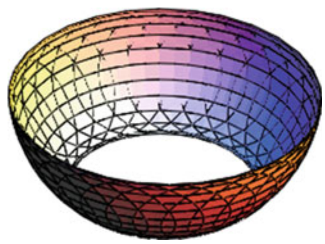
$b = 2$ m; $a = 1$ m; $\theta = \omega = 0$

Fig. 1 Circular torus



$b = 2$ m; $a = 1$ m; $\omega = \pi/3$

Fig. 2 $\theta = 0$



$b = 2$ m; $a = 1$ m; $\omega = \pi/2$

Fig. 3 $\theta = \pi/6$

Coefficients of the fundamental forms of the surface:

$$E = b^2 + 2ab(\cos\theta\cos v - sin\theta\sin\omega\sin v) + a^2\left(1 - \cos^2\omega\sin^2 v\right)$$
$$F = -ab(\sin\theta\sin v - \cos\theta\sin\omega\cos v) + a^2\sin\omega;\ \ G = b^2;$$
$$L = \frac{b(\sin\theta\cos v + \cos\theta\sin\omega\sin v)T_1 + (b\cos\theta + a\cos v)\cos\omega}{\sigma};$$
$$M = -\frac{a\cos\omega}{\sigma}[(b\cos\theta + a\cos v)\sin\omega$$
$$-b(\sin\theta\cos v + \cos\theta\sin\omega\sin v)\sin v];$$
$$N = -\frac{a}{\sigma}(b\cos\theta + a\cos v)\cos\omega;$$
$$T_1 = -[b\sin\theta + a(\cos\omega - 2\sin\omega)\sin v]\cos\omega;$$
$$T_2 = -b(cos\theta\cos v - \sin\theta\sin\omega\sin v) - a\left(1 - \cos^2\omega\sin^2 v\right);$$
$$\sigma^2 = b^2(\sin\theta\cos v + \cos\theta\sin\omega\sin v)^2 + (b\cos\theta + a\cos v)^2\cos^2\omega.$$

If $\theta = \omega = 0$, the cyclic surface degenerates into a circular torus (Fig. 1).

Additional Literature

1. *Aseev VI, Aseev VV*. Cyclic surfaces of revolution. The Materials of the Scientific-and-Technical Conference of Novomoskovsk. Filial of Moscow Chemistry-and-Technological Institute. Novomoskovsk, February 6-11, 1984, Moscow. 1984; Part 3, p. 174-178 (4 refs.), Ruk. dep. v VINITI, November 28, 1984, No. 7581-84 Dep.
2. *Aseev VI, Aseev VV*. Graphical solution of the problem of Villarso for cyclic surfaces of revolution. The Materials of the Scientific-and-Technical Conference of Novomoskovsk. Filial of Moscow Chemistry-and-Technological Institute. Novomoskovsk, February 6-11, 1984, Moscow. 1984; Part 3, p. 179-184 (2 refs.), Ruk. dep. v VINITI, November 28, 1984, No. 7581-84 Dep.
3. *Aseev VI., Aseev VV*. Generalized cross sections of Perseus of cyclic surfaces of revolution. The Materials of the Scientific-and-Technical Conference of Novomoskovsk. Filial of Moscow Chemistry-and-Technological Institute. Novomoskovsk, February 6-11, 1984, Moscow. 1984; Part 3, p. 191-194 (3 refs.), Ruk. dep. v VINITI, November 28, 1984, No. 7581-84 Dep.

■ Cyclic Surface of Revolution with the Rotation Axis Parallel to the Planes with Generatrix Circles

A cyclic surface of revolution with the rotation axis parallel to the planes with generatrix circles of a constant radius a is a special case of cyclic surface of revolution when $\omega = 0$ (see also a Sect. "17.5. Cyclic Surfaces of Revolution").

Forms of definition of a cyclic surface of revolution (when $\omega = 0$)

(1) Vector equation:

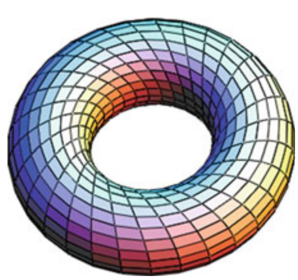
$$\boldsymbol{r} = \boldsymbol{r}(u, v) = b\boldsymbol{h}(u) + a\boldsymbol{e}(u, v),$$

where

$$\boldsymbol{e}(u, v) = \boldsymbol{h}\cos\theta\cos v + \boldsymbol{p}\sin\theta\cos v + \boldsymbol{k}\sin v;\ \ \boldsymbol{p}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u.$$

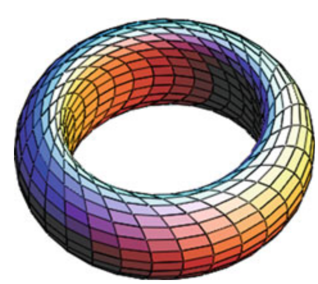
(2) Parametrical equations (Figs. 1, 2 and 3):

$$x(u, v) = b\cos u + a[\cos\theta\cos u - \sin\theta\sin u]\cos v,$$
$$y(u, v) = b\sin u + a[\cos\theta\sin u + \sin\theta\cos u]\cos v,$$
$$z(u, v) = a\sin v.$$



$b = 2$ cm; $a = 1$ cm; $\theta = \pi/6$

Fig. 1



$b = 2$ cm; $a = 1$ cm; $\theta = \pi/3$

Fig. 2

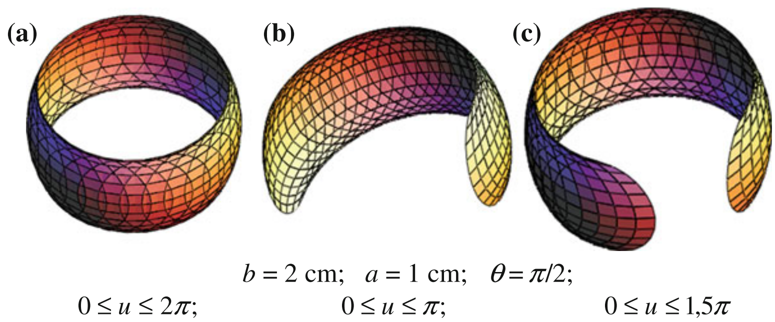


Fig. 3

Coefficients of the fundamental forms of the surface may be obtained from the general formulas given in a Sect. "17.5. Cyclic Surfaces of Revolution," substituting in them $\omega = 0$.

The surface shown in Fig. 3a is called "Wedding-ring." The surface shown in Fig. 3c is called "Bracelet."

■ **Cyclic Surface of Revolution with the Rotation Axis Intersecting the Planes with Generatrix Circles at the Constant Angle**

A cyclic surface of revolution with the rotation axis intersecting the planes with generatrix circles at the constant angle (ω = const) is a special case of *cyclic surfaces of revolution* when $\theta = 0$ (see also a Sect. "17.5. Cyclic Surfaces of Revolution").

Forms of definition of the cyclic surface of revolution (when $\theta = 0$)

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = b\boldsymbol{h}(u) + a\boldsymbol{e}(u, v),$$

where

$$\boldsymbol{e}(u, v) = \boldsymbol{h}\cos v + \boldsymbol{p}\sin\omega\sin v + \boldsymbol{k}\cos\omega\sin v;$$
$$\boldsymbol{p}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u.$$

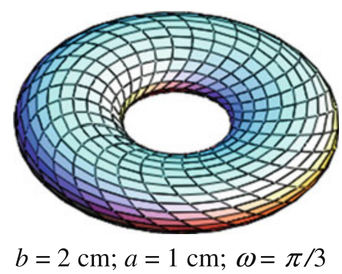


Fig. 2

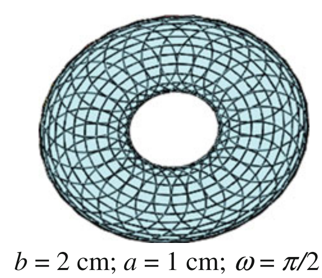


Fig. 3 The annulus

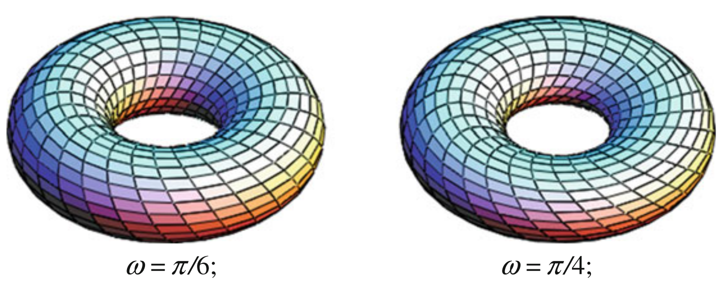


Fig. 1 $b = 2$ cm; $a = 1$ cm

(2) Parametrical equations (Figs. 1, 2 and 3):

$$x(u,v) = b\cos u + a[\cos v\cos u - \sin\omega\sin v\sin u];$$
$$y(u,v) = b\sin u + a[\cos v\sin u + \sin\omega\sin v\cos u].$$
$$z(u,v) = a\cos\omega\sin v$$

Coefficients of the fundamental forms of the surface may be obtained from the general formulas given in a Sect. "17.5. Cyclic Surfaces of Revolution." But it is necessary to substitute in them $\theta = 0$.

One-Sided Surfaces

18

One-sided and *two-sided* surfaces are two types of surfaces differing in the way of their disposition in the space. To be more correct, one-sided and two-sided surfaces are two types of varieties differing in the method of putting of them into complete space. For example, a cylinder is a two-sided surface and a Möbius strip is a one-sided surface, in spite of this, their physical models may be made from the same long rectangular strip. The main difference of these surfaces is the following: the boundary of a cylinder consists of two curves but the boundary of a Möbius strip consists of only one curve.

The two-sidedness and one-sidedness are connected with *orientability* and *non-orientability* but in contrast to the lasts, they are not inherent properties of the surface and depend on the complete space. The present view on orientation is given within the bounds of the generalized theories of *cohomology*. At classic case, the orientation is a choice of one class of the systems of coordinates connected with themselves positively in a certain sense.

Let a normal vector is served round to a closed curve on a smooth surface, put in some space so that it remains a normal one. If the direction of the normal vector after returning into the point of departure coincides with the initial one irrespective of the choice of the curve, then the surface is called a two-sided surface, in the opposite case, it will be a one-sided surface.

D. Hilbert has considered that any one-sided surface must intersect itself. Closed two-sided surfaces of the even bundle do not exist. At the same time, the closed one-sided surfaces may be both with *even bundle* and *odd bundle*. On a closed h-coherent surface, one may draw $h - 1$ closed curves not dividing the surface, but every system consisting of h similar curves divides the surface certainly. A model of a projective plane may be obtained from a globe surface. Generally, every one-sided surface may be brought to conformity with any two-sided surface. It is proved, that for any one-sided surface, a two-sided surface being a two-sheeted covering surface for the first surface, exists.

A Möbius strip and a heptahedron which is a two-coherent surface under the Euler theorem on polyhedron are the simplest one-sided surfaces. Two engravings of the Holland artist Maurits Cornelis Escher are devoted to Möbius strip. O. Roeschel has made a Möbius machinery using the properties of a physical model of a Möbius strip without self-crossing. The model was put together from plane plates. He has proved that not all of the existing physical models of a Möbius strip may be used as the basic form for creation of his mechanism. *A heptahedron* is constructed on the base of a *regular octahedron* with the addition of tree squares disposed in the three planes defined by the diagonals of the octahedron. After this, four triangles are removed. Two removed triangles are at the top half and two triangles are at the lower half. The figure obtained has seven sides, twelve ribs and six vertexes. Heptahedron has $h = 2$. It may be deformed into a *Roman surface* (see also "The Roman Surface").

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■ One-Sided Ruled Surface (Möbius Strip)

The strip twisted ones is called a *ribbon of Möbius* or *Möbius strip, also Mobius or Moebius band*. In contrast to its model, a *Möbius strip* has no thickness. *Möbius strip* is a non-orientable surface, has *zero Euler characteristic* and its edge is a closed line. Hence, it is a surface with only one side and only one boundary component.

Non-orientability of Möbius strip means that in the process of motion of a plane not- symmetrical figure inside a *Möbius strip,* this figure, having returned into the initial point, transforms into its own mirror image. An attempt to paint the only one side of the surface is condemned to failure.

Möbius strip in the Euclidian space E^3 is a *one-sided surface*. Möbius strip was discovered independently by the German mathematicians August Ferdinand Möbius and Johann Benedict Listing in 1858–1965. Near the Museum of Science and Technology in Washington, the steel Möbius strip is rotating on the pedestal.

Let us slit a Möbius strip along its long axis and then we shall have an oriented strip twisted at 2π. Let us slit the obtained strip along the axis for the second time, then two strips engaged in R^3 will be. Möbius strip cannot be realized everywhere as a smooth surface of positive Gaussian curvature.

In particular, the twisted paper model is a developable surface. M.Ya. Gromov (1963) has designed the forms of one-sided developable surfaces with segments that are similar to Möbius strip, bounded by closed geodesic lines transforming, in general case, after developing into parallel sides of isosceles trapezoids and in special case, transforming into parallel sides of rectangles.

A Möbius strip must be a closed regular system of torses and its edge must intersect twice every of the generatrixes of the system of the torses.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(t, \varphi) = [1 + t\sin(\varphi/2)]\cos\varphi,$$
$$y = y(t, \varphi) = [1 + t\sin(\varphi/2)]\sin\varphi,$$
$$z = z(t, \varphi) = t\cos(\varphi/2).$$

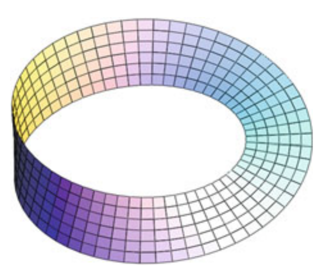

Fig. 1

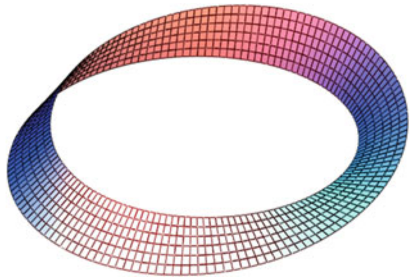

Fig. 2

When $-1/4 \leq t \leq 1/4$, the mapping of Möbius strip into the coordinate space is an *embedding*. Möbius strip of infinite width with the plane metric does not embed into R^3.

Coefficients of the fundamental forms of the surface:

$$A = 1,\ \ F = 0,\ \ B^2 = \frac{t^2}{4} + \left(1 + t\sin\frac{\varphi}{2}\right)^2,$$
$$L = 0,\ \ M = \frac{-1}{2B},\ \ N = \frac{1}{B}\cos\frac{\varphi}{2}\left[\frac{t^2}{2} + \left(1 + t\sin\frac{\varphi}{2}\right)^2\right],$$
$$K = \frac{-1}{4B^4},\quad H = \frac{N}{2B^2}.$$

(2) Parametrical form of the definition of a Möbius strip that is an elliptical in plan:

$$x = x(t,\varphi) = \left(a + t\sin m\frac{\varphi}{2}\right)\cos\varphi,$$
$$y = y(t,\varphi) = \left(b + t\sin m\frac{\varphi}{2}\right)\sin\varphi,$$
$$z = z(t,\varphi) = t\cos m\frac{\varphi}{2},$$

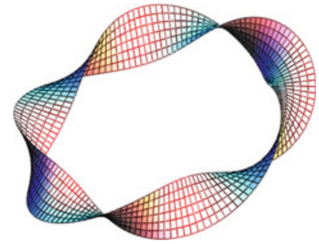

Fig. 3

where m is an integer. When m = 1, we receive a usual elliptical *Möbius strip* (Fig. 2), when m = 3, 5, …, we obtain a *Möbius strip* twisted several times (Fig. 3, m = 5).

(3) Implicit equation (W. Dedonder 1987):

$$y(x^2 + y^2 + z^2 - a^2) - 2z(x^2 + y^2 + ax) = 0.$$

In this case, the Möbius strip is located on the circular plan.

It is straightforward to find algebraic equations the solutions of which have the topology of a Möbius strip, but in general, these equations do not describe the same geometric shape that one gets from the twisted paper model.

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■ Cross Cap

A cross cap is a one-sided surface. The word "cross-cap" is sometimes also written without the hyphen as the single word "*crosscap*." A cross cap is a model of Möbius surface having the circular boundary. In spite of the one-sidedness, a cross cap obviously may be a top of a vessel. This is possible, because this surface has a line of self-crossing.

Let us cut a cross cap along the line of self-crossing, then one may obtain a circle with the quadrangular or circular hole using special deformation. A model in question of the projective plane has two singular points that are two end points of the line of self-crossing. *A sphere with one cross-cap* has traditionally been called a real projective plane. *A sphere with two cross-caps* having coinciding boundaries is topologically equivalent to *the Klein bottle*.

W. Boy has constructed another model of the projective plane which has no singular points and is bent everywhere (see also "The Boy Surface").

Forms of definition of the surface

(1) The surface with particularities given by parametrical equations (Fig. 1):

$$x = x(\theta,\varphi) = (1 + \cos 2\theta)\cos 2\varphi\prime,$$
$$y = y(\theta,\varphi) = (1 + \cos 2\theta)\sin 2\varphi$$
$$z = z(\theta,\varphi) = \sin 2\theta\sin\varphi,$$

where $-\pi/2 \leq \theta \leq \pi/2;\ 0 \leq \varphi \leq 2\pi$. This surface is one of the models of the projective plane. If one cut by a plane a not large disk from this model of the projective plane, then the remained segment will be a cross cap.

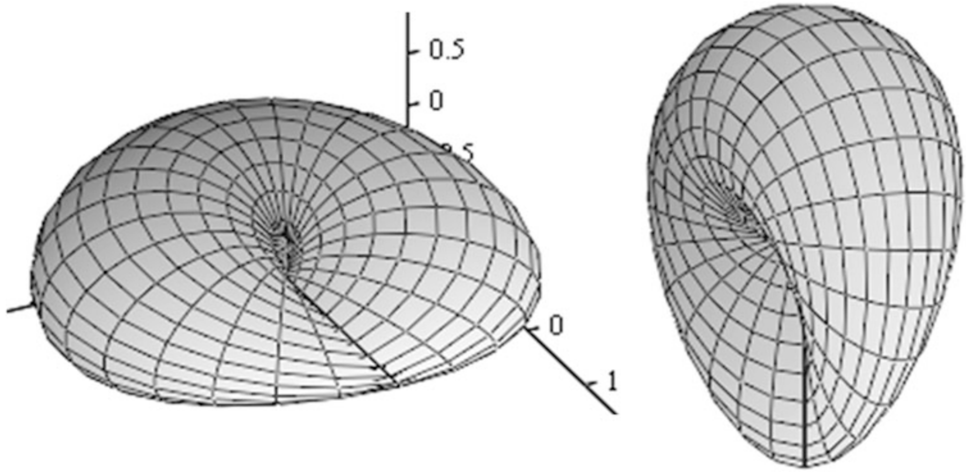

Fig. 1

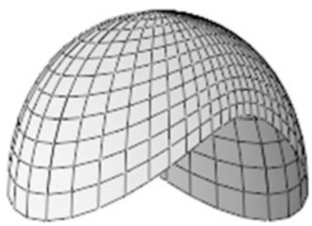

Fig. 2

The line of self-crossing of the cap has $\varphi = 0$. *Pinch points* are the points with $(\theta, \varphi) = (0; 0)$ and $\theta = \pi/2$ (Fig. 1).

(2) Implicit equation:

$$\left(k_1x^2 + k_2y^2\right)\left(x^2 + y^2 + z^2\right) = 2z\left(x^2 + y^2\right),$$

where $k_1 \neq k_2$.

(3) Parametrical equations:

$$x = x(\theta, \varphi) = \frac{\cos\theta\cos\varphi}{k_1\cos^2\varphi + k_2\sin^2\varphi},$$
$$y = y(\theta, \varphi) = \frac{\cos\theta\sin\varphi}{k_1\cos^2\varphi + k_2\sin^2\varphi},$$
$$z = z(\theta, \varphi) = \frac{1 + \sin\theta}{k_1\cos^2\varphi + k_2\sin^2\varphi}, \quad k_1 \neq k_2.$$

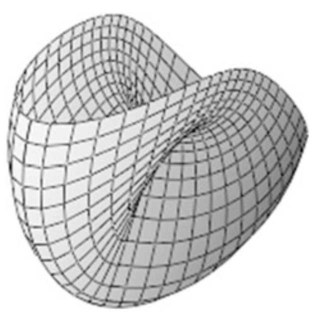

Fig. 3

The formulas presented above define a smooth regular parameterization out of the singular points.

In Fig. 2, the segment of cross cap having the boundaries $-\pi \leq \theta \leq 0$ and $0 \leq \varphi \leq \pi$ is shown when $k_1 = 1$, $k_2 = 2$.

In Fig. 3, another segment of the cross cap with $k_1 = 1$; $k_2 = 2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq \pi$ is given.

The segment of the cross cap limited by the line $-\pi/2 \leq \theta \leq \pi/2$ is presented in Fig. 4.

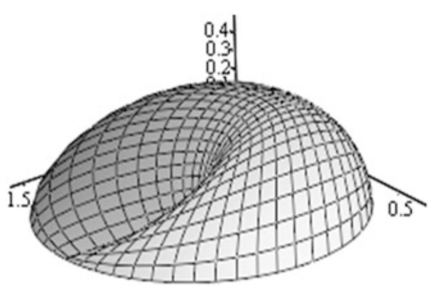

Fig. 4

Additional Literature

Gray A. Modern Differential Geometry of Curves and Surfaces with Mathematica: 2nd ed. Boca Raton, FL: CRC Press. 1998; 1053 p.

■ The Roman Surface

Working with *the Roman surface*, it is necessary to remember that it goes in *a tetrahedron* and touches all of its faces along the circle. Near these four circles, the total curvature of surface changes its sign. At the locality of the point of positive curvature, the surface is bent at one side from its tangent plane. In the point of negative curvature, the surface intersects its tangent plane.

A. Coffman ascribes the Roman surface to the first type of Steiner surfaces. Steiner surfaces contain ten types. The Roman surface, also called *the Steiner surface* (not to be confused with *the class of Steiner surfaces* of which the Roman surface is a particular case), is *a quartic non-orientable surface.* Jakob Steiner (1796–1863) was a Swiss mathematician who became a professor at the University of Berlin. He visited Rome in 1844 where he developed the concept of a surface that we now call Steiner's Roman Surface or the Roman surface.

The central point of the Roman surface is a usual triple point with coordinates (±1, 0, 0) = (0, ±1, 0) = (0, 0, ±1). Six endpoints of tree lines of self-crossing of the surface are singular points of pinch.

Projections of *the Veronese surface* into three-space necessarily have local self-intersection known as pinch points. One such projection is the cross cap, and another is Steiner's Roman surface.

Forms of definition of the surface

(1) Implicit equation:

$$x^2 + y^2 + z^2 + 2xy - 2xz - 2yz = (x + y - z)^2 \geq 0.$$

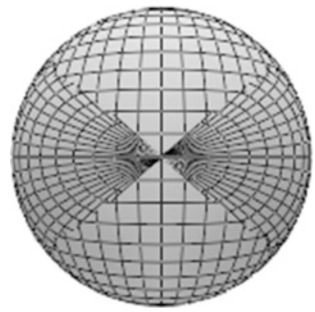

Fig. 1

(2) Implicit equation:

$$y^2z^2 + z^2x^2 + x^2y^2 + 2kxyz = 0.$$

The simplest construction is as the image of a sphere centered at the origin under the map $f(x, y, z) = (yz,\ xz,\ xy)$.

(3) Implicit equation:

$$\left(x^2 + y^2 + z^2 - k^2\right)^2 = \left[(z - k)^2 - 2x^2\right]\left[(z + k)^2 - 2y^2\right].$$

(4) Parametrical equations (Fig. 1):

$$x = x(u, v) = \frac{1}{2}\sin 2u \sin^2 v,$$
$$y = y(u, v) = \frac{1}{2}\sin u \cos 2v,$$
$$z = z(u, v) = \frac{1}{2}\cos u \sin 2v,$$

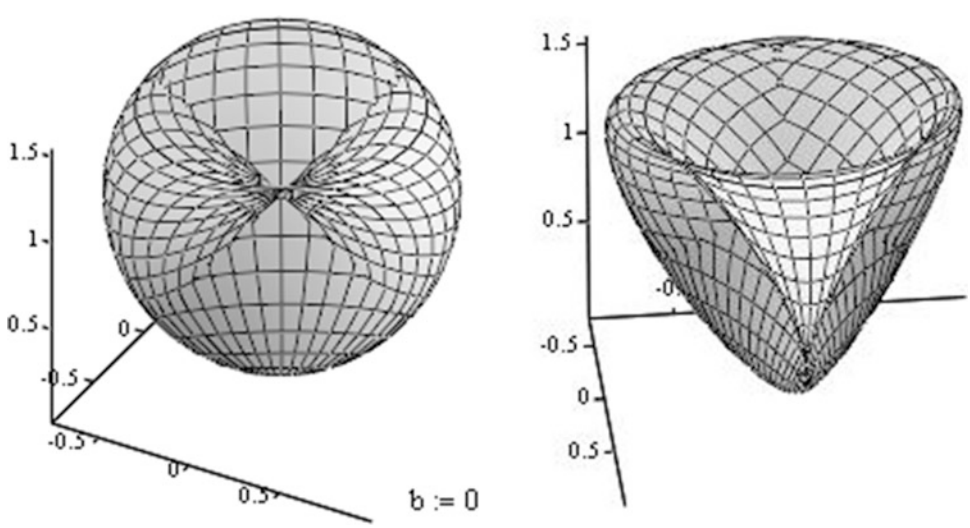

Fig. 2

where $0 \le u \le 2\pi$; $-\pi/2 \le v \le \pi/2$.

(5) The Roman surface and the Boy surface may be given by the same parametrical equations:

$$x = x(\varphi, \theta) = \frac{\sqrt{2}\cos^2\theta\cos 2\varphi + \sin 2\theta\cos\varphi}{2 - b\sqrt{2}\sin 3\varphi\sin 2\theta},$$
$$y = y(\varphi, \theta) = \frac{\sqrt{2}\cos^2\theta\sin 2\varphi - \sin 2\theta\sin\varphi}{2 - b\sqrt{2}\sin 3\varphi\sin 2\theta},$$
$$z = z(\varphi, \theta) = \frac{3\cos^2\theta}{2 - b\sqrt{2}\sin 3\varphi\sin 2\theta},$$

where $-\pi/2 \le \varphi \le \pi/2$; $0 \le \theta \le \pi$.

When $b = 0$, we can obtain the Roman surface (Fig. 2) and if $b = 1$, we have the Boy surface (see also "Boy Surface"). Changing a parameter b from 0 to 1, it is possible to transform smoothly the Roman surface into the Boy surface.

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■ The Klein Surface (The Klein Bottle)

The most known variant of the Klein bottle presents a bent tube of a variable diameter (Fig. 1). The narrower end goes through the wall of the tube and goes out through another wider opening of the tube, so that both boundary circles dispose concentrically. Bending the wide end of the tube inside and the narrow end outside one can join the both ends of the tube without peculiarities.

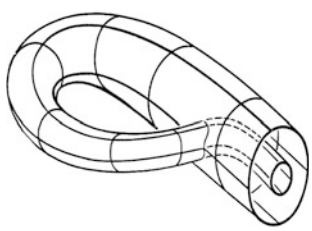

Fig. 1

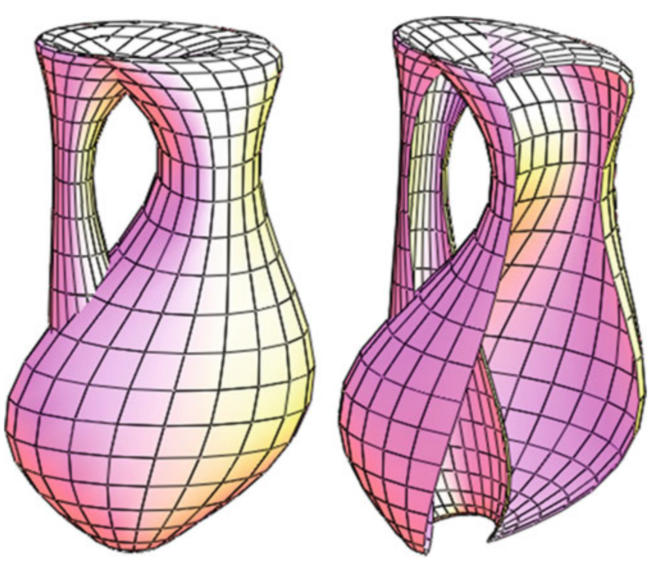

Fig. 2

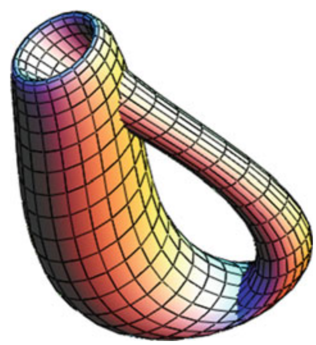

Fig. 3

A Möbius strip is a surface with boundary; the Klein bottle has no boundary. *The Klein bottle* (*the Klein surface*) is formed by splicing of two *Möbius strips* at their boundaries (Fig. 2). In other words, cutting the Klein bottle along the proper circle, one can obtain two *Möbius strips*.

The lemma: the Klein bottle is homeomorphous to a sphere stuck by two films of Möbius. The Klein bottle is an example of a non-orientable surface.

The Klein surface is a closed one-sided surface without singular points. It is topologically put into the 4-dimensional Euclidian space but cannot be embedded in E^3. This surface (Fig. 3) is a 3-coherent surface as a torus ($h = 3$). The Klein bottle may be called an alternative to a torus.

The Klein bottle was first described in 1882 by the German mathematician Felix Klein.

Forms of definition of the Klein surface

(1) Parametrical equations:

$$x = x(u, v) = \left(a + \cos\frac{u}{2}\sin v - \sin\frac{u}{2}\sin 2v\right)\cos u,$$
$$y = y(u, v) = \left(a + \cos\frac{u}{2}\sin v - \sin\frac{u}{2}\sin 2v\right)\sin u,$$
$$z = z(u, v) = \sin\frac{u}{2}\sin v + \cos\frac{u}{2}\sin 2v,$$

where $a > 2,\ \ 0 \le u \le 2\pi;\ \ 0 \le v \le 2\pi$

(2) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = (R_x, R_y, R_z),$$

where R_x, R_y, R_z are components at Cartesian coordinates of the following vector-function

$$\boldsymbol{R}(u, v) = \boldsymbol{R}_0(u) + \rho(u)[e_1(u)\cos v + e_2 \sin v],$$
$$\boldsymbol{R}_0(u) = \begin{pmatrix} a\sin 2u \\ 0 \\ b\cos u \end{pmatrix}, \quad \boldsymbol{e}_1(u) = \begin{pmatrix} b\sin u \\ 0 \\ a\cos 2u \end{pmatrix}\frac{1}{|R_0'|},\ \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$
$$\rho = \rho(u) = \rho_0 + \rho_1 \sin^{2n}(u - u_0),$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the unit vectors orthogonal to $\boldsymbol{R}'_0$. Here, for example, we may take

$$\rho_0 = 0.1;\ \rho_1 = 0.6; \quad a = 0.25;\ b = 1;$$
$$u_0 = 0.3;\ n = 4;$$
$$0 \le u \le \pi;\ 0 \le v \le 2\pi.$$

In Fig. 4, some modifications of the Klein surface are shown.

A one-sided topological surface in question has no inside or outside.

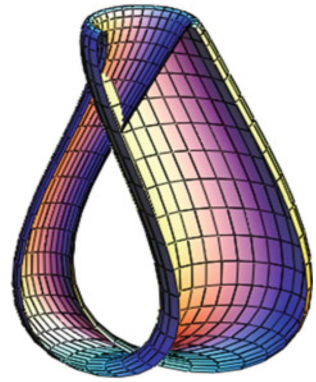

Fig. 4

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■ The Boy Surface

The Boy surface (*Boy's surface*) is *non-orientable one-sided surface*. This is one of three surfaces, which are obtained by joining of a Möbius band to an edge of the disk. Two other surfaces are *a cross cap* and *the Roman surface*. The Boy surface is a model of *the projective plane* that does not have singular points and is twisted everywhere continuously, i.e. Boy surface is an immersion of the real projective plane in 3-dimensional space. Boy surface is one of the two possible immersions of the real projective plane which have only a single triple point. The Boy surface has threefold symmetry and can be cut into three mutually congruent pieces.

The Boy surface has everywhere a continuous spherical image. On account of the Boy surface one-sidedness, a pair of diametrically opposite points of the globe corresponds with every point of the surface when spherical mapping.

The surface may be formed by superposing of the opposite points of the side of a hexagon cut from a surface of the globe. The curve of the self-crossing of the Boy surface consists of three loops passing through the same point. Three cavities of the surface (Fig. 1) pass through this point. These three cavities have the continuous tangent plane in one point. That is why, six ends of the loops converging to a common point must have three pairwise perpendicular tangents in this point. This is necessary and sufficient condition.

The Boy surface was named after Werner Boy who discovered it in 1901. A mathematical model of the Boy

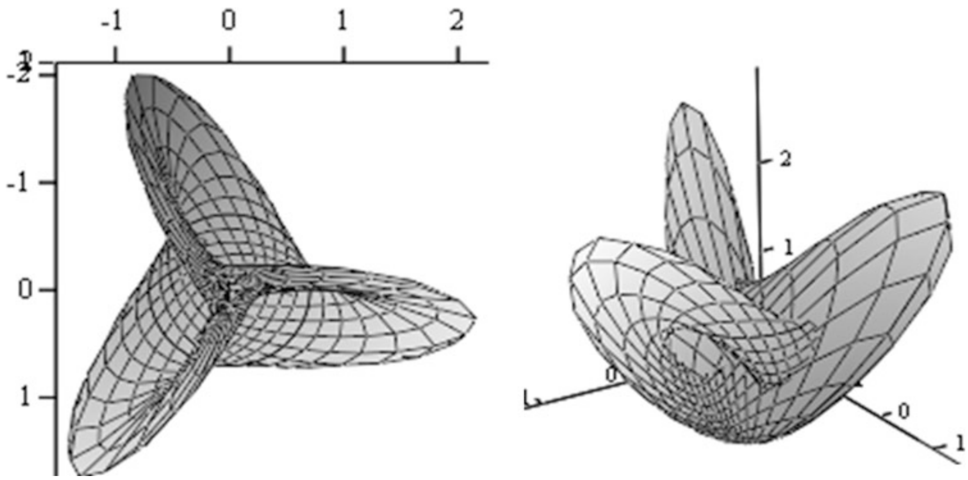

Fig. 1

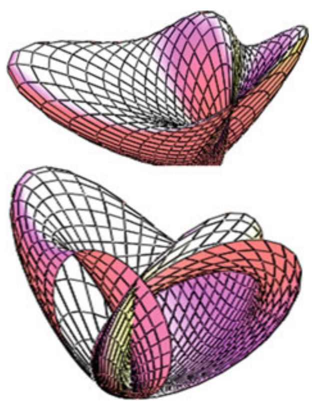

Fig. 2

surface with dimension of 2 m × 2 m made by students from wire is placed at McConnell Hall, Smith College. The Mathematical Research Institute of Oberwolfach has also a large model of a Boy surface outside the entrance, constructed, and donated by Mercedes-Benz in January 1991.

Forms of the definition of the Boy surface

(1) The Roman surface and the Boy surface may be given by the same parametrical equations:

$$x = x(\varphi, \theta) = \frac{\sqrt{2}\cos^2\theta\cos 2\varphi + \sin 2\theta \cos\varphi}{2 - b\sqrt{2}\sin 3\varphi \sin 2\theta},$$
$$y = y(\varphi, \theta) = \frac{\sqrt{2}\cos^2\theta\sin 2\varphi - \sin 2\theta \sin\varphi}{2 - b\sqrt{2}\sin 3\varphi \sin 2\theta},$$
$$z = z(\varphi, \theta) = \frac{3\cos^2\theta}{2 - b\sqrt{2}\sin 3\varphi \sin 2\theta}, \quad -\pi/2 \le \varphi \le \pi/2; \quad 0 \le \theta \le \pi.$$

When $b = 0$, we can obtain the Roman surface (see also "The Roman Surface") and if $b = 1$, we have the Boy surface (*Apery parameterization*). Changing a parameter b from 0 to 1, it is possible to transform smoothly the Roman surface into the Boy surface.

Figure 2 shows the Boy surface when $b = 0.75$.

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■ The Lemniscate One-Sided Surfaces

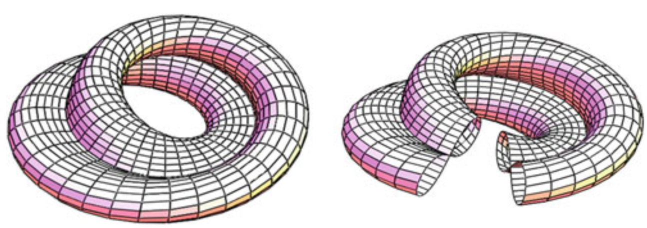
The lemniscate one-sided surface is formed by a lemniscate generatrix curve moving along a directrix circle and, at the same time, rotating at its normal plane, so that, when the generatrix lemniscate returns to initial plane, the cavities of the lemniscate change their position. That means that after returning at initial point (plane), the generatrix lemniscate was rotated at the normal plane of the directrix circle at the angle equals $m\pi$, where m is an odd number.

The lemniscate one-sided surfaces with a directrix circle may be related to a class of s*urfaces of congruent cross-sections*.

Form of definition of the lemniscate one-sided surfaces

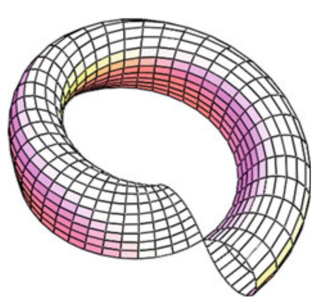
(1) Vector equation (Figs. 1 and 2)

$$\begin{aligned}\boldsymbol{r}(u,v) = {} & [a + X(v)\cos(mu/2) - Y(v)\sin(mu/2)]\,\boldsymbol{e}(u) \\ & + [X(v)\sin(mu/2) + Y(v)\cos(mu/2)]\,\boldsymbol{k},\end{aligned}$$



$a = 1.5$; $b = 1$

Fig. 1



$a = 1.5$; $b = 1$

Fig. 2

where $\boldsymbol{r}(u,v)$ is a radius-vector of the surface, u is a polar angle of the directrix circle; $\boldsymbol{e}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u$ is the vector-function of the circle of the unit radius;

$$X(v) = b\frac{\cos v}{1+\sin^2 v}, \quad Y(v) = b\frac{\cos v}{1+\sin^2 v}\sin v$$

are parametric equations of the lemniscate; $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors of the Cartesian coordinate system; m is an odd number characterizing the number of rotations of the generatrix lemniscate at the normal plane of the directrix circle; $0 \le u \le 2\pi$.

Figure 2 shows one cavity of the lemniscate surface, $-\pi/2 \le v \le \pi/2$.

If one uses a polar form of the definition of a lemniscate

$$\rho(\theta) = b\sqrt{\cos 2\theta},$$

then a vector equation of the lemniscate one-sided surface may be written as:

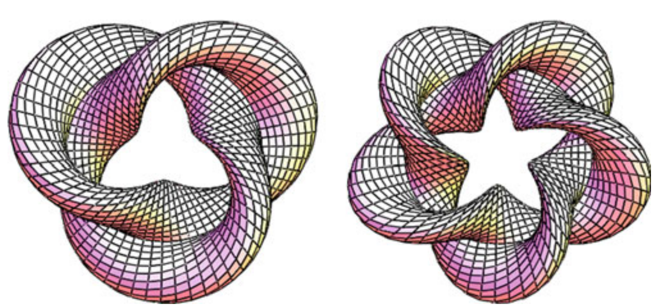
$$\begin{aligned}\boldsymbol{r}(u,\theta) = {} & [a + \rho(\theta)\cos(\theta + mu/2)]\,\boldsymbol{e}(u) \\ & + \rho(\theta)\sin(\theta + mu/2)\,\boldsymbol{k}\end{aligned}$$

where $0 \le u \le 2\pi$, $-\frac{\pi}{4} \le v \le \frac{\pi}{4}$ and $\frac{3}{4}\pi \le v \le \frac{5}{4}\pi$.

Coefficients of the first fundamental form of the surface:

$$\begin{aligned}A^2 &= \left[a + b\sqrt{\cos 2\theta}\cos(\theta + mu/2)\right]^2 \\ &\quad + \frac{m^2}{4}b^2\cos^2 2\theta, \\ B &= \frac{b}{\sqrt{\cos 2\theta}}, \quad F = \frac{mb^2}{2}\cos 2\theta \\ A^2B^2 - F^2 &= b^2\frac{\left[a + b\sqrt{\cos 2\theta}\cos(\theta + mu/2)\right]^2}{\cos 2\theta} \\ &\quad + \frac{m^2b^4}{4}\sin^2\theta\cos 2\theta\end{aligned}$$

The surface is given at non-orthogonal non-conjugate system of the coordinates u, θ.



$m = 3$ $m = 5$

Fig. 3

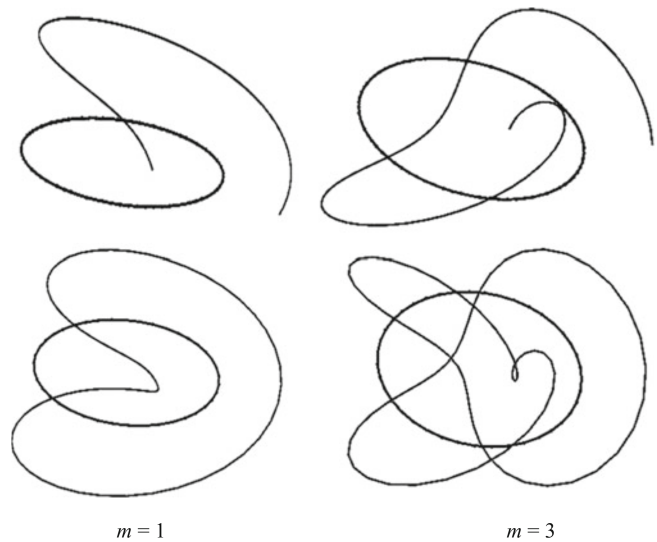


Fig. 4

In Fig. 3, the twisted lemniscate one-sided surfaces is presented when $m = 3$ and $m = 5$.

In Fig. 4, the coordinate lines of the surface when $v = 0$ (space line) and $v = \pi/2$ (circle) are shown. The coordinate line with $v = \pi/2$ forms the directrix circle of the surface. At the first row, the coordinate parameter v varies from 0 to 2π. At the second row, the parameter v varies from 0 to 4π and the coordinate lines $v = 0$ forms the closed space line and the surface repeats itself twice.

19 Minimal Surfaces

Minimal surface is a surface having the mean curvature H equal to zero at all points. Hence, minimal surface is a surface of negative Gaussian curvature. The extensive information on the initial stages of the investigations of minimal surfaces is given at Mathematical encyclopedias, monographs and at numerous courses of differential geometry. It is known that the first investigations of minimal surfaces have been fulfilled by G. Lagrange who has regarded the following variation problem: find a surface of a minimal area pulled over the given contour. Having given the sought surface as $z = z(x, y)$, G. Lagrange has derived that the function $z(x, y)$ must satisfy an equation (*equation of Euler–Lagrange*):

$$\left[1+\left(\frac{\partial z}{\partial y}\right)^2\right]\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x\,\partial y}+\left[1+\left(\frac{\partial z}{\partial x}\right)^2\right]\frac{\partial^2 z}{\partial y^2} = 0.$$

The given expression is a nonlinear partial differential equation and it describes all minimal surfaces in the Cartesian system of coordinates when corresponding contour is given. The minimal surfaces may be formed on the Jordan base contours of arbitrary forms not having self-intersections.

Later, G. Monge (1776) has proved that the condition of minimality of the area leads to the condition $H = 0$. The first common methods of integration of the Euler–Lagrange equation were offered by G. Monge (1784) and A.M. Legendre (1787). S. Poisson (1832) announced the solution of the variation problem of Lagrange for the case when the edge of the surface was close to a plane curve.

The honor of the discovery of the first minimal surfaces in the form of *catenoid* belongs to L. Euler (1774) and J. Meusnier (1776). J. Meusnier discovered also a minimal surface in the form of *a right helicoid* (1776). Then, the third minimal surface called the Scherk minimal surface was discovered (1834). E. Catalan (1842) proved that right helicoid is the only ruled minimal surface, and O. Bonnet (1850) presented his findings that a catenoid is *the only minimal surface of revolution*.

The parametric presentations of minimal surfaces were offered by B. Riemann (1860), A. Enneper (1864), K.M. Peterson (1866), and by others. J. Plateau in his experiments (1849) realized in practice minimal surfaces in the form of soap films pulled over metal wire works of different forms and for this reason, the finding of a minimal surface when its contour is given they became to call *the problem of Plateau*. Due to solution of this problem, *the surface of Riemann–Shwartz* was discovered. If one surface of a set of *parallel surfaces* is a minimal surface then all of the rest surfaces are minimal.

The *Gergonne problem* (1816) is a problem of finding of a minimal surface when a part of its boundary is given but the rest part must be placed at some given surface. This problem is called also *a problem about minimal surfaces with free boundary*.

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■ Catenoid

A catenoid is formed by the revolution of a catenary

$$X = a\text{ch}(z/a)$$

about the axis *Oz*. The word "catenoid" was generated from two words, videlicet, "catena" in Latin (this means "chain") and "eidos" in Greek (this means "form"). Catenoid is the only *minimal surface of revolution*, i.e., at all points, the mean curvature of its surface is equal to zero:

$$H = (k_1 + k_2)/2 = 0.$$

A form of a catenoid is erected by a soap film pulled over two wire circles, the planes of which are perpendicular to the rotation axis. The honor of the discovery of the first minimal surfaces in the form of *catenoid* belongs to L. Euler (1774) and J. Meusnier (1776).

The function of Weierstrass for catenoid has the following form:

$$F(g) = -1/\left(2g^2\right), \quad \text{see also "Schwarz Surface".}$$

Forms of definition of the surface

(1) Explicit form of the definition:

$$z = a\,\text{Arcosh}\sqrt{\frac{x^2+y^2}{a^2}},$$

where a is the radius of a parallel (*waist circle*) lying at the plane *xOy* (z = 0).

(2) Parametrical equations:

$$x = x(r,\beta) = r\cos\beta,$$
$$y = y(r,\ \beta) = r\sin\beta,$$
$$z = z(r) = \pm a\,\text{Arcosh}\,(r/a),$$

where β is an angle read from the axis *Ox* in the direction of the axis *Oy*.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{r}{\sqrt{r^2-a^2}},\quad F = 0,\quad B = r;$$
$$L = -\frac{a}{r^2-a^2},\quad M = 0,\quad N = a,$$
$$k_2 = -k_1 = \frac{a}{r^2}.$$

The coordinate lines r and β (parallels and meridians) are the lines of the principle curvatures (Fig. 1).

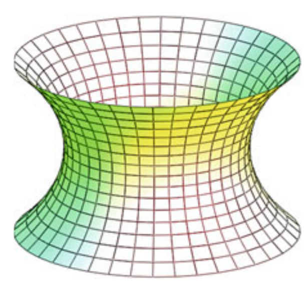

Fig. 1

(3) Parametrical equations (Fig. 1):

$$x = x(z, \beta) = a \cosh \frac{z}{a} \cos \beta,$$
$$y = y(z, \beta) = a \cosh \frac{z}{a} \sin \beta,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \cosh(z/a), \quad F = 0,$$
$$B = a \cosh(z/a),$$
$$L = -1/a, \quad M = 0, \quad N = a,$$
$$k_2 = -k_1 = 1/\left(a \cosh^2(z/a)\right).$$

The coordinate lines z and β (parallels and meridians) are the lines of the principle curvatures (Fig. 1).

(4) Parametrical equations:

$$x = x(u, \beta) = \sqrt{a^2 + u^2} \cos \beta, \quad y = (u, \beta) = \sqrt{a^2 + u^2} \sin \beta,$$
$$z = z(u) = a \operatorname{Arsinh} \frac{u}{a}.$$

where $a^2 + u^2 = r^2$; u is the arc length of the meridian (catenary). The line $u = 0$ is the waist radius.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1, \quad F = 0, \quad B^2 = a^2 + u^2,$$
$$L = -a/(a^2 + u^2),$$
$$M = 0, \quad N = a,$$
$$k_1 = -k_2 = L.$$

(5) Parametrical equations:

$$x = x(t, \beta) = a \cosh t \cos \beta,$$
$$y = y(t, \beta) = a \cosh t \sin \beta;$$
$$z = z(t) = at.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = B = a \cosh t, \quad F = 0;$$
$$L = -a = -N, \quad M = 0,$$
$$k_t = k_1 = -1/\left(a \cosh^2 t\right),$$
$$k_\beta = k_2 = 1/\left(a \cosh^2 t\right),$$
$$K < 0, \quad H = 0.$$

Here is shown that substituting $z/a = t$, into the parametrical equations of a catenoid dealt with in the paragraph 3 of the same page, we can reduce a linear element of the surface of the catenoid to *the isothermic form*.

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■ Right Helicoid

Right helicoid is called a helical ruled surface formed by a straight line, which intersects the *axis of the helicoid* at the right angle, rotates with the constant angular velocity about this axis, and executes a translation along the same axis with constant speed. The speeds of these motions are proportional. If the lifting coincides with the anticlockwise rotation about the axis, then a right helicoid is called *a right-side right helicoid* (Figs. 1 and 2), and at the opposite case, it called *a left-side right helicoid*.

The straight generatrixes of a right helicoid are parallel to its *plane of parallelism* that is perpendicular to the axis of the helicoid. So, right helicoid may be related to a family of conoids and may be called *a right helical conoid*. Generally, any point of a generatrix straight line traces a helical line. A helicoid is formed by the principle normals of a helical line lying on it.

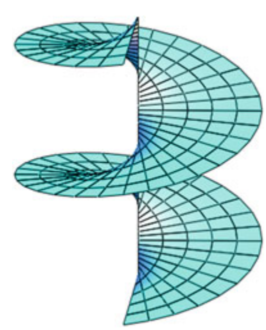

Fig. 1

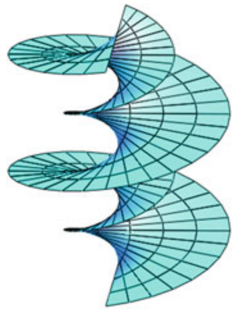

Fig. 2

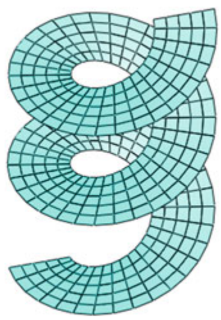

Fig. 3

The honor of the discovery of this minimal surface belongs to J. Meusnier (1776). E. Catalan (1842) has proofed that right helicoid is the only *ruled minimal surface* ($H = 0$). Moreover, a ruled surface which is minimal is necessarily a part of a right helicoid. Assume that a right helicoid is formed with the help of two coaxial helical lines and the plane of parallelism, and then it may be called *a helical cylindroid* (Fig. 3).

It is possible to make an approximate development of the turn of a right helicoid using its bending at surface of the catenoid and at this case, the helical lines of the right helicoid is put in at the parallels of the catenoid and the straight generatrixes of the helicoid is put in at the meridians of the catenoid. Firstly, this was noted by U. Dini in 1865.

The helicoid and the catenoid are *locally isometric surfaces.*

Forms of definition of the surface

(1) Explicit equation:

$$z = c \arctan\frac{y}{x},$$

where c is the displacement of a generatrix straight line under its rotation on one radian.

(2) Parametrical equations:

$$x = x(r, v) = r\cos v, \quad y = y(r, v) = r\sin v, \quad z = cv.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned}
&A = 1,\ \ F = 0,\ \ B^2 = r^2 + c^2;\\
&L = N = 0,\ \ M = -c/B;\\
&k_r = 0,\ \ k_v = 0,\\
&k_1 = k_2 = \pm\, \mathrm{c}\big/\big(r^2 + c^2\big);\\
&K = -c^2\Big/\big(r^2 + c^2\big)^2 < 0,\\
&H = 0,\ \ \chi = \pi/2.
\end{aligned}$$

where χ is the angle between the coordinate lines r and v. The main directions divide the angle between the direction of a generatrix straight line v and a helical line r in half-and-half. In Fig. 1, the right helicoid is limited by the coordinate line $r = r_1$ = const, in Fig. 2, by the lines $r = r_1$ and $r = -r_1$, in Fig. 3, by the lines $r = r_1$ and $r = r_2$, $r_2 > r_1 > 0$.

(3) Vector equation:

$$\boldsymbol{r}(u, v) = r\mathbf{e}_r + cv\mathbf{e}_z,$$

where the value c is connected with the lead L of a helical surface by the condition $L = 2\pi c$.

(4) Parametrical equations:

$$\begin{aligned}
&x = x(u, v) = c\sinh u\cos v,\\
&y = y(u, v) = c\sinh u\sin v,\\
&z = z(v) = cv.
\end{aligned}$$

Coefficients of the fundamental forms of the surface:

$$\begin{aligned}
&A = B = c\,\cosh u, \quad F = Nt = N = 0,\\
&M = 1/\big(c\cosh^2 u\big).
\end{aligned}$$

Using this definition of the surface, we obtain *isothermic coordinate lines* u, v.

The coordinate lines r, v and u, v on a right helicoid are mutually orthogonal and are *asymptotical*. Right helicoid is a surface of negative Gaussian curvature.

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■ Scherk's Minimal Surface (The First One)

Scherk's first minimal surface (H.F. Scherk) was discovered at 1834. This is the only minimal surface belonging to a class of *translation surfaces*.

Forms of definition of the surface

(1) Explicit equation (Fig. 1):

$$z = a\ln\frac{\cos\frac{y}{a}}{\cos\frac{x}{a}} = a\left[\ln\left(\cos\frac{y}{a}\right) - \ln\left(\cos\frac{x}{a}\right)\right],$$

where a parameter $a = b/(n\pi)$ is connected with the dimension b of the square projection of the Scherk's surface at the coordinate plane xOy; n is an arbitrary integer. Figure 1 shows the Scherk's surface with the following geometric parameters:

$$n = 1;\quad -0.475b \le x \le 0.475b;\quad -0.475b \le y \le 0.475b.$$

When $x = \pm b/2$, we find that $z \to \infty$, when $y = \pm b/2$, we find that $z \to -\infty$, but if $x = y = 0$ or $x = y = \pm b/2$, then $z = 0$.

The Scherk's minimal surface presented at the page "Translation surfaces presented in the encyclopedia" has $n = 1; 2; 3$. Scherk's first surface is asymptotic to two infinite families of parallel planes, orthogonal to each other, that meet near $z = 0$ in a checkerboard pattern of bridging arches. It contains an infinite number of straight vertical lines.

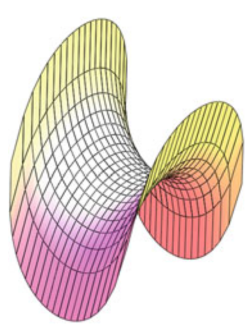

Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \frac{1}{\cos\frac{x}{a}},\quad F = -\tan\frac{x}{a}\tan\frac{y}{a},$$
$$B = \frac{1}{\cos\frac{y}{a}};\quad L = \frac{1}{a\cos^2\frac{x}{a}\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{-1}{a\cos^2\frac{y}{a}\sqrt{A^2B^2 - F^2}},\quad M = 0;$$
$$k_x = \frac{\cos\frac{x}{a}\cos\frac{y}{a}}{a\sqrt{1 - \sin^2\frac{x}{a}\sin^2\frac{y}{a}}} = -k_y,$$
$$k_1 = -k_2 = \frac{\cos\frac{x}{a}\cos\frac{y}{a}}{a\left(1 - \sin^2\frac{x}{a}\sin^2\frac{y}{a}\right)}.$$

The surface is related to nonorthogonal conjugate system of the curvilinear coordinates x, y.

(2) Explicit equation (Fig. 2):

$$z = \ln\frac{\cos y}{\cos x}.$$

In Fig. 2, the surface is plotted for x from $-\pi/4$ to $\pi/4$ $(-\pi/4 \le x \le \pi/4)$, y from $-\pi/4$ to $\pi/4$ $(-\pi/4 \le y \le \pi/4)$.

(3) Parametrical equations:

$$x = x(u, v) = \arctan u + \operatorname{arccotan} v,$$
$$y = y(u, v) = \arctan cu + \operatorname{arccotan} cv,$$
$$z = z(u, v) = \ln\frac{c\sqrt{1+u^2}\sqrt{1+v^2}}{\sqrt{1+c^2u^2}\sqrt{1+c^2v^2}}.$$

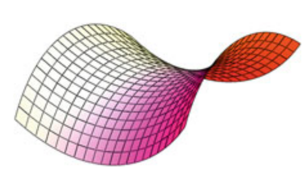

Fig. 2

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{1+c^2}{(1+u^2)(1+c^2u^2)},$$
$$F = \frac{(1+c^2)(1-uv)(uvc^2-1)-2c^2(u+v)^2}{(1+u^2)(1+v^2)(1+c^2u^2)(1+c^2v^2)},$$
$$B^2 = \frac{1+c^2}{(1+v^2)(1+c^2v^2)};$$
$$L = \frac{-c(1-c^2)^2(u+v)^2(A^2B^2-F^2)^{-1/2}}{(1+u^2)^2(1+v^2)(1+c^2u^2)^2(1+c^2v^2)}, \quad M=0,$$
$$N = \frac{c(1-c^2)^2(u+v)^2(A^2B^2-F^2)^{-1/2}}{(1+u^2)(1+v^2)^2(1+c^2u^2)(1+c^2v^2)^2};$$
$$K<0, \quad H=0.$$

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Strubecker Karl. Über die isotropen Gegenstücke der Minimalfläche von Scherk. J. reine und angew. Math. 1977, No. 293-294, p. 22-51.
Scherk H.F. Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen, J. für die reine und angewandte Mathematik. 1835; Vol. 13, p. 185–208.

■ Enneper's Surface

Enneper's surface is the only algebraic minimal surface of the 9th order having plane lines of the principle curvatures. It may be imposed at *a surface of revolution.*

The function of Weierstrass for Enneper's surface has the following form:

$F(g) = 1$, see also "Schwarz Surface".

The Enneper surface is a self-intersecting surface. It was introduced by Alfred Enneper (1864) in connection with minimal surface theory.

Forms of definition of the Enneper's surface

(1) Parametrical equations (Fig. 1):

$$x = x(u,v) = 3u + 3uv^2 - u^3;$$
$$y = y(u,v) = v^3 - 3v - 3u^2v;$$
$$z = z(u,v) = 3(u^2 - v^2).$$

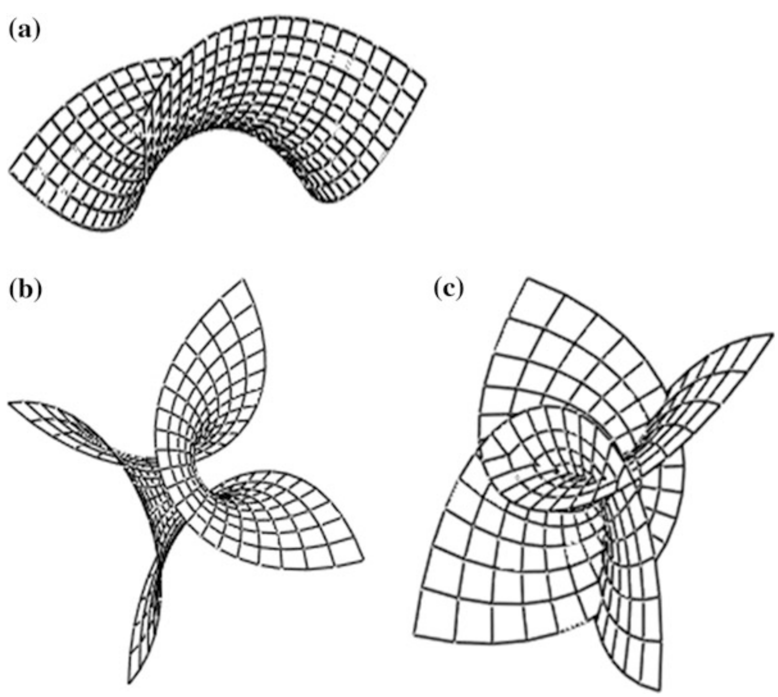


Fig. 1

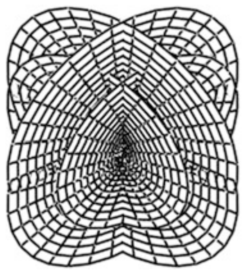


Fig. 2

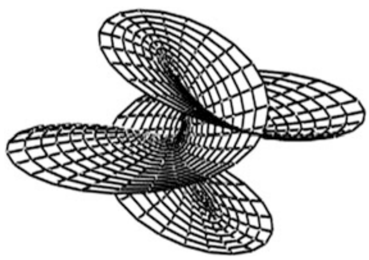


Fig. 3

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = B^2 = 9\left(u^2 + v^2 + 1\right)^2;$$
$$F = 0;\ \ L = -N = -6;\ \ M = 0;$$
$$k_1 = k_u = L/A^2 = -k_2 = -k_v;$$
$$K = -4/\left[9\left(u^2 + v^2 + 1\right)^4\right] < 0;\ \ H = 0.$$

The surface is given in the lines of principle curvatures u, v. In Fig. 1a, the Enneper's surface has $-0.5 \le u \le 0.5$ and $-0.5 \le v \le 0.5$; in Fig. 1b, the surface is formed in the limits $-1.5 \le u \le 1.5$ and $-1.5 \le v \le 1.5$; Fig. 1c shows the surface in question when $-3 \le u \le 3$ and $-3 \le v \le 3$.

Figure 1b shows the contour of the surface in the form of *astroid* if one looks at the surface from one side.

(2) Vector equation (Figs. 2 and 3):

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{r}(r,\theta)\\ &= \left(r\cos\theta - \frac{r^3}{3}\cos 3\theta\right)\boldsymbol{i} - \left(r\sin\theta + \frac{r^3}{3}\sin 3\theta\right)\boldsymbol{j}\\ &\quad + r^2\cos 2\theta\boldsymbol{k}.\end{aligned}$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1 + r^2,\ \ F = 0,\ \ B = r(1 + r^2);$$
$$L = -2\cos 2\theta,\ \ M = 2r\sin 2\theta,$$
$$N = 2r^2\cos 2\theta;$$
$$k_r = -\frac{2\cos 2\theta}{(1 + r^2)^2},\ \ k_\theta = \frac{2\cos 2\theta}{(1 + r^2)^2},$$
$$k_{1,2} = \pm\frac{2}{(1 + r^2)^2};\ \ H = 0.$$

The Enneper's surface is given at curvilinear orthogonal nonconjugate coordinates r, θ, so $H = k_r + k_\theta = k_1 + k_2 = 0$.

Assume $-3 \le r \le 3$ and $0 \le \theta \le 2\pi$, then we obtain the surface presented in Fig. 2. If $0 \le r \le 3$ and $0 \le \theta \le 2\pi$, then the surface has the shape shown in Fig. 3.

Additional Literature

Gaidar OG. Surfaces of shells, related to the set of lines of principle curvatures. Architecture of Shells and Strength Analysis of Thin-Walled Civil-Engineering and Machine-Buildings Constructions of Complex Form: Proc. of International Conference, Moscow, June 4-8, 2001. Moscow: Izd-vo RUDN, 2001; p. 65-69.

Nitshe Johannes. Vorlesungen über Minimalflächen. Berlin-Heidelberg-New York: Springer-Verlag. 1975.

Dan Dumitru. Minimal surfaces that generalize the Enneper's surface. Novi Sad J. Math. 2010; Vol. 40, No. 2, p. 17–22

■ Schwarz Surface

Schwarz surface may be designed if we have four ribs of a regular tetrahedron as a boundary contour (Fig. 1), i.e., this minimal surface id bounded by four straight segments. Firstly, the surface in question was discovered by H.A. Schwarz (1866) and independently of him by B. Riemann and so very often in the scientific literature of the subject, the Schwarz surface is called also *Riemann–Schwartz surface*. H.A. Schwarz used a method of Weierstrass and Riemann.

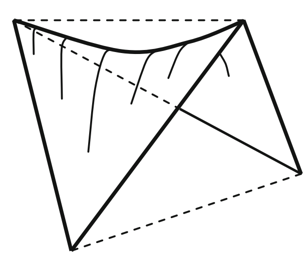


Fig. 1

As all minimal surfaces, the Schwarz surface satisfies the following two properties:

(1) if a part of the boundary of a minimal surface M contains in some straight line, then the reflection of the surface M' relatively to this line is also a minimal surface and the consolidation of M and M' generates a smooth minimal surface without a break at the straight section of the boundary of the surface;

(2) if a minimal surface M meets a plane at the right angle, then its mirror reflection M' relatively to this plane is also a minimal surface and the consolidation of M with M' generates a smooth minimal surface.

Using these properties for the Schwartz surface, it is possible to create the *triply periodic minimal surface*. *The basic cell* of the Schwarz surface is placed in a regular cube with openings in all sides of the cube and has the cubic symmetry (see also a Sect. "19.8 Embedded Triply Periodic Minimal Surfaces").

Today, this property is used widely at the architecture. For example, the Olympic stadium at Munich (1972) was created with the application of minimal surfaces close to the Schwarz surfaces.

The classical definition of Weierstrass–Enneper for a minimal surface has the following view:

$$x = \mathrm{Re}\int \mathrm{F}(g)(1-g^2)\,\mathrm{d}g + a_1,$$
$$y = \mathrm{Re}\,i\int \mathrm{F}(g)(1+g^2)\,\mathrm{d}g + a_2,$$
$$z = \mathrm{Re}\int 2\mathrm{F}(g)g\,\mathrm{d}g + a_3,$$

where $\mathrm{F}(g)$ is the Weierstrass function; g is a complex analytic function at the equatorial plane. If Weierstrass function is taken as

$$\mathrm{F}(g) = 1\Big/\sqrt{1-14g^4+g^8},$$

then we may obtain the parametrical equations of the Schwarz surface as

$$x = \mathrm{Re}\int \frac{(1-g^2)\,\mathrm{d}g}{\sqrt{1-14g^4+g^8}},$$
$$y = \mathrm{Re}\int \frac{i(1+g^2)\,\mathrm{d}g}{\sqrt{1-14g^4+g^8}},$$
$$z = \mathrm{Re}\int \frac{2g\,\mathrm{d}g}{\sqrt{1-14g^4+g^8}}.$$

The asymptotic lines of the minimal surface cross at the right angle. The Schwarz surface is well approximated by a surface with an implicit equation

$$\cos x + \cos y + \cos z = 0.$$

Additional Literature

Schwarz HA. Gesammelte Mathematische Abhandlungen. Julius Springer, Berlin, 1890; Vol. 1.

■ Neovius' Surface

Neovius' surface may be constructed at all the space without singularity. It does not intersect itself and possesses the same symmetry as the space atomic lattice of diamond. The unit cubic sell represents itself a cavity with the exits from the center of every edge of the cube (Fig. 1).

Forms of definition of the Neovius' surface

(1) Implicit equation:

$$3(\cos x + \cos y + \cos z) + 4\cos x \cdot \cos y \cdot \cos z = 0.$$

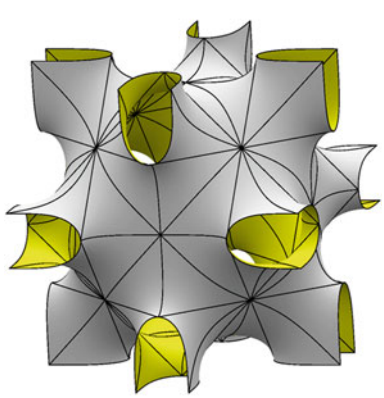


Fig. 1

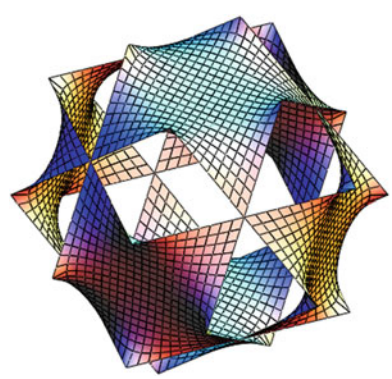


Fig. 2

(2) Parametrical equations (Fig. 2):

$$x = u, \quad y = v, \quad z = \pm \arccos\left(-3\frac{\cos\, u + \cos\, v}{3 + 4\cos\, u \cos\, v}\right).$$

The Neovius' surface given in Fig. 2 is formed when $-\pi/2 \le u \le \pi/2$; $-\pi/2 \le v \le \pi/2$. The surface shown in Fig. 1 is taken at the internet site indicated as http://www.susqu.edu/brakke/evolver/examples/periodic/periodic.html.

Additional Literature

Neovius ER. Bestimmung Zweier Speciellen Periodishe Minimalflächen. Helsingfors, 1883

■ Catalan's Surface

Minimal Catalan's surface was put in practice by E. Catalan in 1855.

Forms of the definition of the Catalan's surface

(1) Parametrical equations (Figs. 1, 2, 3, 4 and 5):

$$x = x(u, v) = u - \sin u \cosh v,$$
$$y = y(u, v) = 1 - \cos u \cosh v,$$
$$z = z(u, v) = 4\sin(u/2)\sinh(v/2).$$

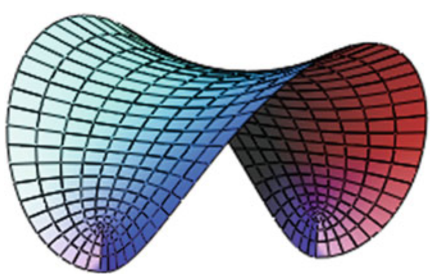
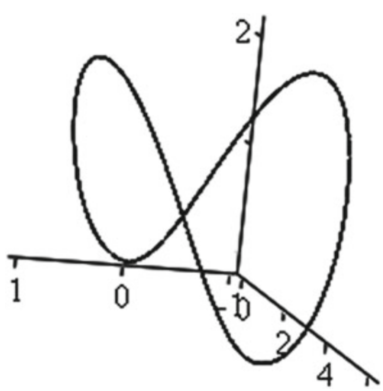
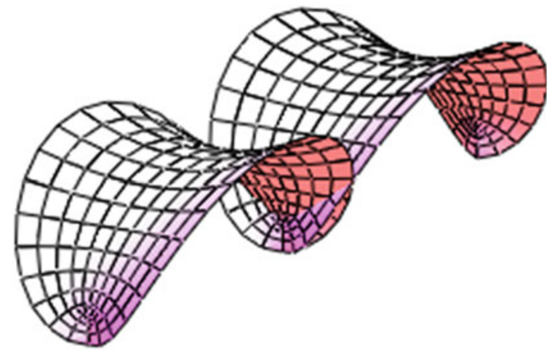


Fig. 1



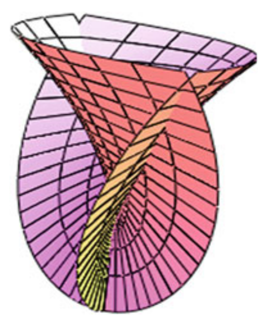
$v = 0.5$;
$0 \le u \le 2\pi$

Fig. 2



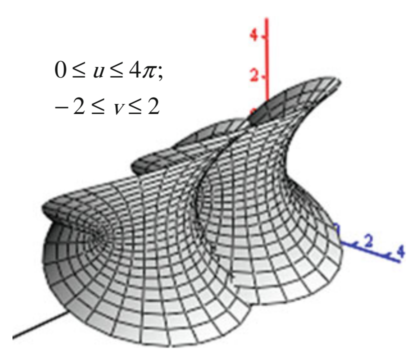
$-2\pi \le u \le 2\pi$;
$-1 \le v \le 1$

Fig. 3



$-0.5\pi \le u \le 0.5\pi$;
$-1 \le v \le 1$

Fig. 4



$0 \le u \le 4\pi$;
$-2 \le v \le 2$

Fig. 5

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = B^2 = (1 + \cosh v)(\cosh v - \cos u), \quad F = 0,$$
$$L = \frac{\cosh v - \cos u}{A^2} \sin\frac{u}{2} \left(2\cosh v \cosh\frac{v}{2} - \sinh v \sinh\frac{v}{2}\right),$$
$$N = -L,$$
$$M = -\frac{\sinh v}{A^2}\left[\left(2\sin u \sin\frac{u}{2} + \cos u \cos\frac{u}{2}\right)\cosh\frac{v}{2} - \left(\cosh v \cosh\frac{v}{2} - 2\sinh v \sinh\frac{v}{2}\right)\cos\frac{u}{2}\right],$$
$$k_u = -k_v = \frac{\sin(u/2)}{(1+\mathrm{ch}v)^2(\mathrm{ch}v - \cos u)}\left(2\mathrm{ch}v\,\mathrm{ch}\frac{v}{2} - \mathrm{sh}v\,\mathrm{sh}\frac{v}{2}\right),$$
$$k_1 = -k_2 = \frac{\sqrt{L^2+M^2}}{A^2},$$
$$K = -\frac{L^2+M^2}{A^4} < 0, \quad H = 0.$$

The surface is related to *isothermic* orthogonal nonconjugate coordinates u, v. The minimal surface presented in Fig. 1 has the boundaries $0 \le u \le 2\pi$; $-1 \le v \le 1$. In Fig. 2, the contour of the surface, limited by the line $v = 0.5$, is shown and $0 \le u \le 2\pi$. The Catalan's surface at boundaries $-2\pi \le u \le 2\pi$; $-1 \le v \le 1$ is represented in Fig. 3. Figure 4 shows the blown up fragment of the surface represented at Fig. 3.

(2) Parametrical equations (Fig. 6):

$$x = x(r, \varphi) = a\left(\sin 2\varphi - 2\varphi + \frac{v^2}{2}\cos 2\varphi\right),$$
$$y = y(r, \varphi) = -a\left(1 + \frac{v^2}{2}\right)\cos 2\varphi,$$

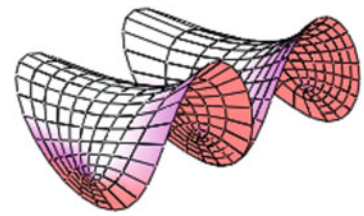


Fig. 6

$$z = z(r, \varphi) = 2av\sin\varphi,$$

where $v = 1/r - r$.

In Fig. 6, the minimal surface is constructed when $0.5 \le r \le 2$; $-\pi \le \varphi \le \pi$.

Minimal Catalan's surface has the special property of being the minimal surface that contains *a cycloid* as a geodesic.

Additional Literature

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do Carmo MP. Mathematical Models from the Collections of Universities and Museums. Ed. G. Fischer. Braunschweig, Germany: Vieweg, 1986; p. 45-46.

Gray A. Catalan's Minimal Surface. Modern Differential Geometry of Curves and Surfaces with Mathematica: 2nd ed., Boca Raton, FL: CRC Press, 1997; p. 692-693.

■ Bour's Minimal Surface

Bour's minimal surface is defined by parametric equations:

$$x = x(r, \theta) = r\cos\theta - \frac{r^2}{2}\cos 2\theta,$$
$$y = y(r, \theta) = -r\sin\theta - \frac{r^2}{2}\sin 2\theta,$$
$$z = z(r, \theta) = \frac{4}{3}r^{3/2}\cos\frac{3\theta}{2}.$$

Coefficients of the fundamental forms of the surface:

$$A = 1 + r, \quad F = 0, \quad B = r(1 + r);$$
$$A^2B^2 - F^2 = r^2(1+r)^4;$$
$$L = -\frac{1}{\sqrt{r}}\cos\frac{3\theta}{2},$$

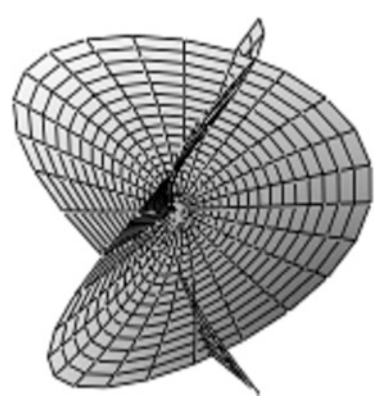


Fig. 1

$$M = \sqrt{r}\sin\frac{3\theta}{2}, \quad N = r^{3/2}\cos\frac{3\theta}{2},$$
$$K = \frac{-1}{r(1+r)^4} < 0, \quad H = 0.$$

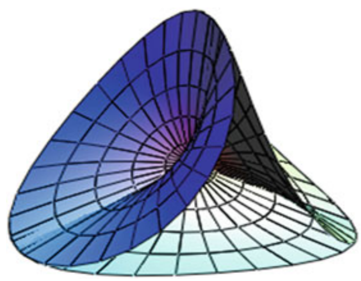


Fig. 2

Fig. 3

Bour's minimal surface is related to the orthogonal nonconjugate curvilinear coordinates r, θ. In Fig. 1, the Bour's minimal surfaces is shown, when $0 \le r \le 5$; $0 \le \theta \le 2\pi$. The surface shown at Fig. 2 has $0 \le r \le 1$; $0 \le \theta \le 4\pi$. The segment of the surface with the boundaries $0 \le r \le 2$; $-\pi \le \theta \le \pi$ is given in Fig. 3.

Additional Literature

Maeder R. Programming in Mathematica: 3rd ed. Addison-Wesley, 1997; 29-30.

■ Costa Minimal Surface

Using computer graphics, D. Hoffmann and W. Meeks III, has proved that Costa surface is an embedded minimal surface. It united the classical examples of a plane, a catenoid and a right helicoid that are the only known examples of the complete embedded minimal surfaces of the finite topological type. The surface does not cross itself. It was discovered by A. Costa in 1984.

For the first time, the representation of the Costa minimal surface appeared at the cover of a book of R. Osserman in 1986. A. Gray (1996) has derived parametrical equations of Costa minimal surface in the following form:

$$x = x(u,v) = \frac{1}{2}\mathrm{Re}\left\{-\varsigma(u+iv) + \pi u + \frac{\pi^2}{4e_1} + \frac{\pi}{2e_1}\left[\varsigma(u+iv-\frac{1}{2}) - \varsigma(u+iv-\frac{1}{2}i)\right]\right\},$$

$$y = y(u,v) = \frac{1}{2}\mathrm{Re}\left\{-i\varsigma(u+iv) + \pi v + \frac{\pi^2}{4e_1} - \frac{\pi}{2e_1}\left[i\varsigma(u+iv-\frac{1}{2}) - i\varsigma(u+iv-\frac{1}{2}i)\right]\right\},$$

$$z = z(u,v) = \frac{1}{4}\sqrt{2\pi}\ln\left|\frac{\wp(u+iv)-e_1}{\wp(u+iv)+e_1}\right|$$

where $\zeta(z)$ is the ζ—Weierstrass function; $\wp$ $(g_2, g_3; z)$ is the elliptic $\wp$—Weierstrass function with the (g_2, g_3) = (189.07 2772, …, 0) invariants that correspond to half-periods 1/2 and $i/2$; $e_1 = \wp$ (1/2; 0; g_3) = (1/2|1/2; i/2) ≈ 6.87519 is the first root, but $\wp$ $(z; g_2, g_3) = \wp$ $(z|\omega_1, \omega_2)$ is the elliptic $\wp$—function of Weierstrass.

P.S.: All formulas represented at this section and Fig. 1 are taken without check and changing at the internet site of Eric W. Weisstein.

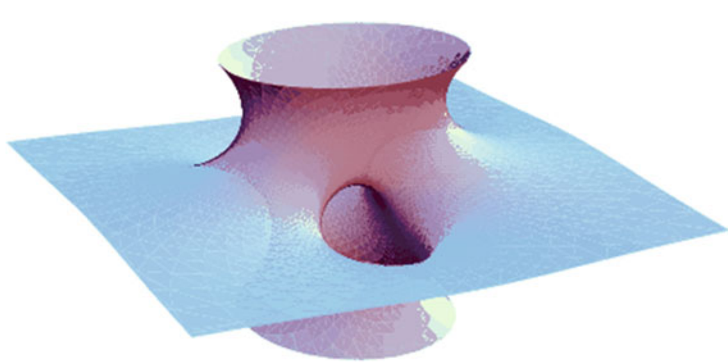


Fig. 1

Additional Literature

Hoffman D, Meeks W. A complete embedded minimal surface with genus one, three ends and finite total curvatures. J. Differ. Geom. 1985; No 21, p.109-127.

Costa A. Example of a complete minimal immersion in R^3 of genus one and three embedded ends. Bull. Soc. Bras. Mat. 1984; No. 15, p. 47-54.

Eric W. Weisstein. http://mathworld.wolfram.com/CostaMinimalSurface.html.

■ Gyroid

Gyroid is a continuously extending minimal surface not containing the straight lines. The surface was discovered by A. Shoen at the end of 1960.

Gyroid is the only of the known embedded triply periodic minimal surface with the triple crossing. K. Groβe-Brauckmann has established that the gyroid does not have any reflectional symmetries.

Sometimes, a gyroid is called *the G Surface*.

Gyroid is well approximated by the surface

$$\cos x\ \sin y + \cos\ y \sin z + \cos z\ \sin x = 0.$$

The presented implicit equation may be written in an explicit form:

$$z = \arcsin \frac{\cos x \cos y \sin y \pm \sin x \sqrt{\sin^2 x + \cos^2 y - \cos^2 x \sin^2 y}}{\sin^2 x + \cos^2 y}.$$

Using the last equation, one may construct some segments of the approximating surface represented in Figs. 1, 2 and 3 which give the insight on the structure of the minimal surface of the Gyroid.

The gyroid is the unique nontrivial embedded member of the associate family *of the Schwarz P and D surfaces* with angle of association approximately 38.01°. The gyroid is similar to the *Lidinoid*.

The real image of the Gyroid is given on a page “Examples of minimal surfaces presented in sites of Internet.”

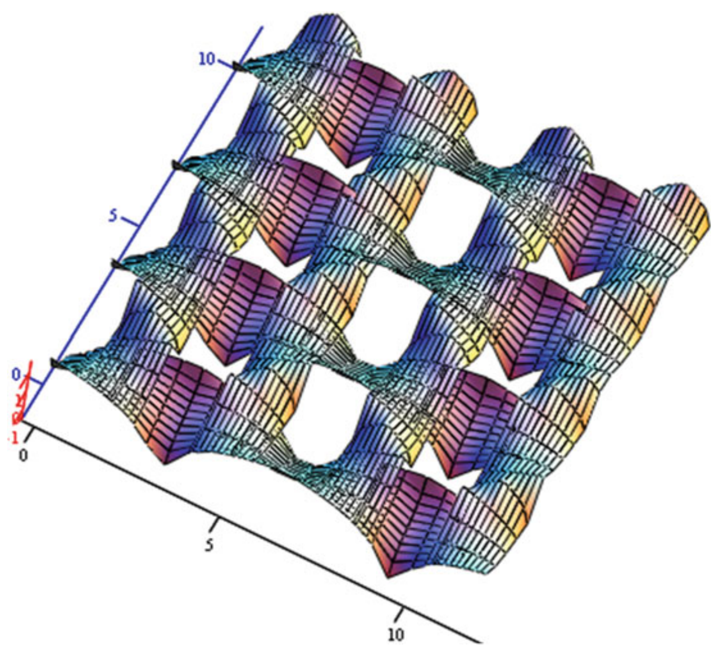
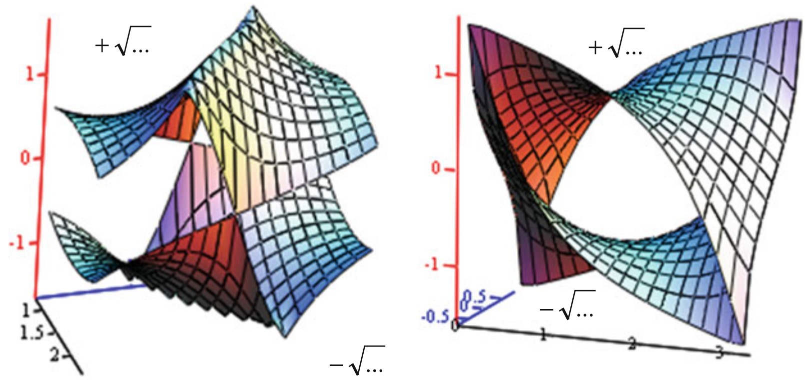
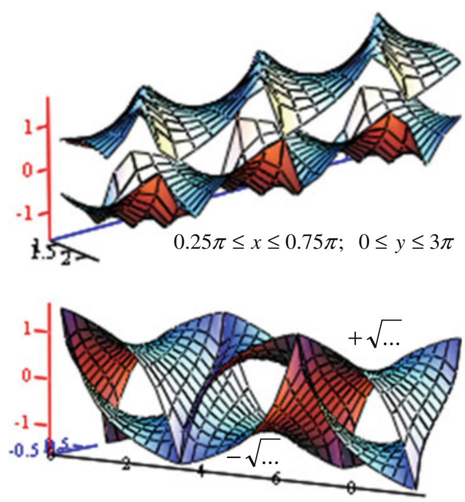


Fig. 1



$0.25\pi \le x \le 0.75\pi;\ \ 0 \le y \le \pi$ $\qquad -0.25\pi \le y \le 0.25\pi;\ \ 0 \le x \le \pi$

Fig. 2



$-0.875\pi \le y \le 0.875\pi;\ \ 0 \le x \le 3\pi$

Fig. 3

Additional Literature

Schoen AN. Infinitive periodic minimal surfaces without selfintersection. NASA Tech. Note. 1970; No D-5541, Washington.
Große-Brauckmann K. Gyroids of constant mean curvature. Experiment. Math. 1997; No 6, p. 33-50.
Paul JF Gamdy, Jacek Klinowski. Exact computation of triply periodic *G* (Gyroid) minimal surface. Chemical Physics Letters. 2000; Vol. 321, No 5-6, May, p. 363-371.
Garstecki P. and Hołyst R. Scattering Patterns of Self-Assembled Gyroid Cubic Phases in Amphiphilic Systems. J. Chem. Phys. 2001; 115, p. 1095-1099.
Damian A. Hajduk, Paul E. Harper, Sol M. Gruner, Christian C. Honeker, Gia Kim, Edwin L. Thomas, Lewis J. Fetters. The Gyroid: A New Equilibrium Morphology in Weakly Segregated Diblock Copolymers. Macromolecules. 1994; 27 (15), p. 4063-4075.

■ Henneberg Minimal Surface

The Weierstrass function for a minimal nonorientable surface of Henneberg has a following form:

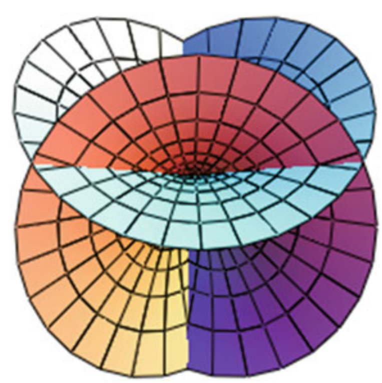
$$F(g) = 1 - 1/g^4,$$

see also "Schwarz Surface." This surface is a double algebraic surface of the 15th order.

Forms of definition the Henneberg minimal surface

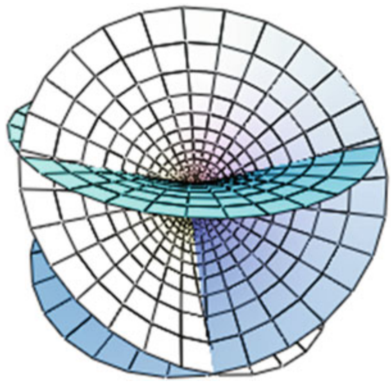
(1) Parametrical equations (Fig. 1, 2, and 3):

$$x = x(u, v) = 2\sinh u \cos v - \frac{2}{3}\sinh 3u \cos 3v,$$



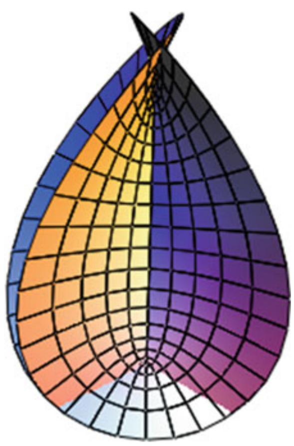
$0 \le u \le 1;\ \ -\pi \le v \le \pi$

Fig. 1



$0 \leq u \leq 1.5; \quad -\pi \leq v \leq \pi$

Fig. 2



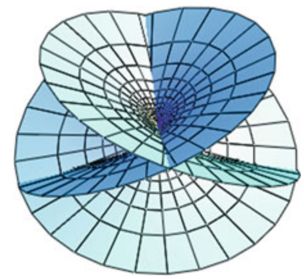
$0 \leq u \leq 0{,}5;$
$-\pi \leq v \leq \pi$

Fig. 3

$$y = y(u, v) = 2 \sinh u \sin v + \frac{2}{3} \sinh 3u \sin 3v,$$
$$z = z(u, v) = 2\cosh 2u \cos 2v.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 16\cosh^2 u(\sinh^2 2u + \sin^2 2v),$$
$$F = 0, \quad B^2 = 16\cosh^2 u(\sinh^2 2u + \sin^2 2v),$$
$$L = 4 \cos 2v \sinh 2u, \quad M = -4\cosh 2u \sin 2v,$$
$$N = -4 \cos 2v \sinh 2u,$$
$$k_u = -k_v = \frac{\cos 2v \sinh 2u}{4\cosh^2 u(\sinh^2 2u + \sin^2 2v)},$$



$0.2 \leq r \leq 1; \; -\pi \leq \varphi$

Fig. 4

$$k_1 = -k_2 = \frac{1}{\cosh uA},$$
$$K = -\frac{1}{A^2 \mathrm{ch}^2 u} < 0, \quad H \equiv 0.$$

The minimal surface is given in the orthogonal nonconjugate system of curvilinear coordinates u, v. The Henneberg minimal surface is a surface of the strictly negative Gaussian curvature.

(2) Parametrical equations (Fig. 4):

$$x = x(r, \varphi) = \frac{2(r^2 - 1) \cos \varphi}{r} - \frac{2(r^6 - 1) \cos 3\varphi}{3r^3},$$
$$y = y(r, \varphi) = -\frac{6r^2(r^2 - 1) \sin \varphi + 2(r^6 - 1) \sin 3\varphi}{3r^3},$$
$$z = z(r, \varphi) = \frac{2(r^4 + 1) \cos 2\varphi}{r^2}.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 4\left(\frac{r^2 + 1}{r^2}\right)^2 \left[\left(\frac{r^4 - 1}{r^2}\right)^2 + 4 \sin^2 2\varphi\right], \quad F = 0,$$
$$B^2 = 4\left(\frac{r^2 + 1}{r}\right)^2 \left[\left(\frac{r^4 - 1}{r^2}\right)^2 + 4 \sin^2 2\varphi\right],$$
$$L = -4 \frac{(r^2 - 1)^2}{r^4(r^2 + 1)} \cos 2\varphi,$$
$$M = -2 \frac{(r^2 - 1)^3}{r^3(r^2 + 1)} \sin 2\varphi,$$
$$N = -2 \frac{(r^2 - 1)^2}{r^2(r^2 + 1)} \cos 2\varphi, \quad H \equiv 0.$$

The surface contains *a semicubical parabola* (Neile's parabola") and can be derived from solving the corresponding *Björling problem*.

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■ Trinoid

Eric W. Weisstein noted that this minimal surface was discovered by L.P.M. Jorge and W. Meeks III at 1983. The parameterization of Enneper–Weierstrass for a trinoid has the following form:

$$f = \frac{1}{(\varsigma^3 - 1)^2}, \quad g = \varsigma^2.$$

The surface is given by the parametric equations:

$$x = x(r,\theta) = \mathrm{Re}\left[\frac{re^{i\theta}}{3(1 + re^{i\theta} + r^2e^{2i\theta})} - \frac{4\ln(re^{i\theta} - 1)}{9} + \frac{2\ln(1 + re^{i\theta} + r^2e^{2i\theta})}{9}\right],$$

$$y = y(r,\theta) = -\frac{1}{9}\mathrm{Im}\left[-\frac{3re^{i\theta}(1 + re^{i\theta})}{r^3e^{3i\theta} - 1} + 4\sqrt{3}\cdot \mathrm{arctg}\frac{1 + 2re^{i\theta}}{\sqrt{3}}\right],$$

$$z = z(r,\theta) = \mathrm{Re}\left[-\frac{2}{3} - \frac{2}{3(r^3e^{3i\theta} - 1)}\right],$$

where $0 \le \theta \le 2\pi$; $0 \le r \le 4$.

The fragments of the trinoid presented in Figs. 1, 2, and 3 are constructed for different boundaries. The entire surface is given at a page "Examples of minimal surfaces presented in sites of Internet."

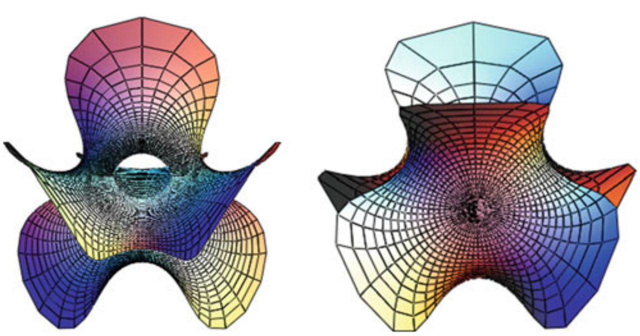
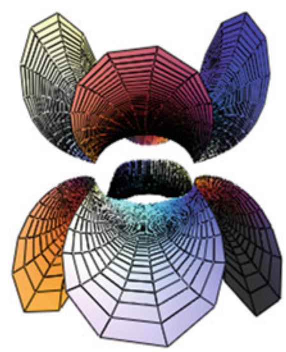


Fig. 1 $0 \le r \le 0.8$; $1.2 \le r \le 4$; $0 \le \theta \le 2\pi$

$0 \le r \le 4$; $0 \le \theta \le 2\pi$

Fig. 2



$0 \le r \le 0.88$; $1.12 \le r \le 4$; $0 \le \theta \le 2\pi$

Fig. 3

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Sterling I and Wente H. Existence and classification of constant mean curvature multibubbletons of finite and infinite type, Indiana Univ. Math. J. 1993; 42, No. 4, p. 1239-1266.
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■ Lichtenfels Minimal Surface

The geodesic lines of *Lichtenfels minimal surface* have the form of *lemniscates*.

Parametrical equations of the Lichtenfels minimal surface (Fig. 1) are

$$x = x(u,v) = \mathrm{Re}\left(\sqrt{2}\cos\frac{\varsigma}{3}\sqrt{\cos\frac{2}{3}\varsigma}\right),$$

$$y = y(u,v) = \mathrm{Re}\left(-\sqrt{2}\sin\frac{\varsigma}{3}\sqrt{\cos\frac{2}{3}\varsigma}\right),$$

$$z = z(u,v) = \mathrm{Re}\left[-i\sqrt{2}F\left(\sqrt{\frac{\varsigma}{3}},2\right)\right],$$

where $F(x, x)$ is an incomplete elliptical integral of the first type; $\varsigma = u + iv$ is a complex parameter. A lemniscate is a line of the intersection of the surface with the xOy plane. The minimal surface is a periodical surface in the direction of its axis with a period

$$\omega = 2\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\frac{t^2}{2}}} = 2K\left(\frac{1}{2}\right),$$

where $K(x)$ is a complete elliptic integral of the first type.

P.S.: The parametrical equations of the surface and the illustrations are taken without changing at the site of Eric W. Weisstein.

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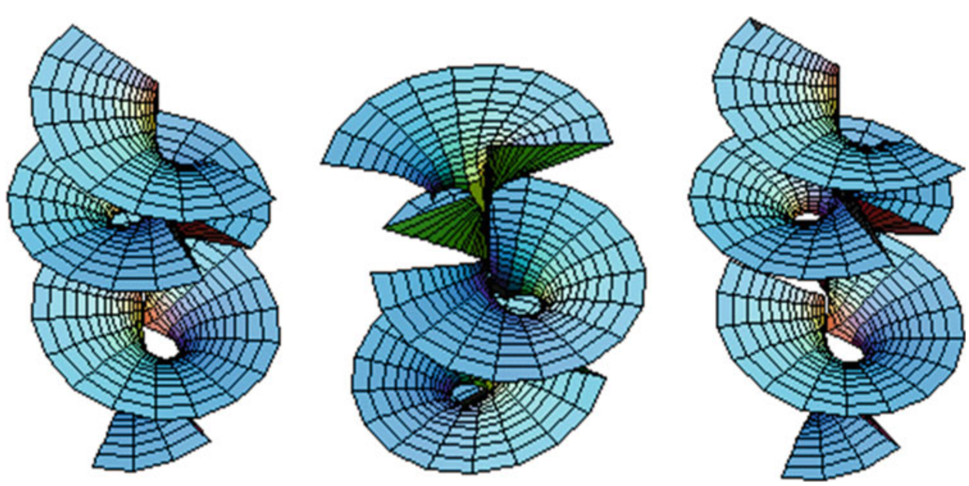

Fig. 1

■ The Second Scherk's Minimal Surface

In 1834, H.F. Scherk discovered two minimal surfaces. These were two new minimal surfaces presented after L. Euler (1774) and J. Meusnier (1776).

The first Scherk's minimal surface was considered before at the section "Scherk's minimal surface (the first one)." His second surface is singly periodic.

Parametrical equations of the surface in question may be written as:

$$\begin{aligned}
x = x(r,\theta) &= 2\mathrm{Re}\left[\ln\left(1 + re^{i\theta}\right) - \ln\left(1 - re^{i\theta}\right)\right] \\
&= \ln\frac{1 + r^2 + 2r\cos\theta}{1 + r^2 - 2r\cos\theta}, \\
y = y(r,\theta) &= \mathrm{Re}\left[4i\arctan\left(re^{i\theta}\right)\right] \\
&= \ln\frac{1 + r^2 - 2r\sin\theta}{1 + r^2 + 2r\sin\theta}, \\
z = z(r,\theta) &= \mathrm{Re}\left\{2i\left[\ln\left(1 + r^2e^{2i\theta}\right) - \ln\left(1 - r^2e^{2i\theta}\right)\right]\right\} \\
&= 2\arctan\frac{2r^2\sin\theta}{r^4 - 1},
\end{aligned}$$

where $0 \le \theta \le 2\pi$; $0 \le r \le 1$. Figure 1 shows the segments of the second Scherk's minimal surface.

The surface was generalized by H. Karcher into *the saddle tower family of periodic minimal surfaces*.

The parametrical equations of the surface were taken at the site of Eric W. Weisstein without control. As it is seen from the given illustrations (Fig. 1), the Scherk's minimal surface has four axes of the symmetry.

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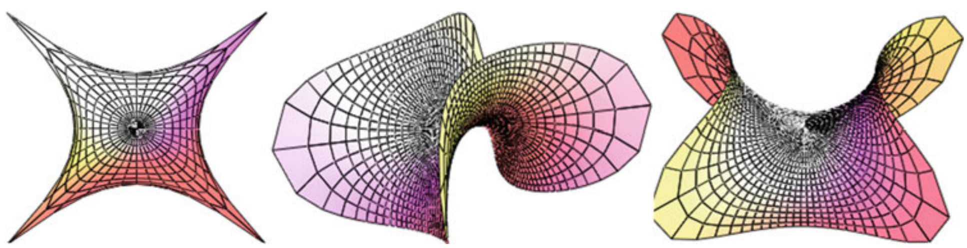


$0 \le r \le 0.95$; $0 \le \theta \le 2\pi$

Fig. 1

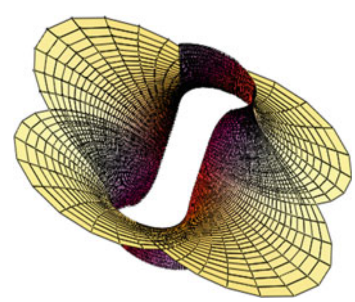
■ Richmond's Minimal Surface

Richmond's minimal surface is a surface with Gaussian curvature tending to zero when outer contour of the surface moves off, i.e., hyperbolic points of the surface near to the outer contour tend to become *plane points*.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$\begin{aligned}
x = x(u,v) &= \frac{-3u - u^5 + 2u^3v^2 + 3uv^4}{6(u^2 + v^2)}, \\
y = y(u,v) &= \frac{-3v - 3u^4v - 2u^2v^3 + v^5}{6(u^2 + v^2)}, \\
z = z(u) &= u.
\end{aligned}$$



$-1 \le u \le 1$; $0.2 \le v \le 1$

Fig. 1

(2) Parametrical equations in cylindrical coordinates (Fig. 2):

$$\begin{aligned}
x = x(r,\theta) &= -\frac{\cos\theta}{2r} - \frac{r^3\cos 3\theta}{6}, \\
y = y(r,\theta) &= -\frac{\sin\theta}{2r} + \frac{r^3\sin 3\theta}{6},
\end{aligned}$$

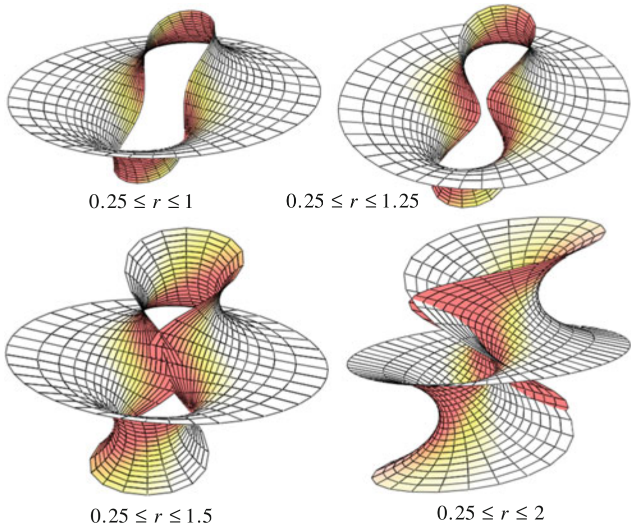


Fig. 2 $0 \le \theta \le 2\pi$

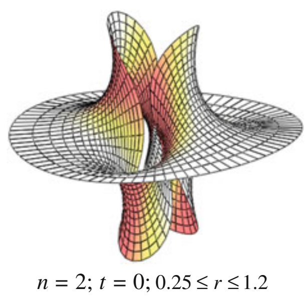


Fig. 3 $0 \le \theta \le 2\pi$

$$z = z(r, \theta) = r\cos\theta.$$

For determination of parametric equations of the Richmond's minimal surface in cylindrical coordinates, the cylindrical coordinates r, θ were introduced into use:

$$u = r\cos\theta; \quad v = r\sin\theta; \quad u^2 + v^2 = r^2.$$

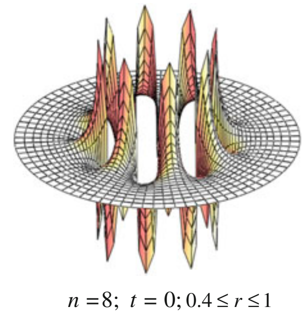


Fig. 4 $0 \le \theta \le 2\pi$

(3) Parametrical equations in polar coordinates r, θ (Figs. 3 and 4):

$$x = x(r, \theta) = -\frac{\cos(t+\theta)}{2r} - \frac{r^{1+2n}\cos[t-(1+2n)\theta]}{2+4n},$$
$$y = y(r, \theta) = -\frac{\sin(t+\theta)}{2r} + \frac{r^{1+2n}\sin[t-(1+2n)\theta]}{2+4n},$$
$$z = z(r, \theta) = \frac{r^n\cos(t-n\theta)}{n},$$

where n = const is a power of the surface, t is a constant parameter defining the given family of minimal surfaces.

Coefficients of the first fundamental form of the surface:

$$A^2 = \frac{1}{4r^4}\left[1 - 2r^4\cos(4\theta) + r^8 + 4r^4\cos^2\theta\right];$$
$$B^2 = \frac{1}{4r^2}\left[1 - 2r^4\cos(4\theta) + r^8 + 4r^4\sin^2\theta\right];$$
$$F = -\frac{1}{4}r\sin 2\theta, \text{ when } t = 0, n = 1.$$

Additional Literature

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19.1 Minimal Surfaces Pulled Over a Rigid Support Contour and Given by Point Frame

The determination of the form of a *minimal surface pulled over a rigid support contour* comes to the integrating of a differential equation

$$\left(1 + z_y^2\right)z_{xx} - 2z_x z_y z_{xy} + \left(1 + z_x^2\right)z_{yy} = 0,$$

where $z = z(x, y)$ is an equation of the unknown minimal surface with a boundary condition $z|_C = \varphi(x, y)$. A problem of the determination of a minimal surface with a given contour is called *the first boundary problem* or *the problem of Dirichlet*. In general case, it is impossible to obtain the function $z = z(x, y)$

by quadratures; therefore, they seek discrete values $z_{i,j}$ by solution of the given equations with the help of a finite difference method. Substituting the difference analogies instead the derivatives into the differential equation, it is possible to derive a nonlinear finite difference operator

$$\begin{aligned}&\left[4h_2^2+\left(z_{i,k+1}-z_{i,k-1}\right)^2\right]\left(z_{i+1,k}-2z_{i,k}+z_{i-1,k}\right)\\&\quad-\left(z_{i+1,k}-z_{i-1,k}\right)\left(z_{i,k+1}-z_{i,jk1}\right)\\&\quad\times\left(z_{i+1,k+1}-z_{i-1,jk1}+z_{i-1,jk1}-z_{i+1,k-1}\right)/2\\&\quad+\left[4h_1^2+\left(z_{i+1,k}-z_{i-1,k}\right)^2\right]\left(z_{i,k+1}-2z_{i,k}+z_{i,jk1}\right)=0,\end{aligned}$$

where h_1, h_2 are intervals of the lattices in the directions of x, y. The rectangle of the finite difference set envelops the contour C but a broken line passing through the nodes of the set substitutes the contour C.

For the decision of a system of the nonlinear algebraic equations, it is possible to use a method of continuation of decision due to parameter in combination with a method of Newton–Kantorovich. The error of the numerical design of a point frame of a minimal surface may be evaluated due to a principle of C. Runge. In this case, the calculation is carried out on two sets with a step h and $2h$. The error of the solution determines by a formula

$$\varepsilon h(x,y)\ = z_h - z_{2h}.$$

Using this error, one may determine the more correct values of the ordinates z of the minimal surface:

$$z = 2z_h - z_{2h}.$$

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19.2 Minimal Surfaces with Free Boundaries

A problem on *minimal surface with free boundaries* (*a problem of Gergonne J.*) has appeared in 1816. It formulates the following way: find a minimal surface if any part of its boundary is given but the rest part must be placed at some given in advance surface. This problem is more difficult than the Plateau problem. The first results were obtained when the given part of a boundary consists of the straight line segments but the rest part must be placed at the given planes (Courante F. 1953). F. Courant described method for proving the existence of a nontrivial and minimizing minimal surface whose boundary lies on a given closed supporting surface.

Much information on minimal surfaces with free boundaries contains in the review lectures of S. Hildebrandt (1984) where the review of the works of the author at the sphere of the investigations of the boundary problems for minimal surfaces is presented. Here, the description of minimal surfaces with free surfaces takes many pages. A problem of the finding of minimal surfaces with the boundaries disposed at the supporting variety and a problem about a minimal surface with the moving boundary of the given length are considered. Minimal surfaces having utterly or partially free boundaries at the given supporting surfaces and a boundary problem with a free surface for surfaces of constant mean curvature are studied. The last results obtained at the sphere of investigation of the behavior of a surface on its free boundary are described and, in particular, the evaluation of the length of the trace of the minimal surface on the free part of the boundary is pointed out.

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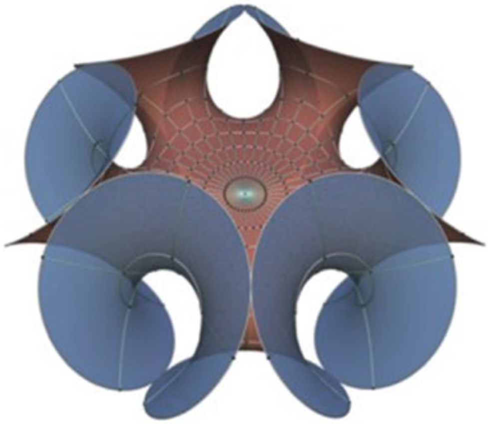
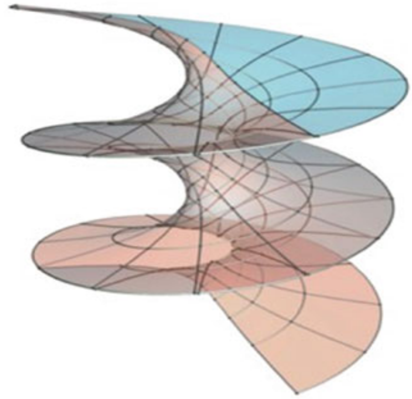
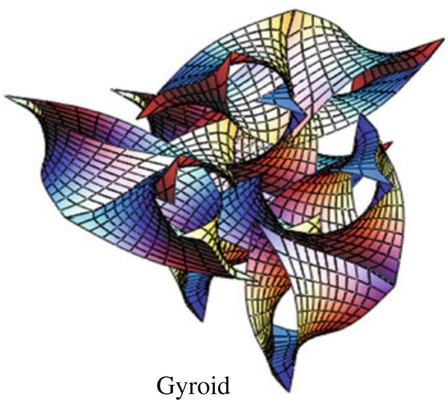
■ **Minimal Surfaces Presented in Internet Sites**



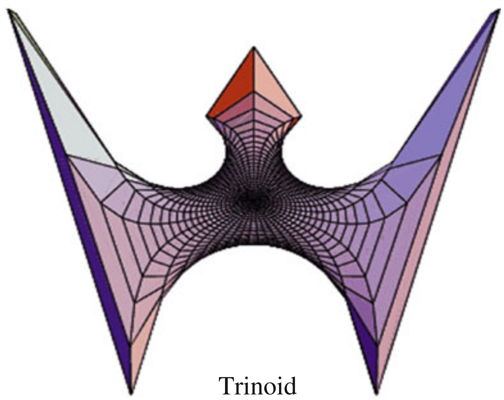
Enneper's surface



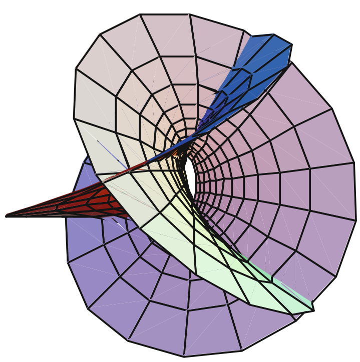
Minimal surface
of the helicoidal type



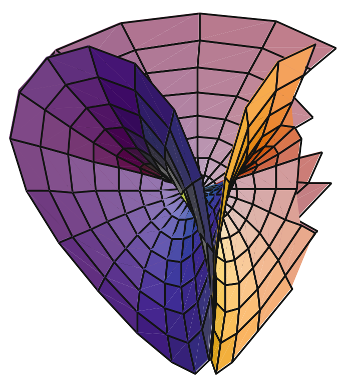
Gyroid



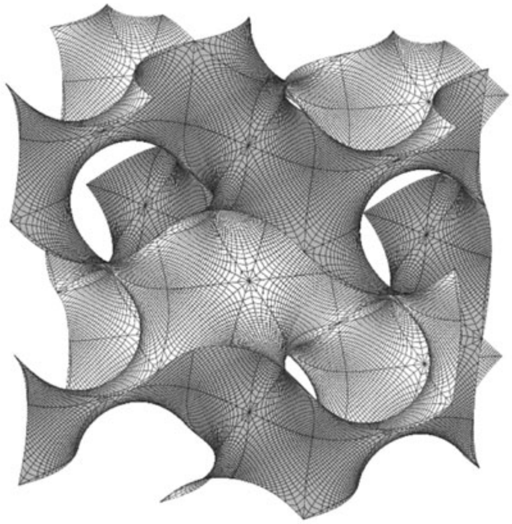
Trinoid



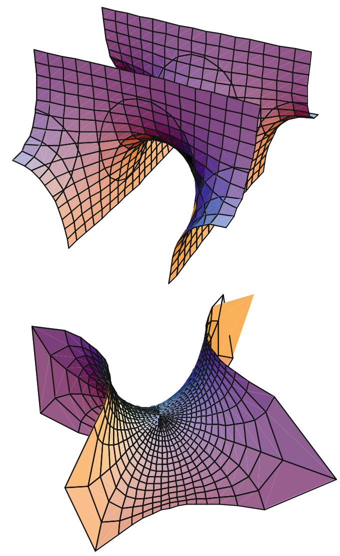
Henneberg minimal surface



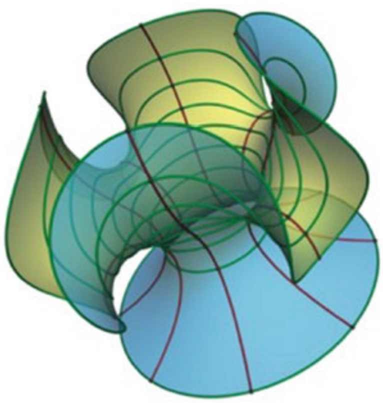
Bour's minimal surface



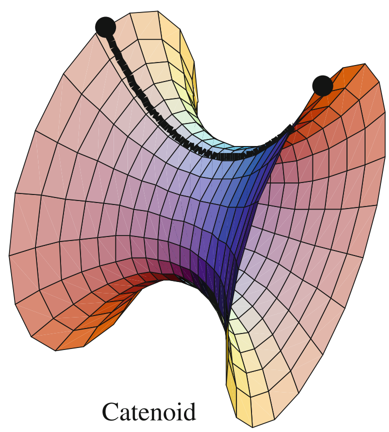
Gyroid



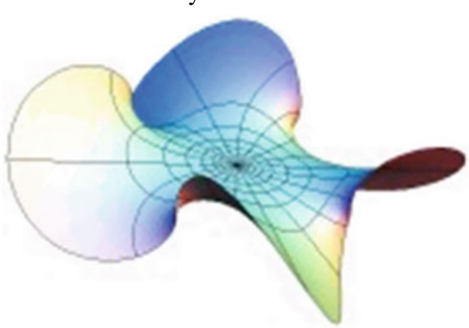
Scherk's minimal surfaces



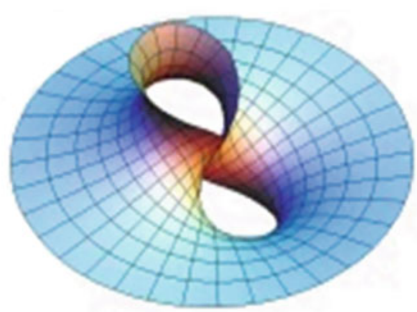
Wavy catenoid



Catenoid

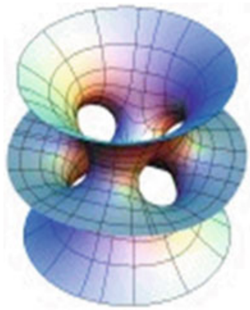


Kusner (Dihedral Symmetric)

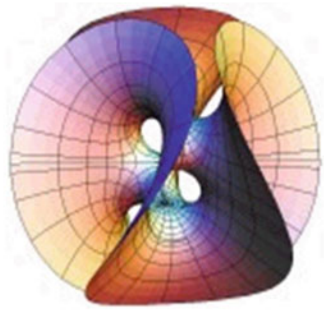


Planar Enneper

■ Minimal Surfaces Presented in Internet Sites (Sequel)

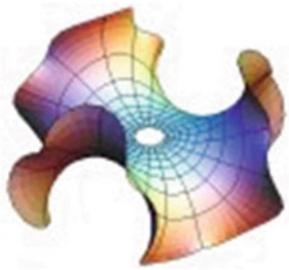


Minimal surface of Costa –Hoffman–Meeks

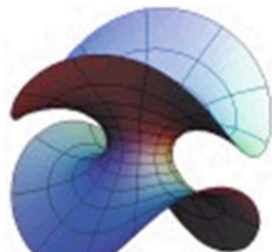


Minimal surface of Chen –Gackstatter

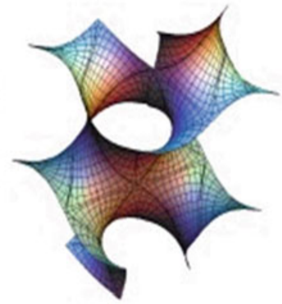
The first Scherk's minimal surface



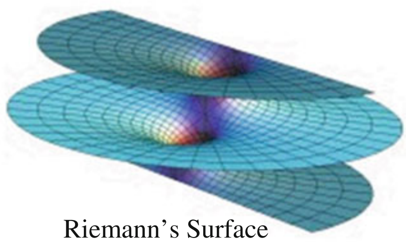
Wavy Enneper surface



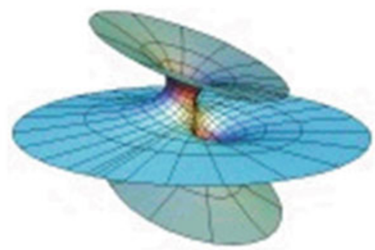
Double Enneper surface



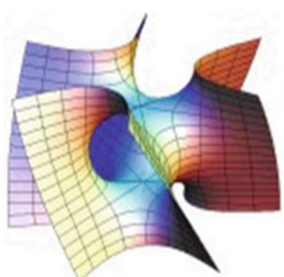
Lidinoid



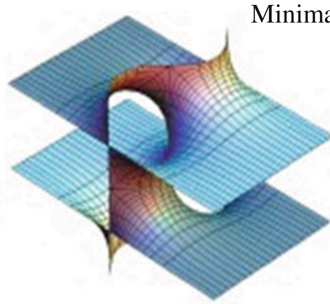
Riemann's Surface



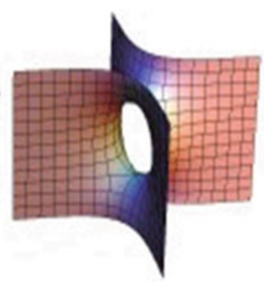
Minimal Lopez –Ros Surface



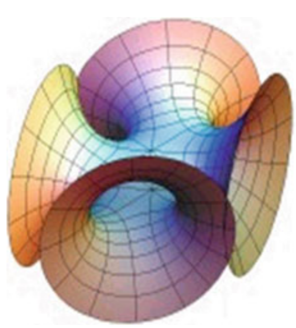
Twisted Scherk surface



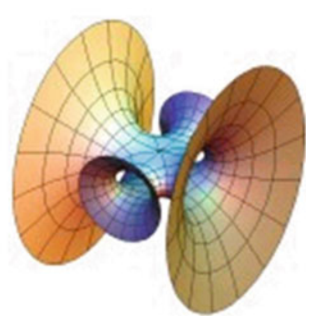
Karcher J.D. Saddle



Saddle Tower



Skew 4-noid



Symmetric 4-noid

The models of the minimal surfaces presented at the last 4 pages “Minimal Surfaces Presented In Internet Sites” are taken at sites:

Eric W. Weisstein “Giroid”, “Trinoid”. From MathWorld: http://mathworld.wolfram.com/Gyroid.html and at http://rsp.math.brandeis.edu/surface/gallery.html.

In scientific literature, *Lopez Minimal Surface* and *Oliveira’s Minimal Surface* [Weisstein EW] are mentioned.

A minimal embedded surface consisting of a helicoid with an opening and handle discovered in 1992 was called *a minimal surface of Hoffman* (Karcher H et al. 1991–1993). This surface has the same topology with *a punctured sphere with a handle*. This is the second (after helicoid) known complete embedded minimal surface of finite topology and infinite total curvature.

A family of complete orientable minimal surfaces named *Chen–Gackstatter Surfaces* was discovered in 1982 (Chen CC, Gackstatter F, 1982). This family contains, as a special case, *Enneper minimal surface*.

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19.3 Complete Minimal Surfaces

Complete minimal surfaces are minimal surfaces that are complete as the metric spaces relatively to their internal metrics. *Completeness* has a strong influence on the topology, conformal structure, and other geometrical properties of a minimal surface.

The complete minimal surfaces may be *compact* (without boundary) and *noncompact* or open.

Complete minimal surfaces remain the focus of intensive current research. In investigations of them, the interest is directed at the research of the connections between the global, metric, geometrical, and topological properties of the surfaces.

The investigation of the two-dimensional complete minimal surfaces at E^3 when $n \geq 3$ is the more advanced. The most parts of the results are obtained when methods of a theory of complex variable are used. They discovered surfaces completely defined by their integral curvature and by topological type. These are *catenoid* and *Enneper’s surface*.

The results on the compact minimal surfaces relate, in general, to the complete minimal surfaces, disposed in the spheres $S^n \subset E^{n+1}$. The interest just to these minimal surfaces is explained, besides the difficulties of the general Riemann space, also by existence of the significant connections between minimal cones at E^{n+1} and minimal surfaces at S^n. Every hypercone at E^{n+1}, determined by its vertex O and intersections with the sphere S^n with the center in O, is a minimal one if and only if its intersection with S^n will be by a minimal surface at S^n. Additional information on the investigations in this sphere may be taken from the additional literature presented at the end of this section.

The *complete Abelian minimal surfaces* were studied by Gackstatter Fritz (1976) who has found the geometric indications of these surfaces and ascertained some their global properties. He has proved a theorem: spherical mapping of the considered surfaces covers the whole of a sphere with the possible exception not more than 4 points.

Rosenberg Harold and Toubiana Eric (1986) proved that complete minimal surface at E^3 with presentation of Weierstrass

$$g(z) = z^3, \quad \omega = \mathrm{d}z \Big/ \left(z^4 - 1\right)^2$$

is an isolated surface. They has shown an example of a nonisolated complete minimal surface at E^3 with presentation of Weierstrass:

$$g(z) = -\frac{z}{2(z^3 - 1)}, \quad \omega = \frac{\mathrm{d}z(z^3 - 1)^2}{(z^3 + 1/2)^2}$$

This surface is modeled at a sphere dotted at the points a_1, a_2, a_3, ∞, where a_k are the cubic roots from $-1/2$.

Rosenberg (1986) conforms that a catenoid, Enneper’s surface and a complete minimal surface M of the finite total curvature modeled at n-dotted sphere and having Weierstrass’ presentation in the form:

$$g(z) = z^{n-1}, \quad \omega = \frac{\mathrm{d}z}{(z^n - 1)^2}$$

are isolated surfaces. The surface M is a subvariety of the Riemann variety N. The detail proof is given for $n = 2$, when M is a catenoid and for $n = 3$.

López F.J. and Martín F. (1999) studied complete minimal surfaces which are bounded as subsets of R^3 (*Naridashvili's theorem*), and some questions related with *Calabi's problem for minimal surfaces*.

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19.4 Minimal Surfaces of Peterson

Peterson surface is a surface with a conjugate set of the conic or cylindrical lines which are the main base of bending. For example, *Monge surfaces with a circular cylindrical directrix surface*, corresponding *translation surfaces* and *surfaces of revolution* are Peterson surfaces. *Indicatrix of revolution* of Peterson surfaces is a right conoid, in particular, this is a right helicoid for the carved Monge surface; for translation surface, this is an equilateral hyperbolic paraboloid (see also Chap. "14. Peterson surfaces").

If a Peterson surface carries a conic set on itself, then the conjugate set is formed by the lines of touching of the cones circumscribed about the surface. The vertexes of these cones lie at two space curves G_1 and G_2. If the both curves are plane but one of them lies at plane of infinity, then the set becomes cylindrical-and-conical one. It is proved that *catenoid* is the only minimal surface among Peterson surfaces of this type. E.A. Korolyov and T.N. Fomina (1979) proposed to write an equation of a surface with conjugate set of the conic lines (Peterson surfaces) in the following form:

$$x = \frac{a_1(u) - b_1(v)}{a_4(u) - b_4(v)}, \quad y = \frac{a_2(u) - b_2(v)}{a_4(u) - b_4(v)},$$
$$z = \frac{a_3(u) - b_3(v)}{a_4(u) - b_4(v)}.$$

In this case, the equations of the lines G_1 and G_2 may be written as:

$$\Gamma_1 : x = \frac{a_1'(u)}{a_4'(u)}, \quad y = \frac{a_2'(u)}{a_4'(u)}, \quad z = \frac{a_3'(u)}{a_4'(u)},$$
$$\Gamma_2 : x = \frac{b_1'(v)}{b_4'(v)}, \quad y = \frac{b_2'(v)}{b_4'(v)}, \quad z = \frac{b_3'(v)}{b_4'(v)}.$$

If we have cylindrical-and-conical set, then $a_4' \neq 0, b_4' = 0$, and so $a_4 \neq$ const, b_4 = const. Not breaking community, they may consider that $a_4 = u; b_1 = v; b_4 = 0$. Then, the equation of the surface with a conjugated net of conic lines becomes simpler:

$$x = \frac{a_1(u) - v}{u}, \quad y = \frac{a_2(u) - b_2(v)}{u}, \quad z = \frac{a_3(u) - b_3(v)}{u}.$$

On condition that this surface must be minimal, i.e., the coefficients of its fundamental forms must satisfy an equation

$$EN - GL = 0\left(A^2N\ -\ B^2L =\ 0\right),$$

which for the considered case will is written as:

$$(\boldsymbol{a}'u - \boldsymbol{a} + \boldsymbol{b})^2(\boldsymbol{b}'', \boldsymbol{a}'u - \boldsymbol{a} + \boldsymbol{b}, \boldsymbol{b}') \\ = \boldsymbol{b}'^2(\boldsymbol{a}''u^2, \boldsymbol{a}'u - \boldsymbol{a} + \boldsymbol{b}, \boldsymbol{b}').$$

The fulfillment of this functional equation is necessary and sufficient condition for a considered surface to be minimal.

If the function $b_2(v)$ is linear, then the surface satisfying the functional equation is a developable surface and exactly *a plane*. For nonlinear function $b_2(v)$, the parametric equations

$$x = -v/u; \quad y = -\sqrt{1-v^2}\Big/xu, \quad z =\ \operatorname{arcosh}(1/u)$$

were obtained and these equations define a *catenoid*. Other variants of the solutions of the functional equation lead to developable surfaces only (Korolyov E.A. and Fomina T.N. 1979).

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19.5 Minimal Surfaces of Thomsen

G. Thomsen has determined minimal surfaces of the Euclidean space, which are *metrical* and *affine minimal surfaces* simultaneously (Blaschke 1923).

If the surface has two of the three following properties:

(1) surface is minimal; (2) it is affine minimal; (3) its asymptotic lines are the slope lines, then it possesses the third property also (Schaal 1973).

Barthel et al. (1980) transformed the parametrical equations of Thomsen surfaces into the form giving an opportunity to join the known properties of these surfaces into one-parametrical cycles differing by the transformation of similitude only and describing by an angular parameter γ, $0 \le \gamma \le 2\pi$. The cycle includes a plane that is a Thomsen surface infinitely moved away, a right helical surface, the Enneper's surface, a left helical surface and then the cycle returns back to the plane. At the paper of Barthel W. et al., the cycle of Thomsen surfaces is presented by visual demonstration with the help of the sequence of the images obtained because of computer.

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■ Minimal Surface of Thomsen Permitting Transition to Enneper's Surface

A minimal surface of Thomsen permitting transition to Enneper's surface is an affine minimal surface simultaneously.

Form of the definition of a minimal surface of Thomsen

(1) Parametrical equations:

$$x = x(u,v) = \frac{cu + \sinh u \cos v}{k^2\sqrt{1-c^2}},$$
$$y = y(u,v) = -\frac{v + c\cosh u \sin v}{k^2\sqrt{1-c^2}},$$
$$z = z(u,v) = -\frac{\sinh u \sin v}{k^2},$$

where k, $|c| < 1$ are constants. When $c = 0$, the surface in question degenerates into a right helicoid.

Comparing the parametrical equations of a Thomsen minimal surface when $k^2 = 1$ and the parametric equations of *a transcendental affine minimal surface* (see also "20. Affine Minimal Surfaces"), it is possible to come to the conclusion that they describe one and the same surface.

Coefficients of the fundamental forms of the surface:

$$A = B = \frac{c\cos v + \cosh u}{k^2\sqrt{1-c^2}}, \quad F = L = N = 0, \quad M = \frac{1}{k^2},$$
$$k_u = k_v = 0, \quad k_1 = k_2 = \pm\frac{1}{k^2A^2},$$
$$K = -\frac{1}{k^4A^4} < 0, \quad H = 0.$$

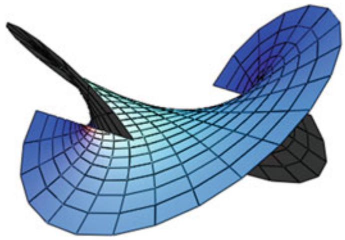

Fig. 1

Schaal Hermann has shown that *the Enneper's minimal surface* turns out when one uses a proper limiting transition under the constants that are contained at the equations of the Thomsen surface. Figure 1 shows the segment of the Thomsen minimal surface when $c = 1/\sqrt{2}$; $k^2 = 1$; $-2/3 \le u \le 2/3$; $-\pi \le v \le \pi$.

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19.6 Chen–Gackstatter Surfaces

A class of *complete orientable minimal surfaces* of R^3 derived from *Enneper's minimal surface*. They are named for the mathematicians who found the first two examples in 1982.

The complete information on the Chen-Gackstatter surfaces is represented in MathWorld—A Wolfram Web Resource (*Barile Margherita*). The data given at this page is taken from this resource.

The Chen-Gackstatter surfaces form a double-indexed collection M_{ij}, where $i \ge 0$ and $j \ge 1$. $M_{0,1}$ is Enneper's minimal surface, and M_{i1} is obtained from $M_{0,1}$ by adding i handles so that it has topological genus equal to i. It has one Enneper end with winding order three, meaning that, like Enneper's minimal surface, it has a symmetric threefold shape which tends to coincide with a triple plane far away from the center.

In general, M_{ij} has total curvature $c = -4\pi(i + 1)j$, topological genus ij, and one Enneper end of winding order $2j + 1$. This property distinguishes it from other surfaces such as *the catenoid* which have two ends of winding order 1.

The first Chen–Gackstatter surface $M_{1,1}$ has topological genus $p = 1$ and total curvature -8π. Its *Enneper–Weierstrass parameterization* is given by

$$g(z) = \frac{A\wp'(z)}{\wp(z)}, \quad f(z) = \wp(z),$$

where $\wp(z)$ is the Weierstrass elliptic function with parameters

$$g_2 = 60 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m+ni)^4}, \quad g_3 = 0,$$

with i the imaginary unit (and where g_2 turns out to be real and positive), and the constant A given by

$$A = \sqrt{(3\pi)/(2g_2)}.$$

In 1992, F.J. López has shown that $M_{1,1}$ is the only genus one orientable complete minimal surface of total curvature -8π.

In a neighborhood of the origin $M_{1,1}$ can be approximated by the following parametrical equations:

$$x = \frac{4A^2}{3r^3}\cos(3\theta), \quad y = -\frac{4A^2}{3r^3}\sin(3\theta), \quad z = \frac{2A^2}{3r^2}\cos(2\theta),$$

where r is a small positive constant and $0 \le \theta \le 2\pi$.

At the site shown in "References," the second Chen–Gackstatter surface is also described and its Enneper–Weierstrass parameterization is given.

Reference

Barile Margherita. Chen-Gackstatter Surfaces. From MathWorld—A Wolfram Web Resource, created by Eric W. Weisstein: http://mathworld.wolfram.com/Chen-GackstatterSurfaces.html.

19.7 Algebraic Minimal Surfaces

For the definition of *an algebraic minimal surface* by parametrical equations, it is possible to use *the Weierstrass method*. According to this method, any analytical function $f(\tau)$ satisfying the condition

$$f'''(\tau) \neq 0$$

determines a minimal surface, the generalized parametrical equations of which may be written as:

$$\begin{aligned} x &= x(u,v) = \mathrm{R}\{(1-\tau^2)f''(\tau) + 2\tau f'(\tau) - 2f(\tau)\},\\ y &= y(u,v) = \mathrm{R}\{i(1+\tau^2)f''(\tau) - 2i\tau f'(\tau) + 2if(\tau)\},\\ z &= z(u,v) = \mathrm{R}\{2\tau f''(\tau) - 2f'(\tau)\}, \end{aligned}$$

where R is a real part of the function of the complex variable.

For the determination of an algebraic minimal surface, it is necessary to give an analytical function $f(\tau)$ in the form of a degree polynomial with the complex coefficients

$$f(\tau) = \alpha_1 + \alpha_2\tau + \alpha_3\tau^2 + \alpha_4\tau^3 + \cdots + \alpha_j\tau^{j-1},$$

where

$$\tau = u + iv; \quad \alpha_j = a_j + ib_j; \quad j = 1,2,3,4,\ldots,k.$$

Substituting the values $f(\tau)$, $f'(\tau)$, and $f''(\tau)$ into the generalized parametrical equations, one may derived the parametric equations of a family of algebraic minimal surfaces that are in conformity with the chosen polynomial.

For illustration of the Weierstrass method, let us define the analytical function $f(\tau)$ by a polynomial of the third degree with the complex coefficients:

$$f(\tau) = \alpha_1 + \alpha_2\tau + \alpha_3\tau^2 + \alpha_4\tau^3,$$

where $\tau = u + iv$; $\alpha_j = a_j + ib_j$; $j = 1, 2, 3, 4$;

$$f'(\tau) = \alpha_2 + 2\alpha_3\tau + 3\alpha_4\tau^2, \quad f''(\tau) = 2\alpha_3 + 6\alpha_4\tau.$$

Putting the values $f(\tau)$, $f'(\tau)$ and $f''(\tau)$ into the generalized parametrical equations, one may obtain the parametric equations of a family of algebraic minimal surfaces that are in conformity with a polynomial of the third degree:

$$\begin{aligned} x &= x(u,v) = (-2u^3 + 6uv^2 + 6u)a_4\\ &\quad + (6u^2v - 2v^3 - 6v)b_4 - 2a_1 + 2a_3,\\ y &= y(u,v) = (2v^3 - 6vu^2 - 6v)a_4\\ &\quad + (6v^2u - 2u^3 - 6u)b_4 - 2b_1 - 2b_3,\\ z &= z(u,v) = (6u^2 - 6v^2)a_4 - 12uvb_4 - 2a_2. \end{aligned}$$

A change of the parameters α_1 and α_3 leads to the motion parallel to the plane *xOy* without deformation of a minimal surface. A change of the real part of the parameter α_2 leads to the motion of a minimal surface parallel the axe *Oz*. On the base of the properties given above, one may conclude about the form of a minimal surface that is in conformity with the chosen polynomial of the third order and depends of the coefficient at the highest degree of the polynomial. For all of the rest cases, one obtains one and the same minimal surface determined at the space accurate within the translation and the similitude. Finally, when the third degree polynomial is chosen, the parametric equations of a minimal surface can be written as:

$$\begin{aligned} x &= x(u,v) = (-2u^3 + 6uv^2 + 6u)a_4 + (6u^2v - 2v^3 - 6v)b_4,\\ y &= y(u,v) = (2v^3 - 6vu^2 - 6v)a_4 + (6v^2u - 2u^3 - 6u)b_4,\\ z &= z(u,v) = (6u^2 - 6v^2)a_4 - 12uvb_4, \end{aligned}$$

For a minimal surface with the polynomial $f(\tau) = \tau^3$, the equations are written in the following form:

$$\begin{aligned} x &= x(u,v) = -2u^3 + 6uv^2 + 6u,\\ y &= y(u,v) = 2v^3 - 6vu^2 - 6v,\\ z &= z(u,v) = 6u^2 - 6v^2. \end{aligned}$$

The presented equations show that an analytic minimal surface, obtained in this case, is *an Enneper's minimal surface*.

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19.8 Embedded Triply Periodic Minimal Surfaces

Embedded triply periodic minimal surfaces are minimal surfaces symmetrical relative to three independent directions. The general property of these surfaces is the following: the existence of basic piece of the surface which may be disposed within a polyhedron so that the boundaries of this piece lie at the sides of the polyhedron and along the whole of the boundary of the contact, the minimal surface intersects the polyhedron at right angle. Several basic pieces of the minimal surface may be joined for *a basic sell*. This sell also is disposed into the polyhedron, which, as usual, is *a regular polyhedron*.

Schwarz minimal surface is the first example of an embedded triply periodic minimal surface. Figure 1 taken without change at the site [http://www.susqu.edu/brakke/evolver/examples/periodic/periodic.html] of internet shows this surface. Twelve minimal surfaces of the considered type were proposed and realized at the plastic models by A.H. Schoen in 1970. Later in 1988, he submitted additionally for consideration eight new embedded triply periodic minimal surfaces, designed with using of a principle of the reflection of H.A. Schwarz. The existence of these surfaces was corroborated by the investigations of H. Karcher, who introduced in circulation many new minimal surfaces of the considered type using the design from the conjugated surfaces. H. Karcher studied also the bending of embedded triply periodic minimal surfaces into the embedded triply periodic minimal surfaces of constant mean curvature.

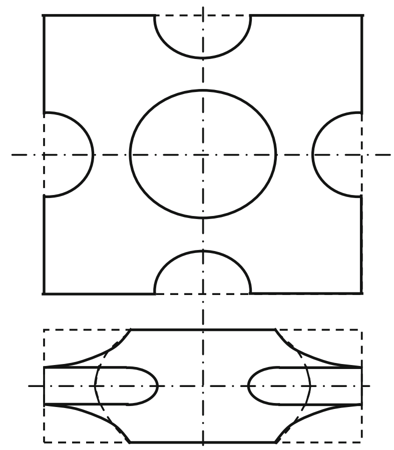

Fig. 2

In Fig. 2, a basic cell of one of the minimal surfaces of A. Shoen disposing into *a cuboid* and consisting of 16 congruent basic parts of the surface. The cuboid has the top ant low square sides. Changing the height of the cuboid, it is possible to derive the one-parametric family of the minimal surfaces. K. Polthier considers that the embedded triply periodic minimal surfaces dividing the space into two non-uniting segments are the most interesting for research. These segments may be represented as two labyrinths penetrating into each other but divided by the minimal surface.

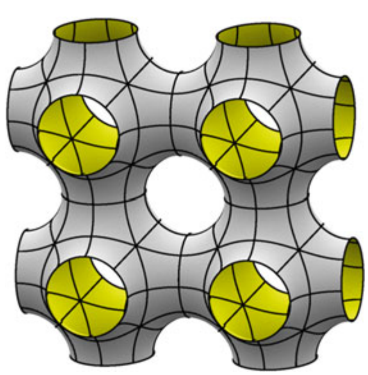

Fig. 1

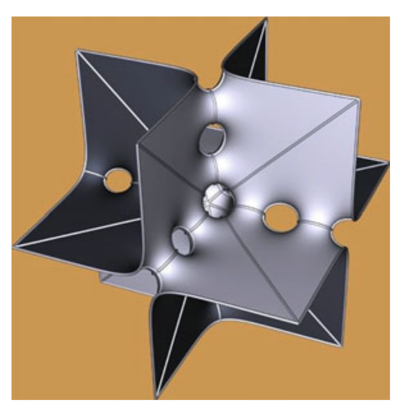

Fig. 3

At present time, the majority of the known embedded triply periodic minimal surfaces was obtained by experimental methods with an application of the soap film, by the vacuum forming of models from a plastic flat blank or was formed at a spatial contour frame from the thermoplastic polymer materials.

In 2000, James Hoffman takes an active part in creation of computer programs for scientific visualization of minimal surfaces. He was chosen by mathematician David Hoffman for special scientific work with minimal surfaces.

Computer program "*Surface Evolver*" written by K. Brakke gives an opportunity to research different types of minimal surfaces. In Fig. 3, the embedded triply periodic minimal surface designed with the help of this program is shown.

Embedded triply periodic minimal surfaces are of interest for the crystallography and chemistry under investigation of the spatial structures of the atoms and their units in the crystal lattices. A.H. Schoen notes the likeness among the surfaces in question and the surfaces of the lime skeleton of some sea microscopic creatures.

Additional Literature

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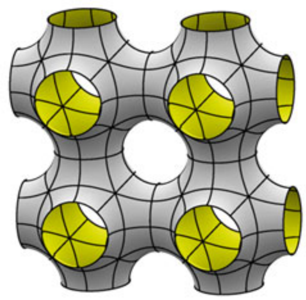
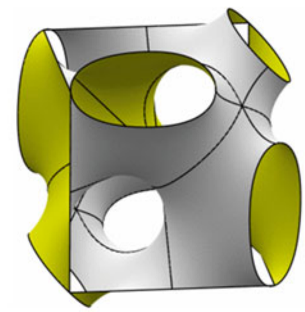
■ **Several Examples of Embedded Triply Periodic Minimal Surfaces**

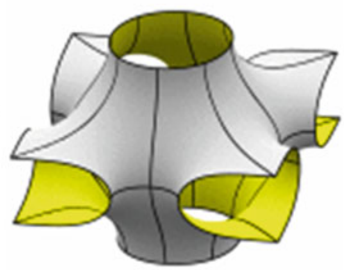


Schwarz' P Surface

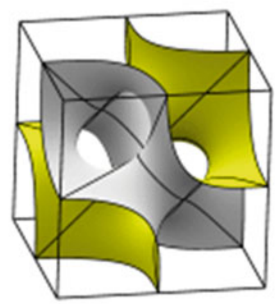


Starfish Minimal Surfaces

Schoen's F-RD Surface

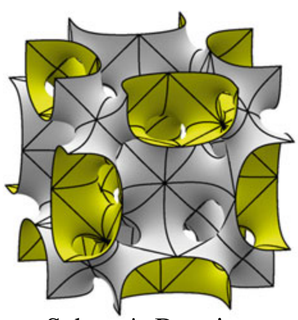


Hybrids

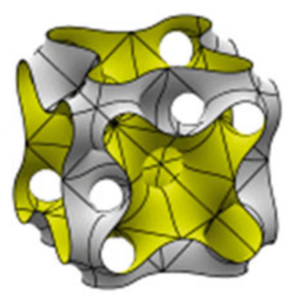


Batwing Family

Schoen's Complementary D Surface

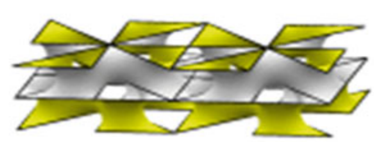


Schoen's Batwing Surface

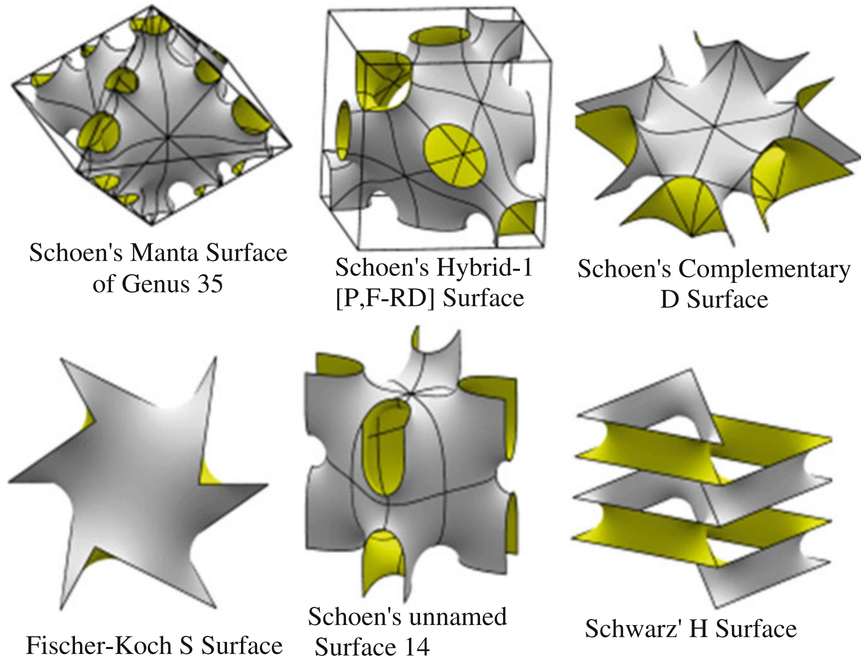


Schoen's Manta Surface of Genus 19

Fischer-Koch Y Surface



Schoen's I-9 Surface



P.S. The surfaces are taken at the internet site http:\\www.susqu.edu/facstaff/b/brakke/examples/periodic/periodic.html.

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20 Affine Minimal Surfaces

Affine minimal surface is a surface with the affine mean curvature equal to zero. In contrast to ordinary minimal surfaces consisting only of the saddle points, the affine minimal surface may contain elliptic points. Thus, *the elliptical paraboloid* consists only of the elliptic points but it is an affine minimal surface.

The Enneper's surface is the most known typical minimal surface and simultaneously *the affine minimal surface*, the sculpture representation of which is exhibited in the new building of the Mathematical Institute of the University of the Wartburg.

The *minimal surfaces of Thomsen* are also simultaneously the affine minimal surfaces. A family of these surfaces includes *the plane* (infinitely removed surface of Thomsen), the right-hand right helicoidal surface, the Enneper's surface, and the left-hand right helicoidal surface. Affine minimal surfaces which are also Euclidean minimal surfaces have first been derived by G. Thomsen.

Along any asymptotic line of an affine minimal surface, tangents to the asymptotic lines of another family are parallel to some plane. For the surface related to the asymptotic coordinates, one uses the conception of the covariant vector

$$\boldsymbol{v}_{uv} = \frac{\lambda\lambda_{uv} - 2\lambda_u\lambda_v}{\lambda^4}[\boldsymbol{n}_u\boldsymbol{n}_v],$$

where

$$\lambda = \sqrt{|\boldsymbol{n}_u|^2}$$

is a scalar parameter equal to the length of the vector $\boldsymbol{n}_u(u,v)$, that is the coefficient of a linear element of a sphere

$$\mathrm{d}s^2 = \lambda^{-2}\left(\mathrm{d}u^2 + \mathrm{d}v^2\right),$$

having the isothermic set.

For the affine minimal surfaces, the covariant vector $v_{uv} = 0$. So, the spherical images of the asymptotic lines are plane curves consisting of the orthogonal net of circles.

Karl Strubecker has determined all of minimal surfaces Φ in the coordinate space, which simultaneously are affine minimal surfaces. The isotropic spherical image of the asymptotic lines of the surfaces Φ generates two orthogonal pencils of isotropic circles. Depending on types of these pencils, the surfaces Φ divide into three classes.

Krauter P. has proved that only four types of the affine minimal surfaces exist in three-dimensional affine space A^3. The affine minimal surface of revolution at A^3 has an affine area rigorously more than any other affine surface of revolution with the same axis and with the same boundary.

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■ Transcendental Affine Minimal Surface

Transcendental affine minimal surface is simultaneously an ordinary minimal surface. The surface in question is called also *the minimal surface of Thomsen permitting transition to Enneper's surface* (see also a Sect. "19.5. Thomsen Minimal Surfaces").

The form of the definition of the transcendental affine minimal surface

(1) Parametrical equations:

$$x = x(u,v) = \frac{cv + \sinh v \cos u}{\sqrt{1-c^2}}$$
$$y = y(u,v) = \frac{u + c \sin u \cosh v}{\sqrt{1-c^2}}$$
$$z = z(u,v) = -\sin u \sinh v,$$

where $c \neq 1$.

If $c = 0$, then a transcendental affine minimal surface degenerates into *the right helicoid* (see also the Chap. "19. Minimal Surfaces").

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = B^2 = \frac{(\cosh v + c\cos u)^2}{1-c^2}, \quad F = 0,$$
$$A^2B^2 - F^2 = A^4,$$
$$L = N = 0, \ M = 1,$$
$$k_u = k_v = 0, \quad k_{1,2} = \pm\frac{(1-c^2)}{(\cosh v + c\cos u)^2}.$$
$$K = -\frac{(1-c^2)^2}{(\cosh v + c\cos u)^4} < 0, \quad H = 0.$$

The surface is given at the curvilinear orthogonal non-conjugate coordinates u, v. The coefficients of the second fundamental form of surface show that the coordinate lines u, v are plane curves.

The coefficients of the first fundamental form of surface confirm that the coordinate set is *isothermic*, i.e.

$$ds^2 = A^2(du^2 + dv^2), \text{ or } \quad A = B.$$

The unit vector of the normal $\boldsymbol{n}(u,v)$ to the surface may be given as:

$$\boldsymbol{n}(u,v) = -\frac{\sqrt{1-c^2}}{\cosh v + c\cos u} \times \left\{\sin u\boldsymbol{i} + \sinh v\boldsymbol{j} + \frac{\cos u + c\cosh v}{\sqrt{1-c^2}}\boldsymbol{k}\right\}.$$

Then

$$\boldsymbol{n}_u(u,v) = \frac{d\boldsymbol{n}}{du} = -\frac{\sqrt{1-c^2}}{(\cosh v + c\cos u)^2} \times \left\{(c + \cos u\cosh v)\boldsymbol{i} + c\sinh v \sin u\boldsymbol{j} - \sqrt{1-c^2}\sin u\cosh v\boldsymbol{k}\right\}.$$

A scalar parameter λ equal to the length of the vector $\boldsymbol{n}_u(u,v)$ is determined by the following formula:

$$\lambda = \sqrt{|\boldsymbol{n}_u|^2} = \frac{\sqrt{1-c^2}}{\cosh v + c\cos u} = \frac{1}{A}.$$

Having determined λ_u, λ_v, λ_{uv}, one can satisfy himself that

$$\boldsymbol{v}_{uv} = \frac{\lambda\lambda_{uv} - 2\lambda_u\lambda_v}{\lambda^4}[\boldsymbol{n}_u\boldsymbol{n}_v] = 0.$$

Thus, the transcendental affine minimal surface in question relates to a class of the affine minimal surfaces.

In Figs. 1, 2 and 3, the fragments of the surfaces in question limited by the coordinate lines u and v are shown. All figures are limited by the contour lines $v = -2$ and $v = 2$. The limits of changing of the coordinate line u and the values of the parameter c are given at the corresponding pictures.

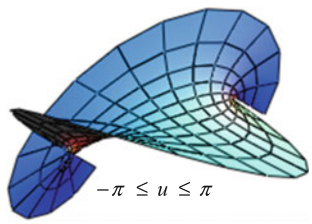

Fig. 1 $c = 0.8$

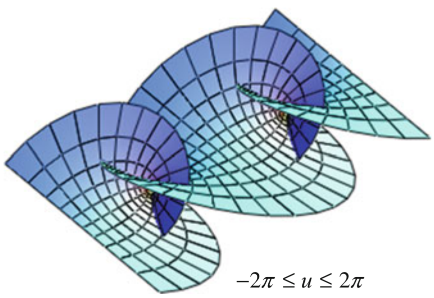

Fig. 2 $c = 0.8$

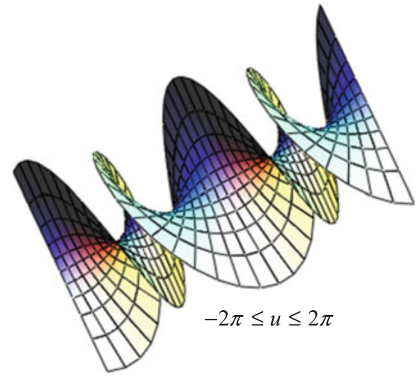

Fig. 3 $c = 0.25$

Reference

Nartova LG, Dzhuraev TK. On one property of the Enneper's minimal surface. Kibernetika Grafiki i Prikl. Geometriya Poverchnostey. Moscow: MAI, 1971; Vol. VIII, Iss. 231, p. 86-88 (2 refs.)

21 Surfaces with Spherical Director Curve

Surfaces with spherical director curve have a spherical curve

$$\boldsymbol{E}_0(u) = a\boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\sin\omega,$$

at the surface of a sphere of a radius a as a director curve; $\omega = \omega(u)$. The unit vector $\boldsymbol{e}_0(u)$ is a normal of the sphere, at which the director curve is disposed.

A generatrix plane curve is given in the local system of coordinates with the origin of the coordinates on the spherical director line:

$$X = X(v),\ Y = Y(v).$$

Usually, they use two types of surfaces with a spherical generatrix curve:

(1) the generatrix curves are disposed at the planes of the meridians of the sphere (type 1);
(2) the generatrix curves lie at the normal planes of the spherical director curve (type 2).

Forms of definition of the surfaces with a generatrix curve in the planes of the meridians of the sphere

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = [a\cos\omega + \varphi(u,v)]\boldsymbol{h}(u) + [a\sin\omega + \psi(u,v)]\boldsymbol{k},$$

where

$$\varphi(u,v) = X(v)\cos\theta - Y(v)\sin\theta;$$
$$\psi(u,v) = X(v)\sin\theta + Y(v)\cos\theta;$$

and $\omega = \omega(u) = pu + c$; p and c are constants; $\theta = \theta(u) = d + tu$ is the angle characterizing the turn of the local system of coordinates relatively to the axis of the sphere; d and t are constants.

(2) Parametrical equations:

$$x = x(u,v) = [a\cos\omega + \varphi(u,v)]\cos u,$$
$$y = y(u,v) = [a\cos\omega + \varphi(u,v)]\sin u,$$
$$z = z(u,v) = a\sin\omega + \psi(u,v).$$

Coefficients of the fundamental forms of the surface (type 1):

$$A^2 = a^2\omega'^2 + \left(X^2 + Y^2\right)\theta'^2 + (a\cos\omega + \varphi)^2 + 2a\omega'\theta'\xi,$$
$$F = a\omega'\zeta + \theta'(X\dot{Y} - \dot{X}Y),\ B^2 = \dot{X}^2 + \dot{Y}^2,$$
$$L = -\Big\{(a\cos\omega + \varphi)\Big[a\omega'^2\zeta + \theta'^2(X\dot{Y} - \dot{X}Y) + (a\cos\omega + \varphi)\dot{\psi}\Big] + 2(a\omega'\sin\omega + \theta'\psi) \times \left[a\omega'\dot{\xi} + \theta'(X\dot{X} - \dot{Y}Y)\right]\Big\}/\sqrt{A^2B^2 - F^2},$$
$$M = \left\{[a\omega'\varphi + \theta'(X\dot{X} - \dot{Y}Y)]\dot{\varphi} - \theta'(a\cos\omega + \varphi)B^2\right\}/\sqrt{A^2B^2 - F^2},$$
$$N = -\left\{(a\cos\omega + \varphi)(\dot{X}\ddot{Y} - \ddot{X}\dot{Y})\right\}/\sqrt{A^2B^2 - F^2}$$

where $\xi = \dot{\varphi}\cos\omega + \dot{\psi}\sin\omega$; $\zeta = -\dot{\varphi}\sin\omega + \dot{\psi}\cos\omega$.

Forms of definition of the surfaces with a generatrix curve in the normal planes of the spherical director line

(1) Vector form of assignment:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = ae_0(u) + \varphi(u,v)e_0(u) + \psi(u,v)\boldsymbol{g}(u),$$

where

$$\boldsymbol{e}_0 = \boldsymbol{e}_0(u) = \boldsymbol{h}(u)\cos\omega + \boldsymbol{k}\sin\omega;$$
$$\boldsymbol{g} = \boldsymbol{g}(u) = \left[\boldsymbol{e}_0'/s \times \boldsymbol{e}_0\right]$$

is a unit vector, orthogonal to the vector $\boldsymbol{e}_0(u)$;

$$\boldsymbol{h} = \boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u;$$
$$\varphi(u,v) = X(v)\cos\theta - Y(v)\sin\theta;$$
$$\psi(u,v) = X(v)\sin\theta + Y(v)\cos\theta;$$

$\theta = \theta(u) = d + tu$; d and t are constant; θ is the angle characterizing the turn of the local system of coordinates relatively to the axis of the sphere;

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$$s = \left[\omega'^2 + \cos^2\omega\right]^{1/2};$$

$\omega = \omega(u)$ is a function chosen in advance.

(2) Parametric form of assignment:

$$\begin{aligned} x = x(u,v) &= [a + \varphi(u,v)]\cos\omega\,\cos u \\ &\quad + \psi(u,v)(\sin\omega\,\cos\omega\,\cos u - \omega'\,\sin u)/s, \\ y = y(u,v) &= [a + \varphi(u,v)]\cos\omega\,\sin u \\ &\quad + \psi(u,v)(\sin\omega\,\cos\omega\,\sin u + \omega'\,\cos u)/s, \\ z = z(u,v) &= [a + \varphi(u,v)]\sin\omega - \psi(u,v)\,\cos^2\omega/s. \end{aligned}$$

Coefficients of the fundamental forms of the surface (type 2):

$$A = s(a+\varphi) + \left[\left(1 + \frac{\omega'^2}{s^2}\right)\psi\,\sin\omega + \frac{\omega''}{s^2}\cos\omega\right]\psi,$$

$$F = 0,\; B = \sqrt{\dot{X}^2 + \dot{Y}^2},$$

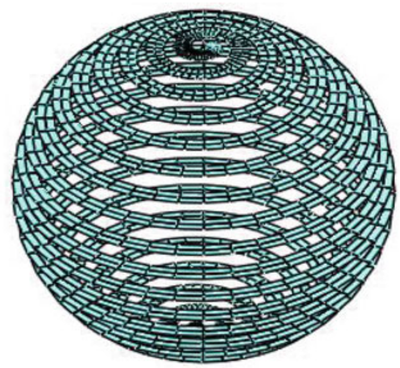
$$L = \frac{A}{B}\left\{s\dot{\psi} - \left[\left(1 + \frac{\omega'^2}{s^2}\right)\psi\,\sin\omega + \frac{\omega''}{s^2}\cos\omega\right]\dot{\varphi}\right\},$$

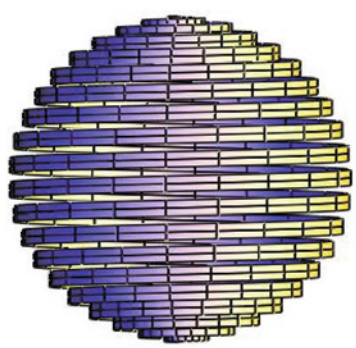
$$M = 0,\; N = \frac{\dot{\varphi}\ddot{\psi} - \ddot{\varphi}\dot{\psi}}{B} = \frac{\dot{X}\ddot{Y} - \ddot{X}\dot{Y}}{B}.$$

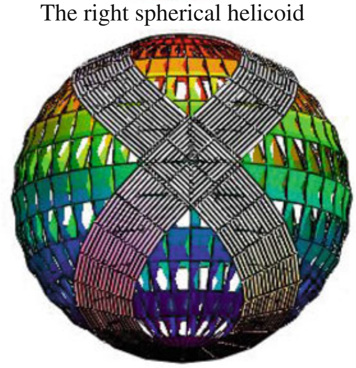
At all formulas, a symbol $\ldots'$ means differentiation with respect to u and a symbol $\dot{\ldots}$ is differentiation with respect to v.

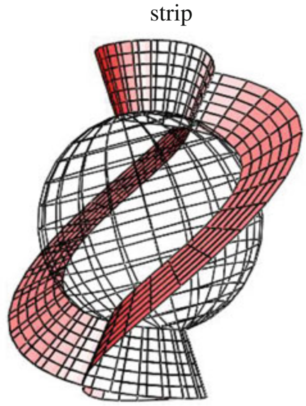
■ Surfaces with a Spherical Director Curve Presented in the Encyclopedia

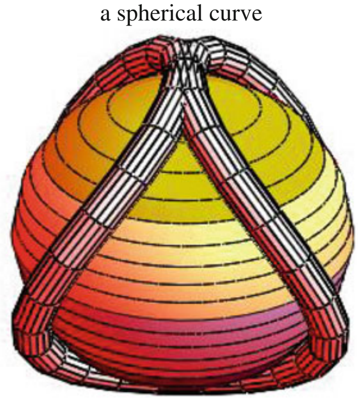

The spherical helicoid

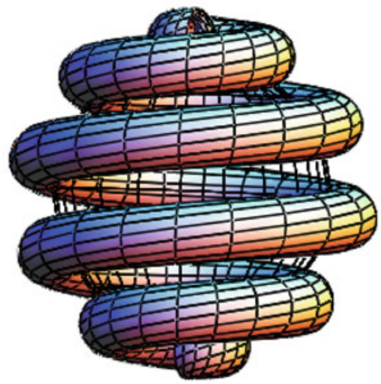

The right spherical helicoid


The cylindrical-and-spherical spiral-shaped strip

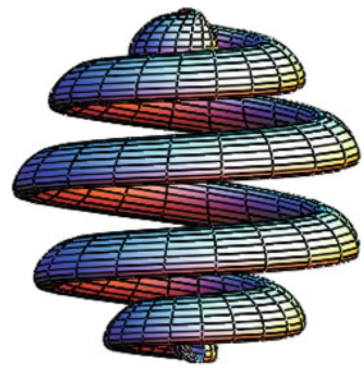

The torse with generating straight lines lying in the normal planes of a spherical curve


The conical surface with a directrix curve on a sphere

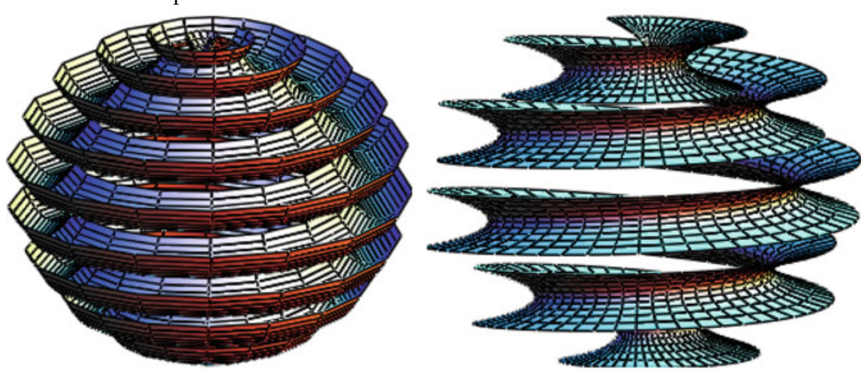

The tubular surface on the sphere



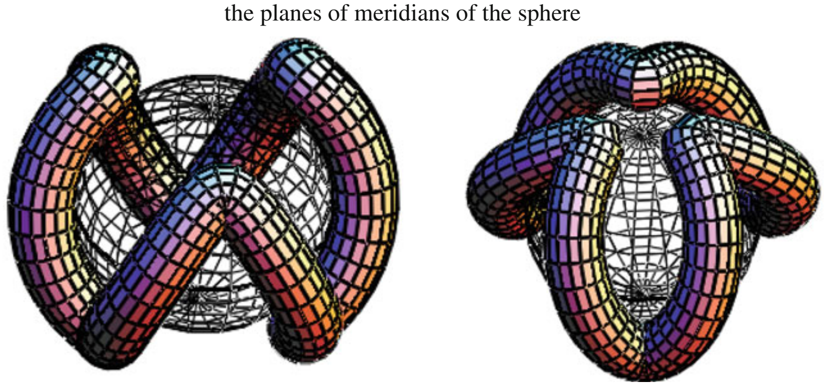
The cyclic surface with circles in the planes of meridians of the sphere and with a center-to-center line on the same sphere



A surface with a spherical director curve and with the elliptical generatrix in the planes of meridians of the sphere



Surfaces with a spherical director curve and with the parabolic generatrix in the planes of meridians of the sphere



The wave-shaped torus on the sphere

■ Surfaces with a Spherical Director Curve and with a Parabolic Generatrix in the Planes of Meridians of the Sphere

A surface with a spherical director curve and with a parabolic generatrix in the planes of meridians of the sphere has a spherical curve

$$\boldsymbol{E}_0(u) = a\boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\,\sin\omega,$$

as a directrix curve at the surface of the sphere of a radius *a;* $\omega = \omega(u)$. The generatrix parabola is given at the local system of the coordinates with the origin at the spherical director line:

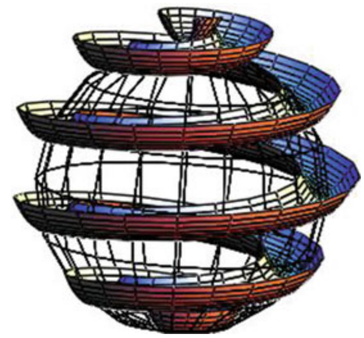
$$X = X(v) = v, \quad Y = Y(v) = bv^2.$$

Parametrical equations of the surface (Figs. 1, 2 and 3) can be written as:

$$x = x(u,v) = [a\,\cos\omega + \varphi(u,v)]\cos u,$$
$$y = y(u,v) = [a\,\cos\omega + \varphi(u,v)]\sin u,$$
$$z = z(u,v) = a\,\sin\omega + \psi(u,v),$$

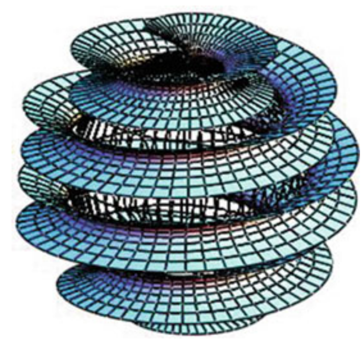
$a = 5;\ b = 1;\ p = 1/5;\ \theta = \pi/2;$
$-1 \le v \le 1;\ -5\pi/2 \le u \le 5\pi/2;$

Fig. 1



$a = 5;\ b = 1;\ p = 1/10;\ \theta = \pi/2;$
$-1 \le v \le 1;\ -5\pi/2 \le u \le 5\pi/2;$

Fig. 2



$a = 10;\ b = 1;\ p = 1/10;\ \theta = \pi/2;$
$-2 \le v \le 2;\ -5\pi \le u \le 5\pi$

Fig. 3

where $\omega = \omega(u) = pu;\ p = \text{const};\ \theta = d = \text{const};$

$$\varphi(u,v) = v\cos\theta - bv^2\sin\theta;$$
$$\psi(u,v) = v\sin\theta + bv^2\cos\theta;$$

θ is the angle characterizing the turn of the local system of coordinates relatively to the axis of the sphere.

The coefficients of the fundamental forms of surface may be obtained by the formulas given at the page entitled as "21. *Surfaces with spherical director curve*".

■ Surface with a Spherical Director Curve and with an Elliptical Generatrix in the Planes of Meridians of the Sphere

A surface with a spherical director curve and with an elliptical generatrix in the planes of meridians of the sphere has a spherical curve

$$\boldsymbol{E}_0(u) = a\boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\sin\omega,$$

as a directrix curve at the surface of the sphere of a radius a; $\omega = \omega(u)$. A generatrix ellipse is given at the local system of Cartesian coordinates with the origin of the coordinate system at the spherical director line:

$$X = X(v) = b\cos v,$$

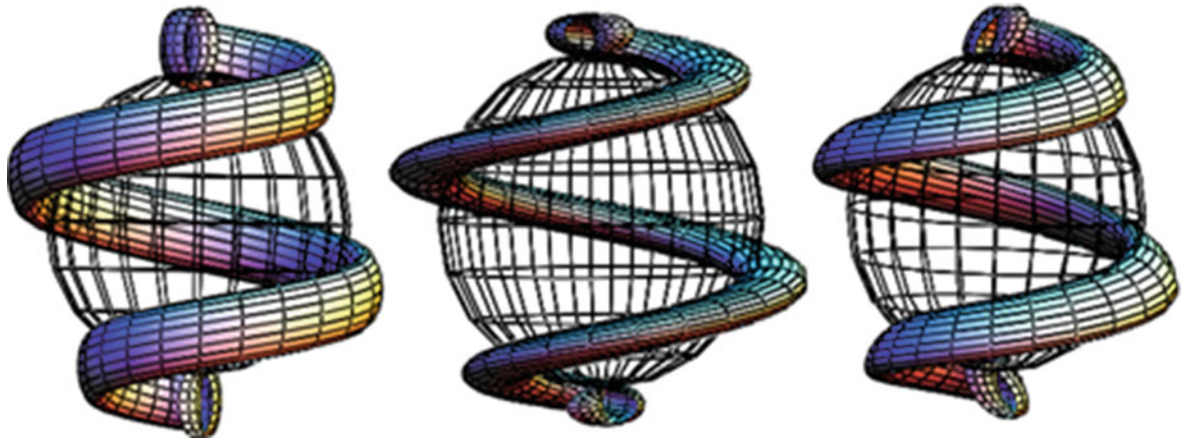
$$Y = Y(v) = c\sin v.$$

Parametrical equations of the surface (Fig. 1) can be written as:

$$x = x(u,v) = [a\cos\omega + \varphi(u,v)]\cos u,$$
$$y = y(u,v) = [a\cos\omega + \varphi(u,v)]\sin u,$$
$$z = z(u,v) = a\sin\omega + \psi(u,v),$$

where

$$\varphi(u,v) = b\cos v\cos\theta - c\sin v\sin\theta;$$
$$\psi(u,v) = b\cos v\sin\theta + c\sin v\cos\theta;$$
$$\omega = pu;\ p = \text{const};$$



$p = 1/5;\ \theta = 0;$ $a = 10;\quad b = 1;\quad c = 2;$

$p = 1/5;\quad \theta = \pi/2;$ $-5\pi/2 \le u \le 5\pi/2;$

$p = 1/5;\quad \theta = \pi/4;$ $-1 \le v \le 1$

Fig. 1

$\theta = d = \text{const}$ is the angle characterizing the turn of the local system of coordinates relatively to the axis of the sphere.

The coefficients of the fundamental forms of surface may be obtained by the formulas given at the page entitled as "21. *Surfaces with spherical director curve.*"

■ Tubular Loxodrome

The tubular loxodrome has a spherical line

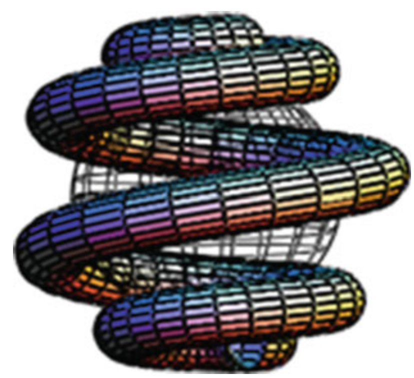
$$\boldsymbol{E}_0(u) = a\boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\sin\omega,$$

as a director line at the surface of the sphere of radius a;

$$\omega = \omega(u) = -\pi/2 + 2\arctan e^{pu},\ p = c\tan\alpha.$$

Here α is the angle of the *loxodrome* with a meridian of the sphere. The unit vector $\boldsymbol{e}_0(u)$ is the normal of the sphere, at which a directrix curve is disposed. The generatrix circle of the constant radius b is given at the local system of coordinates with the origin at the spherical directrix line:

$$X = X(v) = b\cos v,\ \ Y = Y(v) = b\sin v.$$



$a = 10;\ b = 2;\ \alpha = 0.45\pi$

Fig. 1

The circles lie at the normal planes of the spherical directrix line (Fig. 1). So, parametrical equations of the tubular loxodrome with $\theta = 0$ are:

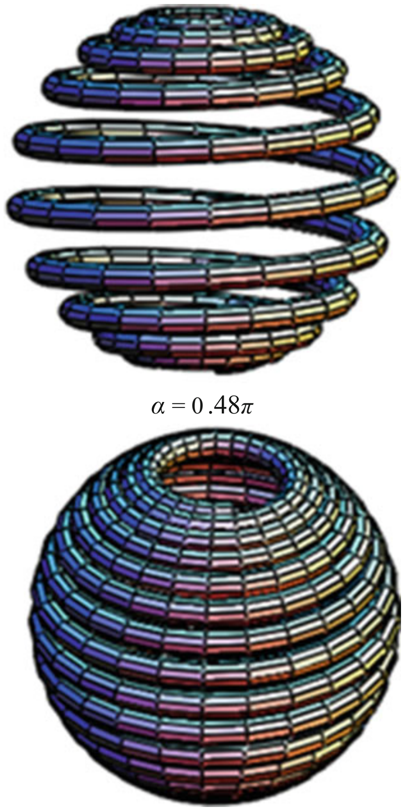
$$x = x(u,v) = [a + b\cos v)]\cos\omega\cos u$$
$$+ b\sin v(\sin\omega\cos\omega\cos u - \omega'\sin u)/s,$$
$$y = y(u,v) = [a + b\cos v]\cos\omega\sin u$$
$$+ b\sin v(\sin\omega\cos\omega\sin u + \omega'\cos u)/s,$$
$$z = z(u,v) = [a + b\cos v]\sin\omega - b\sin v\cos^2\omega/s,$$

where $s = \sqrt{\omega'^2 + \cos^2\omega}$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = s(a + b\cos v) + [(1 + \frac{\omega'^2}{s^2})\sin\omega + \frac{\omega''}{s^2}\cos\omega]b\sin v,$$
$$F = 0,\ B = b;\ L = \frac{A - as}{b}A,\ M = 0,\ N = b;$$
$$k_u = k_1 = \frac{A - as}{bA},\ k_v = k_2 = \frac{1}{b}.$$

The tubular loxodrome is given at the lines of principle curvatures. The coordinate lines v coincide with the generatrix circles (Figs. 1 and 2).



$\alpha = 0.48\pi$

$\alpha = 0.49\ \pi$

Fig. 2

■ Tubular Surface Winding the Sphere

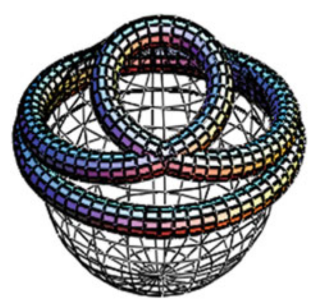
The tubular surface winding the sphere of a radius a has a *spherical line*

$$\boldsymbol{E}_0(u) = a\boldsymbol{e}_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\ \cos\omega + \boldsymbol{k}a\ \sin\omega,$$
$$\omega = \omega(u) = d + \varepsilon\ \arctan(pu),$$

as a director line; d, p and ε are constant parameters. The unit vector $\boldsymbol{e}_0(u)$ is the normal of the sphere, on which a director curve is placed. The generatrix circle of the constant radius b is given at the local system of coordinates with the origin at the spherical director line:

$$X = X(v) = b\cos v, \quad Y = Y(v) = b\sin v.$$

The circles lie at the normal planes of the spherical director line.

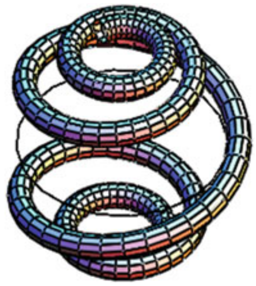


$a = 10;\ b = 1;\ \varepsilon = 1;\ p = 1;$
$d = 0.25\pi;\ -8 \le u \le 8$

Fig. 1

Parametrical equations of the surface in question with $\theta = 0$ (Figs. 1 and 2) can be written in the following form:

$$x = x(u,v) = [a + b\cos v)]\cos\omega\cos u$$
$$+ b\sin v(\sin\omega\cos\omega\cos u - \omega'\sin u)/s,$$



$a = 10$; $b = \varepsilon = 1$; $p = 1/5$;
$d = 0$; $-15 \leq u \leq 15$

Fig. 2

$$y = y(u, v) = [a + b\ \cos v] \cos\omega\ \sin u$$
$$+ b\ \sin v(\sin\omega\ \cos\omega\ \sin u + \omega'\ \cos u)/s,$$
$$z = z(u, v) = [a + b\ \cos v] \sin\omega - b\ \sin v \cos^2\omega/s,$$

where

$$s = \sqrt{\omega'^2 + \cos^2\omega}.$$

The tubular surface is given at the lines of principle curvatures. The coordinate lines v coincide with the generatrix circles. Varying the constant parameters, it is possible to design different forms of the tubular surfaces at the sphere (Figs. 1 and 2).

22 Weingarten Surfaces

The Weingarten surface is a surface, the mean curvature H of which is connected with its Gaussian curvature K by a functional relation:

$$f(H, K) = 0.$$

The set of solutions of this equation is also called *the curvature diagram* or *the W-diagram* (Hopf H. 1951) of the surface (Fig. 1). The study of Weingarten surfaces is a classical topic in differential geometry.

The Weingarten surface is called often *a W surface*. One may say also that W surface is characterized by the certain relation:

$$\varphi(R_1, R_2) = 0,$$

where R_1, R_2 are the radiuses of the principle curvatures. The lines of the principle curvatures of the Weingarten surface are defined by quadratures.

If the curvature diagram degenerates to exactly one point then the surface has two constant principal curvatures which is possible only for a piece of a plane, a sphere, or a circular cylinder. If the curvature diagram is contained in one of the coordinate axes through the origin then the *surface is developable*. If the curvature diagram is contained in the main diagonal $k_1 = k_2$ then the surface is a piece of a plane or a sphere because every point is an umbilic. The curvature diagram is contained in a straight line parallel to the diagonal $k_1 = -k_2$ if and only if the mean curvature is constant. It is contained in a standard hyperbola $k_1 = c/k_2$ if and only if the Gaussian curvature is constant.

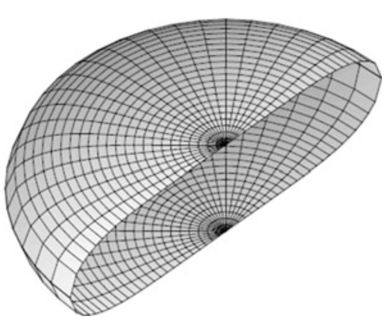


Fig. 1 A *W*-surface with $k_1 = 5k_5$ (*Hopf surface*) (The figure is taken in W. Kühnel and M. Steller 2005)

Locally there are the following five main classes of Weingarten surfaces:

(1) surfaces of revolution,
(2) tubes around curves where one principal curvature is constant,
(3) helicoidal surfaces,
(4) surfaces of constant Gaussian curvature,
(5) surfaces of constant mean curvature (*cmc* surfaces).

This list is, of course, not exhausting. It is not difficult to obtain closed smooth Weingarten surfaces of arbitrary genus by gluing together pieces of spheres, other surfaces of revolution and tubes (W. Kühnel and M. Steller 2005).

Depending on of the sign of the derivative dk_2/dk_1, where k_1 and k_2 are the principle curvatures, Weingarten surfaces were divided into three classes (Van-Brunt B. and Grant K. 1994)

In order to be the Weingarten surface, it is necessarily and sufficiently that both sheets of its evolute superimpose on surfaces of revolution and the cuspidal edges of the normals of the lines of the principle curvatures of the W surface superimpose on the meridians. For example, the surfaces of *the constant mean curvature* or *the constant Gaussian curvature* are superimposed.

In order to be the Weingarten surface, it is necessarily and sufficiently that the asymptotic lines on the sheets of its evolute will be corresponding. In this case, the W surface is a *surface of revolution* either it belongs to the more general class of *surfaces with one family of plane lines of principle curvatures*.

The product of the Gaussian curvatures of two sheets of the evolute (see also a section "Surfaces") of the Weingarten surface is inversely to the fourth degree of the distance between the corresponding points of these sheets.

If the lines of the principle curvatures on two sheets of an evolute of a surface S correspond, than the surface S is a Weingarten surface. In this case, the distance between corresponding points of the sheets is constant but the sheets themselves have the equal constant Gaussian curvatures.

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A *W* surface shows that its fourth quadratic form has a zero Gaussian curvature.

It was proved the sufficient condition that a *W* surface of the three-dimensional Euclidian space with a boundary from the umbilical points is a part of a sphere (Afwat M. 1977). This condition means that four functions exist satisfying together with the mean and the total curvatures of surface one differential equation and one inequality.

The *W* surface has been introduced by Weingarten in 1861 in connection with the problem of the finding of all surfaces that are isometric with the given surface of revolution. This problem reduces to a problem of finding of all Weingarten surfaces of the same class.

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23 Surfaces of the Constant Gaussian Curvature

Gaussian curvature of surface K is determined by a formula:

$$K = k_1 k_2 = \frac{LN - M^2}{A^2 B^2 - F^2}.$$

The Gaussian curvature K is defined fully when the first fundamental form of a surface is given and hence, K belong to the internal geometry of surface and remains invariant under bending.

It is impossible to notice this assertion directly from the determination of the Gaussian curvature $K = k_1 k_2$ because k_1 and k_2 change separately under the bending. So, the surfaces of the constant Gaussian curvature in the process of bending transform only into the surfaces of the constant Gaussian curvature with the same value of curvature K. Thus, two surfaces S and S^* of the same constant Gaussian curvature K can be bent always one into another so that a point M_0 given in advance on the surface S with the direction t_0 on it goes over to S^* in any point M_0^* given in advance with the direction t_0^* on it.

The coordinate system at the surface is called a semi-geodesic if the coordinate lines of the different families are orthogonal in pairs and one of the families consists of geodesic lines.

Let L is any smooth curve on a surface S passing through a point X. Let only one geodesic curve u forming the right angle with the curve L in the point of intersection passes through every point of the L curve. The curve L lying on the surface S is a geodesic line only when the principal normal in every point of the curve L, where its curvature is not equal to zero, coincides with the normal of the surface S at this point. The second family of the coordinate lines v will consist of the orthogonal trajectories constructed geodesic lines u. At this case, the first fundamental form of a surface given in the semi-geodesic system of the coordinates may be lead to the following form:

$$ds^2 = du^2 + B^2(u, v)dv^2.$$

Having used this parameterization, one may calculate the Gaussian curvature of the surface with the help of a formula

$$K = -B_{uu}/B,$$

where the parameter Lame B must satisfy the differential equation:

$$B_{uu} + KB = 0.$$

For surfaces of the constant Gaussian curvature, three variants are possible:

1. $\mathcal{K} = 0$, then $B = 1$ and $ds^2 = du^2 + dv^2$.
2. $K = k^2 > 0$, then $B = \cos ku$ and $ds^2 = du^2 + \cos^2 ku\, dv^2$.
3. $K = -k^2 < 0$, then $B = \cosh ku$ and $ds^2 = du^2 + \cosh^2 ku\, dv^2$.

All surfaces of the constant positive Gaussian curvature are superimposed on a sphere, all surfaces the constant negative Gaussian curvature are superimposed on a pseudosphere but surfaces of zero Gaussian curvature (see also a Subsect. "1.1.1. Torse Surfaces") are locally isometric to a plane. Now we study sufficiently small pieces of surfaces of the constant curvature.

The Gauss-Bonne theorem asserts that if α, β, γ are the internal angles of the triangular region Δ bounded by three geodesic lines on a smooth surface, then

$$\alpha + \beta + \gamma = \pi + \iint_{\Delta} K dS,$$

i.e. the second term of the right part of the equation is equal to the product of the Gaussian curvature K into the area of the triangle Δ. The sum of the angles of a triangle on a surface of positive curvature will exceed π, while the sum of the angles of a triangle on a surface of negative curvature will be less than π. On a surface of zero curvature, such as the Euclidean plane, the angles will sum to precisely π.

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
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Minding's theorem (1839) states that all surfaces with the same constant curvature K are locally isometric.

Hilbert's theorem (1901) states that there exists no complete analytic regular surface in R^3 of constant negative Gaussian curvature.

A surface on which the Gaussian curvature is everywhere positive is called *synclastic*, while a surface on which is everywhere negative is called *anticlastic*.

The non-developable ruled smooth regular surface cannot have the constant Gaussian curvature.

A plane is the only minimal surface of the constant Gaussian curvature.

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23.1 Surfaces of the Constant Positive Gaussian Curvature

For the determination of Gaussian (Gauss) curvature of a surface, we have the following formula:

$$K = k_1 k_2 = \frac{LN - M^2}{A^2B^2 - F^2}$$

but if a surface is given in an explicit form $z = z(x, y)$, one can use a formula:

$$K = \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}.$$

So, the problem of finding of an equation of a surface $z = z(x, y)$ of the constant positive Gaussian curvature ($K > 0$) is reduced to the solution of the nonlinear Monge-Ampere differential equation in the partial derivatives

$$z_{xx}z_{yy} - z_{xy}^2 = k^2(1 + z_x^2 + z_y^2)^2, \quad \text{where it is taken } k^2 = K.$$

The surfaces with constant Gauss curvature K are called also *K-surfaces. A spherical surface* is the most known surface of constant positive Gaussian curvature but sufficiently small pieces of all of the rest of the surfaces of constant positive Gaussian curvature superimpose on sphere. In other words, a surface of constant positive curvature is locally isometric to the sphere, which means that every point on the surface has an open neighborhood that is isometric to an open set on the unit sphere in with its intrinsic Riemannian metric.

The translation surfaces of constant Gaussian curvature, i.e. when $K = \text{const} \neq 0$ do not exist.

Liebmann's theorem (1900) states that the sphere is the only surface (embedded in 3-space) without boundary or singularities with constant positive Gaussian curvature.

If x is an embedding, H. Rosenberg (1993) proved the following inequality:

$$h\sqrt{K} \leq 2,$$

where by h we denote the maximum height that $x(S)$ can rise above the plane containing a boundary curve Γ in a plane P; S is a compact K-surface with a connected boundary.

Two surfaces S and $S*$ having one and the same constant Gaussian curvature K may be bent always one into another so that a point M_0 given in advance on the surface S with the direction t_0 on it goes over to S^* in any point M_0^* given in advance with the direction t_0^* on it. It means that the surface S is bent not compulsory "as whole" but it can be realized for some neighborhood of the point M_0 into some neighborhood of the point M_0^*. So all surfaces of the given constant positive curvature K have "in small" one and the same internal geometry. Hence, bending a neighborhood of one surface into another neighborhood, it is possible to map the first neighborhood into any other place of the surface with the preservation of all lengths and angles.

For the surfaces of the constant positive Gaussian curvature given in semi-geodesic coordinates u, v, the first fundamental form of a surface is written as:

$$ds^2 = du^2 + \left[dv \cos\left(\sqrt{K}u\right)\right]^2,$$

i.e., it coincides with the first fundamental form of a sphere with the radius $1/\sqrt{K}$ and $\sqrt{K}u$ is a latitude and $\sqrt{K}v$ is a longitude.

The three-dimensional quasi-Riemann space of the constant curvature is the Galilean, quasi-elliptical or quasi-hyperbolic space depending on the sign of the curvature.

N.E. Maryukova (2000) proved that a radius of curvature of the special lines ant the angle between the asymptotic lines on a surface of the constant positive Gaussian curvature in the quasi-hyperbolic space are the solutions of *the one-dimensional Klein-Gordon equation*.

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■ Spherical Surface (Sphere)

A spherical surface (*sphere*) is generated by rotation of a circle of a constant radius R about its axis. Sphere may be considered as a special case of ellipsoid of revolution (see also "Ellipsoid of Revolution" in the Chap. "2. Surfaces of Revolution") or circular torus (see also "Circular Torus").

The total area of the sphere with a radius R is

$$S = 4\pi R^2,$$

the volume of the sphere is

$$V = 4\pi R^3/3.$$

Forms of definition of a spherical surface

(1) Implicit equation:

$$x^2 + y^2 + z^2 = R^2.$$

This is an equation of a sphere with the center at the point $C(0, 0, 0)$.

(2) Implicit equation:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

This is an equation of a sphere with the center at the point $D(a, b, c)$.

(3) Parametrical equations (Figs. 1 and 2):

$$x = x(\alpha,\beta) = R\sin\alpha\sin\beta,\ y = y(\alpha,\beta) = R\sin\alpha\cos\beta,$$
$$z = z(\alpha) = R\cos\alpha$$

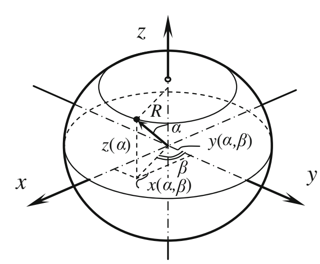

Fig. 1

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = R,\ F = 0,\ B = R\sin\alpha,\ L = R,\ M = 0,\ N = R\sin^2\alpha$$
$$k_\alpha = k_\beta = k_1 = k_2 = 1/R, \quad K = 1/R^2$$

(4) Parametrical equations (Fig. 2):

$$x = x(u,v) = 2R\sin u\cos u\cos v,$$
$$y = y(u,v) = 2R\sin u\cos u\sin v,$$
$$z = z(u) = 2R\sin^2 u$$

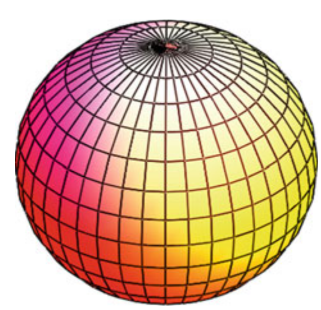

Fig. 2

Here the angles u and v shown in Fig. 3 are introduced as the curvilinear coordinates. The center of the sphere is placed at the point with the coordinates (0, 0, R).

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 2R,\ F = 0,\ B = 2R\sin u\cos u = R\sin 2u,$$
$$L = 4R,\ M = 0,\ N = R\sin^2 2u,$$
$$k_u = k_1 = k_v = k_2 = 1/R, \quad K = 1/R^2,\ H = 1/R.$$

(5) Parametrical equations:

$$x = x(\beta,\gamma) = -\sqrt{R^2-\gamma^2}\sin\beta,\ y = y(\beta,\gamma) = \sqrt{R^2-\gamma^2}\cos\beta,$$
$$z = z(\gamma) = \gamma.$$

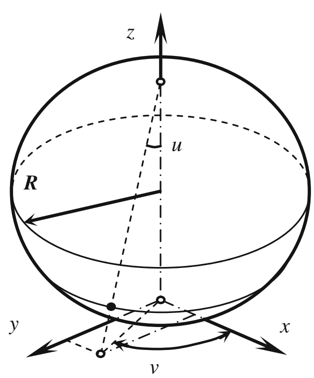

Fig. 3

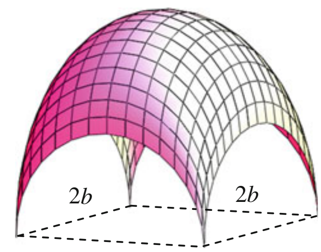

Fig. 4

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = \sqrt{R^2-\gamma^2},\ F = 0,\ B = R\Big/\sqrt{R^2-\gamma^2};$$
$$L = -(R^2-\gamma^2)/R,\ M = 0,\ N = -R/(R^2-\gamma^2);$$
$$k_\beta = k_\gamma = k_1 = k_2 = -1/R.$$

(6) Explicit equation:

$$z = \sqrt{R^2 - x^2 - y^2}.$$

With the help of this surface, it is possible to cover a rectangular plan $2a \times 2b$ (Fig. 4), but it is necessary to fulfill the condition: $a^2 + b^2 \le R^2$. Here the system of the curvilinear coordinates on the surface is non-orthogonal and non-conjugate.

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■ Rembs' Surface

The Rembs' surface is a *surface of constant positive Gaussian curvature* ($K = \text{const},\ K > 0$).

Parametrical equations of the Rembs' surface are written as

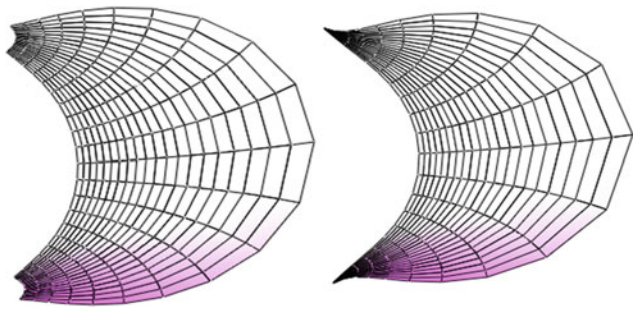
$$x = x(u,v) = a(U\cos u - U'\sin u),$$
$$y = y(u,v) = -a(U\sin u + U'\cos u),$$
$$z = z(v) = v - aV',$$

where

$$U \equiv \frac{\cosh(u\sqrt{C})}{\sqrt{C}},$$
$$V \equiv \frac{\cos(v\sqrt{C+1})}{\sqrt{C+1}},$$
$$a \equiv \frac{2V}{(C+1)(U^2-V^2)},$$
$$U' = \frac{\mathrm{d}U}{\mathrm{d}u} = \sinh(u\sqrt{C}),$$
$$V' = \frac{\mathrm{d}V}{\mathrm{d}v} = -\sin(v\sqrt{C+1}),\quad |v| \le v_0,$$
$$v_0 \equiv \frac{\pi}{2\sqrt{C+1}},\ v_C = \frac{\pi}{2C}.$$

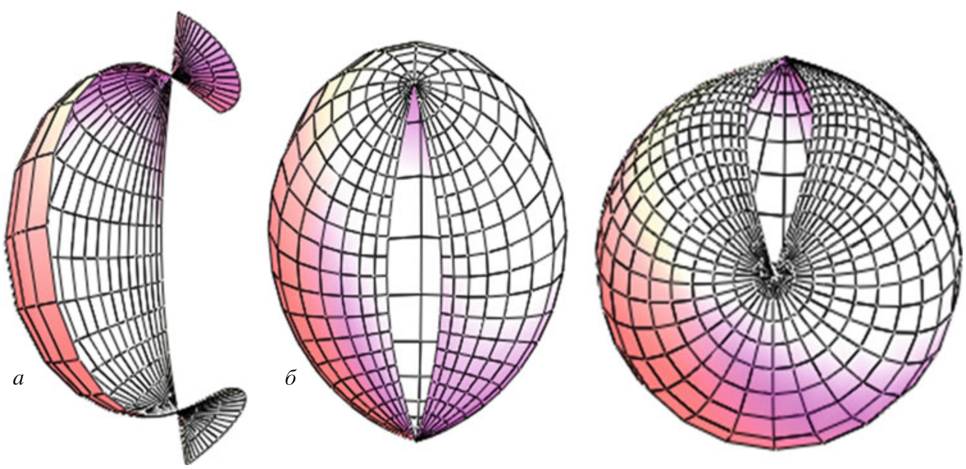
The values $v = \pm v_0$ correspond to the ends of the cleft in the surface.

Figures 1, 2 and 3 show the Rembs' surface with different geometric parameters which are shown under the corresponding surface.



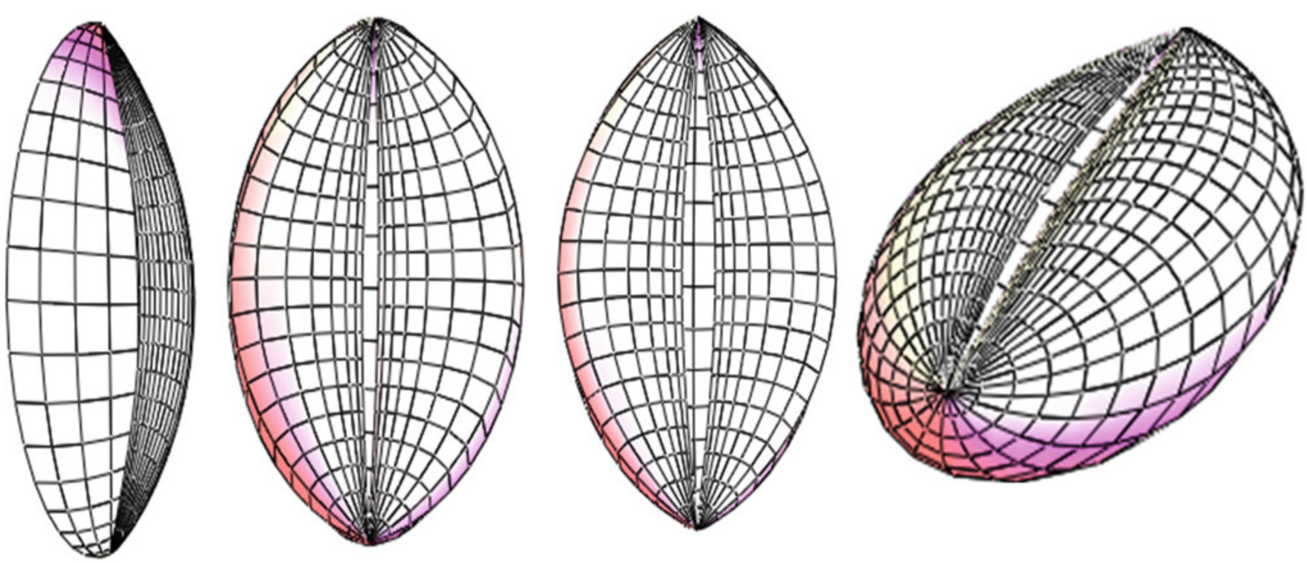
$-v_0 \le v \le v_0$; $-v_C \le v \le v_C$;

Fig. 1 $C = 1$; $0 \le u \le 1.2$



$0.5 \le u \le 1.2$; $-v_0 \le v \le v_0$; $-1.5 \le u \le 1.5$; $-v_C \le v \le v_C$;

Fig. 2 $C = 2$



$C = 5$; $-v_C \le v \le v_C$; $C = 10$; $-v_C \le v \le v_C$;

$0.5 \le u \le 1.2$; $-1.5 \le u \le 1.5$; $0 \le u \le 1.2$; $-1.2 \le u \le 1.2$

Fig. 3

Coefficients of the first fundamental form of the surface:

$$A^2 = \frac{16C(1+C)\cos^2(v\sqrt{C+1})\cosh^2(u\sqrt{C})}{[1 - C\cos(2v\sqrt{C+1}) + (C+1)\cosh(2u\sqrt{C})]^2}, \quad F = 0,$$

$$B^2 = \frac{[1 + 2C + C\cos(2v\sqrt{C+1}) + (C+1)\cosh(2u\sqrt{C})]^2}{[1 - C\cos(2v\sqrt{C+1}) + (C+1)\cosh(2u\sqrt{C})]^2}, \quad K = 1.$$

Hence an area element is

$$\begin{aligned}&\sqrt{A^2B^2 - F^2}\mathrm{d}u\mathrm{d}v\\ &= \mathrm{d}u\mathrm{d}v\,[4\sqrt{C(1+C)}\cos(v\sqrt{C+1})\cosh(u\sqrt{C})]\\ &\quad\times[1 + 2C + C\cos(2v\sqrt{C+1})\\ &\quad+ (C+1)\cosh(2u\sqrt{C})]/[1 - C\cos(2v\sqrt{C+1})\\ &\quad+ (C+1)\cosh(2u\sqrt{C})]^2.\end{aligned}$$

All formulas given at this page are taken without changing and without verification at the internet site of *Eric W. Weisstein* “Remb’s Surface”.

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■ Sievert’s Surface

The Sievert’s surface is a surface of *constant positive Gaussian curvature* (K = const, $K > 0$).

The form of definition of the Sievert’s surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = r(u, v)\cos\varphi(u); \quad y = y(u, v) = r(u, v)\sin\varphi(u);$$

$$z = z(u, v) = \frac{\ln\tan(v/2) + a(u, v)(C+1)\cos v}{\sqrt{C}},$$

where

$$\varphi(u) = -\frac{u}{\sqrt{C+1}} + \arctan(\sqrt{C+1}\cdot\tan u),$$

$$a(u, v) = \frac{2}{C + 1 - C\sin^2 v\cos^2 u},$$

$$r(u, v) = \frac{a(u, v)}{\sqrt{C}}\sqrt{(C+1)(1 + C\sin^2 u)}\sin v,$$

and $-\pi/2 < u < \pi/2$; $0 < v < \pi$. Self-intersecting surfaces of *umbrella type* (Figs. 2 and 3) can be obtained if we shall assume another limit of changing of the u parameter.

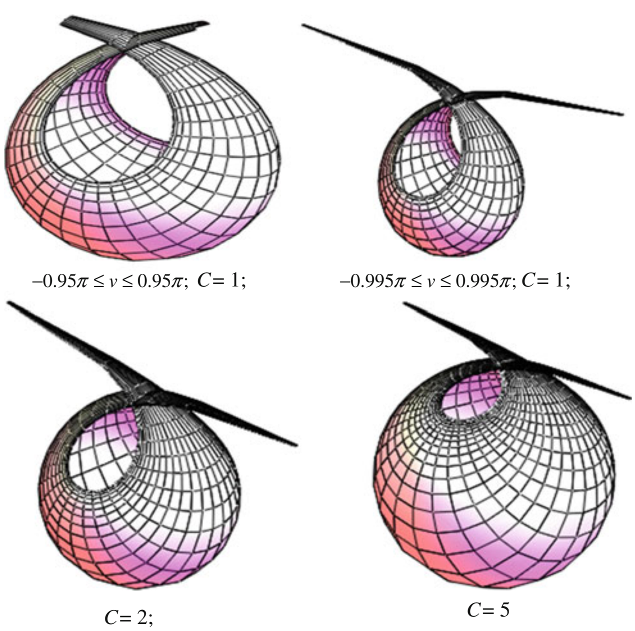
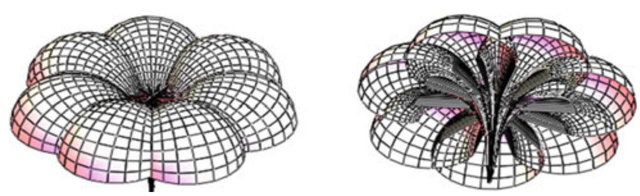
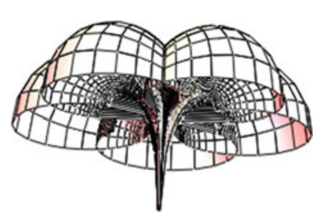


Fig. 1



$C = 2$; $-3.5\pi \le u \le 3.5\pi$; $0.005\pi \le v \le \pi/2$

Fig. 2



$C = 5$;
$-2.5\pi \le u \le 2.5\pi$

Fig. 3

In Figs. 1, 2 and 3, the Sievert's surfaces with different geometric parameters which are given under the corresponding figures are presented.

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{64a\cos^2 u \cos^2 v}{[4+3a-a\cos(2u)+2a\cos^2 u\cos^2(2v)]^2},$$

$$F = 0,$$

$$B^2 = \frac{64[(1+a)\csc v + a\cos^2 u \sin v]^2}{4a[4+3a-a\cos(2u)+2a\cos^2 u\cos^2(2v)]^2},$$

$$L = \sqrt{\frac{a}{a+1}}\frac{8a\cos^3 u\sin(3v) - 4\cos u[8+11a+3a\cos(2u)]}{[4+3a-a\cos(2u)+2a\cos^2 u\cos^2(2v)]^2},$$

$$M = 0,$$

$$N = \sqrt{\frac{a+1}{a}}\frac{[4+5a+a\cos(2u)-2a\cos^2 u\cos(2v)]\csc\left(\frac{v}{2}\right)\sec\left(\frac{v}{2}\right)}{[4+3a-a\cos(2u)+2a\cos^2 u\cos^2(2v)]^2},$$

$$K = 1; \quad H = \frac{1}{1+(a+1)\tan^2 u}.$$

All formulas, given at this page, are taken without changing at internet site of *Eric W. Weisstein*.

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23.2 Surfaces of the Constant Negative Gaussian Curvature

Gaussian curvature of surface ***K*** is determined by a formula:

$$K = k_1 k_2 = \frac{LN - M^2}{A^2B^2 - F^2},$$

so, for *a surface of the constant negative Gaussian curvature*, the condition

$$\mathcal{K} = -k^2 = \text{const}$$

must be fulfilled. All surfaces of the given constant negative Gaussian curvature are locally isometric to *the pseudosphere* with a radius $1/k$, because the first fundamental form of a surface of the constant Gaussian curvature depends only on the value which this curvature takes. The pseudosphere has constant negative Gaussian curvature except at *its singular cusp*. The pseudosphere is a model for a *limited portion* of *the hyperbolic plane*.

The *surfaces of revolution of the constant negative Gaussian curvature* obtained by F.H. Minding have not inner geometry at their regular segments coinciding with the geometry of the parts of the Lobachevski plane but they have singularities: ribs, cusps and, the main thing, in the whole, they do not represent the whole plane of Lobachevski. "Flashes", "Bobbins", and the pseudospheres are classified among *Minding surfaces of the constant negative curvature*. Minding's results on the geometry of geodesic triangles on a surface of constant curvature (1840) anticipated Beltrami's approach to the foundations of non-Euclidean geometry (1868). *Minding's theorem* (1839) states that all surfaces with the same constant curvature *K* are *locally isometric*.

In 1901, D. Hilbert has proved that at space E^3, there is no entire and regular surface, the inner geometry of which presents the geometry of *the entire Lobachevski plane* (*the entire hyperbolic plane*). The proof of Hilbert is based on the analysis of the properties of the solution of *the sine-Gordon equation*

$$\frac{\partial^2 z}{\partial x \partial y} = \sin z.$$

Hilbert's theorem (1901) states that there exists no complete analytic (class C^ω) regular surface in R^3 of constant negative Gaussian curvature.

If the coordinate lines u = const and v = const are the asymptotic lines of a surface of the constant Gaussian curvature with $K = -1$ then for this coordinate system, the first fundamental form of a surface has the following form:

$$ds^2 = du^2 + 2\cos\omega\, du\, dv + dv^2,$$

i.e., $F = \cos\omega$, where ω is the angle between the asymptotic lines. The angle ω is called also a *system angle*. In this case, the net of the coordinate lines u, v is the *Chebyshev net* and the "sine-Gordon equation" may be written as

$$\omega_{uv} = \sin\omega.$$

The solution of the sine-Gordon equation with geometrical point of view is connected with a problem of the design of the Chebyshev nets on the surfaces with the Gaussian curvature equal to –1. And what is more, the Chebyshev net answers the every solution of the sine-Gordon equation on such surface. If the Gaussian curvature of a surface is equal to a negative constant $-k^2$, then one always may obtain a transformed surface with Gaussian curvature equal to –1 using the similar transformation of the space.

The relationship between solutions of *the sinh-Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = \pm \sinh u$$

and the determination of various kinds of surfaces of constant Gaussian curvature, both positive and negative, was investigated by Paul Bracken (2005). He approves that the relationship between the solutions of this type of partial differential equations and the determination of various kinds of surfaces of constant curvature has generated many results which have applications to the areas of both pure and applied mathematics.

For a surface of the constant negative Gaussian curvature, for example with $K = -1$, it is possible to introduce such local regular coordinates p and q, that its coefficients of the first fundamental form will satisfy the relationships:

$$\frac{\partial E}{\partial q} = \frac{\partial E}{\partial p}, \text{ that is } E = A^2 = E(p), \quad G = B^2 = G(q).$$

Surfaces of constant negative Gaussian curvature, surfaces of constant mean curvature, minimal surfaces, affine spheres form a class of *Hashimoto surfaces*.

A.G. Popov has suggested the physical interpretation of the surfaces of the constant negative Gaussian curvature. The main idea concludes in the next. The surfaces of the constant negative Gaussian curvature equal to –1 are used as the phase surface describing the evolution of a physic process given by the sine-Gordon equation. The phase surface is an analog of the phase space in classic mechanics i.e., the every point of the phase surface characterizes in full the state of the investigated physical values for the corresponding values of the coordinates of the phase surface.

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■ Kuen's Surface

The Kuen's surface has the constant negative Gaussian curvature $K = -1$ (Figs. 1, 2 and 3).

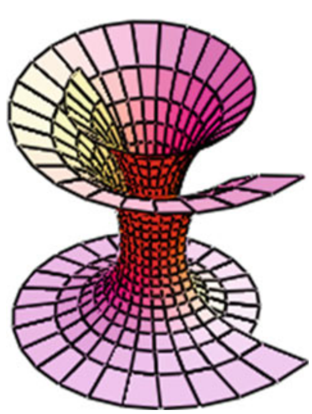
Forms of definition of the Kuen's surface

(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \frac{2}{1 + u^2 \sin^2 u} [\boldsymbol{e}(u, v) - u\boldsymbol{n}(u)] + \ln\left(\tan\frac{v}{2}\right)\boldsymbol{k};$$

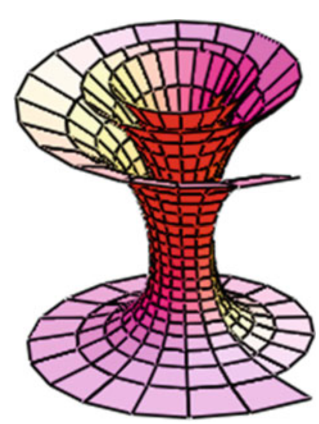
where

$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u;$$
$$\boldsymbol{n}(u) = -\boldsymbol{i}\sin u + \boldsymbol{j}\cos u;$$
$$\boldsymbol{e}(u, v) = \boldsymbol{h}(u)\sin v + \boldsymbol{k}\cos v.$$



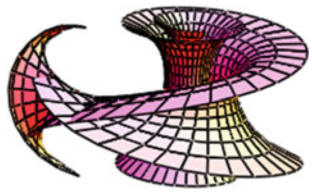
$2\pi \le u \le 5\pi;$
$-0.15\pi \le v \le 0.85\pi$

Fig. 1



$2\pi \le u \le 8\pi;$
$-0.15\pi \le v \le 0.85\pi$

Fig. 2



$0 \le u \le 5\pi; \quad -0.2\pi \le v \le 0.8\pi$

Fig. 3

(2) Parametrical equations:

$$x = x(u,v) = \frac{2(\cos u + u \sin u)\sin v}{1 + u^2 \sin^2 v},$$
$$y = y(u,v) = \frac{2(\sin u - u \cos u)\sin v}{1 + u^2 \sin^2 v},$$
$$z = z(u,v) = \ln \tan \frac{v}{2} + \frac{2\cos v}{1 + u^2 \sin^2 v},$$

where $0 \le v \le \pi$; $0 \le u \le 2\pi$.

(3) Parametrical equations:

$$x = x(u,v) = \frac{2\sqrt{1+u^2}\cos(u - \arctan u)\sin v}{1 + u^2 \sin^2 v},$$
$$y = y(u,v) = \frac{2\sqrt{1+u^2}\sin(u - \arctan u)\sin v}{1 + u^2 \sin^2 v},$$
$$z = z(u,v) = \ln \tan \frac{v}{2} + \frac{2\cos v}{1 + u^2 \sin^2 v},$$

where $0 \le v \le \pi$; $0 \le u \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = \frac{4u^2 \sin^2 v}{(1 + u^2 \sin^2 v)^2}, \quad F = 0, \quad B^2 = \frac{(1 - u^2 \sin^2 v)^2}{(1 + u^2 \sin^2 v)^2 \sin^2 v},$$
$$L = -\frac{(1 - u^2 \sin^2 v) u \sin v}{(1 + u^2 \sin^2 v)^2}, \quad M = 0, \quad N = \frac{4u(1 - u^2 \sin^2 v)}{(1 + u^2 \sin^2 v)^2 \sin v},$$
$$k_1 = \frac{(1 - u^2 \sin^2 v)}{4u \sin v}, \quad k_2 = -\frac{-4u \sin v}{(1 - u^2 \sin^2 v)},$$
$$K = -1; \quad H = \frac{\csc v}{8u} - \frac{u}{8} \sin v \left[1 + \frac{16}{(1 - u^2 \sin^2 v)}\right].$$

P.S.: Principal curvatures k_1 and k_2 of the Kuen's surface are taken at the internet site of Eric W. Weisstein "Kuen Surface".

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Additional Literature

P.S.: Additional literature is given at corresponding pages of the Sect. "23.2. Surfaces of the Constant Negative Gaussian Curvature".

24 Surfaces of the Constant Mean Curvature

"Soap bubble" may be called a physical system which is modeled by a surface of constant mean curvature (CMC) in Euclidian three-dimensional space R^3. *The mean curvature* H of surface is calculated by the formulas:

$$H = \frac{k_1 + k_2}{2} = \frac{1}{2}\frac{LB^2 - 2MF + NA^2}{A^2B^2 - F^2}.$$

The surfaces of the CMC are *isothermic*. They include *minimal surfaces* as a subset. The ruled surfaces, with the exception of a plane and a right helicoid, cannot have the CMC. If a closed surface has the CMC not equal to zero and the positive Gaussian curvature, then this is a sphere. Sphere is the most known representative of surfaces with H = const. The complete mean curvature of sphere is the least among the convex surfaces of the same area. The surface of the CMC in a sphere S^3 having one family of the lines of principal curvatures lying in quite geodesic two-spheres is *a surface of revolution*.

The torus of the CMC in R^3 was discovered by H.C. Wente in 1984. His examples decided the Hopf's problem that stayed on the agenda for long time. This problem consisted of the following: must a torus surface of the CMC in R^3 be only a round sphere? It is proved in the classic theorem of A.D. Aleksandrov that the only compact embedded surface of the CMC in R^3 is a standard sphere. H. Hopf showed that this surface must be round if the surface is a sphere in topologic sense. It is noted in the theorem of Barbosa and do Carmo that the compact stable surface of the CMC is a standard sphere. In 1985, U. Abresch classified all torus of the CMC having one family of the plane lines of the principal curvature. New examples of the twisted torus were discovered by H.C. Wente and studied by U. Abresch who showed that these new examples were derived by the solution of the finite system of the elliptic integrals.

A complete, embedded surface with CMC $H \in$ R in a Riemannian three-manifold N is called *a complete embedded H-surface*.

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25 Wave-Shaped, Waving, and Corrugated Surfaces

Wave-shaped surfaces are formed by translational-and-oscillatory motion of a rigid generatrix curve vibrating about a basic surface, a plane, or a line taken in advance. Hence, the generatrix curves of the wave-shaped surfaces are congruent to each other. Consequently, these surfaces may be included into a class of *surfaces of the congruent cross sections* as well. At literature, there are received another names of the waved-shaped surfaces, for example, *wave surfaces*.

Waving surfaces are formed by translational-and-oscillatory motion of the generatrix curves which do not only vibrate about basic surfaces, planes, or lines chosen in advance but they deform themselves remaining at one and the same class of curves (Fig. 1).

Riffled surfaces (*riffle-shaped surfaces*) have taken this name due to the English word "*riffle.*" So the riffled surfaces are surfaces with hollows or bulges disposed on them in order. The riffled surfaces are widely used in machine building (Fig. 2).

The corrugated products in the form of *corrugated surfaces* are obtained by bending of the metal or nonmetal sheets in order to give them the wave-shaped form of different profiles for increasing of their strength.

Fig. 1 The example of the waving surface: Sage Gateshead, UK, 2004 (N. Foster and Partners)

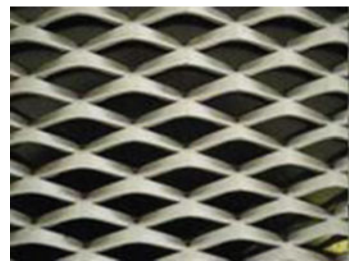

Fig. 2 The riffled surface

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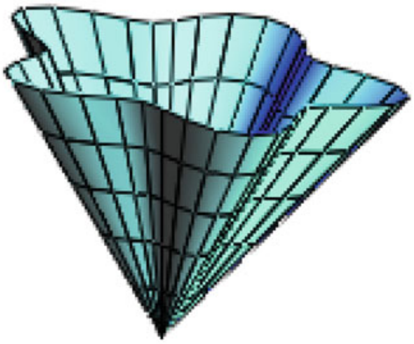
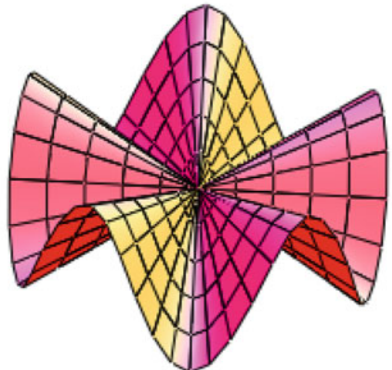
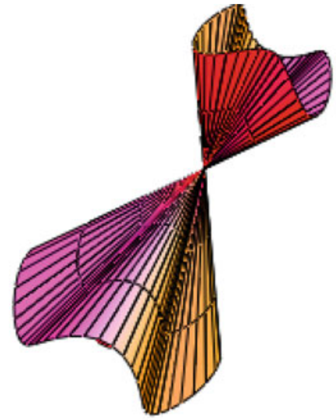
Additional Literature

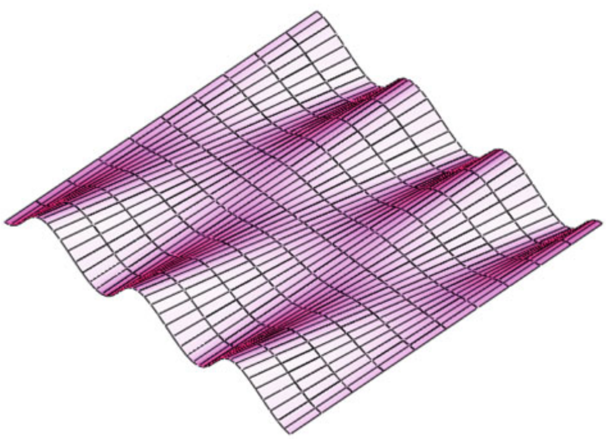
P.S.: Additional literature is given at the corresponding pages of the Chap. "25. Wave-Shaped, Waving, and Corrugated Surfaces".

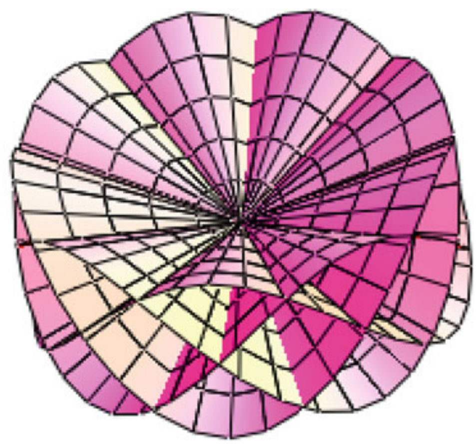
■ **Wave-Shaped, Waving, and Corrugated Surfaces Presented in the Encyclopedia**

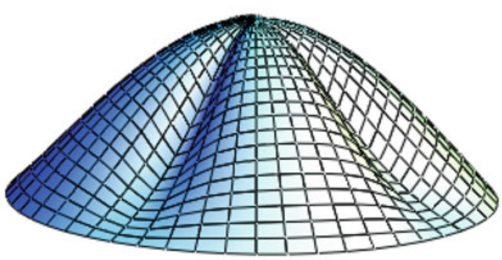

Right conical surface with a plane director curve in the form of the circular sinusoid

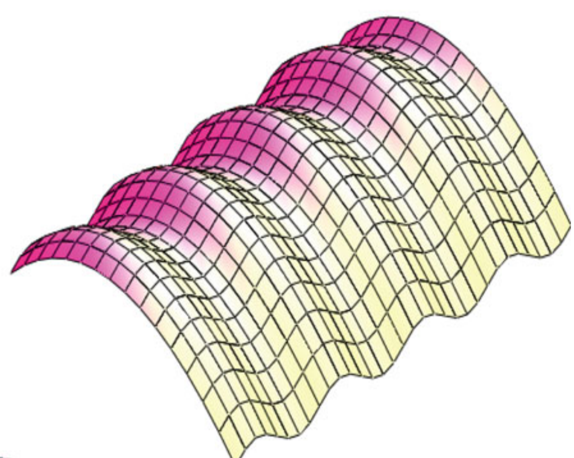

The waving conical surface in the lines of principle curvature with the inner vertex

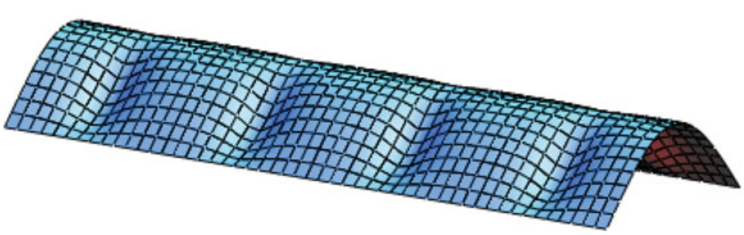

The Right conical sinusoidal wave-shaped surface

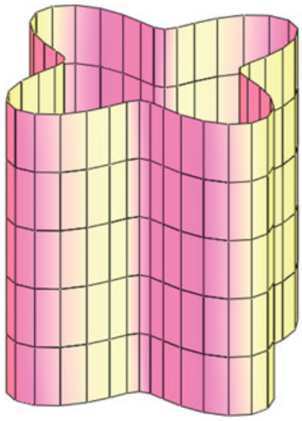

The right sinusoidal conoid


The honeycomb conical surface

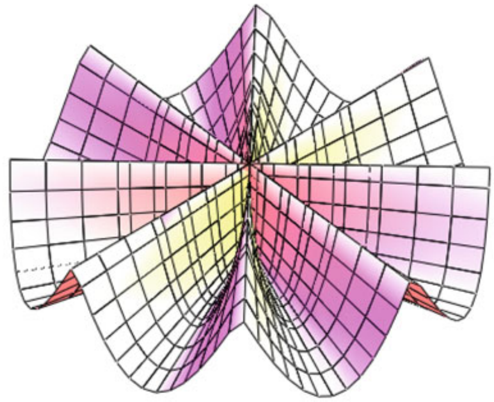

The waving surface with the pseudo Agnesi curl on the circular plan


The surface of translation of the sinusoid along the parabola

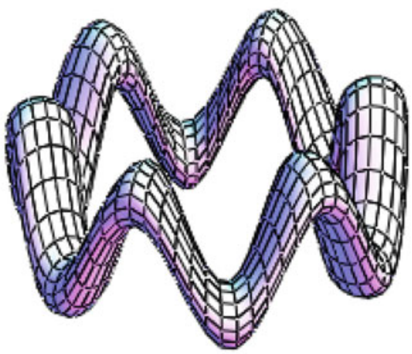

The waving surface with the pseudo Agnesi curl of cylindrical type



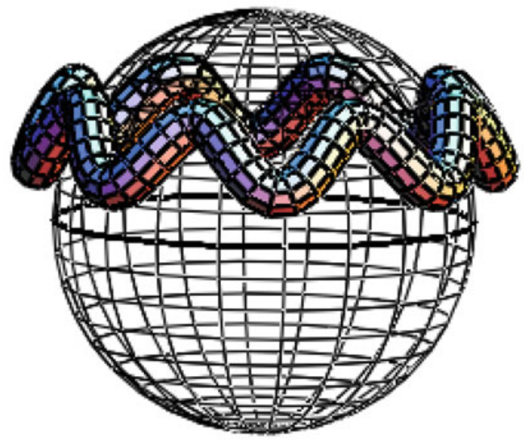
The right wave cylindrical surface



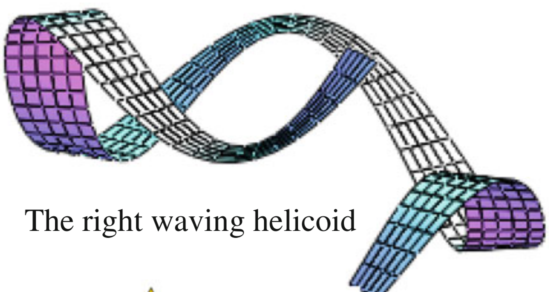
The Skidan's ruled surface



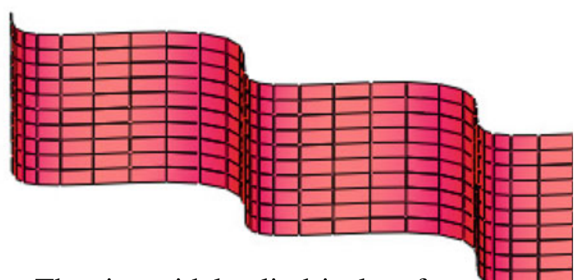
The wave-shaped torus



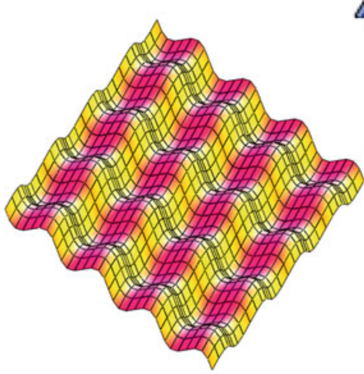
The wave-shaped torus on the sphere



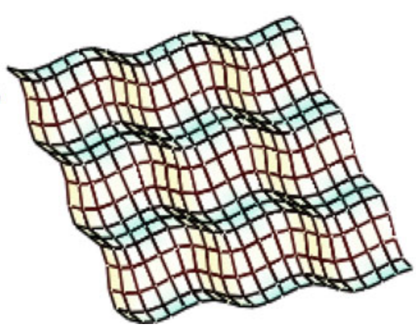
The right waving helicoid



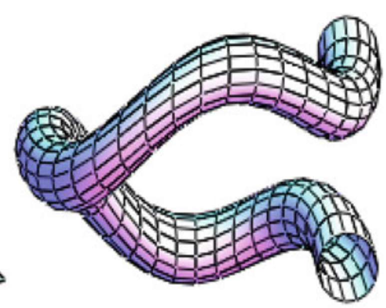
The sinusoidal cylindrical surface



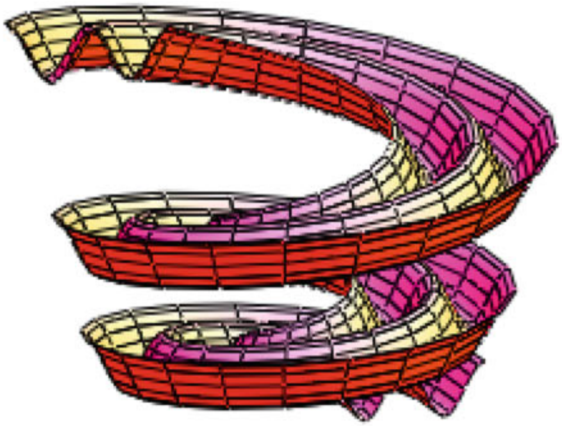
The surface of translation of the sinusoid along the sinusoid



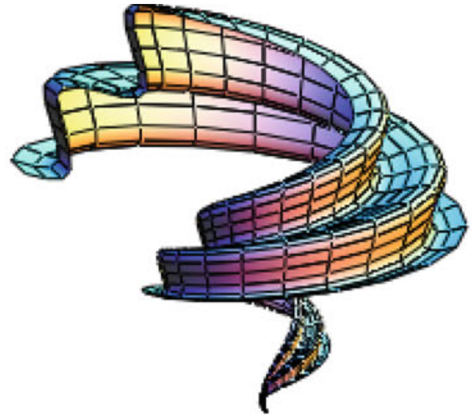
The carved sinusoidal surface



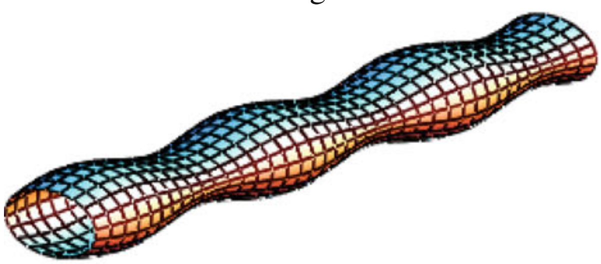
Cyclic surface with the circles in the planes of pencil and with the waving line of centers on the cylinder



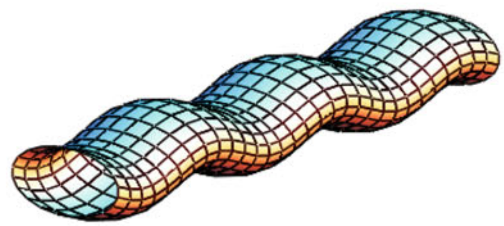
The spiral surface with the sinusoidal generatrix



The spiral-shaped surface with the generatrix sinusoids and with the directrix line of constant pitch on a circular cone

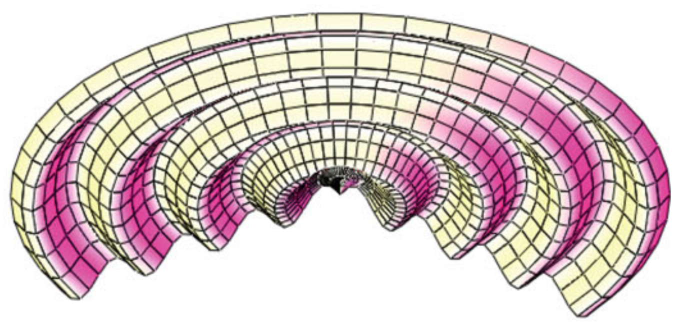


The carved surface with the directrix ellipse and the generatrix sinusoid

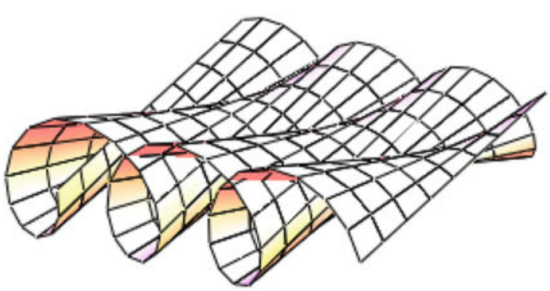


The carved surface with the directrix sinusoid and the generatrix ellipse

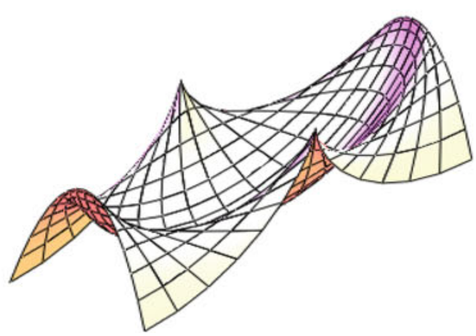
The carved surface with the directrix sinusoid and the generatrix cycloid



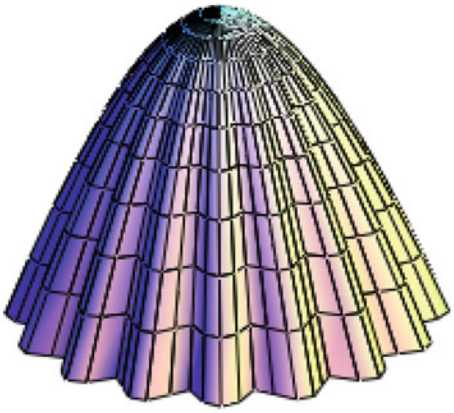
Surface of revolution of the general sinusoid (the riffled surface)



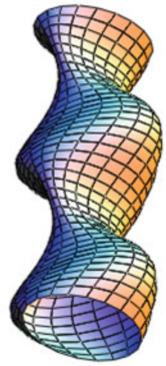
The ruled rotational surface with the axoids “plane – cylinder”



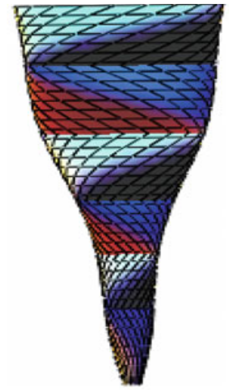
The parabolic rotational surface with the axoids “plane – cylinder”



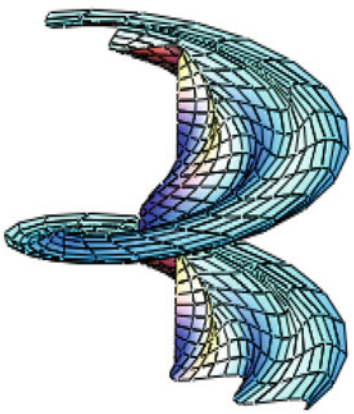
The corrugated paraboloid of revolution (with external crimps)



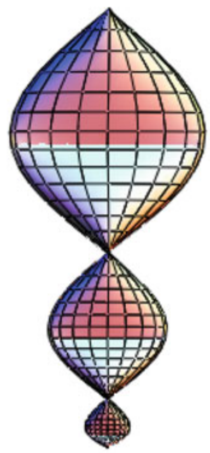
Waving chain with the elliptical cross sections limited by the elliptical cylinder



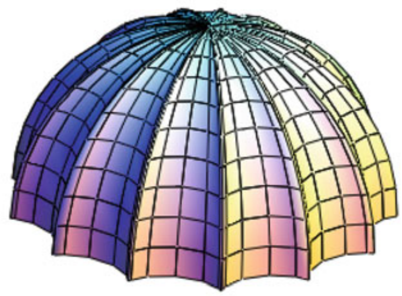
Waving chain with the elliptical cross sections limited by the elliptical cone



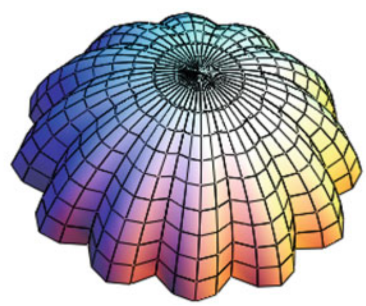
The sinusoidal helicoid



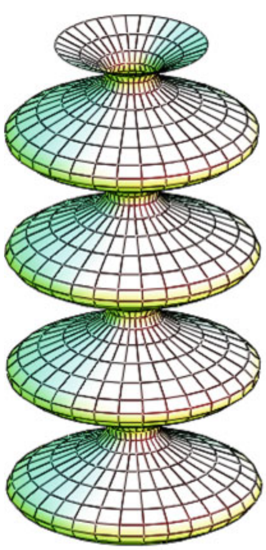
Waving chain with the elliptical cross sections limited by the elliptical cone



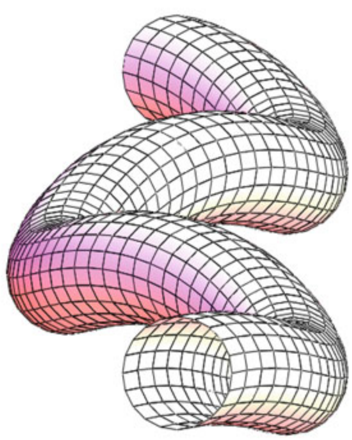
The sphere with cycloidal crimps



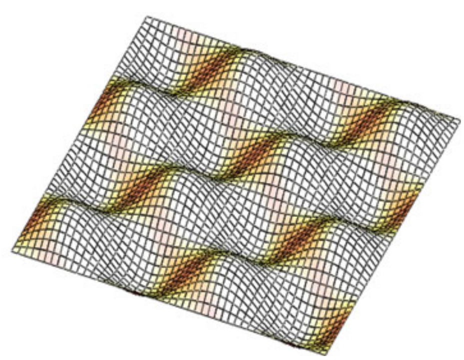
The corrugated sphere



The corrugated surface of revolution of the general sinusoid



The right circular surface on the cylinder



The waving sinusoidal velaroid

■ Sinusoidal Helicoid

A sinusoidal helicoid is a *twisted surface with the congruent sinusoids at the parallel planes*. The surface in question is formed by the rotation of a sinusoid

$$X = v, \quad Y = Y(v) = b \sin mv,$$

placed at a plane that is perpendicular to the axis of rotation. The origin of the mobile system of coordinates oXY is disposed at the stationary coordinate axis Ox at a distance of a from the rotation axis Oz. The mobile coordinate axis oX all the time intersects the Oz axis of rotation.

The form of definition of the surface of the sinusoidal helicoid

(1) Parametrical equations:

$$\begin{aligned} x &= x(u, v) = (a + X) \cos u - Y \sin u, \\ y &= y(u, v) = (a + X) \sin u + Y \cos u \\ z &= z(u) = ut \end{aligned}$$

Using this form of definition, one may take any plane line including a sine curve $X = v$, $Y = Y(v) = b \sin mv$ (Figs. 1 and 2) as a directrix curve.

The surfaces represented in Figs. 1 and 2 have

$$\begin{aligned} &b = 1\,\text{m}; \quad m = 2; \quad t = 1\,\text{m}; \\ &0 \le u \le 3\pi; \quad 0 \le v \le 2\pi. \end{aligned}$$

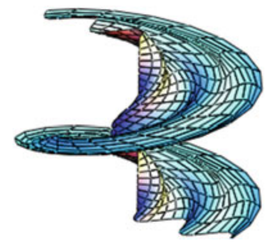


Fig. 1

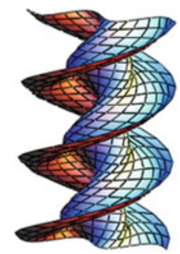


Fig. 2

The coefficients of the fundamental forms of a surface may be calculated by the general formulas given at the page "Twisted surface with congruent ellipses in parallel planes" of the Chap. "5. Surfaces of Congruent Sections." The sinusoidal helicoid may be included into a class of *surfaces of congruent sections*.

■ Waving Sinusoidal Velaroid

A surface of translation on a plane rectangular plane with a generatrix curve of the changing curvature is called *a velaroidal surface* (see also a Sect. "3.4. Velaroidal Surfaces"). The surface is bounded by four mutually orthogonal contour straight lines lying on one plane. The parametric form of definition of *a sinusoidal velaroid* gives an opportunity to consider several velaroids joined along the corresponding edges (Fig. 1).

The surface designed in such way may be related to a group of *waving surfaces*. In German, this surface is called "Die Wellen."

Forms of definition of the surface

(1) Parametrical equations:

$$\begin{aligned} &x = x(u) = u, \; y = y(v) = v, \\ &z = z(u, v) = a \cos(bu) \cos(cv), \end{aligned}$$

$0 \le u \le 2\pi$, $0 \le v \le 2\pi$; a, b and c are any constants (Figs. 2 and 3).

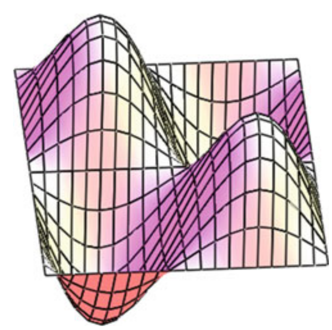


Fig. 1

(2) Explicit equation:

$$z = f \sin \frac{\pi x}{a} \sin \frac{\pi y}{b},$$

where a, b are the dimensions of little plane contour in plane (Fig. 4). The maximum rise f of the surface about the plane $z = 0$ is in the point with coordinates

$$x = a/2; \quad y = b/2.$$

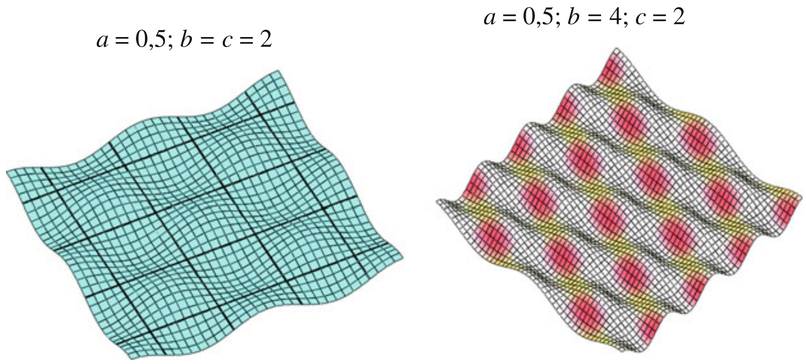


Fig. 2

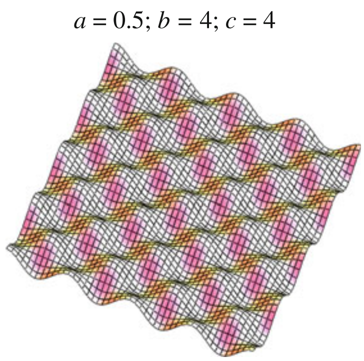


Fig. 3

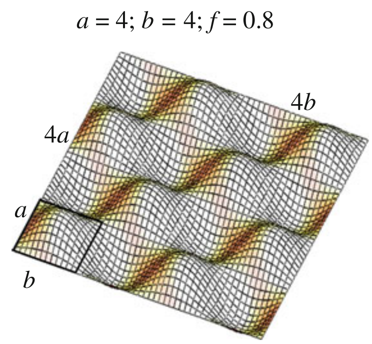


Fig. 4

■ Right Conical Sinusoidal Wave-Shaped Surface

A right conical sinusoidal wave-shaped surface is formed by the motion of a straight line lying at the planes of a pencil and executing oscillatory motions around a circular cone with the slope angle θ_o between the generatrix line and the axis. When $u = 0$, the generatrix straight line is inclined to the axis of the cone at the θ_o angle; u is the angle determining the disposition of the plane of the pencil (Fig. 1); $0 \le u \le 2\pi$.

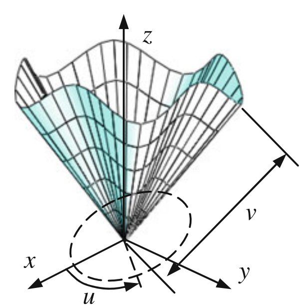


Fig. 1

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = v \sin\theta \cos u,$$
$$y = y(u, v) = v \sin\theta \sin u,$$
$$z = z(u, v) = v \cos\theta,$$

where

$$\theta = \theta_o S(u);\;\; S(u) = 1 + \mu \sin(mu);$$

$\mu\theta_0$ is the amplitude of the oscillation of the slope angle of the generatrix straight line of the wave-shaped cone, m is a number of the waves of the oscillation.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned}
&A^2 = v^2\left[\cos^2\theta + \theta_o m^2\mu^2\cos^2(mu)\right],\\
&F = 0,\quad B = 1;\\
&L = \frac{v}{\sigma}\left[\cos\theta\sin\theta - \theta_o m^2\mu\sin(mu)\right.\\
&\qquad \left.+2\theta_o m\mu\cos(mu)\cos\theta\right],\\
&M = N = 0;\\
&\sigma = \cos^2\theta + \theta_o^2 m^2\mu^2\cos^2(mu),\\
&k_1 = \frac{1}{v\sigma^3}\left[\cos\theta\sin\theta - \theta_o m^2\mu\sin(mu)\right.\\
&\qquad\qquad \left.+2\theta_o m\mu\cos(mu)\cos\theta\right],\\
&k_2 = 0.
\end{aligned}$$

Figure 1 shows the right conical wave-shaped surface with $m = 5$; in Fig. 2, the surface has $m = 4$; in Fig. 3, $m = 6$.

(2) Parametrical equations (Fig. 4):

$$\begin{aligned}
x &= x(u,v) = v\cos\theta\cos u,\\
y &= y(u,v) = v\cos\theta\sin u,\\
z &= z(u,v) = v\sin\theta,
\end{aligned}$$

where

$$\theta = \theta_1 S(u);\ \ \theta_1 = \pi/2 - \theta_0;\ \ S(u) = 1 + \mu\sin(mu);$$

μ is a coefficient of the amplitude of oscillation of the slope angle of the generatrix straight line of the right wave-shaped cone; m is a number of the waves of oscillation.

In Fig. 4, there is shown the conical wave-shaped surface with $m = 2$, $\theta_o = 0$, i.e., when $\theta_1 = \pi/2;\ \ \mu = 0.35$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned}
&A^2 = v^2\left[\sin^2\theta + \theta_1 m^2\mu^2\cos^2(mu)\right];\\
&F = 0,\quad B = 1;\\
&L = \frac{v}{\sigma}\left[\cos\theta\sin\theta - \theta_1 m^2\mu\sin(mu)\right.\\
&\qquad \left.+\,2\theta_1 m\mu\cos(mu)\sin\theta\right],\\
&M = 0,\quad N = 0;\\
&k_1 = \frac{1}{v\sigma^3}\left[\cos\theta\sin\theta - \theta_1 m^2\mu\sin(mu)\right.\\
&\qquad \left.+\,2\theta_1 m\mu\cos(mu)\sin\theta\right],\\
&k_2 = 0,\\
&\sigma = \sin^2\theta + \theta_1^2 m^2\mu^2\cos^2(mu).
\end{aligned}$$

The surface is given in orthogonal conjugate curvilinear system of coordinates u, v. The cross sections by the plane passing through the Oz axis of the cone (u = const) coincide with generatrix straight lines of the surface. At the cross sections by the planes that are perpendicular to the axis of the cone, the sinusoidal circular plane curves lie and they do not coincide with the coordinate curves v = const.

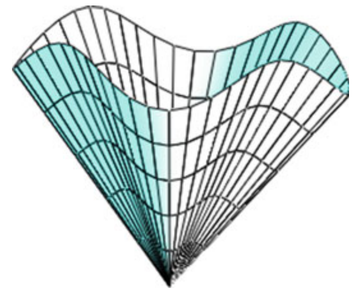


Fig. 2

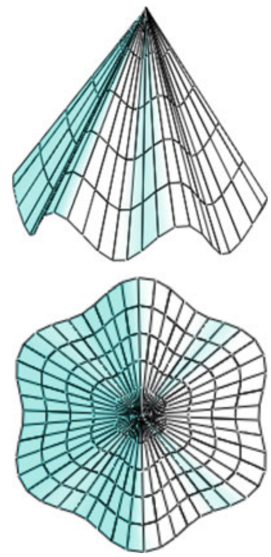


Fig. 3

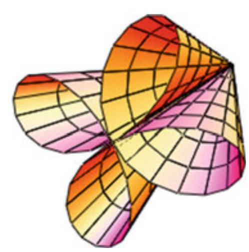


Fig. 4

Reference

Ivanov VN. Geometry and design of shells on the base of surfaces with the system of coordinates lines in the planes of pencil. Prostranstv. Konstruktzii Zdaniy i Sooruzheniy: Sb. Nauchn. Rabot. MOO "Prostranstv. Konstruktzii". Moscow: "Devyatka Print", 2004; Iss. 9, p. 26-35 (13 refs.).

■ Waving Conical Surface in Lines of Principle Curvatures with Inner Vertex

A waving conical surface in lines of principle curvatures with inner vertex is formed by the motion of a straight line passing through a fixed point (the vertex of the conical surface) and executing the oscillatory motions about the plane in which the vertex of the conical surface is placed. The vertex of the conical surface lies in the point with coordinates $x = y = z = 0$.

The form of definition of the waving conical surface in lines of principle curvatures with inner vertex

(1) Parametrical equations (Fig. 1):

$$x = x(u,v) = v\sin\theta\cos u,$$
$$y = y(u,v) = v\sin\theta\sin u,$$
$$z = z(u,v) = v\cos\theta,$$

where

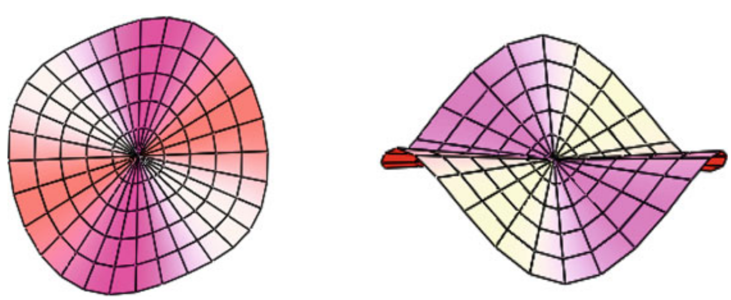
$$\theta = (\pi/2)S(u); \quad S(u) = 1 + \mu\sin(mu);$$

μ is a coefficient of the amplitude of oscillations of the slope angle of the generatrix straight line of the waving cone, m is a number of the waves of oscillations (the whole numbers). When $u = 0$, the generating straight line lies at the plane with the vertex, u is an angle determining the disposition of the plane of a pencil passing through the axis Oz and containing the generatrix line (Figs. 1, 2, 3 and 4); $0 \le u \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

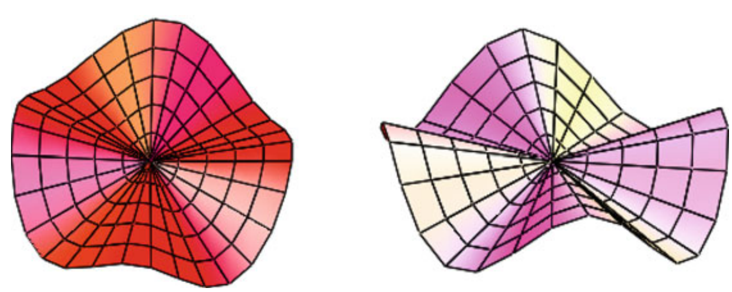
$$A^2 = v^2\left[\cos^2\theta + (\pi/2)m^2\mu^2\cos^2(mu)\right];$$
$$F = 0, \quad B = 1;$$
$$L = \frac{v}{\sigma}\left[\cos\theta\sin\theta - \frac{\pi}{2}m^2\mu\sin(mu) + \pi m\mu\cos(mu)\cos\theta\right],$$
$$M = N = 0,$$
$$\sigma = \cos^2\theta + \left(\frac{\pi}{2}\right)^2 m^2\mu^2\cos^2(mu),$$
$$k_1 = \frac{1}{v\sigma^3}\left[\cos\theta\sin\theta - \frac{\pi}{2}m^2\mu\sin(mu) + \pi m\mu\cos(mu)\cos\theta\right],$$
$$k_2 = 0.$$

In Fig. 1, the conical waving surface in the lines of principle curvatures is shown, when $m = 3$; $\mu = 0.08$; Fig. 2 shows the surface with $m = 4$; $\mu = 0.08$; in Fig. 3, the surface has $m = 6$; $\mu = 0.2$; Fig. 4 shows the surface when $m = 7$; $\mu = 0.05$. With even value of the parameter m at every plane of the pencil, two generatrix lines of the conical surface are placed (Fig. 2 and 3), but having the odd value of m, we shall have the only one straight generatrix in every plane of the pencil (Figs. 1 and 4).



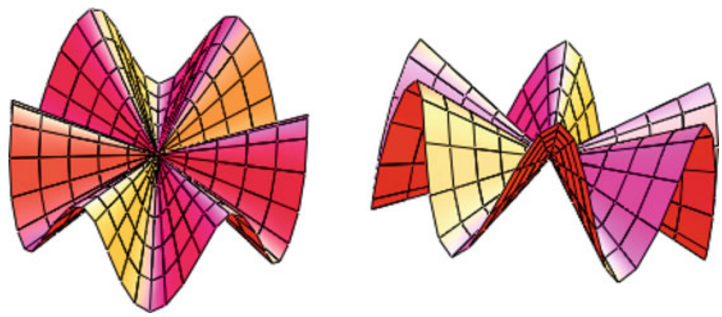
$\mu = 0.08$; $m = 3$

Fig. 1



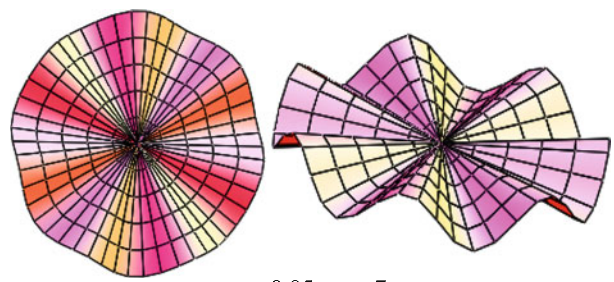
$\mu = 0.08$; $m = 4$

Fig. 2



$\mu = 0.2$; $m = 6$

Fig. 3



$\mu = 0.05$; $m = 7$

Fig. 4

■ Honeycomb Conical Surface

A honeycomb conical surface in the lines of principle curvatures with an inner vertex is formed by a straight line passing through the fixed point (the vertex of the conical surface) and executing the oscillatory motions about the plane at which the vertex of the conical surface lies. The vertex of the conical surface is placed at the point with coordinates $x = y = z = 0$.

Forms of definition of the honeycomb conical surface

(1) Parametrical form of the definiton (Figs. 1, 2 and 3):

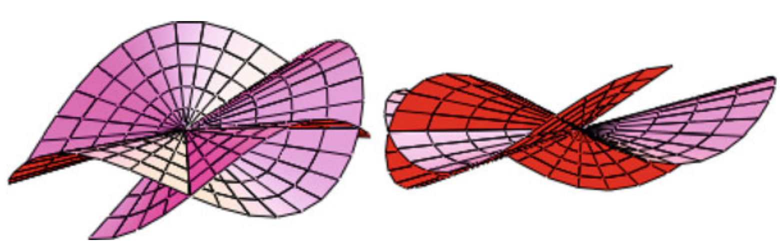
$$x = x(u, v) = v \sin\theta \cos u,$$
$$y = y(u, v) = v \sin\theta \sin u.$$
$$z = z(v) = v \cos\theta,$$

where

$$\theta = (\pi/2)S(u); \quad S(u) = 1 + \mu \sin(mu);$$

μ is a coefficient of the amplitude of oscillation of the slope angle of the generatrix straight line of the waved-shaped cone; m is a number of the waves of the oscillations. The parameter m may be by any nonintegral number. Assume that a parameter m is a whole number, then we shall obtain *a waving conical surface in lines of principle curvatures with inner vertex*. If $u = 0$, then the generatrix straight line lies at the plane with the vertex; u is an angle determining the disposition of the plane of the pencil passing through the axis Oz and containing the generatrix straight line (Figs. 1, 2, 3 and 4); $0 \le u \le \infty$.

The first honeycombs become to form when $u > 2\pi$ (Fig. 1).



$\mu = 0.08$; $m = 2.5$; $0 \le u \le 3\pi$

Fig. 1

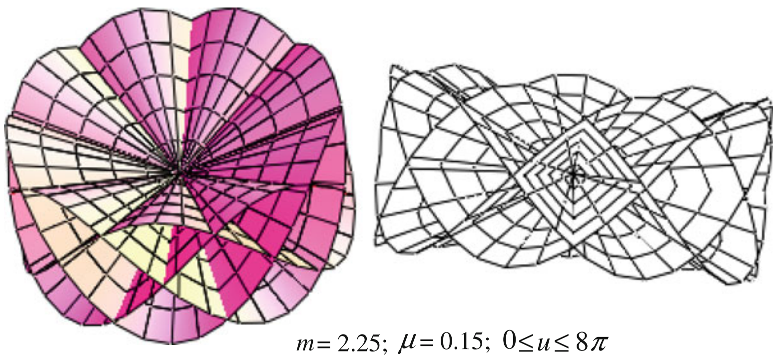


Fig. 2

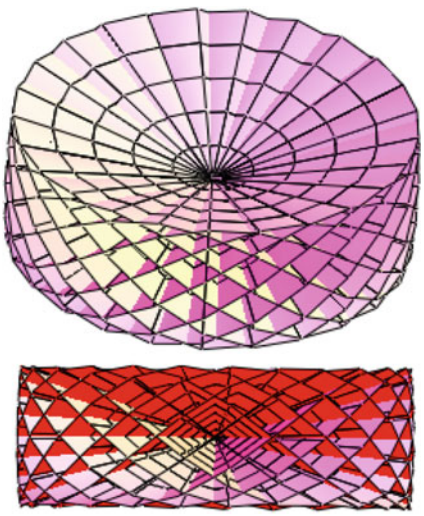


Fig. 3

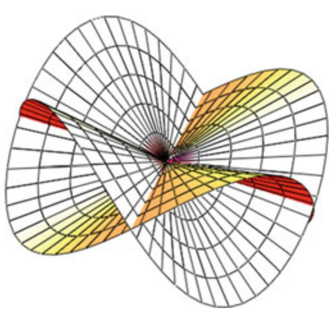


Fig. 4

In Fig. 1, the honeycomb conical surface at lines of principle curvatures is given when $m = 2.5$; $\mu = 0.08$; $0 \le u \le 3\pi$; Fig. 2 shows the surface with $m = 2.25$; $\mu = 0.15$; $0 \le u \le 8\pi$.

In Fig. 3, the surface has $m = 2.1$; $\mu = 0.12$; $0 \le u \le 20\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned}
A^2 &= v^2\left[\cos^2\theta + (\pi/2)m^2\mu^2\cos^2(mu)\right],\\
F &= 0,\quad B = 1;\\
L &= \frac{v}{\sigma}\left[\cos\theta\sin\theta - (\pi/2)m^2\mu\sin(mu)\right.\\
&\quad \left. + \pi m\mu\cos(mu)\cos\theta\right],\\
M &= N = 0;\\
k_1 &= \frac{1}{v\sigma^3}\left[\cos\theta\sin\theta - (\pi/2)m^2\mu\sin(mu)\right.\\
&\quad \left. + \pi m\mu\cos(mu)\cos\theta\right],\\
k_2 &= 0,\\
\sigma &= \cos^2\theta + (\pi/2)^2 m^2\mu^2\cos^2(mu).
\end{aligned}$$

(2) Parametrical equations:

$$\begin{aligned}
x &= x(u,v) = ku\cos nv,\\
y &= y(u,v) = ku\cos(nv),\\
z &= z(u,v) = u\sin v,
\end{aligned}$$

where k = const, n are the whole numbers (Fig. 4).

The lines u = const are not the lines of the principle curvatures $(F \neq 0)$.

■ Right Wave-Shaped Cylindrical Surface

A right wave-shaped cylindrical surface is generated by the straight generatrix line crossing the *directrix closed waving line* designed on the basis of the base circle. The straight directrix lines remain parallel to the axial direction of the cylindrical surface and form the right angles with the base of the cylinder (Figs. 1, 2, 3, 4, 5, 6 and 7).

The closed waving line given in polar coordinates r and u in the following form:

$$r = a + b\sin(nu)$$

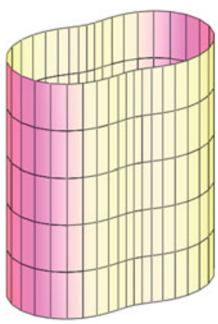


Fig. 1

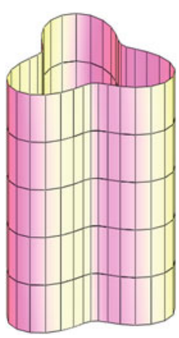


Fig. 2

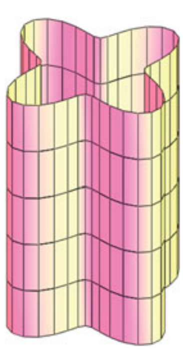


Fig. 3

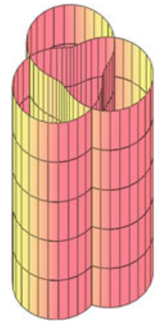


Fig. 4

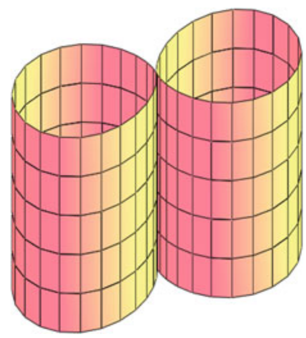


Fig. 5

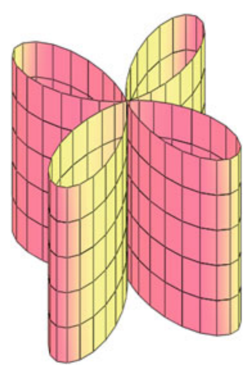


Fig. 6

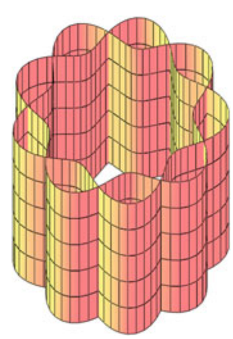


Fig. 7

has a circle of a radius a as a base circle. The amplitude of oscillations of the waving line about the base circle is denoted by b; n is a number of the whole waves of the sinusoid placed at the arc of the length of $2\pi a$; u is an angle taken from the axis Ox in the direction of the axis Oy.

Forms of definition of the surface

(1) Parametrical equations (Figs. 1, 2, 3, 4, 5, 6 and 7):

$$x = x(u) = [a + b\sin(nu)]\cos u,$$
$$y = y(u) = [a + b\sin(nu)]\sin u,$$
$$z = z.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = b^2n^2\cos^2(nu) + [a + b\sin(nu)]^2,$$
$$F = 0, \quad B = 1;$$
$$L = -\frac{A^2 + bn^2[b + a\sin(nu)]}{A}, \quad M = N = 0;$$
$$k_u = -\frac{A^2 + bn^2[b + a\sin(nu)]}{A^3},$$
$$k_z = 0; \quad K = 0.$$

At the cross sections of the surface by the planes z = const, the directrix closed waving lines lie. The curvilinear coordinate lines u, z are orthogonal and conjugate.

Figures 1, 2 and 3 show the cylindrical surfaces with $n = 2$, $n = 3$, and $n = 4$ accordingly. This surfaces have $b < a$.

The cylindrical surfaces shown in Figs. 5 and 6 have $a = b$, but $n = 2$ and $n = 4$ correspondingly.

In Figs. 1, 2, 3, 5 and 6, the surfaces contain the whole number of the waves at the circular direction and $0 \le u \le 2\pi$. It is possible to design the cylindrical surfaces with nonintegral numbers of the waves at the circular direction. In this case, the cylindrical surface will have the lines of self-intersection. For example, at Fig. 7, there is shown the right wave-shaped cylindrical surface with $b < a$ and $n = 4.5$; $0 \le u \le 4\pi$; and at Fig. 4, the surface has $n = 1.5$; $0 \le u \le 4\pi$.

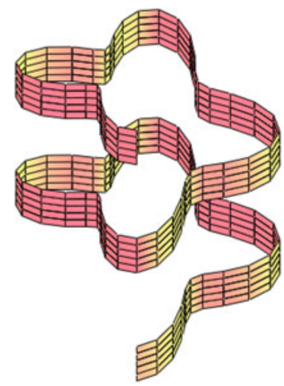


Fig. 8

The right wave-shaped cylindrical surface may be related both to a class of *wave-shaped surfaces* and to a class of *cylindrical surfaces*.

(2) Parametrical equations of a strip on the surface (Fig. 8):

$$x = x(u) - [a + b\sin(nu)]\cos u,$$
$$y = y(u) - [a + b\sin(nu)]\sin u,$$
$$z = z(u, v) = cu + v.$$

(3) Implicit form of the definition:

$$x^2 + y^2 = \left[a + b\sin\left(n\arctan\frac{y}{x})\right)\right]^2.$$

The strip is created by the motion of the straight line of the given length along the wave director curve and by its translation along the axis of the cylindrical base surface at the same time.

■ Skidan's Ruled Surface

I.A. Skidan issupposed to use the special parameterization for definition of surfaces:

$$x = f(u, v, t); \quad y = \varphi(u, v, t); \quad z = \psi(u, v, t),$$

where $u = u(t)$; $v = v(t)$; $v = v(u)$. In general case, *the ruled Skidan's surface* is a surface of *negative Gaussian curvature* and may be added to *a class of wave-shaped surfaces* or to *a class of surfaces of umbrella type*. For definition of the surfaces, they use *the normal conical coordinates*.

The form of definition of the Skidan's ruled surface

(1) Parametrical equations of the surface in normal conical coordinates:

$$x = x(u, t) = (u\sin\alpha + v\cos\alpha)\cos t,$$
$$y = y(u, t) = (u\sin\alpha + v\cos\alpha)\sin t,$$
$$z = z(u, t) = u\cos\alpha - v\sin\alpha,$$

where $v = h|\cos nt|$; α is the angle of the straight generatrix lines with the axis Oz; n is a number of waves of the surface, h = const.

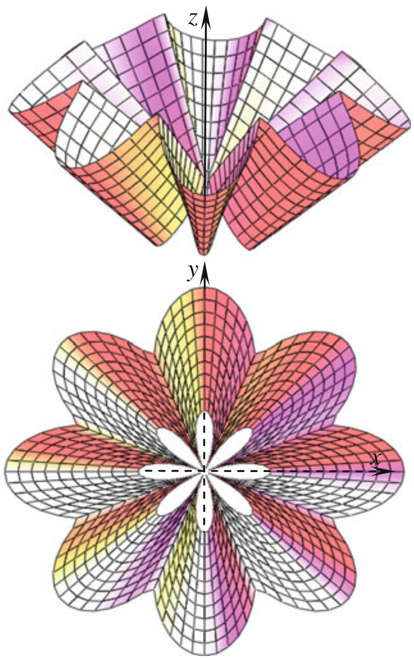


Fig. 1

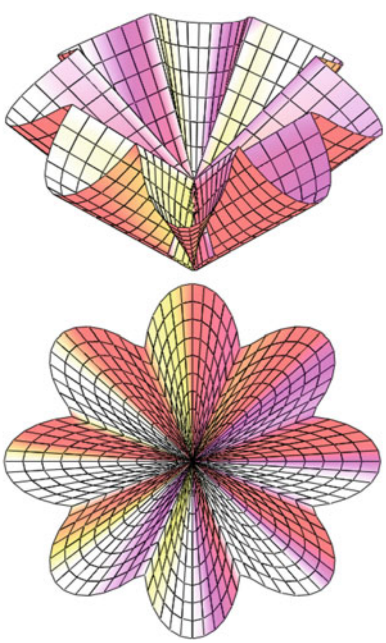


Fig. 2

In Figs. 1, 2, 3 and 4, the Skidan's ruled surfaces are given when $h = 2$ m; $n = 4$; $0 \le t \le 2\pi$; but in Fig. 1, $\alpha = \pi/4$; $0 \le u \le 4$ m; in Fig. 2, we have $\alpha = \pi/4$; $-2\text{m} \le u \le 4$ m; in Fig. 3, $\alpha = \pi/2$; $0 \le u \le 6$ m; in Fig. 4, there is $\alpha = 0$; $0 \le u \le 6$m. When $\alpha = 0$, the Skidan's ruled surface degenerates into a cylindrical surface (Fig. 4).

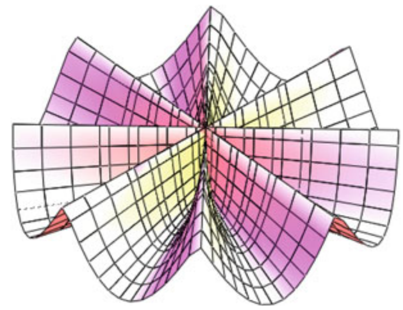


Fig. 3

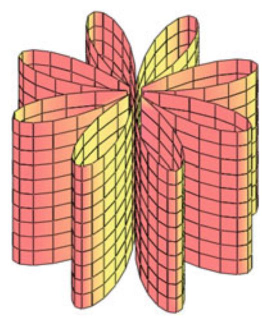


Fig. 4

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1, \quad F = 0,$$
$$B^2 = h^2 n^2 \sin^2 nt + (u \sin\alpha + h|\cos nt| \cos\alpha)^2;$$
$$L = 0, \quad M = \frac{hn \sin\alpha |\sin nt|}{B},$$
$$N = \frac{1}{B}\Big[2n^2h^2\sin^2 nt \cos\alpha - hn^2|\cos nt|(u\sin\alpha + h|\cos nt|\cos\alpha) + (u\sin\alpha + h|\cos nt|\cos\alpha)^2\cos\alpha\Big];$$
$$K = -\frac{n^2h^2\sin^2\alpha \sin^2 nt}{B^4} < 0.$$

The coordinate lines u coincide with the straight generatrix lines of the surface. Hence, the surfaces are given in orthogonal nonconjugate system of the curvilinear coordinate u, v.

Additional Literature

Skidan IA. General analytical theory of applied formation. The 10th International Conference on Geometry and Graphics. July 28 – August 2, 2002, Kyiv, Ukraine, 2002; Vol. 1, p. 104-107 (4 refs.).

Kolomiets EA. Mathematical and Computer Models of Surfaces in Special Normal Coordinates. Thesis of PhD, 1999; 16 p.

Skidan IA. Analytical theory of the forming of shells. Shells in Architecture and Mathematical Analysis of Thin-Walled Civil-Engineering and Machine-Building Constructions of Complex Form: Trudy Mezhdunarodnoy Konf., Moscow, June 4-8, 2001. Moscow: Izd-vo RUDN, 2001; p. 366-371.

■ Right-Waving Helicoid

A right-waving helicoid is a ruled surface traced by a straight line which intersects the axis of the helicoid at the right angle and rotates with a constant angular velocity about this axis and simultaneously translates along the same axis passing simultaneously through the given waving line

$$X = X(u) = R\cos u,$$
$$Y = Y(u) = R\sin u,$$
$$Z = Z(u) = bu + a\sin pu,$$

lying on the circular cylinder of the radius R; $2\pi b$ is a lead of a waving line; u is an angle read from the axis x in the direction of the axis y; p = const; a is the amplitude of oscillations of the directrix waving line relatively to the *base helical line of the constant slope:*

$$X_h = X_h(u) = R\cos u,$$
$$Y_h = Y_h(u) = R\sin u,$$
$$Z_h = Z_h(u) = bu$$

on the cylinder of the radius R.

The straight generatrix lines of the *right-waving helicoid* are parallel to its plane of parallelism that is perpendicular to the axis of the helicoid; therefore the right-waving helicoid may be attached to a family of *Catalan surfaces*.

Paying attention to the method of design of this surface, we may relate it to a group of *conoids*.

The form of definition of the right-waving helicoid

(1) Parametrical equations:

$$x = x(u, v) = v\cos u,$$
$$y = y(u, v) = v\sin u,$$
$$z = z(u) = bu + a\sin pu,$$

where $R \le v \le +\infty$. The line $v = 0$ is the axis of the helicoid. The coordinate lines v coincide with the straight generatrix lines of the right-waving helicoid.

Coefficients of the fundamental forms of the surface:

$$A^2 = v^2 + (b + ap\cos pu)^2, \quad F = 0, \quad B = 1;$$
$$A^2B^2 - F^2 = A^2 = v^2 + (b + ap\cos pu)^2;$$
$$L = \frac{ap^2 v\sin pu}{A}, \quad M = \frac{b + ap\cos pu}{A}, \quad N = 0;$$
$$k_u = \frac{ap^2 v\sin pu}{A^3}, \quad k_v = 0$$
$$K = -\frac{(b + ap\cos pu)^2}{A^4} < 0.$$
$$H = \frac{ap^2 v\sin pu}{2A^3} \neq 0.$$

The coefficients of the fundamental forms of this surface show that the right-waving helicoid is given in the orthogonal nonconjugate system of curvilinear coordinates u, v.

Right-waving helicoid is a surface of *strictly negative Gaussian curvature*. If a = 0 or p = 0, then the surface degenerates into *a right helicoid* (see also the Chap. "19. Minimal Surfaces").

In Fig. 1, the right waving helicoid is designed for p = 3; b = 0.5 m; a = 1 m; R = 5 m; $5\ \text{m} \le v \le 7\ \text{m}$; $0 \le u \le 2\pi$.

The surface presented in Fig. 2 has $a = b$ = 1.5 m; p = 4; $5\ \text{m} \le v \le 12\ \text{m}$; $0 \le u \le 4\pi$.

If we shall assume b = 0, then the right-waving helicoid will be placed between two parallel planes $z = a$ and $z = -a$. When p is a whole number, then this surface does not have the self-intersections and there are p straight generatrixes of the helicoid at every boundary plane (Fig. 3).

Fig. 1

Fig. 2

Fig. 3

The projection of the surface of the right-waving helicoid, constructed in the limits $r_1 \le v \le r_2$, where $r_1 < r_2$, represents an annulus and the projections of the curvilinear coordinates lines u are the concentric circles (Fig. 4).

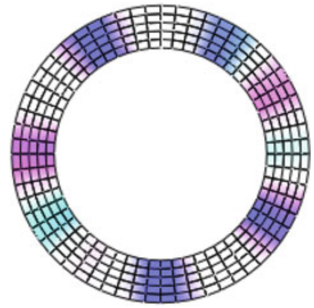


Fig. 4

G. Brankov (1961) considered that it was necessary a more detailed study of the problem of the shape of waving helicoids.

Additional Literature

Krivoshapko SN. Classification of ruled surfaces. Structural Mechanics of Engineering Construction and Buildings. 2006; No. 1, p. 10-20.

Brankov G. Corrugated Shell Constructions. Bulgarian Academy of Science. Sofia, 1961; 80 p.

■ Wave-Shaped Torus on the Sphere

A wave-shaped torus on the sphere has *a spherical curve*

$$E_0(u) = ae_0(u) = a(\boldsymbol{i}\cos u + \boldsymbol{j}\sin u)\cos\omega + \boldsymbol{k}a\sin\omega,$$

as a *directrix curve*, where

$$\omega = \omega(u) = d + \varepsilon\sin pu,\ d = \alpha\pi/2;\ \varepsilon = \beta\pi/2,$$

α, β, and p are constants. The directrix spherical curve is disposed at the surface of the sphere of radius a. The unit vector $\boldsymbol{e}_0(u)$ is the normal of the sphere at which the directrix curve is placed. The generatrix circle of a constant radius b is given at a local system of coordinates with the origin on the spherical directrix curve:

$$X = X(v) = b\cos v,\quad Y = Y(v) = b\sin v.$$

The circles lie in the normal planes of the spherical directrix line.

Wave-shaped torus on the sphere besides *wave-shaped surfaces* may be added to *surfaces with the spherical director curve* or to *tubular surfaces* as well.

Forms of definition of the surface

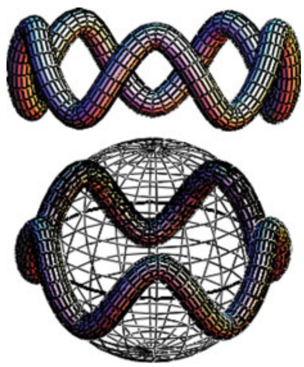
(1) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = a\boldsymbol{e}_0(u) + b\cos v\boldsymbol{e}_0(u) + b\sin v\boldsymbol{g}(u),$$

where

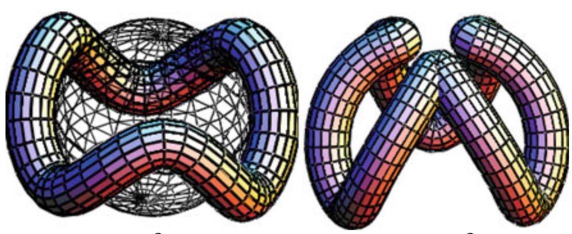
$$\boldsymbol{g} = \boldsymbol{g}(u) = \left[\boldsymbol{e}_0'/s \times e_0\right]$$

is the unit vector that is orthogonal to the unit vector $\boldsymbol{e}_0(u)$ at the normal plane of the spherical directrix curve. The vector equation of the wave-shaped torus on the sphere is derived on the base of the general vector equation of *surfaces with the generatrix curve in the normal planes of the spherical directrix line* (see also the Chap. "21. Surfaces with Spherical Director Curve") under the condition that $\theta = 0$.



$a = 10; b = 1; p = 5; \beta = 0.2$

Fig. 1 $a = 0;\ -\pi \leq u \leq \pi$



$\alpha = 0; \beta = 0.25;$ $\alpha = 0.25; \beta = 0.5$

Fig. 2 $a = 10;\ b = 2;\ p = 3;\ \pi \leq u \leq \pi;\ 0 \leq v \leq 2\pi$

(2) Parametrical equations (Figs. 1, 2 and 3):

$$\begin{aligned} x &= x(u,v) = [a + b\cos v]\cos\omega\cos u \\ &\quad + b\sin v(\sin\omega\cos\omega\cos u - \omega'\sin u)/s, \\ y &= y(u,v) = [a + b\cos v]\cos\omega\sin u \\ &\quad + b\sin v(\sin\omega\cos\omega\sin u + \omega'\cos u)/s, \\ z &= z(u,v) = [a + b\cos v]\sin\omega - b\sin v\cos^2\omega/s, \end{aligned}$$

where

$s = \sqrt{\omega'^2 + \cos^2\omega}$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned} A &= s(a + b\cos v) \\ &\quad + \left[\left(1 + \frac{\omega'^2}{s^2}\right)\sin\omega + \frac{\omega''}{s^2}\cos\omega\right]b\sin v, \\ F &= 0,\ B = b; \\ L &= \frac{A - as}{b}A,\ M = 0,\ N = b; \\ k_u &= k_1 = \frac{A - as}{bA},\ k_v = k_2 = \frac{1}{b}. \end{aligned}$$

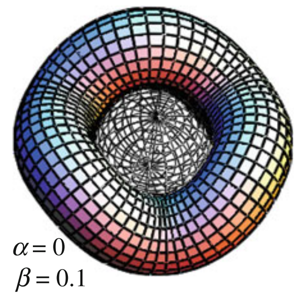


Fig. 3 $-\pi \leq u \leq \pi$

The wave-shaped torus on the sphere is given at lines of principle curvatures *u, v*. The coordinate lines *v* coincide with the generatrix circles.

Taking into account, that $\omega = \omega(u) = d + \varepsilon\sin pu$, where *d*, *p,* and ε are constants, parametrical equations of the wave-shaped torus on the sphere may be written in the more detailed form:

$$\begin{aligned} x &= x(u,v) = [a + b\cos v]\cos\omega\cos u \\ &\quad + b\frac{\sin\omega\cos\omega\cos u - \cos pu\sin u}{s}\sin v, \\ y &= y(u,v) = [a + b\cos v]\cos\omega\sin u \\ &\quad + b\frac{\sin\omega\cos\omega\sin u + \varepsilon p\cos pu\cos u}{s}\sin v, \\ z &= z(u,v) = [a + b]\sin\omega - b\frac{\cos^2\omega}{s}, \end{aligned}$$

where

$$s = \sqrt{\omega'^2 + \cos^2\omega} = \sqrt{\varepsilon^2 p^2\cos^2 pu + \cos^2\omega}.$$

The surface in question touches the sphere of a radius $(a - b)$ along the closed spherical line (Fig. 3):

$$\mathbf{E}_1(u) = (a - b)\boldsymbol{e}_0(u).$$

Reference

Ivanov VN. Spherical curves and geometry of the surfaces on a supporting sphere. Sovremennie Problemy Geometricheskogo Modelirovaniya. Ukraine-Russia Nauchno-Prakt. Konf., Harkiv, April 19-22, 2005. Harkiv, 2005; p. 114-120 (4 refs.).

■ Sinusoidal Cylindrical Surface

A sinusoidal cylindrical surface is formed by the motion of a straight line intersecting the given plane sinusoid (Fig. 1). And the straight generatrixes in the process of their motion remain perpendicular to the plane in which the sinusoid is placed.

The form of definition of the surface

Explicit equation (Fig. 1):

$$y = a\sin(n\pi x/b),$$

where n is a number of the whole semi-waves of the sinusoid located on the section of the b width; a is an amplitude of the sinusoid. The directrix sinusoid has the period

$$T = 2b/n.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{a^2n^2\pi^2}{b^2}\cos^2\frac{n\pi}{b}x, \quad F = 0, \quad B = 1;$$

$$L = \frac{an^2\pi^2}{Ab^2}\sin\frac{n\pi}{b}x, \quad M = N = 0;$$

$$k_x = k_1 = \frac{an^2\pi^2}{A^3b^2}\sin\frac{n\pi}{b}x, \quad k_z = k_2 = 0, \quad K = 0.$$

The surface is given in lines of principle curvatures x and z. In the cross sections of the sinusoidal cylindrical surface by the planes

$$z = x\tan\varphi + c,$$

the sinusoids with a period $T_\varphi = T/\cos\varphi$, increased in comparison with the given directrix sinusoid, lie. The amplitude of the sinusoids in these cross sections remains the same and equal to a. In the cross sections of the cylindrical surface by the planes z = const, the directrix sinusoids are placed.

The lines of the intersections of the cylindrical surface with the planes x = const coincide with the straight generatrixes of this surface.

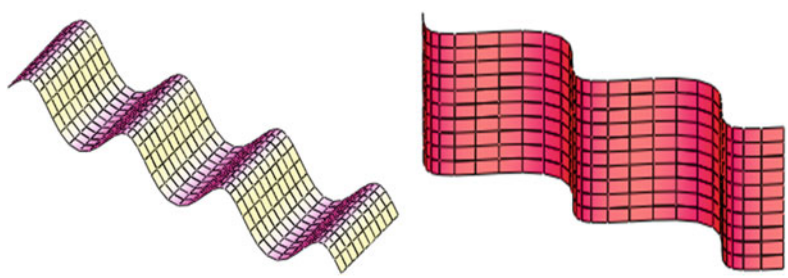


Fig. 1

■ Spiral-Shaped Surface with Generatrix Sinusoids and with a Directrix Line of Constant Pitch on a Circular Cone

The surface is formed by sinusoids lying in the planes of a pencil with a fixed straight line passing through the axis of the cone.

The directrix line of the constant lead on the circular cone is projected on the plane, that is perpendicular to the axis of the cone, as the *spiral of Archimedes*.

Forms of definition of the surface

(1) Vector equation:

$$\mathbf{r} = \mathbf{r}(u, v) = [au + \varphi(v)]\mathbf{h}(u) + [a\lambda u + \psi(v)]\mathbf{k},$$

where

$$\mathbf{h}(u) = \mathbf{i}\cos u + \mathbf{j}\sin u$$

is the unit vector lying in the plane *xOy; ε(u)* is a ratio of similitude of the generatrix sinusoids chosen according to the given condition of design;

$$\varphi(v) = X(v)\cos\theta - Y(v)\sin\theta,$$
$$\psi(v) = X(v)\sin\theta + Y(v)\cos\theta.$$

$X = X(v)$, $Y = Y(v)$ are parametrical equations of generatrix curves given in a local system of Cartesian coordinates. The origin of the local coordinate system is disposed on the directrix curve; θ is an angle of turning of the local axis oZ

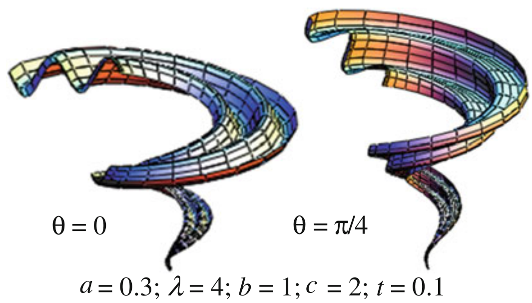


Fig. 1 $\varepsilon = tu$

relative to the Oz axis. For the surface with the generatrix sinusoids, one may write:

$$X = v, \quad Y = Y(v) = b \sin cv.$$

(2) Parametrical equations (Fig. 1):

$$x = x(u, v) = [au + \varepsilon(u)\varphi(v)] \cos u,$$
$$y = y(u, v) = [au + \varepsilon(u)\varphi(v)] \sin u;$$
$$z = z(u, v) = [a\lambda u + \varepsilon(u)\psi(v)],$$

where $\varepsilon(u)$ is a ratio of similitude of the generatrix sinusoids chosen according to the given condition of design.

Additional Literature

Ivanov VN. Geometry and design of shells on the base of surfaces with the system of coordinates lines in the planes of pencil. Prostranstv. Konstruktzii Zdaniy i Sooruzheniy: Sb. Nauchn. Rabot. MOO “Prostranstv. Konstruktzii”. Moscow: “Devyatka Print”, 2004; Iss. 9, p. 26-35 (13 refs.).

■ Waving Surface with the Pseudo Agnesi Curls of Cylindrical Type

The principle of waviness of surfaces of some natural thin-walled bio-structures was assumed as a basis of design of *waving surface with the pseudo Agnesi curls of cylindrical type.*

The surface is formed by the motion of continuously changing pseudo Agnesi curl m lying in a plane parallel to the coordinate plane xOz (Figs. 1, 2 and 3). The vertex of the curve moves along the axis Oy and one of its points moves along the given line h lying at the plane $z = -d$.

The waviness of the surface is achieved with the help of the control by a free parameter R of the form of the pseudo Agnesi curl. In the considered case, the function of control by a parameter of the base circle of the pseudo Agnesi curl

$$R = A + B \cos(py)$$

is chosen.

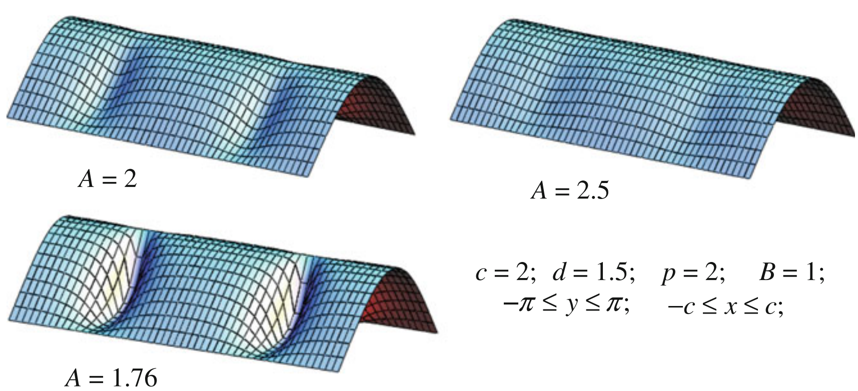


Fig. 1

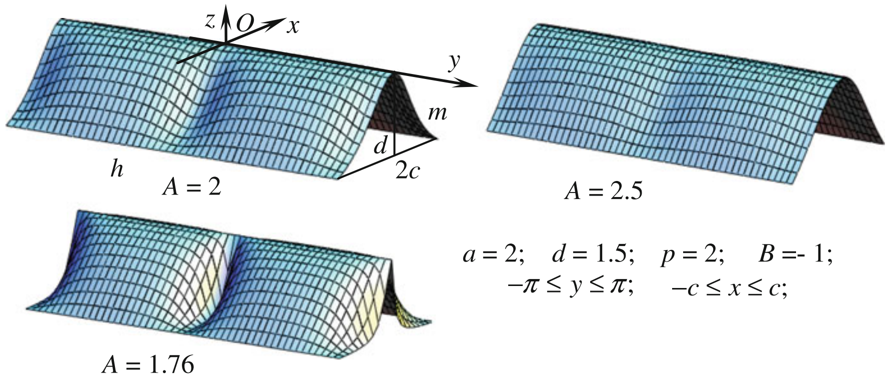


Fig. 2

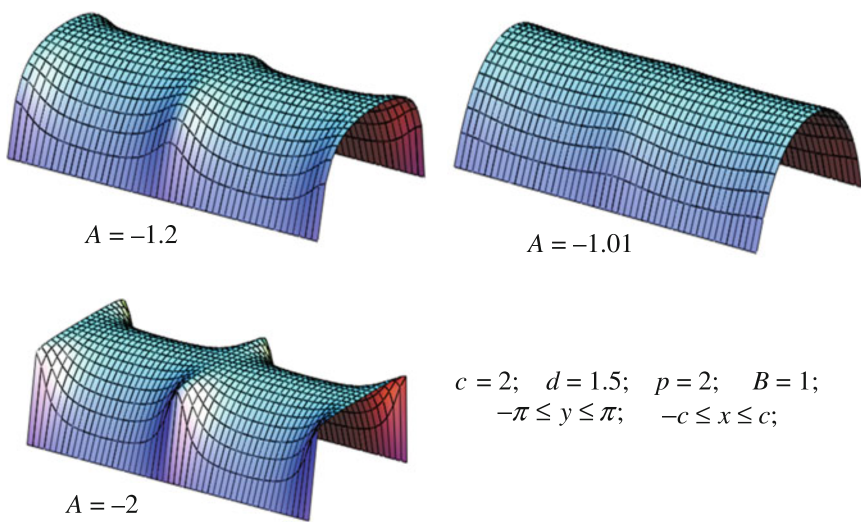


Fig. 3

Forms of definition of the waving surface with the pseudo Agnesi curls of cylindrical type

(1) Explicit equation (Figs. 1, 2 and 3):

$$z = -\frac{2dx^2[A + B\cos(py)]}{d(x^2 - c^2) + 2c^2[A + B\cos(py)]},$$

where d is a rise of the surface; $2c$ is a span of the surface in the direction of the axis Ox (Fig. 2); A, B, p are constants; $-c \le x \le c$; $-d \le z \le 0$; $-\infty \le y \le \infty$.

In Figs. 1, 2 and 3, the waving surfaces with the pseudo Agnesi curls of cylindrical type with different values of the geometrical parameters are shown. These parameters are presented under the corresponding figures.

Designing the surfaces, it is necessary to take into account that

$$A > d/2 - |B|,$$

if $A > 0$ and

$$A < |B|$$

if $A < 0$

It will permit to avoid division by zero.

References

Kaschenko AV. The geometrical modeling of surfaces of some bio-forms of structures. Prikl. Geom. i Ingen. Grafika. Kiev. 1978; Iss. 26, p. 46-48 (4 refs.).

Utishev EG. Apparatus for drawing of Agnesi curl. Patent 1546282 of Russia, 04.04. 1988.

Utishev EG. Apparatus for drawing of pseudo Agnesi curl. Patent 1564008 of Russia, 15.07. 1988.

■ Waving Surface with the Pseudo Agnesi Curls on a Circular Plan

The principle of waviness of surfaces of natural thin-walled bio-structures viz bivalve shellfishes was assumed as a basis of design of *waving surface with the pseudo Agnesi curls on a circular plan.*

The surface is formed by rotation of continuously changing pseudo Agnesi curl m lying at the planes of a pencil with the fixed straight line coinciding with the axis Oz.

The waviness of the surface is achieved with the help of the control by a free parameter R of the form of the pseudo Agnesi curl. In the considered case, the function of control by a parameter of the base circle of the pseudo Agnesi curl

$$R = A + B\cos(p\varphi).$$

Forms of definition of the waving surface with the pseudo Agnesi curls on a circle plan

(1) Parametrical equation (Figs. 1 and 2):

$$x = x(\rho, \varphi) = \rho \sin\varphi,$$
$$y = y(\rho, \varphi) = \rho \cos\varphi,$$
$$z = z(\rho, \varphi) = -\frac{2d\rho^2[A + B\cos(p\varphi)]}{d(\rho^2 - c^2) + 2c^2[A + B\cos(p\varphi)]}.$$

where d is the maximum rise of the surface above the plane $z = -d$; c is a radius of the circular plan of the surface on the plane $z = -d$; A, B, p are constants; $0 \le \rho \le c$; $0 \le \varphi \le 2\pi$; $-d \le z \le 0$; ρ and φ are cylindrical coordinates of the waving surface in question. The origin of the system of coordinates is placed at the vertex of the surface.

Designing the surfaces, it is necessary to take into account that

$$A > d/2 - |B|,$$

if $A > 0$ and

$$A < |B|$$

if $A < 0$. It will permit to avoid division by zero.

The waving surface in question may be also added to a class of *surfaces of umbrella type* (see also the Chap. "26. Surfaces of Umbrella Type").

Additional Literature

Kaschenko AV. The geometrical modeling of surfaces of some bio-forms of structures. Prikl. Geom. i Ingen. Grafika. Kiev. 1978; Iss. 26, p. 46-48 (4 refs.).

Utishev EG. Apparatus for drawing of pseudo Agnesi curl. Patent 1564008 of Russia, 15.07. 1988.

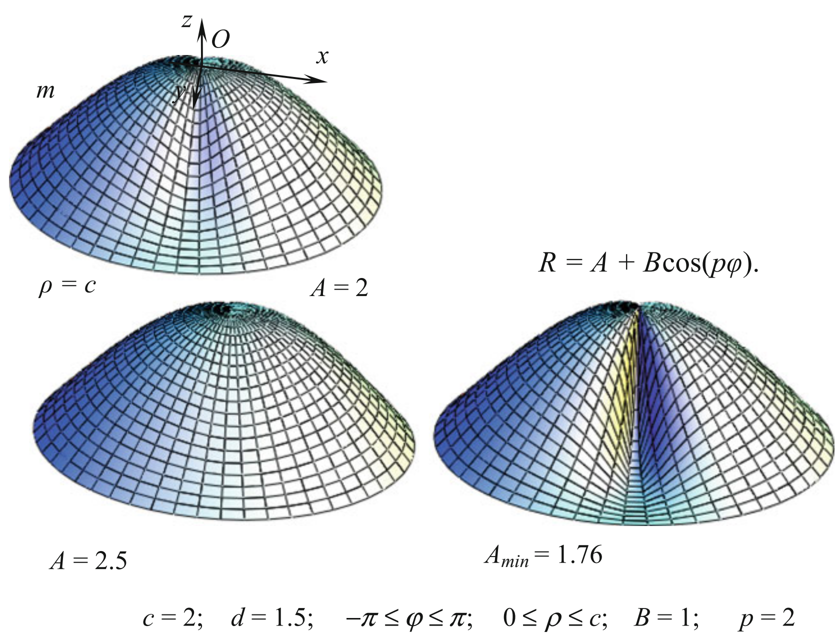


Fig. 1

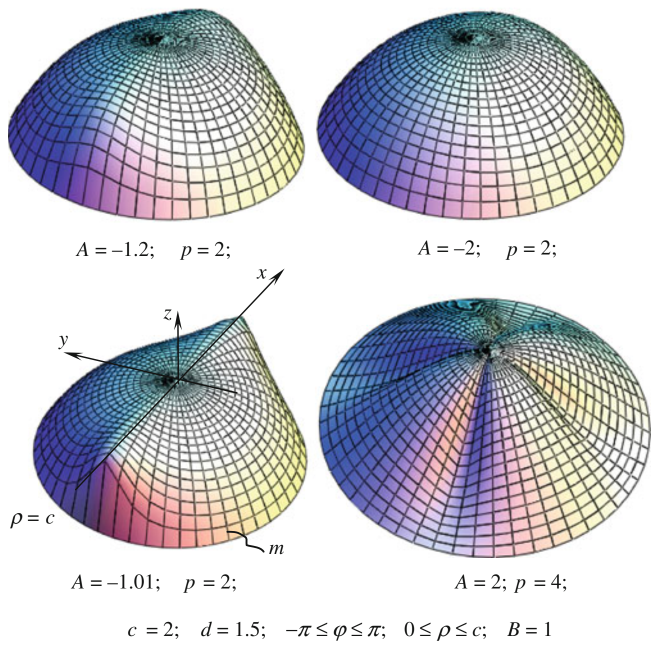


Fig. 2

■ Corrugated Paraboloid of Revolution

A corrugated paraboloid of revolution (Fig. 1) has *a circular sinusoid*

$$x = (R + a\cos n\varphi)\cos\varphi,$$
$$y = (R + a\cos n\varphi)\sin\varphi,$$
$$z = 0$$

as the foot. Here, n is a number of the vertexes of the sinusoid on a circular plan, a is an amplitude of the crimps at the foot of the surface, R is a radius of the base circle of the corrugated paraboloid in the foot relative to which, the circular sinusoid is constructed.

A corrugated paraboloid of revolution with the external crimps (Fig. 2) has a circular waving curve

$$x = (R + a|\cos n\varphi|)\cos\varphi,$$
$$y = (R + a|\cos n\varphi|)\sin\varphi,$$
$$z = 0$$

Fig. 1

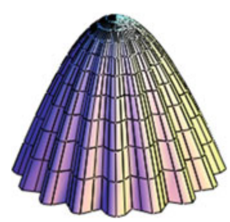


Fig. 2

at the foot with the vertexes directed out of the center of the circular base.

The *corrugated paraboloid of revolution with the internal crimps* (Fig. 3) has a circular waving curve

$$x = (R - a|\cos n\varphi|)\cos\varphi,$$
$$y = (R - a|\cos n\varphi|)\sin\varphi,$$
$$z = 0$$

at the foot with the vertexes directed only inside the circular base.

Fig. 3

Forms of definition of the surface

(1) Parametrical form of the definition of the corrugated paraboloid of revolution (Fig. 1):

$$x = x(r,\varphi) = r\left(1 + \frac{ar\cos n\varphi}{R^2}\right)\cos\varphi,$$
$$y = y(r,\varphi) = r\left(1 + \frac{ar\cos n\varphi}{R^2}\right)\sin\varphi,$$
$$z = z(r) = h\left(1 - \frac{r^2}{R^2}\right),$$

where $0 \le z \le h;\ \ 0 \le \varphi \le 2\pi;\ \ 0 \le r \le R$, h is the height of a corrugated paraboloid of revolution.

Coefficients of the fundamental forms of the surface:

$$A^2 = \left(1 + \frac{2ar\varphi}{R^2}\right)^2 + \frac{4r^2h^2}{R^4},$$
$$F = -\frac{ar^2 n}{R^2}\left(1 + \frac{2ar\cos n\varphi}{R^2}\right)\sin n\varphi,$$
$$B^2 = r^2\left[\left(1 + \frac{ar\cos n\varphi}{R^2}\right)^2 + \frac{a^2r^2n^2}{R^4}\sin^2 n\varphi\right];$$
$$A^2B^2 - F^2 = r^2\left[A^2\left(1 + \frac{ar\cos n\varphi}{R^2}\right)^2 + \frac{4r^4n^2h^2a^2}{R^8}\sin^2 n\varphi\right];$$
$$L = -\frac{2rh}{R^2\sqrt{A^2B^2 - F^2}}\left(1 + \frac{ar\cos n\varphi}{R^2}\right),$$
$$M = -\frac{2r^3 han\sin n\varphi}{R^4\sqrt{A^2B^2 - F^2}},$$
$$N = -\frac{2r^3h}{R^2\sqrt{A^2B^2 - F^2}}\left[\frac{2a^2r^2n^2\sin^2 n\varphi}{R^4} + \frac{arn^2\cos n\varphi}{R^2}\left(1 + \frac{ar\cos n\varphi}{R^2}\right) + \left(1 + \frac{ar\varphi}{R^2}\right)^2\right].$$

(2) Parametrical form of the definition of the corrugated paraboloid of revolution with the external crimps (Fig. 2):

$$x = x(r,\varphi) = r\left(1 + \frac{ar|\cos n\varphi|}{R^2}\right)\cos\varphi,$$
$$y = y(r,\varphi) = r\left(1 + \frac{ar|\varphi|}{R^2}\right)\sin\varphi,$$
$$z = z(r) = h\left(1 - \frac{r^2}{R^2}\right),$$

where $0 \le z \le h;\ \ 0 \le \varphi \le 2\pi;\ \ 0 \le r \le R$.

(3) Parametrical form of the definition of the corrugated paraboloid of revolution with the internal crimps (Fig. 3):

$$x = x(r,\varphi) = r\left(1 - \frac{ar|\cos n\varphi|}{R^2}\right)\cos\varphi,$$
$$y = y(r,\varphi) = r\left(1 - \frac{ar|\cos n\varphi|}{R^2}\right)\sin\varphi,$$
$$z = z(r) = h\left(1 - \frac{r^2}{R^2}\right).$$

Reference

Krivoshapko SN. Geometrical investigations of surfaces of umbrella type. Structural Mechanics of Engineering Constructions and Buildings. 2005; No. 1, p. 11-17.

■ Corrugated Sphere

A corrugated sphere (Fig. 1) has a circular sinusoid

$$x = (R + a\cos n\varphi)\cos\varphi,$$
$$y = (R + a\cos n\varphi)\sin\varphi,\ \ z = 0$$

as the foot, where n is a number of the vertexes of the sinusoid on a circular plan, a is a maximum amplitude of the crimps at the base of the surface (on equator), R is a radius of the sphere on the equator relative to which, the circular sinusoid is constructed, φ is the angle read from the Ox axis in the direction of the Oy axis.

The corrugated sphere with the external crimps (Fig. 2) has the circular waving curve

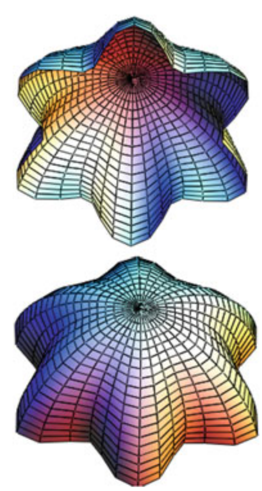


Fig. 1

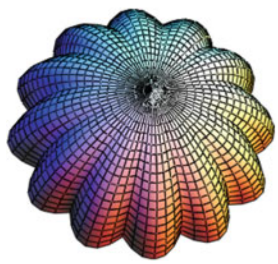


Fig. 2

$$x = (R + a|\cos n\varphi|)\cos\varphi,$$
$$y = (R + a|\cos n\varphi|)\sin\varphi,$$
$$z = 0$$

as the foot with the vertexes directed out of the center of the circular base.

The corrugated sphere with the internal crimps (Fig. 3) has the circular waving curve

$$x = (R - a|\cos n\varphi|)\cos\varphi,$$
$$y = (R - a|\cos n\varphi|)\sin\varphi,$$
$$z = 0$$

as the foot with the vertexes directed only inside the circular base.

Forms of definition of the corrugated sphere

(1) Parametrical form of the definition of the corrugated sphere (Fig. 1):

$$x = x(\varphi, v) = [R\cos v + a(1 - \sin v)\cos n\varphi]\cos\varphi,$$
$$y = y(\varphi, v) = [R\cos v + a(1 - \sin v)\cos n\varphi]\sin\varphi,$$

where v is the angle of the plane xOy with the axis Oz; $0 \le z \le R$; $0 \le \varphi \le 2\pi$; $0 \le v \le \pi$. In the cross sections of the corrugated sphere by the planes z = const, i.e., when $v = v_o$ = const, the circular sinusoids

$$x = x(\varphi) = [R\cos v_o + a(1 - \sin v_o)\cos n\varphi]\cos\varphi,$$
$$y = y(\varphi) = [R\cos v_o + a(1 - \sin v_o)\cos n\varphi]\sin\varphi,$$
$$z = R\sin v_o$$

are placed.

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2n^2(1 - \sin v)^2\sin^2 n\varphi + [R\cos v + a(1 - \sin v)\varphi]^2,$$
$$F = an(1 - \sin v)[R\sin v + a\cos v\varphi]\sin n\varphi,$$

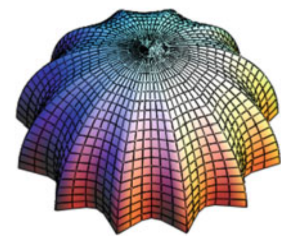


Fig. 3

$$B^2 = R^2 + 2aR\sin v\cos v\varphi + a^2\cos^2 v\cos^2 n\varphi,$$
$$L = \frac{-R\cos v}{\sqrt{A^2B^2 - F^2}}\Big\{2a^2n^2(1 - \sin v)^2\sin^2 n\varphi + [R\cos v + a(1 - \sin v)\cos n\varphi] \times \left[a(1 - \sin v)(1 + n^2)\cos n\varphi + R\cos v\right]\Big\}$$
$$M = \frac{R^2 an\cos v\sin n\varphi}{\sqrt{A^2B^2 - F^2}}(1 - \sin v),$$
$$N = \frac{-R^2[R\cos v + a(1 - \sin v)\cos n\varphi]}{\sqrt{A^2B^2 - F^2}}$$

(2) Parametrical form of the definition of the corrugated sphere with the external crimps (Fig. 2):

$$x = x(\varphi, v) = [R\cos v + a(1 - \sin v)|\cos n\varphi|]\cos\varphi,$$
$$y = y(\varphi, v) = [R\cos v + a(1 - \sin v)|\cos n\varphi|]\sin\varphi$$
$$z = z(v) = R\sin v,$$

where $0 \le z \le R$; $0 \le \varphi \le 2\pi$; $0 \le v \le \pi$.

(3) Parametrical form of the definition of the corrugated sphere with internal crimps (Fig. 3):

$$x = x(\varphi, v) = [R\cos v - a(1 - \sin v)|\cos n\varphi|]\cos\varphi,$$
$$y = y(\varphi, v) = [R\cos v - a(1 - \sin v)|\cos n\varphi|]\sin\varphi,$$
$$z = z(v) = R\sin v,$$

where

$0 \le z \le R$; $0 \le \varphi \le 2\pi$; $0 \le v \le \pi$.

The corrugated spheres shown in Figs. 1, 2 and 3 have R = 1 m; a = 0.24 m; n = 6; $0 \le v \le \pi/2$.

References

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■ Ruled Rotational Surface with Axoids "Plane–Cylinder"

A rotational surface with axoids "*plane–cylinder*" is formed by any curve L when the cylinder with a radius r rolls without sliding on the plane. The generatrix curve L given in the mobile system of coordinates $O_1X_1Y_1Z_1$ by the parametric equations

$$X_1 = X_1(u),$$
$$Y_1 = Y_1(u),$$
$$Z_1 = Z_1(u)$$

is rigidly connected with the loose axoid that is a circular cylinder. If the generatrix line L is a straight line then the parametric equations of this line L may be written as:

$$X_1(u) = \frac{c-a}{H}u + a,$$
$$Y_1(u) = \frac{d-b}{H}u + b,$$
$$Z_1(u) = u,$$

where a, b, c, d are constants shown in Fig. 1. In this case, the surface is called *a ruled rotational surface* with axoids "*plane cylinder*".

Forms of definition of the rotational surface

(1) Parametrical equations:

$$x(u,\varphi) = r\varphi + X_1(u)\cos\varphi + Y_1(u)\sin\varphi,$$
$$y(u,\varphi) = r - X_1(u)\sin\varphi + Y_1(u)\cos\varphi,$$
$$z(u) = Z_1(u),$$

where φ is the angle read from the positive direction of the axis Ox in the direction of the mobile axis O_1X_1; x, y, z are Cartesian coordinates of any point of the curve L relative to the stationary system of coordinates $Oxyz$.

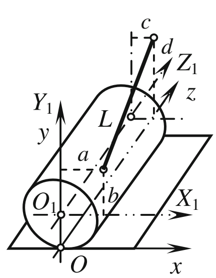

Fig. 1

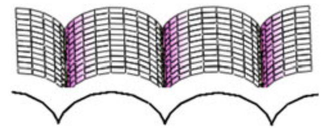

Fig. 2

(2) Assume a generatrix straight line given by parametrical equations:

$$X_1 = 0; Y_1 = r; Z_1 = u,$$

then we shall obtain *a right cylindrical surface with the directrix cycloid* (Fig. 2). The generating straight line will be one of the straight generatrix lines of the circular mobile cylinder (loose axoid). In this case, it is taken: $b = d = r$, $a = c = 0$.

The parametric equations of the surface (Fig. 2) have the following form:

$$x = x(u,\varphi) = r(\varphi + \sin\varphi),$$
$$y = y(u,\varphi) = r(1 + \cos\varphi),$$
$$z = z(u) = u.$$

(3) Figure 3 represents the ruled rotational surface with

$$X_1 = 0;\quad Y_1 = nr;\quad Z_1 = u;\quad n > 1.$$

The represented *right cylindrical surface* has *an elongated cycloid* as the directrix curve.

If $n < 1$, then we obtain a right cylindrical surface with a directrix curve in the form of *a shortened cycloid* (Fig. 4).

The surface shown in Fig. 3 has

$$b = d = 2r,\quad a = c = 0.$$

The surface given in Fig. 4 was constructed when

$$b = d = nr = r/2,\quad a = c = 0.$$

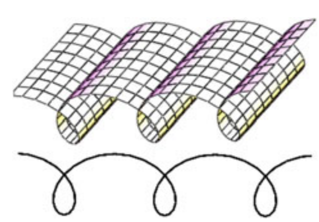

Fig. 3

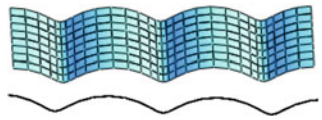

Fig. 4

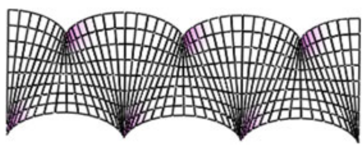

Fig. 5

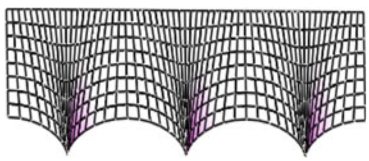

Fig. 6

(4) The ruled rotational surface of the negative Gaussian curvature with axoids "plane–cylinder" having geometrical parameters $b = -r$, $d = r$, $a = c = 0$ is represented in Fig. 5.

In Fig. 6, the surface has

$$b = r, \quad d = a = c = 0.$$

(5) Having $a = d = 0$, $b = r$, $c = -r$, one may design the ruled rotational surface of a negative Gaussian curvature

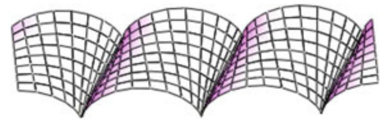

Fig. 7

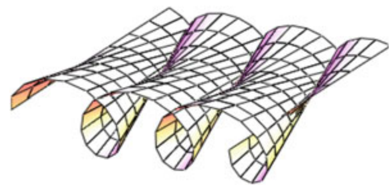

Fig. 8

with axoids "plane–cylinder" (Fig. 7), that is equivalent to a definition of the generatrix straight line by the equations:

$$X_1 = -ru/H;$$
$$Y_1 = -ru/H + r;$$
$$Z_1 = u.$$

(6) The new form of a ruled rotational surfaces of a negative Gaussian curvature may be obtained if one will assume $a = d = 0$ and $b = r$, $c = 2r$ (Fig. 8). The surface is formed by a straight line

$$X_1 = 2ru/H;$$
$$Y_1 = -ru/H + r;$$
$$Z_1 = u.$$

■ Parabolic Rotational Surface with Axoids "Plane–Cylinder"

A parabolic rotational surface with axoids "plane–cylinder" is created by a plane parabola m when a cylinder with a radius r rolls without sliding on the plane. The generatrix parabola m given in the mobile system of coordinates $oXYZ$ by the parametric equations

$$X = X(u), \quad Y = Y(u), \quad Z = Z(u)$$

is rigidly connected with the loose axoid that is a circular cylinder. If the generatrix parabola m is disposed in the mobile coordinate plane oYZ and the axis of the parabola coincides with the axis oY, then the parametric equations of this parabola m (Fig. 1) may be written in the following form:

$$Y = Y(u) = a + bu^2,$$
$$Z = Z(u) = u,$$

where a, b are constants.

Forms of definition of the parabolic rotational surface

(1) Parametrical equations:

$$x(u, \varphi) = r\varphi + X(u)\cos\varphi + Y(u)\sin\varphi,$$
$$y(u, \varphi) = r - X(u)\sin\varphi + Y(u)\cos\varphi,$$
$$z(u) = Z(u),$$

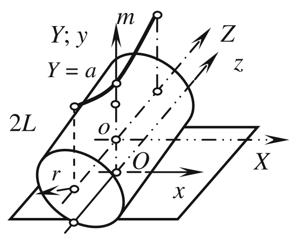

Fig. 1

where φ is the angle read from the positive direction of the axis Ox in the direction of the mobile axis oX; r is the radius of the rolling cylinder; x, y, z are the Cartesian coordinates of any point of the curve m relative to the fixed system of coordinates $Oxyz$. The given parametrical equations may be used for the definition of a rotational surface with axoids "plane–cylinder" with arbitrary generatrix curve rigidly connected with a mobile cylinder.

(2) Parametrical form of definition of a rotational surface, when a generatrix curve is any plane curve lying in a movable plane oYZ:

$$x = x(u, \varphi) = r\varphi + Y(u)\sin\varphi,$$
$$y = y(u, \varphi) = r + Y(u)\cos\varphi,$$
$$z = z(u) = Z(u).$$

Coefficients of the fundamental forms of the surface:

$$A^2 = Y'^2 + Z'^2, \quad F = rY'\sin\varphi,$$
$$B^2 = (r\cos\varphi + Y)^2 + r^2\sin^2\varphi;$$
$$A^2B^2 - F^2 = A^2(r\cos\varphi + Y)^2 + Z'^2\sin^2\varphi;$$
$$L = \frac{(r\cos\varphi + Y)}{\sqrt{A^2B^2 + F^2}}(Y''Z' - Y'Z''),$$
$$M = \frac{-rY'Z'\sin\varphi}{\sqrt{A^2B^2 + F^2}}$$
$$N = -\frac{(r\cos\varphi + Y)}{\sqrt{A^2B^2 - F^2}}YZ'.$$

The derivatives with respect to parameter u are denoted by primes.

(3) Parametrical form of definition of a rotational surface, when a generatrix curve is a plane parabola

$$Y = a + bu^2, Z = u,$$

lying in the mobile plane oYZ (Figs. 1, 2, 3, 4, 5 and 6):

$$x = x(u, \varphi) = r\varphi + (a + bu^2)\sin\varphi,$$
$$y = y(u, \varphi) = r + (a + bu^2)\cos\varphi,$$
$$z = z(u) = u.$$

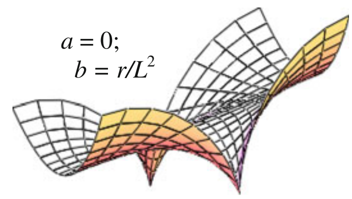

Fig. 2

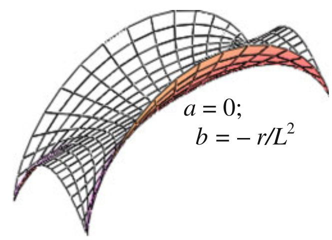

Fig. 3

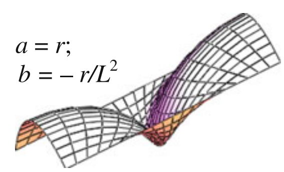

Fig. 4

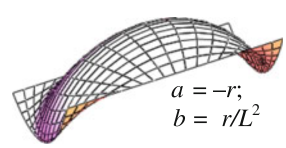

Fig. 5

Coefficients of the fundamental forms of the surface:

$$A^2 = 4b^2u^2 + 1, \; F = 2bru\sin\varphi,$$
$$B^2 = (r\cos\varphi + a + bu^2)^2 + r^2\sin^2\varphi;$$
$$A^2B^2 - F^2 = A^2(r\cos\varphi + a + bu^2)^2 + \sin^2\varphi;$$
$$L = 2b\frac{(r\cos\varphi + a + bu^2)}{\sqrt{A^2B^2 - F^2}},$$
$$M = \frac{-2bru\sin\varphi}{\sqrt{A^2B^2 - F^2}}$$
$$N = -\frac{(r\cos\varphi + a + bu^2)(a + bu^2)}{\sqrt{A^2B^2 - F^2}};$$
$$K = -\frac{2b[(a + bu^2)(r\cos\varphi + a + bu^2)^2 + 2br^2u^2\sin^2\varphi]}{(A^2B^2 - F^2)^2}.$$

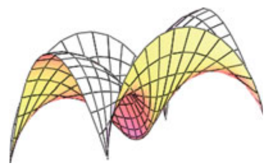

Fig. 6

All rotational surfaces represented in Figs. 2, 3, 4, 5 and 6 are constructed for $L = r$; $r = 1$ m; $0 \le \varphi \le 2\pi$; $-L \le u \le L$. The geometrical parameters a and b are given under the figures. The additional parabolic rotational surfaces with $a = -r$; $b = 2r/L^2$ is shown in the page "Wave-shaped, waving, and corrugated surfaces presented in the encyclopedia."

■ Sphere with Cycloidal Crimps

They have two types of *spheres with cycloidal crimps*.

A sphere with external cycloidal crimps (Fig. 1) has *an epicycloid*

$$x = x(\varphi) = (R + r)\cos\varphi - r\cos(1 + n)\varphi,$$
$$y = y(\varphi) = (R + r)\sin\varphi - r\sin(1 + n)\varphi,$$
$$z = 0$$

at the foot, where n is a number of the vertexes of the epicycloid on the circular plane; $n = R/r$; $2r$ is the maximum amplitude of the crimps at the base of the surface (on the equator), R is a radius of the equator of the sphere, outside of which the circle of a radius r rolls and arbitrary point of which traces the epicycloid; φ is the angle read from the axis Ox in the direction of the axis Oy.

A sphere with internal cycloidal crimps (Fig. 2) has *a hypocycloid*

$$x = x(\varphi) = (R - r)\cos\varphi + r\cos(n - 1)\varphi,$$
$$y = y(\varphi) = (R - r)\sin\varphi - r\sin(n - 1)\varphi,$$
$$z = 0.$$

at the foot with the vertexes directed only inside the circular base, n is a number of the vertexes of the hypocycloid on the circular plane; $n = R/r$; $2r$ is the maximum amplitude of the crimps at the base of the surface (on the equator), R is a radius of the equator of the sphere, inside of which the circle of a radius r rolls and arbitrary point of which traces a hypocycloid; φ is the angle read from the axis Ox in the direction of the axis Oy.

Forms of definition of the surface

(1) Parametrical form of definition of the sphere with external cycloidal crimps (Fig. 1):

$$x = x(u, \varphi) = [(R + r)\cos\varphi - r\cos(n + 1)\varphi]\cos u,$$
$$y = y(u, \varphi) = [(R + r)\sin\varphi - r\sin(n + 1)\varphi]\cos u,$$
$$z = z(u) = R\sin u,$$

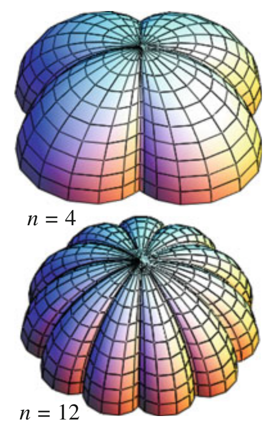

Fig. 1

where u is the angle read from the plane xOy in the direction of the axis Oz; $0 \le z \le R$; $0 \le \varphi \le 2\pi$; $0 \le u \le \pi/2$. In the cross sections of the surface in question by the planes z = const, i.e., when $u = u_o$ = const, we have the epicycloids:

$$x = x(\varphi) = [(R + r)\cos\varphi - r\cos(n + 1)\varphi]\cos u_o,$$
$$y = y(\varphi) = [(R + r)\sin\varphi - r\sin(n + 1)\varphi]\cos u_o$$

with n = const.

Coefficients of the fundamental forms of the surface:

$$A^2 = 2r(R + r)(1 - \cos n\varphi)\sin^2 u + R^2,$$
$$F = -R(R + r)\sin u\cos u\sin n\varphi$$
$$B^2 = 2(R + r)^2(1 - \cos n\varphi)\cos^2 u;$$
$$L = \frac{Rr(R + r)(2 + n)\cos u}{\sqrt{A^2B^2 - F^2}}(1 - \cos n\varphi),$$
$$M = 0.$$
$$N = \frac{R(R + r)^2(2 + n)\cos^3 u}{\sqrt{A^2B^2 - F^2}}(1 - \cos n\varphi);$$
$$K = \frac{R^2r(R + r)^3(2 + n)^2\cos^4 u}{(A^2B^2 - F^2)^2}(1 - \cos n\varphi)^2 \ge 0.$$

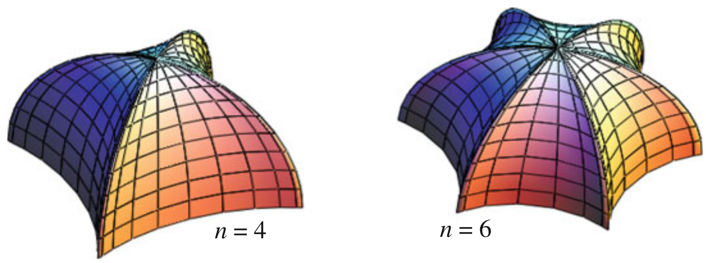

Fig. 2

(2) Parametrical form of definition of the sphere with internal cycloidal crimps:

$$x = x(u, \varphi) = [(R - r)\cos\varphi + r\cos(n-1)\varphi]\cos u,$$
$$y = y(u, \varphi) = [(R - r)\sin\varphi - r\sin(n-1)\varphi]\cos u,$$
$$z = z(u) = R\sin u,$$

where $0 \le z \le R$; $0 \le \varphi \le 2\pi$; $0 \le v \le \pi/2$.

The corrugated spheres shown in Figs. 1 and 2 have $R = 1$ m; $0 \le v \le \pi/2$. The parameter n is presented in the corresponding figures.

Additional Literature

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Churkin GM. The properties of points of hypocycloid. In-t Him. Kinet. i Goreniya SO AN SSSR, Novosibirsk, 1989, 10 p. (3 refs.), Dep. v VINITI, 06.01.89, No. 155-B89.

■ Waving Surface with Cubical Parabolas

A waving surface with cubical parabolas has the waving line given in the polar coordinates

$$v(\varphi) = \sqrt{R^2 + p(1 - \cos n\varphi)},$$

at the foot cross section $z = 0$, where $v(\varphi)$ is a polar radius, $n = R/r$; n is a whole number (integer); φ is the polar angle.

Assume $p = 2r(R + r)$, then the internal vertexes of the waving line is placed at the foot circle that is $R \le v(\varphi) \le R + 2r$. Assume $p = -2r(R - r)$, then the external vertexes of the waving line is placed at the foot circle that is $R - 2r \le v(\varphi) \le R$, Fig. 1.

Besides, when $p = 2r(R + r)$, all vertexes of the waving line lie at the epicycloid that is obtained in the process of external rolling of a circle with a radius r above the circle with the radius R but when $p = -2r(R - r)$, all vertexes lie on the hypocycloid obtained in the process of internal rolling of a circle with a radius r along the circle with the radius R.

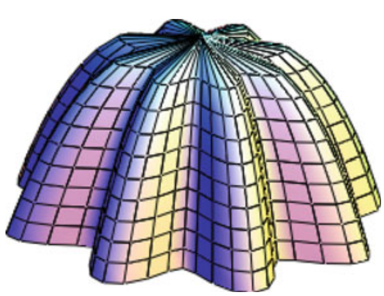


Fig. 1

The form of definition of the surface

(1) The parametrical generalized form of the definition:

$$x = x(\varphi, u) = u^{1/3}v(\varphi)\cos\varphi,$$
$$y = y(\varphi, u) = u^{1/3}v(\varphi)\sin\varphi,$$
$$z = z(u) = h(1 - u),$$

where u is a dimensionless parameter, $0 \le u \le 1$; $0 \le z \le h$; $0 \le \varphi \le 2\pi$.

There is *a cubic parabola* in any cross section of the waving surface by the plane passing through the axis Oz. The surface presented in Fig. 1 has $h = 1$ m; $R = 1$ m; $n = 8$.

Coefficients of the fundamental forms of the surface:

$$A^2 = u^{2/3}\left[\frac{p^2n^2\sin^2 n\varphi}{4v^2(\varphi)} + v^2(\varphi)\right],$$
$$F = \frac{pn\sin n\varphi}{6u^{1/3}},\ B^2 = \frac{v^2(\varphi)}{9u^{4/3}} + h^2;$$
$$L = \frac{hu^{2/3}}{\sqrt{A^2B^2 - F^2}}\left[\frac{3p^2n^2\sin^2 n\varphi}{4v^2(\varphi)} - \frac{pn^2}{2}\cos n\varphi + v^2(\varphi)\right],$$
$$M = 0,\ N = \frac{2hv^2(\varphi)}{9u^{4/3}\sqrt{A^2B^2 - F^2}}.$$

At the vertex of the surface with $u = 0$, we have $K = 0$, $H = 0$.

■ Waving Surface with Semi-cubical Parabolas

A waving surface with semi-cubical parabolas has the same foot line that the *waving surface with cubical parabolas* has.

The form of definition of the waving surface

(1) The parametrical generalized form of the definition:

$$x = x(\varphi, u) = u^{3/2}v(\varphi)\cos\varphi,$$
$$y = y(\varphi, u) = u^{3/2}v(\varphi)\sin\varphi,$$
$$z = z(u) = h(1 - u),$$

where u a dimensionless parameter;

$$0 \le u \le 1;\quad 0 \le z \le h;\quad 0 \le \varphi \le 2\pi.$$

The rest of the conventional signs are presented above in the part "Waving Surface with Cubical Parabolas". At the cross section of the waving surface by a plane passing through the axis Oz, *a semi-cubical parabola* lies. There is *a singular point* at the vertex of the surface. The surface presented in Fig. 1 has h = 3 m; R = 1 m; n = 5;

$$p = 2r(R + r).$$

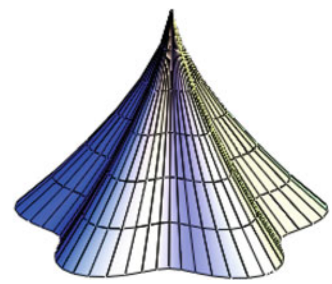


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = u^3\left[\frac{p^2n^2\sin^2 n\varphi}{4v^2(\varphi)} + v^2(\varphi)\right],\ F = \frac{3pnu^2\sin n\varphi}{4},$$
$$B^2 = \frac{9uv^2(\varphi)}{4} + h^2,$$
$$L = \frac{hu^3}{\sqrt{A^2B^2 - F^2}}\left[\frac{3p^2n^2\sin^2 n\varphi}{4v^2(\varphi)} - \frac{pn^2}{2}\cos n\varphi + v^2(\varphi)\right],$$
$$M = 0,\ N = \frac{-3huv^2(\varphi)}{4\sqrt{A^2B^2 - F^2}}.$$

25.1 Waving Chains with Elliptical Cross Sections Limited by Surfaces of the 2nd Order

Waving chains with elliptical cross sections are formed by the translational motion of an ellipse along a straight line brought into coincidence with the coordinate axis Oz. The ellipse all the time lies at the plane that is perpendicular to the axis Oz. The values of the semi-axes (b and c) of the ellipses and their ratio (b/c) change by the law chosen in advance in the process of motion of the ellipse along the axis Oz.

The forms of the definition of the surface

(1) Parametrical equations:

$$x = x(u, v) = [a + b\varphi(v)]\cos u,$$
$$y = y(u, v) = [a + c\psi(v)]\sin u,$$
$$z = z(v) = \omega(v),$$

where a, b, and c are constants. Having taken $(v) = \varphi\ (v) = r\ (v)$, we may obtain *the classical surfaces of the 2nd order* with elliptical cross sections. Depending on the type of the functions $r(v)$, it is possible to design an elliptical cylinder, an elliptical cone, an ellipsoid, and so on.

Coefficients of the fundamental forms of the surface:

$$A^2 = (a+b\varphi)^2\sin^2 u + (a+c\psi)^2\cos^2 u;$$
$$F = [-b(a+b\varphi)\varphi' c(a+c\psi)\psi']\sin u;$$
$$B^2 = b^2\varphi'^2\cos^2 u + c^2\psi'^2\sin^2 u + \omega'^2;$$
$$A^2B^2 - F^2 = \left[(a+b\varphi)^2\sin^2 u + (a+c\psi)^2\cos^2 u\right]\omega'^2 + \left[c(a+b\varphi)\psi'\sin^2 u + b(a+c\psi)\varphi'\cos^2 u\right]^2$$
$$L = -\frac{\omega'}{\sqrt{A^2B^2-F^2}}(a+b\varphi)(a+c\psi),$$
$$M = \frac{1}{\sqrt{A^2B^2-F^2}}[c(a+b\varphi)\psi' - b(a+c\psi)\varphi']\sin u\cos u,$$
$$N = \frac{1}{\sqrt{A^2B^2-F^2}}\left[c(a+b\varphi)(\psi''\omega' - \psi'\omega'')\sin^2 u + b(a+c\psi)(\varphi''\omega' - \varphi'\omega'')\cos^2 u\right].$$

The differentiation with respect to the v parameter is denoted by primes.

■ Waving Chain with Elliptical Cross Sections Limited by an Elliptical Cylinder

A waving chain with elliptical cross sections limited by an elliptical cylinder can be obtained if one will substitute the functions

$$\varphi = \varphi(v) = r\cos pv,$$
$$\psi = \psi(v) = r\sin pv, \quad \omega = tv,$$

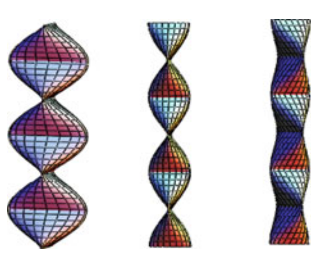
into the general parametric formulas of *waving chains with elliptical cross sections limited by surfaces of the 2nd order.* Here r, p, t are constants.

The form of the definition of the surface

(1) Parametrical equations:

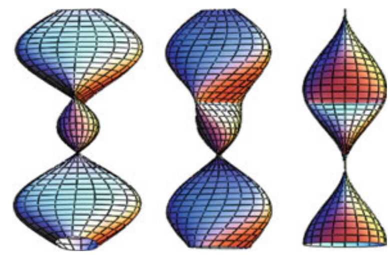
$$x = x(u,v) = [a + br\cos pv]\cos u,$$
$$y = y(u,v) = [a + cr\sin pv]\sin u,$$
$$z = z(v) = tv,$$

where a, b, and c are constants. The different forms of waving chain with elliptical cross sections limited by an elliptical cylinder are represented in Figs. 1, 2, 3 and 4. The geometrical parameters are shown under the corresponding figures. The values of the coefficients of the fundamental forms of the surface may be derived with the help of the general formulas given above in the part "Waving Chain with Elliptical Cross Sections Limited by Surfaces of the 2nd Order."



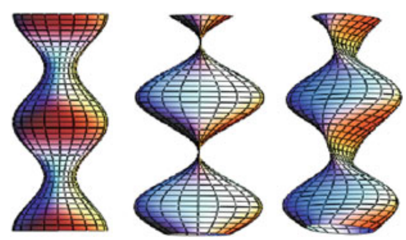
$a = 0$; $b = 2$; $c = 1$;
$0 \le v \le 15$; $t = 1$; $p = \pi/5$

Fig. 1



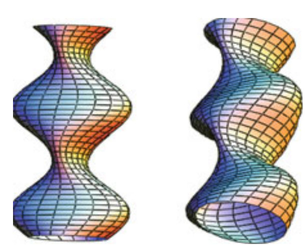
$a = 1$; $b = 2$; $c = 1$;
$0 \le v \le 15$; $t = 1$; $p = \pi/5$

Fig. 2



$a = 2$; $b = 2$; $c = 1$;
$0 \le v \le 20$; $t = 1$; $p = \pi/5$

Fig. 3



$a = 3$; $b = 2$; $c = 1$;
$0 \le v \le 20$; $t = 1$; $p = \pi/5$

Fig. 4

■ Waving Chain with Elliptical Cross Sections Limited by an Elliptical Cone

A waving chain with elliptical cross sections limited by an elliptical cone can be obtained if one will substitute the functions

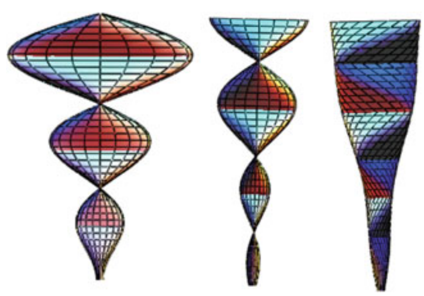
$$\begin{aligned} \varphi &= \varphi(v) = r(v)\cos pv, \\ \psi &= \psi(v) = r(v)\sin pv, \\ \omega &= tv, \\ r(v) &= \mathrm{d}v, \end{aligned}$$

into the general parametric formulas of *waving chains with elliptical cross sections limited by surfaces of the 2nd order*. Here p, t, d are constants.

The forms of the definition of the surface

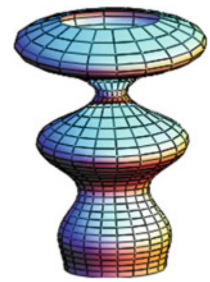
(1) Parametrical equations:

$$\begin{aligned} x &= x(u,v) = [a + b\mathrm{d}v\cos pv]\cos u, \\ y &= y(u,v) = [a + c\mathrm{d}v\sin pv]\sin u, \\ z &= z(v) = tv, \end{aligned}$$



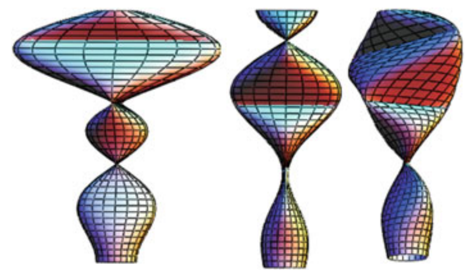
$a = 0;\ b = 2;\ c = 1;\ 0 \le v \le 15;$
$t = 1;\ p = \pi/5;\ d = 0.2$

Fig. 1



$a = 5;\ b = 2;\ 0 \le v \le 25;$
$c = t = 1;\ p = \pi/5;\ d = 0.2$

Fig. 2



$a = 1;\ b = 2;\ c = 1;\ 0 \le v \le 15;$
$t = 1;\ p = \pi/5;\ d = 0.2$

Fig. 3

where a, b, and c are constants. The different forms of the waving chain with the elliptical cross sections bounded by the elliptical cone are presented in Figs. 1, 2 and 3. The geometrical parameters are shown under the corresponding figures. The values of the coefficients of the fundamental forms of the surface may be derived with the help of the general formulas given above in the part "Waving Chain with Elliptical Cross Sections Limited by Surfaces of the 2nd Order".

■ Waving Chain with Elliptical Cross Sections Limited by an Elliptical Paraboloid

A waving chain with elliptical cross sections limited by an elliptical paraboloid can be obtained if one will substitute the functions

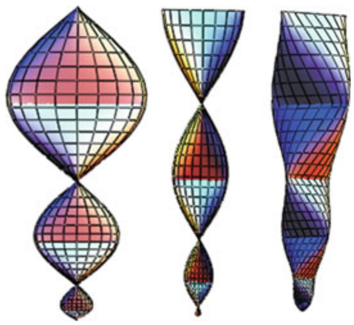
$$\begin{aligned} \varphi &= \varphi(v) = r(v)\cos pv, \\ \psi &= \psi(v) = r(v)\sin pv, \\ \omega &= tv^2, \\ r(v) &= dv, \end{aligned}$$

into the general parametric formulas of *waving chains with elliptical cross sections limited by surfaces of the 2nd order*. Here p, t, d are constants.

The form of the definition of the surface

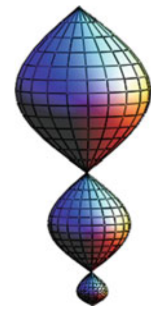
(1) Parametrical equations:

$$\begin{aligned} x &= x(u,v) = [a + bdv\cos pv]\cos u, \\ y &= y(u,v) = [a + cdv\sin pv]\sin u, \\ z &= z(v) = tv^2, \end{aligned}$$



$a = 0$; $b = 2$; $c = 1$; $0 \le v \le 15$;
$t = 0.5$; $p = \pi/5$; $d = 1$

Fig. 1



$a = 0$; $b = 2$; $c = 1$;
$0 \le v \le 15$; $t = 0.3$; $p = \pi$; $d = 1$

Fig. 2

where a, b and c are constants. The different forms of the waving chains with elliptical cross sections bounded by the elliptical paraboloid are presented in Figs. 1 and 2.

The geometrical parameters are shown under the corresponding figures.

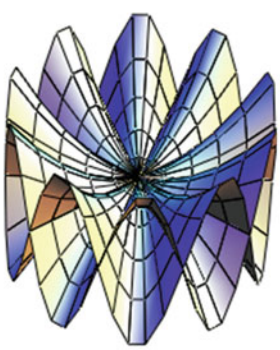
The values of the coefficients of the fundamental forms of the surface may be derived with the help of the general formulas given above in the part "Waving Chain with Elliptical Cross Sections Limited by Surfaces of the 2nd Order."

26 Surfaces of Umbrella Type

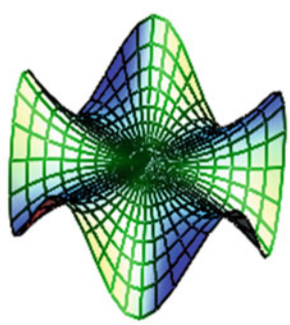
A cyclic symmetrical spatial structure formed from several identical elements is called *an umbrella dome*. Curves obtained as a result of the intersection of their middle surfaces are the generatrix curves of any dome-shaped surface of revolution. A dome-shaped surface of revolution, on which the contour curves of the elements of a dome are placed, is called *a contour surface*. The contour curves of the element are the curves bounding the contour of the middle surface of the element of the dome. The umbrella shells possess the increased rigidity, stability, architectural expressiveness.

The cyclic symmetrical surfaces consisting from several identical elements are called *surfaces of umbrella type*. But unlike an umbrella surface, the whole surface of umbrella type and all surfaces of the identical elements forming the whole surface are determined by one and the same explicit, implicit or parametrical equations.

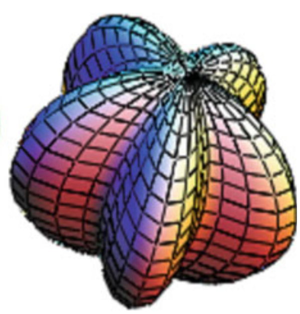
■ Surfaces of Umbrella Type Presented in the Encyclopedia



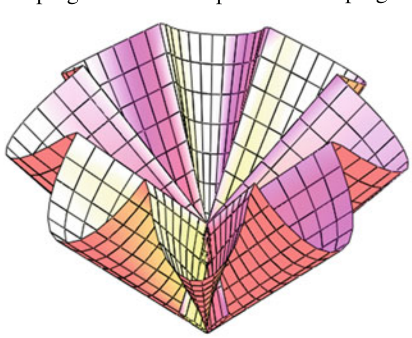
The surface formed by parabolas but with radial waves damping in the central point



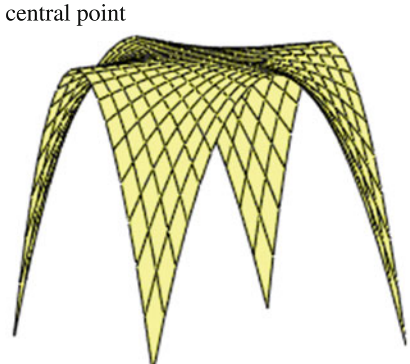
The surface formed by cubic parabolas but with radial waves damping in the central point



The waving ellipsoidal surface

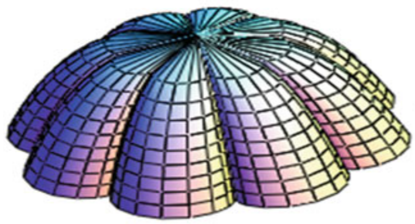


The Skidan's ruled surface

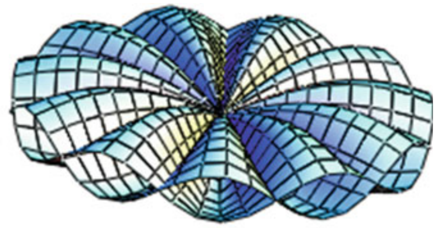


The crossed trough

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_26, © Springer International Publishing Switzerland 2015

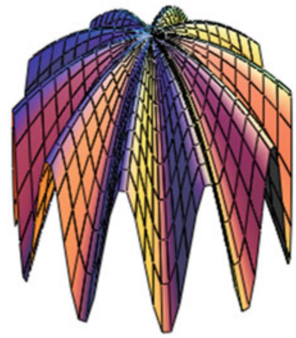


The surface of umbrella type formed by cubic parabolas on the cycloidal plan

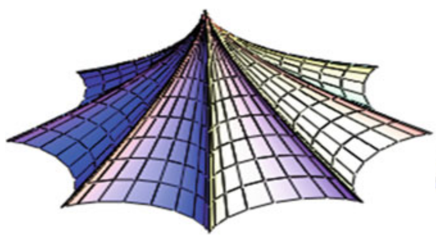


The surface of umbrella type with the sinusoidal generatrix

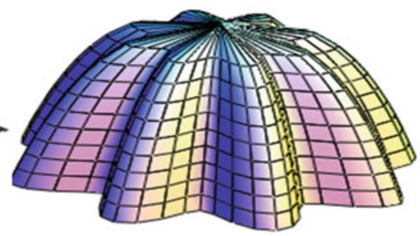
The corrugated paraboloid of revolution (with internal crimps)



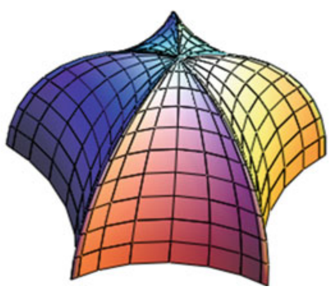
The paraboloid of revolution with radial waves



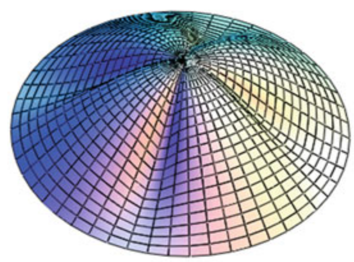
The surface of umbrella type formed by semi-cubic parabolas on the cycloidal plan



The waving surface with cubical parabolas



The sphere with cycloidal crimps



The waving surface with the pseudo Agnesi curls on the circular plan

The Literature on Geometry, the Application, and Analysis of Umbrella Shells and Shells of Umbrella Type

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Romanova VA. Features of the image of process of formation of surfaces in AutoCad system. Structural Mechanics of Engineering Constructions and Buildings. 2014; No. 3, p. 19-22.
PS: Additional literature is given at the corresponding pages of the Chap. "26. Surfaces of Umbrella Type".

■ Surface of Umbrella Type with Parabolic Generatrixes and with the Opening at the Vertex

The surface in question is created by the one-parametric family of parabolas lying at the planes of a pencil passing through the coordinate axis Oz. By the way, the axes of parabolas are placed in the horizontal plane $z = 0$ but the vertexes of parabolas are disposed at the circle of a radius a and with the center in the point O (0; 0; 0). At the vertex $z = h$, the surface has an *epicycloid* (Fig. 1):

$$x = x(\varphi) = (R + r)\cos\ \varphi - r\cos(1 + n)\varphi,$$
$$y = y(\varphi) = (R + r)\sin\ \varphi - r\sin(1 + n)\varphi$$

where n is a number of the external vertexes of the epicycloid; $n = R/r$; R is a radius of a circle, along which from the outside, a circle of radius r rolls and any point lying on it traces the epicycloid; φ is the angle of the Ox axis with the Oy axis (Fig. 2).

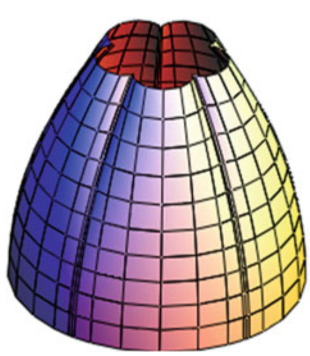

Fig. 1

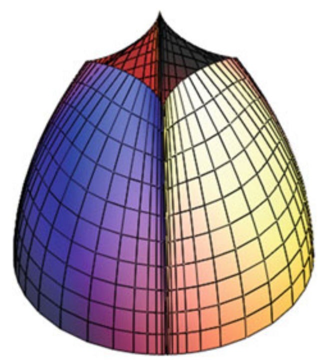

Fig. 2

At the vertex $z = h$, the surface may have *a hypocycloid*:

$$x = x(\varphi) = (R - r)\cos\ \varphi + r\cos(n - 1)\varphi,$$
$$y = y(\varphi) = (R - r)\sin\ \varphi - r\sin(n - 1)\varphi,$$

where n is a number of the vertexes of the hypocycloids; R is a radius of a circle inside of which, a circle of a radius r rolls and any point lying on it traces the hypocycloid; φ is the angle of the axis Ox with the axis Oy.

The form of the definition of the surface

(1) Parametrical equations:

$$x = x(z,\varphi) = \frac{a - [a - v(\varphi)]z^2}{h^2}\cdot\frac{x(\varphi)}{v(\varphi)},$$
$$y = y(z,\varphi) = \frac{a - [a - v(\varphi)]z^2}{h^2}\cdot\frac{y(\varphi)}{v(\varphi)},$$
$$z = z,$$

where

$$v(\varphi) = \sqrt{R^2 + p(1 - \cos\ n\varphi)}.$$

It is necessary to take

$$p = 2r(R + r); \quad a > R + 2r,$$

if the opening at the vertex is assumed in the form of an epicycloid (Fig. 1), and

$$p = -2r(R - r)\, a > R,$$

if the opening has the form of a hypocycloid.

■ Surface Formed by Parabolas but With Radial Waves Damping in the Central Point

A surface formed by parabolas but with radial waves damping in the central point is generated by the plane parabolas, the vertexes of which coincide with *a central fixed point* and another point belonging to the parabola moves along a circle changing the z-coordinate by a sine law.

The tangents put to the parabolas at the central point must remain in one plane all the time. The plane formed by the tangents to the parabolas at the central point is called *a datum plane*.

At every cross-section of the surface by a plane passing through the central point perpendicular to the datum plane, a parabola is located.

The form of the definition of the surface with the radial waves

(1) Parametrical equations:

$$x = x(u, v) = u\cos v,$$
$$y = y(u, v) = u\sin v,$$
$$z = z(u, v) = au^2 \sin(nv),$$

where v is the angle read from the Ox axis in the direction of the Oy axis; a = const; n is any number.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + 4a^2u^2\sin^2(nv),$$
$$F = 2a^2u^3 n \sin(nv)\cos(nv),$$
$$B^2 = u^2\left[1 + n^2a^2u^2\cos^2(nv)\right];$$
$$A^2B^2 - F^2 = u^2\left[1 + 4a^2u^2\sin^2(nv) + n^2a^2u^2\cos^2(nv)\right],$$
$$L = \frac{2au\sin(nv)}{\sqrt{A^2B^2 - F^2}}, \quad M = \frac{au^2 n\cos(nv)}{\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{au^3(2 - n^2)\sin(nv)}{\sqrt{A^2B^2 - F^2}};$$
$$K = \frac{a^2\left[2(2 - n^2)\sin^2(nv) - n^2\cos^2(nv)\right]}{\left[1 + 4a^2u^2\sin^2(nv) + n^2a^2u^2\cos^2(nv)\right]^2},$$
$$H = a\frac{\left[(4 - n^2)(1 + 2a^2u^2\sin^2(nv)) - 2a^2u^2n^2\right]}{2\left[1 + 4a^2u^2\sin^2(nv) + n^2a^2u^2\cos^2(nv)\right]^{3/2}}\sin(nv).$$

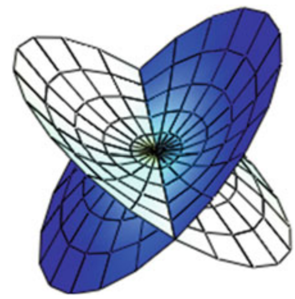

Fig. 1 $n = 1$

The surface is related to non-orthogonal, non-conjugate system of the curvilinear coordinates u, v.

Let n is an even number, then the surface does not have the points of self-intersection of surface when $v > 2\pi$.

Assume that n is an odd number, then the surface does not have the self-intersection of surface when $0 \le u \le u_0$ but *honeycombs* will appear if $-u_0 \le u \le u_0$.

In Fig. 1, the surface with $n = 1$, $0 \le v \le 2\pi$, $-1 \le u \le 1$ is shown.

Having taken $n = 2$, we can obtain *a flat saddle in the drum* (see also the Chap. “33. Saddle Surfaces”).

The surface with $n = 3$, $0 \le v \le 2\pi$, $-1 \le u \le 1$ is shown in Fig. 2.

Figure 3 represents the surface with $n = 3$, $0 \le v \le 2\pi$, $0 \le u \le 1$.

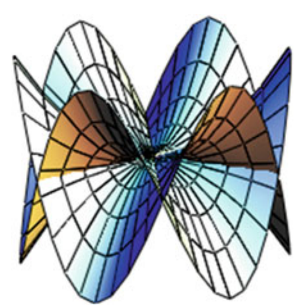

Fig. 2 $n = 3$

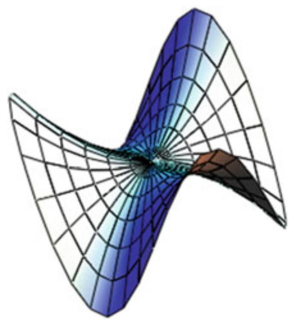

Fig. 3 $n = 3$

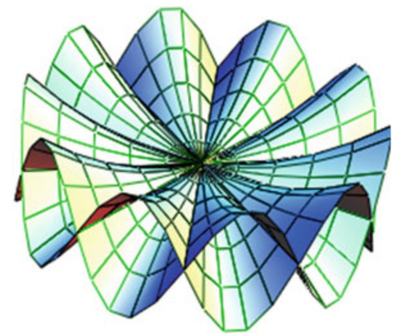

Fig. 4 $n = 5$

Let us assume $n = 5$; $0 \le v \le 2\pi$, $-1 \le u \le 1$, then we shall design the surface shown in Fig. 4.

Honeycombs will be also in the surfaces with fractional values of a parameter n when $v > 2\pi$.

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■ Waving Ellipsoidal Surface

Waving ellipsoidal surfaces are formed by the rotation of an ellipse with the simultaneous changing of its axial ratio. One of the principal axes of the ellipse coincides with the axis of rotation and remains by invariable and an extreme point of the second axis of the ellipse traces a circular sinusoid:

$$r(u) = a[1 + \mu \cos(pu)],$$

where μ is a ratio of the amplitude of the sinusoid to the radius a of the circle; p is a number of waves of the sinusoid.

A vector equation of the waving ellipsoidal surface may be written as:

$$\boldsymbol{\rho}(u, v) = a\{[1 + \mu \cos(pu)] \cos v\boldsymbol{h}(u) + \gamma \sin v\boldsymbol{k}\},$$

where $\boldsymbol{h}(u) = \boldsymbol{i} \cos u + \boldsymbol{j} \sin u$; $\gamma = b/a$ is a ratio of the semi-axes of the generatrix ellipses. The waving ellipsoidal surfaces under different parameters γ, μ, p are presented in Fig. 1.

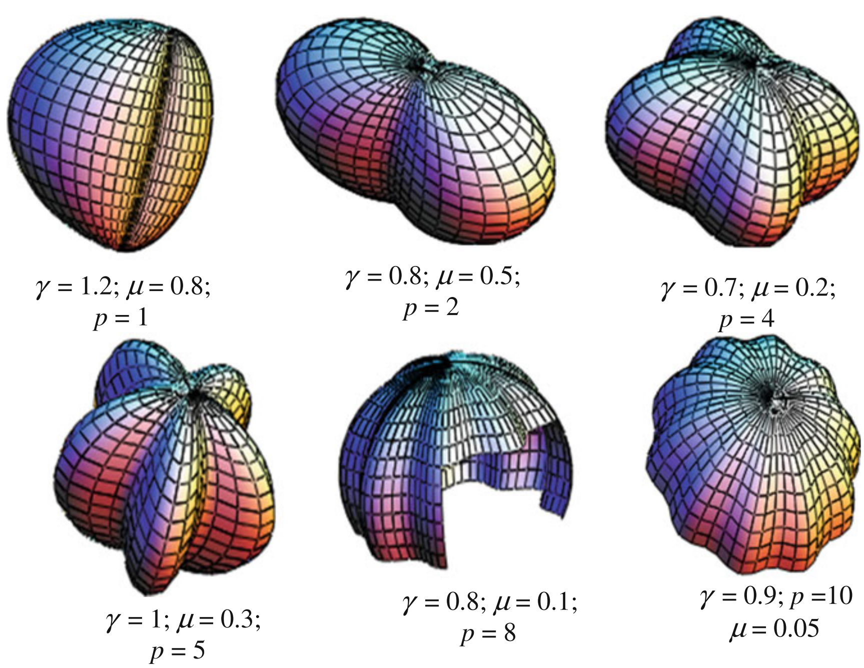


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2\left(\tilde{r}^2 + \tilde{r}'^2\right)\cos v,$$
$$F = a^2\tilde{r}\tilde{r}'\sin v\cos v,$$
$$B^2 = a^2\left(\tilde{r}^2\sin^2 v + \gamma^2\cos^2 v\right);$$
$$L = \frac{a\gamma}{\sigma}\left[\tilde{r}(\tilde{r}'' - \tilde{r}) - 2\tilde{r}'^2\right]\cos^2 v,\ M = 0,\ N = -\frac{ar^2}{\sigma}\gamma,$$

where

$$\sigma^2 = \gamma\left(\tilde{r}^2 + \tilde{r}'^2\right)\cos^2 v + \tilde{r}^4\sin^2 v,$$
$$\tilde{r} = \tilde{r}(u) = 1 + \mu\cos(pu).$$
$$\tilde{r}' = \partial\tilde{r}/\partial u = -\mu p\sin(pu),$$
$$\tilde{r}'' = \partial^2\tilde{r}/\partial u^2 = -\mu p\sin(pu).$$

The surface is related to non-orthogonal, conjugate system of the curvilinear coordinates.

The waving ellipsoidal surface may be given in the spherical system of coordinates:

$$\boldsymbol{\rho}(u,v) = R(u,v)\boldsymbol{e}(u,v),$$

where

$$e(u,v) = \boldsymbol{h}(u)\cos v + \boldsymbol{k}\sin v,$$

but

$$R(u,v) = b/\sqrt{\xi(u,v)},$$

$$\xi(u,v) = \sin^2 v + \frac{\gamma^2}{\tilde{r}^2}\cos^2 v.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = \left(\frac{\eta^2\tilde{r}'^2}{\xi^2\tilde{r}^2}\cos^2 v + 1\right)R^2\cos^2 v,$$
$$F = -\frac{\eta\tilde{r}'R^2}{\xi^2\tilde{r}}(1-\eta)\sin v\cos^3 v,$$
$$B^2 = \frac{R^2}{\xi^2}\left(\sin^2 v + \eta^2\cos^2 v\right);$$
$$L = \frac{\eta R}{\sigma\xi}\left[\left(1 - \frac{\tilde{r}''}{\tilde{r}}\right)\xi + \frac{3\tilde{r}'^2}{\tilde{r}^2}\sin^2 v\right]\cos^2 v,$$
$$M = \frac{\eta\tilde{r}'R}{\sigma\xi\tilde{r}}\sin v\cos v,\ N = \frac{\eta R}{\sigma\xi};$$
$$\sigma^2 = \sin^2 v + \eta^2\left(1 + \frac{\tilde{r}'^2}{\tilde{r}^2}\right)\cos^2 v$$

In this way of the definition of the surface, non-orthogonal non-conjugate system of the curvilinear coordinates u, v is used.

Additional Literature

Nasr Unes Abbushi. Wave-shaped domes. Structural Mechanics of Engineering Constructions and Buildings. 2002; No. 11, p. 49-58 (7 refs).

■ Paraboloid of Revolution with Radial Waves

A paraboloid of revolution with radial waves is formed by the plane parabolas the vertexes of which coincide with *a central fixed point*.

The tangents put to the parabolas at the central point must remain in one plane all the time. There is a parabola in every cross-section of the surface by a plane passing through the axis Oz.

The form of the definition of a surface of paraboloid of revolution with radial waves

(1) Parametrical equations:

$$x = x(u,v) = u\cos v,$$
$$y = y(u,v) = u\sin v,$$
$$z = z(u,v) = [a\sin(nv) + b]u^2,$$

where v is the angle read from the axis Ox in the direction to the axis Oy; a = const is an amplitude of a wave; n is a number of the vertexes of the waves; b is a constant parameter of the datum paraboloid of revolution.

Coefficients of the fundamental forms of the surface:

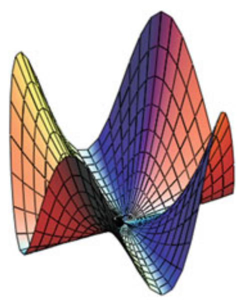
$$A^2 = 1 + 4u^2[a\sin(nv) + b]^2,$$
$$F = 2au^3n[a\sin(nv) + b]\cos(nv),$$
$$B^2 = u^2\left[1 + n^2a^2u^2\cos^2(nv)\right];$$
$$A^2B^2 - F^2 = u^2\left[A^2 + n^2a^2u^2\cos^2(nv)\right];$$
$$L = \frac{2u[a\sin(nv) + b]}{\sqrt{A^2B^2 - F^2}},$$
$$M = \frac{au^2n\cos(nv)}{\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{u^3[2(a\sin nv + b) - an^2\sin nv]}{\sqrt{A^2B^2 - F^2}};$$
$$K = \frac{2(a\sin nv + b)[2(a\sin nv + b) - an^2\sin nv] - a^2n^2\cos^2 nv}{\left[A^2 + n^2a^2u^2\cos^2(nv)\right]^2}.$$

When b = 0, the surface in question becomes *a surface formed by parabolas but with radial waves damping in the central point* (see also the corresponding page presented before).

Let $a = b$, then the lower vertexes of the waves of the surface will lie at the plane xOy (Fig. 1). Assume $b > a$, then the surface in question will be lying at the area of the positive values of the z ordinate (Fig. 2). If $b < a$, then the surface has both the positive and the negative values of the z ordinate (Fig. 3).

When $a = 0$, a paraboloid of revolution with radial waves degenerates into *a paraboloid of revolution.* If n is an even integer, then the surface does not have the points of self-intersection of surface when $v > 2\pi$. But if n is an odd integer, then the surface does not have the points of self-intersection of surface, when $0 \le u \le u_0$, but *honeycombs* will appear, when $-u_0 \le u \le u_0$. Figure 4 represents the surface with $n = 2$, $0 \le v \le 2\pi$, $-1 \le u \le 1$. The surface with $n = 3$, $0 \le v \le 2\pi$, $-1 \le u \le 1$. is given in Fig. 5.

If $n = 6$, $0 \le v \le 2\pi$, $0 \le u \le 1$, the surface has the form shown in Fig. 6. In Fig. 7, the surface has $n = 3$, $0 \le v \le 2\pi$, $0 \le u \le 1$.

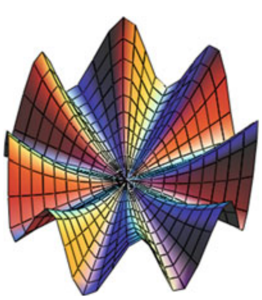


$a = b = 1; n = 4$

Fig. 1 $0 \le u \le 1$

$a = 1; b = 1.5; n = 5$

Fig. 2 $0 \le u \le 1$

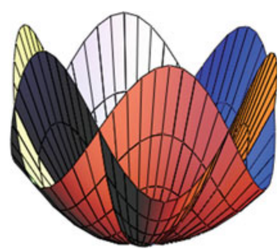


$a = 0.8; b = 0.5; n = 8$

Fig. 3 $0 \le u \le 1$

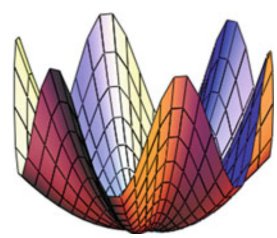
$a = 0.8; b = 1.5; n = 2$

Fig. 4 $-1 \le u \le 1$



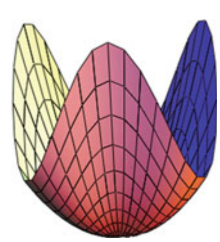
$a = 0.5; b = 1; n = 3$

Fig. 5 $-1 \le u \le 1$



$a = 0.8; b = 1; n = 6$

Fig. 6 $0 \le u \le 1$



$a = 1; b = 1.5;$
$n = 3$

Fig. 7 $0 \le u \le 1$

Honeycombs will be in the surfaces with fractional values of a parameter n when $v > 2\pi$.

All surfaces shown in Figs. 4, 5, 6 and 7 have $b > a$.

Additional Literature

Basilevich IA. Drawing of surfaces with the help of computer. Prikl. Geom. i Ingenern. Grafika. Kiev. 1975; Iss. 20, p. 157-162 (2 refs).

■ Surface of Umbrella Type with Parabolic Generatrixes and With a Circular Opening at the Vertex

The surface in question has *an epicycloid* (Fig. 1a)

$$X = X(\varphi) = (R + r)\cos\ \varphi - r\cos(1 + n)\varphi,$$
$$Y = Y(\varphi) = (R + r)\sin\ \varphi - r\sin(1 + n)\varphi,\ Z = 0,$$

at the foot cross-section $z = 0$, where n is a number of the external vertexes of the epicycloid; $n = R/r$; $2r$ is the maximum amplitude of the crimps at the foot of the surface, R is a radius of the circle outside of which a circle of a radius r rolls and any point lying on it traces the epicycloid; φ is an angle read from the axis Ox in the direction of the axis Oy.

The surface may have *a hypocycloid* (Fig. 1b)

$$X = X(\varphi) = (R - r)\cos\ \varphi + r\cos(n - 1)\varphi,$$
$$Y = Y(\varphi) = (R - r)\sin\ \varphi - r\sin(n - 1)\varphi, Z = 0$$

at the foot contour $z = 0$ with the vertexes turned only inside the circular base, n is a number of the vertexes of the hypocycloid; R is a radius of a circle inside of which, a circle of a radius r rolls and any point lying on it traces the hypocycloid; φ is the angle of the axis Ox with the axis Oy.

At any cross-section of the surface in question by a plane passing through the axis of the surface coinciding with the coordinate axis Oz, *the parabolas*

$$z = h - b(\varphi)(x - a)^2$$

lie, $0 \le z \le h$; $b(\varphi)$ is a changing parameter of the parabolas. The top boundary of the surface is a circle of a radius a with the center placed in the point with the coordinates (0; 0; h). Hence the vertexes of the generatrix parabolas are placed on the boundary circle.

The form of the definition of the surface of umbrella type

(1) Parametrical equations:

$$x = x(z, \varphi) = \left\{a + [v(\varphi) - a]\sqrt{\frac{h - z}{h}}\right\}\frac{X(\varphi)}{v(\varphi)},$$
$$y = y(z, \varphi) = \left\{a + [v(\varphi) - a]\sqrt{\frac{h - z}{h}}\right\}\frac{Y(\varphi)}{v(\varphi)},\ z = z,$$

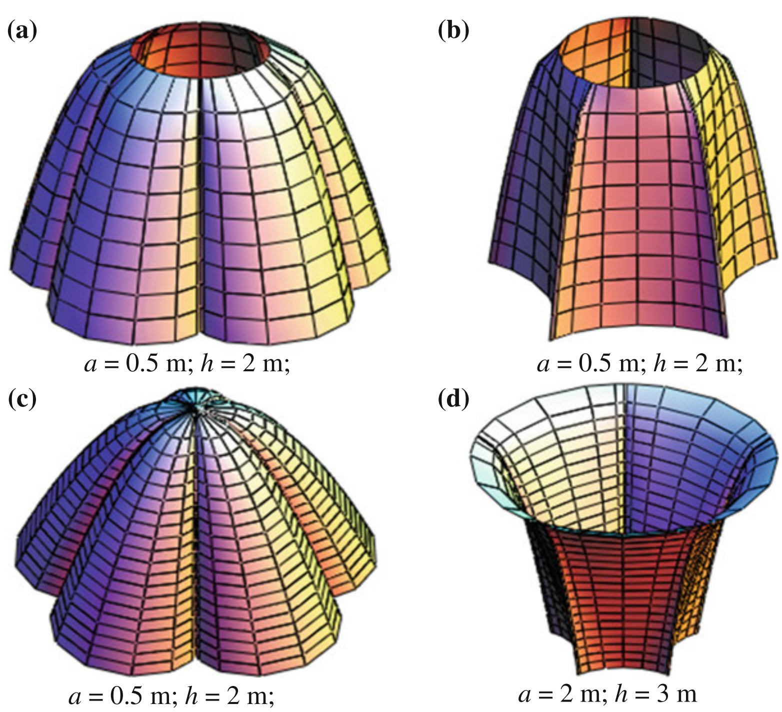


Fig. 1 $R = 1$ m; $n = 5$

where

$$v(\varphi) = \sqrt{R^2 + p(1 - \cos\ n\varphi)}$$

is a polar radius of the hypocycloid or epicycloid lying at the foot of the surface;
and

$$v(\varphi) = \sqrt{X^2(\varphi) + Y^2(\varphi)};$$

$0 \le z \le h$; but the angle φ is not a polar angle; $0 \le \varphi \le 2\pi$; h is a maximum height of the surface of umbrella type.

It is necessary to assume $p = 2r(R + r)$, $a < R$, if the base is taken in the form of epicycloid (Fig. 1a) or $p = -2r(R-r)$, $a < R-2r$, if the base has the form of hypocycloid (Fig. 1b). So, at the cross-section of the surface by a plane $z = 0$, the epicycloid or hypocycloid is placed depending on of the value of the p parameter and on the coordinates $X(\varphi)$and $Y(\varphi)$.

The boundary circle $x^2 + y^2 = a^2$ lies at the cross-section of the surface by a plane $z = h$.

If $a = 0$, then the surface in question degenerates into *a paraboloid of revolution with the cycloidal crimps* (Fig. 1c). But if one assumes $p = -2r(R-r)$, $a > R-2r$, then he will obtain a surface with a hypocycloid at the foot (Fig. 1d).

Reference

Krivoshapko SN. New examples of surfaces of umbrella type and their coefficients of fundamental forms of surface. Structural Mechanics of Engineering Constructions and Buildings. 2005; No. 2, p. 6-14.

26.1 Surfaces of Umbrella Type with the Central Plane Point

■ Surface Formed by Cubic Parabolas but with Radial Waves Damping in the Central Point

A Surface formed by cubic parabolas but with radial waves damping in the central point is generated by the plane cubic parabolas, *an inflection point* of which coincides with the *central fixed point* and the other point belonging to the cubic parabola moves in a circle changing the ordinate by a sine law. The tangents passed to the cubic parabolas at the central point must remain in one plane all time. The plane formed by the tangents to the cubic parabolas at the central point is called *a datum* (*base*) *plane*. At every cross-section of the surface by a plane passing through the central point perpendicular to the datum plane, a cubic parabola is located.

The form of the definition of the surface

(1) Parametrical equations:

$$x = x(u, v) = u\cos v,$$
$$y = y(u, v) = u\sin v,$$
$$z = z(u, v) = au^3 \sin(nv),$$

where v is an angle read from the axis Ox into the direction of the axis Oy; a = const; n is any number.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + 9a^2u^4\sin^2(nv),$$
$$F = 3a^2u^5n\sin(nv)\cos(nv),$$
$$B^2 = u^2 + n^2a^2u^6\cos^2(nv);$$
$$A^2B^2 - F^2 = u^2\left[1 + 9a^2u^4\sin^2(nv) + n^2a^2u^4\cos^2(nv)\right];$$
$$L = \frac{6au^2\sin(nv)}{\sqrt{A^2B^2 - F^2}},$$
$$M = \frac{2au^3n\cos(nv)}{\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{au^4(3 - n^2)\sin(nv)}{\sqrt{A^2B^2 - F^2}};$$
$$K = 2a^2u^6\frac{\left[3(3 - n^2)\sin^2(nv) - 2n^2\cos^2(nv)\right]}{(A^2B^2 - F^2)^2};$$
$$H = \frac{0,5au^4\sin(nv)}{(A^2B^2 - F^2)^{3/2}}\left[9 - 6a^2n^2u^4 - n^2 + 27a^2u^4\sin^2(nv) - 3a^2u^4n^2\sin^2(nv)\right].$$

The surface is related to the non-orthogonal non-conjugate system of the curvilinear coordinates u, v.

At the central point with the coordinate $u = 0$, the Gaussian and mean curvatures of the surface are equal to zero $(K = H = 0)$. Hence, the central point is *isolated plane point*. In Fig. 1, the surface with $n = 2, 0 \le v \le 2\pi$ is shown. When $n = 3$, we obtain *the monkey saddle* (see also “Saddle Surfaces”).

The surface with $n = 4, 0 \le v \le 2\pi$ is represented in Fig. 2. Figure 3 shows the surface with $n = 5$.

Fig. 1

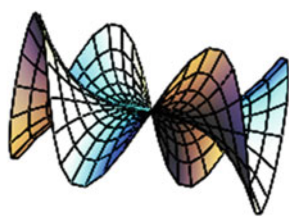


Fig. 2

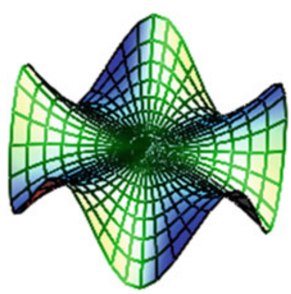


Fig. 3

Let n is an odd integer, then the surface does not have the points of self-intersection of surface when $v > 2\pi$. Assume that n is an even integer, then the surface does not have the self-intersection of surface when $0 \le u \le u_0$, but *honeycombs* will appear if $-u_0 \le u \le u_0$ (Fig. 4; $n = 2; -1 \le u \le 1$).

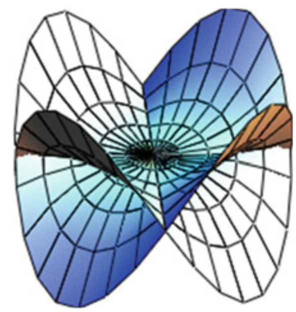


Fig. 4

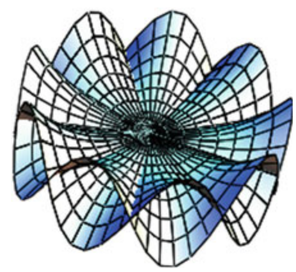


Fig. 5

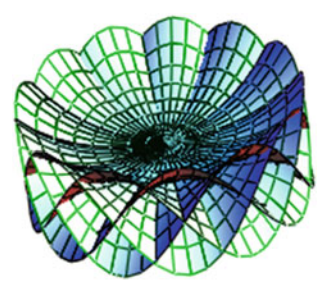


Fig. 6

Honeycombs will be also in the surfaces with fractional values of a parameter n, when $v > 2\pi$. Fig. 5 shows the surface with $n = 4.5$; Fig. 6 is with $n = 3.25$.

■ Crossed Trough

A crossed trough consists of four identical petals divided by the mutually perpendicular straight lines lying in one plane. The crossed trough is a surface of the forth order.

Forms of definition of the crossed trough

(1) Explicit form of the definition (Fig. 1):

$$z = cx^2y^2.$$

At the cross-sections of the surface by the coordinate planes $x = 0$ and $y = 0$, the straight lines coinciding with coordinate axes Oy and Ox are placed. In the sections of the trough by the planes $z = z_0 = \text{const}$, *the equilateral hyperbolas*

$$y = \pm\sqrt{\frac{z_0}{c}}\frac{1}{x} = \pm\frac{k}{x}$$

are disposed.

Coefficients of the fundamental forms of the surface:

$$\begin{aligned}
A^2 &= 1 + 4c^2x^2y^4,\\
F &= 4c^2x^3y^3,\\
B^2 &= 1 + 4c^2x^4y^2;\\
A^2B^2 - F^2 &= 1 + 4c^2x^2y^2(x^2 + y^2);\\
L &= \frac{2cy^2}{\sqrt{1 + 4c^2x^2y^2(x^2 + y^2)}},\\
M &= \frac{4cxy}{\sqrt{1 + 4c^2x^2y^2(x^2 + y^2)}},\\
N &= \frac{2cx^2}{\sqrt{1 + 4c^2x^2y^2(x^2 + y^2)}};\\
K &= \frac{-12c^2x^2y^2}{[1 + 4c^2x^2y^2(x^2 + y^2)]^2} \le 0,\\
H &= \frac{c(x^2 + y^2 - 8c^2x^4y^4)}{[1 + 4c^2x^2y^2(x^2 + y^2)]^{3/2}}.
\end{aligned}$$

At the central point with the coordinates $x = 0,\ y = 0$, the surface has

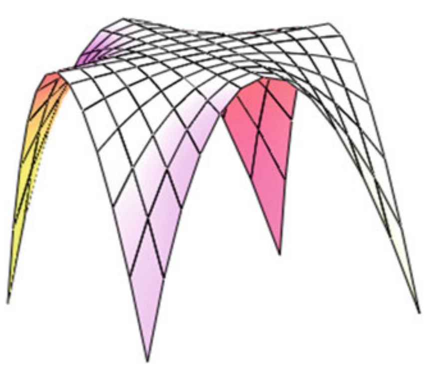


Fig. 1

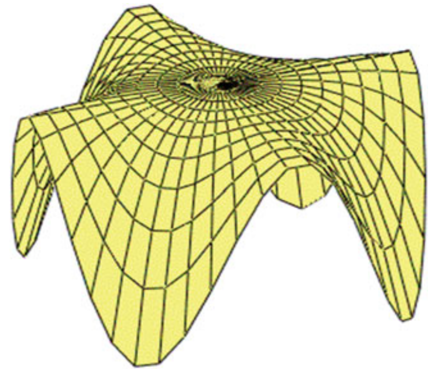


Fig. 2

$$K = H = 0$$

and hence this point is a plane point. Along the coordinate axes Ox and Oy, the Gaussian curvature of the surface is equal to zero.

(2) Parametrical equations (Fig. 2):

$$x = x(u, v) = u\sin v, \quad y = y(u, v) = u\cos v,$$
$$z = z(u, v) = \frac{cu^4\sin^2(2v)}{4}$$

where $0 \le v \le 2\pi$; c is a constant.

Coefficients of the fundamental forms of the surface:

$$\begin{aligned}
A^2 &= 1 + c^2u^6\sin^4(2v),\\
F &= c^2u^7\sin^3(2v)\cos(2v),\\
B^2 &= u^2 + c^2u^8\sin^2(2v)\cos^2(2v);\\
A^2B^2 - F^2 &= u^2\left[1 + c^2u^6\sin^2(2v)\right];\\
L &= \frac{-3cu^2\sin^2(2v)}{\sqrt{1 + c^2u^6\sin^2(2v)}},\\
M &= \frac{-3cu^3\sin(2v)\cos(2v)}{\sqrt{1 + c^2u^6\sin^2(2v)}},\\
N &= \frac{cu^4\left[\sin^2(2v) - 2\cos^2(2v)\right]}{\sqrt{1 + c^2u^6\sin^2(2v)}};\\
K &= -\frac{3c^2u^4\sin^2(2v)}{\left[1 + c^2u^6\sin^2(2v)\right]^2} \le 0.
\end{aligned}$$

The curvilinear coordinates lines $v = 0$ and $v = \pi/2$ are the straight lines. The Gaussian curvature of surface is equal to zero along these lines.

Additional Literature

von Seggern D. CRC D. CRC. Standard Curves and Surfaces. Boca Raton, FL: CRC Press. 1993; p. 286.

■ Umbrella Surface Formed by Biquadratic Parabolas and with Astroidal Level Line

Assume that an astroid

$$x = a\cos^3 t,\ y = a\sin^3 t$$

is disposed at the plane xOy, then its polar radius can be written in the following form:

$$\rho = a\sqrt{\sin^6 t + \cos^6 t},$$

where parameter t is equal to the angle of the axis Ox with the straight line joining the centers of the fixed circle with a radius R and a circle with a radius r rolling inside the circle of the radius R. A point belonging to the circle of the radius r traces the astroid; $R/r = 4$.

Forms of definition of the surface

(1) Parametrical equations:

$$x = x(t,u) = au^{1/4}\cos^3 t,$$
$$y = y(t,u) = au^{1/4}\sin^3 t,$$
$$z = z(u) = H(1-u),$$

where u is a dimensionless parameter; $0 \le u \le 1; 0 \le t \le 2\pi$; H is the rise of the surface, i.e. the distance the base of the surface from the highest point of the surface taken along the axis Oz.

Coefficients of the fundamental forms of the surface and its principle curvatures:

$$A^2 = \frac{9}{4}a^2\sqrt{u}\sin^2 2t,$$
$$F = -\frac{3a^2}{16\sqrt{u}}\sin 4t,$$
$$B^2 = \frac{a^2}{16u^{3/2}}(\sin^6 t + \cos^6 t) + H^2;$$
$$A^2B^2 - F^2 = \frac{9}{4u}a^2\sin^2 t\cos^2 t\left[\frac{a^2}{16}\sin^2 t\cos^2 t + H^2u^{3/2}\right];$$
$$L = -\frac{9a^2H\sqrt{u}}{4\sqrt{A^2B^2-F^2}}\sin^2 2t,$$
$$M = 0,\ N = \frac{9a^2H}{64u^{3/2}\sqrt{A^2B^2-F^2}}\sin^2 2t;$$
$$k_t = \frac{-9H}{\sqrt{A^2B^2-F^2}},$$
$$k_u = \frac{9H\sin^2 2t}{4(\sin^6 t + \cos^6 t + 16H^2u^{3/2}/a^2)\sqrt{A^2B^2-F^2}},$$
$$K = \frac{-16^2uH^2}{(a^2\sin^2 t\cos^2 t + 16H^2u^{3/2})^2} \le 0.$$

The surface in question of the negative Gaussian curvature is given in non-orthogonal, conjugate system of the curvilinear coordinate u, t. Only at the point $u = 0$, the surface has the zero values of the Gaussian and mean curvatures, hence the vertex of the surface is *a plane point*.

In Fig. 1, the surface with $a = 1$ m, $H = 2$ m, $0.05 \le u \le 1, 0 \le t \le 2\pi$ is shown.

(2) Explicit form of the definition:

$$z = \left\{1 - \left[\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3}\right]^6\right\}H.$$

(3) Parametrical equations:

$$x = x(r,\varphi) = r\cos\varphi,\ y = y(r,\varphi) = r\sin\varphi,$$
$$z = z(r,\varphi) = \left[1 - \left(\frac{r}{a}\right)^4\left(\cos^{2/3}\varphi + \sin^{2/3}\varphi\right)^6\right]H.$$

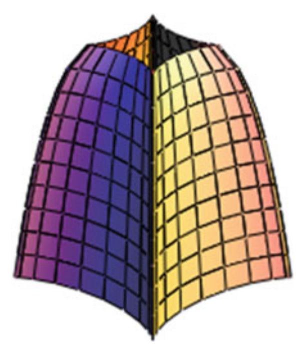


Fig. 1

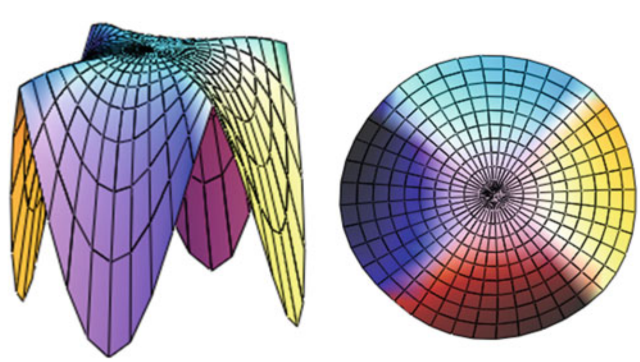


Fig. 2

Figure 2 shows the surface with a = 1 m, H = 2 m, $0 \le r \le a; 0 \le \varphi \le 2\pi$; φ is the angle read from the Ox axis in the direction of the axis Oy; $\varphi \ne t$.

The surface is given in non-orthogonal, conjugate curvilinear coordinate system.

Additional Literature

Ch.A. Bock Hyeng, Krivoshapko SN. Umbrella-Type Surfaces in Architecture of Spatial Structures. IOSR Journal of Engineering (IOSRJEN). 2013; Vol. 3, Iss. 3, p. 43-53.

■ Surface of Umbrella Type Formed by Cubic Parabolas on a Cycloidal Plan

A surface of umbrella type formed by cubic parabolas on a cycloidal plan has *an epicycloid* (Fig. 1)

$$x = x(\varphi) = (R + r)\cos\varphi - r\cos(1 + n)\varphi,$$
$$y = y(\varphi) = (R + r)\sin\varphi - r\sin(1 + n)\varphi,$$
$$z = 0$$

at the foot cross-section z = 0. Here n is a number of the external vertexes of the epicycloid; $n = R/r$; $2r$ is a maximal amplitude of the crimps at the base of the surface, R is a radius of the circle outside of which a circle of a radius r rolls and any point lying on it traces the epicycloid; φ is an angle read from the axis Ox in the direction of the axis Oy.

The surface may have *a hypocycloid*

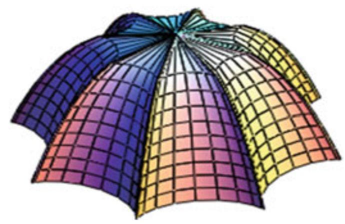
$$x = x(\varphi) = (R - r)\cos\varphi + r\cos(n - 1)\varphi,$$
$$y = y(\varphi) = (R - r)\sin\varphi - r\sin(n - 1)\varphi,$$
$$z = 0$$

at the foot contour $z = 0$ with the vertexes turned only inside the circular base (Fig. 2), n is a number of the vertexes of the hypocycloid; R is a radius of a circle inside of which, a circle of a radius r rolls and any point lying on it traces the hypocycloid; φ is the angle of the axis Ox with the axis Oy.

At any cross-section of the surface in question by a plane passing through the axis of the surface coinciding with the coordinate axis Oz, *the cubic parabolas*

$n = 8$; $h = 2$; $R = 1$

Fig. 1



$n = 8$; $h = 1$; $R = 1$

Fig. 2

$$z = h - a(\varphi)x^3$$

lie; $0 \le z \le h$; $a(\varphi)$ is a changing parameter of the cubic parabolas.

Forms of definition of the surface of umbrella type

(1) Parametrical form of the definition of the surface of umbrella type with the epicycloid at the base (surface with the external crimps Fig. 1):

$$x = x(u,\varphi) = u^{1/3}[(R + r)\cos\varphi - r\cos(n + 1)\varphi],$$
$$y = y(u,\varphi) = u^{1/3}[(R + r)\sin\varphi - r\sin(n + 1)\varphi],$$
$$z = z(u) = h(1 - u),$$

where $0 \le u \le 1$; u is a dimensionless parameter; $0 \le z \le h$; h is a maximum height of the surface; $0 \le \varphi \le 2\pi$. At any cross-section of the surface by the plane u = const, an epicycloid is placed. The coordinate line u = 1 coincides with the foot epicycloid.

Coefficients of the fundamental forms of the surface:

$$A^2 = u^{-4/3}\left[R^2 + 2r(R + r)(1 - \cos n\varphi)\right]/9 + h^2,$$
$$F = -u^{-1/3}R(R + r)\sin n\varphi/3,$$
$$B^2 = 2u^{2/3}(R + r)^2(1 - \cos n\varphi);$$
$$L = \frac{-2h(R + r)(R + 2r)}{9u^{4/3}\sqrt{A^2B^2 - F^2}}(1 - \cos n\varphi),$$
$$M = 0,\ N = -\frac{hu^{2/3}(R + r)^2(R + 2r)}{r\sqrt{A^2B^2 - F^2}}(1 - \cos n\varphi);$$
$$K = \frac{2h^2(R + r)^3(R + 2r)^2(1 - \cos n\varphi)^2}{9u^{2/3}r(A^2B^2 - F^2)^2} \ge 0;$$
$$K(u = 0) = 0,\ H(u = 0) = 0.$$

(2) Parametrical form of the definition of the surface of umbrella type with a hypocycloid at the base of the surface (surface with the internal crimps, Fig. 2):

$$x = x(u,\varphi) = u^{1/3}[(R-r)\cos\varphi + r\cos(n-1)\varphi],$$
$$y = y(u,\varphi) = u^{1/3}[(R-r)\sin\varphi - r\sin(n-1)\varphi,$$
$$z = z(u) = h(1-u),$$

$0 \le u \le 1;\ 0 \le z \le h$; h is a maximal height of the surface; $n \ne 2;\ 0 \le \varphi \le 2\pi$.

At any cross-section of the surface by the plane u = const, a hypocycloid lies.

Coefficients of the fundamental forms of the surface:

$$A^2 = u^{-4/3}[R^2 - 2r(R-r)(1-\cos n\varphi)]/9 + h^2,$$
$$F = -u^{-1/3}R(R-r)\sin n\varphi/3,$$
$$B^2 = 2u^{2/3}(R-r)^2(1-\cos n\varphi);$$
$$L = \left[-2u^{-4/3}h(R-r)(R-2r)/9\right](1-\cos n\varphi)/\sigma,$$
$$M = 0,\ N = \left[u^{2/3}h(R-r)^2(n-2)\right](1-\cos n\varphi)/\sigma;\ K \le 0,$$

where $\sigma^2 = A^2B^2 - F^2$. The plane point is located at the vertex of the surface (Fig. 2).

Reference

Krivoshapko S.N. Geometrical investigation of the surfaces of umbrella type. Structural Mechanics of Engineering Constructions and Buildings. 2005; No. 1, p.11-17 (4 refs).

26.2 Surfaces of Umbrella Type with Singular Central Point

■ Surface of Umbrella Type with a Sinusoidal Generatrix

*A surface of umbrella type with a sinusoidal generatrix*is formed by the sinusoids *the inflection points* of which pass through the fixed point at the center of the surface but at the same time, another point of the generatrix sinusoid moves in a circle changing the ordinate and its amplitude by the law of sine. The plane formed by the numerical straight lines of the generatrix sinusoids and passing through the central point is called *a datum* (*base*) *plane* of the surface. Any plane passing through the central point perpendicular to the base plane crosses the surface along a sinusoid.

The form of the definition of the surface

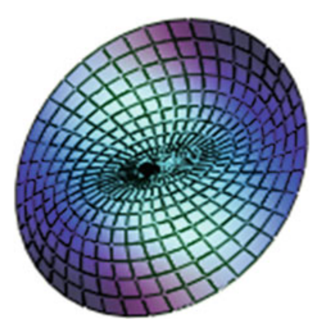
(1) Parametrical equations:

$$x = x(u,v) = u\cos v,$$
$$y = y(u,v) = u\sin v,$$
$$z = z(u,v) = a\sin u\sin(nv),$$

where a is a maximal amplitude of the generatrix sinusoid; $-a \le z \le a$.

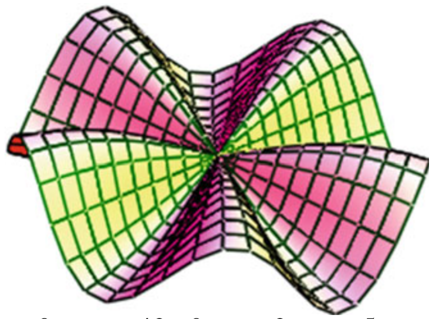
Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + a^2\cos^2 u\sin^2(nv),$$
$$F = 0.25na^2\sin(2v)\sin(2nv),$$
$$B^2 = u^2 + n^2a^2\sin^2 u\cos^2(nv);$$
$$A^2B^2 - F^2 = u^2\left[1 + a^2\cos^2 u\sin^2(nv)\right] + n^2a^2\sin^2 u\cos^2(nv);$$
$$L = \frac{-au\sin u\sin(nv)}{\sqrt{A^2B^2 - F^2}},$$
$$M = \frac{an\cos(nv)(u\cos u - \sin u)}{\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{au\sin(nv)}{\sqrt{A^2B^2 - F^2}}(u\cos u - n^2\sin u);$$
$$K = \left[-u^2(u\cos u - n^2\sin u)\sin u\sin^2(nv) - n^2(u\cos u - \sin u)^2\cos^2(nv)\right]\frac{a}{\Sigma^2},$$
$$\Sigma = u^2\left[1 + a^2\cos^2 u\sin^2(nv)\right] + n^2a^2\sin^2 u\cos^2(nv).$$



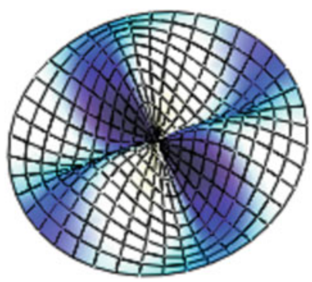
$0 \le u \le \pi,\ 0 \le v \le 2\pi,\ n = 1$

Fig. 1



$0 \le u \le \pi/2,\ 0 \le v \le 2\pi,\ n = 5$

Fig. 2

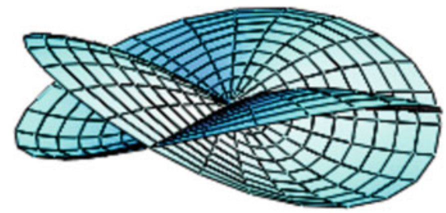


$0 \le u \le \pi,\ 0 \le v \le 2\pi,\ n = 3$

Fig. 3

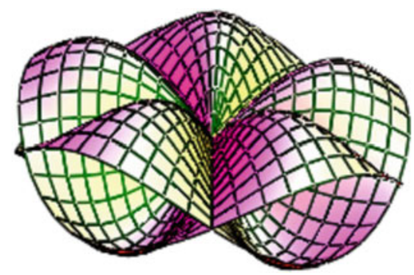
If n is an odd number, then the surface in question does not have the self-intersections when $v > 2\pi$ (Figs. 1, 2 and 3). If n is an even number or a not whole number, then the surface crosses itself when $v > 2\pi$ (Figs. 4 and 5).

At the central point of the surface ($u = 0$), the Gaussian curvature of surface becomes indefinite.



$0 \le u \le \pi/2,\ 0 \le v \le 4\pi;\ n = 1.5$

Fig. 4



$0 \le u \le 2\pi/5,\ 0 \le v \le 4\pi,\ n = 2.5$

Fig. 5

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■ Surface of Umbrella Type Formed by Semi-Cubic Parabolas on a Cycloidal Plan

A surface of umbrella type formed by semi-cubic parabolas on a cycloidal plan has *an epicycloid* (Fig. 1).

$$x = x(\varphi) = (R + r)\cos\varphi - r\cos(1 + n)\varphi,$$
$$y = y(\varphi) = (R + r)\sin\varphi - r\sin(1 + n)\varphi, \quad z = 0$$

at the foot cross-section $z = 0$. Here n is a number of the external vertexes of the epicycloid; $n = R/r$; $2r$ is a maximum amplitude of the crimps at the foot of the surface; R is a radius of the circle outside of which a circle of a radius r rolls and any point lying on it traces the epicycloid; φ is an angle read from the axis Ox in the direction of the axis Oy.

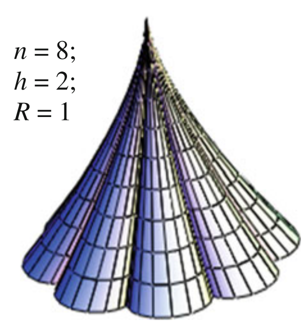

Fig. 1

The surface may have *a hypocycloid* (Fig. 2)

$$x = x(\varphi) = (R - r)\cos\varphi + r\cos(n - 1)\varphi$$
$$y = y(\varphi) = (R - r)\sin\varphi - r\sin(n - 1)\varphi, \quad z = 0.$$

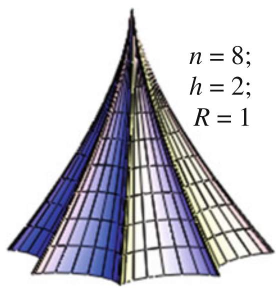

Fig. 2

at the foot contour $z = 0$ with the vertexes turned only inside the circular base, n is a number of the vertexes of the hypocycloid; R is a radius of the circle inside of which, a circle of a radius r rolls and any point lying on it traces the hypocycloid.

At any cross-section of the surface in question by the plane passing through the axis of the surface coinciding with the coordinate axis *Oz,* the *semi-cubic parabolas*

$$x = a(\varphi)(h - z)^{3/2}$$

lie; $0 \le z \le h$; $a(\varphi)$ is a changing parameter of the semi-cubic parabolas.

Forms of definition of the surface of umbrella type

(1) Parametrical form of definition of a surface of umbrella type with the epicycloid at the foot (surface with the external crimps, Fig. 1):

$$x = x(u, \varphi) = u^{3/2}[(R + r)\cos\varphi - r\cos(n + 1)\varphi],$$
$$y = y(u, \varphi) = u^{3/2}[(R + r)\sin\varphi - r\sin(n + 1)\varphi],$$
$$z = z(u) = h(1 - u),$$

where $0 \le u \le 1$; u is a dimensionless parameter; $0 \le z \le h$; h is a maximum height of the surface; $0 \le \varphi \le 2\pi$. At any cross-section of the surface by the plane $u = \text{const}$, an epicycloid is located. The coordinate line $u = 1$ coincides with the foot epicycloid. *The singular point* is at the vertex of the surface.

Coefficients of the fundamental forms of the surface:

$$A^2 = 9u[R^2 + 2r(R + r)(1 - \cos n\varphi)]/4 + h^2,$$
$$F = 3u^2R(R + r)\sin n\varphi/2,$$
$$B^2 = 2u^3(R + r)^2(1 - \cos n\varphi);$$
$$L = \frac{3uh(R + r)(R + 2r)}{4\sqrt{A^2B^2 - F^2}}(1 - \cos n\varphi),\ M = 0,$$
$$N = -\frac{u^3h(R + r)^2(R + 2r)}{r\sqrt{A^2B^2 - F^2}}(1 - \cos n\varphi);$$
$$K = -\frac{3u^4h^2(R + r)^3(R + 2r)^2(1 - \cos n\varphi)^2}{4r(A^2B^2 - F^2)^2} < 0.$$

(2) Parametrical form of definition of a surface of umbrella type with a hypocycloid at the foot of the surface (surface with the internal crimps, Fig. 2):

$$x = x(u, \varphi) = u^{3/2}[(R - r)\cos\varphi + r\cos(n - 1)\varphi],$$
$$y = y(u, \varphi) = u^{3/2}[(R - r)\sin\varphi - r\sin(n - 1)\varphi],$$
$$z = z(u) = h(1 - u),$$

$0 \le u \le 1$; $0 \le z \le h$; h is a maximum height of the surface; $n \ne 2$; $0 \le \varphi \le 2\pi$. At any cross-section of the surface by plane $u = \text{const}$, a hypocycloid lies.

Coefficients of the fundamental forms of the surface:

$$A^2 = 9u[R^2 - 2r(R - r)(1 - \cos n\varphi)]/4 + h^2,$$
$$F = -3u^2R(R - r)\sin n\varphi/2,$$
$$B^2 = 2u^3(R - r)^2(1 - \cos n\varphi);$$
$$L = [3uh(R - r)(R - 2r)/4](1 - \cos n\varphi)/\sigma,\ M = 0,$$
$$N = \left[u^3h(R - r)^2(n - 2)\right](1 - \cos n\varphi)/\sigma,\ K > 0,$$

where $\sigma^2 = A^2B^2 - F^2$. There is the singular point at the vertex of the surface (Fig. 2).

27 Special Profiles of Cylindrical Products

Cylindrical products with various cross sections, i.e., *profiles*, are widely used in civil engineering and different branches of machine building.

Under conditions of application, all profiles are divided into two groups: *profiles of the general purpose* and *profiles of the special purpose*.

Profiles of the general purpose find widespread use in modern engineering. I-section beams, channels, Z-profiles, C-profiles, fluted and tee sections, angles, pipes, and also sheet, universal, corrugated and waving steel form this group.

Profiles, the form and dimensions of which are defined by functional purpose and characteristic properties of designs, form the profile of special purpose. This group contains corrugated profiles, the I-beams for the tracks of suspended transport, crane rails, and other products.

For the manufacture of the different profiles (corrugated iron, thin-walled angles, beams, channels, and others) from the sheet metal by the longitudinal cold bending, the roller-bending and profile-bending machines are used.

The forms of some profiles were obtained by experimental methods, for the determination of the forms of other profiles, it is necessary to know the methods of differential geometry or the methods of the aerodynamics.

In this part, some profiles of the special purpose are presented, that are interesting with a mathematical point of view.

■ Triangular Profile of Cylindrical Fragment of a Shaft for the Profile Detachable Joint

Detachable joints having the contact of a hub and a shaft which realizes along smooth not round surface of the shape are called *profile joints*.

In comparison with keyed and splined joints, the profile joints have some advantages: they provide the better centering of the connecting details and differ by the more fatigue strength because of the absence of stress concentration.

The shortcoming of these joints is necessary in special equipment for making of the profile holes.

A triangular profile of cylindrical fragment of a shaft for the profile detachable joint is given by parametrical equations:

$$x = x(t) = \frac{D}{2}\cos t - e\cos 3t\cos t - 3e\sin 3t\sin t,$$
$$y = y(t) = \frac{D}{2}\sin t - e\cos 3t\sin t + 3e\sin 3t\cos t,$$

where the geometrical parameters D and e are shown in Fig. 1.

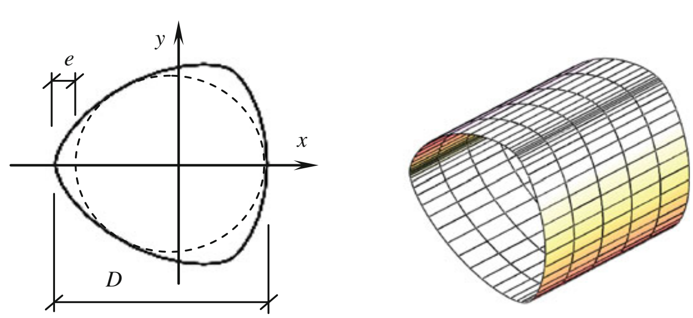

Fig. 1

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_27, © Springer International Publishing Switzerland 2015

Figure 1 presents the triangular profile of the cylindrical fragment of a shaft for the profile detachable joint, when $0 \leq t \leq 2\pi$, $D = 1$ m; $e = 0.05$ m.

Additional Literature

Druzhinskiy IA. Complex Surfaces: Mathematical Description and Technological Description: Reference Book. Leningrad: «Mashinostroenie», 1985; 263p.

■ Superellipses

A superellipse is a closed curve given in the Cartesian system of coordinates by an equation:

$$\left|\frac{x}{a}\right|^r + \left|\frac{y}{b}\right|^r = 1.$$

A superellipse may be given by the following parametric equations:

$$x = x(t) = a\cos^{2/r} t.$$
$$y = y(t) = b\sin^{2/r} t.$$

Superellipses with $a = b$ are called *the Lame curves* or *ovals of Lame*. A superellipse with $r = 2/3$ and $a = b$ is called *an astroid*; with $r = 2$ and $a \neq b$ is called *an ellipse*; with $r = 2$ and $a = b$ is called *a circle*.

A superellipse with $r = 5/2$ is called *the Piet Hein's superellipse*. Increasing the value of the parameter r, we bring the superellipse nearer the rectangular contour. Superellipses may be seen in the outlines of the profiles of some industrial cylindrical products. Figure 1 shows the forms of the superellipses with the following values of the parameter r:

$$r = 2/5; \quad r = 2/3; \quad r = 10/7; \quad r = 10/3; \quad r = 10$$

considering from the center, but with $a = 1$ m; $b = 1.5$ m.

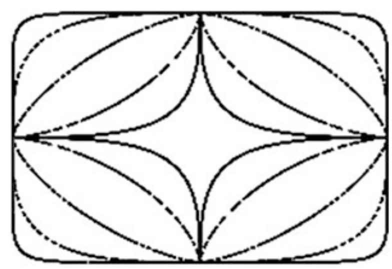

Fig. 1

■ Two Types of Aerodynamic Cylindrical Profiles

Aerodynamics is a section of *aeromechanics* studying the laws of motion of the gaseous medium and its force interaction with the flowed solid bodies moving in it. The aerodynamics is a theoretical base of the aviation and the meteorology. The main problems solved by the aeromechanics are the determination of the hydrodynamic lift and the force of resistance, the distribution of the pressure and the direction of the flows on the surface of solid bodies which are in the air flow.

The aerodynamic profile is a form of the outlines of the body which gives an opportunity for the appearance of the hydrodynamic lift which exceeds the force of resistance to motion in the air. *An airfoil* (in American English) or *aerofoil* (in British English) is the shape of a wing. The lift on an airfoil is primarily the result of its *angle of attack* and shape. When oriented at a suitable angle, the airfoil deflects the oncoming air, resulting in a force on the airfoil in the direction opposite to the deflection. This force is known as *aerodynamic force* and can be resolved into two components: lift and drag.

The optimal aerodynamic profiles are different for different velocities of the motion, i.e., various airfoils serve different flight regimes. The lift and drag curves are obtained in wind tunnel testing. They can also be represented by mathematical functions. For example, the airfoils' both top and bottom curves may be modeled by two *cubic Beziér curves* in parametric form.

The suction surface (upper surface) is generally associated with higher velocity and lower static pressure. *The pressure surface* (lower surface) has a comparatively higher static pressure than the suction surface. The pressure gradient between these two surfaces contributes to the lift force generated for a given airfoil.

Many types of aerodynamic profile were described in special literature.

Consider only two of them. The aerodynamic profile represented in Figs. 1 and 2 is described by the parametric equations:

$$\begin{aligned} x = x(t) &= 1 - \cos t - 0.05(1 - \cos 2t) \\ &\quad - 0.0166(1 - \cos 4t), \\ y = y(t) &= 0.09(1 - \cos 2t) - 0.004(1 - \cos 4t) \\ &\quad - (0.1114\sin t + 0.0156\sin 2t) \\ &\quad - 0.014\sin 3t + 0.0022\sin 4t, \end{aligned}$$

where $0 \leq t \leq 2\pi$.

The aerodynamic profile represented in Figs. 3 and 4 is given by the following parametrical equations:

Fig. 1

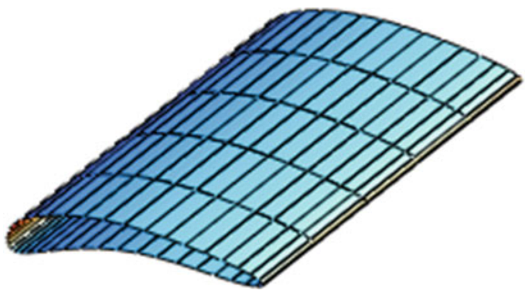

Fig. 2

$$x = x(\alpha) = 10 \sin \alpha,$$
$$y = y(\alpha) = 0.509 \cos \alpha - 0.133 \sin 2\alpha + 0.05 \cos 3\alpha,$$

where $0 \leq \alpha \leq 2\pi$.

Fig. 3

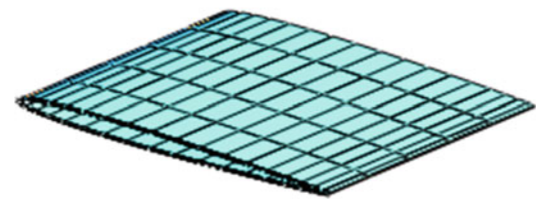

Fig. 4

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■ Generalized Superellipses

J. Gielis has put into practice the closed curves with several axes of the symmetry which were called *generalized superellipses* later.

The equation of a generalized superellipses in the polar system of coordinates has the following form:

$$r(\theta) = \left[\left|\frac{\cos\frac{m\theta}{4}}{a}\right|^{n_2} + \left|\frac{\sin\frac{m\theta}{4}}{b}\right|^{n_3}\right]^{-1/n_1}.$$

The presented formula called "*superformula*" in some scientific publications gives the possibility to approximate some biological forms by the mathematical curves. The presence of six free parameters allows creating great number of the closed curves with the given numbers of the axes of symmetry (Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11). In Figs. 1, 2, 3, 4, 5, 6, 8, 9 and 10 it is assumed that $a = b = 1$.

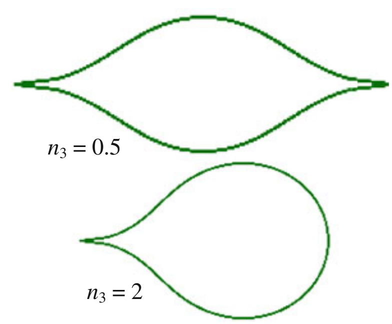


Fig. 1 $m = 2$; $n_1 = n_2 = 0.5$

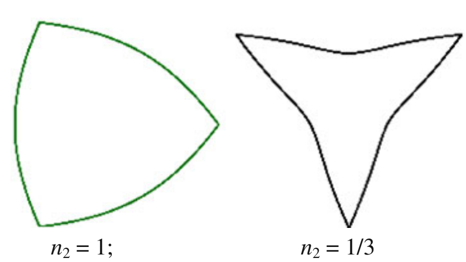
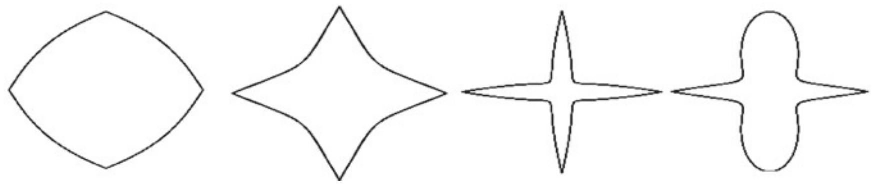
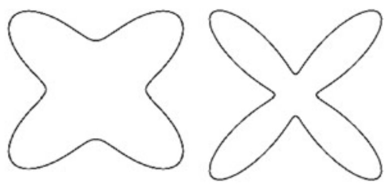


Fig. 2 $m = 3$; $n_1 = n_3 = 1$



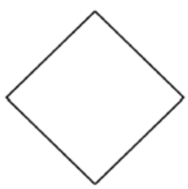
$n_1 = 2; n_3 = 1;$ $n_1 = 1/2; n_3 = 3;$ $n_1 = 1/5; n_3 = 1;$ $n_1 = 1/5; n_3 = 2$

Fig. 3 $m = 4; n_2 = 1$



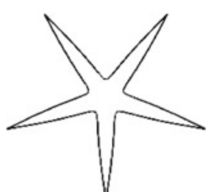
$n_3 = 2;$ $n_3 = 3$

Fig. 4 $m = 4; n_1 = 1/5; n_2 = 3$



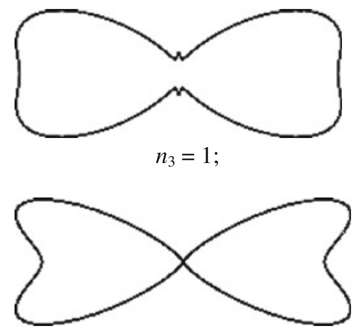
$n_1 = n_2 = n_3 = 1$

Fig. 5 $m = 4$



$n_1 = 1/5;\ n_2 = n_3 = 1$

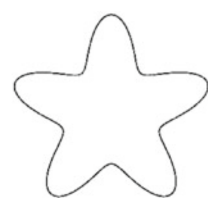
Fig. 6 $m = 5$



$n_3 = 1;$

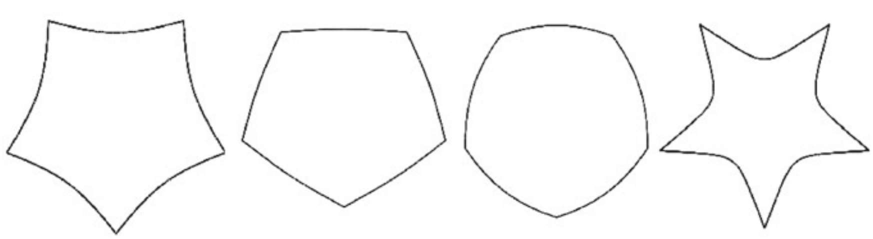
$n_3 = 3$

Fig. 7 $m = 4; n_1 = 1/5; n_2 = 3; a = 1; b = 1.5$



$n_1 = 1/2; n_2 = 3$

Fig. 8 $m = 5; n_3 = 3$



$n_1 = 1;$ $n_1 = 2;$ $n_1 = 4;$ $n_1 = 1/2;$

Fig. 9 $m = 5; n_2 = n_3 = 1$

Fig. 10 $m = 8$

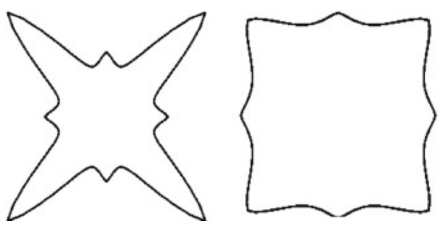


Fig. 11 $m = 8$; $a = 1$; $b = 1.2$

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■ Compound Profiles Formed by a Curve and Its Mirror Reflection

In some scientific publications, *the superellipses* are called a special case of *the compound profiles formed by a curve and its mirror reflection*.

Forms of definition of compound profiles formed by a curve and its mirror reflection

(1) Implicit form of the definition:

$$|x|^p + |y|^q = 1,$$

where $p > 0$, $q > 0$ are arbitrary constants.

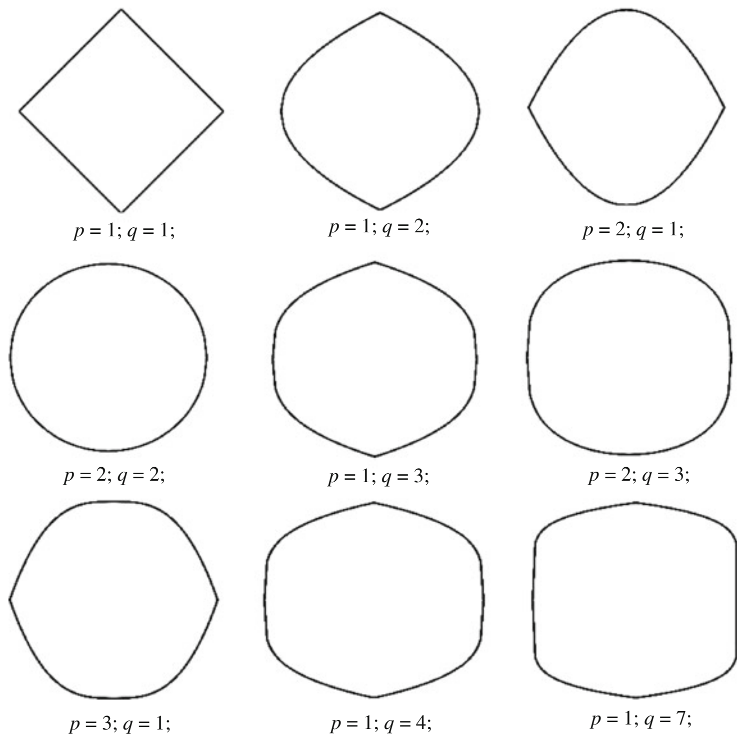
(2) Explicit form of the definition (Figs. 1 and 2):

$$y = \pm(1 - |x|^p)^{1/q},$$

where $-1 \leq x \leq 1$.

As it is seen from the figures, the presented profiles have two or four axes of symmetry. These profiles may find the application at machine-building details of the cylindrical type.

Changing the parameters p and q, it is possible to obtain a great number of the different forms of the profiles selecting the necessary profiles using either criterion.


$p = 1; q = 1;$
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$p = 3; q = 1;$
$p = 1; q = 4;$
$p = 1; q = 7;$

Fig. 1

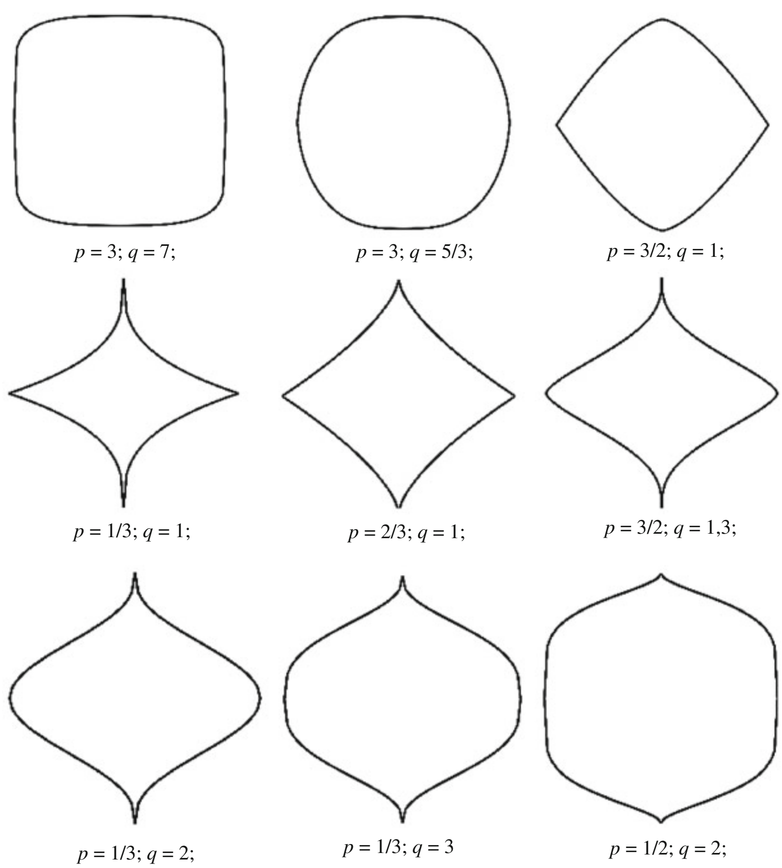
$p = 3; q = 7;$
$p = 3; q = 5/3;$
$p = 3/2; q = 1;$
$p = 1/3; q = 1;$
$p = 2/3; q = 1;$
$p = 3/2; q = 1,3;$
$p = 1/3; q = 2;$
$p = 1/3; q = 3$
$p = 1/2; q = 2;$

Fig. 2

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■ Profiles of Products of Trochoidal Rotary Machines

The trochoid (*trochos* in Greek, *wheel* in English, and *eidos* in Greek, *form* in English) is a curtate or prolate cycloid (epicycloid or hypocycloid).

A trochoid traces by a point lying on a circle or on an extension to the line of the radius of this circle rolling without sliding above a straight line or a circle. If the circle rolls on the convex side of the circle, a curve is called *an epitrochoid*, and if it rolls on the concave side, then a curve is called *a hypotrochoid*.

Forms of the definition of the profile line

(1) Let a circle of a radius R rolls by its concave side along a convex side of the circle of a radius $r(R > r)$, then the parametrical equations of the epitrochoid formed by a point disposed at the distance of $d = \mu R$ from the center of the rolling circle (Fig. 1) are

$$x = x(\varphi) = R\left(\frac{\cos p\varphi}{p} + \mu\cos\varphi\right),$$
$$y = y(\varphi) = R\left(\frac{\sin p\varphi}{p} + \mu\sin\varphi\right),$$

where

$$p = R/(R - r), \quad r = R - a \quad \text{or} \quad p = R/a;$$
$$\varphi = \theta - \omega;$$

θ is the angle of the coordinate axis Ox with the line joining the center O of the fixed circle with a point C of the touching of two circles; ω is the angle of the coordinate axis Ox with the line joining the center O_1 of the mobile circle with a C point of the touching of two circles; $0 \le \varphi \le 2\pi$.

(2) Let a circle of a radius r rolls by its convex side along a concave side of the circle of radius $R(R > r)$, then the parametrical equations of the hypotrochoid formed by a point disposed at the distance of $d = \mu r$ from the center of the rolling circle are (Fig. 2):

$$x = x(\varphi) = R\left(-\frac{\cos p\varphi}{p} + \mu\cos\varphi\right),$$
$$y = y(\varphi) = R\left(\frac{\sin p\varphi}{p} + \mu\sin\varphi\right),$$

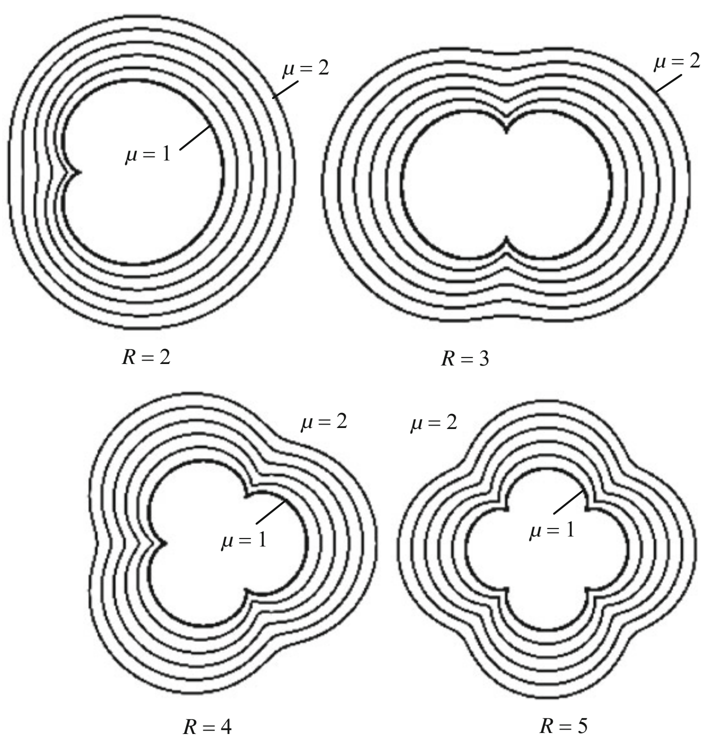

Fig. 1 $a = 1$; $r = R - a$; $1 \le \mu \le 2$; with the step of $\Delta\mu = 0.2$

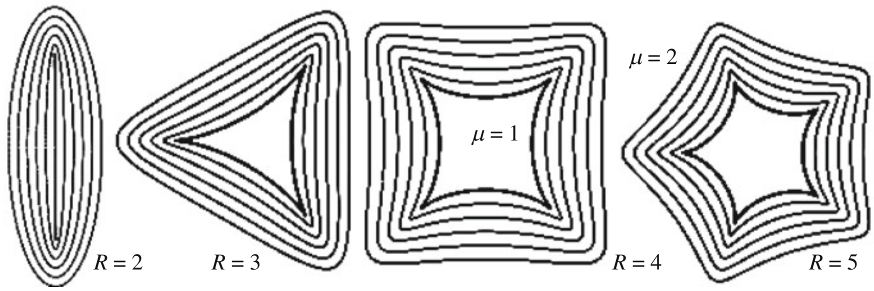

Fig. 2 $a = 1$; $r = R - a$; $1 \le \mu \le 2$; with the step of $\Delta\mu = 0.2$

where

$$p = r/(R - r), \quad r = R - a \quad \text{or} \quad p = (R - a)/a;$$
$$\varphi = \omega - \theta;$$

θ is the angle of the coordinate axis Ox with the line joining the center O of the fixed circle with a point C of the touching of two circles; ω is the angle of the coordinate axis Ox with the line joining the center O_1 of the mobile circle with a point C of the touching of two circles; $0 \le \varphi \le 2\pi$.

The profiles in question are used in the design of the theoretic contours of details (products) of throchoidal rotary machines [1].

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28 Bonnet Surfaces

A surface permitting the isometric transformation with the preservation of the mean curvature is called *a Bonnet surface*. V. Lalan (1949) was the first who used the term "*Bonnet surface*".

Under isometric transformation of a surface, the distance between the corresponding points of the surface remains constant. *Isometric surfaces* are surfaces in Euclidian or in Riemann space and it is possible to find mutually single point conformity between them, when every straightened curve of one of the surfaces has also the straightened curve of the same length as its image. The whole complex of the surfaces obtained by the bending of a given surface is the most impotent example of the isometric surfaces.

If the square of a linear element of surface given in the isometric coordinates u, v has the following form:

$$\mathrm{d}s^2 = \lambda(\mathrm{d}u^2 + \mathrm{d}v^2),\ \lambda(u, v) > 0,$$

then the satisfaction to differential equations in the partial derivatives:

$$\begin{aligned}
&(X^2 + Y^2)e^{2R} - R_{uu} - R_{vv} = 0,\\
&X_u - Y_v = 0,\\
&X_v + Y_u = 0,
\end{aligned}$$

is a necessary and sufficient condition of belonging of a surface to the Bonnet surfaces [2]. Here

$$\begin{aligned}
X &= H_u/\lambda(H^2 - K),\\
Y &= H_v/\lambda(H^2 - K),\\
R &= \lg(\lambda\sqrt{H^2 - K}),
\end{aligned}$$

H and K are the mean and Gaussian curvatures of surface.

For *a developable surface* related to the geodesic coordinates u, v, the square of a linear element of surface may be written as

$$\mathrm{d}s^2 = \mathrm{d}u^2 + g^2\mathrm{d}v^2,\ g(u, v) > 0.$$

Taking into account, that for developable surfaces, we have

$$K = 0; \quad LN - M^2 = 0; \quad H = (g^2L + N)/2g^2,$$

Richard Blum proved (1972), that a necessary and sufficient condition of belonging of a developable surface to the Bonnet surfaces is the satisfaction to the system of the equations:

$$\begin{aligned}
H_{uu} &= 2H_u^2/H,\\
H_{uv} &= 2H_uH_v/H + (g_u/g)H_v,\\
H_{vv} &= 2H_v^2/H + (g_v/g)H_v - gg_uH_u,
\end{aligned}$$

and that leads to the relations:

$$1/H = \alpha u + \beta,$$
$$g = \frac{\alpha_v u + \beta_v}{\sqrt{c^2 - \alpha^2}},$$

where α and β are the functions of a parameter v, and c is a real constant.

The subsequent investigations have shown that developable surfaces with the nondegenerated cuspidal edge (see also a Subsect. "1.1.1. Torse Surface (torses)") cannot be by the Bonnet surfaces.

A conical surface may be by a Bonnet surface if a curvature and torsion of its spherical mapping are equal to

$$k_s = \frac{\sqrt{a^2 + \cos^2 v}}{\cos v}, \quad \kappa_s = \frac{a \sin v}{a^2 + \cos^2 v},$$

where a is a constant; v is the length of an arc, $-\pi/2 < v < \pi/2$.

A cylindrical surface is a Bonnet surface, if its orthogonal cross section is *the circle* or *the logarithmic spiral*.

A surface of constant Gaussian curvature not equal to zero cannot be a Bonnet surface.

I. Roussos (1988, 1999) has confirmed that *helical surfaces* may belong to a class of the Bonnet surfaces.

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E. Cartan proved that in R^3 any Bonnet surface is a *Weingarten surface.* W. Chen and H. Li (1997) showed that there exist always Bonnet surfaces which are not *Weingarten surfaces*, if the ambient space is not $R^3(c)$ for $c \geq 0$.

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29 Edlinger's Surfaces

An oblique ruled surface the osculating hyperboloids of which are *hyperboloids of revolution* (Fig. 1) is called *an Edlinger's surface*. The Edlinger's surfaces are characterized by the constant parameter of the distribution and have the lines of principal curvature as striction lines.

R. Edlinger was the first who carried out the detailed research of these surfaces in 1924. These surfaces were mentioned also by H. Brauner (1961).

The Edlinger's surfaces may be given by a vector equation:

$$\boldsymbol{r}(u,v) = \boldsymbol{s}(u) + v\boldsymbol{e}(u),$$

where u is an arc length and $\boldsymbol{s}(u)$ is a radius-vector of a directrix striction line; $\boldsymbol{e}(u)$ are the unit vectors of the generatrix straight lines. The unit vectors $\boldsymbol{e}(u)$ are placed at the tangent plane of the striction line and generate a variable angle σ with the tangents to the striction line; $-\pi/2 \le \sigma \le \pi/2$ and besides that

$$\begin{aligned} \boldsymbol{s}'^2(u) &= 1; \\ \boldsymbol{s}'(u)\cdot\boldsymbol{e}'(u) &= 0; \\ \boldsymbol{e}^2(u) &= 1. \end{aligned}$$

The first derivatives of the unit vectors are written as:

$$\begin{aligned} \boldsymbol{e}'(u) &= \kappa\boldsymbol{n}, \\ \boldsymbol{n}' &= -\kappa\boldsymbol{e}(u) + \tau\boldsymbol{b}(u), \\ \boldsymbol{b}'(u) &= -\tau\boldsymbol{n}(u). \end{aligned}$$

Sachs Hans (1973) has shown the general properties and the characteristics of the Edlinger's surfaces and has proved one of conditions formulated by R. Edlinger without any analytical proof:

$$\delta = \frac{\sin\sigma}{\kappa} = \text{const},$$

$$\tau\cos\sigma + \kappa\sin\sigma = 0,$$

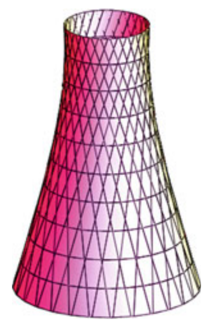

Fig. 1

where $\kappa(u)$, $\tau(u)$, $\sigma(u)$ are tree geometrical characteristics defining an Edlinger's surface. Sachs Hans (1973) has also obtained new conditions for ruled surfaces to be a Edlinger's surface. For example, he has proved, that one family of the lines of principal curvature may be given by an equation:

$$v = \varphi(u) + C$$

if it satisfies the condition:

$$M(v'^2 - E) + L(v' + F) = 0,$$

where E, F are the coefficients of the first fundamental form of surface; M, L are the coefficients of the second fundamental forms of surface.

Coefficients of the fundamental forms of the Edlinger's surfaces:

$$\begin{aligned} E &= 1 + v^2\kappa^2, \\ F &= \cos\sigma, \\ G &= 1, \\ v' &= \mathrm{d}v/\mathrm{d}u, \\ W^2 &= A^2B^2 - F^2 = \kappa^2v^2 + \sin^2\sigma \ne 0, \\ WL &= -\kappa^2\tau v^2 + (\kappa'\sin\sigma - \kappa\sigma'\cos\sigma)v \\ &\quad + \sin\sigma(\kappa\cos\sigma - \tau\sin\sigma), \\ WM &= \kappa\sin\sigma, \\ N &= 0. \end{aligned}$$

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DOI 10.1007/978-3-319-11773-7_29, © Springer International Publishing Switzerland 2015

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30 Coons Surfaces

A Coons surface on any four given lines of the contour is determined by a sum of two linear surfaces which are constructed by the motion of a straight line above two corresponding opposite contour lines with the deduction of the oblique plane passed through the angular points of the contour (Fig. 1).

Let $\boldsymbol{\rho}_i(t_i),\ t_{ib} \le t_i \le t_{ie}\ (i = 1; 2; 3; 4)$ are the vector equations of the contour lines. The numerations of the contour lines are assumed anticlockwise. Additionally,

$$\boldsymbol{\rho}_1(t_{1b}) = \boldsymbol{\rho}_4(t_{4b}) = \boldsymbol{p}_1; \quad \boldsymbol{\rho}_1(t_{1e}) = \boldsymbol{\rho}_2(t_{2b}) = \boldsymbol{p}_2;$$
$$\boldsymbol{\rho}_2(t_{2e}) = \boldsymbol{\rho}_3(t_{3e}) = \boldsymbol{p}_3;$$
$$\boldsymbol{\rho}_3(t_{3b}) = \boldsymbol{\rho}_4(t_{4e}) = \boldsymbol{p}_4,$$

where $\boldsymbol{p}_i (i = 1; 2; 3; 4)$ are the vectors of the angular points of the contour. The opposite lines of the contour are parameterized by the common parameters $u, v \in (0, 1)$:

$$t_i = t_i(u) = t_{ib}(1-u) + t_{ie}u, \quad i = 1; 3;$$

and

$$t_j = t_j(v) = t_{jb}(1-v) + t_{je}v, \quad j = 2; 4.$$

In this case, the vector equation of the Coons surface may be represented as:

$$\boldsymbol{r}(u,v) = \boldsymbol{r}_1(u,v) + \boldsymbol{r}_2(u,v) - \boldsymbol{r}_p(u,v),$$

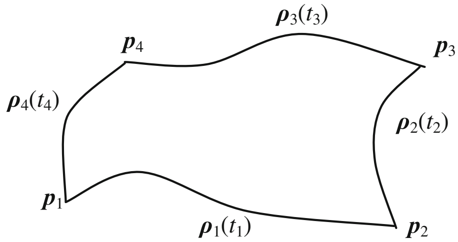

Fig. 1

where

$$\boldsymbol{r}_1(u,v) = \boldsymbol{\rho}_1(t_1(u))(1-v) + \boldsymbol{\rho}_3(t_3(u))v;$$
$$\boldsymbol{r}_2(u,v) = \boldsymbol{\rho}_2(t_2(v))(1-u) + \boldsymbol{\rho}_4(t_4(u))u$$

are the equations of the ruled surfaces created by the motion of the straight line above the opposite contour lines;

$$\boldsymbol{r}_p(u,v) = [\boldsymbol{p}_1(1-u) + \boldsymbol{p}_2u](1-v) + [\boldsymbol{p}_4(1-u) + \boldsymbol{p}_3u]\,v$$

is an equation of the oblique plane.

Figure 2 shows the Coons surface bounded by the plane contour lines:

$\boldsymbol{\rho}_2 = \boldsymbol{i} + v\boldsymbol{j} + 1.5\sin(\pi v/2)\cdot\boldsymbol{k}$, $\boldsymbol{\rho}_3 = u\boldsymbol{i} + \boldsymbol{j} + [1.25 + 0.25\cos(2\pi u)]\boldsymbol{k}$ are sinusoids;

$\boldsymbol{\rho}_1 = u\boldsymbol{i} + (1-u^2)\boldsymbol{k}$, $\boldsymbol{\rho}_4 = v\boldsymbol{j} + (1+u^2/2)\boldsymbol{k}$ are parabolas.

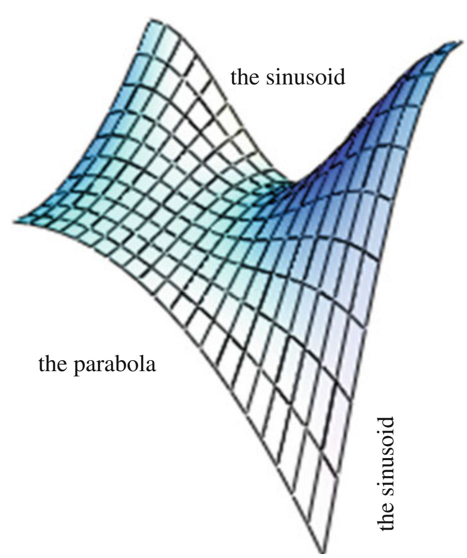

Fig. 2

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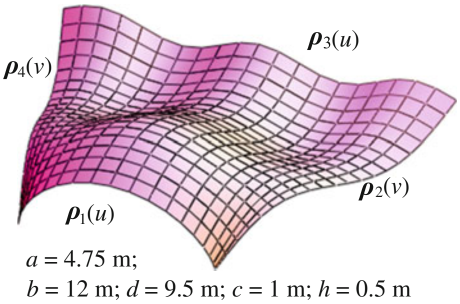

Fig. 3

Figure 3 represents the Coons surface on the trapezoidal plane with the contour lines in the form of the cycloids:

$$x_1 = 2a(u - 0.5 - \sin(2\pi u)), \quad y_1 = 0,$$
$$z_1 = (a/\pi)(1 - \cos(2\pi u))$$

and the sinusoids:

$$x_2 = a + (d - a)v, \quad y_2 = bv, \quad z_2 = c\,\sin(2\pi v);$$

$$x_3 = 2d(u - 0.5), \quad y_3 = b, \quad z_3 = h\,\sin(5\pi u);$$

$$x_4 = -(a + (d - a)v),$$
$$y_4 = bv, \quad z_4 = c\,\sin(2\pi v).$$

The supporting points: $p_1(-a, 0, 0); p_2(a, 0, 0); p_3(d, b, 0); p_4(-d, b, 0)$.

A vector equation of the Coons surface with the boundary curves in the form of the rational curves of Bézier, the control points of which are derived with the help of the methods of interpolation of the given points of the surface, is written as:

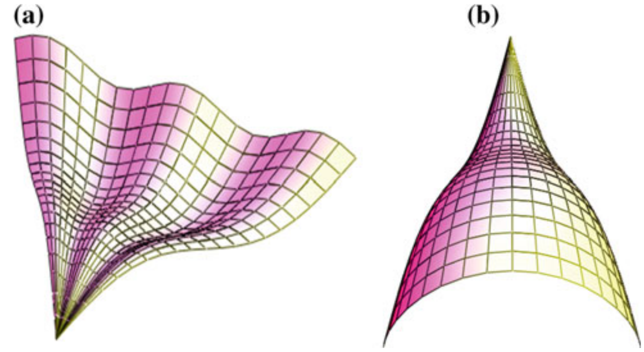

Fig. 4

$$\boldsymbol{r}(u,v) = \frac{\sum_{i=0}^{m} w_i B_i^m(u) P_i}{\sum_{i=0}^{m} w_i B_i^m(u)} + \frac{\sum_{j=0}^{n} w_j B_j^n(u) P_j}{\sum_{j=0}^{n} w_j B_j^n(u)} - \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij} B_i^m(u) B_j^n(u) P_{ij}}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij} B_i^m(u) B_j^n(u)},$$

$0 \le u \le 1$; $0 \le v \le 1$; B_i^m, B_j^n are Bernstein polynomials (see also the Chap. "15. Surfaces of Bézier"); w_{ij} are *a scale* (*balance*).

Assume that two angular points of a contour are at one point, and then we shall obtain the Coons surface on a triangular plane. Figure 4 shows the Coons surface with the contour used in Fig. 3; in Fig. 4a, the first and the second angular points were moved at the center of the first side and the parameter a is put equal to zero; in Fig. 4b, the third and fourth angular points were moved at the center of the third side (Fig. 3) and the amplitude of the sinusoid was taken equal to zero.

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■ Coons Surfaces on a Curvilinear Quadrangular Plane

A Coons surface on any four given lines of the contour is determined by a sum of two linear surfaces which are constructed by the motion of a straight line above two corresponding opposite contour lines with the deduction of the oblique plane passed through the angular points of the contour (Fig. 1).

Let $\boldsymbol{\rho}_i(t_i)$, $t_{ib} \le t_i \le t_{ie}$ $(i = 1; 2; 3; 4)$ are the vector equations of the contour lines. The numerations of the contour lines are assumed anticlockwise. Additionally,

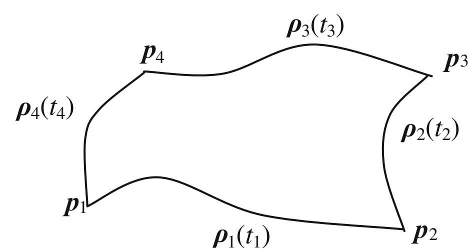

Fig. 1

$$\boldsymbol{\rho}_1(t_{1b}) = \boldsymbol{\rho}_4(t_{4b}) = \boldsymbol{p}_1; \boldsymbol{\rho}_1(t_{1e}) = \boldsymbol{\rho}_2(t_{2b}) = \boldsymbol{p}_2;$$
$$\boldsymbol{\rho}_2(t_{2e}) = \boldsymbol{\rho}_3(t_{3e}) = \boldsymbol{p}_3; \boldsymbol{\rho}_3(t_{3b}) = \boldsymbol{\rho}_4(t_{4e}) = \boldsymbol{p}_4,$$

where $\boldsymbol{p}_i (i = 1; 2; 3; 4)$ are the vectors of the angular points of the contour. The opposite lines of the contour are parameterized by the common parameters $u, v \in (0, 1)$:

$$t_i = t_i(u) = t_{ib}(1 - u) + t_{ie}u, \quad i = 1;\ 3; \text{ and}$$
$$t_j = t_j(v) = t_{jb}(1 - v) + t_{je}v, \quad j = 2; 4.$$

In this case, the vector equation of the Coons surface may be represented as:

$$\boldsymbol{r}(u, v) = \boldsymbol{r}_1(u, v) + \boldsymbol{r}_2(u, v) - \boldsymbol{r}_p(u, v),$$

where

$$\boldsymbol{r}_1(u, v) = \boldsymbol{\rho}_1(t_1(u))(1 - v) + \boldsymbol{\rho}_3(t_3(u))v;$$
$$\boldsymbol{r}_2(u, v) = \boldsymbol{\rho}_2(t_2(v))(1 - u) + \boldsymbol{\rho}_4(t_4(u))u$$

are the equations of the ruled surfaces created by the motion of the straight line above the opposite contour lines;

$$\boldsymbol{r}_p(u, v) = [\boldsymbol{p}_1(1 - u) + \boldsymbol{p}_2 u]\,(1 - v) + \ [\boldsymbol{p}_4(1 - u) + \boldsymbol{p}_3 u]v$$

is an equation of the oblique plane.

Usually, Coons surfaces are designed with the plane contour lines lying in the vertical planes (see also this chapter). But the equation of Coons surfaces does not change if one uses the inclined plane contour lines or the space contour lines. In this case, we can obtain Coons surfaces on a curvilinear quadrangular plane. Three of four contour lines may also lie in the horizontal plane.

If all four contour lines lie in one horizontal plane, the Coons surface degenerates into a plane with the curvilinear coordinate lines including the contour lines. This variant may be used for the design of surfaces on curvilinear quadrangular planes.

Figure 2 represents six variants of Coons surfaces with the contour lines:

1 is the parabola: $x = -2a(0.5-u)$;
$y = 0;\ z(u) = cu(1 - u)$;
2 is the sinusoid: $x_2 = a + (d - a)v,\ y_2 = bv$,
$z_2 = c\sin(1.5\pi v)$;
3 is the sinusoid: $x_3 = 2d(u - 0.5),\ y_3 = b$,
$z_3 = h\sin(4\pi u)$;
4 is the sinusoid: $x_4 = -[a + (d - a)v]$,
$y_4 = bv,\ z_3 = h\sin(1.5\pi u)$;
$a = 1.5;\ b = 3;\ c = 5;\ d = 1;\ h = 0.25.$

The coordinates of the angle points are

$$p_1 = (-1.5, 0, 0);\ \ p_2 = (1.5, 0, 0);$$
$$p_3 = (1, 3, 0);\ \ p_3 = (-1, 3, 0).$$

The form of the Coons surface depends on the angles of inclinations φ_i of the contour lines with respect to the vertical plane; if $\varphi_i\ > 0$, then a contour line inclines out the quadrangular plane; if $\varphi_i < 0$, it inclines inside the quadrangular plane. The curves given above are placed in the vertical planes.

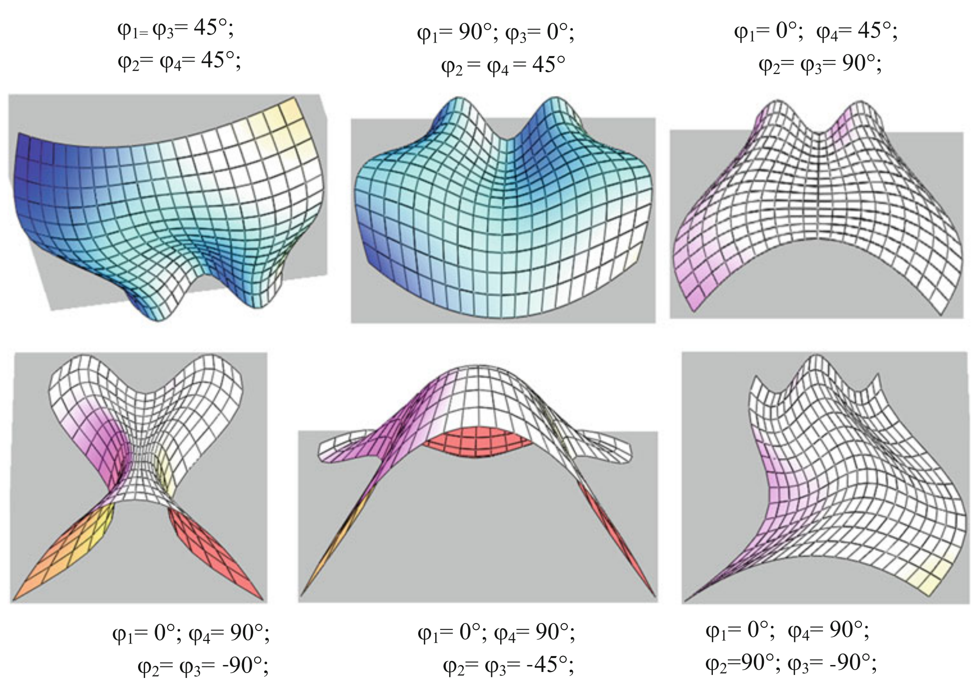

Fig. 2

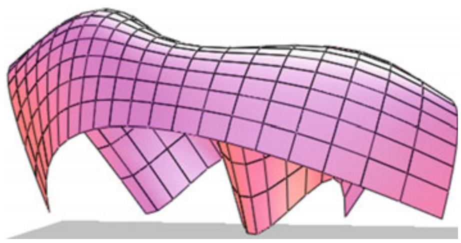

Fig. 3

All angular points of a Coons surface with inclined contour lines may be placed on one line (Fig. 3).

Additional Literature

Ivanov VN. Geometry and design of normal surfaces with a family of the plane coordinate lines. Structural Mechanics of Engineering Constructions and Buildings. 2011; No. 4, p. 6-14.

Surfaces Given by Harmonic Functions

31

A group of the surfaces of negative Gaussian curvature given in the explicit form by an equation

$$z = f(x, y),$$

where $f_{xx} + f_{yy} = 0$ relates to a class of *surfaces given by harmonic functions*. The real function $f(x, y)$, determined at the points x, y of the domain G having continuous partial derivatives with respect to two variables till the 2nd other inclusive (class $C^2(G)$) and being by a decision of Laplace's equation:

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

where x, y, z are Cartesian orthogonal coordinates of arbitrary point, is called *harmonic function at the domain* G. Any harmonic function is continuously differentiable, i.e., it has the derivatives of all orders which are the harmonic functions in one's turn.

A harmonic function

$$z = \ln\sqrt{x^2 + y^2}$$

is the most known harmonic function.

Two functions, that are harmonic in a closed-bounded domain $\bar{G}$ with piecewise-and-smooth boundary taking the same values at the boundary, coincide everywhere in the $\bar{G}$. If the function $f(x, y)$ is a harmonic one in the domain G, then it cannot have the minimal or the maximum value inside this domain with the exception of the case when $f(x, y) = \text{const}$. The harmonic function is considered *a regular one at the infinity*, if $f(x, y) = 0$ at the infinity. There is a close connection between the harmonic function of two variables (x_1, x_2) and the analytical functions of a complex variable $z = x_1 + ix_2$. The real and imaginary parts of the analytical function are the multiple-valued conjugate harmonic functions, i.e., they are connected by Cauchy–Riemann equations.

Coefficients of the first fundamental forms of surface given in the explicit form as $z = f(x, y)$, can be calculated by formulas:

$$A^2 = 1 + f_x^2, \quad F = f_x f_y, \quad B^2 = 1 + f_y^2.$$

Coefficients of the second fundamental forms of surface, given in the explicit form as $z = f(x, y)$, can be calculated by formulas:

$$L = \frac{f_{xx}}{\sqrt{1 + z_x^2 + z_y^2}}, \quad M = \frac{f_{xy}}{\sqrt{1 + z_x^2 + z_y^2}},$$
$$N = \frac{f_{yy}}{\sqrt{1 + z_x^2 + z_y^2}},$$

which for the surfaces given by harmonic functions take the form:

$$L = -N = \frac{f_{xx}}{\sqrt{1 + z_x^2 + z_y^2}} = \frac{-f_{yy}}{\sqrt{1 + z_x^2 + z_y^2}},$$
$$M = \frac{f_{xy}}{\sqrt{1 + z_x^2 + z_y^2}}.$$

The surfaces given by harmonic functions have the negative Gaussian curvature:

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2} = -\frac{f_{xx}^2 + f_{xy}^2}{(1 + f_x^2 + f_y^2)^2} < 0,$$

and the mean curvature may be calculated by a formula:

$$H = \frac{(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy}}{2(1 + f_x^2 + f_y^2)^{3/2}}$$
$$= \frac{(f_y^2 - f_x^2) f_{xx} - 2 f_x f_y f_{xy}}{2(1 + f_x^2 + f_y^2)^{3/2}}.$$

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■ Surface of Revolution Given by the Harmonic Function $z = \ln[x^2 + y^2]^{1/2}$

A surface, described by a harmonic function

$$z = \ln\sqrt{x^2 + y^2},$$

is a surface of revolution. It is generated by the rotation of a curve

$$z = \ln x$$

about the coordinate axis Oz.

Forms of definition of the surface of revolution given by the harmonic function $z = \ln\sqrt{x^2 + y^2}$.

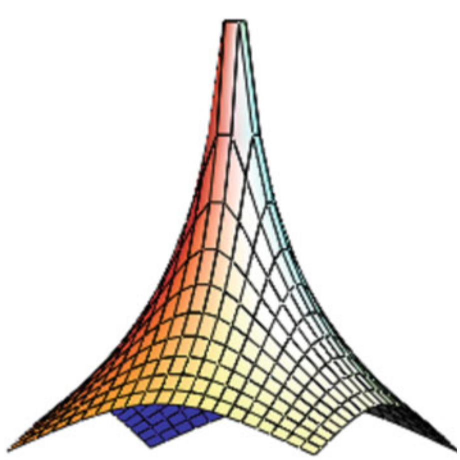


Fig. 1

(1) Explicit equation (Fig. 1):

$$z = \ln\sqrt{x^2 + y^2}.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{x^2}{(x^2 + y^2)^2},$$
$$F = \frac{xy}{(x^2 + y^2)^2},\ B^2 = 1 + \frac{y^2}{(x^2 + y^2)^2},$$
$$A^2B^2 - F^2 = \frac{1 + x^2 + y^2}{(x^2 + y^2)},$$
$$L = \frac{(y^2 - x^2)}{(x^2 + y^2)^{3/2}\sqrt{1 + x^2 + y^2}},$$
$$M = \frac{-2xy}{(x^2 + y^2)^{3/2}\sqrt{1 + x^2 + y^2}},\ N = -L,$$
$$K = \frac{-1}{(1 + x^2 + y^2)^2} < 0,$$
$$H = \frac{1}{2\sqrt{x^2 + y^2}(1 + x^2 + y^2)^{3/2}} \neq 0.$$

The surface of revolution in question is a surface of the strictly negative Gaussian curvature, but its mean curvature does not become equal to zero.

(2) Parametric form of assignment (Fig. 2):

$$x = x(r, \varphi) = r\cos\varphi,$$
$$y = y(r, \varphi) = r\sin\varphi,$$
$$z = z(r) = \ln r.$$

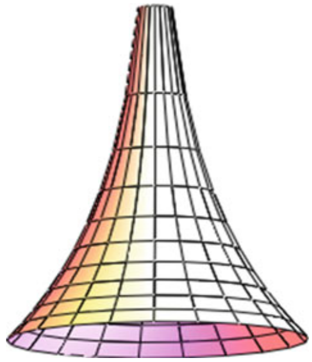


Fig. 2

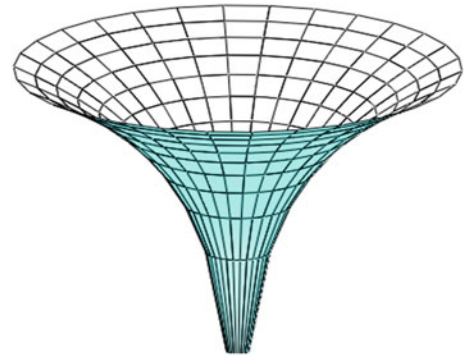


Fig. 3

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{1+r^2}{r^2}, \quad F = 0, \quad B = r;$$
$$A^2B^2 - F^2 = 1 + r^2;$$
$$L = \frac{-1}{r\sqrt{1+r^2}}, \quad M = 0, \quad N = \frac{r}{\sqrt{1+r^2}};$$
$$k_r = k_1 = \frac{-r}{(1+r^2)^{3/2}}, \quad k_\varphi = k_2 = \frac{1}{r\sqrt{1+r^2}},$$
$$K = \frac{-1}{(1+r^2)^2} < 0, \quad H = \frac{1}{2r(1+r^2)^{3/2}} \neq 0.$$

Under this method of the definition, the surface of revolution is related to the lines of the principal curvatures r, φ.

The area of a circle segment of the surface can be obtained by a formula:

$$S = 2\pi \left(\frac{r\sqrt{1+r^2}}{2} + \frac{1}{2} \ln \left| r + \sqrt{1+r^2} \right| \right) \Bigg|_{r_0}^{r}.$$

In scientific literature, the surface (Fig. 3) given by the parametric equations

$$x = x(r,\varphi) = r\cos\varphi,$$
$$y = y(r,\varphi) = r\sin\varphi,$$
$$z = z(r) = \log r$$

is also called *Funnel*.

Additional Literature

Dao Chong Thi, Fomenko AT. Minimal Surfaces and the Plato's Problem. Moscow: «Nauka», 1987; 312 p. (428 refs).

■ Harmonic Surface of Right Translation of a Sinusoid with Changing Amplitude

At present time, the surfaces related to classes of helical surfaces, surfaces of revolution and translation surfaces are the most studied surfaces given by harmonic functions.

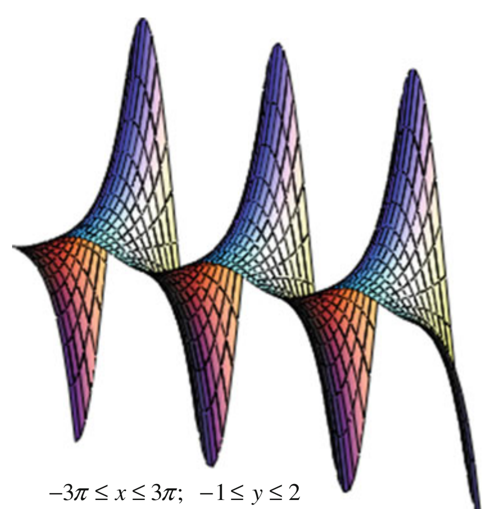
One of the surfaces which may be conditionally called a special translation surface is *a harmonic surface of right translation of a sinusoid with the changing amplitude*.

Form of the definition of the harmonic surface

(1) Explicit equation (Fig. 1):

$$z = e^y \cos x.$$

The function describing the surface is a solution of Laplace's equation (see also the Chap. "31. Surfaces Given by Harmonic Functions").



$-3\pi \le x \le 3\pi; \; -1 \le y \le 2$

Fig. 1

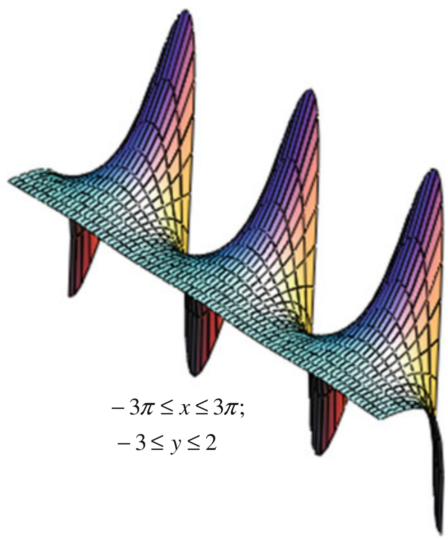


Fig. 2

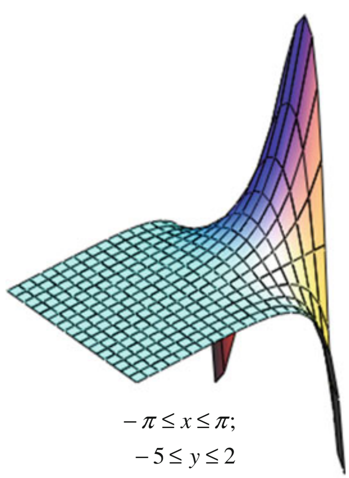


Fig. 3

The cross sections of the harmonic surface in question by the planes $y = y_o$ = const contain the sinusoids:

$$z = e^{y_o} \cos x$$

with the equal period, but with different values of the amplitudes equal to e^{y_o}. When $y \to \infty$, the values of amplitudes of the generatrix sinusoids tend to the infinity. When $y \to -\infty$, the value of amplitude tends to zero, i.e., the sinusoid degenerates into a straight line (Figs. 2 and 3).

At the cross sections of the harmonic surface by the planes $x = x_o$ = const, the curves

$$z = e^y \cos x_o$$

are placed that are given by the exponential functions.

The cross sections of the surface by the planes

$$x = x_o = \pm\pi k/2 = \text{const}\ (k = 1; 3; 5;\ \ldots)$$

represent themselves as straight lines (Fig. 1) disposed at the plane $z = 0$.

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} A^2 &= 1 + e^{2y}\sin^2 x, \\ F &= -e^{2y}\sin x \cos x, \\ B^2 &= 1 + e^{2y}\cos^2 x; \\ A^2B^2 - F^2 &= 1 + e^{2y}; \\ L &= \frac{-e^y \cos x}{\sqrt{1+e^{2y}}},\ M = \frac{-e^y \sin x}{\sqrt{1+e^{2y}}}, \\ N &= \frac{e^y \cos x}{\sqrt{1+e^{2y}}}; \end{aligned}$$

that is

$$\begin{aligned} L &= -N; \\ K &= -\frac{e^{2y}}{(1+e^{2y})^2} < 0, \\ H &= -\frac{e^{3y}\cos x}{2(1+e^{2y})^{3/2}}. \end{aligned}$$

The harmonic surface in question is a surface of *strictly negative Gaussian curvature* ($K < 0$). Along the parallel straight lines $z = 0$, $x = x_o = \pm\pi k/2 = \text{const}$ $(k = 1; 3; 5;\ \ldots)$, the surface has a zero mean curvature.

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Dorfmeister J, McIntosh J, Pedit F, Wu H. On the meromorphic potential for a harmonic surface in a *k*-symmetric space. Manuscripta Math. 1997; 92, p. 143-152.

■ Harmonic Surface of Right Translation

St. Stefanova (1973) determined that the only *surfaces of the right translation* relating simultaneously to a class of *surfaces given by harmonic functions* are surfaces given by an explicit equation:

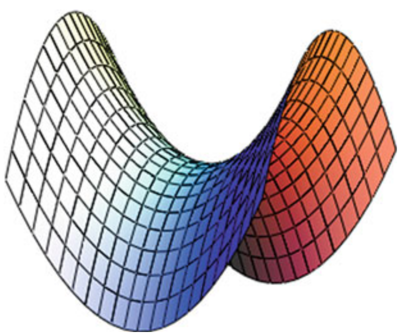
$$z = x^2 - y^2.$$

This equation defines a surface of *the hyperbolic paraboloid* (Fig. 1).

St. Stefanova (1973) investigated also an equation of a surface written in the following form:

$$z = \frac{1}{1-mn} \ln \left| \frac{\cos(y+mx)\sqrt{1+n^2}}{\cos(x+ny)\sqrt{1+m^2}} \right|,$$

where n and m are constants. Having substituted the derivatives $\partial^2 z/\partial x^2$ and $\partial^2 z/\partial y^2$ into the Laplace's equation



$-2\,\mathrm{m} \le x \le 2\,\mathrm{m};\ \ -2\,\mathrm{m} \le y \le 2\,\mathrm{m}$

Fig. 1

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

she obtained the relation:

$$\frac{\cos^2(x+ny)\sqrt{1+m^2}}{\cos^2(y+mx)\sqrt{1+n^2}} = 1,$$

which may lead to the form:

$$\left(\sqrt{1+m^2} - m\sqrt{1+n^2}\right)x + \left(n\sqrt{1+m^2} - \sqrt{1+n^2}\right)y - 2k\pi = 0,$$

where k = 0; 1; 2; … . Hence, St. Stefanova proved that surfaces satisfying the last condition given at this section above and the hyperbolic paraboloids $z = x^2 - y^2$ are the only harmonic translation surfaces.

In the same paper, Stefanova noted without corresponding references to the literature, that N.I. Kovantzov has found *the harmonic minimal surfaces*.

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32 Surfaces of Joachimsthal

If one family of plane lines of the principal curvatures v of surface lies at the planes of a pencil, then this surface is called *a surface of Joachimsthal*. The second family of the lines of the curvatures w of the surface consists of the lines of its contact with the cones, the vertexes of which belong to the axis of the pencil. The lines w are orthogonal trajectories of the generatrixes of their cones and therefore they lie on the spheres with the centers at the different points of the axis of the pencil. The lines of the curvature v are orthogonal trajectories of these spheres. It is possible to find surfaces of Joachimsthal as surfaces formed by the trajectories of single-parametrical family of the spheres with the centers on one straight line. Assume this straight line as the Oz axis, write u for the z-coordinate of the center of the sphere, and denote a radius of the sphere by $R = R(u)$, then a radius-vector $r = r(u)$ of arbitrary point M of the sphere may be written as:

$$\boldsymbol{r} = \boldsymbol{r}(u) = u\boldsymbol{k} + R(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}), \tag{1}$$

where $x = x(u)$, $y = y(u)$, $z = z(u)$ are the coordinates of the unit vector directed from the center of the sphere into the point M that must satisfy the equality

$$x^2 + y^2 + z^2 = 1.$$

The finding of the orthogonal trajectories of a family of spheres leads to the solution of an equation

$$\frac{d\boldsymbol{r}}{d\boldsymbol{u}} = \lambda(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) \;\text{ or }\; \frac{x'}{x} = \frac{y'}{y} = \frac{z' + 1/R}{z}.$$

From this, we get $y = cx$ (c = const). The coordinates of the basis vector directed from the center of the sphere, we may find using the formulas:

$$\begin{gathered} z = \tanh\tau,\ \tau = \int \frac{du}{R} + c_1,\ c_1 = \text{const}, \\ x = \frac{1}{\cosh\tau\sqrt{1 + c^2}},\ y = \frac{c}{\cosh\tau\sqrt{1 + c^2}}. \end{gathered} \tag{2}$$

The Eqs. (1) and (2) determine the two-parametrical family of the orthogonal trajectories. Constants c and c_1 are the parameters. Assume $c = \tan v$, $c_1 = V(v)$, then one can obtain the parametrical equations of any surface of Joachimsthal depending on two arbitrary functions $R = R(u)$ and $V = V(v)$:

$$x = \frac{R\cos v}{\cosh\tau},\ y = \frac{R\sin v}{\cosh\tau},\ z = u + R\tanh\tau,$$

where $\tau = \int \frac{du}{R} + V(v)$.

The surfaces were studied by F. Joachimsthal in 1846.

The surface having one family of lines of the principal curvature which consists of circles is called *a canal surface*. If these circles lie at the planes of a pencil, then this cyclic surface may be called a *canal surface of Joachimsthal*.

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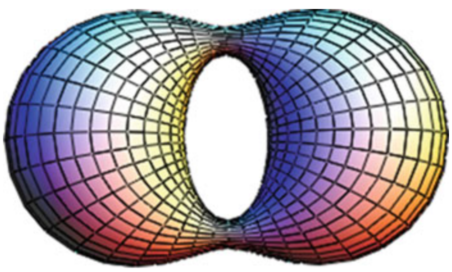
DOI 10.1007/978-3-319-11773-7_32, © Springer International Publishing Switzerland 2015

■ Virich Cyclic Surface

Virich cyclic surface is a closed surface with three planes of symmetry. The surface is formed by circles of variable radius lying in the planes of a pencil passing through the fixed axis. The fixed straight line of the planes of pencil with the circles passes over the point of intersection of tree planes of symmetry perpendicularly to one plane of the symmetry in which a plane line of the centers of the generatrix circles is disposed.



$a = 1.5$ m; $b = 3$ m; $c = 2$ m; $d = 4$ m

Fig. 1 $0 \le t \le 2\pi; 0 \le v \le 2\pi$

Forms of definition of the surface

(1) Parametrical equations:

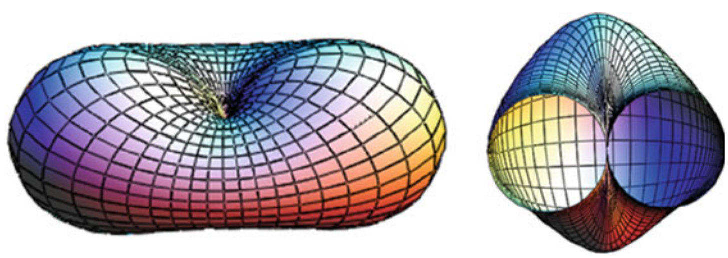
$$x = x(t, v) = \frac{1}{2}\left[f(v)(1 + \cos t) + (d^2 - c^2)\frac{1 - \cos t}{f(v)}\right]\cos v,$$
$$y = y(t, v) = \frac{1}{2}\left[f(v)(1 + \cos t) + (d^2 - c^2)\frac{1 - \cos t}{f(v)}\right]\sin v,$$
$$z = z(t, v) = \frac{1}{2}\left[f(v) - \frac{d^2 - c^2}{f(v)}\right]\sin t,$$

where

$$f(v) = \frac{ab}{\sqrt{a^2 \sin^2 v + b^2 \cos^2 v}},$$

$0 \le t \le 2\pi$; $0 \le v \le 2\pi$; a, b, c, d are constants.

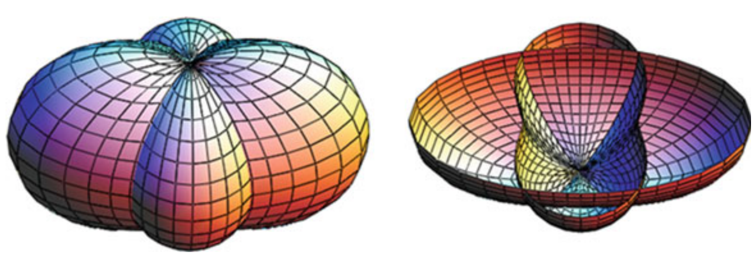
In Figs. 1, 2 and 3, the Virich cyclic surfaces and their fragments with different values of the constants c and d are shown.



$0 \le t \le 2\pi$; $0 \le t \le \pi$;

$a = 1.5$ m; $b = 3$ m; $c = d = 2$ m

Fig. 2 $0 \le v \le 2\pi$



$0 \le v \le 2\pi$; $0 \le v \le \pi$;

$a = 1.5$ m; $b = 3$ m; $c = 2$ m; $d = 1$ m

Fig. 3 $0 \le t \le 2\pi$

(2) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(t, v) = \frac{1}{2}[R(t, v)\boldsymbol{h}(v) + \psi(v)\boldsymbol{k}],$$

where

$$\begin{aligned}
\boldsymbol{h}(v) &= \boldsymbol{i}\cos v + \boldsymbol{j}\sin v;\\
p &= d^2 - c^2;\\
R(t, v) &= \varphi(v) + \psi(v)\cos t;\\
\varphi &= \varphi(v) = f(v) + p/f(v);\\
\psi &= \psi(v) = f(v) - p/f(v).
\end{aligned}$$

Coefficients of the fundamental forms of the surface:

$$\begin{aligned}
A^2 &= [(\varphi + \psi\cos t)^2 + \varphi'^2 + \psi'^2 + 2\varphi'\psi'\cos t]/4,\\
F &= -\varphi'\psi/4;\ B^2 = \psi^2/4,\\
L &= \{(\varphi + \psi)[\varphi'' + \psi'' - (\varphi + \psi\cos t)]\cos t\\
&\quad + 2(\varphi' + \psi'\cos t)(\psi' + \varphi'\cos t)\}/(2\sigma);\\
M &= \psi(\psi' + \varphi'\cos t)\sin t/(2\sigma);\\
N &= -\psi(\varphi + \psi\cos t)/(2\sigma),
\end{aligned}$$

where

$$\sigma^2 = (\varphi + \psi\cos t)^2 + (\psi' + \varphi'\cos t)^2$$

The coordinate lines v (t = const) coincide with the generatrix circles and form one family of the lines of principal curvatures. Hence Virich cyclic surface may be related to *canal surfaces of Joachimsthal*. The coordinate lines t (v = const) are not the lines of principal curvatures.

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33 Saddle Surfaces

Saddle surfaces are the generalization of *surfaces of negative Gaussian curvature*. A part of arbitrary surface of three-dimensional Euclidean space cut off by arbitrary plane with compact form closure of a contour section is called *a crust*. If we cannot cut off a crust by any plane, then this surface is a saddle surface. For a twice continuously differentiable surface to be a saddle surface, it is necessary and sufficient that at each point of the surface its Gaussian curvature is nonpositive. There are no *closed surfaces* among saddle surfaces in E^3.

It is known several additional definitions of saddle surfaces. For example, a surface, all points of which are saddle, is a saddle surface. *A saddle point* is a point of a smooth surface near which, the surface lies in various side from its tangent plane. A saddle surface is a smooth surface containing one or more saddle points. The term derives from the peculiar shape of historical horse saddles, which curve both up and down.

Saddle point is the generalization of *hyperbolic point* (see also "Surfaces"). One-sheet hyperboloid of revolution, hyperbolic paraboloid (see also a Sect. "1.2. Ruled Surfaces of Negative Gaussian Curvature"), minimal surfaces, surfaces of constant negative Gaussian curvature (see also a Sect. "23.2. Surfaces of the Constant Negative Gaussian Curvature"). A.L. Werner (1973) presented the classification of *spherical one-sheet saddle surfaces*.

In this chapter of the encyclopedia, surfaces having a term "*saddle*" in their names are gathered; the rest saddle surfaces are pointed out in the "Contents."

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■ Saddle Surface of the ε Class

A saddle surface of the ε class given by an implicit equation

$$a^2x^2 + b^2y^2 = e^{lx+my+z},$$

where a, $b \neq 0$ and l, $m \neq 0$ are parameters, is used for the solution of one transcendental equation.

Forms of definition of the surface

(1) Implicit equation:

$$a^2x^2 + b^2y^2 = e^{lx+my+z},$$

where a, $b \neq 0$ and l, $m \neq 0$ are parameters.

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
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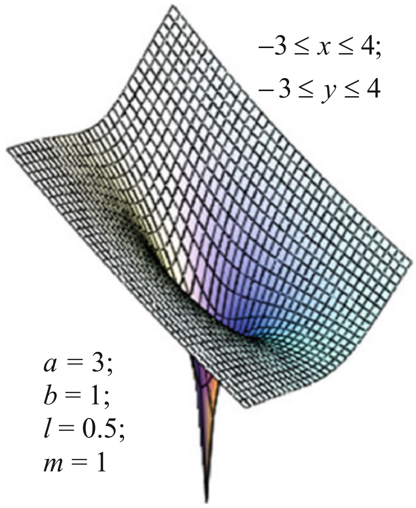

Fig. 1

(2) Obvious form of assignment (Fig. 1):

$$z = \ln(a^2x^2 + b^2y^2) - lx - my.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \left(\frac{2a^2x}{a^2x^2 + b^2y^2} - l\right)^2,$$

$$F = \left(\frac{2a^2x}{a^2x^2 + b^2y^2} - l\right)\left(\frac{2b^2y}{a^2x^2 + b^2y^2} - m\right)$$

$$B^2 = 1 + \left(\frac{2b^2y}{a^2x^2 + b^2y^2} - m\right)^2;$$

$$L = \frac{-2a^2(a^2x^2 - b^2y^2)}{(a^2x^2 + b^2y^2)^2\sqrt{A^2 + B^2 - 1}},$$

$$M = \frac{-4a^2b^2xy}{(a^2x^2 + b^2y^2)^2\sqrt{A^2 + B^2 - 1}},$$

$$N = -\frac{b^2}{a^2}L;\ K = \frac{-4a^2b^2}{(a^2x^2 + b^2y^2)^2(A^2 + B^2 - 1)^2} < 0.$$

When $a = b$, the saddle surface in question of strictly negative Gaussian curvature becomes *a surface given by the harmonic function* (see also the Chap. “31. Surfaces Given by Harmonic Functions”).

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■ Peano Saddle

Peano saddle is an analytic smooth surface of the fourth order (Fig. 1). The plane yOz is a plane of symmetry of the surface. In the explicit form, the Peano saddle may be defined by an equation:

$$z = (y - x^2)(y - 3x^2) = (y - 2x^2)^2 - x^4.$$

There is the fourth order parabola $z = 3x^4$ in the cross section of the surface by the coordinate plane $y = 0$. The plane $x = 0$ crosses the surface along the parabola $z = y^2$. In the cross section of the surface by the coordinate plane $z = 0$, two parabolas $y = 3x^2$ and $y = x^2$ with the common vertex lie.

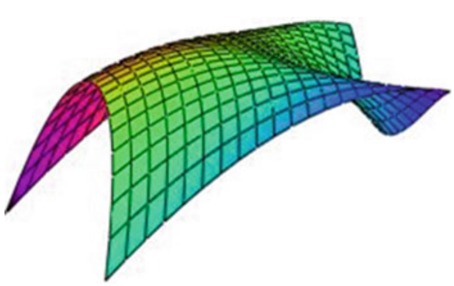

Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + 16x^2(3x^2 - 2y)^2,$$
$$F = 8x(y - 2x^2)(3x^2 - 2y),$$
$$B^2 = 1 + 4(y - 2x^2)^2;$$
$$L = \frac{4(9x^2 - 2y)}{\sqrt{A^2 + B^2 - 1}};$$
$$M = \frac{-8x}{\sqrt{A^2 + B^2 - 1}},\ N = \frac{2}{\sqrt{A^2 + B^2 - 1}};$$
$$K = \frac{8(x^2 - 2y)}{(A^2 + B^2 - 1)^2}.$$

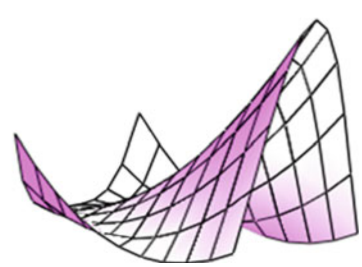

Fig. 2

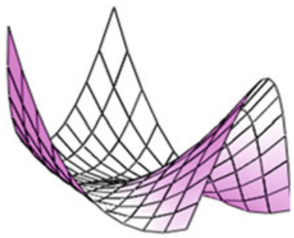

Fig. 3

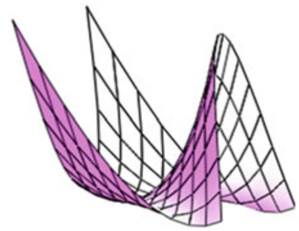

Fig. 4

The Peano saddle contains the parts both of positive and negative Gaussian curvatures. The Gaussian curvature changes the sign at the points of the parabola $2y = x^2$. Figure 2 shows the *Peano saddle* when $-0.5 \leq x \leq 0.5;\ 0 \leq y \leq 1$. In Fig. 3, the surface is limited by the coordinate lines $x = -0.5;\ x = 0.5$ and $y = -0.4;\ y = 1$; in Fig. 4, the contour is bounded by the coordinate lines $x = -0.8;\ x = 0.8$ and $y = -0.5;\ y = 1.5$.

■ Narrowing Saddle Surface of Rosendorn

E.R. Rosendron was the first who presented the examples of *the narrowing saddle surfaces*. The complete saddle surfaces, all tubes on which are the horns, are called *narrowing*. For example, the surface F_0 of the forth order (Fig. 1; $a = 16$; $-4 \leq x,\ y \leq 4$):

$$x^2y^2 + y^2z^2 + z^2x^2 = a^2$$

are the narrowing saddle surfaces.

Its Gaussian curvature at the point $(x,\ y,\ z) \in F_0$ calculates by the formula:

$$K = -\frac{2a^2[a^2(x^2+y^2+z^2) - 9x^2y^2z^2]}{[a^2(x^2+y^2+z^2) + 3x^2y^2z^2]^2} \leq 0$$

and it turns into zero only at the eight points, where

$$|x| = |y| = |z|.$$

In Fig. 2, the narrowing surfaces of Rosendorn is given in coordinates $r,\ v$;

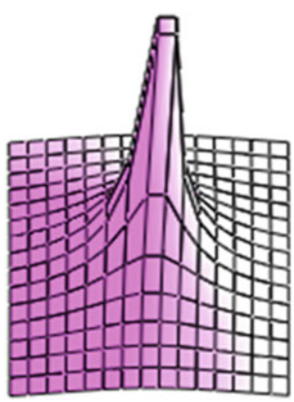

Fig. 1

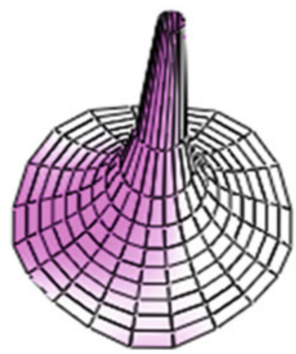

Fig. 2

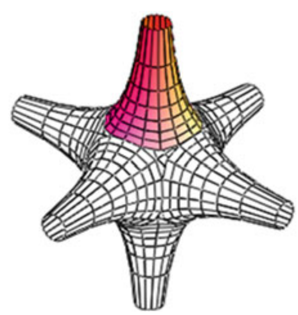

Fig. 3

$$x = x(r,v) = r\cos v; \quad y = y(r,v) = r\sin v;$$
$$z = z(r,v) = \sqrt{a^2 - r^4\cos^2 v\sin^2 v}/r,$$
$$0 \leq v \leq 2\pi; \quad 0.5 \leq r \leq 5; \quad a = 16.$$

The complete surface is represented in Fig. 3. The narrowing saddle surfaces cannot have the one-to-one correspondence spherical mapping.

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■ Flat Saddle in the Drum

A flat saddle in the drum reminds a saddle with practically plane segment at the center of the surface (Fig. 1). At the view from above, the surface is bounded by the circular contour and so it can be placed into a cylinder. At the center, the surface has a point which is both *parabolic* and *umbilic*.

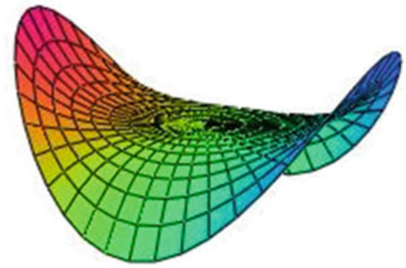

Fig. 1

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$x = x(r, v) = r \sin v, \quad y = y(r, v) = r \cos v,$

$z = z(r, v) = 0.5r^4 \sin(2v),$

where $0 \le r \le \infty,\ 0 \le v \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principle curvatures:

$$A^2 = 1 + 4r^6 \sin^2(2v),$$
$$F = r^7 \sin(4v),$$
$$B^2 = r^2 + r^8 \cos^2(2v);$$
$$L = \frac{-6r^3 \sin(2v)}{\sqrt{A^2B^2 - F^2}},$$
$$M = \frac{-3r^4 \cos(2v)}{\sqrt{A^2B^2 - F^2}},\ N = 0;$$
$$k_r = \frac{-6r^3 \sin(2v)}{A^2\sqrt{A^2B^2 - F^2}},$$
$$k_v = 0,\ K = \frac{-9r^4 \cos^2(2v)}{\left\{1 + r^6\left[1 + 3\sin^2(2v)\right]\right\}^2} \le 0.$$

This surface is a surface of *negative Gaussian curvature* and only along the lines $v = \pm\pi/4$, the parabolic points with $\mathcal{K} = 0$ are disposed and the point with the coordinates $v = r = 0$ is a parabolic and umbilical one.

(2) Explicit equation: $z = xy(x^2 + y^2)$.

Coefficients of the fundamental forms of the surface and its principle curvatures:

$$A^2 = 1 + y^2(3x^2 + y^2)^2,$$
$$F = xy(x^2 + 3y^2)(3x^2 + y^2),$$
$$B^2 = 1 + x^2(x^2 + 3y^2)^2;$$
$$L = N = \frac{6xy}{\sqrt{A^2B^2 - F^2}},\ M = \frac{3(x^2 + y^2)}{\sqrt{A^2B^2 - F^2}};$$
$$K = \frac{-9(x^2 - y^2)^2}{(A^2B^2 - F^2)^2} \le 0,$$
$$H = \frac{6xy\left[1 - (x^2 - y^2)^2(x^2 + y^2)\right]}{(A^2B^2 - F^2)^{3/2}},$$

Under this case of the definition of the surface, the lines $x = \pm y$ contain the parabolic points and the point with the coordinates $x = y = z = 0$ lying on the surface is simultaneously a parabolic and umbilical. Along the lines $x = 0$ and $y = 0$, the mean curvature is equal to zero.

■ Monkey Saddle

A monkey saddle has a point, which at the same time is parabolic and umbilical, three ranges and three slopes so the surface after turning at an angle of $2\pi/3$ coincides with itself. Obviously, that in this case, a slope is disposed opposite every range. Every normal cross section has *a point of inflection* and consequently, zero curvature.

A man needs in only two slope of a saddle but a monkey needs additionally in the third slope for tail. So a name of the surface in question appeared.

The surface possesses parallel normals at the points which are disposed diametrically opposite with respect to the point of inflection. Hence a closed curve on the sphere twice going round a spherical image of the parabolic isolated point corresponds with the closed not having double points of the curve run out a parabolic isolated point.

The surface with the isolated parabolic point may be designed by the rotation of a curve $f(u)$ passing through the origin of the coordinates $f(0) = 0$ and transforming in the process of rotation by the sine law with odd number of the waves:

$$f(u)\sin(\mathrm{n}v); \quad n = 1, 3, 5, \ldots; \quad u \ge 0 \text{ or } u \le 0; \quad 0 \le v \le 2\pi.$$

Assuming $n = 3$, we can obtain a monkey saddle.

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$x = x(u, v) = u\cos v, \quad y = y(u, v) = u \sin v,$$
$$z = z(u, v) = au^3 \sin(3v).$$

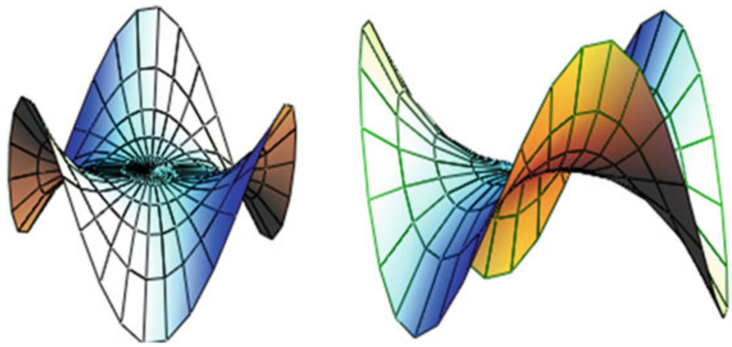

Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + 9a^2u^4\sin^2(3v),$$
$$F = 9a^2u^5\sin(3v)\cos(3v),$$
$$B^2 = u^2\left[1 + 9a^2u^4\cos^2(3v)\right];$$
$$A^2B^2 - F^2 = u^2(1 + 9a^2u^4),$$
$$L = \frac{6au\sin(3v)}{\sqrt{1+9a^2u^4}}$$
$$M = \frac{6au^2\cos(3v)}{\sqrt{1+9a^2u^4}},\ N = \frac{-6au^3\sin(3v)}{\sqrt{1+9a^2u^4}};$$
$$k_u = \frac{6au\sin(3v)}{A^2\sqrt{1+9a^2u^4}},$$
$$k_v = \frac{-6au\sin(3v)}{[1+9a^2u^4\cos^2(3v)]\sqrt{1+9a^2u^4}};$$
$$K = -\frac{36a^2u^2}{(1+9a^2u^4)^2} \leq 0,$$

$$H = \frac{-27a^3u^5\sin(3v)}{(1+9a^2u^4)^{3/2}}.$$

The monkey saddle is a surface of negative Gaussian curvature. The values of the coefficients of the fundamental forms show that the surface is given at nonorthogonal nonconjugate curvilinear coordinates u, v. Only at the isolated point $u =$ the, Gaussian K and mean H curvatures of the surface are equal to zero.

(2) Explicit equation:

$$z = x^3 - 3xy^2,$$

and it is equivalent to a definition of the surface in parametrical form:

$$x = x(u,v) = u\cos v, \quad y = y(u,v) = u\sin v,$$
$$z = z(u,v) = u^3\cos(3v).$$

■ Saddle Surface with Zero Rotation of Horn

The saddle surface with zero rotation of horn is a complete surface F of negative Gaussian curvature with a horn T (Fig. 2). The complete saddle surface is unlimited surface in E^3. For the surface in question $\omega(T) = 0$, where $\omega(T)$ is *the rotation of the horn T*. It will be equal to the rotation of the field of the tangents on every cross section of the horn T by a plane Q, if Q does not intersect the edges of the horn T.

The form of the definition of the surface

(1) The definition of the saddle surface, having a horn, in the cylindrical coordinates z, ρ, φ:

$$\rho = e^{-z}\frac{\sqrt{\cos^2\varphi - e^{2z}\sin^2\varphi}}{(\cos^2\varphi + e^{2z}\sin^2\varphi)}.$$

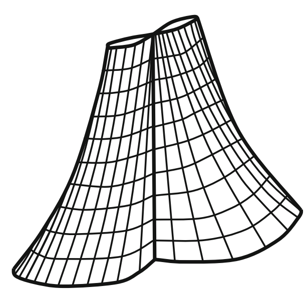

Fig. 1

The saddle surface (Fig. 1) has one-sheet spherical image. The plane belts on the horn T have self-intersections.

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Bakel'man IYa, Werner AL, Kantor BE. Introduction in Differential Geometry "in the Large". Moscow. 1973; 440 p.

34 Kinematical Surfaces of General Type

A generatrix curve of *a kinematical surface of general type* transferring from one position to another can keep a certain character of motion but parameters of movements, positions of axes and the direction of infinitesimal displacements of the generatrix line simultaneously change.

These displacements can be of the following types:

(1) translational motion of a varying direction;
(2) rotational motion with continuously changing position and direction of the rotation axis in the space;
(3) helical motion with continuously changing position and direction of the helix axis and with continuously changing parameter of the helical motion.

Depending on the type of displacements of a generatrix curve, kinematical surfaces of general type are subdivided in (1) *translation surfaces*, (2) *rotational surfaces*, and (3) *spiroidal surfaces*.

A translation surface may be given by a generatrix curve in the initial position and by some directrix curve determining the direction of translation (see also Chap. "3. Translation Surfaces". If one of the datum curves is a straight line, then the translation surface has the form of *a cylindrical surface*. If the datum lines are two skew lines, then the translation surface degenerates into *a plane*.

The motion of a generatrix curve is called *a rotational motion* if its infinitesimal step-by-step displacements are rotational about axes intersecting under infinitesimal angles. Rotational motion may be obtained when rolling of a developable surface on another developable surface. Cuspidal edges of both torses must have the equal curvatures in corresponding points i.e. the torse must roll on its bending. Assume that a generatrix curve is rigidly connected with a mobile developable surface. A rotational surface of general type is generated in the process of rolling of this mobile developable surface without slipping.

The rolling (mobile) and stationary developable surfaces are called *axoids*. So, a rotational surface may be defined by two osculating axoids and a generatrix curve rigidly connected with the rolling axoid.

A rotational surface is called *a regular rotational surface* if its moveable axoid is a plane. A regular rotational surface is called *a limaçon of revolution* if a generatrix plane curve belongs to the plane, i.e. to the moveable axoid.

Cylindrical or *conical limaçons of revolution* may be constructed if cylindrical or conical surfaces are assumed as stationary axoids, see also Sect. "4.1. Monge Surfaces with a Circular Cylindrical Directrix Surface" and Sect. "4.2. Monge Surfaces with a Conic Directrix Surface". If one assumes a straight line or a circle as generatrix curves, then it is possible to obtain *ruled* or *cyclic limaçons of revolution,* correspondingly, see also "Tubular surface with a plane line of centers in the form of an evolvent of the circle".

A motion of generatrix curve is called *spiroidal motion* if its infinitesimal step-by-step displacements are helical motion and the axes of two step-by-step infinitesimal displacements intersect and set up infinitesimal angles between themselves. If a generatrix curve rigidly connected with a mobile axoid, then in the process of its rolling with slipping on a stationary developable surface (on a unmovable axoid), we may obtain a general case of helical (spiroidal) motion of the generatrix curve. A surface formed by spiroidal motion of generatrix line is called *a spiroidal surface*; see also Sect. "34.2. Spiroidal Surfaces".

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34.1 Rotational Surfaces

A rotational surface (*rotative surface*) is formed by arbitrary space curve L rigidly connected with rolling mobile developable surface (*a mobile axoid*) and this rolling without sliding takes place on a stationary developable surface (*unmovable, motionless axoid*). In this case, they say, that a generatrix curve performs *rotational motion*.

Hence, a rotational motion of a line L is a motion when infinitesimal step-by-step its displacements are displacements of rotation about the continuously changing axis.

Developable surfaces with the help of which a motion of a curve L takes place are called mobile and stationary (unmovable) axoids.

A plane, a cylinder, a cone and a straight line are special forms of developable surface. But not every combination of axoids gives an opportunity to carry out rotational motion. A torse may roll without sliding only on its own bending.

Ten possible combinations of stationary and mobile axoids exist. They are (1) “straight line–plane”, (2) “plane–cone”, (3) “plane–cylinder”, (4) “plane torse”, (5) “cone–plane”, (6) “cone–cone”, (7) “cylinder–plane”, (8) “cylinder–cylinder”, (9) “torse–plane”, (10) “torse–torse”. But cone and cylinder, cylinder and torse cannot set up pairs of axoids for rotational motion.

When rolling of a cone over a stationary plane without sliding takes place, then the vertex of the rolling cone remains unmovable and hence, the cone executes *a spherical motion*.

A family of the normal planes of a spatial curve envelope some surface that is called *a polar torse*. Hence, some point of a plane rolling without sliding on the polar torse will generate the given curve.

Let us take a circular cylinder with the helix lying on it as mobile axoid then every point of the helix will trace the *plane cycloid* at time of the rolling. Geometric locus of the cycloids created by all points of the helix will form *the translation surface* (see also Sect. “3.3. Surfaces of Oblique Translation”).

An oblique helicoid may be related also to a subclass “Rotational Surfaces”. In this case, it is necessary to assume a plane as mobile axoid and this plane must roll over a stationary circular cylinder (motionless axoid).

A generatrix straight line must be parallel to the mobile plane and must intersect the axis of the right circular cylinder under a given acute angle (Fig. 1) and be rigidly connected with the mobile plane Σ. In the process of rolling of the plane Σ on the cylinder, a generatrix straight line m will be intersecting the axis of the cylinder in different its points, i.e. will form the oblique helicoid (see also “Oblique helicoid” in Sect. “1.2. Ruled Surfaces of Negative Total Curvature”). The lines of intersection of an oblique helicoid with coaxial cylinders are helixes. If a generatrix straight line l (Fig. 1) is parallel to the rolling plane Σ but is placed at some distance from the axis of a unmovable axoid that is a circular cylinder Ω, then in the process of rolling of the plane will form *the convolute helicoid* (see also “Convolute Helicoid” in Subsect. “7.1.1. Ruled Helical Surfaces”).

Having taken cones as axoids it is possible to generate *helical surfaces of variable helical parameter*. Considering external and internal rolling of the surface of a stationary cone, it is possible to obtain rotational surfaces of two types.

One of domes of Saint Basil the Blessed, Church constructed on Red Square in Moscow between 1554 and 1560 by Tsar Ivan IV the Terrible, bears a resemblance to the

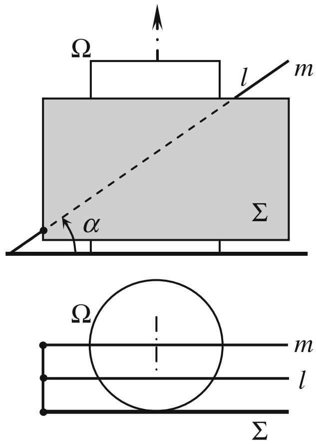

Fig. 1

surface formed by the rotational motion of a generating helical line lying on the surface of revolution in the form of a spindle which is rigidly connected with a mobile cone with the diameter that is many times (integer) less than the diameter of the stationary cone. In spite of all this, the mobile cone must roll on external surface of the stationary cone.

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34.1.1 Rotational Surfaces with Axoids "Cylinder–Plane"

Rotational surfaces with axoids "cylinder–plane" have a circular cylinder as stationary axoid and a plane as mobile axoid. For the definition of a considered rotational surface, let us assume two systems of Cartesian coordinates, one of them is a stationary system of coordinates $Oxyz$ but another one is a mobile system of coordinates $o_1X_1Y_1Z_1$.

An axis Oz of the stationary system of coordinates coincides with the axis of the cylinder (Fig. 1). In its initial position, an origin of the mobile system of coordinates o_1 is disposed at a point with coordinates $x = R$; $y = 0$; $z = 0$, where R is a radius of the stationary axoid that is a circular cylinder, but an axis o_1Z_1 coincides with a rectilinear generatrix of the cylinder.

When rolling of a plane Q over a cylinder without sliding is available, then a generatrix line L given in the mobile system of coordinates with the help of parametrical equations

$$X_1 = X_1(u),\ Y_1 = Y_1(u),\ Z = Z_1(u),$$

will form a rotational surface with axoids "cylinder–plane".

In general case, a curve L may be by arbitrary spatial line.

Forms of definition of the rotative surfaces

(1) Vector equation (Fig. 1):

$$\begin{aligned} \boldsymbol{r} = \boldsymbol{r}(u,\varphi) = R(\cos\varphi\boldsymbol{i} + \sin\varphi\boldsymbol{j}) \\ + R\varphi(\sin\varphi\boldsymbol{i} - \cos\varphi\boldsymbol{j}) + Y_1(u)(\sin\varphi\boldsymbol{i} - \cos\varphi\boldsymbol{j}) \\ + X_1(u)(\cos\varphi\boldsymbol{i} + \sin\varphi\boldsymbol{j}) + Z_1(u)\boldsymbol{k}, \end{aligned}$$

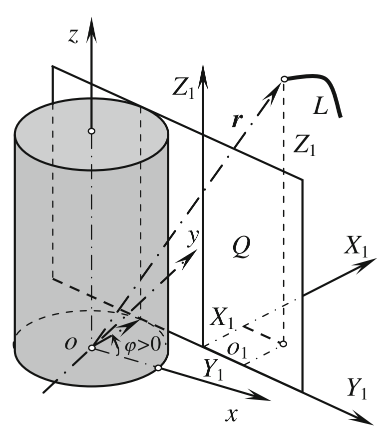

Fig. 1

where φ is the angle read from the coordinate axis Ox in the direction of the axis Oy.

(2) Parametrical equations (Fig. 1):

$$\begin{aligned} x &= x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi) \\ &\qquad + X_1(u)\cos\varphi + Y_1(u)\sin\varphi, \\ y &= y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi) \\ &\qquad + X_1(u)\sin\varphi - Y_1(u)\cos\varphi, \\ z &= z(u) = Z_1(u). \end{aligned}$$

Coefficients of the fundamental forms of the surface:

$$A^2 = X_1'^2 + Y_1'^2 + Z_1'^2, \quad F = R\varphi X_1' + X_1'Y_1 - X_1Y_1',$$
$$B^2 = R^2\varphi^2 + 2R\varphi Y_1 + X_1^2 + Y_1^2,$$
$$L = \frac{1}{\sqrt{A^2B^2 - F^2}}\begin{vmatrix} X_1''\cos\varphi + Y_1''\sin\varphi & X_1''\sin\varphi - Y_1''\cos\varphi & Z_1'' \\ X_1'\cos\varphi + Y_1'\sin\varphi & X_1'\sin\varphi - Y_1'\cos\varphi & Z_1' \\ R\varphi\cos\varphi - X_1\sin\varphi + Y_1\cos\varphi & R\varphi\sin\varphi + X_1\cos\varphi + Y_1\sin\varphi & 0 \end{vmatrix},$$
$$M = \frac{Z_1'}{\sqrt{A^2B^2 - F^2}}(R\varphi X_1' + X_1'Y_1 - X_1Y_1'),$$
$$N = \frac{Z_1'}{\sqrt{A^2B^2 - F^2}}[(R\varphi + Y_1)^2 + X_1(X_1 - R)],$$

where

$$\ldots' = \frac{\mathrm{d}\ldots}{\mathrm{d}u}.$$

If a generating curve L is a plane curve and lies in the plane $Y_1o_1Z_1$, then due to rolling of the plane with the curve L rigidly connected with it over the cylinder, we shall form *a Monge surface with a circular cylindrical directrix surface* (see also Chap. "4. Carved Surfaces").

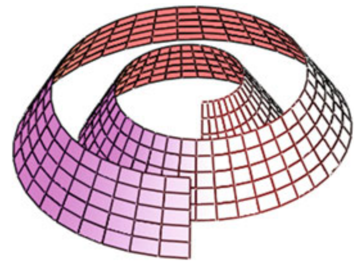

Fig. 2

Let a straight line is a generatrix curve L. In this case, we can obtain four groups of ruled surfaces, viz, *cylindrical surfaces*, *Monge ruled surfaces with a circular cylindrical directing surface* (Fig. 2)*, plane, and ruled rotational surfaces of general type with axoids "cylinder–plane"*

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■ Ruled Rotational Surface with Axoids "Cylinder–Plane"

A ruled rotational surface with axoids ""cylinder–plane"" has stationary axoid in the form of a circular cylinder and a plane as mobile axoid. The surface is traced by a straight line disposed as desired but rigidly connected with the plane rolling without sliding. A ruled rotational surface with axoids "cylinder–plane" is called *a regular ruled cylindrical rotational surface* also.

Forms of definition of the surface

(1) Parametrical equations:

$$\begin{aligned} x &= x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi) + u\cos\varphi, \\ y &= y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi) + u\sin\varphi, \\ z &= z(u) = u\tan\alpha, \end{aligned}$$

where R is the radius of a stationary cylinder, α is the slope angle of the straight generatrix L_1 with the mobile coordinate axis o_1X_1. Under this form of definition of rotational surface, the generatrix straight L_1 (Fig. 1) must be placed in the plane $X_1o_1Z_1$, and parametrical equations of the straight L_1 given in the mobile system of coordinates have the following form:

$$X_1(u) = u,\ Y_1 = 0,\ Z_1(u) = u\tan\alpha.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 1/\cos^2\alpha,\ F = R\varphi,$$
$$B^2 = R^2\varphi^2 + u^2,$$
$$A^2B^2 - F^2 = u^2A^2 + R^2\varphi^2\tan^2\alpha,$$
$$L = 0,\ M = \frac{R\varphi\tan\alpha}{\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{\tan\alpha[R^2\varphi^2 + u^2 - uR]}{\sqrt{A^2B^2 - F^2}},$$
$$K = -\frac{R^2\varphi^2\tan^2\alpha}{(A^2B^2 - F^2)^2}.$$

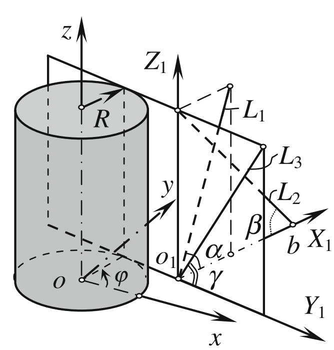

Fig. 1

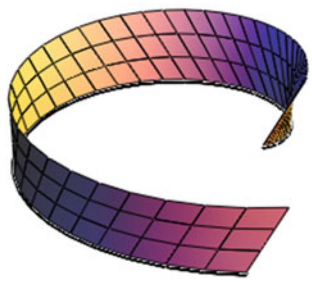

Fig. 2

A ruled surface of negative Gaussian curvature can be designed if there is a generatrix straight in the plane $X_1o_1Z_1$. In Fig. 2, the considered rotational surface is shown when $R = 1$ m, $\alpha = \pi/3$, $0 \le u \le 1$ m; $0 \le \varphi \le 2\pi$.

(2) Parametrical equations:

$$x = x(u, \varphi) = R(\cos\varphi + \varphi\sin\varphi) + u\cos\varphi,$$
$$y = y(u, \varphi) = R(\sin\varphi - \varphi\cos\varphi) + u\sin\varphi,$$
$$z = z(u) = (b - u)\tan\beta,$$

where β is the slope angle of the straight generatrix with the mobile coordinate axis o_1X_1 (Fig. 1). A generatrix straight line L_2 is placed in the plane $X_1o_1Z_1$. Parametrical equations of this straight line have the following form:

$$X_1(u) = u, Y_1 = 0, Z_1(u) = (b - u)\tan\beta.$$

In Fig. 3, the rotational ruled surface is presented with the following geometrical parameters: $\beta = \pi/3$, $R = b = 1$ m, $0 \le u \le 1$ m; $0 \le \varphi \le 2\pi$.

(3) Parametrical equations:

$$x = x(u, \varphi) = R(\cos\varphi + \varphi\sin\varphi) + u\sin\varphi,$$
$$y = y(u, \varphi) = R(\sin\varphi - \varphi\cos\varphi) - u\cos\varphi,$$
$$z = z(u) = u\tan\gamma,$$

where γ is the slope angle of the straight generatrix with the mobile coordinate axis o_1Y_1 (Fig. 1). A generatrix straight line L_3 lies in the plane $Y_1o_1Z_1$ and is given by equations:

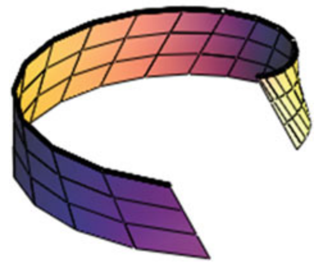

Fig. 3

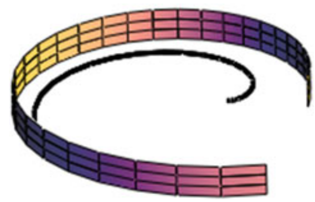

Fig. 4

$$X_1(u) = 0,\ Y_1(u) = u,\ Z_1(u) = u\tan\gamma.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 1/\cos^2\gamma,\ F = 0,\ B^2 = (R\varphi + u)^2,$$
$$L = M = 0,\ N = B\sin\gamma,\ K = 0.$$

Having taken a generatrix straight line L_3 (Fig. 1) in a rolling plane, we may obtain a surface of *zero Gaussian curvature* which is called *a ruled cylindrical limaçon of revolution* or, what is the same, *a Monge ruled surface with a circular cylindrical directrix surface.* Assume $\alpha = \pi/2$ or $\gamma = \pi/2$ (Fig. 1), then the constructed surface will be a cylindrical surface with *the directrix evolvent of a circle* with a radius R (Fig. 4, the heavy line). Taking $\beta = \pi/2$, we can construct *a cylindrical surface with a directrix curve that is equidistant to the evolvent of a circle* with a radius R (Fig. 4). Putting $\alpha = 0$ or $\gamma = 0$, we shall have fragments of a plane.

■ Cyclic Rotational Surface with Axoids "Cylinder–Plane"

A cyclic rotational surface with axoids "cylinder–plane" has stationary axoid in the form of a circular cylinder and a plane as mobile axoid.

The surface is formed by a circle located arbitrarily but rigidly connected with the plane rolling without sliding (Fig. 1).

A cyclic rotational surface with axoids "cylinder–plane" is called also *a regular cyclic cylindrical rotational surface*.

Forms of definition of the surface

(1) Parametrical equations (Fig. 2):

$$x = x(u, \varphi) = R(\cos\varphi + \varphi\sin\varphi) + (b + r\cos u)\cos\varphi,$$
$$y = y(u, \varphi) = R(\sin\varphi - \varphi\cos\varphi) + (b + r\cos u)\sin\varphi,$$
$$z = z(u) = a + r\sin u,$$

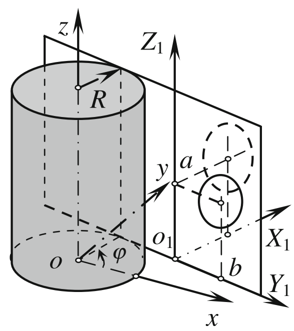

Fig. 1

where R is the radius of a stationary cylinder, r is the radius of a generatrix circle lying in the plane $X_1o_1Z_1$ (Fig. 1); $X_1 = b$, $Z_1 = a$ are the coordinates of the center of the generatrix circle in a mobile system of Cartesian coordinates.

Parametrical equations of the generatrix circle given in the mobile system of coordinates must be written in the following form:

$$X_1(u) = b + r\cos u,\ \ Y_1 = 0,$$
$$Z_1(u) = a + r\sin u.$$

Coefficients of the fundamental forms of the surface:

$$A = r,\ \ F = -rR\varphi\sin u,$$
$$B^2 = (b + r\cos u)^2 + R^2\varphi^2,$$
$$L = \frac{r^2(b + r\cos u)}{\sqrt{A^2B^2 - F^2}},\quad M = -\frac{r^2R\varphi\sin u\cos u}{\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{r\cos u[R^2\varphi^2 - R(b + r\cos u) + (b + r\cos u)^2]}{\sqrt{A^2B^2 - F^2}}.$$

Fig. 2

The surface is given in non-orthogonal non-conjugated system of curvilinear coordinates u, φ. In Fig. 2, the cyclic rotational surface with axoids “cylinder–plane” is shown when

$$r = R = a = b = 1\ \text{m};\ \ 0 \le u \le 2\pi;\ \ 0 \le \varphi \le 2\pi.$$

If $b = r = R$ then coefficients of the fundamental forms of the surface take the following form:

$$A = r,\ \ F = -r^2\varphi\sin u,$$
$$B^2 = r^2[(1 + \cos u)^2 + \varphi^2],$$
$$A^2B^2 - F^2 = r^4[\varphi^2\cos^2 u + (1 + \cos u)^2],$$
$$L = \frac{r(1 + \cos u)}{\sqrt{\varphi^2\cos^2 u + (1 + \cos u)^2}},$$
$$M = -\frac{r\varphi\sin u\cos u}{\sqrt{\varphi^2\cos^2 u + (1 + \cos u)^2}},$$
$$N = \frac{r[\varphi^2 + \cos u(1 + \cos u)]\cos u}{\sqrt{\varphi^2\cos^2 u + (1 + \cos u)^2}}.$$

(2) Parametrical equations:

$$x = x(u, \varphi) = R(\cos\varphi + \varphi\sin\varphi) + (b + r\cos u)\sin\varphi,$$
$$y = y(u, \varphi) = R(\sin\varphi - \varphi\cos\varphi) - (b + r\cos u)\cos\varphi,$$
$$z = z(u) = a + r\sin u,$$

where R is the radius of a stationary cylindrical axoid, r is the radius of a generatrix circle lying in the rolling plane $Y_1o_1Z_1$;

$$Y_1 = b,\ Z_1 = a$$

are the coordinates of the center of the generatrix circle in the mobile system of coordinates. Parametrical equations of the generatrix circle given in the mobile system of coordinates must be written in the following form:

$$X_1 = 0,$$
$$Y_1(u) = b + r\cos u,\ Z_1(u) = a + r\sin u.$$

Using this variant of the disposition of the generatrix circle, we can obtain *a Monge surface with a circular cylindrical directrix surface and with a circular meridian* which may be called *a cyclic cylindrical limaçon of revolution*.

Fig. 3

(3) Parametrical equations:

$$
\begin{aligned}
x = x(u,\varphi) &= R(\cos\varphi + \varphi\sin\varphi) + b\cos\varphi \\
&\quad + (d + r\cos u)\sin\varphi, \\
y = y(u,\varphi) &= R(\sin\varphi - \varphi\cos\varphi) + b\sin\varphi \\
&\quad - (d + r\cos u)\cos\varphi, \\
z = z(u) &= a + r\sin u.
\end{aligned}
$$

In Fig. 3, the surface with $a = d = r = R = 1$ m; $b = 4$ m is shown. The trajectory of motion of the point o_1, which is the evolvent of the circle with the radius R, is denoted by the heavy line

34.1.2 Rotational Surfaces with Axoids "Cylinder–Cylinder"

Assume a pair of axoids, that are the cylinders of revolution with radiuses of the bases equal to R and r, where $R = nr$, and take some line$l$rigidly connected with the loose axoid that is the cylinder Ω_l with the radius r. Rotational surface with the axoids "cylinder–cylinder" is formed by the line l in the process of rolling without sliding of the mobile cylinder Ω_l above the fixed cylinder Ω_f. In Fig. 1, the position of the axoids at the initial moment of the time is shown.

The generatrix line l is given by the parametrical equations:

$$X_1 = X_1(u),\ Y_1 = Y_1(u),\ Z_1 = Z_1(u)$$

in a mobile system of coordinates $O_1X_1Y_1Z_1$, the axis O_1Z_1 of which coincides with the axis of the mobile cylinder Ω_l. The origin of the fixed system of coordinates $Oxyz$ is disposed at the O center of the base of the fixed cylinder Ω_f and the coordinate axis Oz coincides with its axis (Fig. 1). At the beginning of the motion, the axes of the both coordinate systems are parallel.

If a point M belongs to the l curve, then its coordinates in two systems are accordingly $(x,\ y,\ z)$ and (X_1, Y_1, Z_1). After some rotational motion, the loose axoid will occupy a post shown in Fig. 2. Due to this drawing, one may obtain the following geometrical relations:

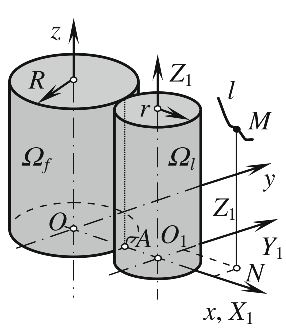

Fig. 1

$$
\begin{gathered}
R\varphi = rt, \text{ or } n\varphi = t, n = R/r, \\
\varphi + t = (1+n)\varphi,\ \overline{NM} = Z_1\boldsymbol{k},
\end{gathered}
$$

where φ is the angle read from the fixed axis Ox in the direction of the rolling till a straight line connecting the O center of the fixed cylinder with a point D. The line of contact of two axoids is projected into the point D; the angle t is shown in Fig. 2.

Forms of definition of the rotational surfaces with axoids "cylinder–cylinder"

(1) Parametrical form of the definition under external rolling of the fixed cylinder (Figs. 1 and 2):

$$
\begin{aligned}
x = x(u,\varphi) &= (R+r)\cos\varphi + X_1(u)\cos(n+1)\varphi \\
&\quad - Y_1(u)\sin(n+1)\varphi \\
y = y(u,\varphi) &= (R+r)\sin\varphi + X_1(u)\sin(n+1)\varphi \\
&\quad + Y_1(u)\cos(n+1)\varphi, \\
z = z(u) &= Z_1(u).
\end{aligned}
$$

Coefficients of the fundamental forms of the surface:

$$
\begin{aligned}
A^2 &= X_1'^2 + Y_1'^2 + Z_1'^2, \\
F &= (R+r)(X_1'\sin n\varphi + Y_1'\cos n\varphi) + (1+n)(X_1Y_1' - X_1'Y_1), \\
B^2 &= (R+r)^2 + (1+n)^2(X_1^2 + Y_1^2) \\
&\quad + 2(1+n)(R+r)(X_1\cos n\varphi - Y_1\sin n\varphi); \\
L &= \{-Z_1'[(R+r)(X_1''\cos n\varphi + Y_1''\sin n\varphi) \\
&\quad + (1+n)(X_1X_1'' + Y_1Y_1'')] + Z_1''[(R+r)(X_1'\cos n\varphi - Y_1'\sin n\varphi) \\
&\quad + (1+n)(X_1X_1' + Y_1Y_1')]\}/\Sigma \\
N &= Z'\Big[(R+r)^2 + (1+n)^3(X_1^2 + Y_1^2) \\
&\quad + (2+n)(1+n)(R+r)(X_1\cos n\varphi - Y_1\sin n\varphi)\Big]/\Sigma;
\end{aligned}
$$

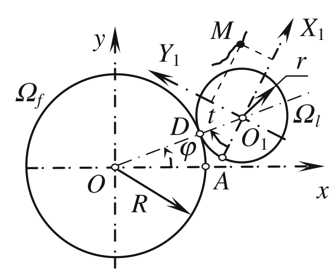

Fig. 2

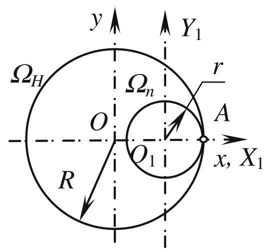

Fig. 3

where

$$\Sigma = \sqrt{A^2B^2 - F^2},$$

$$\ldots' = \frac{\mathrm{d}\ldots}{\mathrm{d}u}, \ \ldots'' = \frac{\mathrm{d}^2\ldots}{\mathrm{d}u^2}.$$

(2) Parametrical form of the definition under internal rolling of the fixed cylinder (in Fig. 3, the initial disposition of the axoids is presented):

$$\begin{aligned} x = x(u,\varphi) &= (R-r)\cos\varphi + X_1(u)\cos(n-1)\varphi \\ &\quad + Y_1(u)\sin(n-1)\varphi; \\ y = y(u,\varphi) &= (R-r)\sin\varphi - X_1(u)\sin(n-1)\varphi \\ &\quad + Y_1(u)\cos(n-1)\varphi, \\ z = z(u) &= Z_1(u). \end{aligned}$$

Here φ is the angle read from the axis Ox till the straight line OD connecting the point O and a point of the contact of the axoids after the beginning of the rolling, t is an angle from the new position of the axis O_1X_1 till the straight line OD.

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■ Ruled Rotational Surface of Lusta

For construction of *a ruled rotational surface of Lusta,* it is necessary to take a fixed circular cylinder of a radius R and a mobile cylinder of a radius $R/2$. The generatrix straight line AB lying at the meridian plane of the mobile cylinder (Fig. 1) under internal rolling on the surface of the fixed axoid will have the constant angle α with a coordinate plane $z = 0$. In the process of rolling without sliding, the point A will be on the top bases of the cylinders all the time and the point B will be on the low bases. But the point A will move along the axis y_1 of the top base, but the point B will move along the axis Ox of the low base.

Forms of the definition of the ruled rotational surface of Lusta

(1) Implicit form of the definition:

$$\frac{x^2}{(R - Rz/H)^2} + \frac{y^2}{(Rz/H)^2} = 1,$$

where H is the height of the cylinders (Fig. 1). Ruled rotational surface of Lusta is *an algebraic surface of the fourth order*. The lengths of the pieces of all its straight generatrixes between two horizontal cross-sections are equal to each other. The horizontal cross-sections of the surface in question are ellipses. The ellipse of the middle cross-section of the surface with $z = H/2$ degenerates into a circle of a radius R. The ellipses of the low and top cross-sections generate into the straight lines with the angle of their crossing equal to 90°. The cross-sections by the horizontal planes equidistant from the middle cross section give the equal ellipses rotated to each other through an angle of 90°. The sum of the lengths of the major and minor axes of the every ellipse obtained by a horizontal cross section is a constant value and is equal to $2R$. The horizontal projection

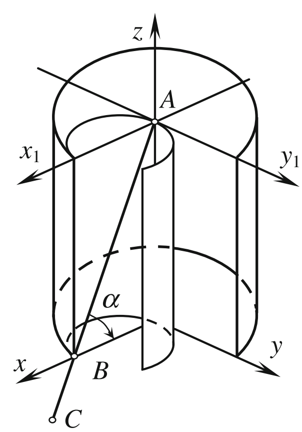

Fig. 1

of the surface falls within *a four-vertex hypocycloid* (*astroid*) which is an envelope of projections of the generatrix straight line. The frontal and profile projections of the surface in question represents by the contour dispositions of the generatrix analogically to contour generatrixes of conical surface. If we shall continue the generatrix straight line in the direction of the points A and B, then every point of the generatrix lying out of the straight-line segment AB will trace an ellipse also. The family of these ellipses are characteristically by the fact that the differences of the axes of every of them are equivalent and equal to $2R$.

(2) Parametrical form of the definition (Fig. 2):

$$x = x(u, v) = R(1 - u/H)\cos v,$$
$$y = y(u, v) = (uR/H)\sin v,$$
$$z = u.$$

The coordinate lines v are ellipses; $0 \le v \le 2\pi$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A^2 = 1 + \frac{R^2}{H^2} = \frac{1}{\sin^2\alpha},$$
$$F = \frac{R^2}{H}\sin v\cos v,$$
$$B^2 = R^2\left[\left(1 - \frac{2u}{H}\right)\sin^2 v + \frac{u^2}{H^2}\right];$$
$$L = 0,$$
$$M = \frac{-R^2\sin v\cos v}{H\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{uR^2}{H\sqrt{A^2B^2 - F^2}}\left(1 - \frac{u}{H}\right);$$
$$k_1 = 0,\ K = \frac{-R^4\sin^2 v\cos^2 v}{H^2(A^2B^2 - F^2)^2} < 0.$$

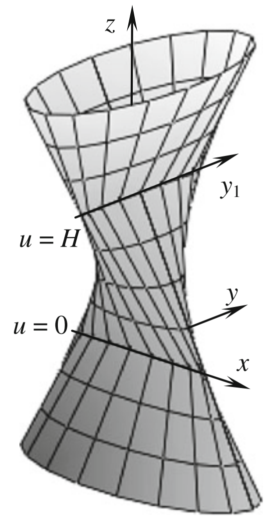

Fig. 2

Ruled rotational surface of Lusta is a surface of negative Gaussian curvature and only along the coordinate lines $v = 0$; $\pi/2$; π; $3\pi/2$, parabolic points are placed.

The fragment of the surface (Fig. 2) between the double straight lines $u = 0$ and $u = H$ is called "Die Milchtüte" in German.

The ruled surface in question may be recommended for the application as a rod structure for transmission of load from one beam to another one lying crosswise. In this case, the rods will work in compression or tension.

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Lusta GI. A review on rotational surfaces. Trudy Moskovskogo Nauchno-Metod. Seminara po Nachertat. Geometrii i Ingenernoy Grafike. Moscow. 1963; Iss. 2, p. 120-124 (4 refs).

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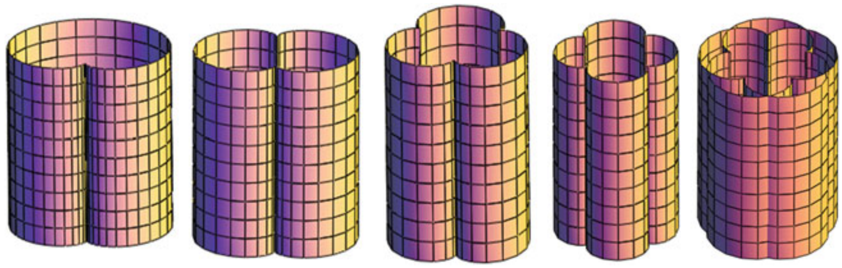
■ Epicycloidal Cylinder

*Epicycloidal cylinder*or *epicycloidal right cylindrical surface* is formed by a straight generatrix of a loose axoid that is a circular cylinder with a radius r, and this cylinder rolls without sliding above the external surface of a fixed axoid that is a cylinder with a radius R.

Using the designations and the formulas given before at the page Sect. "34.1.2. Rotational Surface with Axoids "Cylinder–Cylinder,"" the parametric equations of the generatrix straight line given in the mobile system of Cartesian coordinates may be represented in the following form:

$$X_1 = -r;$$
$$Y_1 = 0;$$
$$Z_1 = u.$$



$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 3.5$

$0 \le \varphi \le 2\pi$ $0 \le \varphi \le 4\pi$

Fig. 1 $R = 1\,\mathrm{m}$; $0 \le u \le 4\,\mathrm{m}$

The form of the definition of the epicycloidal cylinder

(1) Parametrical equations:

$$x = x(u, \varphi) = (R + r)\cos\varphi - r\cos(1 + n)\varphi,$$
$$y = y(u, \varphi) = (R + r)\sin\varphi - r\sin(1 + n)\varphi,$$
$$z = u,$$

where $n = R/r$; φ is the angle read from the fixed axis Ox in the direction of the rolling until the straight line connecting the center O of the fixed cylinder with the D point. The line of contact of two cylindrical axoids is projected into the point D. If a modulus of n is integer, then a cylindrical rotative surface does not intersect itself (Fig. 1).

Coefficients of the fundamental forms of the surface:

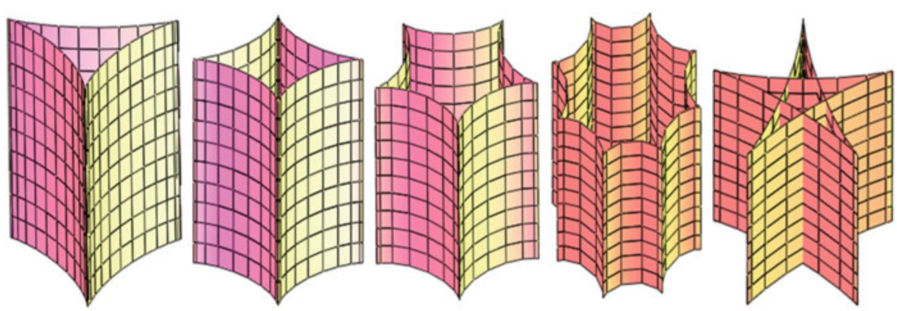
$$A = 1,\ F = 0,$$
$$B^2 = (R + r)^2 + r^2(1 + n)^2 - 2r(R + r)(1 + n)\cos n\varphi;$$
$$L = M = 0,$$
$$N = \left[(R + r)^2 + r^2(1 + n)^3 - r(R + r) \times (1 + n)(2 + n)\cos n\varphi\right]/B;$$
$$K = 0.$$

■ Hypocycloidal Cylinder

Hypocycloidal cylinder or *hypocycloidal right cylindrical surface* is formed by a straight generatrix of a loose axoid that is a cylinder with a radius r rolling without sliding on the internal surface of the fixed axoid that is a cylinder with a radius R.

Using the designations and the formulas given before at the page Sect. "34.1.2. Rotational Surface with Axoids "Cylinder–Cylinder"", the parametrical equations of a generatrix straight line given in a mobile system of Cartesian coordinates may be written in the following form:

$$X_1 = r;$$
$$Y_1 = 0;$$
$$Z_1 = u.$$



$n = 3$; $n = 4$; $n = 5$; $n = 8$ $n = 2.5$

$0 \le \varphi \le 2\pi$ $0 \le \varphi \le 4\pi$

Fig. 1 $R = 1\,\mathrm{m}$; $0 \le u \le 4\,\mathrm{m}$

The form of the definition of the hypocycloidal cylinder

(1) Parametrical equations:

$$x = x(u, \varphi) = (R - r)\cos\varphi + r\cos(n - 1)\varphi,$$
$$y = y(u, \varphi) = (R - r)\sin\varphi - r\sin(n - 1)\varphi,$$
$$z = u$$

where $n = R/r$; φ is the angle read from the fixed axis Ox in the direction of the rolling until the straight line connecting the center O of the fixed cylinder with the D point. The line of contact of two cylindrical axoids is projected into the point D. If a modulus of n is integer, then a cylindrical rotative surface does not intersect itself (Fig. 1). When $n = 2$, the cylindrical rotational surface degenerates into the rectangular segment of the plane $y = 0$ bounded by the straight lines $x = -R$; $x = R$. Taking $n = 4$, we may obtain *the astroidal cylindrical surface*.

Additional literature

Rachkovskaya GS. Construction of lines and surfaces on the base of the rotational transformation. Dis. Kand. Tehn. Nauk, Nizhniy Novgorod, NGAS. 1997.

■ Rotational Surface with Axoids "Cylinder–Cylinder" Formed by a Straight not Intersecting the Axis of a Mobile Cylinder in the Process of External Rolling

Assume that a generatrix straight line is given in the mobile system of coordinates $O_1X_1Y_1Z_1$ as

$$X_1 = u,$$
$$Y_1 = b(1 - u/a),$$
$$Z_1 = H(1 - u/a),$$

where constants a, b, H are shown in Fig. 1. In this case, parametrical equations of a ruled rotational surface formed by the straight generatrix under the external rolling without sliding of a mobile cylinder Ω_l with a radius r above a fixed cylinder Ω_f of radius R may be written in the following form:

$$x = x(u, \varphi) = (R + r)\cos\varphi + u\cos(n + 1)\varphi - b(1 - u/a)\sin(n + 1)\varphi,$$
$$y = y(u, \varphi) = (R + r)\sin\varphi + u\sin(n + 1)\varphi + b(1 - u/a)\cos(n + 1)\varphi,$$
$$z = H(1 - u/a).$$

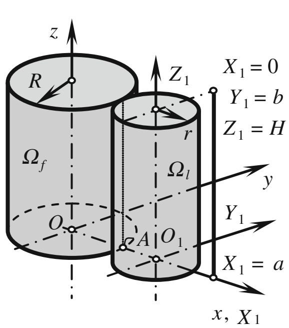


Fig. 1

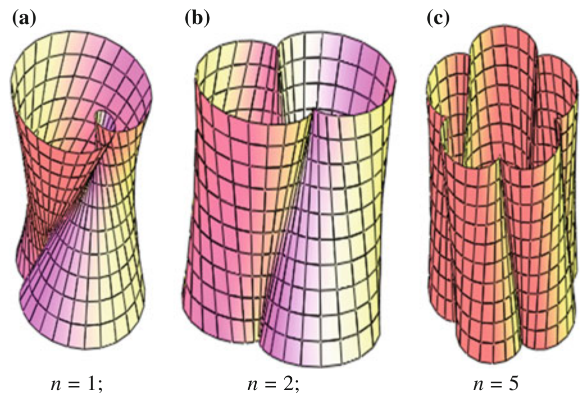
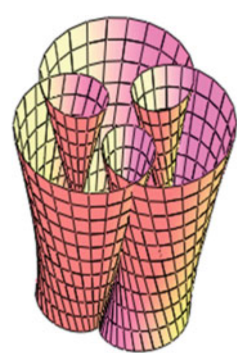


Fig. 2 $R = 1$ m; $H = 4$ m; $a = -b = r$

The presented equations of the surface is obtained with the help of the general equations given before on the page Sect. "34.1.2. Rotational Surfaces with Axoids "Cylinder–Cylinder."" On the same page, there are given the formulas



$n = 3$; $a = r$; $b = -3r$;

Fig. 3 $R = 1$ m; $H = 4$ m

for calculation of the coefficients of the fundamental forms of the surface in question.

The ruled rotational surfaces are surfaces of *negative Gaussian curvature*.

In Fig. 2, the rotational surfaces with the different values of $n = R/r$ are given but Fig. 2a shows the surface, when $-H/2 \le z \le 1.5H$; Figs. 2b, c and 3 show the surfaces if —$0 \le z \le H$.

■ RotationalSurface With Axoids "Cylinder–Cylinder" Formed by a Straight not Intersecting the Axis of a Mobile Cylinder in the Process of Internal Rolling

Assume that a generatrix straight line is given in the movable system of coordinates $O_1X_1Y_1Z_1$ as

$$X_1 = u,$$
$$Y_1 = b(1 - u/a),$$
$$Z_1 = H(1 - u/a),$$

where constants a, b, H are shown in Fig. 1. In this case, parametrical equations of a ruled rotational surface formed by the straight generatrix l under the internal rolling without sliding of a mobile cylinder with a radius r on a fixed cylinder of a radius R may be written in the following form:

$$\begin{aligned} x = x(u,\varphi) &= (R - r)\cos\varphi + u\cos(n-1)\varphi \\ &\quad + b(1 - u/a)\sin(n-1)\varphi; \\ y = y(u,\varphi) &= (R - r)\sin\varphi - u\sin(n-1)\varphi \\ &\quad + b(1 - u/a)\cos(n-1)\varphi, \\ z = H(1 - u/a), & \end{aligned}$$

where $n = R/r$.

The presented equations of the surface is obtained with the help of the general equations given before at the page Sect. "34.1.2. Rotational Surfaces with Axoids "Cylinder–Cylinder."" On the same page, there are given the formulas for calculation of the coefficients of the fundamental forms of the surface in question.

The ruled rotational surfaces are surfaces of *negative Gaussian curvature.*

In Fig. 1, the initial disposition of two cylindrical axoids is shown.

In Fig. 2, the rotational surfaces have different values of b and n; Fig. 2a shows the surface when $0 \le z \le H$, in Fig. 1b, the surface is designed, when $-H/2 \le z \le 1.5H$.

In Fig. 3, the strip section of the ruled rotational surface with $R = 1$ m; $H = 2$ m; $a = -b = r, n = 4$ and $-H \le z \le 0.6H$ is given.

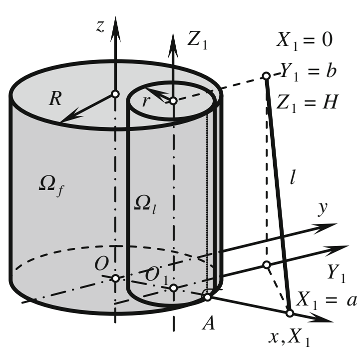


Fig. 1

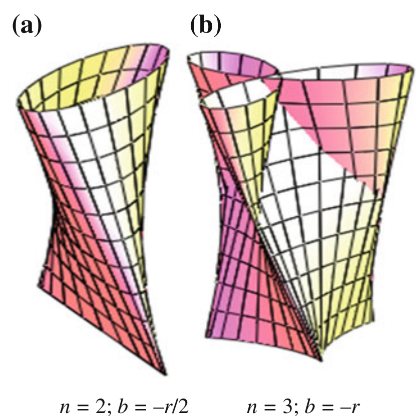


Fig. 2 $R = 1$ m; $H = 2$ m; $a = r$

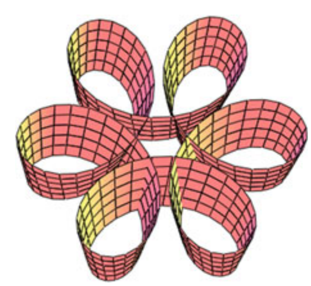


Fig. 3 $a = -b = r, n = 4$

■ Rotational Surface with Axoids "Cylinder–Cylinder" Formed by a Straight Intersecting the Axis of a Mobile Cylinder in the Process of External Rolling

Let a generatrix straight line l is given in the mobile system of coordinates $O_1X_1Y_1Z_1$ as

$$X_1 = u,\ Y_1 = 0,\ Z_1 = H(a-u)/(a+b),$$

where constants a, b, H are shown in Fig. 1. In this case, parametric equations of a ruled rotational surface formed by the straight line l under the external rolling without sliding of a mobile cylinder Ω_l with a radius r above a fixed one Ω_f with a radius R may be written as (Fig. 2):

$$\begin{aligned} x &= x(u,\varphi) = (R+r)\cos\varphi + u\cos(n+1)\varphi;\\ y &= y(u,\varphi) = (R+r)\sin\varphi + u\sin(n+1)\varphi,\\ z &= H(a-u)/(a+b), \end{aligned}$$

where $n = R/r$.

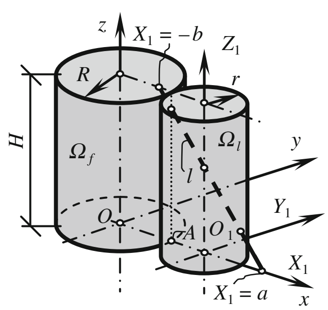


Fig. 1

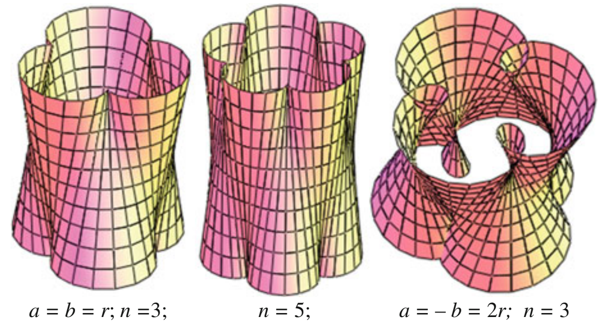


Fig. 2 $R = 1\,\text{m};\ H = 4\,\text{m};\ 0 \le z \le H$

The presented equations of the surface with $K < 0$ are obtained with the help of the general equations given before at the page Sect. "34.1.2. Rotational Surfaces with Axoids "Cylinder–Cylinder.""

Coefficients of the fundamental forms of the surface with $a = b = r$:

$$\begin{aligned} A^2 &= 1 + H^2/(4r^2),\\ F &= (R+r)\sin n\varphi,\\ B^2 &= (R+r)^2 + u^2(n+1)^2\\ &\quad + 2u(n+1)(R+r)\cos n\varphi;\\ L &= 0,\ M = -\frac{(n+1)H(R+r)}{2r\sqrt{A^2B^2-F^2}}\sin n\varphi,\\ N &= -\frac{H\left[(R+r)^2 + u^2(n+1)^3 + u(n+1)(R+r)(2+n)\cos n\varphi\right]}{2r\sqrt{A^2B^2-F^2}}. \end{aligned}$$

■ Rotational Surface with Axoids "Cylinder–Cylinder" Formed by a Straight Intersecting the Axis of a Mobile Cylinder in the Process of Internal Rolling

Let a generatrix straight line l s given in the mobile system of Cartesian coordinates $O_1X_1Y_1Z_1$ in the following form:

$$\begin{aligned} X_1 &= u,\\ Y_1 &= 0,\\ Z_1 &= H(a-u)/(a+b), \end{aligned}$$

where constants a, b, H are shown in Fig. 1. In this case, parametric equations of a ruled rotational surface formed by the straight line l under the internal rolling without sliding of a mobile cylinder Ω_l with a radius r on a fixed cylinder Ω_f with a radius R may be given in the form (Fig. 2):

$$\begin{aligned} x &= x(u,\varphi) = (R-r)\cos\varphi + u\cos(n-1)\varphi;\\ y &= y(u,\varphi) = (R-r)\sin\varphi - u\sin(n-1)\varphi,\\ z &= H(a-u)/(a+b), \end{aligned}$$

where $n = R/r$.

The presented equations of the ruled rotational surface are obtained with the help of the general equations given before at the page Sect. "34.1.2. Rotational Surfaces with Axoids "Cylinder–Cylinder.""

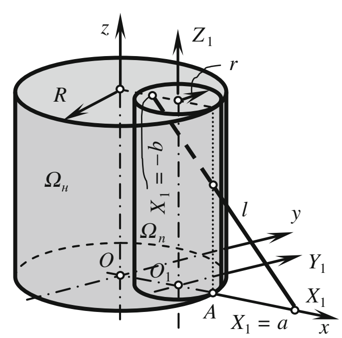


Fig. 1

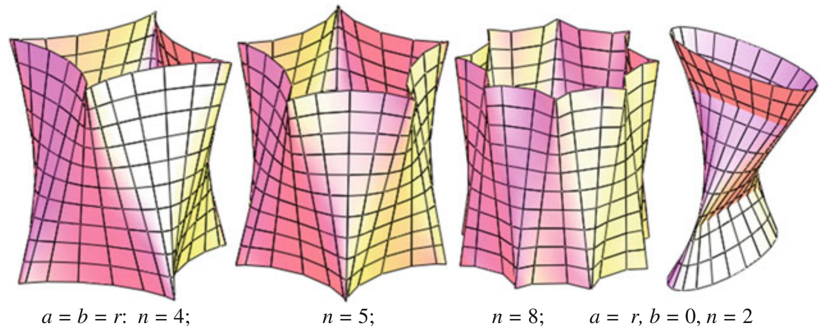


Fig. 2 $R = 1\,\text{m}$; $H = 2\,\text{m}$

The rotational ruled surface with the parameters $a = r$, $b = 0$, $n = 2$ is called *the ruled rotational surface of Lusta* (see also the page "Ruled Rotational Surface of Lusta"). In Fig. 2, it is built at the limits $-H/2 \le z \le 1,5H$.

34.1.3 Rotational Surfaces with Axoids "Cone–Cone"

Assume two cones of revolution as two axoids and some line l rigidly connected with the loose axoid Ω_l (Fig. 1). A rotational surface with axoids "cone–cone" is formed by the line l when the mobile cone Ω_l rolls without sliding over the fixed cone Ω_f. A generatrix line l (Fig. 1) is given by parametric equations

$$x_1 = x_1(u),$$
$$y_1 = y_1(u),$$
$$z_1 = z_1(u)$$

in a mobile coordinate system $O_1x_1y_1z_1$, the axis O_1z_1 of which coincides with the axis of the mobile cone. The origin of a fixed system of Cartesian coordinates $Oxyz$ is disposed at the center O of the base of the fixed circular cone Ω_f. The origin of the mobile system of coordinates $Ox_1y_1z_1$ coincides with the center O_1 of the base of the mobile cone Ω_l.

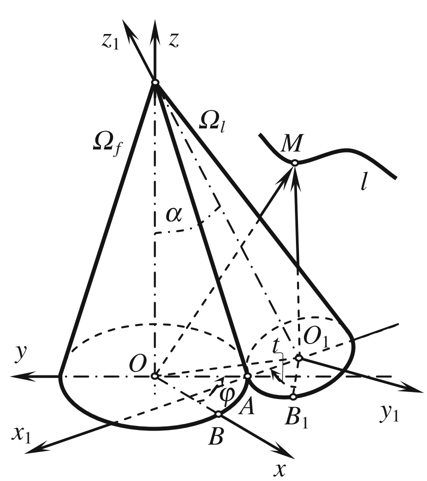


Fig. 1

If a point M belongs to the curve l, then its Cartesian coordinates in two systems are accordingly (x, y, z) and (x_1, y_1, z_1).

Parametrical equations of the curve traced by the point M of the generatrix curve l were derived by D.Ya. Yadgarov in the form:

$$\begin{aligned} x = x(\varphi) &= (R + r\cos\alpha)\cos\varphi - x_1(\cos\alpha\cos n\varphi\cos\varphi - \sin\varphi\sin n\varphi) \\ &\quad + y_1(\cos\alpha\cos\varphi\sin n\varphi + \cos n\varphi\sin\varphi) - z_1\sin\alpha\cos\varphi, \\ y = y(\varphi) &= (R + r\cos\alpha)\sin\varphi - x_1(\cos\alpha\cos n\varphi\sin\varphi + \sin n\varphi\cos\varphi) \\ &\quad + y_1(\cos\alpha\sin n\varphi\sin\varphi - \cos n\varphi\cos\varphi) - z_1\sin\alpha\sin\varphi, \\ z = z(\varphi) &= r\sin\alpha - x_1\sin\alpha\cos n\varphi + y_1\sin\alpha\sin n\varphi + z_1\cos\alpha, \end{aligned}$$

where R is a radius of the base of the fixed cone; r is a radius of the base of the mobile cone; α is the angle between the axes of the cones; $n = R/r$; φ is the angle of the axis Ox with the line connecting the point O with a point A (Fig. 1) read in the plane xOy. For the determination of parametrical equations of the surface traced by the particular curve l, it is enough to substitute

$$x_1 = x_1(u), y_1 = y_1(u), z_1 = z_1(u)$$

into the formulas given before.

If a generatrix curve l is a straight-line segment connecting two arbitrary points of the mobile cone Ω_l, then this straight-line segment forms any *ruled surface*. The ends of this straight line must belong to two different generatrixes of the loose axoid. If a generatrix curve l is a straight-line segment of a generatrix of the loose axoid Ω_l, then in the process of the rotational movement, *a conic surface* will be formed.

V.N. Ivanov suggests disposing of the origins of the both systems of coordinates $oxyz$ and $oXYZ$ at the point o, where the vertexes of two axoids Ω_l and Ω_f (Fig. 2) converge. In this

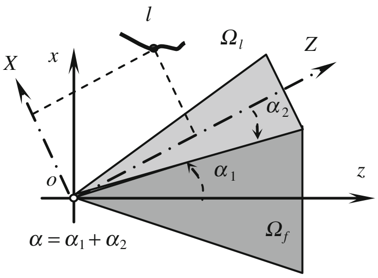


Fig. 2

case, parametrical equations of *the rotational surface with axoids* "*cone–cone*" may be written in the following form:

$$
\begin{aligned}
x = x(u,\varphi) &= X(u)[\cos\alpha\cos\varphi\cos n\varphi \mp \sin\varphi\sin n\varphi] \\
&\quad - Y(u)[\sin\varphi\cos n\varphi \pm \cos\alpha\cos\varphi\sin n\varphi] \\
&\quad + Z(u)\sin\alpha\cos\varphi, \\
y = y(u,\varphi) &= X(u)[\cos\alpha\sin\varphi\cos n\varphi \pm \cos\varphi\sin n\varphi] \\
&\quad + Y(u)[\cos\varphi\cos n\varphi \mp \cos\alpha\cos\varphi\sin n\varphi] \\
&\quad + Z(u)\sin\alpha\sin\varphi, \\
z = z(u,\varphi) &= [-X(u)\cos n\varphi \pm Y(u)\sin n\varphi]\sin\alpha + Z(u)\cos\alpha,
\end{aligned}
$$

where the upper signs are taken if the mobile cone Ω_l rolls over the external (convex) surface of the fixed cone (Fig. 2), but the low signs are taken if the mobile cone Ω_l rolls on the inner (concave) surfaces of the fixed cone Ω_f;

$$n = \frac{\sin\alpha_1}{\sin\alpha_2}.$$

The generatrix curve l is given in the mobile system of Cartesian coordinates $oXYZ$ as:

$$
\begin{aligned}
X &= X(u), \\
Y &= Y(u), \\
Z &= Z(u).
\end{aligned}
$$

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Milousheva V. General rotational surfaces in R^4 with meridians lying in two-dimensional planes. C. R. Acad. Bulg. Sci. 2010; 63, 3, p. 339-348.
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Barros M, Caballero M and Ortega M. Rotational Surfaces in L^3 and Solutions of the Nonlinear Sigma Model, Comm. Math. Phys. 2009; 290, p. 437-477.

■ Rotational Surface with Axoids "Cone–Cone" Generated by a Straight Line Coming Through the Common Vertex of the Axoids (External Rolling)

A rotational surface with axoids "cone–cone" generated by a straight line coming through the common vertex of the axoids (external rolling) is a *conical surface*.

The parametrical equations of all types of these surfaces may be obtained by substitution of the corresponding parametric equations of the chosen straight line given in the mobile system of coordinates $oXYZ$ (Fig. 1) into the general parametrical equations of rotational surfaces with axoids "cone–cone" (see also a Subsect. "34.1.3. Rotational Surfaces with Axoids "Cone–Cone"").

There are four possible variants of disposition of a generatrix straight line passing through a common vertex o of the cones: $\beta < \alpha_2$ (a straight line l_1 in Fig. 1); $\beta > \alpha_2$ (a straight line l_2 in Fig. 1); $\beta = \alpha_2$ (straight line l_3 in Fig. 1) and $\beta = 0$ (the axis of the mobile cone Ω_l); α_2 is the angle of the axis of the mobile cone with its straight generatrixes; β is the angle of the axis of the same cone with the generatrix straight lines l_i.

Parametrical equations of the generatrix straight lines l_i may be written as:

$$
\begin{aligned}
X &= X(u) = u\sin\beta, \\
Y &= 0, \\
Z &= Z(u) = u\cos\beta,
\end{aligned}
$$

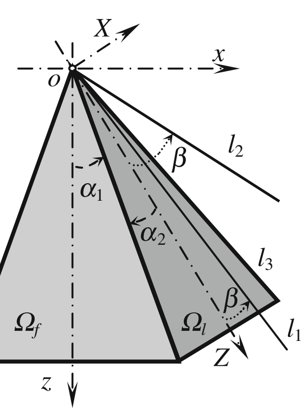


Fig. 1

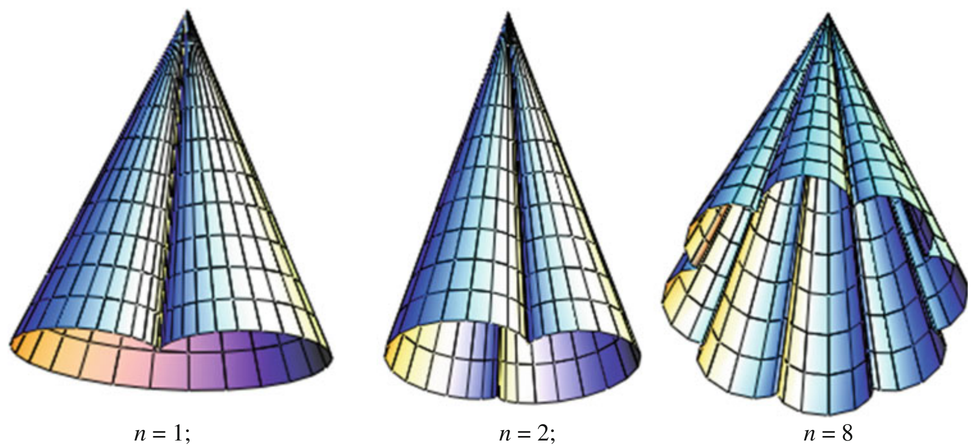


Fig. 2 $\beta = \alpha_2$

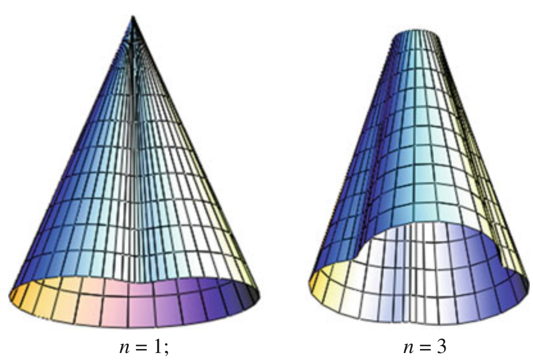


Fig. 3 $\beta < \alpha_2$

where $|u|$ is a length of the section of the generatrix straight line l_i, taken from the o point till any point on the same line.

The forms of the definition of the rotational surface

(1) Parametrical form of the definition (Figs. 1, 2, 3 and 4):

$$\begin{aligned} x = x(u, \varphi) &= u[\cos\alpha\cos\varphi\cos n\varphi - \sin\varphi\sin n\varphi]\sin\beta \\ &\quad + u\cos\beta\sin\alpha\cos\varphi, \\ y = y(u, \varphi) &= u[\cos\alpha\sin\varphi\cos n\varphi + \cos\varphi\sin n\varphi]\sin\beta \\ &\quad + u\cos\beta\sin\alpha\sin\varphi, \\ z = z(u, \varphi) &= -u\sin\beta\cos n\varphi\sin\alpha + u\cos\beta\cos\alpha, \end{aligned}$$

where $n = \sin\alpha_1/\sin\alpha_2; \alpha = \alpha_1 + \alpha_2$; φ is an angle characterizing the rolling of the mobile cone over the fixed cone (see also Fig. 1 of a Subsect. "34.1.3. Rotational Surface with Axoids "Cone–Cone"").

Changing the angle φ in the interval $0 \le \varphi \le 2\pi$, we obtain the full revolving around the fixed cone.

In Fig. 2, three rotational surfaces in question formed by the straight generatrix l_3 of the mobile cone $\Omega_l(\beta = \alpha_2)$ are represented.

Two rotational surfaces with $\beta < \alpha_2$ are shown in Fig. 3.

In Fig. 4, the surface formed by the straight line l_2 (Fig. 1) is given, i.e. when $\beta > \alpha_2$; $u_1 \le u \le u_2$; $u_1 \ne 0$.

Figure 5 show two ruled rotational surfaces: when $\alpha_1 < \alpha_2$; $n = 1/2$, $0 \le \varphi \le 4\pi$ and when $n = 1/3$, $0 \le \varphi \le 6\pi$.

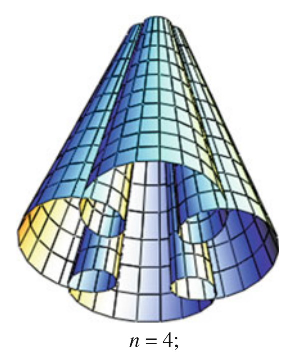


Fig. 4 $\beta < \alpha_2$

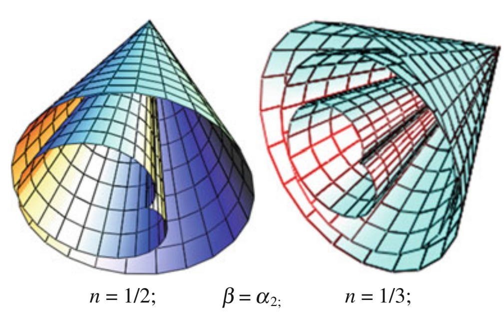


Fig. 5

■ Rotational Surface with Axoids "Cone–Cone" Generated by a Straight Line Coming Through the Common Vertex of the Axoids (Internal Rolling)

A rotational surface with axoids "cone–cone" generated by a straight line coming through the common vertex of the axoids (internal rolling) is a *conical surface*.

The parametrical equations of all types of these surfaces may be obtained by substitution of the corresponding parametric equations of the chosen straight line given in the mobile system of coordinates *oXYZ* (Fig. 1) into the general parametrical equations of rotational surfaces with axoids "cone–cone" (see also a Subsect. "34.1.3. Rotational Surfaces with Axoids "Cone–Cone"").

There are four possible variants of disposition of a generatrix straight line passing through a common vertex o of the cones: $\beta < \alpha_2$ (a straight line l_1 in Fig. 1); $\beta > \alpha_2$ (a straight line l_2 in Fig. 1); $\beta = \alpha_2$ (straight line l_3 in Fig. 1) and $\beta = 0$ (the axis of the mobile cone Ω_l); α_2 is the angle of the axis of the mobile cone with its straight generatrixes; β is the angle of the axis of the same cone with the generatrix straight lines l_i.

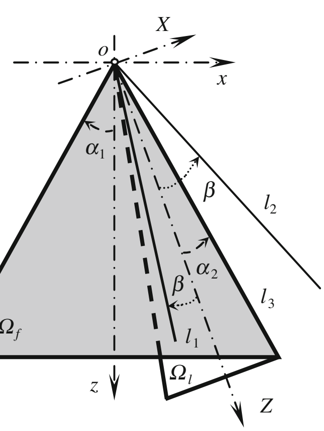
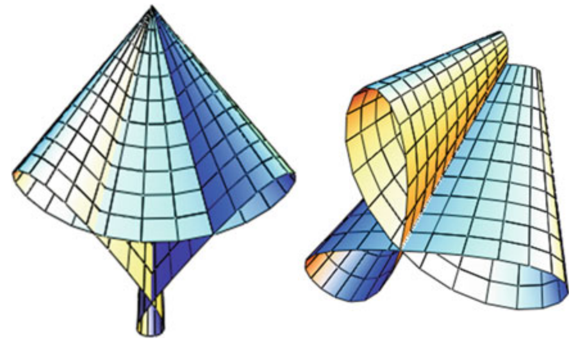


Fig. 1



$n = 3;\ 0 \le u \le u_2;$ $\quad$ $n = 3;\ u_1 \le u \le u_2$

Fig. 3 $\beta = \alpha_2$

Parametrical equations of the generatrix straight lines l_i may be written as:

$$X = X(u) = u \sin\beta,$$
$$Y = 0,$$
$$Z = Z(u) = u\cos\beta,$$

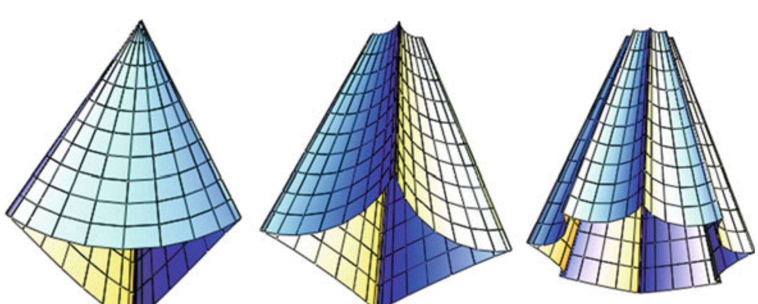
where $|u|$ is a length of the section of the generatrix straight line l_i, taken from the o point till any point on the same line.

The form of the definition of the rotational surface

(1) Parametrical form of the definition (Figs. 1, 2, 3 and 4):

$$x = x(u,\varphi) = u[\cos\alpha\cos\varphi\cos n\varphi + \sin\varphi\sin n\varphi]\sin\beta$$
$$+ u\cos\beta\sin\alpha\cos\varphi,$$
$$y = y(u,\varphi) = u[\cos\alpha\sin\varphi\cos n\varphi - \cos\varphi\sin n\varphi]\sin\beta$$
$$+ u\cos\beta\sin\alpha\sin\varphi,$$
$$z = z(u,\varphi) = -u\sin\beta\cos n\varphi\sin\alpha + u\cos\beta\cos\alpha,$$

where $n = \sin\alpha_1/\sin\alpha_2$; $\alpha = \alpha_1 + \alpha_2$; φ is an angle characterizing the rolling of the mobile cone on the internal (concave) side of fixed cone (see also a Subsectect. "34.1.3. Rotational Surface with Axoids "Cone–Cone"").



$n = 3;\ 0 \le u \le u_2;$ $\quad$ $n = 4;\ u_1 \le u \le u_2;$ $\quad$ $n = 8;\ u_1 \le u \le u_2$

Fig. 2 $\beta < \alpha_2$

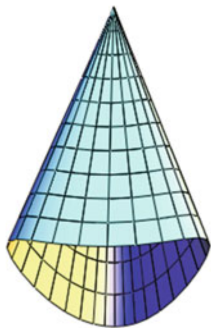


Fig. 4

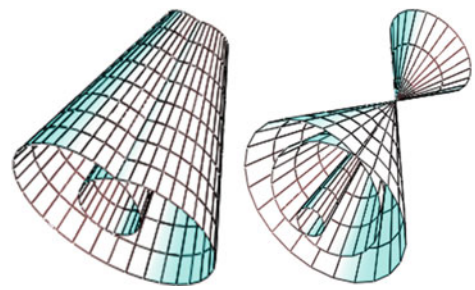


Fig. 5

If one knows the radiuses of the bases of two cones with the same height, then n = R/r, where R is a radius of the fixed cone Ω_f, r is a radius of the mobile cone Ω_l. Changing the angle φ in the interval $0 \le \varphi \le 2\pi$, we obtain the full revolving on the inside of the fixed cone.

In Fig. 2, three of the rotational surfaces, formed by the straight generatrix l_3 of the mobile cone Ω_l ($\beta = \alpha_2$) are represented.

Two rotational ruled surfaces with $\beta > \alpha_2$ are shown in Fig. 3.

In Fig. 4, the rotational surface formed by the straight l_1 is given and it has $0 \le u \le u_2$, $\beta < \alpha_2$. Figure 5 show the ruled surfaces with $\alpha_1 < \alpha_2$, $\beta > \alpha_2$; $n = 1/2$, $0 \le \varphi \le \pi$ and with $n = 1/3$, $0 \le \varphi \le 6\pi$ are presented.

■ Rotational E Surface with Axoids "Cone–Cone" Formed by a Straight Intersecting the Axis of a Mobile Cone in the Process of External Rolling

Let a generatrix straight line l is given in a mobile system of Cartesian coordinates $oXYZ$ (Fig. 1) as:

$$X = X(u) = u \sin\beta,$$
$$Y = 0,$$
$$Z = Z(u) = a + u \sin\beta,$$

where Rotational a is a constant that is equal to the distance the vertexes of the cones from the point of the intersection of the generatrix straight line with the mobile axis oZ; β is the angle of the generatrix straight line l with the axis of the mobile cone (Fig. 1). The rotational surface in question is formed by a generatrix straight line l rigidly connected with the mobile circular cone Ω_l rolling without sliding over the fixed circular cone Ω_f. The vertexes of the cones are at one point o all the time (Fig. 1).

The form of the definition of the rotational surface

(1) Parametrical equations (Figs. 2 and 3):

$$x = x(u,\varphi) = X(u)[\cos\alpha\cos\varphi\cos n\varphi - \sin\varphi\sin n\varphi] + Z(u)\sin\alpha\cos\varphi,$$
$$y = y(u,\varphi) = X(u)[\cos\alpha\sin\varphi\cos n\varphi + \cos\varphi\sin n\varphi] + Z(u)\sin\alpha\sin\varphi,$$
$$z = z(u,\varphi) = -X(u)\cos n\varphi\sin\alpha + Z(u)\cos\alpha,$$

where

$$n = \sin\alpha_1/\sin\alpha_2;\ \alpha = \alpha_1 + \alpha_2;$$

angles α_1, α_2 and β there are shown in Fig. 1.

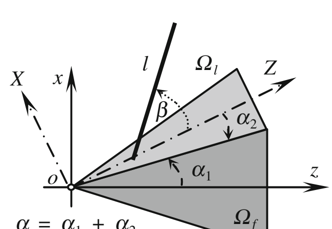


Fig. 1

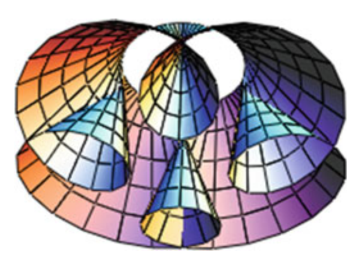


Fig. 2

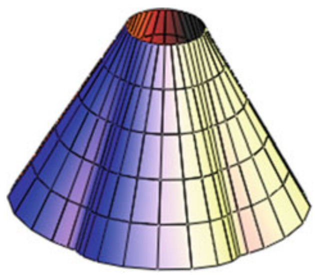


Fig. 3

The ruled rotational surface in question is a *surface of negative Gaussian curvature*.

In Fig. 2, the surface with $\beta = 30°$; $\alpha_1 = 20°$; $n = 4$; $0 \le u \le 3$ m; $a = 3$ m is shown; in Fig. 3, the surface is with $\beta = \alpha_2 = \arcsin(\sin \alpha_1/n)$; $a = 0, 5$ m; $\alpha_1 = 20°$; $n = 4$; $0 \le u \le 1, 2$ m.

So, the surface represented in Fig. 3 is formed by the straight line which is parallel to one of the generatrix lines of the mobile cone.

■ Rotational Surface with Axoids "Cone–Cone" Formed by a Straight Parallel to the Axis of a Mobile Cone in the Process of External Rolling

Let a generatrix straight line l is given in a mobile system of Cartesian coordinates $oXYZ$ as:

$$X = b,$$
$$Y = 0,$$
$$Z = Z(u) = u,$$

where b is a constant equaled to the distance the mobile axis oZ from the generatrix straight l (Fig. 1).

The rotational surface in question is formed by the generatrix straight line l rigidly connected with the mobile circular cone Ω_l rolling without sliding over the fixed circular cone Ω_f. In the process of rolling, the vertexes of the cones are at one point o all the time.

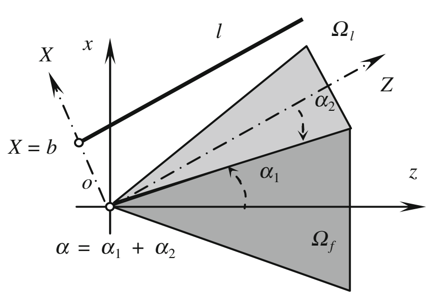


Fig. 1

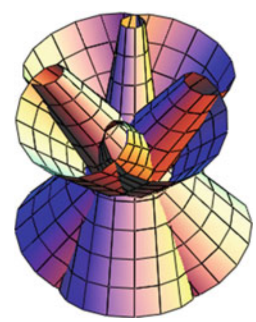


Fig. 2

The form of the definition of the rotational surface

(1) Parametrical equations (Fig. 2):

$$x = x(u, \varphi) = b[\cos\alpha \cos\varphi \cos n\varphi - \sin\varphi \sin n\varphi] + u \sin\alpha \cos\varphi,$$
$$y = y(u, \varphi) = b[\cos\alpha \sin\varphi \cos n\varphi + \cos\varphi \sin n\varphi] + u \sin\alpha \sin\varphi,$$
$$z = z(u, \varphi) = -b \cos n\varphi \sin\alpha + u \cos\alpha,$$

where $n = \sin\alpha_1/\sin\alpha_2$; $\alpha = \alpha_1 + \alpha_2$; the angles α_1 and α_2 are shown in Fig. 1. The ruled rotational surface in question is *a surface of negative Gaussian curvature*.

The surface shown in Fig. 2 has the following geometrical parameters:

$\alpha_1 = 30°$; $n = 4$; $b = 1$ m; $0 \le \varphi \le 2\pi$; -8 m $\le u \le 8$ m.

Additional literature

Yadgarov DYa. On the question of the forming of some rotational surfaces. Prikladnaya Geom. i Inzhen. Grafika. Kiev. 1976; Iss. 22, p. 42-44 (2 refs).

■ Rotational Surface with Axoids "Cone–Cone" Formed by a Parabola in the Process of External Rolling (the First Type)

Let a generatrix parabola l is given in the mobile system of Cartesian coordinates $oXYZ$ as:

$$X = X(u) = au^2, Y = 0, Z = Z(u) = u,$$

where a is a constant (Fig. 1).

The rotational surface in question is formed by the generatrix parabola l rigidly connected with a mobile circular cone Ω_l rolling without sliding over the external surface of a fixed circular cone Ω_f. In the process of rolling, the vertexes of the cones are at one point o all the time. All points of the parabola will trace the *spherical lines*. The top of the parabola is disposed at the origin of the both systems of the Cartesian coordinate, i.e. at the o point (Fig. 1).

The form of the definition of the rotational surface

(1) Parametrical equations (Fig. 2):

$$x = x(u,\varphi) = X(u)[\cos\alpha\cos\varphi\cos n\varphi - \sin\varphi\sin n\varphi] + Z(u)\sin\alpha\cos\varphi,$$

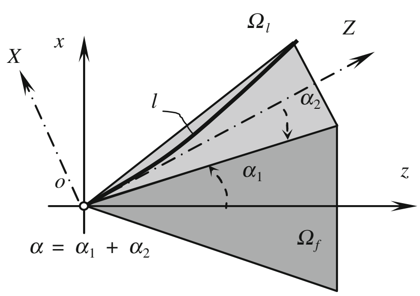


Fig. 1

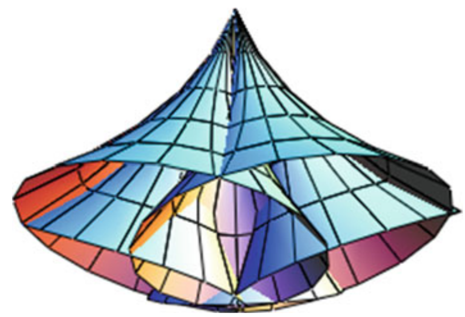


Fig. 2

$$y = y(u,\varphi) = X(u)[\cos\alpha\sin\varphi\cos n\varphi + \cos\varphi\sin n\varphi] + Z(u)\sin\alpha\sin\varphi,$$
$$z = z(u,\varphi) = -X(u)\cos n\varphi\sin\alpha + Z(u)\cos\alpha,$$

where

$$n = \frac{\sin\alpha_1}{\sin\alpha_2}, \alpha = \alpha_1 + \alpha_2;$$

angles α_1 and α_2 are shown in Fig. 1.

In Fig. 2, the rotational surface with the generatrix parabola is shown. The surface has: $\alpha_1 = 20°$; $n = 4$, $a = 0{,}25\ m^{-1}$; $0 \le u \le 2\,m$; $0 \le \varphi \le 2\pi$.

Additional literature

Yadgarov DYa. On the question of the forming of some rotational surfaces. Prikladnaya Geom. i Inzhen. Grafika. Kiev. 1976; Iss. 22, p. 42-44 (2 refs).

■ Rotational Surface with Axoids "Cone–Cone" Formed by a Parabola in the Process of External Rolling (the Second Type)

Let a generatrix parabola l is given in the mobile system of Cartesian coordinates $oXYZ$ as:

$$X = X(u) = -b(u-c)^2/c^2 + b,$$
$$Y = 0,$$
$$Z = Z(u) = a + u,$$

where a, b, c are constants (Fig. 1). The generatrix parabola with $Z = a + c$ has the rise with respect to the axis oZ equal to b, that is

$$X(u = c) = b.$$

Hence the rotational surface in question is formed by the generatrix parabola l rigidly connected with the mobile circular cone Ω_l rolling without sliding over the external surface of the fixed circular cone Ω_f. In the process of rolling, the vertexes of the cones are at one point o all the time. All points of the parabola will trace the *spherical lines*.

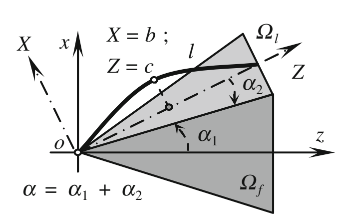


Fig. 1

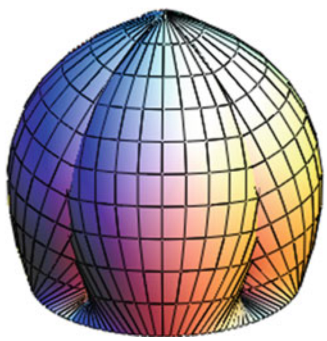


Fig. 2

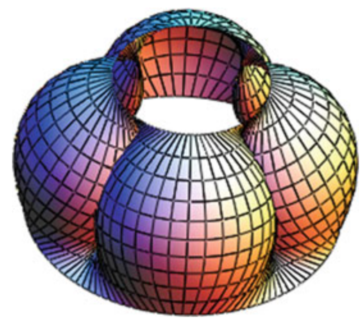


Fig. 3

The form of the definition of the rotational surface

(1) Parametrical equations (Figs. 1 and 2):

$$\begin{aligned} x = x(u,\varphi) &= X(u)[\cos\alpha\cos\varphi\cos n\varphi - \sin\varphi\sin n\varphi] \\ &\quad + Z(u)\sin\alpha\cos\varphi, \\ y = y(u,\varphi) &= X(u)[\cos\alpha\sin\varphi\cos n\varphi + \cos\varphi\sin n\varphi] \\ &\quad + Z(u)\sin\alpha\sin\varphi, \\ z = z(u,\varphi) &= -X(u)\cos n\varphi\sin\alpha + Z(u)\cos\alpha \end{aligned}$$

where $n = \sin\alpha_1/\sin\alpha_2$; $\alpha = \alpha_1 + \alpha_2$; the angles α_1 and α_2 are shown in Fig. 1.

The surface shown in Fig. 2 has the following geometrical parameters:

$$\alpha_1 = 20^{\circ}; n = 4; b = 1 \text{ m}; \quad a = 0; c = 2 \text{ m};$$
$$0 \le \varphi \le 2\pi; 0 \le u \le 2c.$$

If we shall take $a = 5$ m, then we shall design the surface represented in Fig. 3.

34.1.4 Rotational Surfaces with Axoids "Plane–Cone"

Rotational surfaces with axoids "plane–cone" have a plane as a fixed axoid and a right circular cone as a loose axoid. The rotational surface is formed by arbitrary curve rigidly connected with the mobile cone which rolls without sliding on the plane. It rolls on its side on a uniform horizontal plane in such a manner that it returns to its original position in a time τ. In Fig. 1, the both axoids are represented at the initial moment of the time when the rolling of the mobile cone does not begin.

The generatrix line l is given by parametrical equations

$$X_1 = X_1(v), Y_1 = Y_1(v), Z_1 = Z_1(v)$$

in a mobile system of Cartesian coordinates $OX_1Y_1Z_1$. The coordinate axis OX_1 coincides with the axis of the mobile cone. The origin of a fixed system of Cartesian coordinates $Oxyz$ coincides with the origin of the mobile system of coordinates $OX_1Y_1Z_1$. The vertex of the mobile cone (Fig. 1)

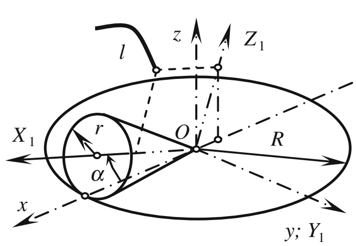


Fig. 1

is placed at the same point O. The angle of the height of the cone with its straight generatrixes is denoted by α. The angle of the fixed axis Ox with the projection of the mobile axis OX_1 on the plane xOy arising when rolling of the cone is denoted by u. When rolling of the cone on the plane, its base with a radius r rotates around the axis of the cone. If the angle of the axis Ox with the projection of the axis OX_1 on the plane xOy has a value u, it means that the base of the cone turned about the axis OX_1 at the angle φ. One may write the following relations between the geometrical parameters:

$$Ru = r\varphi \text{ whence } \varphi = \frac{R}{r}u = nu,$$
$$\text{that is } n = \frac{R}{r} \text{ or } n = \frac{1}{\sin\alpha},$$
$$\sin\alpha = \frac{1}{n}, \ \cos\alpha = \frac{\sqrt{n^2-1}}{n}, \tan\alpha = \frac{1}{\sqrt{n^2-1}}.$$

The form of the definition of the rotational surface "plane–cone"

(1) Parametrical equations:

$$\begin{aligned} x = x(u,v) &= X_1\cos\alpha\cos u + Y_1(\sin\varphi\sin\alpha\cos u - \cos\varphi\sin u) \\ &\quad - Z_1(\cos\varphi\sin\alpha\cos u + \sin\varphi\sin u), \\ y = y(u,v) &= X_1\cos\alpha\sin u + Y_1(\sin\varphi\sin\alpha\sin u + \cos\varphi\cos u) \\ &\quad - Z_1(\cos\varphi\sin\alpha\sin u - \sin\varphi\cos u), \\ z = z(u,v) &= X_1\sin\alpha - Y_1\cos\alpha\sin\varphi + Z_1\cos\varphi\cos\alpha. \end{aligned}$$

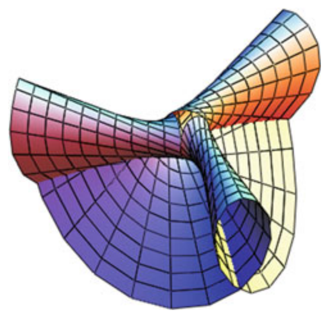


Fig. 2

The distance the center O from arbitrary point on the generatrix curve l both in the mobile and fixed systems of coordinates is the same one. Hence

$$x^2 + y^2 + z^2 = X_1^2 + Y_1^2 + Z_1^2.$$

So in the process of rolling of the circular cone on the plane, every point of the generatrix traces *a spherical line*.

Since no slipping is assumed and the fact that the plane is held fixed all of the points on the line of contact must have zero velocity. From this fact one can also state that the rotation must be along the line of contact

Coefficients of the fundamental forms of the surface:

$$A^2 = \cos^2\alpha \cdot [X_1^2 + Y_1^2(n^2 - \sin^2\varphi) + Z_1^2(n^2 - \cos^2\varphi)$$
$$-2nX_1(Y_1\sin\varphi - Z_1\cos\varphi)\cos\alpha + Y_1 Z_1 \sin(2\varphi)],$$

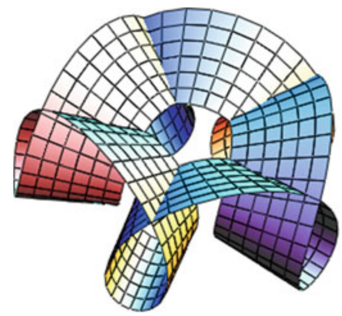


Fig. 4

$$F = \left[(X_1Z_1' - X_1'Z_1)\sin\varphi + (X_1Y_1' - Y_1X_1')\cos\varphi\right.$$
$$\left. -n(Y_1Z_1' - Y_1'Z_1)\cos\alpha\right]\cos\alpha,$$
$$B^2 = X_1^2 + Y_1^2 + Z_1^2.$$

Having assumed different generatrix curves l, it is possible to obtain interesting surfaces. In Figs. 2, 3 and 4, rotational surfaces with the straight generatrix curves l are shown.

Additional literature

Martirosov AL. Rotational transformation of the space. Rostov-Na-Donu: RGSU, 2006; 248 p.

Jackman Henrik. Rolling constraints. Karlstads University. 2008-01-08. 12 p.

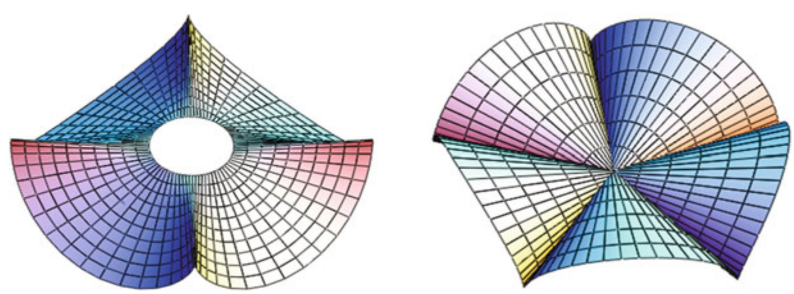


Fig. 3

■ Rotational Surface with Axoids “Plane–Cone” Generated by a Straight Line Coming Through the Vertex of a Mobile Cone

A rotational surface with axoids “plane–cone”, generated by a straight line coming through the vertex of a mobile cone is *a conic surface*. Parametric equations of all varieties of this surface one may obtain substituting parametric equations of a corresponding generatrix curve given in a mobile system of coordinates $OX_1Y_1Z_1$ (Fig. 1) into the general parametrical equations of the rotational surfaces with the axoids “plane-cone” (see also a Subsect. “34.1.4. Rotational Surfaces with Axoids “Plane–Cone”).

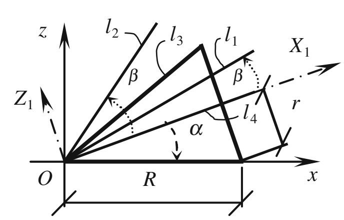


Fig. 1

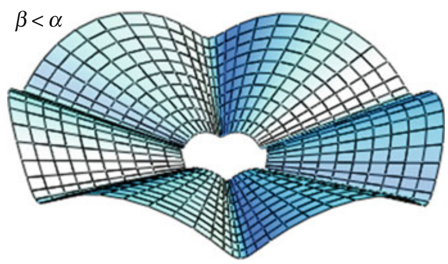


Fig. 2 $n = 6$

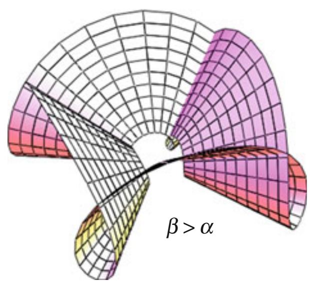


Fig. 3 $n = 4$

There are four possible variants of disposition of a generatrix straight line passing through the vertex O of the cone: $\beta < \alpha$ (a straight line l_1 in Fig. 1); $\beta > \alpha$ (a straight line l_2 in Fig. 1); $\beta = \alpha$ (straight line l_3 in Fig. 1) and $\beta = 0$ (a straight line l_4 that is the axis of the mobile cone Ω_l); α is the angle of the axis of the mobile cone with its straight generatrixes; β is the angle of the axis of the same cone with the generatrix straight lines l_i.

Forms of definition of the ruled rotational surface

(1) Parametrical equations (Figs. 2 and 3):

$$\begin{aligned} x = x(u, v) &= v \cos\alpha \cos u - v(\cos\varphi \sin\alpha \cos u \\ &\qquad + \sin\varphi \sin u) tg\beta, \\ y = y(u, v) &= v \cos\alpha \sin u - v(\cos\varphi \sin\alpha \sin u \\ &\qquad - \sin\varphi \cos u) tg\beta, \\ z = z(u, v) &= v \sin\alpha + v \cos\varphi \cos\alpha tg\beta, \end{aligned}$$

where $\varphi = nu$; α is the angle of the axis of the mobile cone with its straight generatrixes (Fig. 1); $n = 1/\sin\alpha = R/r$; β is the angle of the axis of the mobile cone with the generatrix straight line; u, v are the curvilinear coordinate lines on the rotational surface. In this case, a generatrix straight line (l_1 or l_2) is given in the mobile system of Cartesian coordinates by parametric equations:

$$X_1 = v, Y_1 = 0, Z_1 = v \tan\beta.$$

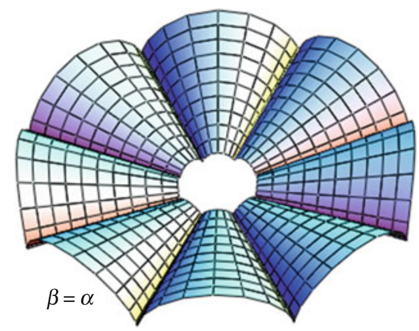


Fig. 4 $n = 8$

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} A^2 &= v^2[(1 + n \tan\beta \cos\varphi \cos\alpha)^2 + n^2 \tan^2\beta \sin^2\varphi] \cos^2\alpha, \\ F &= 0, \; B = \frac{1}{\cos\beta}; \\ L &= \{(1 + n \tan\beta \cos\varphi \cos\alpha)[-\sin\alpha + (n^2 - 2)\tan\beta \cos\varphi \cos\alpha \\ &\qquad + (\sin\alpha \cos^2\varphi - 2n) \tan^2\beta] + n^3 \tan^2\beta \sin^2\varphi\} \frac{v^2 \cos^2\alpha \cos\beta}{A} \\ M &= N = K = 0. \end{aligned}$$

Hence, the chosen system of the curvilinear coordinates u, v is orthogonal and conjugated (Figs. 2 and 3).

(2) Parametrical equations (Fig. 4):

$$\begin{aligned} x = x(u, v) &= v \cos\alpha \cos u - v(\cos\varphi \sin\alpha \cos u \\ &\qquad + \sin\varphi \sin u) \tan\alpha, \\ y = y(u, v) &= v \cos\alpha \sin u - v(\cos\varphi \sin\alpha \sin u \\ &\qquad - \sin\varphi \cos u) \tan\alpha, \\ z = z(u, v) &= v(1 + \cos\varphi) \sin\alpha. \end{aligned}$$

Under this case of the definition, the surface is formed by the generatrix curve l_3 (Fig. 1), i.e. $\beta = \alpha$. The straight generatrix l_3 coincides with the generatrix straight lines of the movable cone.

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} A^2 &= v^2[(1 + \cos\varphi)^2 \cos^2\alpha + \sin^2\varphi], \\ F &= 0, \; B = 1/\cos\alpha; \\ L &= \frac{v^2 \cos^3\alpha}{A} \{(1 + \cos\varphi)[-\sin\alpha - 2 \sin\alpha \cos\varphi \\ &\qquad + \cos^2\varphi \sin\alpha \tan^2\alpha + n \cos\varphi - 2\tan\alpha/\cos\alpha] \\ &\qquad + n \sin^2\varphi / \cos^2\alpha\}, \\ M &= N = K = 0. \end{aligned}$$

(3) Parametrical equations:

$$x(u,v) = v\cos\alpha\cos u,$$
$$y(u,v) = v\cos\alpha\sin u.$$
$$z(v) = v\sin\alpha.$$

Under this variant of the definition, the surface (*right circular cone*) is formed by the straight generatrix l_4 (Fig. 1) coinciding with the axis of the mobile cone OX_1, that is $\beta = 0$.

■ Rotational Surface with Axoids "Plane–Cone" Formed by a Straight Intersecting the Axis of a Mobile Cone

A rotational surface with axoids "plane–cone" formed by a straight intersecting the axis of a mobile cone is a *ruled surface of negative Gaussian curvature*. Figure 1 represents the different possible disposition of a straight generatrix line which would be able to form rotational surfaces in the process of the rolling of a circular cone without slipping on the plane. Under this variant of the rolling, the vertex of the mobile cone will be disposed at the point O all the time and any point of the generatrix straight lines l_i will trace a *spherical line*.

Parametric equations of arbitrary generatrix straight line in a mobile system of coordinates $OX_1Y_1Z_1$ may be written as:

$$X_1 = v, \; Y_1 = 0,$$
$$Z_1 = (v - a)\mathrm{tg}\beta,$$

where β is the angle of the axis of the cone OX_1 with the generatrix straight line l_i (Fig. 1).

The form of the definition of the rotational surface

(1) Parametrical equations:

$$x = x(u,v) = X_1\cos\alpha\cos u - Z_1(\cos\varphi\sin\alpha\cos u + \sin\varphi\sin u),$$

$$y = y(u,v) = X_1\cos\alpha\sin u - Z_1(\cos\varphi\sin\alpha\sin u - \sin\varphi\cos u)$$

$$z = z(u,v) = X_1\sin\alpha + Z_1\cos\varphi\cos\alpha,$$

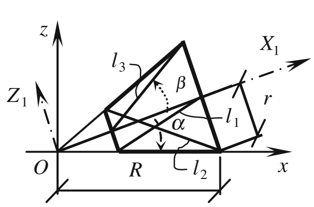


Fig. 1

where $\varphi = nu$; α is the angle of the axis of the mobile circular cone with its straight generatrixes (Fig. 1); $n = 1/\sin\alpha = R/r$. The curvilinear coordinate lines v coincide with the straight generatrixes of the rotational surface.

In Fig. 2, the rotational surface, obtained when the rolling without slipping of the circular cone with the generatrix straight line l_1, is shown; the surface given in Fig. 3 has the generatrix straight line l_2.

Figure 4 shows the surface with the generatrix straight line l_3.

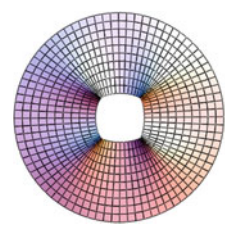


Fig. 2 $n = 4$

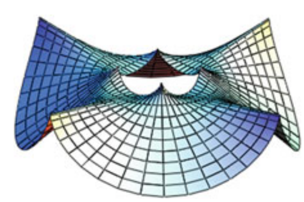


Fig. 3 $n = 4$

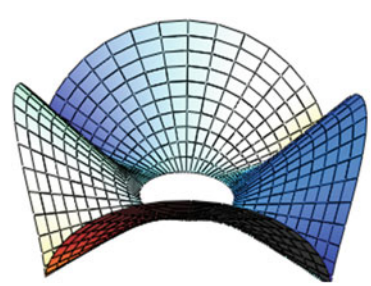


Fig. 4 $n = 4$

■ Rotational Surface with Axoids "Plane–Cone" Formed by a Straight Parallel to the Axis of a Mobile Cone

A rotational surface with axoids "plane–cone" formed by a straight line that is parallel to the axis of a mobile cone is *a ruled surface of negative Gaussian curvature*. Figure 1 represents the disposition of the straight l_1 which forms the rotational surface in question in the process of rolling of the circular cone on the plane without the sliding. Under this variant of the rolling, the vertex of the mobile cone will be all the time at the point O.

Parametrical equations of a generatrix straight line in a mobile system of coordinates $OX_1Y_1Z_1$ may be written as:

$$X_1 = v, Y_1 = 0, Z_1 = a = \text{const}.$$

The form of the definition of the rotational surface

(1) Parametrical equations (Figs. 2, 3, 4 and 5; $0 \leq v \leq 3$):

$$\begin{aligned} x = x(u,v) &= v\cos\alpha\cos u - a(\cos\varphi\sin\alpha\cos u \\ &\qquad + \sin\varphi\sin u), \\ y = y(u,v) &= v\cos\alpha\sin u - a(\cos\varphi\sin\alpha\sin u \\ &\qquad - \sin\varphi\cos u), \\ z = z(u,v) &= v\sin\alpha + a\cos\varphi\cos\alpha. \end{aligned}$$

The parametrical equations represented here were derived by the substitution of the expressions $X_1 = v$, $Y_1 = 0$, $Z_1 = a$ into the general parametrical equations of the rotational surfaces with axoids "plane–cone" given at the page Sect. "34.1.4. Rotational Surface with Axoids "Plane–Cone"".

When $a = 0$, the rotational surface degenerates into *a right conical surface of revolution* with the axis Oz (Fig. 1).

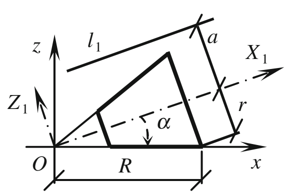


Fig. 1

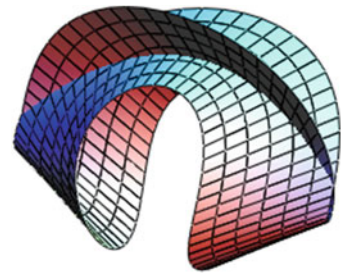


Fig. 2 $n = 2$; $a = 2$

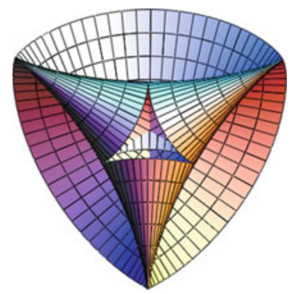


Fig. 3 $n = 3$; $a = 0.75$

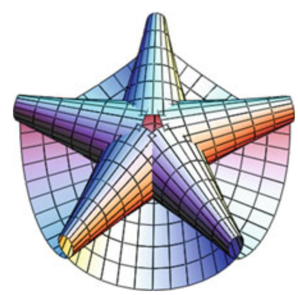


Fig. 4 $n = 5$; $a = 0.6$

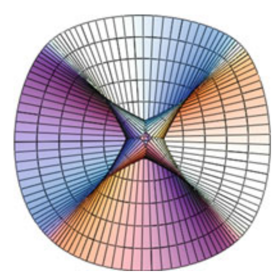


Fig. 5 $n = 5$; $a = 0.25$

34.2 Spiroidal Surfaces

Spiroidal surfaces are formed by a generatrix curve performing a helical motion with continuously changing disposition and direction of the helical axis and continuously changing *parameter of helical motion*. Spiroidal surfaces are a part of *kinematical surfaces of general type* (see also in this Chap. "34. Kinematical Surfaces of General Type").

A spiroidal surface may be given by a fixed and a loose axoids that are developable surfaces contacting along their common generatrix and by a generatrix line rigidly connected with the loose axoid in its initial disposition.

Spiroidal surfaces are called *regular surfaces* if a mobile axoid is a plane. A generatrix curve of a regular spiroidal surface is invariably connected with the *moving trihedral of Frenet* of the cuspidal edge of a fixed axoid, i.e. developable surface. A generatrix curve together with the trihedral performs a helical motion.

A regular spiroidal surface is called *a helical limaçon*, if a generatrix flat curve lies at the plane rolling with the slipping over a fixed axoid that is a developable surface. If a helical limaçon with arbitrary developable surface as a fixed axoid is created by a straight line or by a circle, then this kinematical surface is called *ruled helical limaçon* or *cyclic helical limaçon* correspondingly. Taking into account the kinematical peculiarity of the forming, we may maintain that ruled helical limaçon consists of continuously large number of continuously small oblique helicoids and that is why the area of a ruled helical limaçon may be calculated as a sum of the continuously small segments of surfaces of oblique helicoids. Generatrixes of fixed cylinder, cone, or torse serve by turns as instantaneous helical axes of the oblique helicoids.

Cylindrical or conical helical limaçons may be obtained if one assumes cylindrical or conical surfaces for fixed axoids accordingly.

If a fixed axoid of a helical limaçon is a cylindrical surface and a generatrix curve is a straight line, then a surface is called a *ruled cylindrical helical limaçon*. But if a generatrix curve is a circle of constant radius, then a cylindrical helical limaçon is called *a cyclic cylindrical helical limaçon*.

Spiroidal surface with axoids "cone–plane" and a generator straight line lying at the moving plane is called a *ruled conic helical limaçon*. But if a generatrix curve is a circle, then we have a *cyclic conic helical limaçon*. The *usual helical surfaces* (see also the Chap. "7. Helical Surfaces") may be called *a degenerated cylindrical helical limaçon*, if one assumes that a fixed axoid that is a cylinder degenerated into its axis (into a straight line).

Let a generatrix plane curve is given in a local system of coordinates x_l, y_l as

$$x_l = x_l(t),\; y_l = y_l(t).$$

In this case, parametrical equations of the cylindrical helical limaçon may be written as

$$\begin{aligned} x = x(\alpha, t) &= r\cos\alpha - [r(\alpha_o - \alpha) + x_l(t)\cos\theta_o \\ &\qquad - y_l(t)\sin\theta_o]\sin\alpha, \\ y = y(\alpha, t) &= r\sin\alpha + [r(\alpha_o - \alpha) + x_l(t)\cos\theta_o \\ &\qquad - y_l(t)\sin\theta_o]\cos\alpha, \\ z = z(t) &= x_l(t)\sin\theta_o + y_l(t)\cos\theta_o + f(\alpha), \end{aligned}$$

where r is a radius of a fixed axoid that is a cylinder; θ_o is the slope angle of the local axis x_l with the plane xOy; α is an angle read from the axis Ox in the direction of the axis Oy; α_0 = const; $f(\alpha)$ is a function of the velocity of the translation motion of a generatrix curve along the axis of the fixed axoid, that is a cylinder.

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34.2.1 Spiroidal Surfaces with Axoids "Cylinder–Plane"

Spiroidal surfaces with axoids "cylinder–plane" have a circular cylinder as fixed axoid and a plane as loose axoid. Spiroidal surfaces with axoids "cylinder–plane" may be called also *the regular cylindrical spiroidal surfaces* (see also a Sect. "34.2. Spiroidal Surfaces").

For the definition of the spiroidal surface in question, they introduce two system of Cartesian coordinates: a fixed system of coordinates $Oxyz$ and a mobile system of coordinates $o_1X_1Y_1Z_1$. Axis Oz of the fixed system coincides with the axis of the cylinder (Fig. 1). In initial disposition, the origin o_1 of the moving system of coordinates is placed at a point with the coordinates $x = R$; $y = 0$; $z = 0$, where R is a radius of the fixed axoid, that is a circular cylinder, and the axis o_1Z_1 coincides with the straight generatrix of the cylinder.

In the process of rolling of the plane Q over the cylinder with the slipping (sliding) along the straight generatrixes of the cylinder, a generatrix curve L given in the mobile system of coordinates by the equations

$$X_1 = X_1(u),\quad Y_1 = Y_1(u),\quad Z_1 = Z_1(u)$$

forms a spiroidal surface with axoids "cylinder–plane". In general case, the curve L may be by any space line.

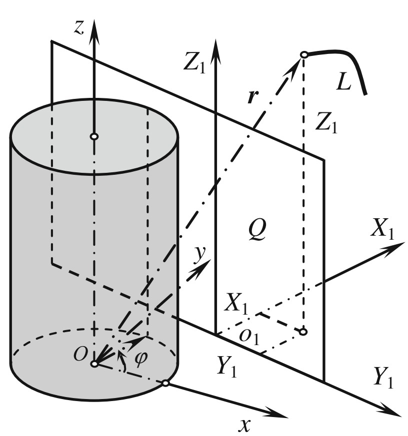

Fig. 1

Forms of definition of the spiroidal surface with axoids "cylinder–plane"

(1) Vector equation (Fig. 1):

$$\begin{aligned}\boldsymbol{r} = \boldsymbol{r}(u,\varphi) &= R(\cos\varphi\,\boldsymbol{i} + \sin\varphi\,\boldsymbol{j}) + R\varphi(\sin\varphi\,\boldsymbol{i} - \cos\varphi\,\boldsymbol{j}) \\ &+ Y_1(u)\,(\sin\varphi\boldsymbol{i} - \cos\varphi\,\boldsymbol{j}) \\ &+ X_1(u)(\cos\varphi\,\boldsymbol{i} + \sin\varphi\,\boldsymbol{j}) + [Z_1(u) + \lambda\varphi]\boldsymbol{k},\end{aligned}$$

where φ is an angle read from the coordinate axis Ox in the direction of the axis Oy; λ = const. Usually

$$\lambda\varphi$$

is called *a helical parameter of spiroidal surface*.

(2) Parametrical equations (Fig. 1):

$$\begin{aligned}x &= x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi) \\ &\quad + X_1(u)\cos\varphi + Y_1(u)\sin\varphi, \\ y &= y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi) \\ &\quad + X_1(u)\sin\varphi - Y_1(u)\cos\varphi, \\ z &= z(u) = Z_1(u) + \lambda\varphi.\end{aligned}$$

Coefficients of the fundamental forms of the surface:

$$A^2 = X_1'^2 + Y_1'^2 + Z_1'^2,$$
$$F = R\varphi X_1' + X_1'Y_1 - X_1Y_1' + \lambda Z_1'$$
$$B^2 = (R\varphi + Y_1)^2 + X_1^2 + \lambda^2;$$
$$L = \frac{1}{\Sigma}\begin{vmatrix} X_1''\cos\varphi + Y_1''\sin\varphi & X_1''\sin\varphi + Y_1''\cos\varphi & Z_1'' \\ X_1'\cos\varphi + Y_1'\sin\varphi & X_1'\sin\varphi + Y_1'\cos\varphi & Z_1' \\ R\varphi\cos\varphi - X_1\sin\varphi + Y_1\cos\varphi & R\varphi\sin\varphi - X_1\cos\varphi + Y_1\sin\varphi & \lambda \end{vmatrix},$$
$$M = \frac{Z_1'(R\varphi X_1' + X_1'Y_1 - X_1Y_1') - \lambda(X_1'^2 + Y_1'^2)}{\Sigma},$$
$$N = \frac{Z_1'[(R\varphi + Y_1)^2 + X_1(X_1 - R)] + \lambda[Y_1'(X_1 - R) - X_1'(R\varphi + Y_1')]}{\Sigma},$$

where

$$\Sigma = \sqrt{A^2B^2 - F^2};$$
$$\ldots' = \frac{\mathrm{d}\ldots}{\mathrm{d}u}.$$

If a generatrix L is a straight line lying at the plane $Y_1o_1Z_1$ ($X_1 = 0$), then in the process of rolling of the plane over the cylinder with the sliding, we shall obtain *a ruled cylindrical helical limaçon*.

A spiroidal surface is called *cyclic cylindrical helical limaçon,* if a generatrix circle of a constant radius lies at the plane Q rolling with the sliding over the fixed cylinder.

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■ Regular Ruled Cylindrical Spiroidal Surface

A ruled spiroidal surface with axoids "*cylinder–plane*" has a fixed axoid in the form of a circular cylinder and a plane as a mobile axoid. The surface is formed by an arbitrary disposed straight line rigidly connected with the plane rolling with sliding (λ = const).

The sliding takes the place along the line of contact of the cylinder and the plane. A ruled spiroidal surface with axoids "cylinder–plane" is called also *a regular ruled cylindrical spiroidal surface*.

Forms of definition of the surface

(1) Parametrical equations:

$$\begin{aligned} x &= x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi) + u\cos\varphi, \\ y &= y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi) + u\sin\varphi, \\ z &= z(u,\varphi) = u\tan\alpha + \lambda\varphi, \end{aligned}$$

where R is a radius of the fixed cylinder, α is the slope angle of the straight generatrix line L_1 with the moving coordinate axis o_1X_1; λ = const. Under this form of the definition of a spiroidal surface, the generatrix straight line L_1 (Fig. 1) is disposed at the plane $X_1o_1Z_1$ and parametrical equations of the straight line L_1 in a mobile system of coordinates have the following form:

$$\begin{aligned} X_1(u) &= u, \\ Y_1 &= 0, \\ Z_1(u) &= u\tan\alpha. \end{aligned}$$

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} &A^2 = \frac{1}{\cos^2\alpha}, \\ &F = R\varphi + \lambda\tan\alpha, \\ &B^2 = R^2\varphi^2 + u^2 + \lambda^2; \\ &A^2B^2 - F^2 = u^2A^2 + (R\varphi\tan\alpha - \lambda)^2; \\ &L = 0, \\ &M = \frac{R\varphi\tan\alpha - \lambda}{\sqrt{A^2B^2 - F^2}},\ N = \frac{\tan\alpha[R^2\varphi^2 + u^2 - uR] - R\varphi\lambda}{\sqrt{A^2B^2 - F^2}}; \\ &K = -\frac{(R\varphi\tan\alpha - \lambda)^2}{(A^2B^2 - F^2)^2} \le 0. \end{aligned}$$

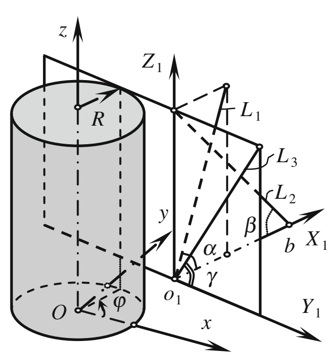


Fig. 1

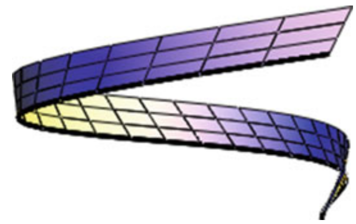


Fig. 2

When a generatrix straight line is placed at the plane $X_1o_1Z_1$, *a ruled surface of negative Gaussian curvature* will be.

In Fig. 2, the spiroidal surface with R = 1 m, $\alpha = \pi/3$, λ = 0.5 m; $0 \le u \le 1m$; $0 \le \varphi \le 2\pi$ is shown.

(2) Parametrical equations:

$$\begin{aligned} x &= x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi) + u\cos\varphi, \\ y &= y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi) + u\sin\varphi, \\ z &= z(u,\varphi) = (b - u)\tan\beta + \lambda\varphi, \end{aligned}$$

where β is the slope angle of a straight generatrix with the moving coordinate axis o_1X_1 (Fig. 1). The generatrix straight line L_2 is disposed at the plane $X_1o_1Z_1$. Parametric equations of this straight line are written as:

$$X_1(u) = u, Y_1 = 0, Z_1(u) = (b - u)\tan\beta.$$

In Fig. 3, the ruled spiroidal surface with $\beta = \pi/3$, $R = b$ = 1 m, λ = 0.5 m; $0 \le u \le$ m; $0 \le \varphi \le 2\pi$ is shown.

(3) Parametrical equations (Fig. 4):

$$\begin{aligned} x &= x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi) + u\sin\varphi, \\ y &= y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi) - u\cos\varphi, \\ z &= z(u,\varphi) = u\tan\gamma + \lambda\varphi, \end{aligned}$$

where γ is the slope angle of the straight generatrix with the mobile coordinate axis o_1Y_1 (Fig. 1). The generatrix straight line L_3 lies at the plane $Y_1o_1Z_1$ and is given by equations:

$$X_1(u) = 0,\ Y_1(u) = u,\ Z_1(u) = u\tan\gamma.$$

Coefficients of the fundamental forms of the surface:

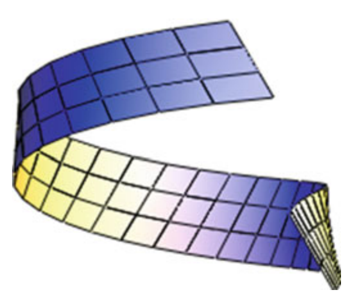


Fig. 3

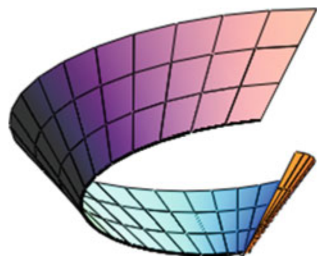


Fig. 4

$$A^2 = 1/\cos^2\gamma,\ F = \lambda\tan\gamma,\ B^2 = (R\varphi + u)^2 + \lambda^2;$$
$$L = 0,\ M = -\lambda/\sqrt{A^2B^2 - F^2},$$
$$N = \left[(R\varphi + u)^2 \mathrm{tg}\gamma - \lambda R\right]/\sqrt{A^2B^2 - F^2},\ K < 0$$

When the generatrix straight line L_3 (Fig. 1) is placed at the rolling plane, they obtain *a ruled cylindrical helical limaçon* (Fig. 4).

■ Regular Cyclic Cylindrical Spiroidal Surface

A cyclic spiroidal surface with axoids "cylinder–plane" has a fixed axoid in the form of a circular cylinder and a plane as a mobile axoid. The surface is formed by an arbitrary disposed circle rigidly connected with the plane rolling with sliding (λ = const).

The sliding takes the place along the line of contact of the cylinder and the plane. A cyclic spiroidal surface with axoids "cylinder–plane" is called also *a regular cyclic cylindrical spiroidal surface*.

Forms of definition of the surface

(1) Parametrical equations (Fig. 2):

$$x = x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi) + (b + r\cos u)\cos\varphi,$$
$$y = y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi) + (b + r\cos u)\sin\varphi,$$
$$z = z(u,\varphi) = a + r\sin u + \lambda\varphi,$$

where R is a radius of the fixed cylinder, r is a radius of the generatrix circle I (Figs. 1 and 2) lying at the plane $X_1o_1Z_1$; $X_1 = b$, $Z_1 = a$ are coordinates of the center of the generatrix circle at a mobile system of the Cartesian coordinates (Fig. 1); λ = const. Parametric equations of the generatrix circle given in the mobile system of coordinates have the following form:

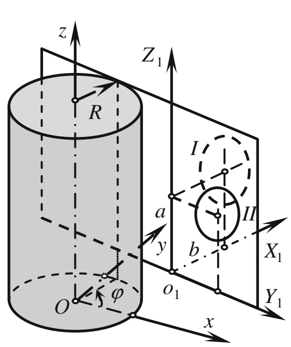


Fig. 1

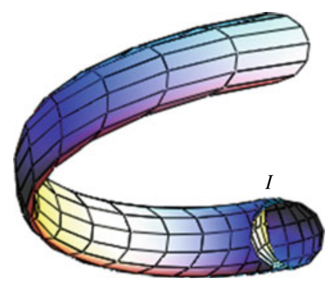


Fig. 2

$$X_1(u) = b + r\cos u,\ Y_1 = 0,$$
$$Z_1(u) = a + r\sin u.$$

Coefficients of the fundamental forms of the surface:

$$A = r,$$
$$F = -rR\varphi\sin u + \lambda r\cos u,$$
$$B^2 = (b + r\cos u)^2 + R^2\varphi^2 + \lambda^2;$$
$$L = \frac{r^2X_1(u)}{\sqrt{A^2B^2 - F^2}},\ M = -\frac{R\varphi\cos u + \lambda\sin u}{\sqrt{A^2B^2 - F^2}}r^2\sin u,$$
$$N = r\frac{[R^2\varphi^2 - RX_1(u) + X_1^2(u)]\cos u + \lambda R\varphi\sin u}{\sqrt{A^2B^2 - F^2}}.$$

Figure 2 shows the cyclic spiroidal surface *with axoids "cylinder–plane"* which has $r = R = a = b = 1$ m; $\lambda = 0.5$ m; $0 \le u \le 2\pi$; $0 \le \varphi \le 2\pi$.

If $b = r = R$, then the coefficients of the fundamental forms of the surface assume the following form:

$$A = r,\ F = -r^2\varphi\sin u + \lambda r\cos u,$$
$$B^2 = (1 + \cos u)^2r^2 + R^2\varphi^2 + \lambda^2;$$
$$L = \frac{r^2X_1(u)}{\sqrt{A^2B^2 - F^2}},\ M = -\frac{r\varphi\cos u + \lambda\sin u}{\sqrt{A^2B^2 - F^2}}r^2\sin u,$$
$$N = r^2\frac{r[\varphi^2 + (1 + \cos u)\cos u]\cos u + \lambda\varphi\sin u}{\sqrt{A^2B^2 - F^2}}.$$

(2) Parametrical equations:

$$
\begin{aligned}
x &= x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi) + (b + r\cos u)\sin\varphi,\\
y &= y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi) - (b + r\cos u)\cos\varphi,\\
z &= z(u,\varphi) = a + r\sin u + \lambda\varphi,
\end{aligned}
$$

where R is a radius of the fixed cylindrical axoid, r is a radius of the generatrix circle II (Fig. 1) lying in the rolling plane $Y_1o_1Z_1$; $Y_1 = b$, $Z_1 = a$ are the coordinates of the center of the generatrix circle in the moving system of coordinates. Parametrical equations of the generatrix circle given in the moving system of coordinates may be written as:

$$X_1 = 0,\ Y_1(u) = b + r\cos u,\ Z_1(u) = a + r\sin u.$$

Under this variant of the disposition of the generatrix circle, we may obtain *a cyclic cylindrical helical limaçon*.

(3) Parametrical equation:

$$
\begin{aligned}
x &= x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi)\\
&\quad + b\cos\varphi + (d + r\cos u)\sin\varphi,\\
y &= y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi)\\
&\quad + b\sin\varphi - (d + r\cos u)\cos\varphi,\\
z &= z(u,\varphi) = a + r\sin u + \lambda\varphi
\end{aligned}
$$

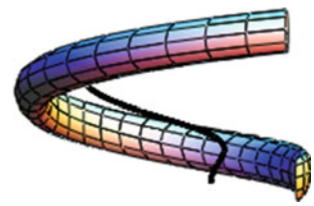


Fig. 3

The surface with $a = d = r = R = 1$ m; $b = 4$ m, $\lambda = 1$ m is shown in Fig. 3. Here the generatrix surface is given by equations

$$
\begin{aligned}
X_1 &= b,\ Y_1(u) = d + r\cos u,\\
Z_1(u) &= a + r\sin u.
\end{aligned}
$$

■ Cylindrical Helical Limaçon with Parabolic Generatrix

A spiroidal surface with axoids "cylinder–plane" and with a parabolic generatrix has a fixed axoid in the form of a circular cylinder and a plane as a mobile axoid. The surface is formed by an arbitrary disposed parabola rigidly connected with the plane rolling with sliding (λ = const). The sliding takes the place along the line of contact of the cylinder and the plane. A spiroidal surface with axoids "cylinder–plane" and with a parabolic generatrix is called also *a cylindrical helical limaçon with a parabolic generatrix* (Fig. 1).

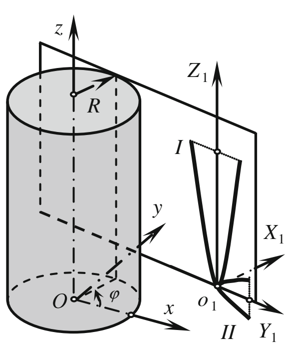


Fig. 1

Forms of definition of the surface

(1) Parametric form of assignment (Fig. 2):

$$
\begin{aligned}
x &= x(u,\varphi) = R(\cos\varphi + \varphi\sin\varphi) + u\sin\varphi,\\
y &= y(u,\varphi) = R(\sin\varphi - \varphi\cos\varphi) - u\cos\varphi,\\
z &= z(u,\varphi) = au^2 + \lambda\varphi,
\end{aligned}
$$

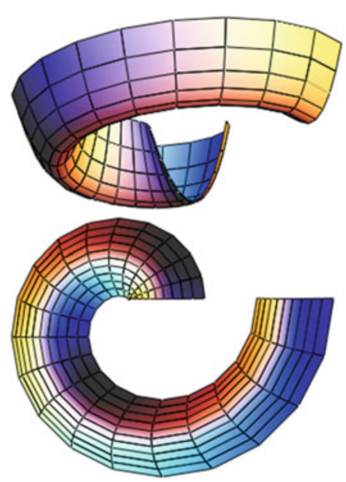


Fig. 2

where R is a radius of the fixed cylinder, λ and a are constants. Parametric equations of the generatrix parabola I given in a moving system of Cartesian coordinates have the following form:

$$X_1 = 0, Y_1(u) = u, Z_1(u) = au^2.$$

So, the axis of the generatrix parabola coincides with the moving coordinate axis o_1Z_1 and the parabola itself is disposed at the coordinate plane $Y_1o_1Z_1$.

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} A^2 &= 1 + 4a^2u^2, \\ F &= 2a\lambda u, \\ B^2 &= (R\varphi + u)^2 + \lambda^2; \\ A^2B^2 - F^2 &= (R\varphi + u)^2A^2 + \lambda^2; \\ L &= \frac{2a(R\varphi + u)}{\sqrt{A^2B^2 - F^2}}, \; M = \frac{-\lambda}{\sqrt{A^2B^2 - F^2}}, \\ N &= \frac{2au(R\varphi + u)^2 - \lambda R}{\sqrt{A^2B^2 - F^2}}. \end{aligned}$$

In Fig. 2, the cylindrical helical limaçon with the parabolic generatrix I is presented. Here one has $R = 1$ m; $a = 1\ \text{m}^{-1}$; $\lambda = 0.5$ m; $-2\,\text{m} \le u \le 2\,\text{m}\,\pi/2 \le \varphi \le 2.5\pi$.

(2) Parametric form of assignment (Fig. 3):

$$\begin{aligned} x &= x(u, \varphi) = R(\cos\varphi + \varphi\sin\varphi) + au^2\sin\varphi, \\ y &= y(u, \varphi) = R(\sin\varphi - \varphi\cos\varphi) - au^2\cos\varphi, \\ z &= z(u, \varphi) = u + \lambda\varphi, \end{aligned}$$

where R is a radius of the fixed cylinder, λ and a are constants. Parametric equations of the generatrix parabola II given in the mobile system of coordinates have the following form:

$$X_1 = 0, \; Y_1(u) = au^2, \; Z_1(u) = u.$$

So, the axis of the generatrix parabola coincides with the mobile coordinate axis o_1Y_1 and the parabola is disposed at the coordinate plane $Y_1o_1Z_1$.

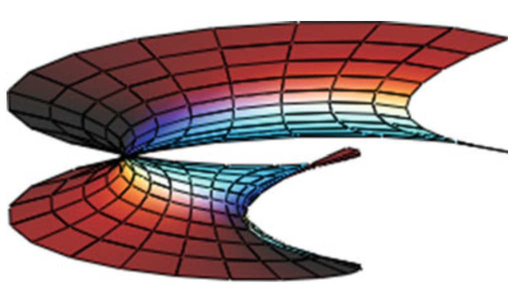


Fig. 3

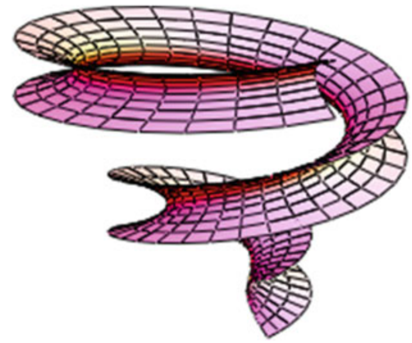


Fig. 4

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} A^2 &= 1 + 4a^2u^2, \\ F &= \lambda, \\ B^2 &= (R\varphi + au^2)^2 + \lambda^2; \\ A^2B^2 - F^2 &= (R\varphi + au^2)^2A^2 + 4a^2u^2\lambda^2; \\ L &= -\frac{2a(R\varphi + au^2)}{\sqrt{A^2B^2 - F^2}}, \; M = \frac{-4a^2u^2\lambda}{\sqrt{A^2B^2 - F^2}}, \\ N &= \frac{(R\varphi + au^2)^2 - 2auR\lambda}{\sqrt{A^2B^2 - F^2}}. \end{aligned}$$

In Fig. 3, the cylindrical helical limaçon with parabolic generatrix II is represented. Here there are taken

$$\begin{aligned} &R = 1\,\text{m}; \; a = 1\,\text{m}^{-1}; \quad \lambda = 0.5\ \text{m}; \\ &-2\,\text{m} \le u \le 2\,\text{m}; \quad 0 \le \varphi \le 2\pi. \end{aligned}$$

In Fig. 4, the cylindrical helical limaçon with parabolic generatrix II having the following geometrical parameters:

$$\begin{aligned} &R = 1\,\text{m}; \; a = 1\,\text{m}^{-1}; \; \lambda = 1.5\,\text{m}; \\ &-2\,\text{m} \le u \le 2\,\text{m}; \; 0 \le \varphi \le 4\pi \end{aligned}$$

is shown.

34.2.2 Spiroidal Surfaces with Axoids "Cylinder–Cylinder"

Assume a pair of axoids that are cylinders of revolution with radiuses of bases R and r, where $R = nr$, and some line l rigidly connected with the loose axoid that is the cylinder Ω_l of the radius r. A spiroidal surface with axoids "cylinder–cylinder" is formed by the line l when rolling with sliding (slipping) of the mobile cylinder Ω_l over the fixed cylinder Ω_f.

The generatrix line l is given by parametrical equations:

$$x_1 = x_1(u),$$
$$y_1 = y_1(u),$$
$$z_1 = z_1(u)$$

in the moving system of Cartesian coordinates $O_1x_1y_1z_1$. The coordinate axis O_1z_1 coincides with the axis of the mobile cylinder Ω_l. The origin of the fixed system of coordinates $Oxyz$ is disposed at the center O of the base of the fixed right cylinder Ω_f but the coordinate axis Oz coincides with its axis.

The origin of the mobile system of coordinates $Ox_1y_1z_1$ coincides with the center O_1 of the base of the mobile right cylinder Ω_l (Fig. 1). Before the beginning of the motion, the axes of the both coordinate systems must be parallel.

If the point M belongs to the curve l, then its coordinates in two systems of coordinates are (x, y, z) and (x_1, y_1, z_1) correspondingly. After some spiroidal motion, the moving axoid will take a position shown in Fig. 1. Researching the Fig. 1, it is possible to obtain the following geometric relation:

$$R\varphi = rt \text{ or } n\varphi = t,$$

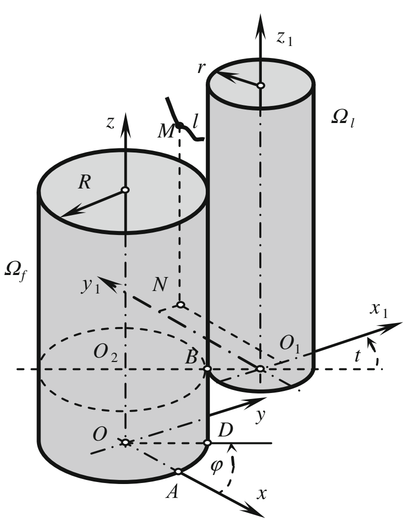


Fig. 1

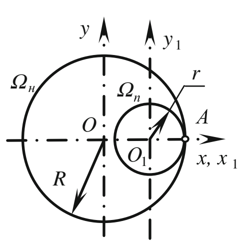


Fig. 2

where λ = const is a coefficient of proportionality; φ is an angle read from the fixed axis x in the direction of the rolling until the straight line connecting the center O of the fixed cylinder with the point D. The projection of the line of the contact of two axoids on the plane $z = 0$ coincides with the point D; the angle t is shown in Fig. 1.

Parametrical equations of the curve traced by the point M of the generatrix curve l are represented as:

$$x = x(\varphi) = (R + r)\cos\varphi + x_1\cos(n+1)\varphi - y_1\sin(n+1)\varphi,$$
$$y = y(\varphi) = (R + r)\sin\varphi + x_1\sin(n+1)\varphi + y_1\cos(n+1)\varphi,$$
$$z = z(\varphi) = \lambda\varphi + z_1.$$

For the determination of parametric equations of the surface formed by the generatrix curve l, it is enough to substitute

$$x_1 = x_1(u), y_1 = y_1(u), z_1 = z_1(u)$$

into the parametrical equations given above. Hence

$$x = x(u,\varphi) = (R + r)\cos\varphi + x_1(u)\cos(n+1)\varphi - y_1(u)\sin(n+1)\varphi,$$
$$y = y(u,\varphi) = (R + r)\sin\varphi + x_1(u)\sin(n+1)\varphi + y_1(u)\cos(n+1)\varphi,$$
$$z = z(u,\varphi) = \lambda\varphi + z_1(u).$$

The rolling of the loose axoid Ω_l on the fixed axoid Ω_f may takes place both over the external (convex) side of the fixed cylinder (Fig. 1) and on its internal (concave) side (Fig. 2). In this case, parametric equations of the spiroidal surface (Fig. 2) may be written as:

$$x = x(u,\varphi) = (R - r)\cos\varphi + x_1(u)\cos(n-1)\varphi + y_1(u)\sin(n-1)\varphi;$$
$$y = y(u,\varphi) = (R - r)\sin\varphi - x_1(u)\sin(n-1)\varphi + y_1(u)\cos(n-1)\varphi,$$
$$z = z(u) = \lambda\varphi + z_1(u).$$

Here φ is the angle of the axis Ox with the straight line OD connecting the point O and the point D of the touching of the axoids after the beginning of the rolling, t is an angle from a new disposition of the axis O_1X_1 till the straight line OD.

Additional literature

Yadgarov DYa. On the question of some spiroidal surfaces. Prikl. Geom. i Ingenern. Grafica. Kiev. 1977; Iss. 23, p. 98-100 (2 refs).

Edward J. Haug. Computer-Aided Kinematics and Dynamics of Mechanical Systems. Allyn and Bacoo, Needham Heights, MA. 1989.

■ Spiroidal Surfaces with Axoids "Cylinder–Cylinder" Formed by a Straight not Intersecting the Axis of a Mobile Cylinder in the Process of External Rolling

Let a generatrix straight line is given in a mobile system of coordinates $O_1X_1Y_1Z_1$ in the form:

$$X_1 = u,$$
$$Y_1 = b(1 - u/a),$$
$$Z_1 = H(1 - u/a),$$

where constants a, b, H are shown in Fig. 1. In this case, parametric equations of the ruled spiroidal surface formed by the straight generatrix when external rolling of a cylinder with a radius r with sliding over a cylinder with a radius R may be written as:

$$x = x(u, \varphi) = (R + r)\cos\varphi + u\cos(n+1)\varphi - b(1 - u/a)\sin(n+1)\varphi,$$

$$y = y(u, \varphi) = (R + r)\sin\varphi + u\sin(n+1)\varphi + b(1 - u/a)\cos(n+1)\varphi,$$
$$z = z(u, \varphi) = H(1 - u/a) + \lambda\varphi,$$

where λ = const; $n = R/r$.

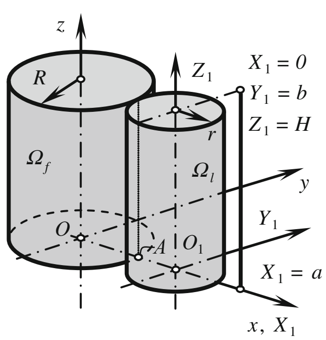


Fig. 1

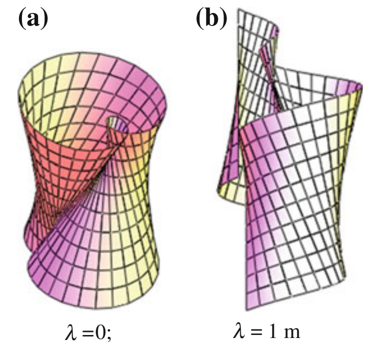


Fig. 2 $n = 1$

These equations of the surface are obtained on the base of the general equations given at the page Sect. "34.2.2. Spiroidal Surfaces with Axoids "Cylinder–Cylinder"".

In Fig. 1, the initial position of two cylindrical axoids is shown. The ruled spiroidal surfaces in question are *surfaces of negative Gaussian curvature*. If $\lambda = 0$, then the spiroidal surface degenerates into a rotational surface, i.e. rolling of a cylinder over a cylinder without sliding will be (Figs. 2a and 3a). A parameter $n = R/r$ is shown at the drawings. Figures 2b and 3b represent the spiroidal surfaces with $\lambda = 1$ m; $R = 1$ m; $a = -b = r$; $H = 4$ m; $-H/2 \le z \le 1,5H$.

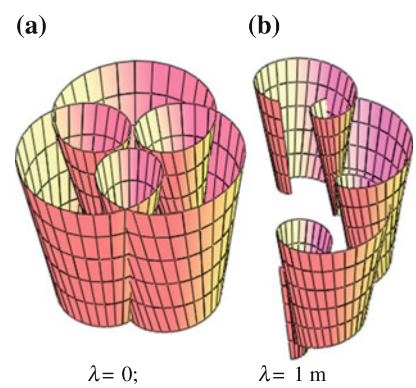


Fig. 3 $n = 3$

■ Spiroidal Surface with Axoids "Cylinder–Cylinder" Formed by a Straight not Intersecting the Axis of a Mobile Cylinder in the Process of Internal Rolling

Let a generatrix straight line is given in a movable system of coordinates $O_1X_1Y_1Z_1$ in the form:

$$X_1 = u,$$
$$Y_1 = b(1 - u/a),$$
$$Z_1 = H(1 - u/a),$$

where constants a, b, H are shown in Fig. 1. In this case, parametric equations of the spiroidal surface formed by the straight generatrix l, when internal rolling with sliding of the mobile cylinder of a radius r on the fixed cylinder with a radius R, may be written as:

$$x = x(u, \varphi) = (R - r)\cos\varphi + u\cos(n-1)\varphi + b(1 - u/a)\sin(n-1)\varphi;$$
$$y = y(u, \varphi) = (R - r)\sin\varphi - u\sin(n-1)\varphi + b(1 - u/a)\cos(n-1)\varphi,$$

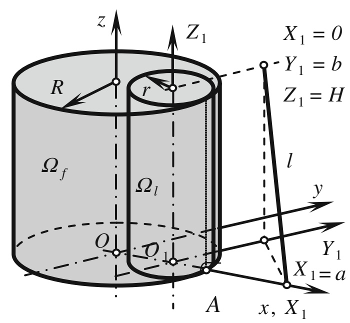


Fig. 1

$$z = z(u,\varphi) = H(1 - u/a) + \lambda\varphi,$$

where $n = R/r$, λ = const.

These equations of the surface are obtained on the base of the general equations given at the page Sect. "34.2.2. Spiroidal Surfaces with Axoids "Cylinder–Cylinder"". In Fig. 1, the initial position of two cylindrical axoids is shown.

If $\lambda = 0$, then the spiroidal surface degenerates into a rotational surface, i.e. rolling of a cylinder on a cylinder without the sliding (Figs. 2a and 3a) will be. In Figs. 2b and 3b, the spiroidal surfaces with the geometrical parameters $\lambda = 0.5$ m; $R = 1$ m; $a = r$; $H = 2$ m; $b = -r/2$; $0 \le z \le H$ are shown. In Fig. 2, the surfaces have $n = 2$; $0 \le \varphi \le 2\pi$; in Fig. 3, the surface have $n = 2.5$; $0 \le \varphi \le 4\pi$.

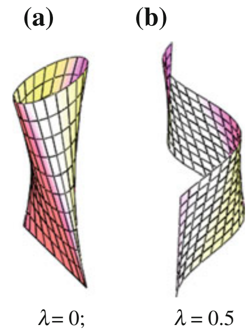


Fig. 2 $n = 2$

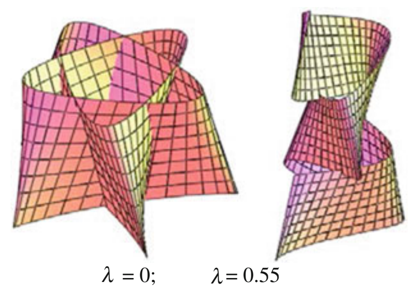


Fig. 3 $n = 2.5$

■ **Spiroidal Surface with Axoids "Cylinder–Cylinder" Formed by a Straight Intersecting the Axis of a Mobile Cylinder in the Process of External Rolling**

Let a generatrix straight line l is given in a mobile system of coordinates $O_1x_1y_1z_1$ as:

$$x_1 = u,$$
$$y_1 = 0,$$
$$z_1 = H(a - u)/(a + b),$$

where constants a, b, H are represented in Fig. 1. In this case, parametric equations of a spiroidal surface formed by the straight generatrix l when external rolling of a mobile cylinder Ω_l with a radius r over a fixed cylinder Ω_f with a radius R, may be given in the following form (Figs. 2 and 3):

$$x = x(u,\varphi) = (R + r)\cos\varphi + u\cos(n + 1)\varphi;$$
$$y = y(u,\varphi) = (R + r)\sin\varphi + u\sin(n + 1)\varphi,$$
$$z = z(u,\varphi) = H(a - u)/(a + b) + \lambda\varphi,$$

where $n = R/r$, λ = const.

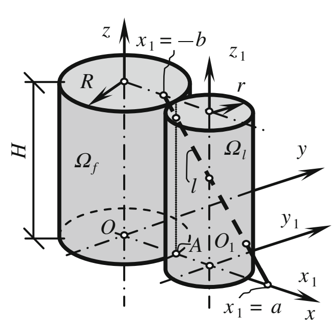


Fig. 1

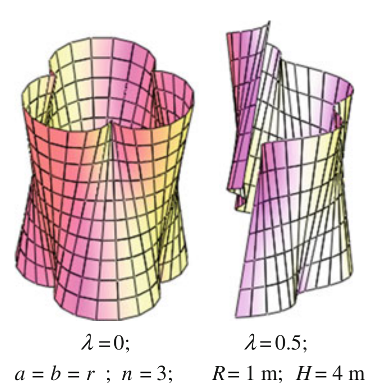
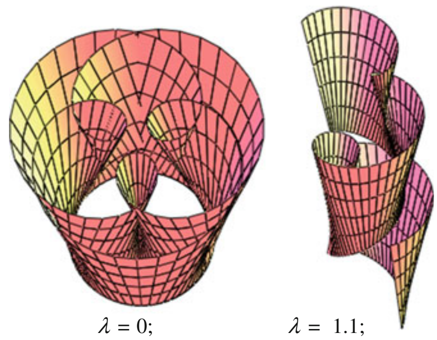


Fig. 2 $0 \le z \le H$



$\lambda = 0$; $\lambda = 1.1$;
$a = b = r$; $n = 1.5$; $R = 1$ m; $H = 4$ m

Fig. 3 $H/2 \leq z \leq H$

The represented equations of the spiroidal surface are obtained on the base of the general equations given at the page Sect. "34.2.2. Spiroidal Surfaces with Axoids "Cylinder–Cylinder"". In Fig. 1, the initial position of two cylindrical axoids is shown. The sliding with rolling of the cylinder Ω_l over the cylinder Ω_f takes place along the line of their contact.

The geometrical parameters of the surfaces are given under the corresponding figures. In Fig. 2, the surfaces have $n = R/r = 3$, $0 \leq \varphi \leq 2\pi$; Fig. 3 shows the surfaces with $n = 1.5$; $0 \leq \varphi \leq 4\pi$;

■ Spiroidal Surface with Axoids "Cylinder–Cylinder" Formed by a Straight Intersecting the Axis of a Mobile Cylinder in the Process of Internal Rolling

Let a generatrix straight line l is given in a mobile system of coordinates $O_1x_1y_1z_1$ in the following form:

$$x_1 = u,\ y_1 = 0,$$
$$z_1 = H(a - u)/(a + b),$$

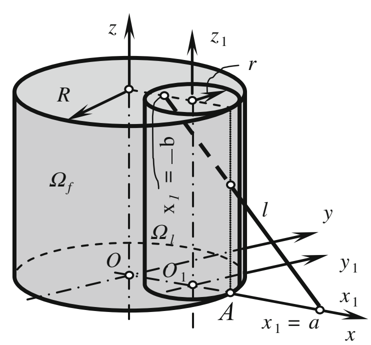


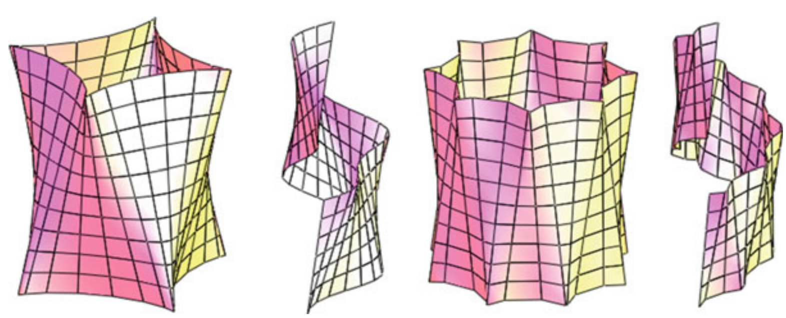
Fig. 1

where constants a, b, H are shown in Fig. 1. In this case, parametrical equations of the spiroidal surface, formed by the straight generatrix l when internal rolling with sliding of a mobile cylinder Ω_l of a radius r on a fixed cylinder Ω_f of a radius R, may be written as (Fig. 1):

$$x = x(u, \varphi) = (R - r)\cos\varphi + u\cos(n - 1)\varphi;$$
$$y = y(u, \varphi) = (R - r)\sin\varphi - u\sin(n - 1)\varphi,$$
$$z = z(u, \varphi) = H(a - u)/(a + b) + \lambda\varphi,$$

where $n = R/r$, λ = const.

The equations of the surface are written on the base of the general equations given at the page Sect. "34.2.2. Spiroidal Surfaces with Axoids "Cylinder–Cylinder"". In Fig. 1, the initial position of two cylindrical axoids is shown. The sliding with rolling of the cylinder Ω_l on the cylinder Ω_f take place along the line of their contact. In Fig. 2, the surfaces are built in the limits $0 \leq z \leq H$. If $\lambda = 0$, the spiroidal surfaces degenerate into the rotational surfaces. The spiroidal surfaces with the axoids "cylinder–cylinder" may be related to a class of *helix-shaped surfaces* (Fig. 2) or to a class of *wave-shaped, waving and corrugated surfaces* (see also the Chap. "25. Wave-Shaped, Waving and Corrugated Surfaces").



$n = 4$; $\lambda = 0$; $n = 4$; $\lambda = 0.5$; $n = 8$; $\lambda = 0$; $n = 8$; $\lambda = 0.5$

Fig. 2 $R = 1$ m; $H = 2$m; $a = b = r$

34.2.3 Spiroidal Surfaces with Axoids "Plane–Cone"

Spiroidal surfaces with axoids "*plane–cone*" are generated by arbitrary curves rigidly connected with a mobile cone, which rolls with sliding over a plane that is a fixed axoid. The sliding (slipping) takes place along the line of contact of the cone and the plane. Figure 1 shows both axoids at the initial moment of the time, when the rolling of the cone did not yet begin.

A generatrix line l is given by parametrical equations:

$$X_1 = X_1(v), Y_1 = Y_1(v), Z_1 = Z_1(v)$$

in the mobile system of Cartesian coordinates $O_1X_1Y_1Z_1$. The coordinate axis O_1X_1 coincides with the axis of the mobile cone. At the initial moment of the time, the origin of the fixed system of Cartesian coordinates $Oxyz$ coincides with the origin of the mobile coordinate system $O_1X_1Y_1Z_1$. In the process of rolling with sliding, the vertex of the cone is traced *the spiral of Archimedes*. The angle of the height of the cone with its straight generatrixes is denoted by α. The angle of the fixed axis Ox with the projection of the mobile axis O_1X_1 on the plane xOy arising in the process of rolling of the cone is denoted by u. When rolling of the cone on the plane, its base of a radius r rotates about the axis of the cone. When the angle of the axis Ox with the projection of the axis O_1X_1 on the plane xOy is equal to u, it means that the base of the cone turns around the axis O_1X_1 through an angle of φ.

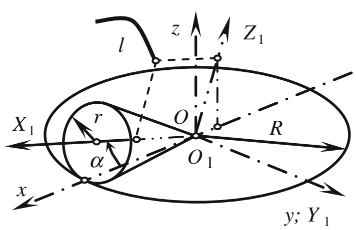


Fig. 1

Between the geometric parameters, the next relations exist:

$Ru = r\varphi$, so it follows that $\varphi = Ru/r = nu$, i.e. $n = R/r = 1/\sin\alpha$, then

$$\sin\alpha = \frac{1}{n}, \cos\alpha = \frac{\sqrt{n^2-1}}{n}.$$

Forms of definition of the spiroidal surface with axoids "plane–cone"

(1) Parametrical equations:

$$\begin{aligned} x = x(u,v) &= X_1\cos\alpha\cos u + Y_1(\sin\varphi\sin\alpha\cos u - \cos\varphi\sin u) \\ &\quad - Z_1(\cos\varphi\sin\alpha\cos u + \sin\varphi\sin u) + \lambda\varphi\cos u, \\ y = y(u,v) &= X_1\cos\alpha\sin u + Y_1(\sin\varphi\sin\alpha\sin u + \cos\varphi\cos u) \\ &\quad - Z_1(\cos\varphi\sin\alpha\sin u - \sin\varphi\cos u) + \lambda\varphi\sin u, \\ z = z(u,v) &= X_1\sin\alpha - Y_1\cos\alpha\sin\varphi + Z_1\cos\varphi\cos\alpha, \end{aligned}$$

where λ is a parameter characterizing the sliding of the cone along the line of its contact with the plane.

■ Spiroidal Surface with Axoids "Plane–Cone" Formed by a Straight Passing Through the Vertex of a Mobile Cone

If a generatrix straight line is given in a mobile system of Cartesian coordinates $O_1X_1Y_1Z_1$ by parametric equations (Fig. 1)

$$X_1 = v,\ Y_1 = 0,\ Z_1 = v\tan\beta$$

then the straight line will generate *the ruled spiroidal surface with axoids* "*plane–cone*" *formed by a straight line passing through the vertex of a mobile cone* in the process of rolling with sliding of a circular cone over a plane. Depending on the value of the angle β, it is possible to obtain four variants of disposition of the generatrix straight line passing thorough the vertex of the cone: $\beta > \alpha$, $\beta < \alpha$, $\beta = \alpha$, and $\beta = 0$, where α is the angle of the height of the cone with its straight generatrixes. Parametrical equations of the spiroidal surface in question may be obtained from the general equations given at the page Sect. "34.2.3. Spiroidal Surfaces with Axoids "Plane–Cone"".

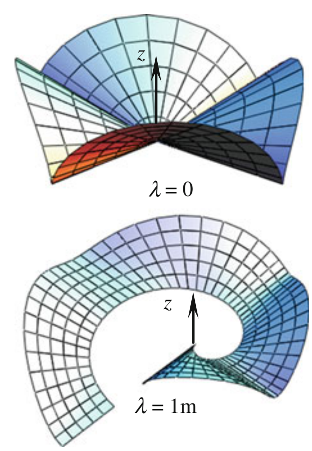


Fig. 2

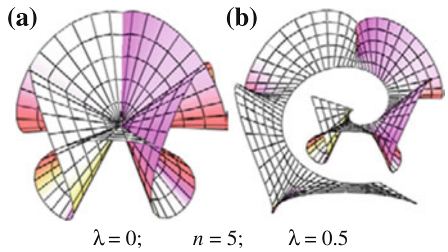


Fig. 3 $\beta > \alpha$

$$
\begin{aligned}
x = x(u, v) &= v\cos\alpha\cos u - v(\cos\varphi\sin\alpha\cos u \\
&\quad + \sin\varphi\sin u)tg\beta + \lambda\varphi\cos u, \\
y = y(u, v) &= v\cos\alpha\sin u - v(\cos\varphi\sin\alpha\sin u \\
&\quad - \sin\varphi\cos u)tg\beta + \lambda\varphi\sin u, \\
z = z(u, v) &= v\sin\alpha + v\cos\varphi\cos\alpha tg\beta.
\end{aligned}
$$

Figure 2 show the ruled spiroidal surface formed by the straight generatrix of the loose axoid that is a circular cone ($\beta = \alpha$), $n = 4$.

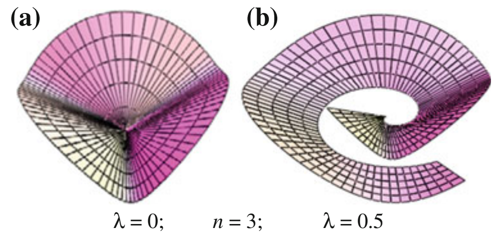


Fig. 4 $\beta < \alpha$

In Fig. 3, the ruled spiroidal surfaces with axoids "plane–cone" are presented, when $\beta > \alpha$, $n = 5$. In Fig. 4, the surfaces have $\beta < \alpha$, $n = 3$. When $\lambda = 0$, the surfaces are constructed in the limit $0 \le u \le 2\pi$, but when $\lambda > 0$, the surfaces shown in Figs. 3b and 4b have $0 \le u \le 2.5\pi$.

When $\lambda > 0$, the coordinate line $v = 0$ is *the spiral of Archimedes*.

34.2.4 Spiroidal Surfaces with Axoids "Plane–Cylinder"

Spiroidal surfaces with axoids "plane–cylinder" are formed by arbitrary curves L when rolling with sliding of a cylinder of a radius r on a plane. The sliding takes place along the line of contact of the cylinder and the plane. The generator curve L given in a mobile system of Cartesian coordinates $O_1X_1Y_1Z_1$ by parametric equations

$$X_1 = X_1(u), Y_1 = Y_1(u), Z_1 = Z_1(u)$$

is rigidly connected with the loose axoid that is a circular cylinder. If a generator line L is a straight line, then parametric equations of a straight line L may be written as:

$$
\begin{aligned}
&X_1(u) = \frac{c-a}{H}u + a,\; Y_1(u) = \frac{d-b}{H}u + b, \\
&Z_1(u) = u,
\end{aligned}
$$

where a, b, c, d are the coordinates of the ends of a straight-line segment in the mobile system of coordinates $X_1O_1Y_1$ (Fig. 1). In this case, the surface is called *the ruled spiroidal surface with axoids "plane–cylinder"*. Figure 1 shows the initial disposition of the cylinder, when the rolling did not yet begin.

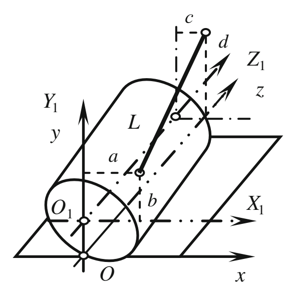


Fig. 1

The form of the definition of the spiroidal surface

(1) Parametrical equations:

$$
\begin{aligned}
x(u, \varphi) &= r\varphi + X_1(u)\cos\varphi + Y_1(u)\sin\varphi, \\
y(u, \varphi) &= r - X_1(u)\sin\varphi + Y_1(u)\cos\varphi, \\
z(u, \varphi) &= Z_1(u) + \lambda\varphi,
\end{aligned}
$$

where φ is an angle read from the positive direction of the axis Ox in the direction of the mobile axis O_1X_1; λ = const is a parameter characterizing the sliding of the cylinder along the line of its contact with the plane; x, y, z are the Cartesian coordinates of any point of the curve L relatively to the fixed system of coordinates $Oxyz$.

Coefficients of the fundamental forms of the surface:

$$A^2 = X_1'^2 + Y_1'^2 + Z_1'^2,$$
$$F = r(X_1' \cos\varphi + Y_1' \sin\varphi) - X_1 Y_1' + Y_1 X_1' + \lambda Z',$$
$$B^2 = r^2 + X_1^2 + Y_1^2 + 2r(Y_1 \cos\varphi - X_1 \sin\varphi) + \lambda^2,$$

$$L = \{\lambda(Y_1'X_1'' - X_1'Y_1'') + r[(Z_1'Y_1'' - Y_1'Z_1'')\cos\varphi + (X_1'Z_1'' - Z_1'X_1'')\sin\varphi] + Z'(X_1X_1'' + Y_1Y_1'') - Z''(X_1X_1' + Y_1Y_1')\}/\sqrt{A^2B^2 - F^2},$$
$$M = \frac{\lambda(X_1'^2 + Y_1'^2) + Z_1'[X_1Y_1'^2 - X_1'Y_1 - r(X_1'\cos\varphi + Y_1'\sin\varphi)]}{\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{\lambda(X_1'Y_1 - X_1Y_1') + Z_1'[r(X_1\sin\varphi - Y_1\cos\varphi) - X_1'^2 - Y_1'^2]}{\sqrt{A^2B^2 - F^2}}.$$

■ Spiroidal Surface with Axoids "Plane–Cylinder" Formed by a Straight Parallel to the Axis of a Rolling Cylinder

Assume a generator line

$$X_1 = 0; Y_1 = kr; Z_1 = u$$

where k is a constant, then a strip of a constant width of *a right cylindrical surface with the directrix cycloid* is available if $k = 1$; if $k > 1$, then *the directrix extended cycloid* (Figs. 1 and 2) will be available; or we may obtain *the directrix curtate cycloid* taking $k < 1$. If $k = 1$, then one of the generator straight line of the circular mobile cylinder will be by the straight generatrix of the spiroidal surface. Parametrical equations of the surfaces in question are

$$x = x(u,\varphi) = r(\varphi + k\sin\varphi),$$
$$y = y(u,\varphi) = r(1 + k\cos\varphi),$$
$$z = z(u,\varphi) = u + \lambda\varphi,$$

for which

$$A = 1,\ F = \lambda,\ B^2 = r^2(1 + k^2 + 2k\cos\varphi) + \lambda^2,\ L = M = 0,\ K = 0.$$

If $\lambda = 0$, we shall have a rotational surface (Fig. 1).

Figure 3 show the spiroidal surfaces with different values of parameter k. When $k = 4.6$, the braches of cycloids touch each other, but when $k > 4.6$, the branches cross themselves.

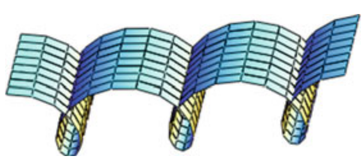


Fig. 1

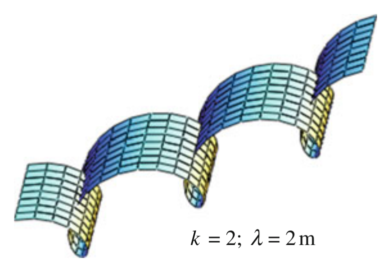


Fig. 2

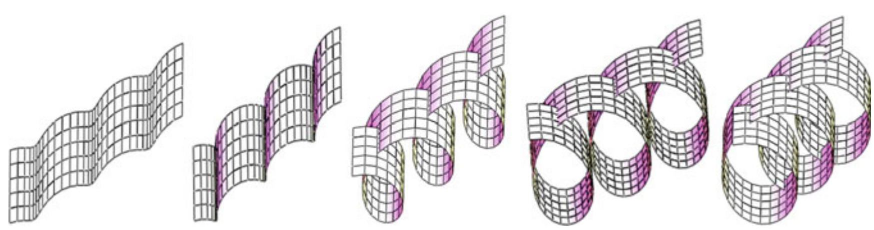


Fig. 3 $\lambda = 2$

■ Spiroidal Surface with Axoids "Plane–Cylinder" Formed by a Straight Intersecting the Axis of a Rolling Cylinder

All necessary geometric characteristics of *a spiroidal surface with axoids «plane–cylinder» formed by a straight line intersecting the axis of a rolling cylinder* may be obtained from the general formulas given at the Subsect. "34.2.4. Spiroidal Surfaces with Axoids "Plane–Cylinder"". For example, the generatrix straight line may be given by the following parametrical equations:

$$X_1 = 0, Y_1(u) = u \tan\beta, Z_1(u) = u,$$

where β is the angle of the generatrix straight line with the axis of the rolling with sliding cylinder of a radius r. The axis of the mobile cylinder coincides with the mobile axis O_1Z_1. Before the beginning of the process of the rolling, the straight line crosses the axis of the mobile cylinder at the point with the coordinates: $x = 0$, $y = r$, $z = 0$, or with the coordinates $X_1 = Y_1 = Z_1 = 0$, if one considers the mobile system of coordinates.

The form of the definition of the ruled spiroidal surface
(1) Parametrical equations:

$$\begin{aligned} x(u,\varphi) &= r\varphi + u\tan\beta\sin\varphi, \\ y(u,\varphi) &= r + u\tan\beta\cos\varphi, \\ z(u) &= u + \lambda\varphi, \end{aligned}$$

where φ is an angle read from the positive direction of the axis Ox in the direction of the positive direction of the mobile axis O_1X_1; λ = const is a parameter characterizing the sliding of the rolling cylinder along the line of its contact with the fixed plane; x, y, z are the Cartesian coordinates of arbitrary point of the generatrix straight line relatively to the fixed system of coordinates $Oxyz$.

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} A &= 1/\cos\beta, \\ F &= \lambda + r\tan\beta\sin\varphi, \\ B^2 &= r^2 + u(u\tan\beta + 2r\cos\varphi)\tan\beta + \lambda^2, \\ L &= 0, \\ M &= \frac{\lambda\tan\beta - r\sin\varphi}{\sqrt{A^2B^2 - F^2}}\tan\beta, \\ N &= -\frac{ru\cos\varphi + \tan\beta}{\sqrt{A^2B^2 - F^2}}\tan\beta; \\ K &= \frac{-M^2}{(A^2B^2 - F^2)^2} < 0 \end{aligned}$$

In Fig. 1, the spiroidal ruled surface with $\beta = \pi/16$, $r = 1$ m, $\lambda = 0.5$ m is shown but in Fig. 1a, we have $u = 0 \div 5$, in Fig. 1b we have $u = 0 \div 10$.

When $\lambda = 0$, the spiroidal surface degenerates into the rotational surface, i.e. the rolling of a cylinder over a plane without sliding takes place.

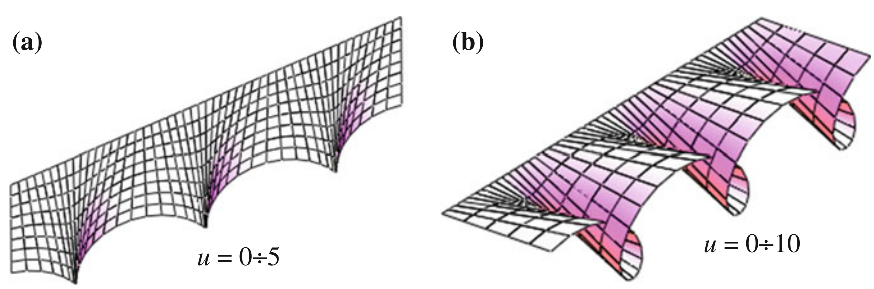


Fig. 1

■ Parabolic Spiroidal Surface with Axoids "Plane–Cylinder"

A parabolic spiroidal surface with axoids "plane–cylinder" is formed by a *plane parabola m* when the rolling with sliding of a cylinder of a radius r over a plane. The sliding takes place along the line of the contact of the plane with the cylinder. The generator parabola m given in a mobile system of Cartesian coordinates $O_1X_1Y_1Z_1$ by parametric equations

$$\begin{aligned} X_1 &= u, \\ Z_1 &= 0, \\ Y_1 &= Y_1(u) = a + bu^2, \end{aligned}$$

is rigidly connected with the loose axoid, that is a circular cylinder; a, b are constants. The parabola m (Fig. 1) is disposed in the plane perpendicular to the axis of the cylinder rolling with sliding.

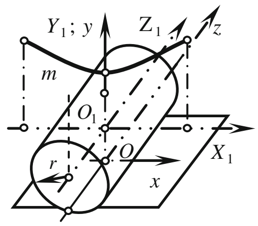


Fig. 1

The form of definition of the surface
Parametrical equations of the surface in question are

$$x(u,\varphi) = r\varphi + u\cos\varphi + (a + bu^2)\sin\varphi,$$
$$y(u,\varphi) = r - u\sin\varphi + (a + bu^2)\cos\varphi,$$
$$z(\varphi) = \lambda\varphi,$$

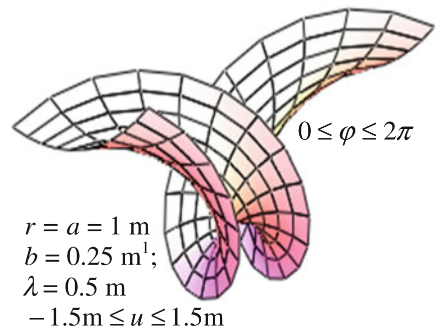


Fig. 2

where φ is the angle between the axes Ox and O_1X_1 rising after the beginning of the process of the rolling (Fig. 2).

34.2.5 Spiroidal Ruled Surfaces of Rachkovskaya: Kharabaev

Spiroidal ruled surfaces of Rachkovskaya — Kharabaev are formed by a selected straight generatrix of a given arbitrary cone taken as a loose axoid which rolls with sliding above other developable surface that is a fixed axoid. The vertex of the cone is on the cuspidal edge of the fixed developable axoid all the time and two ruled surfaces touch each other along their common straight generatrix at every moment of the time (Fig. 1).

The motion of a conical surface over the developable fixed axoid may be represented as a translational motion of the vertex of the cone along the cuspidal edge (f) of the fixed axoid with the simultaneous rotation of the cone about its axis. The condition of the co-ordination of these two motions is expressed with the help of a formula:

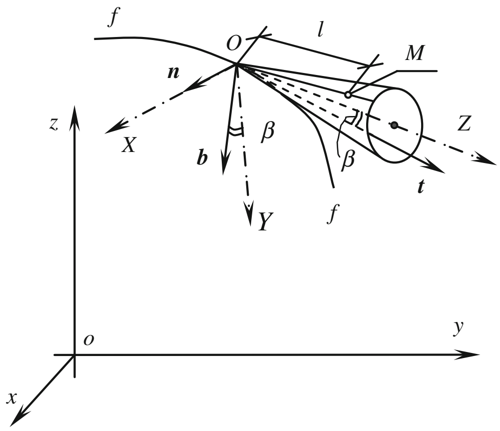


Fig. 1

$$d\varphi = \frac{ds}{R\sin\beta},$$

where ds is the arc length of the cuspidal edge passed by the vertex of the cone; $d\varphi$ is an angle of rotation of the cone about its axis when its rolling; R is a radius of the curvature of the cuspidal edge f of the fixed axoid in the point O; β is the angle of the axis of the cone with the straight generatrix that is the common one for the cone and the fixed axoid.

The torse surface (fixed axoid) is given in a general fixed system of coordinates denoted by $oxyz$ by parametrical equations of its cuspidal edge:

$$x = x(t),\ y = y(t),\ z = z(t).$$

The conical surface is given in a subsidiary mobile system of the Cartesian coordinates $OXYZ$, the origin of which is disposed in the vertex of the mobile cone (the point O). The directions of the axes OX, OY and OZ are given with the help of the corresponding Frenet trihedral $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ connected with the cuspidal edge f of the torse surface. The axis OX is directed along the principle normal $\boldsymbol{n}$ to the cuspidal edge f of the fixed axoid that is a torse surface. The axis OZ coincides with the axis of the mobile cone and is formed the angle β with the tangent $\boldsymbol{t}$ to the curve f. The axis OY is determined by the choice of the right Cartesian system of coordinates $OXYZ$ and constitutes the angle β with the binormal $\boldsymbol{b}$.

With the help of differential geometry methods, an equation of the kinematical curve $f_M[x_M(t),\ y_M(t),\ z_M(t)]$ forming the trajectory of the motion of arbitrary point M fixed at the surface of the mobile cone at a distance l from its vertex, was derived. These equations may be written in parametrical form:

$$x_M(t) = x(t) + X_x X_M + Y_x Y_M + Z_x Z_M,$$
$$y_M(t) = y(t) + X_y X_M + Y_y Y_M + Z_y Z_M$$
$$z_M(t) = z(t) + X_z X_M + Y_z Y_M + Z_z Z_M,$$

where $x(t)$, $y(t)$, $z(t)$ are the coordinates of the vertex of the cone (point O); X_x, Y_x, Z_x, X_y, Y_y, Z_y, X_z, Y_z, Z_z are the direction cosines of the axes OX, OY, OZ in the system of the fixed axes ($oxyz$); X_M, Y_M, Z_M are the coordinates of the moving point M in the system of the mobile axes $OXYZ$.

On the base of the kinematical curve $f_M(t)$ and assumed curve $f(t)$, it is possible to design a kinematical ruled surface as the set of the straight-line segments connecting the corresponding points O and M on the curves $f(t)$ and $f_M(t)$ for every value of the parameter t.

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■ Spiroidal Ruled Surface of Rachkovskaya–Kharabaev with Axoids "Evolvent Helicoid–Right Circular Cone"

Analytic form of definition of the kinematical surface formed as a result of the *complex motion* of a cone over a developable surface (see also a Subsect. "34.2.5. Spiroidal Ruled Surfaces of Rachkovskaya–Kharabaev") for the most simple case, i.e. for the case of the motion of a right circular cone over a torse surface with the cuspidal edge in the form of *the circular cylindrical helical line*.

The design of the kinematical curve f_M [$x_M(t)$, $y_M(t)$, $z_M(t)$] describing the trajectory of the motion of any point M fixed at the surface of the cone at the distance of l from the vertex of the cone carries out on the base of parametric definition of the cuspidal edge of the torse helicoid $f(t)$ in the system of fixed coordinates $oxyz$:

$$f[x(t) = a\cos t,\ y(t) = a\sin t,\ z(t) = ct],$$

where $x(t)$, $y(t)$, $z(t)$ are coordinates of the vertex of the loose cone. For the determination of the direction cosines of the axes OX, OY, OZ in the system of coordinates $oxyz$ one may use the following formulas:

$$X_x = -\cos t;\ Y_x = \frac{c\cos\beta + a\sin\beta}{\sqrt{a^2+c^2}}\sin\ t;$$
$$Z_x = \frac{c\sin\beta - a\cos\beta}{\sqrt{a^2+c^2}}\sin\ t;$$
$$X_y = -\sin t;\ Y_y = -\frac{c\cos\beta + a\sin\beta}{\sqrt{a^2+c^2}}\cos t;$$
$$Z_y = -\frac{C\sin\beta - a\cos\beta}{\sqrt{a^2+c^2}}\cos t;$$
$$X_z = 0;\ Y_z = -\frac{c\sin\beta - a\cos\beta}{\sqrt{a^2+c^2}};$$
$$Z_z = \frac{c\cos\beta + a\sin\beta}{\sqrt{a^2+c^2}}.$$

The coordinates of the rotating point M in the mobile system of coordinates $OXYZ$ are determined by the formulas:

$$X_M = l\sin\beta\cos(\varphi + \varphi_0);$$
$$Y_M = l\sin\beta\sin(\varphi + \varphi_0);$$
$$Z_M = l\cos\beta,$$

where φ is an angle of rotation of the cone about its axis in the system $OXYZ$ [$\varphi_0 = \varphi(t = 0)$].

In this case, parametrical equations of the curve f_M [$x_M(t)$, $y_M(t)$, $z_M(t)$] in the system of Cartesian coordinates $oxyz$ have the following form:

$$x_M(t) = a\cos t - \Bigg[\cos t\cdot\cos(\varphi+\varphi_0) - \frac{c\cos\beta + a\sin\beta}{\sqrt{a^2+c^2}}\sin t\cdot\sin(\varphi+\varphi_0) - \frac{c\sin\beta - a\cos\beta}{\tan\sqrt{a^2+c^2}}\sin t\Bigg] l\sin\beta;$$

$$y_M(t) = a\sin t - \Bigg[\sin t\cdot\cos(\varphi+\varphi_0)| + \frac{c\cos\beta + a\sin\beta}{\sqrt{a^2+c^2}}\cos t\cdot\sin(\varphi+\varphi_0) + \frac{c\sin\beta - a\cos\beta}{\tan\beta\sqrt{a^2+c^2}}\cos t\Bigg] l\sin\beta;$$

$$z_M(t) = ct - \left[\frac{c\sin\beta - a\cos\beta}{\sqrt{a^2+c^2}}\sin(\varphi+\varphi_0) - \frac{c\cos\beta + a\sin\beta}{\tan\beta\sqrt{a^2+c^2}}\right] l\sin\beta$$

The parameter φ, that is an angle of rotation of the cone about its axis in the system *OXYZ,* is determined from the condition of co-ordination of two motions, i.e. of translational motion of the vertex of the cone along the cuspidal edge of the developable helicoid and of the rotary motion of the cone about its axis:

$$\varphi = \frac{-at}{\sin\beta\cdot\sqrt{a^2+c^2}};\ (\varphi_0 = 0).$$

With the help of the kinematical curve $f_M(t)$ obtained in the analytical form and the assumed curve $f(t)$, the kinematical ruled surface (Fig. 1) was constructed as the combination of the straight-line segments connecting the corresponding points O and M on the curves $f(t)$ and $f_M(t)$ for every values of t.

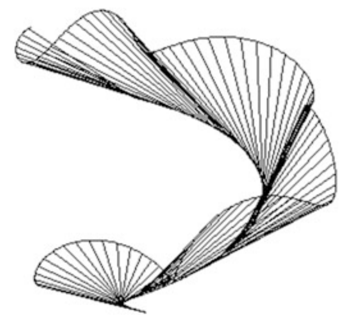

Fig. 1

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■ Spiroidal Ruled Surface of Rachkovskaya–Kharabaev with Axoids "Developable Conical Helicoid–Right Circular Cone"

The computer design of kinematical ruled surfaces generated when the rolling of a cone with the variable director curve F above a torse surface with a complex cuspidal edge $f(t)$ gives an opportunity to go over from analytical form of the definition of the curve $f_M(t)$ to its design with the help of points obtained by numerical methods of differentiation and integration of the corresponding equations. The symbolic designation and the method giving the opportunity to obtain the trajectory of the motion of any point M at the surface of the mobile cone is described in the Subsect. "34.2.5. Spiroidal ruled surfaces of Rachkovskaya–Kharabaev". In Fig. 1, the spiroidal ruled surface obtained for the case of the complex motion of the right circular cone over the developable surface with the cuspidal edge in the form of *a conical helical line*.

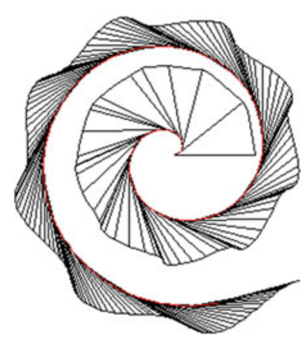

Fig. 1

In accordance with the geometrical model of the complex motion of a right circular cone on a torse, the condition of the co-ordination of two motions, that are the translational motion of the vertex of the cone along the cuspidal edge of the torse (the curve $f(t)$) and the rotary motion of the cone about its axis, is determined at a finite segment of the curve $f(t)$ by the following manner:

$$\varphi = \int_0^s \frac{1}{R\sin\beta}\mathrm{d}s = \int_0^t \frac{k_1}{R\sin\beta}\mathrm{d}t,$$

where φ is an angle of rotation of the cone about its axis given in the mobile system of coordinates *OXYZ*; *R* is a radius of the curvature of the curve *f*(*t*) given by parametric equations: $x = x(t)$, $y = y(t)$, $z = z(t)$. *R* is determined by the formula:

$$R = \frac{\sqrt{\left(x'^2 + y'^2 + z'^2\right)^3}}{\sqrt{\left(y'z'' - z'y''\right)^2 + \left(z'x'' - x'z''\right)^2 + \left(x'y'' - y'x''\right)^2}};$$

$$k_1 = \sqrt{x'^2 + y'^2 + z'^2};$$

x', y', z' and x'', y'', z'' are the values of the first and the second derivatives of the functions $x(t)$, $y(t)$, $z(t)$ at the point *O* correspondingly.

The values of a parameter φ calculated for every value of the parameter *t* allow determining the coordinates of any rotating point *M* fixed on the surface of the cone at the distance of *l* from the vertex of the cone in the system *OXYZ*:

$$X_M = l\sin\beta\cos(\varphi + \varphi_0);$$
$$Y_M = l\sin\beta\sin(\varphi + \varphi_0);$$
$$Z_M = l\cos\beta.$$

With the help of the kinematical curve $f_M(t)$ and the assumed curve *f*(*t*), the kinematical ruled surface (Fig. 1) was constructed as the combination of the straight-line segments connecting the corresponding points *O* and *M* on the conical helical line *f*(*t*) and on the trajectory $f_M(t)$ of the motion of the *M* point, belonging to the rolling cone, for every values of *t*.

■ Spiroidal Ruled Surface of Rachkovskaya–Kharabaev with Axoids “Evolvent Helicoid–Right Elliptical Cone”

In Fig. 1, the example of the kinematical surface received with the using of the numerical methods of differentiation and integration for the case of the spiroidal motion of *the right elliptical cone* over the torse surface with the cuspidal edge in the form of *the helix*. In the case of arbitrary conical surface (Fig. 2), given by the vertex of the cone *O* and by arbitrary convex generatrix, that is a curve *F* [$X = X(p)$, $Y = Y(p)$, $Z = Z(p)$], in the first instance, re-determination of this surface is carried out with the help of a subsidiary spherical directrix, that is the curve F_1 [$X = X_1(p)$, $Y = Y_1(p)$, $Z = Z_1(p)$], the point of which lie on the intersection of the chosen arbitrary conical surface and the sphere of a arbitrary radius *l* with the center at the vertex of the cone *O*. In the second instance, it is necessary to re-determine the selected generatrix going through the *N* point with the help the point *O* and the point *M* lying on the intersection of the generatrix *ON* with the subsidiary spherical directrix F_1. The length of the segment *OM* of the selected generatrix equals *l*. For the determination of an equation of the kinematical curve $f_M(x_M(t), y_M(t), z_M(t))$, it is necessary to find a value of a parameter *p* for every value of the *t* parameter:

$$\frac{1}{l}\int_{p_0}^{p} K_1(p)\mathrm{d}p = \int_0^t \frac{1}{R(t)}k_1(t)\mathrm{d}t,$$
$$p_0 = p(t = 0);$$
$$K_1 = \sqrt{X_1'^2 + Y_1'^2 + Z_1'^2},$$
$$k_1 = \sqrt{x'^2 + y'^2 + z'^2},$$

where $R(t)$ is a radius of the curvature of the cuspidal edge of the torse $f(x = x(t), y = y(t), z = z(t))$. The parameter *p* determines the coordinates of the point *M* in the system *OXYZ,* the current value of the angle $\beta(p)$, the direction cosines and the coordinates of the *M* point in the system *oxyz* for every value *t*.

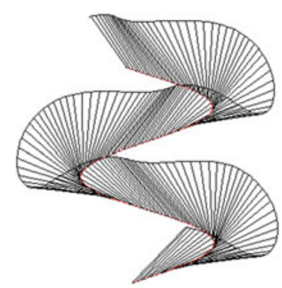

Fig. 1

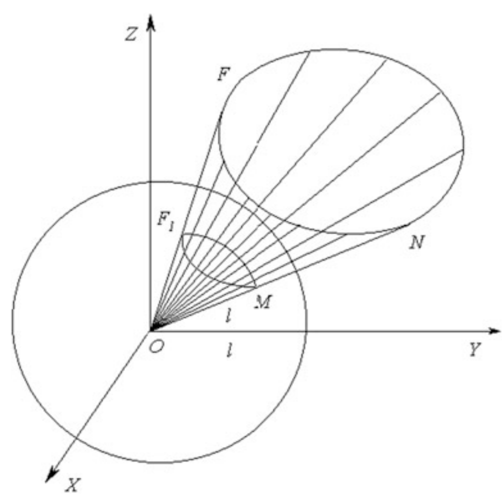

Fig. 2

■ Kinematic Ruled Surface on the Base of One-Sheet Hyperboloidal Surfaces of Revolution as Fixed and Loose Axoids (One Axoid Is Located Outside Another)

Kinematic ruled surfaces can be constructed by the motion of one of the generating lines of one (mobile) ruled surface in the process of its motion on another (fixed) ruled surface.

The main condition for constructing kinematic ruled surfaces is: the mobile ruled surface contacts with the fixed ruled surface along their common generatrix line in each of their positions when the motion of one axoid along the other. This condition of contact of one-sheet hyperboloidal surface of revolution with another one along their common generatrix line can be satisfied on the base of *complex moving* of one axoid on another.

Geometrical model of complex moving of one-sheet hyperboloidal surface of revolution on another is represented as a superposition of three interrelated movements:

(1) rotational motion of the loose axoid ***2*** around its axis described in the mobile system of Cartesian coordinates *OXYZ* (Fig. 1);
(2) rotational motion of the axis *OZ* coinciding with the axis of the mobile axoid ***2*** about the axis *oz* coinciding with the axis of the fixed axoid ***1*** described in the fixed coordinate system *oxyz* connected with the fixed axoid ***1*** (Fig. 1);
(3) sliding motion of the loose axoid ***2*** along a generatrix line shared by both ***1*** and ***2*** axoids.

In the case of two different fixed (***1***) and loose (***2***) axoids ($a_1 \neq a_2$, $c_1 \neq c_2$), construction of kinematic surfaces becomes possible subject to the following parametric condition:

$$a_1^2 + c_1^2 = a_2^2 + c_2^2$$

where a_i, c_i are parameters of the canonical equation:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1.$$

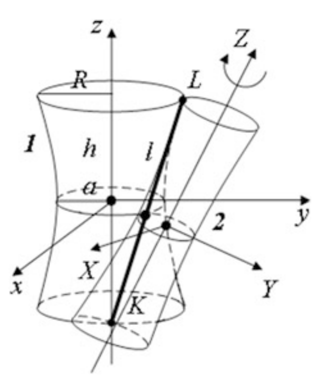

Fig. 1

Parametric representation of a kinematic ruled surface is based on the following transition equations from the system *OXYZ* to the system *oxyz*:

$$x = (Z\sin\theta + X\cos\theta)\cos u - (Y + a_1 + a_2)\sin u;$$
$$y = (Z\sin\theta + X\cos\theta)\sin u + (Y + a_1 + a_2)\cos u;$$
$$z = Z\cos\theta - X\sin\theta,$$

where

$$\theta = \arctan\frac{a_1}{c_1} + \arctan\frac{a_2}{c_2}.$$

The coordinates *X, Y, Z* are determined by parametric equations of surface, generated by one of the generatrix lines of the mobile one-sheet hyperboloid of revolution (axoid ***2***) in the mobile coordinate system *OXYZ*:

$$X = -a_2\sin\varphi + a_2 v\cos\varphi;$$
$$Y = a_2\cos\varphi + a_2 v\sin\varphi;$$
$$Z = c_2 v,$$

where

$$\varphi = (a_1/a_2)u + \pi.$$

Parametrically matched pairs of contacting axoids (Figs. 2, 3 and 4) and corresponding kinematic ruled surfaces (Figs. 2a, 3 and 4a) obtained for three cases ($a_1 = a_2$, $a_1 > a_2$, and $a_1 < a_2$) are shown.

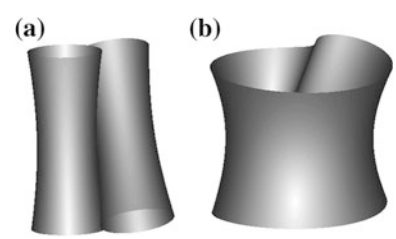

Fig. 2

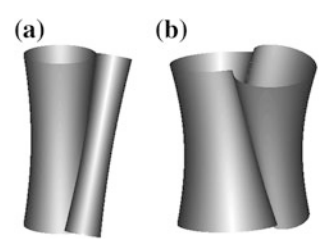

Fig. 3

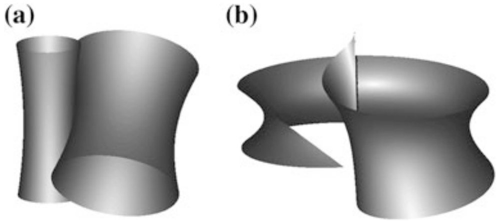

Fig. 4

Reference

Rachkovskaya GS, Kharabayev YuN. Geometric modeling and computer graphics of kinematic ruled surfaces on the base of complex moving one axoid along another (one-sheet hyperboloid of revolution as fixed and moving axoids). Proceedings of the 17-th International Conference in Central Europe on Computer Graphics, Visualization and Computer Vision, Plzen, Czech Republic. 2009; p. 31-34 (4 refs).

■ Kinematic Ruled Surface on the Base of One-Sheet Hyperboloidal Surfaces of Revolution as Fixed and Loose Axoids (One Axoid Is Located in the Interior of Another)

Two possible variants of mutual arrangement of loose and fixed axoids can be realized in the geometrical model of constructing kinematic ruled surfaces on the base of complex motion of one-sheet hyperboloidal surface of revolution on another one for the case when one axoid is located in the interior of another.

In the first variant, the loose axoid is located in the interior of the fixed axoid as shown in Fig. 1.

In the second variant, the fixed axoid is located in the interior of the loose axoid as shown in Fig. 2. The axes of the fixed axoids are located vertically.

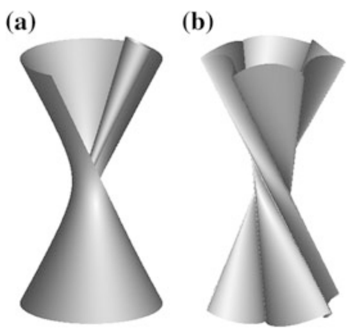

Fig. 1

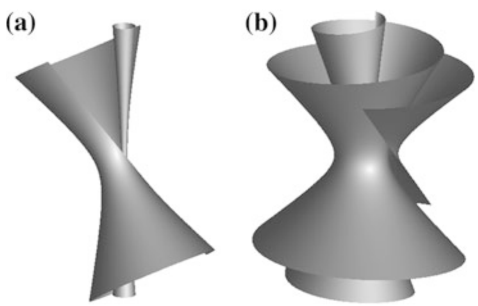

Fig. 2

In the first variant, the outside surface of the moving axoid revolves around the interior surface of the fixed axoid.

In the second variant, the interior surface of the moving axoid revolves around the outside surface of the fixed axoid.

The first variant corresponds to the case when $a_1 > a_2$, and the second variant corresponds to the case when$a_1 < a_2$. Here a_1 and a_2 are the waist radius of fixed and moving axoids.

In concordance with the geometrical model of constructing kinematic ruled surfaces on the base of *complex motion* of one-sheet hyperboloidal surface of revolution on another described before in a previous Subsect. "34.2.5. Spiroidal Ruled Surfaces of Rachkovskaya–Kharabaev" in the case when one axoid is located in the interior of another parametric representation of kinematic ruled surface is based on the following transition equations from the mobile coordinate system *OXYZ* to the fixed coordinate system *oxyz*:

$$x = (Z \sin\theta + X\cos\theta)\cos u - (Y + a_1 - a_2)\sin u;$$
$$y = (Z \sin\theta + X\cos\theta)\sin u + (Y + a_1 - a_2)\cos u;$$
$$z = Z\cos\theta - X\sin\theta,$$

where

$$\theta = \left|\arctan\frac{a_1}{c_1} - \arctan\frac{a_2}{c_2}\right|.$$

X, Y, Z are defined by parametric equations of the surface generated by one of the generatrix lines of the moving one-sheet hyperboloid of revolution (axoid ***2***) in the mobile coordinate system *OXYZ*:

$$X = -a_2\sin\varphi + a_2 v\cos\varphi;$$
$$Y = a_2\cos\varphi + a_2 v\sin\varphi;$$
$$Z = c_2 v$$

where

$$\varphi = -(a_1/a_2)u.$$

Parametrically matched pairs of contacting axoids (Figs. 1 and 2) satisfied parametrical condition

$$a_1^2 + c_1^2 = a_2^2 + c_2^2,$$

and the corresponding kinematic ruled surfaces, obtained for two cases ($a_1 > a_2$ (Fig. 2a) and ($a_1 < a_2$ (Fig. 2a) are shown.

References

Rachkovskaya GS, Kharabayev YuN. Geometric modeling and computer graphics of kinematic ruled surfaces on the base of complex moving one axoid along another (one-sheet hyperboloid of revolution as fixed and moving axoids). Proceedings of the 17-th International Conference in Central Europe on Computer Graphics, Visualization and Computer Vision, Plzen, Czech Republic. 2009; p. 31-34 (4 refs).

Rachkovskaya GS, Kharabayev YuN. Geometric model of kinematic surfaces on the base of one-sheet hyperboloidal surfaces of revolution (one axoid is located in the interior of another axoid). Proceedings of the 14-th International Conference on Geometry and Graphics, Kyoto, Japan. 2010; p. 320-321 (4 refs).

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35 The Second Order Surfaces

The second order surfaces are defined by algebraic equations of the second order relatively to the Cartesian coordinates. The second order surfaces are called also *quadratic surfaces* or *quadrics*. A general equation of the second order has the following form:

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0,$$

where $a_{ik} = a_{ki}$; $i, k = 1, 2, 3, 4$. This equation may not determine the real geometrical image. In this case, they say that the equation defines *imaginary surface of the second order*. Depending on the values of coefficients of the general equation, it may describe 17 surfaces:

Nondegenerating irreducible surfaces
(1) Ellipsoid ($x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$),
(2) Imaginary ellipsoid ($x^2/a^2 + y^2/b^2 + z^2/c^2 = -1$),
(3) One-sheet hyperboloid ($x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$),
(4) Two-sheet hyperboloid ($-x^2/a^2 - y^2/b^2 + z^2/c^2 = 1$),
(5) Elliptic paraboloid ($x^2/p + y^2/q = 2z$; $p, q > 0$),
(6) Hyperbolic paraboloid ($x^2/p - y^2/q = 2z$; $p, q > 0$)
Degenerating irreducible surfaces
Cylindrical surfaces of the second order:
(7) Elliptic cylinder ($x^2/a^2 + y^2/b^2 = 1$),
(8) Imaginary elliptical cylinder ($x^2/a^2 + y^2/b^2 = -1$),
(9) Hyperbolic cylinder ($x^2/a^2 - y^2/b^2 = 1$),
(10) Parabolic cylinder ($y^2 = 2px$)
Conical surfaces of the second order:
(11) Elliptical conical surface ($x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$),
(12) Imaginary conical surface ($x^2/a^2 + y^2/b^2 + z^2/c^2 = 0$)
Degenerating irreducible surfaces
(13) A pair of intersecting planes ($x^2/a^2 - y^2/b^2 = 0$),
(14) A pair of imaginary intersecting planes ($x^2/a^2 + y^2/b^2 = 0$),
(15) A pair of parallel planes ($x^2/a^2 = 1$),
(16) A pair of imaginary parallel planes ($x^2 + a^2 = 0$),
(17) A pair of coinciding planes ($x^2 = 0$)

When intersection of the second order surface with a plane, only the following cases may appear:

(a) a surface intersects with a plane along the second order curve;
(b) a surface intersects with a plane along a straight line;
(c) a surface disintegrates into a pair of planes, one of them is the given plane;
(d) a surface has not any common point with the plane.

If a straight line have nonasymptotic direction, then it intersects a surface at two points that are different (real or imaginary) or coinciding.

The main invariants of the second order surface

$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0,$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{24} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0,$$

$$T = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix},$$

$$S = a_{11} + a_{22} + a_{33}$$

define the properties of a surface not depending on its disposition at the space. The determinant Δ is called *a discriminant* of the general equation.

The second order surfaces having the only center of symmetry are called *the central surfaces* for which $\delta \neq 0$. The coordinates of the center are defined as the solution to the system

$$a_{11}x + a_{12}y + a_{13}z + a_{14} = 0,$$
$$a_{21}x + a_{22}y + a_{23}z + a_{24} = 0,$$
$$a_{31}x + a_{32}y + a_{33}z + a_{34} = 0.$$

A second order surface without a center of symmetry or with an indeterminate center is called *a noncentral surface*.

S.N. Krivoshapko and V.N. Ivanov, *Encyclopedia of Analytical Surfaces*,
DOI 10.1007/978-3-319-11773-7_35, © Springer International Publishing Switzerland 2015

The second order surfaces having *singular points* are:

(1) conical surface (the vertex is the only singular point);

(2) a pair of intersecting planes (a set of singular points lying at the straight line of the intersection of these planes);

(3) a pair of coinciding planes all consists of the singular points and the every of them is a center of the surface.

Additional Literature

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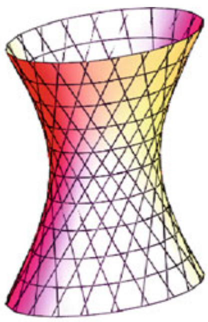
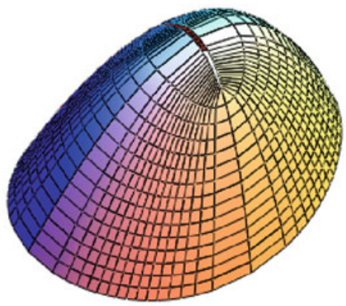
Efimov NV. Quadratic Forms and Matrixes. Moscow: Fizmatizdat, 1962; 160 p.

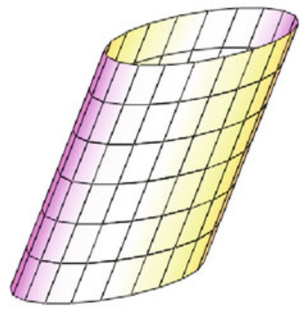
■ Quadratic Surfaces Presented in the Encyclopedia



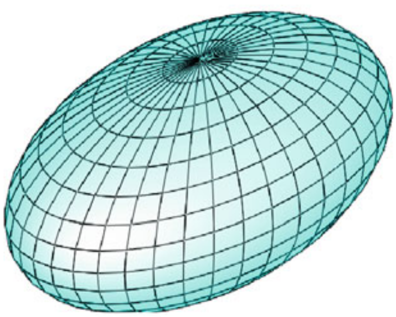
The one-sheet hyperboloid



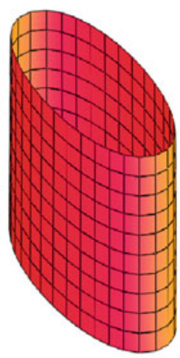
The elliptic paraboloid in the lines of curvatures



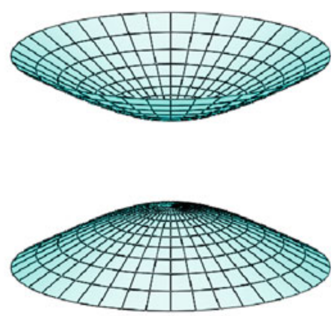
The oblique elliptical cylinder



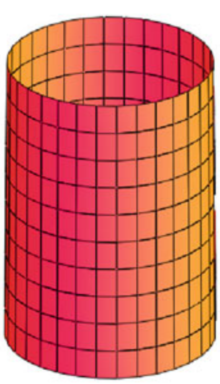
The ellipsoid



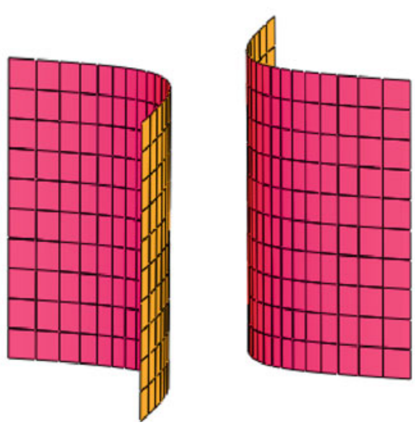
The elliptic cylinder



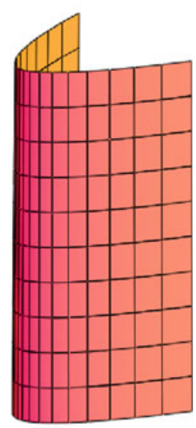
The two-sheet hyperboloid



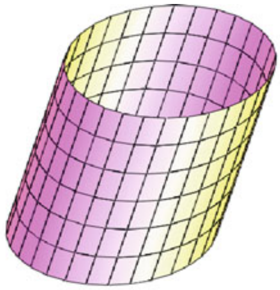
The cylindrical surface of revolution



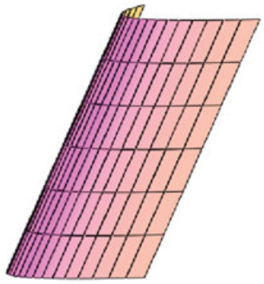
The right hyperbolic cylinder



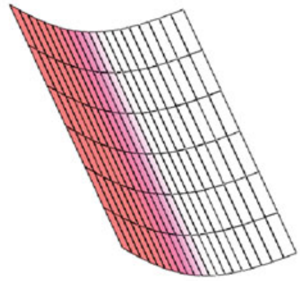
The right parabolic cylinder



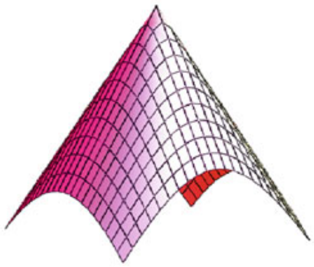
The oblique circular cylinder



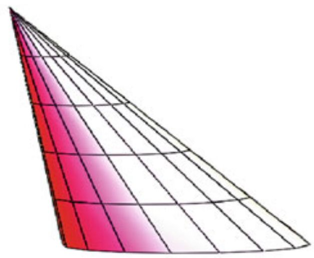
The oblique parabolic cylinder



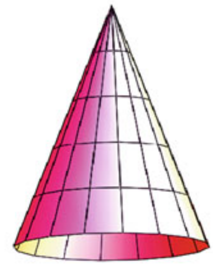
The oblique hyperbolic cylinder



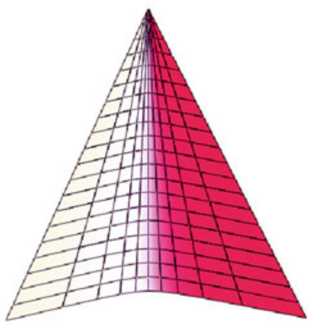
The elliptic conical surface



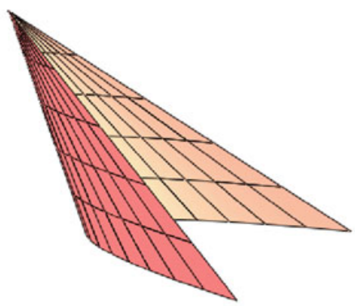
The oblique circular conic surface



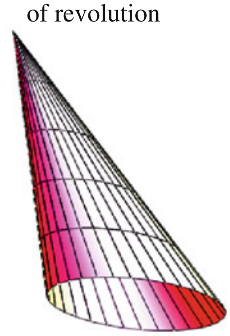
The conic surface of revolution



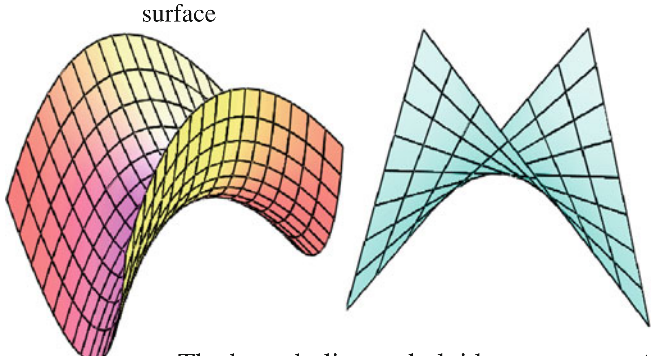
The hyperbolic conic surface



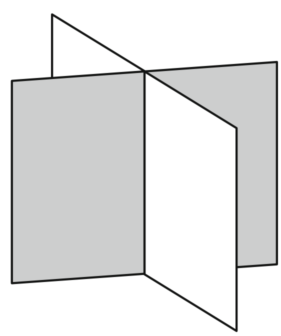
The parabolic conic surface



The oblique elliptic conic surface



The hyperbolic paraboloid



A pair of intersecting planes

■ Hyperbolic Paraboloid (Hypar)

A hyperbolic paraboloid is a twice ruled surface of negative Gaussian curvature and is presented *a geometric locus* belonging to the straight lines intersecting three fixed *skew lines* that are parallel to one plane.

Any two straight generatrix of a hypar belonging to different sets intersect and that is why they are *coplanar*. If the angle between the straight generatrixes of two sets equals 90^{o}, then a hyperbolic paraboloid is called *a right one*, but if the angle is not equal to 90^{o}, then it is called *an oblique hyperbolic paraboloid*. Hyperbolic paraboloid is *conoid with* *two axes* (see also "Conoids" in a Subsect. "1.2.1. Catalan Surfaces").

Forms of definition of the surface of a hyperbolic paraboloid

(1) Explicit equation (Fig. 1):

$$z = \frac{x^2}{2p} - \frac{y^2}{2q}, \quad p > 0, \; q > 0 \,(\text{the canonical equation}).$$

The presented formula shows that hyperbolic paraboloids may be related to a class of *surfaces of right translation*. So, hyperbolic paraboloids can be generated by the translation of a mobile parabola $y^2 = -2qz$ along a fixed parabola $x^2 = 2pz$ or on the contrary. The cross sections of a hyperbolic paraboloid by the planes z = const are hyperbolas.

(2) Explicit equation:

$$z = a_o xy + a_1 x + a_2 y + a_3.$$

Coefficients of the fundamental forms of the surface:

$$\begin{aligned}
A^2 &= 1 + (a_0 y + a_1)^2,\\
F &= (a_0 y + a_1)(a_0 x + a_2),\\
B^2 &= 1 + (a_0 x + a_2)^2;\\
L &= N = 0, \quad M = a_0/\left[A^2B^2 - F^2\right]^{1/2}.
\end{aligned}$$

Having taken the elevations of four corners of the rectangular contour of a hyperbolic paraboloid, one may calculate the coefficients c_i using the values of these elevations and the dimensions of the hyperbolic paraboloid on the plan.

(3) Explicit equation (Fig. 2):

$$z = axy.$$

At the point $x = y = 0$ (*the center of the surface*), mean curvature $H = 0$ and Gaussian curvature $K = -a^2$.

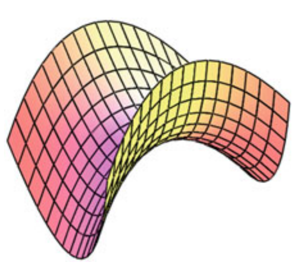


Fig. 1

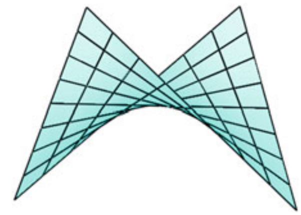


Fig. 2

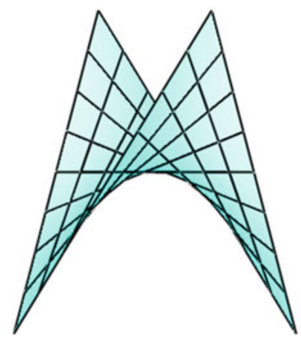


Fig. 3

The equations of the families of the lines of the principle curvatures on the hyperbolic paraboloid may be written as:

$$\ln\left(ay + \sqrt{1 + a^2y^2}\right) \pm \ln\left(ax + \sqrt{1 + a^2x^2}\right) = \text{const}.$$

(4) Parametrical equations (Fig. 3):

$$x = \sqrt{p}(u + v), \qquad y = \sqrt{q}(u - v), \qquad z = 2uv.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned}
&A^2 = p + q + 4v^2, \quad F = p - q + 4uv,\\
&B^2 = p + q + 4u^2;\\
&L = N = 0,\\
&M = -4\sqrt{pq}/\sqrt{A^2B^2 - F^2};\\
&k_u = k_v = 0,\\
&k_1 = M/(AB + F), \quad k_2 = -M/(AB - F);\\
&K = -16pq/(A^2B^2 - F^2)^2, \quad H = -MF/\left(A^2B^2 - F^2\right).
\end{aligned}$$

(5) Parametrical equations (Fig. 4):

$$x = \sqrt{p}u \cosh v, \qquad y = \sqrt{q}u \sinh v, \qquad z = u^2/2.$$

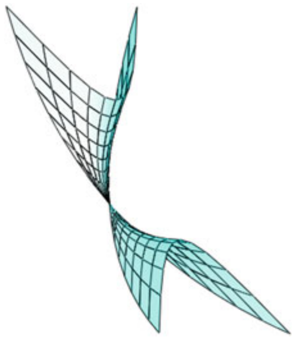


Fig. 4

Coefficients of the fundamental forms of the surface:

$$
\begin{aligned}
A^2 &= p\cosh^2 v + q\sinh^2 v + u^2,\\
F &= u\sinh v\cosh v(p+q),\\
B^2 &= u^2(p\sinh^2 v + q\cosh^2 v);\\
L &= u\sqrt{pq}/\sqrt{A^2B^2 - F^2},\quad M = 0,\quad N = -u^2 L.
\end{aligned}
$$

Additional Literature

Mileykovskiy I.E., Kupar A.K. Hypars. Analyses and Design of Shallow Shells of Coverings in the Form of Hyperbolic Paraboloids. Moscow: “Stroyizdat”, 1977; 223 p. (97 refs).

■ Parabolic Conic Surface

A parabolic conic surface is a *nonclosed conic surface* of *the second order* formed by the motion of a straight line passing through a given point and intersecting a directrix parabola (Fig. 1). A parabolic conic surface may be obtained as the envelope of a single parametrical family of the planes simultaneously touching two directrix parabolas with parameters of a and b:

$$x = 0,\ z = \frac{y^2}{2a} \quad \text{and} \quad x = l,\ z = \frac{(y-m)^2}{2b} + n,$$

where l is a distance between the planes of the directrix parabolas (Fig. 2). The directrix parabolas have the parallel axes. A perpendicular to the coordinate plane $x = 0$ dropped from the vertex of the parabola laying at the plane $x = l$ passes through the point C with coordinates $C(0, m, n)$. An equation of the single parametrical family of the tangent planes has the following form:

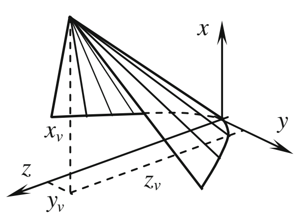


Fig. 1

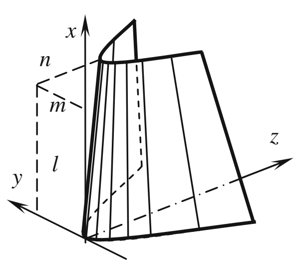


Fig. 2

$$M(x,y,z,\beta) = x\left[\beta m - na - \frac{(a-b)}{2a}\beta^2\right] + zal - ly\beta + \frac{l}{2}\beta^2 = 0,$$

where $\beta = y$ of the directrix parabola laying at the plane $x = 0$;

$$\gamma = \frac{b}{a}\beta + m,$$

where $\gamma = y$ of the directrix parabola laying at the plane $x = l$.

The coordinates of the vertex of a conical surface may be determined by the formulas:

$$x_v = \frac{al}{a-b},\quad y_v = \frac{am}{a-b},\quad z_v = \frac{an}{a-b}.$$

It is obviously, when $a = b$, a conic surface degenerates into a cylindrical parabolic surface, because the vertex of the cone goes to infinity.

A slope angle φ of the generatrix straight line, passing through the vertexes of the directrix parabolas (Fig. 2), with the plane $x = 0$ is determined by the formula:

$$\tan\varphi = \frac{l}{\sqrt{m^2 + n^2}}.$$

Forms of definition of the parabolic conic surface

(1) Implicit equation:

$$(ly - xm)^2 + 2(nx - lz)[al - x(a-b)] = 0.$$

In the cross sections of a conical surface by the planes $x = h$, parabolas are placed:

$$z - \frac{nh}{l} = \frac{l}{2[a(l-h)+bh]}\left(y - \frac{m}{l}h\right)^2,$$

and the vertexes of the obtained parabolas lie on the straight line connecting the vertexes of two directrix parabolas.

(2) Parametrical equations (Figs. 1 and 3):

$$\begin{aligned} x &= x,\\ y &= y(x,\beta) = \frac{a(l-x)+bx}{al}\beta + \frac{m}{l}x,\\ z &= z(x,\beta) = \frac{a(l-x)+bx}{2a^2l}\beta^2 + \frac{n}{l}x. \end{aligned}$$

In Fig. 2, the truncated conic surface when $0 \le x \le l$, $l < x_v$ is shown. Having assumed $0 \le x \le x_v = al/(a-b)$, we can design a conic surface with the vertex (Fig. 3).

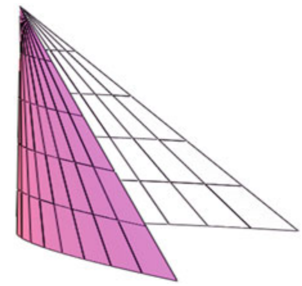


Fig. 3

(3) Parametrical equations (Figs. 2 and 3):

$$\begin{aligned} x &= x,\\ y &= y(x,\beta) = \frac{x_v - x}{x_v}\beta + \frac{y_v}{x_v}x,\\ z &= z(x,\beta) = \frac{x_v - x}{2ax_v}\beta^2 + \frac{z_v}{x_v}x. \end{aligned}$$

This form of definition is comfortable, if the coordinates of the vertex of a conic surface and a parameter a of the directrix parabola lying in the plane $x = 0$ are known.

■ Cylindrical Surface of Revolution

A cylindrical surface of revolution is formed by the rotation of a straight line that is parallel to the axis Ox about the axis Ox (Fig. 1). A radius of a *circular cylinder* equals the distance the generatrix straight line from the axis of revolution ($r = a$). The term “circular cylinder” is commonly used to describe a right circular cylinder. *A right circular cylinder* is a circular cylinder that has perpendicular base and height. Cylindrical surface is a surface of zero Gaussian curvature. If a plane inclined with respect to the caps of a right circular cylinder intersects a cylinder, it does so in *an ellipse*.

Forms of definition of the side surface of a circular cylinder

(1) Implicit equation:

$$y^2 + z^2 = a^2.$$

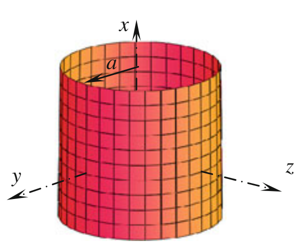


Fig. 1

(2) Explicit equation:

$$z = \pm\sqrt{a^2 - y^2},$$

where $-a \le y \le a$, $-\infty \le x \le \infty$.

(3) Parametrical equations (Fig. 1):

$$\begin{aligned} x &= x,\\ y &= y(\alpha) = a\cos\alpha,\\ z &= z(\alpha) = a\sin\alpha. \end{aligned}$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$\begin{aligned} &A = 1, \quad F = 0, \quad B = a;\\ &L = M = 0, \quad N = a;\\ &k_1 = 0, \quad k_2 = 1/a. \end{aligned}$$

Figure 1 shows *the closed cylindrical surface of revolution*, when $0 \le \alpha \le 2\pi$.

The coordinate lines x, α are the curvilinear coordinates in the lines of principle curvatures and the lines x = const coincide with the circular cross sections of a cylindrical surface and the lines α = const are its straight generatrixes.

(4) Parametrical equations (Fig. 2):

$$\begin{aligned} x &= x,\\ y &= y(s) = a\sin\left(\varphi - \frac{s}{a}\right),\\ z &= z(s) = a\cos\left(\varphi - \frac{s}{a}\right), \end{aligned}$$

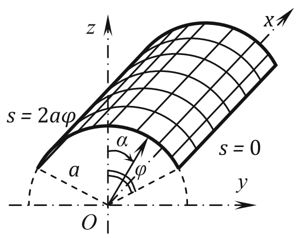


Fig. 2

where $-\varphi \leq \alpha \leq \varphi$; s is an arc length of a circular cross section, $0 \leq s \leq 2a\varphi$; $\varphi = \alpha_{\max} = \text{const}$; $\alpha = \varphi - s/a$ is the angle read from the axis z until the radius vector of a point lying at the lateral surface of the cylinder. The area of the side (lateral) surface is

$$S = 2a\varphi l,$$

where l is the length of the straight generatrixes of the surface.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1, \quad F = 0, \quad B = 1;$$
$$L = M = 0, \quad N = 1/a;$$
$$k_1 = 0, \quad k_2 = 1/a.$$

The coordinate lines x, s are the lines of the principal curvatures.

(5) Parametrical equation (Fig. 3):

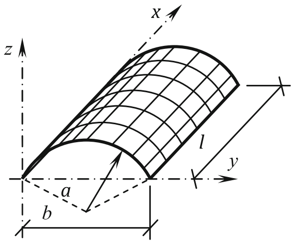


Fig. 3

$$x = x, \quad y = y,$$
$$z = \sqrt{a^2 - (y - b/2)^2} - \sqrt{a^2 - b^2/4},$$

such that $0 \leq x \leq l$; $0 \leq y \leq b$.

The circular cylindrical surface covers the rectangular plan $l \times b$.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1, \quad F = 0, \quad B^2 = a^2/\left[a^2 - (y - b/2)^2\right];$$
$$L = M = 0, \quad N = -a/\left[a^2 - (y - b/2)^2\right];$$
$$k_1 = 0, \quad k_2 = -1/a.$$

Additional Literature

Weisstein, Eric W. "Cylinder". From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Cylinder.html

Maan H. Jawad. Design of Plate & Shell Structures. NY: ASME PRESS, 2004; 476 p.

■ Conical Surface of Revolution

A conical surface of revolution is formed by the rotation of a straight line

$$x = az/c$$

about the axis z. The straight generatrix intersects the axis of rotation at *the* φ *angle of slope of a cone* and

$$\tan\varphi = a/c;$$

2φ is the angle of a cone

Forms of definition of the lateral conical surface of revolution

(1) Implicit equations:

$$x^2 + y^2 - z^2\tan^2\varphi = 0,$$

or

$$(x^2 + y^2)/a^2 - z^2/c^2 = 0.$$

(2) Explicit equation (Fig. 1):

$$z = \pm\frac{c}{a}\sqrt{x^2 + y^2}.$$

The signs correspond with two nappes of a circular cone (Fig. 2).

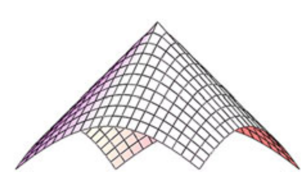


Fig. 1

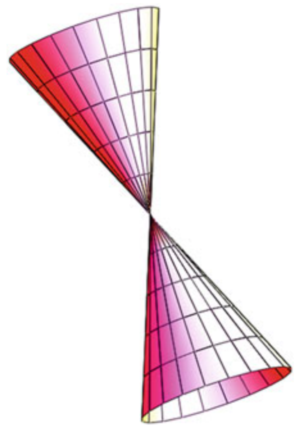


Fig. 2

In the cross sections of a circular cone by the planes $z = h$ ($h \neq 0$) = const, we have the circles with radiuses

$$r = ah/c = h \tan \varphi.$$

The line of intersection of a cone with the plane $x = b$ = const is a hyperbola

$$a^2 z^2/(bc)^2 - y^2/b^2 = 1$$

but the line of intersection of a cone with the plane $y = d$ = const is a hyperbola (Fig. 1)

$a^2 z^2/(dc)^2 - x^2/d^2 = 1.$

Ellipses lie in the cross sections of a cone by a plane not passing through the vertex of the cone.

If the inclined plane is parallel to any straight generatrix of a cone, then we shall obtain a parabola in the cross section.

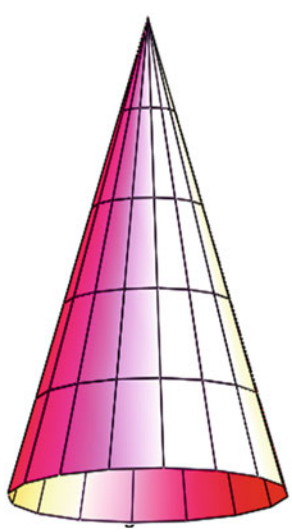


Fig. 3

(3) Parametrical equations (Fig. 3):

$$x = x(\alpha, \beta) = \alpha \sin \varphi \cos \beta,$$
$$y = y(\alpha, \beta) = \alpha \sin \varphi \sin \beta,$$
$$z = z(\alpha) = \alpha \cos \varphi,$$

where α is a length of the straight generatrix from the vertex of the cone till a point lying on the surface of the *cone*. The coordinate lines α coincide with the straight generatrixes of the conical surface. There are circles with radiuses $r = \alpha \sin \varphi$ in the cross sections of the cone by the planes z = const.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1, \quad F = 0, \quad B = \alpha \sin \varphi;$$
$$L = M = 0, \quad N = \alpha \sin \varphi \cos \varphi;$$
$$k_1 = 0, \quad k_2 = \operatorname{co\,tan} \varphi/\alpha.$$

A cone of revolution is a surface of zero Gaussian curvature ($K = k_1 k_2 = 0$) and the coordinate lines α, β (meridians and parallels) are lines of the principal curvatures.

(4) Parametrical equations (Fig. 3):

$$x = x(r, \beta) = r \cos \beta, \quad y = y(r, \beta) = r \sin \beta,$$
$$z = r \operatorname{co\,tan} \varphi.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 1/\sin \varphi, \quad F = 0, \quad B = r;$$
$$L = M = 0, \quad N = r \cos \varphi;$$
$$k_1 = 0, \quad k_2 = \cos \varphi / r.$$

A truncated conical surface (Fig. 4) with a small cone angle (2φ) is called *a cone of Morse* (conical shank of tool). *Conicity K* is a ratio of the difference of the diameters of two cross sections of the cone to the distance l between them

$$K = (2r_1 - 2r_2)/l; \quad r_1 > r_2.$$

A perpendicular line drawn from the vertex to the center of the base is called *the height* of *a right circular cone*.

(5) Parametrical equations:

$$x = x(u, \beta) = au \cos \beta, \quad y = y(u, \beta) = au \sin \beta,$$
$$z = z(u) = cu.$$

Coefficients of the fundamental forms of the surface and its principal curvatures:

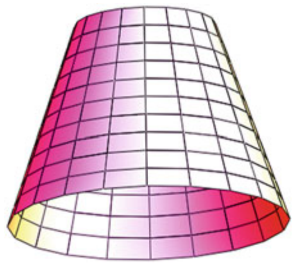


Fig. 4

$$A^2 = a^2 + c^2,\ \ F = 0,\ \ B = au;$$
$$L = M = 0,\ \ N = acu/A;$$
$$k_1 = 0,\ \ k_2 = c/(auA).$$

A theory of shell analysis cannot be used near the vertex of cone.

Additional Literature

Pikul VV. Theory and Analyses of Shells of Revolution. Moscow: "Nauka", 1982; 158 p. (55 refs).

■ One-Sheet Hyperboloid

A one-sheet hyperboloid (*a one-sheeted hyperboloid*) is an unclosed central twice ruled surface of the second order of negative Gaussian curvature. In 1669 Christopher Wren, the architect who designed St. Paul's Cathedral in London, showed that one-sheet hyperboloid is *a ruled surface*.

Forms of definition of the one-sheet hyperboloid

(1) Implicit equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ (canonical equation)},$$

where a, b, c are the semi-axes of *one-sheet hyperboloid*. The coordinate planes of a rectangular system are its planes of the symmetry (Fig. 1). If $a = b = c$, then a hyperboloid is called *a regular one*.

The sections of a one-sheet hyperboloid by the planes $y = h(|h| < b)$ or $x = h(|h| < a)$ give *hyperbolas*. The cross sections of a hyperboloid by the planes $y = \pm b$ or $x = \pm a$ are the pairs of intersecting straight lines. At the cross sections of a hyperboloid by the planes $z = h =$ const, ellipses lie and these ellipses are similar to each other (a ratio of their axes are the same). *The waist ellipse* lies at the cross section of a hyperboloid by the plane $z = 0$. When $a = b$, one-sheet hyperboloid degenerates into *one-sheet hyperboloid of revolution*.

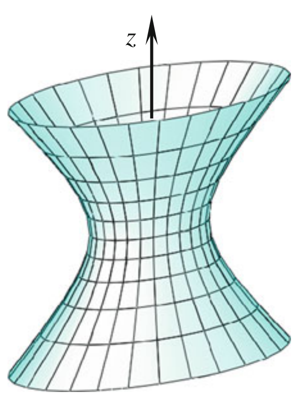


Fig. 1

(2) Parametrical equations (Fig. 1):

$$x = a\cosh u\cos v, \quad y = b\cosh u\sin v, \quad z = c\sinh u.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = \sinh^2 u(a^2\cos^2 v + b^2\sin^2 v) + c^2\cosh^2 u,$$
$$F = (b^2 - a^2)\sinh u\cosh u\sin v\cos v,$$
$$B^2 = \cosh^2 u(a^2\sin^2 v + b^2\cos^2 v);$$
$$L = -abc\cosh u/\sqrt{A^2B^2 - F^2},\ M = 0,\ N = -\cosh^2 uL$$

So, the coordinate net $u,\ v$ is nonorthogonal but conjugate.

(3) Parametrical equations (Fig. 1):

$$x = x(\alpha, \beta) = a\cos\beta/\cos\alpha,$$
$$y = y(\alpha, \beta) = b\sin\beta/\cos\alpha,$$
$$z = z(\alpha) = c\tan\alpha.$$

The surface of this one-sheet hyperboloid is related to *a geographic system of coordinates.*

Coefficients of the fundamental forms of the surface:

$$A^2 = \left[\sin^2\alpha\left(a^2\cos^2\beta + b^2\sin^2\beta\right) + c^2\right]/\cos^4\alpha,$$
$$F = \sin\beta\cos\beta\sin\alpha(b^2 - a^2)/\cos^3\alpha,$$
$$B^2 = \left[a^2\sin^2\beta + b^2\cos^2\beta\right]/\cos^2\alpha;$$
$$L = -\frac{abc}{\cos^4\alpha\sqrt{A^2B^2 - F^2}} = -N,\ \ M = 0.$$

(4) Parametrical equations:

$$x = x(\lambda, \mu) = a\frac{\lambda - \mu}{\lambda + \mu},$$
$$y = y(\lambda, \mu) = b\frac{\lambda\mu + 1}{\lambda + \mu},$$
$$z = z(\lambda, \mu) = c\frac{\lambda\mu - 1}{\lambda + \mu}.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = \frac{4\mu^2 a^2 + b^2(\mu^2-1)^2 + c^2(\mu^2+1)^2}{(\mu+\lambda)^4},$$
$$F = \frac{-4\mu\lambda a^2 + b^2(\mu^2-1)(\lambda^2-1) + c^2(\mu^2+1)(\lambda^2+1)}{(\mu+\lambda)^4}$$
$$B^2 = \frac{4a^2\lambda^2 + b^2(\lambda^2-1)^2 + c^2(\lambda^2+1)^2}{(\mu+\lambda)^4};$$
$$L = N = 0,$$
$$M = \frac{-6abc\mu\lambda}{(\mu+\lambda)^3\sqrt{a^2b^2(\mu\lambda-1)^2 + a^2c^2(\mu\lambda+1)^2 + b^2c^2(\mu-\lambda)^2}},$$
$$k_\mu = k_\lambda = 0.$$

The coordinate lines λ, μ coincide with the straight generatrixes of the hyperboloid.

Additional Literature

Goldenveyzer AL. Membrane theory of shells formed by the second order surfaces. PMM. 1947; Vol. XI, Iss. 2, p. 285-290 (6 refs).

■ Ellipsoid

An ellipsoid is *a closed quadric surface* that is a three-dimensional analog of an ellipse. Mathematical literature often uses "ellipsoid" in place of "tri-axial ellipsoid" or "general ellipsoid" or "three-axial ellipsoid."

Forms of definition of an ellipsoid

(1) Implicit equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ (a canonical equation)},$$

where a, b, c are semi-axes of an ellipsoid.

When $a = b = c$, a tri-axial ellipsoid degenerates into *a sphere* with a radius a, i.e., a sphere is a special case of an ellipsoid. Having assumed $a = b$, we can obtain an ellipsoid of revolution formed by the rotation of an ellipse $x^2/a^2 + z^2/c^2 = 1$ about the axis Oz. The ellipsoids of revolution may be obtained, if one takes $a = c$ or $b = c$.

The three-axial ellipsoid of Krasovskiy with a = 6,379,351 km; b = 6,356,863 km; c = 6,378,139 km describes the form of the Earth very precisely. The clearance between it and the form of the Earth does not go over 100 m.

An ellipsoid as a whole can be placed inside a rectangle parallelepiped with the dimension $2a \times 2b \times 2c$ and with the center at the point (0, 0, 0). The faces are parallel to the coordinate planes. The coordinate planes are the only planes of symmetry of the ellipsoid. There are ellipses at the cross sections of the ellipsoid by the planes parallel to the coordinate planes.

The volume of an ellipsoid can be calculated by a formula:

$$V = \frac{4}{3}\pi abc.$$

(2) Parametrical equations (Figs. 1 and 2):

$$x = x(u,v) = a\sin u\cos v; \quad y = y(u,v) = b\sin u\sin v;$$
$$z = z(u) = c\cos u,$$

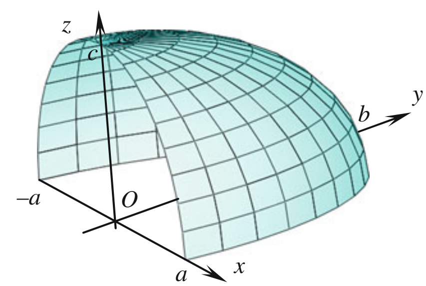


Fig. 1 Fragment of an ellipsoid $-a \le x \le a$; $0 \le y \le b$; $0 \le z \le c$

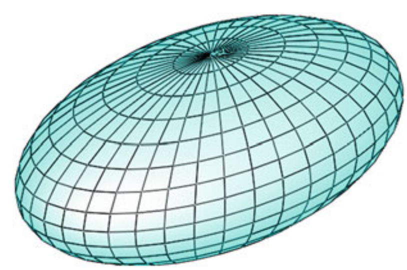


Fig. 2 The definition of an ellipsoid in the geographic curvilinear coordinates

where $0 \le u \le \pi$, $0 \le v \le 2\pi$

Coefficients of the fundamental forms of the surface:

$$A^2 = \cos^2 u\left(a^2\cos^2 v + b^2\sin^2 v\right) + c^2\sin^2 u,$$
$$F = (b^2 - a^2)\frac{\sin 2v\sin 2u}{4},$$
$$B^2 = \sin^2 u\left(a^2\sin^2 v + b^2\cos^2 v\right);$$
$$L = -\frac{abc\sin u}{\sqrt{A^2B^2 - F^2}}; \quad M = 0; \quad N = -\frac{abc\sin^3 u}{\sqrt{A^2B^2 - F^2}};$$
$$K = \frac{a^2b^2c^2\sin^4 u}{(A^2B^2 - F^2)^2} > 0.$$

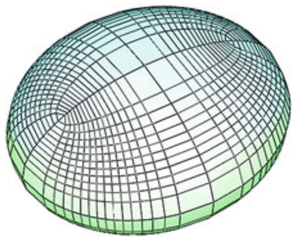


Fig. 3 The lines of principle curvatures

(3) Parametrical equations (*a stereographic form*):

$$x(\alpha,\beta)=\frac{a(1-\alpha^2-\beta^2)}{1+\alpha^2+\beta^2},\quad y(\alpha,\beta)=\frac{2b\alpha}{1+\alpha^2+\beta^2},$$
$$z(\alpha,\beta)=\frac{2c\beta}{1+\alpha^2+\beta^2}.$$

(4) Parametrical equations (*Mercator projection*):

$$x(\gamma,v)=a\,\mathrm{sech}\,\gamma\cos v;\quad y=y(\gamma,v)=b\,\mathrm{sech}\,\gamma\sin v,$$
$$z=z(\gamma)=c\tanh\gamma.$$

(5) Parametrical equations given in the *u, v* lines of principal curvatures (Fig. 3):

$$x^2=\frac{a^2(a^2-u)(a^2-v)}{(a^2-b^2)(a^2-c^2)},\quad y^2=\frac{b^2(b^2-u)(b^2-v)}{(b^2-a^2)(b^2-c^2)},$$
$$z^2=\frac{c^2(c^2-u)(c^2-v)}{(c^2-a^2)(c^2-b^2)},\quad a^2\ge u\ge b^2\ge v\ge c^2.$$

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■ Elliptic Paraboloid

The elliptic paraboloid is *an unclosed noncentral surface of the second order*. On the surface of an elliptical paraboloid, the innumerable set of *the translational nets* exist.

Forms of definition of an elliptic paraboloid

(1) Explicit equation:

$$\frac{x^2}{p}+\frac{y^2}{q}=2z\,(\text{canonical equation}),$$

where p, $q>0$, Fig. 1.

The plane $z=h<0$ does not intersect an elliptical paraboloid; the plane $z = h = 0$ has one common point with the paraboloid. The plane $z=h>0$ intersects the paraboloid along an ellipse with the semi-axes $\sqrt{2hp}$ and $\sqrt{2hq}$. All ellipses are similar with each other, they have one and the same ratio of the semi-axes $\sqrt{q/p}$. The plane $y = h$ intersects the paraboloid along a parabola with the focal parameter p and with the peak at the point $(0, h, h^2/(2q))$. The plane $x = h$ intersects the paraboloid along a parabola with the focal parameter q and with the peak at the point $(h, 0, h^2/(2p))$. The cross sections of an elliptical paraboloid by the planes $y = 0$ and $x = 0$ are *the principle parabolas*: $x^2 = 2pz$, $y = 0$, and $y^2 = 2qz$, $x = 0$ of the paraboloid. An elliptical paraboloid has not any straight line. When $p = q$, an elliptical paraboloid degenerates into a paraboloid of revolution, see also "*Paraboloid of Revolution.*"

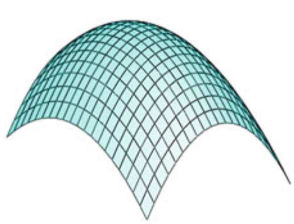

Fig. 1

(2) Explicit equation:

$$z=h-\frac{h}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right),$$

where h is a rise of the surface.

The plane $z = 0$ intersects an elliptical paraboloid along an ellipse with the semi-axes $\sqrt{2}a$ and $\sqrt{2}b$.

(3) Parametrical equations (Fig. 2):

$$x=x(u,v)=\sqrt{p}u\cos v;\quad y=y(u,v)=\sqrt{q}u\sin v;$$
$$z=z(u)=u^2/2.$$

Coefficients of the fundamental forms of the surface:

$$A^2=p\cos^2v+q\sin^2v+u^2,\quad F=u\sin 2v(q-p)/2,$$
$$B^2=u^2(p\sin^2v+q\cos^2v);$$

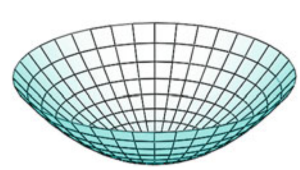

Fig. 2

$$L = u\sqrt{pq}\Big/\sqrt{A^2B^2 - F^2}, \quad M = 0, \quad N = u^2L,$$
$$K = u^4pq\Big/\left(A^2B^2 - F^2\right)^2 > 0.$$

The curvilinear coordinates *u, v* are nonorthogonal but conjugate.

(4) Parametrical equations (Fig. 1):

$$x = x, \quad y = y, \quad z = z(x, y) = x^2/(2p) + y^2/(2q).$$

The parametric equations show that an elliptical paraboloid is a *surface of the right translation,* i.e., is formed by the motion of one principle parabola along the other.

(5) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(\alpha, v) = ae^{\alpha}\cos v\boldsymbol{i} + be^{\alpha}\sin v\boldsymbol{j} + e^{2\alpha}\boldsymbol{k}/2.$$

The surface is related to the *isometric conjugate* coordinates α, v ($L = N$; $M = 0$).

(6) Parametrical form of definition of an elliptical paraboloid in the *u* and *v* lines of principal curvatures:

$$x^2 = \frac{b^2(b^2 - u)(b^2 - v)}{c^2 - b^2}, \quad y^2 = -\frac{c^2(c^2 - u)(c^2 - v)}{b^2 - c^2},$$
$$z^2 = \frac{u + v - b^2 - c^2}{2},$$

where $c^2 < u < b^2 < v$, $b^2 = p$, $c^2 = q$.

The elliptical paraboloid in the lines of principal curvatures is shown at the page 856 "Quadratic surfaces presented in the encyclopedia."

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■ Hyperboloid of Two Sheets

A two-sheet hyperboloid (*a two-sheeted hyperboloid*) is an unclosed central surface of the second order, which consists of two parts not linked with each other. The surface does not have straight generatrixes and so, it consists of *elliptical points*.

Forms of definition of the surface

(1) The implicit canonical equation:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The plane $z = h$ with $|h| < c$ does not intersects a hyperboloid of two sheets; when $|h| = c$, the plane $z = h$ has one common point (0, 0, c) or (0, 0, $-c$) with the hyperboloid. The plane $z = h$, where $|h| > c$, intersects a two-sheet hyperboloid along an ellipse with the semi-axes: $a\sqrt{h^2/c^2 - 1}$, $b\sqrt{h^2/c^2 - 1}$. Any plane $y = h$ intersects a hyperboloid along a hyperbola with semi-axes $c\sqrt{1 + h^2/b^2}$, $a\sqrt{1 + h^2/b^2}$ but the plane $x = h$ intersects a hyperboloid along a hyperbola with the semi-axes $c\sqrt{1 + h^2/a^2}$, $b\sqrt{1 + h^2/a^2}$.

The coordinate planes are the planes of symmetry of the hyperboloid and the origin of the coordinates is its *center of symmetry*.

A cone defined by an equation

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$$

is called *an asymptotical cone*. When a point moves off on a hyperboloid so that the absolute value of its coordinate z increases unlimitedly, then this point approaches unlimitedly to the asymptotical cone.

When the semi-axes of a two-sheet hyperboloid decrease proportionally, then the two-sheet hyperboloid tightens to the asymptotic cone. A two-sheet hyperboloid has no real straight generatrixes.

Having assumed $a = b$, we shall obtain *a two-sheet hyperboloid of revolution*.

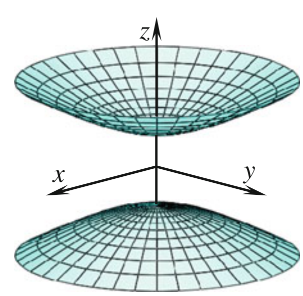

Fig. 1

(2) Parametrical equations (Fig. 1):

$$x = x(u, v) = a \sinh u \cos v,$$
$$y = y(u, v) = b \sinh u \sin v,$$
$$z = z(u) = \pm c \cosh u,$$

where the signs (+) and (–) correspond the two space of the hyperboloid.

Coefficients of the fundamental forms of the surface:

$$A^2 = \cosh^2 u\left(a^2 \cos^2 v + b^2 \sin^2 v\right) + c^2 \sinh^2 u,$$
$$F = \sinh 2u \sin 2v\left(b^2 - a^2\right)/4,$$
$$B^2 = \sinh^2 u\left(a^2 \sin^2 v + b^2 \cos^2 v\right),$$
$$L = \pm abc \sinh u/\Delta; \quad M = 0, \quad N = \pm abc \sinh^3 u/\Delta,$$
$$K = a^2 b^2 c^2 \sinh^4 u/\Delta^4 > 0,$$

where $\Delta = \sqrt{A^2 B^2 - F^2}$.

(3) Vector equation (Fig. 1):

$$\boldsymbol{r} = \boldsymbol{r}(\alpha, v) = -a(\cos v/\sinh\alpha)\boldsymbol{i} - b(\sin v/\sinh\alpha)\boldsymbol{j} - c\operatorname{cotanh}\alpha\boldsymbol{k}.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = \mathrm{cs}\cosh^2\alpha\left[\mathrm{c}\tanh^2\alpha\left(a^2\cos^2 v + b^2 \sin^2 v\right) + c^2\right],$$
$$F = \mathrm{cs}\cosh^2\alpha\,\mathrm{c}\tanh\alpha \sin 2v\left(a^2 - b^2\right)/2,$$
$$B^2 = \mathrm{cs}\cosh^2\alpha\left(a^2 \sin^2 v + b^2 \cos^2 v\right),$$
$$L = \frac{abc}{\Delta \sinh^4\alpha} = N; \quad M = 0, \quad K = \frac{a^2b^2c^2}{\Delta^4 \sinh^8\alpha} > 0,$$

where $\Delta = \sqrt{A^2B^2 - F^2}$; $\mathrm{cs}\cosh\alpha = \mathrm{a}/\sinh\alpha$.

The surface of the hyperboloid is related to the conjugate curvilinear coordinates α, v.

(4) Parametrical form of the definition of a two-sheet hyperboloid in the u and v lines of principle curvatures:

$$x^2 = -\frac{a^2(a^2-u)(a^2-v)}{(a^2+b^2)(a^2+c^2)}, \quad y^2 = -\frac{b^2(b^2+u)(b^2+v)}{(b^2+a^2)(b^2-c^2)},$$
$$z^2 = -\frac{c^2(c^2+u)(c^2+v)}{(c^2+a^2)(b^2-c^2)}, \quad -b^2 \ge u \ge -c^2 \ge v.$$

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■ Quadrics

Quadric is a surface of the second order given in the form:

$$\frac{x^2}{a^2+\rho} + \frac{y^2}{b^2+\rho} + \frac{z^2}{c^2+\rho} - 1 = 0,$$

where $a^2 > b^2 > c^2$. In Euclidean space, quadrics have dimension $D = 2$, and are known as *quadric surfaces*. Quadric with nonzero Gaussian curvature is *Darboux surface* in three-dimensional Euclidean space. Quadrics are also called *quadratic surfaces*, and there are 17 standard-form types.

Fixing a point $M(x, y, z)$ and clearing denominator, it is possible to derive an equation of the third order relatively to *the* ρ *parameter of the quadric*. This equation has three real solution of an equation, that are u, v, and w. They have the following boundaries:

$$+\infty > u > -c^2, \quad -c^2 > v > -b^2, \quad -b^2 > w > -a^2.$$

Three numbers (u, v, w) are called *elliptical coordinates* of the taken point $M(x, y, z)$. One supposes that all three coordinates of the point (x, y, z) are different from zero. In the opposite case, one will obtain an equation below the third order relatively to the ρ parameter. For example, if $z = 0$ but x and y are different from zero, then the equation presented above gives the values of u and v, but it is necessary to take w equal to ($-c^2$).

Let us research the coordinate surfaces given in an elliptical system of coordinates. Substituting $\rho = u$ into the equation given above, we can design the surface:

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1,$$

where u is some number from the limits $(-c^2, \infty)$. Obviously, this surface is an *ellipsoid* (see also “Ellipsoid”), because all three denominators of the last equation are positive due to the first inequality.

Putting $\rho = v$, where v is taken from the interval ($-b^2$, $-c^2$), it is possible to obtain an equation of *one-sheet hyperboloid* (see also a Sect. “1.2. Ruled surfaces of negative Gaussian curvature”):

$$\frac{x^2}{a^2+v} + \frac{y^2}{b^2+v} + \frac{z^2}{c^2+v} = 1,$$

because in this case, $a^2 + v > b^2 + v > 0$ and $c^2 + v < 0$.

And at last, when $\rho = w$, where w is taken from the interval ($-a^2$, $-b^2$), it is possible to obtain an equation of *two-sheet hyperboloid* (see also "Two-Sheet Hyperboloid"):

$$\frac{x^2}{a^2 + w} + \frac{y^2}{b^2 + w} + \frac{z^2}{c^2 + w} = 1.$$

The direction cosines of the normals to surfaces of the ellipsoid and the one-sheet hyperboloid

$$\frac{x}{a^2 + u}, \frac{y}{b^2 + u}, \frac{z}{c^2 + u} \quad \text{and} \quad \frac{x}{a^2 + v}, \frac{y}{b^2 + v}, \frac{z}{c^2 + v}$$

are accordingly proportional. That is why the equality obtained by the term-by-term subtraction of the equation of an ellipsoid from the equation of a one-sheet hyperboloid expresses the condition of the perpendicularity of these normals:

$$\frac{x^2}{(a^2 + u)(a^2 + v)} + \frac{y^2}{(b^2 + u)(b^2 + v)} + \frac{z^2}{(c^2 + u)(c^2 + v)} = 0$$

and this equality gives the proof of the orthogonality of surfaces of the ellipsoid and the one-sheet hyperboloid. By analogy, one may prove the mutual orthogonality of other coordinate surfaces, i.e., three coordinate surfaces considered before are mutually orthogonal.

Using the Dupin's theorem, one may assert that two families of the lines of principal curvatures on ellipsoid with fixed u are obtained as a result of intersection of an ellipsoid with any different one-sheet and two-sheet hyperboloids from the foregoing families.

Hence, quadrics contain 16 forms of real surfaces, five are nondegenerate real: ellipsoid, elliptic paraboloid, hyperbolic paraboloid, hyperboloid of one sheet, and hyperboloid of two sheets; five special cases: spheroid, sphere, paraboloid of revolution, one-sheet hyperboloid of revolution and two-sheet hyperboloid of revolution; and six degenerate surfaces such as cone, circular cone, elliptic cylinder, circular cylinder, hyperbolic cylinder, and parabolic cylinder.

Quadric surfaces are often used as example surfaces since they are relatively simple.

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36 Algebraic Surfaces of the High Orders

Algebraic surface is a two-measured algebraic variety. A theory of algebraic surfaces is one of the sections of algebraic geometry. The main feature of the algebraic geometry is application of polynomial functions only. Further, we shall consider that the surfaces determined in the rectangular coordinates by algebraic equations are called *algebraic surfaces*.

An algebraic equation is called an equation in the form of $F(x, y, z) = 0$, the left part of which is a polynomial relative to x, y, z with numerical coefficients a_{ij}. A degree of a polynomial $F(x, y, z)$ is the largest from the degrees of its monomials and it is called *a degree of the equation*. A surface defined by an algebraic equation with the n degree in some system of rectangular coordinates is called *an algebraic surface of the n order*.

An order of a surface does not depend on a system of the coordinates but defines this surface, i.e., if a surface is represented at one system of rectangular coordinates by an algebraic equation of the order n, then it is represented by the equation of the same degree in another system of the rectangular coordinates too.

A plane is an algebraic surface of the first order. It is defined by an equation

$$ax + by + cz + d = 0,$$

where at least one of the numbers a, b, and c differ from zero (see also a Section "Plane").

Surfaces of the second order (*quadrics*) are given by an algebraic equation of the second degree relative to Cartesian rectangular coordinates. In general form, the equation of the second order can be written as

$$\begin{aligned} F(x, y, z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz \\ + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0, \end{aligned}$$

where $a_{ik} = a_{ki}$; $i, k = 1, 2, 3, 4$. This equation may not determine any real geometric form. In these cases, they say, that the equation defines *an imaginary surface of the second order*. Depending on the values of the a_{ik} coefficients of the general equation, it may describe 17 surfaces (see also the Chap. "35. The Second Order Surfaces").

A rectilinear congruence is a two-parametrical variety of straight lines. The fixation of one parameter picks out a single-parametrical variety from the whole of totality of straight lines that is *a ruled surface*. Immersing of a curve of the n order into a rectilinear congruence build on two skew lines leads to the forming of a ruled surface of the order $2n$ in the general case. So, immersing *a curve of the second order* into the congruence, in general case, they obtain *a ruled surface of the fourth order*. For the design of *a ruled surface of the third order,* it is necessary disintegrating of a surface of the fourth order into a plane and a surface of the third order. This will take place, if one of the directrix straight of the congruence will cross the immersing curve of the second order at some point.

That is quite natural that algebraic surfaces may be related into one or at once into several other classes of surfaces. For example, an algebraic surface of the fourth order that is a circular torus may be related also to surfaces of revolution or to cyclic surfaces.

In present part, algebraic surfaces of the high orders not included into other parts of the encyclopedia are described. The full list of algebraic surfaces described in the encyclopedia is given at the pp. 627–631. This list includes only small part of the known algebraic surfaces. The authors have selected the surfaces which may attract attention of engineers and architects.

A brief additional information on algebraic surfaces such as "Miter Surface" (Fig. 1) "Burkhardt Quartic", "Clebsch Diagonal Cubic", "Cushion", "Desmic Surface", "Endraβ Octic", "Heptic Surface", "Labs Septic", "Nonic Surface", "Octic Surface", "Paraboloid Geodesic", "Septic Surface", "Symmetroid", "Tetrahedroid" is presented by Eric W. Weisstein.

Algebraic surfaces Cayley Cubic, Pretzel Surface, Pilz Surface, Orthocircles given in implicit form can be seen in site "Virtual Math Museum" (2004–2006).

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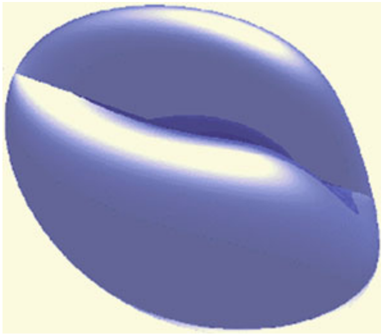

Fig. 1

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■ Algebraic Surfaces of the High Orders Presented in the Encyclopedia

In the corresponding pages, the encyclopedia contains the information on the algebraic surfaces denoted by the symbol (•). The surfaces not marked by this marker are not described in the encyclopedia. Information on them can be found in other sources.

Algebraic surfaces of the third order

- $z^2 = a^2y^3$: right cylindrical surface with directrix semi-cubical parabola;
- $ab^2z + c(y^2 - b^2)(x - a) = 0$: parabolic conoid;
- $2pz(x - a)^2 - a^2y^2 = 0$: right conoid with a directrix parabola the axis of which is parallel to the axis of conoid;
- $zy^2 - x^2 = 0$: Whitney umbrella;
- $z^3 - b^3(x^2 + y^2) = 0$: surface of revolution of a hyperbola $z = b/x$ around the Oz axis;
- $(zl - ax + xf)^2 = x^2(a^2 - y^2)$: conoid with a directrix circle;
- $(x^2 + y^2)z = 2xy$: Plücker conoid;
- $z = xy - x^3/6$: Cayley surface;
- $z = -x^3 - xy$: cubic surface $x^3 + xy + z = 0$;
- $z = x^3 - 3xy^2$: monkey saddle;
- $z(x^2 + y^2 + z^2) + (y^2 + z^2)(l - \mu) - x^2(l + \mu) - (z + l - \mu)(l + \mu)^2 = 0$: Dupin cyclide of the second type (of the third order);
- $z = x^3/3 + xy^2 + 2(x^2 - y^2)$: "Handkerchief surface";
- $z = x^3/3 - y^2/2$: shoe surface; $y(x^2 + y^2 + z^2 - a^2) - 2z(x^2 + y^2 + ax) = 0$—Möbius strip on a circle planнe;
- $x^3 + y^3 + z^3 + t^3 = 0$: diagonal cubic surface of Fermat;
- $x^2 + y^2 + z^2 - 2xyz = 1 + \lambda^2$: the cubic surface with 24 straight lines;
- $y = 2f\left(3x^2/b^2 - 2|x|^3/b^3\right)$: surface of cylindrical flexible hopper-type bin for the keeping of dry materials;
- $z = \frac{2B^2T(L-x)^3}{L[4y^2L^2+B^2(L-x)^2]} - \frac{T(L-x)}{L}$: conical surface with a directrix Agnesi curve;
- $y = \pm 2{,}5426\frac{B}{T}\frac{(Lz-xT/\sqrt{3})}{L}\sqrt{\frac{3\left(TL-Tx-Lz+xT/\sqrt{3}\right)}{\left(TL-xT+3Lz-\sqrt{3}xT\right)}}$: conical surface with a directrix curve in the form of Cartesian folium;
- $z = 1/(xy)$: Tzitzéica surface of the third order with the centroaffine invariant $I = 1/27$;
- $z = z(x, y) = \frac{1}{(x-\varepsilon_1 y)^2+y^2} + \varepsilon_2 x$: Tzitzéica surface of the third order with the centroaffine invariant $I = -4/27$;
- $x^2 + y^2 = (1 - z)z^2$: Ding–Dong surface;
- $x^2 + y^2 = c^2z(z - a)(z - b)$: surface of revolution "Egg" of the third order;
- $y^2z - x^3 - xz^2 = 0$: cubic cone;

  $4(x^2 + y^2 + z^2) + 16xyz - 1 = 0$: Cayley cubic;

  $2(xy + xz + yz) - 5(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) = 0$: Hunt parameterization of Cayley cubic;

  the spherical surface of the third order (Niče Vilko).

Algebraic quartic surfaces

- $z = cx^2y^2$: crossed trough;
- $z = ax^4 + x^2y - y^2$: Menn's surface;
- Parabolic torse (see Sect. "1.1.1. Torse surfaces (torses)")
- $(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$: circular torus;
- $z^2(x^2 + y^2) = b^2$: surface of revolution of a hyperbola $z = b/x$ around the Oz axis;
- $(z^2 + 2ap)^2 = 4p^2(x^2 + y^2)$: surface of revolution of a parabola;
- $cz = (x^2 + y^2)^2$: the 4th order paraboloid of revolution;
- $z^3(a - z) - b^2(x^2 + y^2) = 0$: surface of revolution "Pear";
- $[z + y^2/(2p) + a]^2 = (a^2 - x^2)$: surface of translation of circle along parabola;
- $(z + ax^2)^2 - 2c(z + ax^2) = y^2c^2/b^2$: surface of translation of parabola along hyperbola;
- $z = (3 - x^2 - y^2 - x^2y^2)/2$: Środa's parabolic surface;
- $z = c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{x^2y^2}{a^2b^2}\right)$: parabolic velaroid;
- $z^2 = f^2 - \frac{f^2-c^2}{a^2}\left(x^2 + y^2 + \frac{x^2y^2}{a^2}\right)$: elliptical velaroid;
- $z = (y - x^2)(y - 3x^2) = (y - 2x^2)^2 - x^4$: Peano saddle;
- $x^2y^2 + y^2z^2 + z^2x^2 = a^2$: narrowing saddle surface of Rosendorn;
- $x^2y^2 + y^2z^2 - z^2x^2 - 2kxyz = 0$: Steiner surfaces of the second type;
- $y^2 - 2xy^2 - xz^2 + x^2y^2 + z^2x^2 - z^4 = 0$: Steiner surfaces of the fourth type;
- $y^2z^2 + z^2x^2 + x^2y^2 + 2kxyz = 0$: the Roman surface;
- $(x^2 + y^2 + z^2 - \mu^2w^2)^2 = \lambda\left[(z - w)^2 - 2x^2\right]\left[(z + w)^2 - 2y^2\right]$: Kummer surface;
- $z = xy(x^2 + y^2)$: flat saddle in the drum;
- $c^2b^2x^2 - (a^2c^2 - z^2)(x^2 + y^2) = 0$: Wallis's conical edge;
- $(k_1x^2 + k_2y^2)(x^2 + y^2 + z^2) = 2z(x^2 + y^2)$: cross cap;
- $(x^2 + y^2 + z^2 - 2\mu ax)^2 = 4a^2(x^2 + y^2)$: epitrochoidal surface;
- $(x^2 + y^2 + z^2 - \mu^2 + b^2)^2 = 4(cx - a\mu)^2 + 4b^2y^2$: Dupin cyclide of the first type (of the fourth order);
- $[x^2 + y^2 + z^2 \pm 2rR - 2z(r \pm R)]^2 = 4(r^2 - x^2)(R^2 - y^2)$: circular surface of translation;
- $\left[(z + b + d)^2a^2c^2 - b^2c^2(a^2 - x^2) - a^2d^2(c^2 - y^2)\right]^2 = 4(abcd)^2(a^2 - x^2)(c^2 - y^2)$: elliptic surface of translation;
- $a^2c^2y^2 = (c^2 - z^2)[ab + (x - a)(b - d)]^2$: cylindroid with two directrix ellipses;
- $x^2y^2 = (a + y)^2(r^2 - y^2 - z^2)$: cyclic surface with circles in planes of pencil and with a straight line of centers;
- $x^2(r^2 - y^2) + (a + y)^2(r^2 - y^2 - z^2)$—cyclic surface with circles of variable radius in planes of pencil and with three straight parallel directrixes;
- $25[x^3(y + z) + y^3(x + z) + z^3(x + y)] + 50(x^2y^2 + y^2z^2 + z^2x^2) - 125(x^2yz + y^2xz + z^2xy) + 60xyz - 4(xy + xz + yz) = 0$: Nordstrand's weird surface;
- $(x^2 + y^2 + z^2 - ak^2)^2 - b\left[(z - k)^2 - 2x^2\right]\left[(z + k)^2 - 2y^2\right] = 0$: "Chair";
- $x^4 + y^4 + z^4 = 1$: Euler surface of the fourth order;
  $ax^4 + by^4 + cz^4 + dw^4 = 0$: diagonal quartic surface;
- $x^4 + y^4 + z^4 - 5(x^2 + y^2 + z^2) + 11,8 = 0$: tanglecube;
- $x^4 + y^4 + z^4 - (x^2 + y^2 + z^2) = 0$: tooth surface;
- $x^4 + y^4 + z^4 + a(x^2 + y^2 + z^2)^2 + b(x^2 + y^2 + z^2) + c = 0$: Goursat's surface;
- $(x^2 + y^2 + z^2)(x^2 + y^2) + 2z(x^2 - y^2) = 0$: quartic surface with double straight line and with a triple point;
- $(a - z)^4r^2 - (a - z)^2(x^2 + y^2)a^2 + 2axyz(a - z) - y^2z^2 = 0$: quartic surface with two double straight lines;
- $(x^2 + y^2 + z^2)^2 + 2xyz = 0$: cyclides with a triple point;
- $4T^2y^2 + B^2z^2 = T^2B^2(1 - x^2/L^2)^2$: quartic surface with parabola, ellipse, and parabola in three principal coordinate sections;
- $256B^2T^2x^3(x - L) + 108T^2L^4y^2 + 27B^2L^4z^2 = 0$: quartic surface with the 4th order curve, ellipse, the 4th order curve in three principal coordinate sections;
- $(xz - ab)^2 - (b^2 - z^2)(b - y)^2 = 0$: ruled rotor cylindroid;
- $x^2 + y^2 + z^2 - \frac{1}{2}\left[\frac{a^2b^2(x^2+y^2)}{H(b^2x^2+a^2y^2)} + H\right]z = 0$: surface of circles of Feuerbach;
- $z^2 = (b^2/a^2)[(y_o - y)/y_o]^2\left[a^2 - (x - c)^2\right]$: continuous topographic ruled surface with distributing ellipse;
- $x^2 + y^2 = 4(1 - z^2)z^2$: "Eight surface";
- $z^2 + y^2 = 3x(2a - x)[1 - c^2/(x + a)^2]/4$: surface of revolution "Egg" of the fourth order;
- $\sqrt{x^2 + y^2 + (z + a)^2} - \sqrt{y^2 + (z - a)^2} - R = 0$: equidistant of the system "straight–sphere";
- $(b^2 + z^2)(x^2 + y^2) = a^2z^2$: bullet nose;
- $(x^2 + z^2)^2 - r^2(x^2 - z^2) = 0$: lemniscate cylinder;
- $\sqrt{x^2 + y^2} - \sqrt{y^2 + z^2 + (x - a)^2} - R = 0$: equidistant of the system "point–cylinder";
- $\sqrt{x^2 + y^2} - \sqrt{(lx - z)^2 \Big/ (1 + l^2) + (y - a)^2} - R = 0$: equidistant of the system "straight–cylinder";
- $(x^2 - a)^2 \Big/ 4^2 + az^2 - cz(x^2 - a) = (R \pm r)^2y^2$: equidistant of the system "straight–torus";
  $4x^2(x^2 + y^2 + z^2) - y^2(1 - y^2 - z^2) = 0$: Miter Surface;
  Cyclides of revolution of the fourth order (Krames).

Algebraic quintic surfaces

- $x^2+y^3+z^5=1$: Peninsula surface;
- $x^2+y^2=(1-z)z^4$: surface of revolution "Kiss surface";
- $z=2a-\frac{a^3}{y^2+a^2}-\frac{a^3}{x^2+a^2}$: bi-agnésienne translation surface;
- $d^2xy^4-2a^2d^2xy^2+a^2(b+c)(2dy^2z+a^2z^2-2a^2zd)+a^4d^2x=0$: parabolic surface of conoidal type;
- $y=\frac{B\sqrt{T^2-z^2}}{2}\left[\frac{1}{T}-\frac{x^2}{L^2(z+T)}\right]$: surface of the 5th order with parabola, ellipse, and parabola lying in three principal coordinate sections;

$$64(x-w)[x^4-4x^3w-10x^2y^2-4x^2w^2+16xw^3 \\ -20xy^2w+5y^4+16w^4-20y^2w^2] \\ -5\sqrt{5-\sqrt{5}}(2z-\sqrt{5-\sqrt{5}}w) \\ [4(x^2+y^2+z^2)+(1+3\sqrt{5})w^2]^2:$$

"Dervish".

Algebraic surfaces of the sixth order

- Torse of constant slope with directrix parabola (see a Subsect. "1.1.1. Torse surfaces (torses)");
- $(z^2+4a^2)^2(x^2+y^2)=64a^6$: surface of revolution of the Agnesi curl (the first variant);
- $(x^2+9y^2/4+z^2-1)^3-x^2z^3-9y^2z^3/80=0$; "Heart surface";
- $4(x^2+y^2+z^2-13)^3+27(3x^2+y^2-4z^2-12)^2=0$: Hunt 's surface;
- $a^2(z^2-x^2-y^2)^2+4x^2y^2(z^2-a^2)=0$: sine surface;
- $z^2+y^2-T^2\left(\frac{2L^2}{x^2+L^2}-1\right)^2=0$: surface of revolution of an Agnesi curl (the 2nd variant);
- $z=\frac{x^2}{m^2}\left(\frac{q-h}{t^2}y^2-q+\frac{2Ry^2}{y^2+4R^2}\right)-\frac{2Ry^2}{y^2+4R^2}$: the 6th order surface with the pseudo Agnesi curl and two parabolas lying in three principle coordinate planes;
- $\left[\frac{x^2(z+b)}{2b}+\frac{y^2(b-z)}{2b}-a^2-(b^2-z^2)\right]^2-(b^2-z^2)\left(2a-\frac{xy}{b}\right)^2=0$: ruled rotor cylindroid;
- $y=\frac{L^2(T-z)B\sqrt{T^2-z^2}}{4x^2T+L^2(T-z)}-\frac{B}{2T}\sqrt{T^2-z^2}$: aerodynamic surfaces given by a continuous framework of water lines in the form of generalized Agnesi curls;
- $\pm y=\frac{\sqrt{3}B}{2T}\sqrt{\frac{4}{3}Tz-z^2}\left(\frac{z}{T}-\frac{x^2}{L^2}\right)$: the 6th order surface with parabola, the 4th order curve, parabola lying in three principal coordinate sections;
- $(x^2+L^2)^2\left(\frac{4y^2}{B^2}+\frac{z^2}{T^2}\right)=(L^2-x^2)^2$: the 6th order surface with Agnesi curl, ellipse, Agnesi curl lying in three principal coordinate sections;
  $4(\phi^2x^2-y^2)(\phi^2y^2-z^2)(\phi^2z^2-x^2)-(1+2\phi)(x^2+y^2+z^2-w^2)^2\ w^2=0$:
  Barth Sextic;
- $z^2=b^2[1-(x^2+y^2)/a^2]^3$: "Soucoupoid";
- $(1+z)y=(1+z)\left(\frac{z^5}{5}+\frac{z^4}{2}\right)-(z^3+2z^2)\left(\frac{z^2}{2}+\frac{z^3}{3}-x\right)+\left(\frac{z^2}{2}+\frac{z^3}{3}-x\right)^2$:
  the first algebraic surface of the 6th order with two nets of translation;
- $z^6-\frac{15}{4}z^4-15xz^3-45x^2+45yz+\frac{65}{4}=0$: the second algebraic surface of the 6th order with two nets of translation.

Algebraic surfaces of the seventh order

- $z^3=a(x^3-y^3)^2y+b(x^2+y^2)x$: "Ski hill";
- $\frac{y^2}{\left[\frac{B}{2}-\frac{2B}{L^2}\left(x-\frac{L}{2}\right)^2\right]^2}+\frac{z^2}{\left[2{,}5426\frac{T}{L}x\sqrt{\frac{3(L-x)}{L+3x}}\right]^2}=1$: the 7th order surface with parabola, ellipse, and Cartesian folium lying in three principle coordinate planes;
- $\frac{4Bx^2}{L^2(B-2y)}+\frac{(4y^2+B^2)^2z^2}{T^2(B^2-4y^2)^2}=1$: the 7th order surface with parabola, Agnesi curl, and ellipse lying in three principal coordinate sections;
- $y^2=\frac{3B^2(L^2-x^2)}{2LT^2}\left(\frac{4}{3}Tz-z^2\right)\left(z^2-\frac{T}{L^2}x^2z\right)$: the 7th order surface with parabola, the 4th order curve, parabola lying in three principal coordinate sections;
- $B^2(T^3+z^3)(L^2-x^2)^2=4y^2T^3(x^2+L^2)^2$: the 7th order surface with Agnesi curl, Lame's curve of the third order, and straight lines lying in three principal coordinate sections.

Algebraic surfaces of the 8th order

- $(ac+z^4)^2-c^2(x^2+y^2)=0$: surface of revolution of a biquadrate parabola;
- $x^2y^2\left[x^2y^2+2(a-y)^2(y^2+z^2-2r^2)\right]+(a-y)^4\left[(y^2+z^2)-4r^2y^2\right]=0$: cyclic surface with circles in planes of pencil with a straight directrix and a fixed straight of pencil that are lying on different sides of a plane center-to-center line;
- $\left[x^2y^2+(a+y)^2(y^2+z^2-2r^2)\right]^2=4r^2(a+y)^4(r^2-z^2)$: cyclic surface with circles in planes of pencil, with a

straight directrix and a fixed straight of pencil that are lying on the same side of a plane center-to-center line;

- $\frac{y^2}{\left(\frac{L^2 B}{4x^2+L^2}-\frac{B}{2}\right)^2}+\frac{z^2}{\frac{T^2(L^2-4x^2)}{L^2}}=1$: the 8th order surface with Agnesi curl, ellipse, and ellipse lying in three principal coordinate sections;
- $\frac{16y^4}{\frac{B^4(L^4-x^4)}{L^4}}+\frac{z^4}{\left(\frac{T^2(L^2-x^2)}{L^2}\right)^2}=1$: the 8th order surface with Lame's curve of the 4th order, Lame's curve of the 4th order, and the ellipse lying in three principal coordinate sections;
- $y^2-\frac{64}{27}B^2\left(\frac{z}{T}-\frac{3}{4}\right)^3\left(\frac{1}{4}-\frac{z}{T}\right)\left(1-\frac{x^2}{L^2}\right)^2=0$: the 8th order surface with parabola, the 4th order curve, and the parabola lying in three principal coordinate sections.

Algebraic surfaces of the 9th order

- $\left(\frac{y^2-x^2}{2z}+\frac{2}{9}z^2+\frac{2}{3}\right)^3-6\left[\frac{y^2-x^2}{4z}-\frac{1}{4}\left(x^2+y^2+\frac{8}{9}z^2\right)+\frac{2}{9}\right]^2=0$: Enneper's surface.

Algebraic surfaces of the 10th order

- $\frac{y^2}{\left(\frac{L^2 B}{4x^2+L^2}-\frac{B}{2}\right)^2}+\frac{z^2}{\left(T-\frac{4Tx^2}{L^2}\right)^2}=1$: aerodynamic surfaces given by a continuous framework of elliptical ribs;

$$\begin{aligned}
&8(x^2-\phi^4 y^2)(y^2-\phi^4 z^2)(z^2-\phi^4 x^2)\\
&\times(x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2)\\
&+(3+5\phi)(x^2+y^2+z^2-w^2)^2\\
&\times[x^2+y^2+z^2-(2-\phi)w^2]^2w^2=0:
\end{aligned}$$

Barth Decic.

Algebraic surfaces of the 12th order

- $y^2-C^2B^2(z/T+v_m)^3\left[1-(z/T+v_m)^5\right](1-x^2/L^2)^2$: surface of the 12th order with parabola, the 8th order curve, parabola lying in three principal coordinate sections.

Algebraic surfaces of the 16th order

- $\left\{(z-n)[2n^2(x^2-q^2)+q^4]+q^2nx^2\right\}^2-q^2(q^2-4n^2)\left[x^2n+(z-n)q^2\right]^2=0$: algebraic surface with a continuous net of the pseudo Agnesi curl passing through a parabola and two straights ($n=n(y^2)$).

Additional Literature

Krames Josef. Über Drehzykliden vierter Ordnung. Monatsh. Math. 1975; 80, No. 1, p. 45-60.
Niče Vilko. Über die konstruktive Behandlung einer Art Kugelflächen 3. Ordnung. Glass. mat. 1974; 9, No. 2, p. 303-315.

36.1 Algebraic Surfaces of the Third Order

Assume, that S is a nonsingular cubic surface given by a homogeneous cubic $f=f(X, Y, Z, T)$ and l are straight lines lying on S. It is known, that through any point P belonging to nonsingular cubic surface S, not more than three straight lines pass lying on S. If a surface contains two or three such straights, then they lie on the same plane. Several methods are available proving the existence of not less than one straight line l lying on S.

Additional Literature

Bisztriczky Tibor. On surfaces of order three. Can. Math. Bull. 1979; 22, No. 3, p. 351-355.
Pushkar Joshi, Carlo H. Séquin. An intuitive explanation of third-order surface behavior. *Computer Aided Geometric Design*. 2010; 27(2), p. 150–161.

■ "Handkerchief Surface"

A surface named "Handkerchief surface" is an algebraic surface of the third order that is symmetrical relative to one coordinate plane.

Forms of definition of the "Handkerchief surface"

(1) Parametrical equations (Fig. 1):

$$x=x(u)=u,\quad y=y(v)=v,$$
$$z=z(u,v)=u^3/3+uv^2+2(u^2-v^2).$$

The "Handkerchief surface" defined by these equations is given in the nonorthogonal nonconjugate system of curvilinear coordinates u, v.

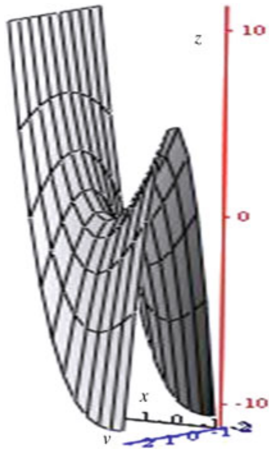


Fig. 1

(2) Explicit equation:

$$z = x^3/3 + xy^2 + 2(x^2 - y^2).$$

The surface in question is symmetrical relative to the coordinate plane xOz. At the cross-section of the surface by the plane $y = 0$, the curve of the third order $z = x^3/3 + 2x^2$ (Fig. 1) is placed.

Coefficients of the fundamental forms of the surface and its principle curvatures:

$$\begin{aligned} A^2 &= 1 + (x^2 + y^2 + 4x)^2, \\ F &= 2y(x-2)(x^2 + y^2 + 4x), \\ B^2 &= 1 + 4y^2(x-2)^2, \\ A^2B^2 - F^2 &= 1 + (x^2 + y^2 + 4x)^2 + 4y^2(x-2)^2 \\ &= A^2 + B^2 - 1 \end{aligned}$$

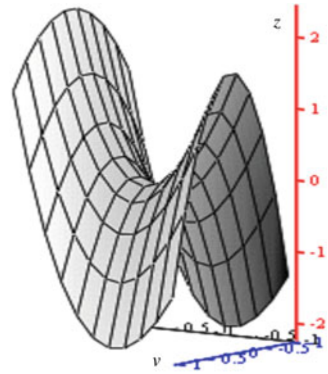


Fig. 2

$$L = \frac{2(x+2)}{\sqrt{A^2 + B^2 - 1}},\; M = \frac{2y}{\sqrt{A^2 + B^2 - 1}},$$
$$N = \frac{2(x-2)}{\sqrt{A^2 + B^2 - 1}},\; k_x = \frac{2(x+2)}{A^2\sqrt{A^2 + B^2 - 1}},$$
$$k_y = \frac{2(x-2)}{B^2\sqrt{A^2 + B^2 - 1}},\; K = \frac{4(x^2 - y^2 - 4)}{(A^2 + B^2 - 1)^2}.$$

The plane coordinate line $y = 0$ lying at the coordinate plane xOz is the line of principle curvature. The surface contains the segments of positive and negative Gaussian curvature. The parabolic points with $K = 0$ are disposed on two lines: $y = v = \pm\sqrt{u^2 - 4} = \pm\sqrt{x^2 - 4}$, which are projected on the coordinate plane xOy in the form of *the rectangular hyperbolas*. The curvilinear coordinate line $u = x = 0$ is *a parabola* $z = -2y^2$, and the coordinate line $u = x = 2$ is *a straight line* $z = 32/3$ with $k_y = 0$ (Fig. 1). At the cross-sections by the planes $y = \pm x$, *the cubic parabolas* lie.

Figure 1 shows the surface build at the boundaries $-2 \le x \le 2$; $-2 \le y \le 2$. In Fig. 2, the surface is limited by the coordinates $-1 \le y \le 1$ and $-1 \le x \le 1$.

■ Cubic Surface $x^3 + xy + z = 0$

The cubic surface $x^3 + xy + z = 0$ is usually considered in papers and books devoted to a theory of catastrophes. This is a ruled surface of negative Gaussian curvature.

The structure of the formula of the cubic surface in question coincides with the structure of the formula of *the Cayley surface* (see also the Chap. "3. Translation surfaces").

Forms of definition of the surface

(1) Implicit equation (Fig. 1):

$$x^3 + xy + z = 0.$$

(2) Explicit equation (Fig. 1):

$$z = -x^3 - xy.$$

Assuming $x = 0$, we obtain $z = 0$ and consequently the straight generatrix of the cubic surface lying in the plane $x = 0$ coincides with the axis Oy.

All straight generatrixes of the cubic surface lie at the planes $x = x_0 = \text{const}$, their projections on the coordinate plane yOz are defined by the equations:

$$z = -x_0^3 - x_0 y.$$

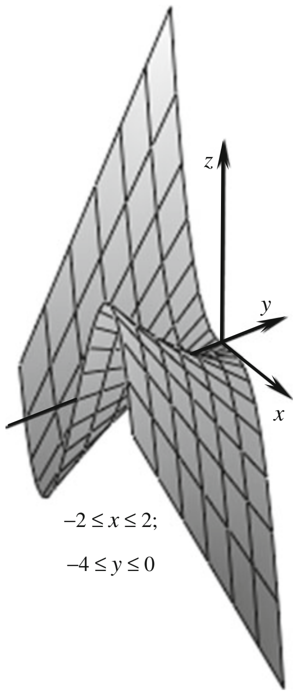


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + (3x^2 + y)^2,\ F = x(3x^2 + y),\ B^2 = 1 + x^2;$$
$$A^2B^2 - F^2 = A^2 + B^2 - 1;$$
$$L = \frac{-6x}{\sqrt{A^2 + B^2 - 1}},\ M = \frac{-1}{\sqrt{A^2 + B^2 - 1}},$$
$$N = 0;\ k_y = 0;$$
$$K = -\left[1 + (3x^2 + y)^2 + x^2\right]^{-2} < 0,$$
$$H = x(y - 3)\left[1 + (3x^2 + y)^2 + x^2\right]^{-3/2}.$$

Additional Literature

Reid M. Algebraic Geometry for Everyone. Moscow: «Mir», 1991; 151 p.

Reid M. Undergraduate Algebraic Geometry. Cambridge University Press, 1988; 129 p.

■ Diagonal Cubic Surface of Fermat

An algebraic surface of the third order, which is called *a diagonal cubic surface of Fermat*, is usually used for the illustration of the assertion on the existence of straight lines on a cubic surface.

Forms of definition of the surface

(1) Implicit equation (Fig. 1):

$$x^3 + y^3 + z^3 + t^3 = 0.$$

(2) Explicit equation (Fig. 1):

$$z = \sqrt[3]{-x^3 - y^3 - t^3},$$

where t = const.

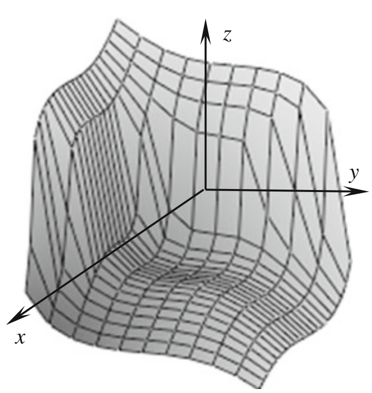


Fig. 1

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{x^4}{(x^3 + y^3 + t^3)^{4/3}},$$
$$F = \frac{x^2y^2}{(x^3 + y^3 + t^3)^{4/3}},$$
$$B^2 = 1 + \frac{y^4}{(x^3 + y^3 + t^3)^{4/3}};$$
$$A^2B^2 - F^2 = 1 + \frac{x^4 + y^4}{(x^3 + y^3 + t^3)^{4/3}};$$
$$L = \frac{-2x(y^3 + t^3)}{(x^3 + y^3 + t^3)^{5/3}\sqrt{A^2B^2 - F^2}},$$
$$M = \frac{2x^2y^2}{(x^3 + y^3 + t^3)^{5/3}\sqrt{A^2B^2 - F^2}},$$
$$N = \frac{-2y(x^3 + t^3)}{(x^3 + y^3 + t^3)^{5/3}\sqrt{A^2B^2 - F^2}};$$
$$K = \frac{4t^3xy(x^3 + y^3 + t^3)^{1/3}}{\left[(x^3 + y^3 + t^3)^{4/3} + x^4 + y^4\right]^2}.$$

The point on the cubic surface with the coordinates $x = y = 0$, $z = -t$ is *a flat point*. The planes $x = -y$, $x = -t$, $y = -t$ intersect the cubic surface along the straight lines. Parabolic points with $K = 0$ are placed on the surface along the lines $x = 0$ and $y = 0$.

■ Cubic Cone

In the dissertation of Ibolya Szilágyi D.M. (2005), a surface of the third order

$$y^2z - x^3 - xz^2 = 0,$$

called "*Cubic cone*" is mentioned.

Forms of definition of the cubic cone

(1) Explicit equation (Fig. 1):

$$y = \pm\sqrt{\frac{x(x^2+z^2)}{z}},$$

where

$$0 \le x \le \infty, \quad 0 < z \le \infty$$

or

$$-\infty \le x \le 0, \quad -\infty \le z < 0.$$

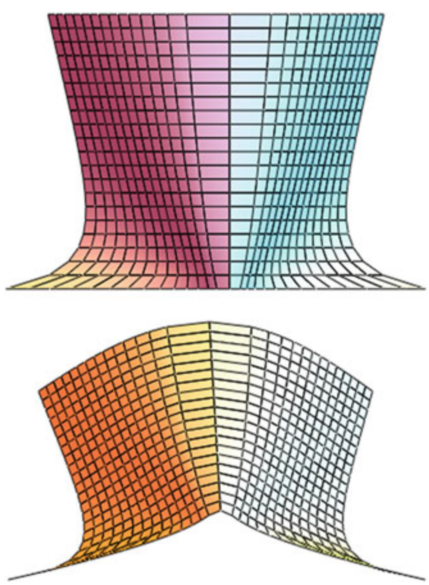


Fig. 1

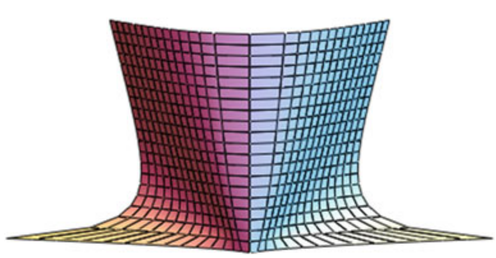


Fig. 2

Figure 1 shows the surface build in the limits $0 \le x \le 5$ m, 0.5 m $\le z \le 15$ m.

In Fig. 2, the surface has

$$0 \le x \le 5 \text{ m}, \quad 0.5\,\text{m} \le z \le 15\,\text{m}$$

The cubic cone has two planes of symmetry: x = 0 and y = 0. But the cubic cone is inverse symmetrical relative to the coordinate plane z = 0.

(2) Parametrical equations (Fig. 3):

$$x = x(u,v) = v\sin u,$$
$$y = \pm v\sqrt{\tan u},$$
$$z = z(u,v) = v\cos u,$$

where

$$0 \le u \le \pi/2, \quad -\infty < v < \infty.$$

In Fig. 3, the surface is designed in the limits

$$0 \le u \le 0.4\pi; \quad -2 \text{ m} \le v \le 2\,\text{m}.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = v^2\left(1 + \frac{1}{4\sin u\cos^5 u}\right),$$
$$F = \frac{v}{2\cos^2 u}, \; B^2 = 1 + \tan u;$$
$$\sigma = \sqrt{A^2B^2 - F^2} = v\sqrt{1 + \tan u\frac{1}{4\sin u\cos^5 u}};$$
$$L = \frac{v}{\sigma}\frac{4\sin^2 u(1+\cos^2 u) - 1}{4\sin u\cos^3 u\sqrt{\tan u}},$$
$$M = N = 0, \; k_v = 0, \; K = 0.$$

The algebraic surface in question of the third order has zero Gaussian curvature

$$K = 0,$$

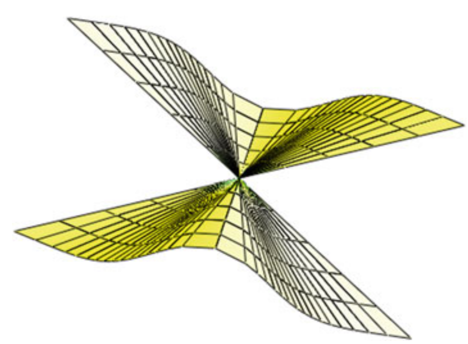


Fig. 3

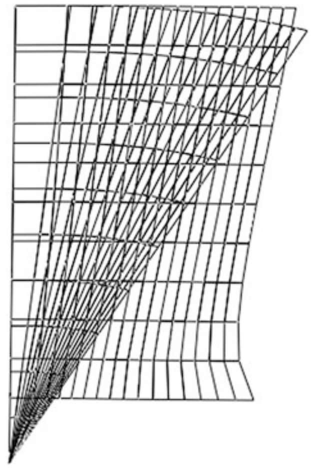


Fig. 4

So, this surface is a conical surface. The coordinate lines v coincide with the generatrix straight lines of the conical surface (Fig. 3).

In Fig. 4, one and the same surface is shown, but it is given in the explicit form and by the parametric equations. Two systems of the curvilinear coordinates are put on each other.

Additional Literature

Ibolya Szilágyi DM. Symbolic–Numeric Techniques for Cubic Surfaces. Juli 2005, Diss. Doctor der Naturwissenschaften, Johannes Kepler Universität Linz, 115 p. (74 refs).

■ The Cubic Surface with 24 Straight Lines

The cubic surface with 24 straight lines is an algebraic surface of the third order with 24 straight lines lying on it. The surface has four spaces going in infinity.

Forms of definition of the surface

(1) Implicit equation:

$$x^2 + y^2 + z^2 - 2xyz = 1 + \lambda^2,$$

where $\lambda = \text{const} > 0$.

(2) Implicit equation:

$$(x - yz)^2 = \left(y^2 - 1\right)\left(z^2 - 1\right) + \lambda^2.$$

(3) Implicit equation:

$$(z - yx)^2 = \left(y^2 - 1\right)\left(x^2 - 1\right) + \lambda^2.$$

(4) Explicit equation (Fig. 1):

$$z = xy \pm \sqrt{(y^2 - 1)(x^2 - 1) + \lambda^2}.$$

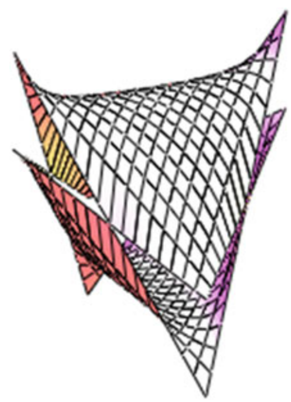


Fig. 1

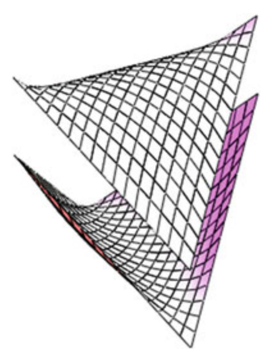


Fig. 2

24 straight lines belonging to the cubic surface lie on the 12 planes, i.e., one pair of the straight lines is placed on every plane.

Assume a new constant $\mu^2 = 1 + \lambda^2$.

Two straight lines $x = (\mu \pm \lambda)y$ lie at the cross-section of the cubic surface by the plane $z = \mu$. Other two straight lines $x = (-\mu \pm \lambda)y$ lie at the cross-section of the surface by the plane $z = -\mu$.

Additional two straight lines $x - y = \pm\lambda$ are placed at the cross-section $z = 1$. At the cross-section by the plane $z = -1$, one pair of the straight lines $x + y = \pm\lambda$ is disposed also.

The next pair of the straight lines $z = (\mu \pm \lambda)y$ lays at the plane $x = \mu$; two straight lines $z = (-\mu \pm \lambda)y$ are placed at the plane $x = -\mu$.

At the planes $x = \pm 1$, two pairs of the straight lines $z - y = \pm\lambda$ and $z + y = \pm\lambda$ are placed also, accordingly.

Two pairs of the straight lines $x = (\mu \pm \lambda)z$ and $x = (-\mu \pm \lambda)z$ are disposed at the cross-sections of the surface by the planes $y = \pm\mu$, accordingly.

The last two pairs of the parallel lines $x - z = \pm\lambda$ and $x + z = \pm\lambda$ lie at the planes $y = \pm 1$, accordingly.

Figure 1 shows the surface build in the boundaries $-\sqrt{1+\lambda^2} \le u \le \sqrt{1+\lambda^2}$ and $-\sqrt{1+\lambda^2} \le v \le \sqrt{1+\lambda^2}$, $\lambda^2 = 0.5$. The same surface is presented in Fig. 2, but in the boundaries $-1 \le u \le 1$.

36.2 Algebraic Quartic Surface

Till the present time, the whole classification of the algebraic quartic surfaces is absent.

Additional Literature

Krylov IP. Forms and types of the geometrical surfaces described by the algebraic equations of the fourth order. Materialy 55 NTK SPbGUT, January 27-31, 2003. SPb: SPbGUT, 2003.

Jessop CM. Quartic surfaces with singular points. The Mathematical Gazette. 1916; Vol. 8, No. 126, p. 337-338.

Martin Bright. Brauer groups of diagonal quartic surfaces. J. Symbolic Computation. May 2006; Vol. 41, Iss. 5, p. 544-558.

Gaffet B. On a class of quartic surfaces and an associated integrable differential system. J. of Physics A: Mathematical and Theoretical. 2007; Vol. 40, p. 6085-6099.

Sir Peter Swinnerton-Dyer. Arithmetic of diagonal quartic surfaces. Proc. of the London Mathematical Society. May 2000; Vol. 80, Cambridge Univ. Press, p. 513-544.

Degtyarev VM, Krylov IP. On classification of cubic and . quartic surfaces. Seventh Intern. Workshop on Nondestructive Testing and Computer Simulations in Science and Engineering: Proc. of the SPIE. 2004; Vol. 5400, p. 287-291.

■ Euler Surface of the Fourth Order

The Euler surface of the fourth order is a closed surface with three coordinate planes of symmetry (with $c = 1$).

Forms of definition of the Euler surface

(1) Implicit form of the definition:

$$x^4 + y^4 + z^4 = c \quad (\text{when } c = 1).$$

(2) Explicit equation (Fig. 1):

$$z = \pm\sqrt[4]{c - x^4 - y^4}.$$

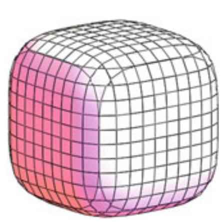


Fig. 1

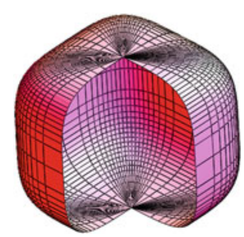


Fig. 2

(3) Parametrical (spherical) form of the definition (Fig. 2):

$$x = \pm\sqrt[4]{c}\sqrt{\cos u \sin v},\ y = \pm\sqrt[4]{c}\sqrt{\sin u \sin v},$$
$$z = \pm\sqrt[4]{c}\sqrt{\cos v}.$$

In Fig. 2, the surface consists of three segments; for example, in the first quadrant: $0 \le u \le \pi/2$, $0 \le v \le \pi/2$.

(4) Parametrical form of the definition:

$$x = \pm\sqrt[4]{c}\sqrt{u \cos v}, \quad x = \pm\sqrt[4]{c}\sqrt{u \sin v},$$
$$z = \pm\sqrt[4]{c}\sqrt[4]{1 - u^2}.$$

■ Surface of Circles of Feuerbach

Fepl St. (1971) has investigated a surface representing the geometrical place of *the Feuerbach circles*. The Feuerbach circles correspond to the axial cross-sections of some *right elliptical cone*.

Forms of definition of the surface of circles of Feuerbach

(1) Implicit equation:

$$x^2 + y^2 + z^2 - \frac{1}{2}\left[\frac{a^2b^2(x^2+y^2)}{H(b^2x^2+a^2y^2)} + H\right]z = 0,$$

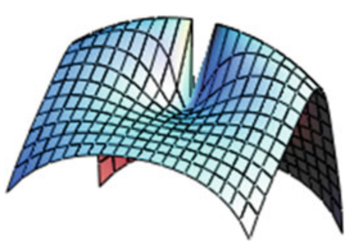
where a, b, and H are constants.

(2) Explicit equation (Figs. 1, 2 and 3; $H = 9.9$; $p = \pm 1$)

$$z = \frac{1}{4}\left(\frac{a^2b^2(x^2+y^2)}{H(b^2x^2+a^2y^2)} + H\right) + p\sqrt{\frac{1}{16}\left(\frac{a^2b^2(x^2+y^2)}{H(b^2x^2+a^2y^2)} + H\right)^2 - x^2 - y^2}.$$

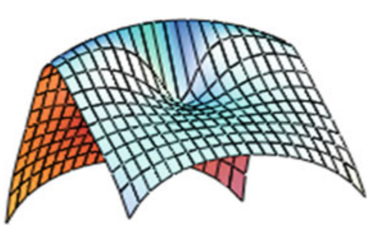
(3) Parametric form of assignment (Figs. 4, 5 and 6; $H = 9.9$; $p = \pm 1$)

$$x = x(r,v) = r\cos v,$$
$$y = y(r,v) = r\sin v,$$



$a = 3;\ b = 6;\ p = +1;$
$-1{,}8 \le x \le 1{,}8\ ;\ -2 \le y \le 2$

Fig. 1

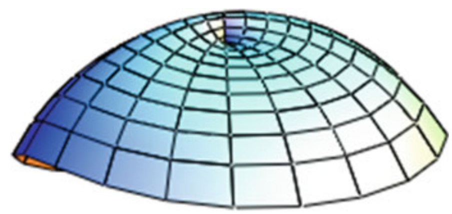


$a = 6;\ b = 3;\ p = +1;$
$-1{,}8 \le x \le 1{,}8\ ;\ -2 \le y \le 2$

Fig. 2

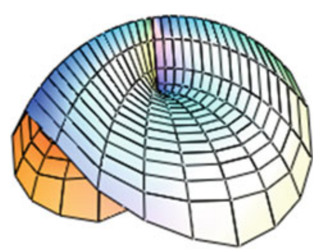
$a = 6;\ b = 3;\ p = \pm 1;$
$-1{,}8 \le x \le 1{,}8\ ;\ -2 \le y \le 2$

Fig. 3



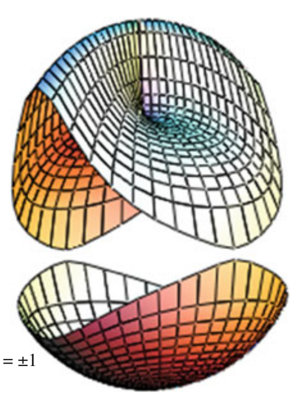
$a = 3;\ b = 4;\ p = +1;$
$0 \le r \le 2{,}5\ ;\ -\pi \le v \le \pi$

Fig. 4



$a = 3;\ b = 7;\ p = +1;\ 0 \le r \le 2{,}7$
$(-\pi \le v \le \pi)$

Fig. 5



$p = \pm 1$

Fig. 6

$$z = z(r, v) = \frac{1}{4}\Psi + p\sqrt{\frac{1}{16}\Psi^2 - r^2},$$

where $\Psi = \frac{a^2b^2}{H(b^2\cos^2 v + a^2\sin^2 v)} + H.$

Reference

Fepl Stanimir. Über eine Fläche. Publ. Elektrotehn. fak. Univ. Beogradu. Ser. Mat. i fiz. 1971; No. 357-380, p. 59-62.

■ Nordstrand's Weird Surface

Nordstrand's weird surface is given by the following implicit equation:

Fig. 1

$$25[x^3(y+z) + y^3(x+z) + z^3(x+y)]$$
$$+ 50(x^2y^2 + x^2z^2 + y^2z^2)$$
$$- 125(x^2yz + y^2xz + z^2xy)$$
$$+ 60xyz - 4(xy + xz + yz) = 0.$$

The equation of the surface and its drawing (Fig. 1) are taken without any changing at the site of Eric W. Weisstein.

References

Eric W. Weisstein. Nordstrand's weird surface. From MathWorld. A Wolfram Web Resource. © 1999 CRC Press LLC, © 1999-2004 Wolfram Research, Inc.: http://mathworld.wolfram.com/NordstrandsWeirdSurface.html

Nordstrand T. Weird cube: http://www.uib.no/people/nfyth/weirdtxt.htm

■ Menn's Surface

The Menn's surface is an algebraic surface of the 4th order that is symmetrical relative to the one of the coordinate plane.

Forms of definition of the Menn's surface

(1) Parametrical equations (Fig. 1):

$$x = x(u) = u,$$

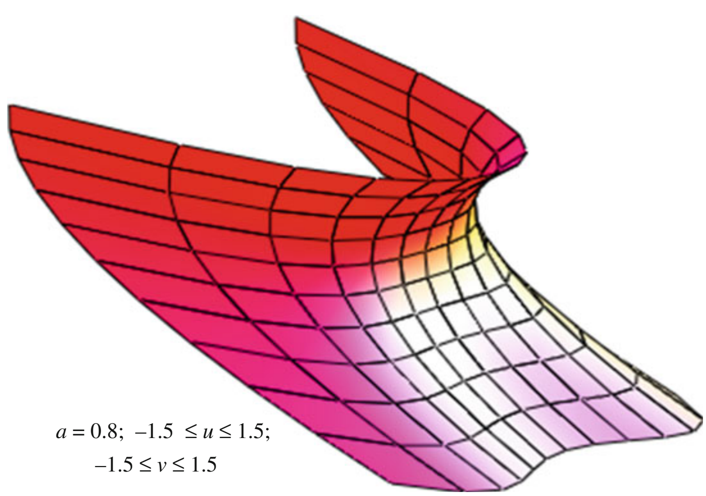


Fig. 1

$$y = y(v) = v,$$
$$z = z(u, v) = au^4 + u^2 v - v^2.$$

The surface is given in nonorthogonal, nonconjugate curvilinear coordinates u, v.

(2) Explicit equation:

$$z = ax^4 + x^2 y - y^2.$$

The surface in question is symmetrical relatively to the coordinate plane yOz. At the cross-section of the surface by the plane $y = 0$, *the biquadrate parabola*

$$z = ax^4$$

lies.

The coordinate plane $x = 0$ intersects the surface along a parabola (Fig. 2):

$$z = -y^2.$$

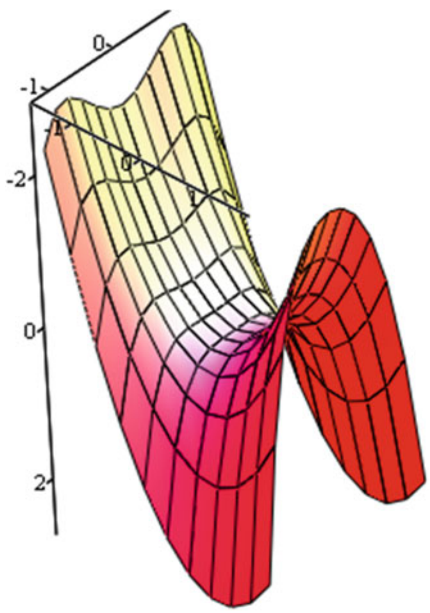


Fig. 2

There are two parabolas

$$y = \frac{-2ax^2}{1 \mp \sqrt{1 + 4a}}$$

at the cross-section of the surface by plane $z = 0$.

Coefficients of the fundamental forms of the surface and its curvatures:

$$A^2 = 1 + 4x^2(2ax^2 + y)^2,$$
$$F = 2x(2ax^2 + y)(x^2 - 2y),$$
$$B^2 = 1 + (x^2 - 2y)^2;$$
$$A^2B^2 - F^2 = 1 + (x^2 - 2y)^2 + 4x^2(2ax^2 + y)^2 = A^2 + B^2 - 1;$$
$$L = \frac{2(6ax^2 + y)}{\sqrt{A^2 + B^2 - 1}}, \quad M = \frac{2x}{\sqrt{A^2 + B^2 - 1}},$$
$$N = \frac{-2}{\sqrt{A^2 + B^2 - 1}}; \quad k_x = \frac{2(6ax^2 + y)}{A^2\sqrt{A^2 + B^2 - 1}},$$
$$k_y = \frac{-2}{B^2\sqrt{A^2 + B^2 - 1}}; \quad K = \frac{-4(x^2 + y + 6ax^2)}{(A^2 + B^2 - 1)^2}.$$

The plane curvilinear coordinate $x = 0$ laying at the coordinate plane yOz is the line of principle curvature. The parabolic points with $K = 0$ are disposed on the line which is projected on the coordinate plane xOy as a parabola:

$$y = -(1 + 6a)x^2;$$

and on the coordinate plane xOz as a bi-quadratic parabola

$$z = -\left(2 + 17a + 36a^2\right)x^4.$$

In Fig. 2, the Menn's surface build in the boundaries

$$-1.2 \le x \le 1.2 \quad -1.5 \le y \le 1.5$$

with $a = 1$ is shown.

Additional Literature

Bleeker D. and *Wilson L.* Stability of Gauss maps. Illinois J. Math. 1978; Vol. 22, p. 279-289.

■ Środa's Parabolic Surface

The Środa's parabolic surface is an algebraic surface of the fourth order that is symmetrical relatively to two coordinate planes and at the cross-sections of the surface by the planes parallel to these coordinate planes, parabolas are placed.

Forms of definition of Środa's parabolic surface

(1) Explicit equation (Fig. 1):

$$z = \left(3 - x^2 - y^2 - x^2y^2\right)/2.$$

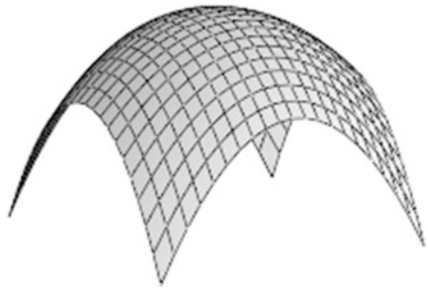


Fig. 1

At the cross-sections of the surface by the coordinate planes $x = 0$ and $y = 0$, the same parabolas $z = (3 - y^2)/2$ and $z = (3 - x^2)/2$ lie; $z_{\max} = 1,5$ is at the point with the coordinates $x = y = 0$.

At the cross-sections of the surface by the planes $x = \pm x_o$, the parabolas

$$z = \left[3 - x_o^2 - (1 + x_o^2)y^2\right]/2$$

are disposed and at the cross-sections by the planes $y = \pm y_o$, the parabolas

$$z = \left[3 - y_o^2 - (1 + y_o^2)x^2\right]/2$$

are placed.

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} A^2 &= 1 + x^2(1 + y^2)^2, \\ F &= xy(1 + x^2)(1 + y^2), \\ B^2 &= 1 + y^2(1 + x^2)^2; \\ L &= -\frac{(1 + y^2)}{\sqrt{A^2 + B^2 - 1}},\ M = \frac{-2xy}{\sqrt{A^2 + B^2 - 1}}, \\ N &= -\frac{(1 + x^2)}{\sqrt{A^2 + B^2 - 1}},\ K = \frac{1 + x^2 + y^2 - 3x^2y^2}{(A^2 + B^2 - 1)^2} \end{aligned}$$

The curvilinear coordinate lines x, y are nonconjugate and nonorthogonal. The only coordinate lines $x = 0$ and $y = 0$ coincide with the lines of principle curvature. The surface has the segments as of the positive Gaussian curvature and the segments of the negative Gaussian curvature as well. For example, at the angular points of the surface with the coordinates (1; −1; 0), (−1; 1; 0), (−1; −1; 0), and (1; 1; 0), the Gaussian curvature of the surface is equal to zero $K = 0$.

In Fig. 1, the surface designed in the boundaries $-1 \le x \le 1$; $-1 \le y \le 1$; $0 \le z \le 1.5$ is given.

The space closed curve with parabolic points $K = 0$ is projected on the coordinate plane xOy as the curve

$$y^2 = \left(1 + x^2\right)/\left(3x^2 - 1\right);$$

at the plane xOz as the curve

$$z = 2\left(x^2 - 1\right)^2/\left(1 - 3x^2\right),$$

at the plane yOz as the curve

$$z = 2\left(y^2 - 1\right)^2/\left(1 - 3y^2\right).$$

Figure 2 shows the surface limited by the coordinate lines $x = \pm 2$; $y = \pm 2$.

(2) Parametrical equations:

$$\begin{aligned} x &= x(v) = 2v - 1, \\ y &= y(u) = 1 - 2u, \\ z &= z(u, v) = 4[(1 - u)u + (1 - v)v - 2(1 - u)u(1 - v)v]. \end{aligned}$$

Coefficients of the fundamental forms of the surface:

$$\begin{aligned} A^2 &= 4 + 16(1 - 2u)^2[1 - 2v(1 - v)]^2, \\ F &= 16(1 - 2u)(1 - 2v)[1 - 2v(1 - v)] \\ &\quad \times [1 - 2u(1 - u)] \\ B^2 &= 4 + 16(1 - 2v)^2[1 - 2u(1 - u)]^2; \\ A^2B^2 - F^2 &= 16 + 64(1 - 2v)^2[1 - 2u(1 - u)]^2 \\ &\quad + 64(1 - 2u)[1 - 2v(1 - v)]^2; \\ L &= -\frac{32[1 - 2v(1 - v)]}{\sqrt{A^2B^2 - F^2}}, \\ M &= -\frac{32(1 - 2u)(1 - 2v)}{\sqrt{A^2B^2 - F^2}}, \\ N &= -\frac{32[1 - 2u(1 - u)]}{\sqrt{A^2B^2 - F^2}}; \end{aligned}$$

Fig. 2

$$K = 2\frac{32^2[u(1-u) + v(1-v)(1-6u+6u^2)]}{(A^2B^2 - F^2)^2}.$$

As it is seen, the surface is given in the curvilinear, nonorthogonal, and nonconjugate coordinates u, v.

Additional Literature

Środa P, Bożek B. Constructing of Bézier's curves and surfaces. Konferencja o Geometrii. Częstochowa (Poland), 24-25 września 1999, Częstochowa, 1999; p. 219-226.

Milnor J. On deciding whether a surface is parabolic or hyperbolic. Amer. Math. Monthly. 1977; 84, p. 43-46.

■ Overlimited Parabolic Velaroid

The algebraic surfaces of the fourth order having the name "*parabolic velaroid*" is widely known and described in detail in the scientific and technical literature (see also a Sect. "3.4. Velaroidal Surfaces"). *Parabolic velaroid* is related to a class of translation surfaces of the positive Gaussian curvature with the rectangular plan. Parabolic velaroid is generated by parabolas with the changing curvature in the process of translation motion and lying in the parallel planes. The surface is limited by four mutually orthogonal contour straight lines (Fig. 1):

$$x = \pm a; \quad y = \pm b.$$

But at the same time, it is interesting to research the segments of the surface which are placed outside of the rectangular boundaries of a parabolic velaroid.

An overlimited parabolic velaroid can be designed at arbitrary boundaries ($-\infty \le x \le \infty$, $-\infty \le y \le \infty$) and its segment placed inside the boundaries $-a \le x \le a$, $-b \le$ — $\le b$ is a parabolic velaroid.

Forms of definition of the overlimited parabolic velaroid

(1) Explicit equation (Fig. 2):

$$z = c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{x^2y^2}{a^2b^2}\right),$$

where c is a rise of the surface relatively to the plane $z = c$ with four straight lines $x = \pm a$; $y = \pm b$ lying on it.

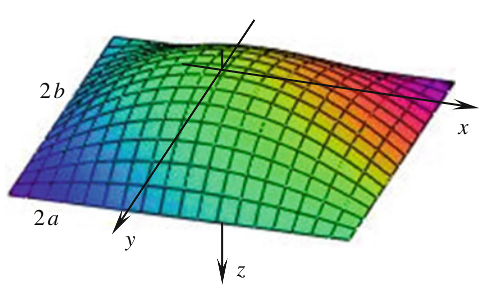


Fig. 1

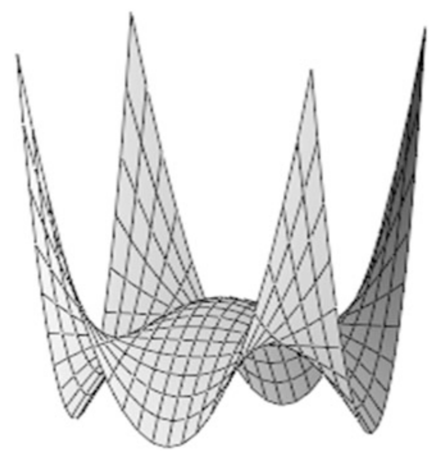


Fig. 2

The coefficients of the fundamental forms of the surface of overlimited parabolic velaroid may be calculated with the help of the same formulas that were obtained for parabolic velaroid, see also "Parabolic velaroid" in a Sect. "3.4. Velaroidal Surfaces".

From the formula for calculation of the Gaussian curvature of the surface, one may obtain:

$$K = \frac{4c^2}{a^2b^2(A^2 + B^2 - 1)^2}\left(1 - \frac{y^2}{b^2}\right)\left(1 - \frac{x^2}{a^2}\right).$$

It is obviously, that the chosen segment of the surface has both positive and negative Gaussian curvatures. The fragments of the surface with equal Gaussian curvature are distributed on the whole surface of the overlimited parabolic velaroid symmetrically relatively to the coordinate planes $x = 0$ and $y = 0$ as it is shown in Fig. 3.

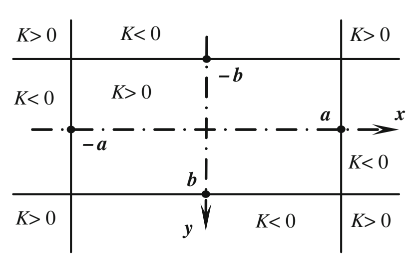


Fig. 3

■ "Chair"

Nordstrand has given the name to this surface in 1970 counting that it resemble an inflatable chair. *The surface "Chair"* has the tetrahedral symmetry.

The implicit equation of this surface is written in the following form:

$$\left(x^2+y^2+z^2-ak^2\right)^2-b\left[(z-k)^2-2x^2\right]$$
$$\times\left[(z+k)^2-2y^2\right]=0.$$

Figure 1 is taken without changing at the site of Nordstrand with $k = 5$; $a = 0.95$ and $b = 0.8$.

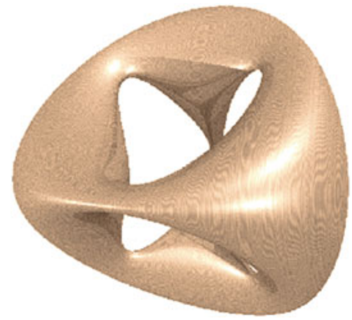


Fig. 1

Reference

Nordstrand T. "Chair": http://www.uib.no/people/nfytn/chairtxt.htm

■ Goursat's Surface

Goursat's surface is a closed algebraic surface of the fourth order consisting of one or two spaces and symmetrical relative to all three coordinate planes.

Forms of definition of the Goursat's surface

(1) Implicit equation (Fig. 1):

$$x^4+y^4+z^4+a\left(x^2+y^2+z^2\right)^2$$
$$+b\left(x^2+y^2+z^2\right)+c=0.$$

(2) Explicit equation (Fig. 1):

$$z=\sqrt{\frac{x^2+y^2-b/2\pm\sqrt{D}}{1+a}-(x^2+y^2)},$$

where $D=\left(x^2+y^2-b/2\right)^2-(1+a)\left[2\left(x^2+y^2\right)^2-2x^2y^2+c\right]$.

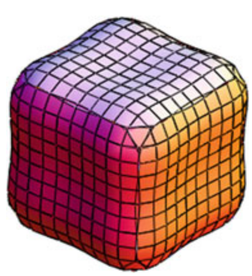
The mutual changing x, z or y, z gives the opportunity to obtain the alternative forms of the definition of the surface.

(3) Parametrical equations in the cylindrical system of coordinates (Fig. 2):

$$x=x(u,t)=t\cos u;$$
$$y=y(u,t)=t\sin u;$$
$$z=z(u,t)=\sqrt{\frac{t^2-b/2\pm\sqrt{D}}{1+a}-t^2},$$

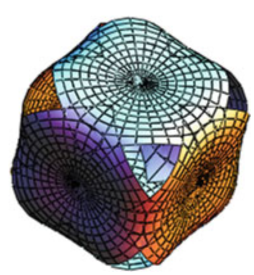
where

$$t^2=x^2+y^2,$$
$$D=\left(t^2-b/2\right)^2-(1+a)$$
$$\times\left[2t^4\left(1-\cos^2 u\sin^2 u\right)+c\right].$$



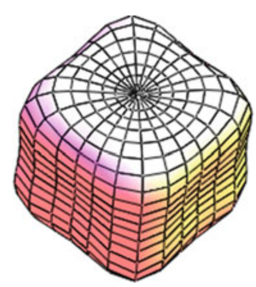
$-1.19 \le x; y \le 1.19;$

Fig. 1



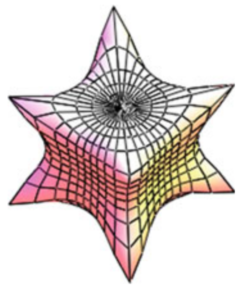
$a = 0; b = -1; c = -1;$
$0 \le t \le 1.313;$

Fig. 2



$0 \le u; v \le 2\pi$

Fig. 3



$c = -3; b = 0.2;$
$a = -0.33;$

Fig. 4

(4) Vector equation:

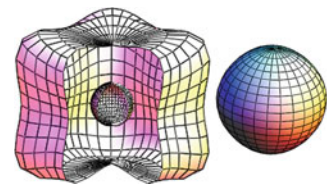
$$\boldsymbol{r} = \boldsymbol{r}(u,t) = t\boldsymbol{h}(u) + z(u,t)\boldsymbol{k};$$
$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u.$$

(5) Parametrical equations in the spherical system of coordinates (Figs. 3, 4 and 5):

$$\begin{aligned} x &= x(u,v) = R(u,v)\cos v\cos u;\\ y &= y(u,v) = R(u,v)\cos v\sin u;\\ z &= z(u,v) = R(u,v)\sin v, \end{aligned}$$

where $x^2 + y^2 + z^2 = R^2$; u is a polar angle in the plane xOy; v is the angle read from the axis Oz in the direction of the plane xOy; $p = \pm 1$;

$$R(u,v) = \sqrt{\frac{-b + p\sqrt{b^2 - 4c[a + D(u,v)]}}{2(a + D(u,v))}};$$



$c = 0.1;\ a = 0;\ b = -1$

Fig. 6

$$D(u,v) = \left(\sin^4 u + \cos^4 u\right)\cos^4 v + \sin^4 v.$$

One, two, or neither one real branches of the surface exist depending on the values of the parameters a, b, and c.

For example, when $a = 0,\ b < 0,\ c \le 0$, or when $a = 0, b > 0, c < 0$, or when $a > 0,\ b > 0,\ c < 0$, one branch of the surface exists; but when $a = 0,\ b < 0, 0 < c < (b/2)^2$ (Fig. 6) and when $b < 0,\ a > -0.3333;\ 0 < c < (b/2)^2/(a + 0.3333)$ (Fig. 5), two branches of the surface will be.

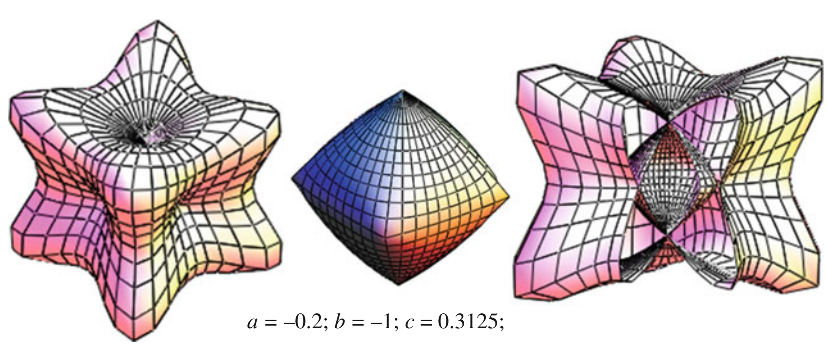
(6) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = R(u,v)e(u,v);$$
$$\boldsymbol{e}(u,v) = \boldsymbol{k}\cos v + h(u)\sin v;$$
$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u.$$

Here, the Goursat's surface is related to the curvilinear nonorthogonal, and nonconjugate coordinates u, v.

Additional Literature

Goursat E. Étude des surfaces qui admettent tours les plans de symétrie d'un polyèdre régulier. Ann. Sci. École Norm. Sup. 1897; 5, p. 159-200.



$a = -0.2; b = -1; c = 0.3125;$

$p = 1;$ $p = -1;$ $p = \pm 1$

Fig. 5

■ Tooth Surface

The tooth surface is a special case of *the Goursat's surface* (see also "Goursat's surface"). It is necessary to substitute $a = c = 0$, $b = -1$ into the formulas of the Goursat's surface, if one wants to obtain the tooth surface. So, the tooth surface is a closed algebraic surface of the fourth order consisting of one space and one isolated point (center of the surface). The surface is symmetrical relatively to all three coordinate planes (Fig. 1).

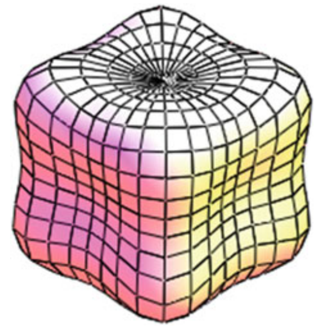


Fig. 1

Forms of definition of the tooth surface

(1) Implicit equation:

$$x^4 + y^4 + z^4 - \left(x^2 + y^2 + z^2\right) = 0.$$

(2) Explicit equation in the Cartesian system of coordinates:

$$z = \pm\sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - (x^4 + y^4) + (x^2 + y^2)}}.$$

The mutual changing x, z or y, z gives the opportunity to obtain the alternative forms of the definition of the surface.

(3) Parametrical equations in the cylindrical coordinates:

$$x = x(u,t) = t\,\cos u; \quad y = y(u,t) = t \sin u;$$
$$z = z(u,t) = \sqrt{1/2 \pm \sqrt{D}},$$

where

$t^2 = x^2 + y^2; D = \left(t^2 + 1/2\right)^2 - 2t^4\left(1 - \cos^2 u \sin^2 u\right).$

(4) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u,t) = t\boldsymbol{h}(u) + z(u,t)\boldsymbol{k};$$
$$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u.$$

(5) Parametrical equations in the spherical system of coordinates (Fig. 1):

$$x = x(u,v) = R(u,v)\cos v \cos u;$$
$$y = y(u,v) = R(u,v)\cos v \sin u;$$
$$z = z(u,v) = R(u,v)\sin v,$$

where $x^2 + y^2 + z^2 = R^2$; u is a polar angle in the plane xOy; v is the angle read from the axis Oz in the direction of the plane xOy;

$$R(u,v) = \pm\sqrt{1/D};$$
$$D(u,v) = \left(\sin^4 u + \cos^4 u\right)\cos^4 v + \sin^4 v$$

and $R(u,v) \equiv 0$. The surface is given in curvilinear, nonorthogonal, and nonconjugate coordinates u, v and consists of the closed space (Fig. 1) and the isolated point (0, 0, 0).

(6) Vector equation:

$$\boldsymbol{r} = \boldsymbol{r}(u,v) = R(u,v)e(u,v);$$
$$\boldsymbol{e}(u,v) = \boldsymbol{k}\cos v + \boldsymbol{h}(u)\sin v;$$
$$h(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u.$$

■ Tanglecube

The tanglecube is a special case of *the Goursat's surface* (see also "Goursat's surface"). It is necessary to substitute $a = 0$, $b = -5$, $c = 11.8$ into the formulas of the Goursat's surface, if one wants to obtain the tanglecube. The surface is symmetrical relatively to all three coordinate planes.

Forms of definition of the tanglecube

(1) Implicit equation:

$$x^4 + y^4 + z^4 - 5\left(x^2 + y^2 + z^2\right) + 11.8 = 0.$$

(2) Parametrical equations in spherical system of coordinates (Fig. 1):

$$x = x(u,v) = R(u,v)\cos v \cos u;$$
$$y = y(u,v) = R(u,v)\cos v \sin u;$$
$$z = z(u,v) = R(u,v)\sin v,$$

and

$$x^2 + y^2 + z^2 = R^2;$$

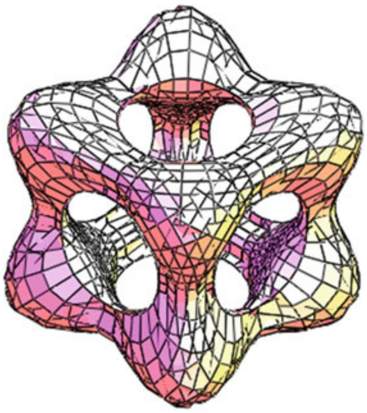


Fig. 1

u is a polar angle in the plane xOy; v is the angle read from the axis Oz in the direction of the plane xOy; $p = \pm 1$;

$$R(u,v) = \sqrt{\frac{5 + p\sqrt{25 - 47{,}2D(u,v)}}{2D(u,v)}};$$
$$D(u,v) = \left(\sin^4 u + \cos^4 u\right)\cos^4 v + \sin^4 v.$$

In Fig. 1, the surface with $0 \le u \le 2\pi$; $0.18\pi \le v \le 0.32\pi$ is shown.

Figure 2 represents the drawings of the tanglecube taken without any changing in the site: http://www.omnigraphic.com.

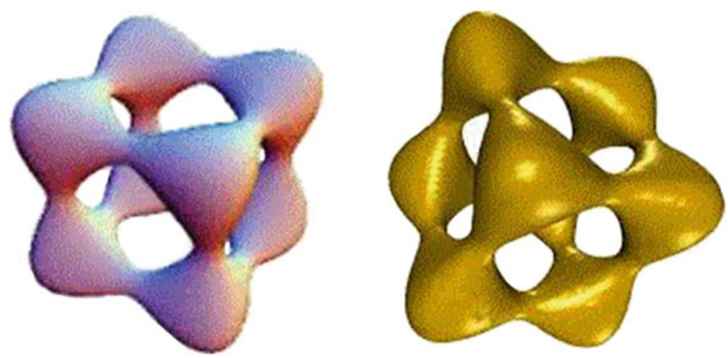


Fig. 2

■ Kummer Surface

The Kummer surface is *an algebraic surface of the fourth order* and of the class and it is self-mutual. The majority of surfaces from a family of the Kummer surfaces consist of 16 usual double points, that is maximal possible for the quartic surfaces. It was discovered by Kummer in 1864.

Forms of definition of the Kummer surface

(1) Implicit equation:

$$\left(x^2 + y^2 + z^2 - \mu^2 w^2\right)^2 - \lambda pqrs = 0,$$

where

$$\lambda = \frac{3\mu^2 - 1}{3 - \mu^2}, \quad \mu = \text{const};$$
$$p = w - z - \sqrt{2}x; \quad q = w - z + \sqrt{2}x;$$
$$r = w + z + \sqrt{2}y; \quad s = w + z - \sqrt{2}y.$$

(2) Implicit equation:

$$\left(x^2 + y^2 + z^2 - \mu^2 w^2\right)^2 - \lambda\left[(z - w)^2 - 2x^2\right] \times \left[(z + w)^2 - 2y^2\right] = 0.$$

(3) Parametrical equations in the polar-cylindrical system of coordinates:

$$x = x(u,v) = r(u,v)\cos u;$$
$$y = y(u,v) = r(u,v)\sin u;$$
$$z = z(v) = v;$$

where

$$r(u,v) = \sqrt{\frac{-b \pm \sqrt{b^2 - ac}}{a}},$$
$$a(u,v) = 1 - \lambda \sin^2 2u;$$
$$c(u,v) = \left(z^2 - \mu^2 w^2\right)^2 - \lambda\left(z^2 - w^2\right)^2;$$

$$b(u,v) = z^2 - \mu^2 w^2 + \lambda\Big[(v-w)^2 \cos^2 u + (v+w)^2 \sin^2 u\Big].$$

Assume w = 1; μ^2 = 1/3, then λ = 0, and the Kummer surface is *a double sphere* with a radius $R = 1/\sqrt{3}$ (Fig. 1).

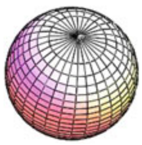


Fig. 1

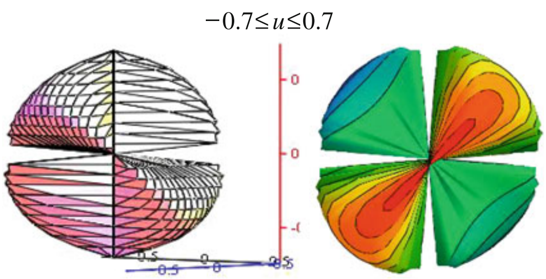


Fig. 2

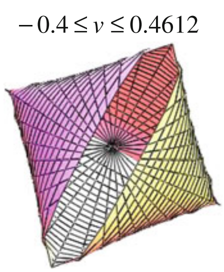


Fig. 3

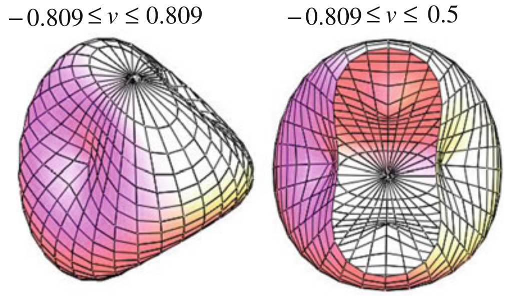


Fig. 4

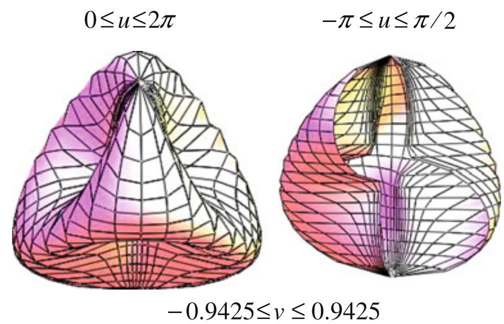


Fig. 5

Assume w = 2/3; μ^2 = 1, then λ = 1, and the Kummer surface degenerates into *a Roman surface* (Fig. 2).

In Fig. 3, a fragment of the Kummer surface with $w = \sqrt{2}$; $\mu^2 = \sqrt{3}$ (λ = 3,3094) is shown.

Assume w = 1; μ^2 = 0.5, then λ = 0.2, and the Kummer surface has the form shown in Fig. 4. Figure 5 represents the surface with $w = 1;\ \mu^2 = 0.8(\lambda = 0.636)$.

In Fig. 6, the surface has $w = 1;\ \mu^2 = 0.95(\lambda = 0.924)$.

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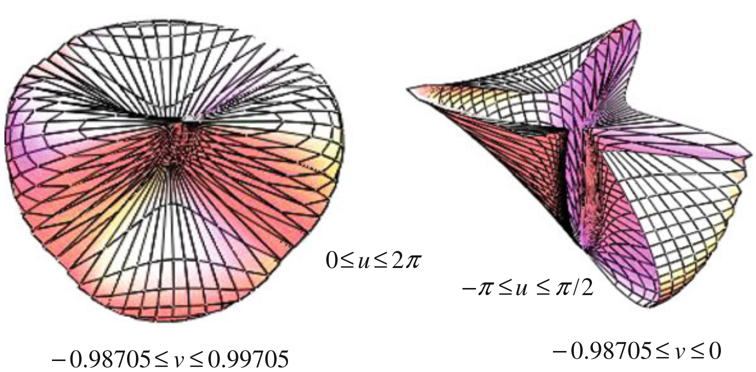


Fig. 6

Kummer E. Über die Flächen vierten Grades mit sechszech sihgulären Punkten. Ges. Werke. 1864; 2, p. 418-432.

■ Quartic Surface with Parabola, Ellipse, Parabola in Three Principal Coordinate Sections

The surface in question has a parabola

$$y = B/2 - Bx^2/\left(2L^2\right)$$

in the cross-section by the plane xOy ($z = 0$), an ellipse

$$4y^2/B^2 + z^2/T^2 = 1$$

at the cross-section by the plane yOz ($x = 0$) and a parabola

$$z = -T + Tx^2/L^2$$

at the cross-section by the plane xOz ($y = 0$). Here T is *a drought* of the surface, B is its maximum width along the axis Oy, and $2L$ is its length along the axis Ox (Fig. 1).

Forms of definition of the surface

(1) Implicit equation:

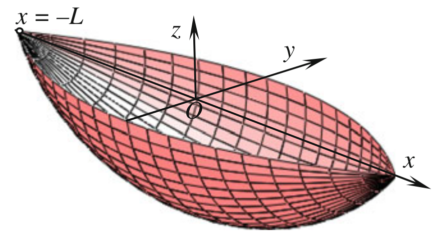


Fig. 1

$$\frac{4y^2}{\left(B - \frac{Bx^2}{L^2}\right)^2} + \frac{z^2}{\left(\frac{Tx^2}{L^2} - T\right)^2} = 1.$$

The surface is formed by a family of the ellipses lying at the planes parallel to the coordinate plane yOz.

(2) Implicit equation:

$$4T^2y^2 + B^2z^2 = T^2B^2\left(1 - x^2/L^2\right)^2.$$

It is seen from the last formula that the surface is *a quartic surface*.

■ Quartic Surface with the 4th Order Curve, Ellipse, the 4th Order Curve in Three Principal Coordinate Sections

The surface in question has a curve of the 4th order

$$y = \pm\frac{8Bx}{3\sqrt{3}L^2}\sqrt{Lx - x^2}$$

at the cross-section by the plane xOy ($z = 0$), an ellipse

$$4y^2/B^2 + z^2/T^2 = 1$$

at the cross-section $x = 3L/4$, and a curve of the 4th order

$$z = \pm\frac{16Tx}{3\sqrt{3}L^2}\sqrt{Lx - x^2}$$

at the cross-section of the surface by the plane xOz ($y = 0$). Here T is a drought of the surface along the axis Oz at the cross-section $x = 3L/4$, B is its maximum width along the axis Oy at the cross-section $x = 3L/4$, and L is its length along the axis Ox.

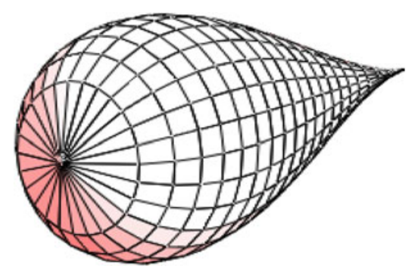


Fig. 1

Forms of definition of the surface

(1) Implicit equation:

$$256B^2T^2x^3(x - L) + 108T^2L^4y^2 + 27B^2L^4z^2 = 0,$$

where $0 \le x \le L$; $-B/2 \le y \le B/2$; $-T \le z \le T$. The surface is a quartic surface. It is formed by a family of the ellipses lying at the planes parallel to the coordinate plane yOz (Fig. 1).

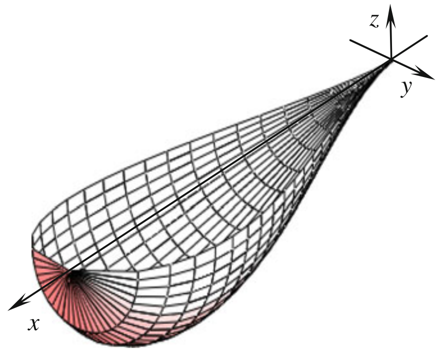


Fig. 2

(2) Parametrical equations (Fig. 2):

$$x = x;$$

$$y = y(x, \alpha) = \frac{8Bx\sqrt{Lx - x^2}}{3\sqrt{3}L^2}\cos\alpha,$$
$$z = z(x, \alpha) = \frac{16Tx\sqrt{Lx - x^2}}{3\sqrt{3}L^2}\sin\alpha,$$

where $0 \le x \le L$; α is an angular parameter; $0 \le \alpha \le 2\pi$. The surface has a conical point with the coordinates (0; 0; 0).

References

Avdon'ev EYa, Protod'yakonov SM. Equations and characteristics of some algebraic surfaces of higher others. Prikl. Geometriya i Inzenernaya Grafika. Kiev. 1976; Iss. 21, p. 108-120 (2 refs).

Avdon'ev EYa, Protod'yakonov SM. Using of the algebraic curves of the high orders for design of the hydro-aerodynamics profile. Prikl. Geometriya i Inzenernaya Grafika. Kiev. 1974; Iss. 18, p. 111-114 (4 refs).

■ Pillow Shape

The Pillow Shape is a special case of an *elliptical velaroid* (see also a Sect. "3.4. Velaroidal Surfaces").

Forms of definition of the surface

(1) Parametric form of assignment:

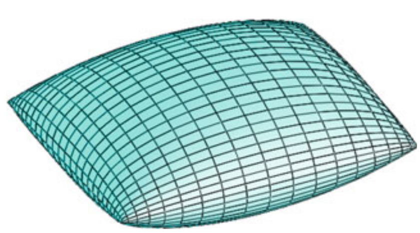
$$x = x(u) = a\cos u,$$
$$y = y(v) = a\cos v,$$
$$z = z(u, v) = f\sin u\sin v,$$

where $0 \le u \le 2\pi$; $0 \le v \le 2\pi$.

The Pillow Shape is a closed surface, its projection on the plane xOy is a square with the sides equal to $2a$ (Figs. 1 and 2).

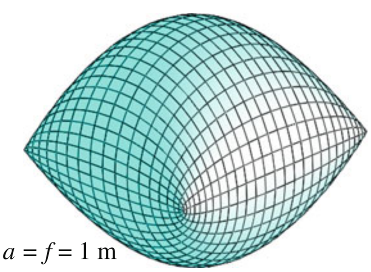
(2) Implicit equation:

$$z^4a^4 - f^2\left(a^2 - x^2\right)\left(a^2 - y^2\right) = 0.$$



$a = 1$ m; $f = 0.5$ m

Fig. 1



$a = f = 1$ m

Fig. 2

Hence, this is an algebraic surface of the fourth order.

(3) Explicit equation:

$$z = \pm\frac{f}{a^2}\sqrt{(a^2 - x^2)(a^2 - y^2)} = 0.$$

At the cross-sections of the surface by the planes $x = \text{const}$ $(-a \le x \le a)$ or $y = \text{const}$ $(-a \le y \le a)$, ellipses are placed. The cross-sections by the plane xOy are four fragments of the straight lines forming the square.

Additional Literature

Parametrische Flächen und Körper: http://www.3d-meier.de/tut3/Seite46.html

Alexandrov AD. Intrinsic geometry of convex surfaces (in Russian). M.-L.: Gostekhizdat, 1948; English translation in Selected Works. Part 2, CRC Press, Boca Raton, FL. 2005.

■ Additional Information on Surfaces of the 4th Order

The algebraic equation of degree 4 contains 35 factors (the a_{ij} coefficients). The type of a surface is certain by conditions at which numerical values of factors of the equation define the concrete geometrical form of a surface (see also the Chap. "36. Algebraic Surfaces of The High Order"). Surfaces of the fourth order were by the subject of research of many mathematicians.

The ruled surfaces of the fourth order have been rather well studied to a considerable extent both by analytical and by synthetic processes. Holgate has supposed that the most important and complete analytical treatment of the surface was by Cayley in his second and third "Memoirs on Skew Surfaces, otherwise Scroll" (Philosophical Transaction, 1864 and 1869, respectively). Additional information on the 4th order torse surfaces can be taken in "The third and fourth order developable surfaces."

Gordana Đukanović and Marija Obradović show a spatial model of the cone and harmonically equivalent surfaces of the 4th order (spindle-shaped cyclide) which was performed by using two systems of circular cross-section. They also presented a harmonic equivalent of cylinder that is Dupin cyclide, parabolic cylinder and its harmonic equivalent that is the 4th order surface with a cusp; the 4th order surface obtained as a harmonic equivalent of a revolving paraboloid.

Adam Logan et al. researched diagonal quartic surfaces defined by the equation $ax^4 + by^4 + cz^4 + dw^4 = 0$ for nonzero a, b, c, and d .

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Utkin GA. Design of a surface of the fourth order consisting of ten components. Izv. Vuzov. Matematika. 1977; No. 2, p. 116–117.
Logan A, Mckinnon D., Van Luijk R. Density of rational points on diagonal quartic surfaces. 2009; arXiv:0812.4779 [math.AG].

36.2.1 The 4th Order Surfaces with Multiply Lines

The 4th order surfaces with multiply lines in three dimensional Euclidian space in the Cartesian system of coordinates $(x : y : z : w); x, y, z \in E^3; w \in \{0, 1\}$, $(x : y : z : w) \neq (0 : 0 : 0 : 0)$ are defined by the homogeneous equations $F^4(x, y, z) = 0$ of degree 4. The 4th order surfaces (*quartics*) may have the following multiple lines:

(1) one triple straight line; (2) one double twisted cubic; (3) one double conic and one double straight line; (4) one double conic; (5) three double straight lines; (6) two double straight lines; and (7) one double straight line.

So, seven types of the surfaces of the 4th order with the multiply lines exist.

The 4th order surfaces with one triple straight line are ruled surfaces. Every cross-section of the surface by the plane passing through the triple straight line contains this line and other straight line. If the triple line coincides with the axis Oz ($x = 0$, $y = 0$), then the equation of the ruled surface of the fourth order may be written in the form:

$$\mathbf{u}_4 + z\mathbf{u}_3 + w\mathbf{v}_3 = 0,$$

where $\mathbf{u}_4$, $\mathbf{u}_3$, and $\mathbf{v}_3$ are homogeneous polynomials in x and y of degree 4, 3, and 3, respectively. Sturm has offered to divide these surfaces in four classes (IX, X, XI, XII), which differ by the number and by the kind of torsal lines.

The quartic with a double twisted cubic curve are always ruled surfaces. They belong to the classes III and IV according to classification of Sturm. At the cross-section of the surface by the plane, the plane curve of the fourth order with three double points lies. The director line is a twisted cubic curve.

The quartics with a double conic and a double straight line are also *ruled quartics* and belong to the classes V and VI. The director curves are two conics intersecting at two points and a straight line which cuts one of them.

If a double line of the quartic is a conic, then the surface is a nonruled surface of *the 4th order with the double conic*. A double conic can be real or imaginary. Cyclides, that are the most famous quartics, belong to that type of the surfaces and the absolute conic

$$x^2 + y^2 + z^2 = 0, \quad w = 0$$

is their double curve. Dupin's cyclides of the first type are the surfaces enveloping a family of the spheres tangent to three given spheres (see also "Dupin's cyclides of the first type (of the fourth order)" in a Subsect. "17.1.2. Dupin cyclides"). *Pedal surface* of the surface Φ for the pole P is the locus of the feet of perpendiculars drown from any fixed point P, called the pole, to the tangent planes of the n-class surface Φ. The perpendicular passes n times through the absolute conic and the pole P. The pedal surfaces of central surfaces of the 2nd order are cyclides with a double point in the pole. The pedal surfaces of sphere, one-sheeted and two-sheeted hyperboloids, are surfaces of revolution of the pedal curves of a circle and a hyperbola. The cyclides with the triple point are given in the Section "Cyclides with a triple point".

The quartics with tree double straight lines can have three double straight lines in two cases. The ruled surfaces of the 4th order (the ruled quartics) with three double lines belong to the class VII according to the classification of Sturm. Here, the director curves are a conic and two straight lines without common points. Non-ruled Steiner surfaces of the 4th order are the second modification of the surfaces of the 4th order with three double straight lines intersecting in one point. They are the reciprocals of four-nodal cubic surfaces.

The quartics with two double straight lines are given in "Quartics with double straight line".

The quartics with a double straight line are divided into two types: the pedal surfaces of (1, 2)-congruences of the rays and the surfaces with double straight line and a triple point (see also "Quartic surface with a double straight line and a triple point").

Reference

Gorjanc Sonja. Quartics with multiple lines in E^3. Proc. of the 10th International Conference on Geometry and Graphics. Ukraine, Kyiv, 2002, July 28 – August 2. Kyiv, 2002; Vol. 1, p. 48-52 (10 refs).

■ Quartic Surface with a Triple Straight Line

Quartic surface with a triple straight line is ruled surface. A triple line is the highest singularity which a quartic can possess. Every plane cross-section of the surface through the triple line consists of that line counted thrice and another straight line.

If the triple line is the axis Oz ($x = 0$, $y = 0$), the equation of the ruled quartic may be written in the form:

$$\mathbf{u}_4 + z\mathbf{u}_3 + w\mathbf{v}_3 = 0,$$

where $\mathbf{u}_4$, $\mathbf{u}_3$, and $\mathbf{v}_3$ are homogeneous polynomials in x and y of degree 4, 3, and 3 accordingly. According to it, there are four classes (IX, X, XI, XII) of ruled quartics with a triple line under the classification of Sturm (Müller and Krames 1931). These groups differ by the number and kind of torsal lines.

Gorjanc Sonja (2002) presents two figures of the surfaces of this type and gives equations of these ruled surfaces (1997). The surfaces represented in these drawings contain two directing nonintersecting conics and one triple straight line which intersects every conic at one point.

Additional Literature

Müller E, Krames JL. Konstruktive Behandlung der Regelflächen. Franc Deuticke, Leipzig und Wien, 1931.
Gorjanc Sonja. Quartics with multiple lines in E^3. Proc. of the 10th International Conference on Geometry and Graphics. Ukraine, Kyiv, 2002, July 28 – August 2. Kyiv, 2002; Vol. 1, p. 48-52 (10 refs).
Gorjanc Sonja. Izvodenje pet tipova pravčastih ploha 4. stupnja, KoG, 1997; No. 2, p. 57-67.

■ Quartic Surface with a Double Conic and with a Double Straight Line

The quartics with a double conic and a double straight line are also *ruled quartics* and belong to the classes V and VI. The director curves are two conics intersecting at two points and a straight line which cuts one of them. Gorjanc Sonja (2002) represented the drawing of one surface in question of the type V with two directing circles in the parallel planes and a double straight line intersecting the double circle.

Reference

Gorjanc Sonja. Quartics with multiple lines in E^3. Proc. of the 10th International Conference on Geometry and Graphics. Ukraine, Kyiv, 2002, July 28 – August 2. Kyiv, 2002; Vol. 1, p. 48-52 (10 refs).

■ Quartic Surface with Two Double Straight Lines

Quartic surface with two double straight lines is formed by the congruence of the straight lines passing through the circle $x^2 + y^2 = r^2$, $z = 0$ and the straight line $z = a$, $y = 0$.

Forms of definition of the surface

(1) Parametrical equations (Simenko 2005):

$$x = x(w,t) = \frac{r}{a}\sin w + r\left(\cos w - \frac{\sin w}{a}\right)t,$$
$$y = y(w,t) = rt\sin w,$$
$$z = z(t) = a(1-t),$$

where $-\infty \le t \le \infty$; $0 \le w \le 2\pi$. In Fig. 1, the surface in question is shown when $a = 1.2$; $r = 1$; $-0{,}5 \le t \le 1{,}2$; $0 \le w \le 2\pi$.

(2) Implicit equation:

$$(a-z)^4 r^2 - (a-z)^2(x^2+y^2)a^2 + 2axyz(a-z) - y^2z^2 = 0.$$

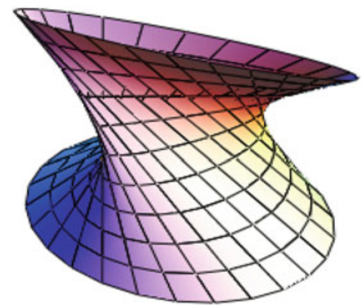


Fig. 1

Additional Literature

Gorjanc Sonja. Quartics with multiple lines in E^3. Proc. of the 10th International Conference on Geometry and Graphics. Ukraine, Kyiv, 2002, July 28 – August 2. Kyiv, 2002; Vol. 1, p. 48-52 (10 refs).
Simenko OV. Forming of the surfaces by the projections of lines of the linear parabolic congruences. Geometrichne ta kompyuterne modelyuvannya: Zb. Nauk. Pr, Harkiv. 2005; Iss. 9, p. 45-52 (6 refs).

■ Quartics with a Double Conic

The general information on the surfaces of the 4th order is given in a Subsect. “36.2.1. The 4th Order Surfaces with Multiply Lines”.

■ Cyclides with a Triple Point

Cyclides with a triple point belong to algebraic surfaces of the fourth order and they may be given by the homogeneous equation

$$(x^2+y^2+z^2)^2 + w\mathbf{u}_3 = 0$$

where $u_3 = 0$ is a homogeneous equation in x, y and z of degree 3 and represents the tangent cone in the triple point (0; 0; 0; 1).

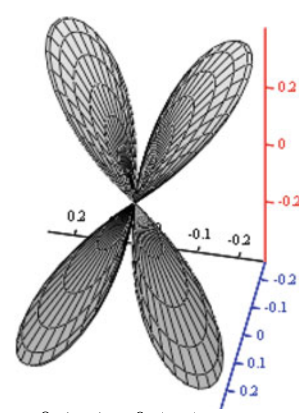
Forms of definition of the surface

(1) Implicit form of assignment:

$$(x^2+y^2+z^2)^2 + 2xyz = 0.$$

The tangent cone of the degree 3 in the point (0; 0; 0; 1) degenerates into three real planes $x = 0$; $y = 0$; and $z = 0$.

(2) Parametrical equations in the polar coordinates (Figs. 1, 2 and 3):



$0 \le v \le \pi; 0 \le u \le \pi$

Fig. 1

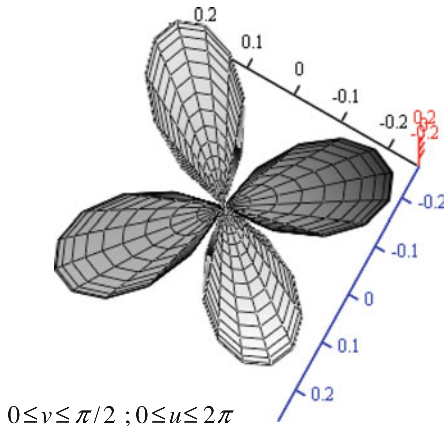


Fig. 2

$$x = x(u, v) = \cos v \sin^3 v \sin 2u \cos u,$$
$$y = y(u, v) = \cos v \sin^3 v \sin 2u \sin u,$$
$$z = z(u, v) = -\cos v \sin^2 v \sin 2u.$$

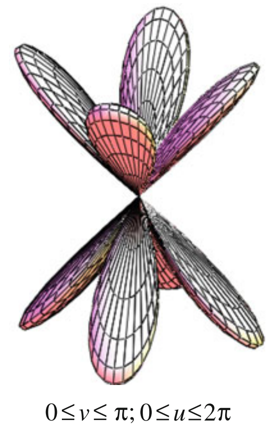


Fig. 3

In Figs. 1, 2 and 3, the cyclides with the triple point designed with the different boundaries of the changing of parameters u and v are shown.

■ Quartics with Three Double Straight Lines

Quartics with three double straight lines can be divided into two groups. The first group contains the ruled surfaces containing two real double straight lines and one double straight line placed at infinity. The second group contains the non-ruled Steiner surfaces with three real double straight lines crossing at one point.

■ Ruled Surfaces of the 4th Order with Three Double Lines

The Wallis's conical edge (see also a Subsect. "1.2.1. Catalan Surfaces") and *the ruled rotational surface of Lusta* (see also "Ruled Rotational Surface of Lusta" in a Subsect. "34.1.2. Rotational surfaces with axoids "cylinder—cylinder"") can be related to *the ruled surfaces of the 4th order with three double lines*.

Forms of the definition of the Wallis's conical edge

(1) Implicit equation of the Wallis's conical edge:

$$c^2b^2x^2 - \left(a^2c^2 - z^2\right)\left(x^2 + y^2\right) = 0,$$

where a, b, and c are constants; $c\sqrt{a^2 - b^2} \le z \le ac$, where $a > b$.

At the cross-sections of the surface by the planes $z = z_o = \text{const}$, two intersecting straight lines

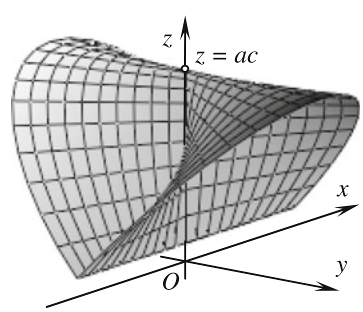


Fig. 1 $a = 2$, $b = 1$, $c = 5$

$$y = \pm\sqrt{\frac{c^2b^2 - a^2c^2 + z_o^2}{a^2c^2 - z_o^2}}\,x$$

are placed and a curve

$$z = c \cdot \sqrt{a^2 - \frac{b^2x_o^2}{x_o^2 + y^2}}$$

with an asymptote $z = ac$ lies at the cross-sections of the surface by the planes $x = x_o$ = const. At the plane $z = ac$, the third double straight line $y = \infty$ is disposed.

(2) Parametrical equations of the Wallis's conical edge (Figs. 1 and 2):

$$x = x(u, v) = v\cos u,\ y = y(u, v) = v\sin u,$$
$$z = c\sqrt{a^2 - b^2\cos^2 u},$$

where $0 \le u \le 2\pi$; $0 \le v \le \infty$; a, b, c are constants; $c\sqrt{a^2 - b^2} \le z \le ac$; $a > b$.

Coefficients of the fundamental forms of the surface

$$A^2 = \frac{c^2b^4}{4}\frac{\sin^2 2u}{a^2 - b^2\cos^2 u} + v^2, \quad B = 1, \quad F = 0,$$
$$L = -\frac{cb^2}{4}\frac{4\cos 2u(a^2 - b^2\cos^2 u) - b^2\sin^2 2u}{(a^2 - b^2\cos^2 u)^{3/2}}\frac{v}{A},$$
$$M = \frac{cb^2}{2A}\frac{\sin 2u}{\sqrt{(a^2 - b^2\cos^2 u)}}, \quad N = 0.$$

The surface is given in orthogonal nonconjugate system of coordinates u, v.

(3) Vector equation of the Wallis's conical edge:

$$\boldsymbol{r}(u, v) = v\boldsymbol{h}(u) + c\sqrt{\varphi}\boldsymbol{k},$$

$\varphi = c^2 - b^2\cos^2 u$, $\varphi' = b^2\sin 2u$, $\boldsymbol{h} = \boldsymbol{i}\cos u + \boldsymbol{j}\sin u$, $\boldsymbol{h}' = \boldsymbol{n}$, $\boldsymbol{n}' = -\boldsymbol{h}$, $\boldsymbol{h}$, $\boldsymbol{n}$, and $\boldsymbol{k}$ are orthogonal unit vectors, $\boldsymbol{r_u} = v\boldsymbol{n} + \psi\boldsymbol{k}$, $\boldsymbol{r}_v = \boldsymbol{h}$,

$$\psi = \left(c\sqrt{\varphi}\right)' = \frac{cb^2}{2}\frac{\sin 2u}{\sqrt{\varphi}},$$
$$\boldsymbol{m} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_{vu}}{\sigma} = \frac{\psi\boldsymbol{n} - v\boldsymbol{k}}{\sigma}, \ \sigma = A, \text{ so}$$
$$A^2 = \psi^2 + v^2,\ B = 1,\ F = 0;$$

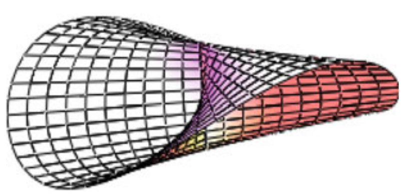


Fig. 2 $a = 5, b = 1, c = 5$

$$L = -v\frac{\psi'}{A},\ M = \frac{\psi}{A},\ N = 0.$$

Forms of the definition of the ruled rotational surface of Lusta

(1) Implicit equation:

$$\frac{x^2}{(R - Rz/H)^2} + \frac{y^2}{(Rz/H)^2} = 1,$$

where R and H are constants.

(2) Parametrical equations (Fig. 3):

$$x = x(u, v) = R(1 - u/H)\cos v,$$
$$y = y(u, v) = (uR/H)\sin v,$$
$$z = u$$

Gorjanc considers that the third double line lies at the infinity on the horizontal plane. She affirms that this double line joins the points of intersection of two director straight lines with the plane of a directing conic.

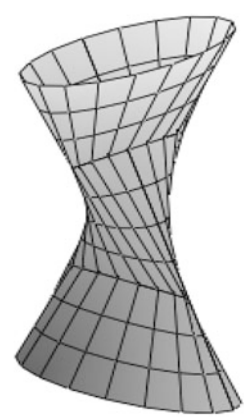


Fig. 3

Additional Literature

Gorjanc Sonja. Quartics with multiple lines in E^3. Proc. of the 10th International Conference on Geometry and Graphics. Ukraine, Kyiv, 2002, July 28 – August 2. Kyiv, 2002; Vol. 1, p. 48-52 (10 refs).

■ Steiner Surfaces of the First and Second Types

According to the classification of Coffman, *Steiner's surfaces of the first and second types* have three double straight lines intersecting at one point. These surfaces are the algebraic surfaces of the 4th order.

Forms of definition of the Steiner surface of the first type

(1) Implicit equation of the Steiner surface of the first type:

$$y^2z^2 + z^2x^2 + x^2y^2 - 2kxyz = 0.$$

The Steiner surface of the first type is called also *the Roman surface* (see also the Chap. "18. One-sided surface"). Under this method of the definition, the coordinate lines x, y, and z coincide with three double lines (Fig. 1).

(2) Parametrical equations of the Roman surface:

$$x = x(u, v) = \frac{1}{2}\sin 2u \sin^2 v,$$
$$y = y(u, v) = \frac{1}{2}\sin u \cos 2v,$$
$$z = z(u, v) = \frac{1}{2}\cos u \sin 2v$$

where $0 \le u \le 2\pi$; $-\pi/2 \le v \le \pi/2$ (Fig. 1).

Forms of definition of the Steiner surface of the second type

(1) Implicit equation of the Steiner surface of the second type:

$$y^2z^2 - z^2x^2 + x^2y^2 - 2kxyz = 0.$$

(2) Explicit equation:

$$y = \frac{xz}{x^2 + z^2}\left(k \pm \sqrt{k^2 + x^2 + z^2}\right).$$

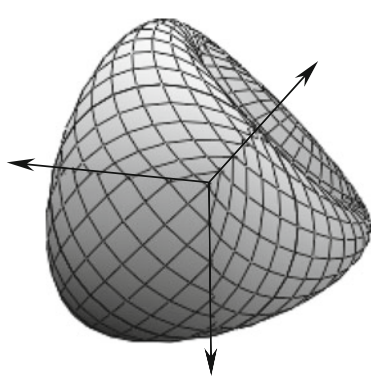


Fig. 1

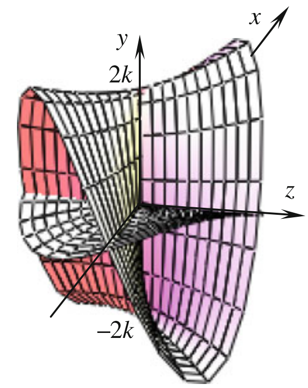


Fig. 2

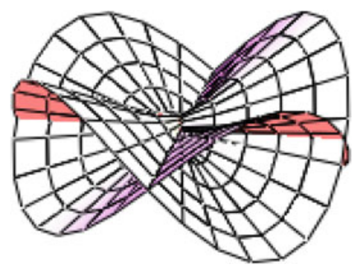


Fig. 3

(3) Parametrical equations (Fig. 2):

$$x = x(r, v) = r\cos v,$$
$$y = y(r, v) = \sin v \cos v\left(k \pm \sqrt{k^2 + r^2}\right),$$
$$z = z(r, v) = r\sin v,$$

The coordinate axes x, y, and z coincide with three double straight lines of the surface. In Fig. 2, the surface with $k = 0.5$ m; $0 \le v \le 2\pi$; $0 \le r \le 0.5$ m is shown. The surface given in Fig. 3 has $k = 0.5$ m; $0 \le v \le 2\pi$; $0 \le r \le 5$ m.

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{r^2\sin^2 2v}{4(k^2 + r^2)},$$
$$F = \frac{r\sin 4v}{4}\left(1 \pm \frac{k}{\sqrt{k^2 + r^2}}\right),$$
$$B^2 = r^2 + \left(k \pm \sqrt{k^2 + r^2}\right)^2\cos^2 2v;$$
$$A^2B^2 - F^2 = B^2 + \frac{r^4\sin^2 2v}{4(k^2 + r^2)};$$
$$L = \mp\frac{rk^2\sin 2v}{2\sqrt{A^2B^2 - F^2}(k^2 + r^2)^{3/2}},$$

$$M = \frac{k\cos 2v}{\sqrt{A^2B^2 - F^2}}\left(1 \pm \frac{k}{\sqrt{k^2 + r^2}}\right),$$
$$N = \frac{r\sin 2v}{\sqrt{A^2B^2 - F^2}}\left[\frac{\mp r^2}{2\sqrt{k^2 + r^2}} + 2\left(k \pm \sqrt{k^2 + r^2}\right)\right].$$

Additional Literature

Coffman A, Schwarts A, and Stanton C. The Algebra and Geometry of Steiner and Other Quadratically Parametrizable Surfaces. Computer Aided Geom. Designю 1996; 13, p. 257-286.

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Gray A. Modern Differential Geometry of Curves and Surfaces with Mathematica. CRC Press, Boca Raton. 1998, p. 331.

Bert Jüttler, Ragni Piene. Geometric Modeling and Algebraic Geometry. Springer. 2008; p. 30.

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■ Quartics with a Double Straight Line

Quartics with a double straight line are divided into *the pedal surfaces of the congruences of the rays* and *surfaces with a double straight line and a triple point*. If a double line of a surface of the fourth order is one straight line, then the quartic is a nonruled surface which contains simple infinity system of the conic lines in the plane going through the double line. In eight planes going through a double line, the conics degenerate into two lines and thus there are 16 simple lines on these surfaces of the fourth order.

The locus of the feet of the perpendiculars drawn from any fixed finite point P, called the pole, to the rays of an (n, m)-congruence is the pedal surface of this congruence for the pole P. If the congruence is of the first and the second class, then the pedal surface is a quartic with a double straight line and contains absolute conic as a simple curve. Besides the absolute conic, these surfaces contain a pair of lines at infinity. Gorjanc Sonja gives the classification of the pedal surfaces according to the number and kind of their singular points.

The second type of the surfaces of the fourth order with a double straight line is represented in a Section "Quartic surface with a double straight line and with a triple point".

References

Gorjanc Sonja. The classification of the pedal surfaces of (1, 2) congruences: Dissertation. Faculty of Natural Sciences, Department of Mathematics, University of Zagreb, 2000.

■ Quartic Surface with a Double Straight Line and a Triple Point

Quartic surface with a double straight line and with a triple point belongs to algebraic surfaces of the fourth order.

This surface is given by an equation:

$$\mathbf{u}_4 + w\mathbf{v}_3 = 0,$$

where $\mathbf{u}_4$ is a homogeneous polynomial of degree 4 in x and y while $\mathbf{v}_3 = 0$ is a homogeneous equation in x, y, and z and represents the tangent cone in the triple point (0: 0: 0: 1). The equations

$$\mathbf{u}_4 = 0, \quad w = 0$$

represent the section with the plane at infinity which breaks up into a conic and a pair of lines through the point (0: 0: 1: 0).

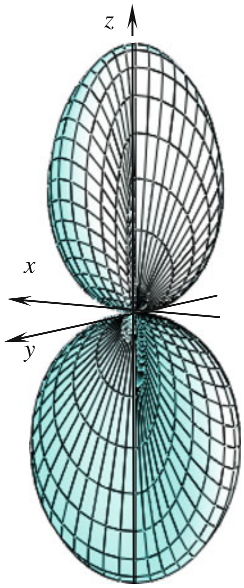
Forms of definition of the surface with a double straight line and a triple point

(1) Implicit equation:

$$\left(x^2 + y^2 + z^2\right)\left(x^2 + z^2\right) + 2z\left(x^2 - y^2\right) = 0.$$

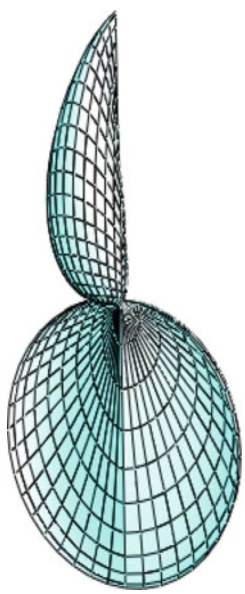
The third degree tangent cone at (0: 0: 0: 1) disintegrates into three real and different planes:

$$x = y; \quad x = -y; \quad z = 0.$$



$0 \le u \le 2\pi, 0 \le v \le \pi$

Fig. 1



$0 \le u \le \pi, \ 0 \le v \le \pi$

Fig. 2

$0 \le u \le 2\pi, 0 \le v \le \pi/2$

Fig. 3

(2) Parametrical equations in the polar coordinates (Fig. 1):

$$x = x(u, v) = \cos u \cos 2u \sin v,$$
$$y = y(u, v) = \sin u \cos 2u \sin v,$$
$$z = z(u, v) = (\cos v - 1) \cos 2u.$$

Figures 1, 2 and 3 show the surface designed for different boundary parameters *u, v*, which are given under the figures.

Additional Literature

Gorjanc Sonja. Quartics with multiple lines in E^3. Proc. of the 10th International Conference on Geometry and Graphics. Ukraine, Kyiv, 2002, July 28 – August 2. Kyiv, 2002; Vol. 1, p. 48-52 (10 refs).
Müller E, Krames JL. Konstructive Behandlung der Regelflächen. Franc Deuticke, Leipzig und Wien, 1931.
Gorjanc sonja. Izvodenje pet tipova pravcastih ploha 4. Stupnja. KoG. 1997; No. 2, p. 57-67.

36.3 Algebraic Quintic Surfaces

The encyclopedia contains the descriptions of six algebraic surfaces of the fifth order (*quantics*) dealing with mathematics and engineering. Two of them are given in this section and the rest are described in other parts.

It is necessary to mention togliatti who showed that quintic surfaces having 31 ordinary double points exist, although he did not explicitly derive equations for such surfaces. Quintic surfaces having 31 ordinary double points are therefore sometimes called *Togliatti surfaces*. In 1994, Barth derived an algebraic equation of one such surface that was called "Dervish".

Additional Literature

Togliatti EG. Una notevole superficie de 5^D ordine con soli punti doppi isolati. Vierteljschr. Naturforsch. Ges. Zürich. 1940; 85, p. 127-132.

■ Peninsula Surface

Peninsula surface is an algebraic surface of the 5th order (algebraic quintic surface) which is symmetrical relative to one of the coordinate planes.

Forms of definition of the Peninsula surface

(1) Implicit equation (Fig. 1):

$$x^2 + y^3 + z^5 = 1.$$

The surface is symmetrical relative to the coordinate plane yOz.

(2) Explicit equation (Fig. 1):

$$y = \sqrt[3]{1 - x^2 - z^5}.$$

(3) Explicit equation (Fig. 2):

$$z = \sqrt[5]{1 - x^2 - y^3}.$$

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{4x^2}{25[1 - x^2 - y^3]^{8/5}},$$
$$F = \frac{6xy^2}{25[1 - x^2 - y^3]^{8/5}},$$
$$B^2 = 1 + \frac{9y^4}{25[1 - x^2 - y^3]^{8/5}};$$

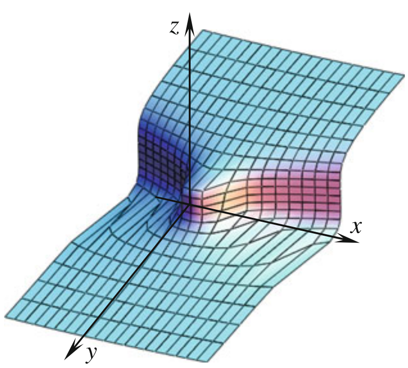


Fig. 1

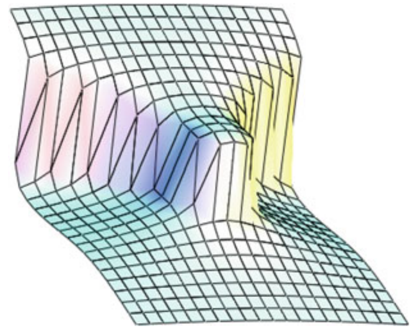


Fig. 2

$$A^2B^2 - F^2 = 1 + \frac{4x^2 + 9y^4}{25[1 - x^2 - y^3]^{8/5}} = A^2 + B^2 - 1;$$
$$L = \frac{2(5y^3 - 3x^2 - 5)}{25[1 - x^2 - y^3]^{9/5}\sqrt{A^2 + B^2 - 1}},$$
$$M = \frac{-24xy^2}{25[1 - x^2 - y^3]^{9/5}\sqrt{A^2 + B^2 - 1}}$$
$$N = \frac{6y(5x^2 - y^3 - 5)}{25[1 - x^2 - y^3]^{9/5}\sqrt{A^2 + B^2 - 1}};$$
$$K = \frac{12y(5y^3 - 3x^2 - 5)(5x^2 - y^3 - 5) - 24^2x^2y^4}{625[1 - x^2 - y^3]^{18/5}(A^2 + B^2 - 1)^2}.$$

The surface is given in the curvilinear, nonorthogonal, and nonconjugate coordinates x, y.

The plane curvilinear coordinate lines $x = 0$ and $y = 0$ lying in the coordinate planes yOz and xOz accordingly are the lines of principle curvatures.

Along the coordinate line $y = 0$, parabolic points with $K = 0$ are disposed.

In Fig. 1, the surface designed in the boundaries $-5 \le x \le 5$; $-3.5 \le z \le 3.5$ is shown. The contours of the Peninsula surface presented in Fig. 2 coincide with the coordinate lines $x = -8$ m; $x = 8$ m; $y = -6$ m and $y = 6$ m.

Additional Literature

Weisstein Eric W. Peninsula Surface. From MathWorld. A Wolfram Web Resource. http://mathworld.wolfram.com/PeninsulaSurface.htm

Weisstein, Eric W. CRC Concise Encyclopedia of Mathematics. CRC Press, 1999.

■ Parabolic Surface of Conoidal Type

A parabolic surface of conoidal type covers a rectangular plan, three straight sides of which coincide with the boundaries of the surface but the fourth boundary of the surface is a parabola.

The surface is formed by a family of parabolas lying in the planes which are parallel to the coordinate plane xOz. This coordinate plane is a plane of symmetry of the surface (Figs. 1, 2 and 3).

The surface in question can be used as a model of shell roof of factories of the machine-building profile with the net of the columns 6 × 12 m (Podgorniy and Hoang Zuy Thang 1978).

Forms of definition of the surface

(1) Implicit equation:

$$\left(\frac{a^2 z}{a^2 - y^2} - d\right)^2 + d^2\left(\frac{x}{b+c} - 1\right) = 0,$$

where $2a \times b$ are the dimensions of the surface at the plan.

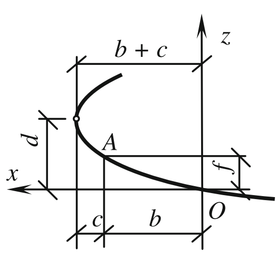


Fig. 1

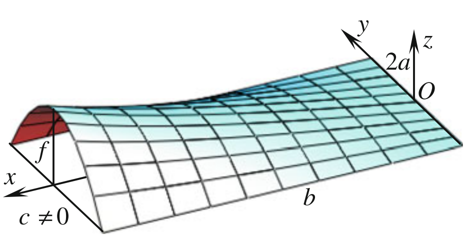


Fig. 2

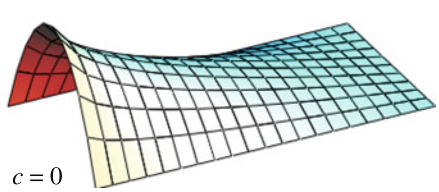


Fig. 3

At the cross-section $y = 0$, *the ridge parabola* (Fig. 1) lies:

$$(z-d)^2 = d^2\left(1 - \frac{x}{b+c}\right) = -\frac{(f-d)^2}{c}(x-b-c).$$

The equation of the ridge parabola is obtained due to a condition of the passing of the curve through the origin of the coordinates:

$$O(x = y = z = 0).$$

The ridge parabola must also go through the point A with the coordinates $x = b;\ \ y = 0;\ \ z = f$ (Fig. 1). This condition gives the following expression:

$$f = d\left(1 - \sqrt{\frac{c}{b+c}}\right), \quad f \le d.$$

At the cross-section of the surface by the plane $x = b$, a parabola is placed:

$$z = \frac{(a^2 - y^2)d}{a^2}\left(1 - \sqrt{\frac{c}{b+c}}\right).$$

At the cross-sections by the planes $y = \pm a$, the contour straight line is disposed. The third contour straight line coincides with the coordinate axis Oy.

The cross-sections of the surface by the planes parallel to the coordinate plane xOz are the parabolas with the horizontal axes parallel to the axis Ox.

(2) Explicit equation (Figs. 2 and 3):

$$z = \frac{(a^2 - y^2)d}{a^2}\left(1 - \sqrt{1 - \frac{x}{b+c}}\right).$$

(3) Explicit equation (Fig. 4):

$$x = (b+c)\left\{1 - \left[\frac{a^2 z}{(a^2 - y^2)d} - 1\right]^2\right\}.$$

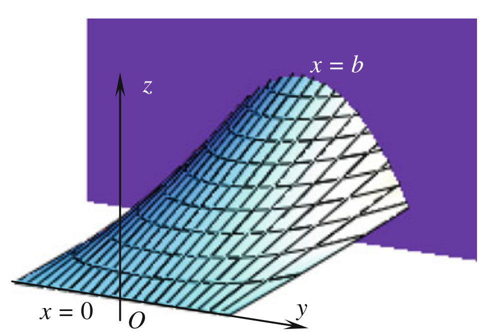


Fig. 4

(4) Implicit equation:

$$d^2xy^4 - 2a^2d^2xy^2 + a^2(b + c) \\ \times \left(2dy^2z + a^2z^2 - 2a^2zd\right) + a^4d^2x = 0.$$

This form of definition shows, that the parabolic surface of conoidal type is an algebraic surface of the 5th order.

Additional Literature

Podgorniy AL, Hoang Zuy Thang. Forming of some shells for the factory building. Prikl. Geometriya i Inzhenernaya Grafika. Kiev. 1978; Iss. 25, p. 12-15.

36.4 Algebraic Surfaces of the Sixth Order

Sine Surface

Sine surface is symmetrical relative to all three coordinate planes and may be put into ϕ sphere of a radius a.

Forms of the definition of the sine surface

(1) Parametrical equations (Fig. 1):

$$x = x(u) = a\sin u,$$
$$y = y(v) = a\sin v,$$
$$z = z(u, v) = a\sin(u + v),$$

where $-a \le x \le a$; $-a \le — \le a$; $-a \le z \le a$.

Coefficients of the fundamental forms of the surface:

$$A^2 = a^2\left[\cos^2 u + \cos^2(u + v)\right],$$
$$F = a^2\cos^2(u + v),$$
$$B^2 = a^2\left[\cos^2 v + \cos^2(u + v)\right];$$
$$A^2B^2 - F^2 = a^4\left[\cos^2 u\cos^2 v + \left(\cos^2 u + \cos^2 v\right)\cos^2(u + v)\right];$$
$$L = \frac{-a\cos v\sin v}{\sqrt{\cos^2 u\cos^2 v + (\cos^2 u + \cos^2 v)\cos^2(u + v)}},$$
$$M = \frac{-a\cos u\cos v\sin(u + v)}{\sqrt{\cos^2 u\cos^2 v + (\cos^2 u + \cos^2 v)\cos^2(u + v)}},$$
$$N = \frac{-a\cos u\sin u}{\sqrt{\cos^2 u\cos^2 v + (\cos^2 u + \cos^2 v)\cos^2(u + v)}};$$

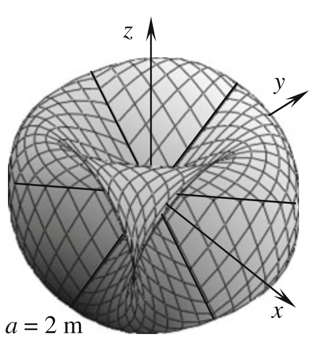


Fig. 1

$$K = \frac{\cos u\cos v\left[\sin u\sin v - \cos u\cos v\sin^2(u + v)\right]}{a^2[\cos^2 u\cos^2 v + (\cos^2 u + \cos^2 v)\cos^2(u + v)]^2};$$
$$H = \frac{\cos^2(u + v)\sin[2(u + v)] - \cos^2 v\sin 2v - \cos^2 u\sin 2u}{4a[\cos^2 u\cos^2 v + (\cos^2 u + \cos^2 v)\cos^2(u + v)]^{3/2}}.$$

The sine surface is related to the nonorthogonal non-conjugate system of the curvilinear coordinates u, v.

The area element is

$$dS = \sqrt{A^2B^2 - F^2}\,\mathrm{d}u\mathrm{d}v \\ = a^2\sqrt{\cos^2 u\cos^2 v + (\cos^2 u + \cos^2 v)\cos^2(u + v)}\,\mathrm{d}u\mathrm{d}v.$$

(2) Implicit equation:

$$a^2(z^2 - x^2 - y^2)^2 + 4x^2y^2(z^2 - a^2) = 0,$$

or

$$a^2z^4 - 2z^2(a^2x^2 + a^2y^2 - 2x^2y^2) + a^2(x^2 - y^2)^2 = 0.$$

The represented formulas show that the sine surface is *an algebraic surface of the sixth order* (*a sextic surface*). All three coordinate planes are the planes of symmetry of the surface.

The cross-section of the sine surface by the coordinate plane $z = 0$ is projected on the plane xOy as two orthogonal straight lines

$$x = \pm y$$

The projection of the cross-section of the surface by the coordinate plane $x = 0$ on the plane zOy is two intersecting orthogonal straight lines (Fig. 1)

$$z = \pm y.$$

And the line of the intersection of the coordinate plane $y = 0$ with the sine surface is projected on the coordinate plane zOx in the form of two intersecting orthogonal straight lines

$$z = \pm x.$$

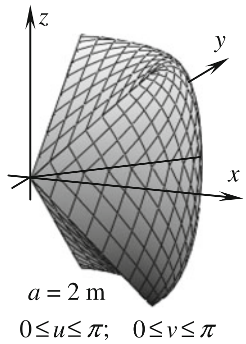


Fig. 2

In Figs. 1 and 2, one and the same sine surface is shown, but with the different boundaries. Geometrical parameters characterizing the surfaces are represented below the corresponding drawings.

Additional Literature

Gray A. Modern Differential Geometry of Curves and Surfaces with Mathematica. 2nd ed. Boca Raton, FL: CRC Press, 1997, p. 315-316.

Arvid Perego. Introduction to Algebraic Surfaces. Lecture Notes for the Course at the University of Mainz. Winter semester of 2009/2010; 208 p.

Weisstein, Eric W. "Sine Surface." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/SineSurface.html

■ Heart Surface

Taubin has introduced into practice algebraic surface of the sixth order which resembles heart and so become known as *Heart surface*.

Form of the definition of the Heart surface

(1) Implicit equation:

$$\left(x^2+\frac{9}{4}y^2+z^2-1\right)^3-x^2z^3-\frac{9}{80}y^2z^3=0.$$

Fig. 1

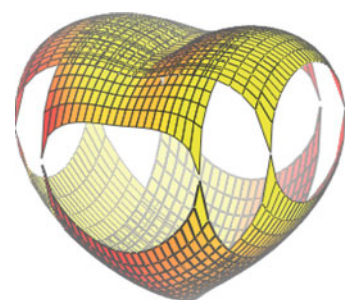


Fig. 2

The Heart surface taken without changing in the site of Weisstein is shown in Fig. 1. Figure 2 shows the surface collected from the segments of the Heart surface.

Additional Literature

Taubin G. An accurate algorithm for rasterizing algebraic curves and surfaces. IEEE Comput. Graphics Appl. 1994; No.14, p.14-23.

Weisstein EW. Heart Surface. http://mathworld.wolfram.com/HeartSurface.html. From MathWorld. A Wolfram Web Resource. © 1999 CRC Press LLC.

■ Hunt's Surface

The Hunt's surface is an algebraic surface of the sixth order:

$$4\left(x^2+y^2+z^2-13\right)^3+27\left(3x^2+y^2-4z^2-12\right)^2=0.$$

The Hunt's surface, shown in Fig. 1, was taken without changing in the site of Nordstrand.

Fig. 1

Additional Literature

Nordstrand T. Hunt's surface: http://www.uib.no/people/nfytn/hunttxt.htm.

Hunt Bruce. A gallery of algebraic surfaces. January 8, 2001; http//arpam.free.fr/hunt.pdf

Hunt B. Algebraic surfaces: http://www.mathematik.uni-kl.de/~wwwagag/E/Galerie.html.

■ The 6th Order Surface with the Pseudo-Agnesi Curl and Two Parabolas Lying in Parallel Planes

An algebraic surface containing *a pseudo-Agnesi curl*

$$z=-\frac{2Ry^2}{y^2+4R^2},\quad \text{where}\quad R=\frac{t^2-t\sqrt{t^2-4q^2}}{4q},$$

and two symmetrical *parabolas* $z=(q-h)y^2/t^2y^2-q$ lying in the planes $x=\pm m$ is *the 6th order surface*. The surface covers the rectangular plan 2m × 2*t*.

Forms of definition of the 6th order surface

(1) Explicit equation (Figs. 1 and 2):

$$z=\frac{x^2}{m^2}\left(\frac{q-h}{t^2}y^2-q+\frac{2Ry^2}{y^2+4R^2}\right)-\frac{2Ry^2}{y^2+4R^2}.$$

There is a parabola $z = -qx^2/m^2$ in the cross-section $y = 0$ of the surface. In the cross-section of the surface by the plane $x = 0$, the ridge line is disposed and this is *a pseudo-Agnesi curl*. At the cross-sections $y = c =$ const, the parabolas (Fig. 1a)

$$z=\frac{x^2}{m^2}\left(\frac{q-h}{t^2}c^2-q+\frac{2Rc^2}{c^2+4R^2}\right)-\frac{2Rc^2}{c^2+4R^2}$$

are disposed.

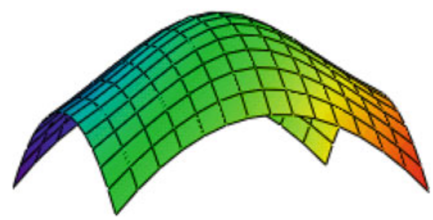


Fig. 2

(2) Implicit equation:

$$(q-h)x^2y^4/t^2+\left[4(q-h)R^2/t^2-q+2R\right]x^2y^2$$
$$-m^2y^2z-4qR^2x^2-2m^2Ry^2-4m^2R^2z=0.$$

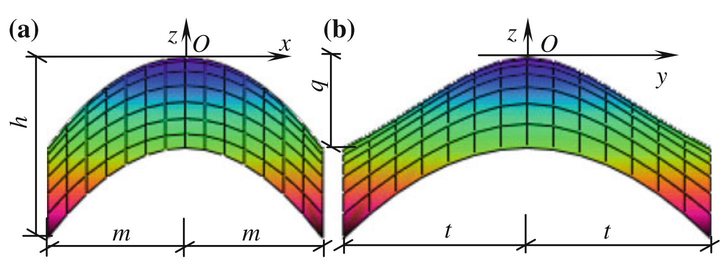


Fig. 1

■ The 6th Order Surface with Parabola, the 4th Order Curve, Parabola Lying in Three Principal Coordinate Sections

The surface in question has a parabola

$$\pm y = \frac{1}{2}B\left(1 - \frac{x^2}{L^2}\right)$$

at the cross-section by the plane $z = T$, a curve of the 4th order

$$\pm y = \sqrt{3}\frac{Bz}{2T^2}\sqrt{\frac{4Tz}{3} - z^2}$$

at the cross-section by the plane yOz ($x = 0$) and a parabola

$$z = \frac{x^2 T}{L^2}$$

at the cross-section of the surface by the plane xOz ($y = 0$). Here T is the draught of the surface, B is its maximal width along the axis Oy, $2L$ is its length along the axis Ox (Fig. 1).

The plane $z = T$ passes through two points of the curve of the 4th order (*midship frame*) with the coordinates $(0; \pm B/2; T)$.

Forms of definition of the surface

(1) Explicit equation (Fig. 1):

$$\pm y = \frac{\sqrt{3}B}{2T}\sqrt{\frac{4}{3}Tz - z^2}\left(\frac{z}{T} - \frac{x^2}{L^2}\right),$$

where $-L \le x \le L;\ -B/2 \le y \le B/2;\ 0 \le z \le T.$

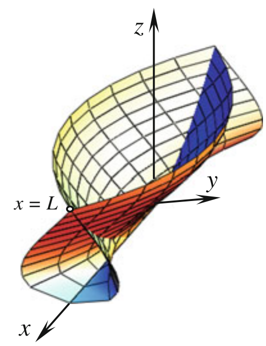


Fig. 1

The surface of the 6th order with a parabola, the 4th order curve, and a parabola lying in three principle coordinate planes has one plane of symmetry coinciding with the coordinate plane yOz.

Aerohydrodynamical surface has two conic points ($\pm L$; 0; T) and a parabolic rib $z = x^2T/L^2$ disposed in the plane $y = 0$.

The surface is formed by a family of parabolas lying in the planes $z = t =$ const, i.e., at the planes parallel to the coordinate plane xOy.

(2) Implicit equation:

$$3B^2\left(\frac{4}{3}Tz - z^2\right)\left(\frac{z}{T} - \frac{x^2}{L^2}\right)^2 = 4T^2y^2.$$

■ Surface of Revolution af an Agnesi Curl (the 2nd Variant)

Surface of revolution of an Agnesi curl (Fig. 1) is formed by the rotation of an Agnesi curl.

$$z = T\left(\frac{2L^2}{x^2 + L^2} - 1\right)$$

about the axis Ox. The surface has two conical points.

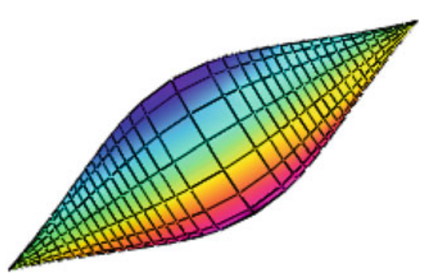


Fig. 1

Forms of definition of the surface

(1) Implicit equation:

$$z^2 + y^2 - T^2\left(\frac{2L^2}{x^2 + L^2} - 1\right)^2 = 0,$$

where $-L \le x \le L$; T is the maximum radius of the parallel of the surface in the plane zOy. Surface of revolution of the Agnesi curl is an algebraic surface of the 6th other. The first variant of definition of this surface is given in the Chap. "2. Surfaces of Revolution".

(2) Parametrical equation (Fig. 1):

$$x = x(r) = \pm L\sqrt{(T - r)/(T + r)},$$
$$y = y(r, \varphi) = r \sin\varphi,$$
$$z = z(r, \varphi) = r \cos\varphi,$$

where $0 \le \varphi \le 2\pi;\ 0 \le r \le T.$

Coefficients of the fundamental forms of the surface:

$$A^2 = 1 + \frac{L^2T^2}{(T^2 - r^2)(T + r)^2},$$
$$F = 0, \quad B = r;$$
$$L = \mp \frac{LT(T - 2r)}{A(T^2 - r^2)^{3/2}(T + r)}, \; M = 0,$$
$$N = \pm \frac{rLT}{A\sqrt{T - r}(T + r)^{3/2}}.$$

(3) Parametrical equations (Fig. 1):

$$x = x, \quad y = y(x, \varphi) = T\left(\frac{2L^2}{x^2 + L^2} - 1\right)\cos\varphi,$$
$$z = z(x, \varphi) = T\left(\frac{2L^2}{x^2 + L^2} - 1\right)\sin\varphi.$$

■ Algebraic Surface of the 6th Order with Two Nets of Translation

Parametrical equations of *an algebraic surface of the sixth order with two nets of translation* in the space E^3 may be written as

$$x = \varphi(u) + \varphi(v),$$
$$y = \psi(u) + \psi(v),$$
$$z = \kappa(u) + \kappa(v),$$

where

$$\varphi(x) = \int \frac{xdx}{F_y}, \quad \psi(x) = \int \frac{ydx}{F_y}, \quad \kappa(x) = \int \frac{dx}{F_y}$$

are the Abel integrals; $F(x,y) = 0$ is an equation of the curve of the fourth order. The translation surface relatively two planes can be algebraic and transcendental as well. The highest order of the algebraic surface equals six. There are two such surfaces of the sixth order.

Forms of the definition of the algebraic surface of the 6th order with two nets of translation

(1) Parametrical equations of the first surface:

$$x = x(u, v) = \frac{1}{2}\left(u^2 + v^2\right) + \frac{1}{3}\left(u^3 + v^3\right),$$
$$y = y(u, v) = \frac{1}{2}\left(u^4 + v^4\right) + \frac{1}{5}\left(u^5 + v^5\right),$$
$$z = z(u, v) = u + v.$$

Eisland (1907) has established, that the curve $F(x,y) = 0$ for this surface is given by an equation

$$(y - x^2)^2 \pm 2xy(y - x^2) + y^3 = 0.$$

(2) Implicit equation of the first surface:

$$(1 + z)y = (1 + z)\left(\frac{z^5}{5} + \frac{z^4}{2}\right) - (z^3 + 2z^2)\left(\frac{z^2}{2} + \frac{z^3}{3} - x\right) + \left(\frac{z^2}{2} + \frac{z^3}{3} - x\right)^2.$$

The surface is symmetrical relative to the point $\left(\frac{1}{3}, -\frac{1}{5}, 1\right)$.

(3) Implicit equation of the second surface:

$$z^6 - \frac{15}{4}z^4 - 15xz^3 - 45x^2 + 45yz + \frac{65}{4} = 0.$$

This equation of the second surface was derived from the implicit equation of the first surface by the transfer of the origin of the coordinates into the point

$$\left(\frac{1}{3}, -\frac{1}{5}, 1\right)$$

and by the replacement of y by

$$-x + y - \frac{1}{2}z.$$

Ignatenko (1977) has shown, that the surface obtained by this method is a surface of translation relative to two planes that are *improper plane* and the plane $z = 0$.

(4) Explicit equation of the second surface:

$$y = \frac{z^6 - 15\left(\frac{z^4}{4} + xz^3 + 3x^2\right) + \frac{65}{4}}{45z}.$$

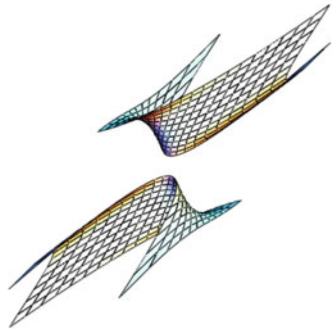


Fig. 1

In Fig. 1, the fragments of the second surface of translation designed in the boundaries:

$$-3 \le x \le 3; \quad -0.6 \le z \le -3; \quad 0.6 \le z \le 3$$

are shown.

(5) Explicit equation of the second surface:

$$x = \frac{1}{6}\left(-z^3 \pm \sqrt{\frac{9}{5}z^6 - 3z^4 + 36yz + 13}\right).$$

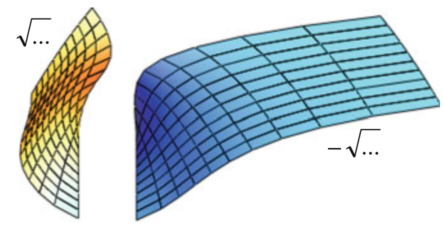


Fig. 2

In Fig. 2, the fragments of the second surface of translation constructed in the boundaries:

$$-0.15 \le y \le 5.5; \quad 0 \le z \le 3.5.$$

are shown.

Additional Literature

Eisland J. On a certain class of algebraic translation-surfaces. Amer. J. of Math. 1907; Vol. 29, p. 363-386.

Ignatenko VF. On the algebraic surface of the 6th order with two nets of translation. Ukrainskiy Geometricheskiy Sbornik. Kharkov. 1977; Iss. 20, p. 46-48 (6 refs).

■ The 6th Order Surface with Agnesi Curl, Ellipse, Agnesi Curl Lying in Three Principal Coordinate Sections

The surface in question has an Agnesi curl $y = \frac{L^2B}{x^2+L^2} - \frac{B}{2}$ at the cross-section by the plane xOy (z = 0), an ellipse $\frac{4y^2}{B^2} + \frac{z^2}{T^2} = 1$ at the cross-section by the plane yOz (x = 0) and an Agnesi curl $z = T - \frac{2L^2T}{x^2+L^2}$ at the cross-section of the surface by the plane xOz (y = 0). Here T is the draught of the surface, B is its maximal width along the axis Oy, $2L$ is its length along the axis Ox (Fig. 1).

Forms of definition of the surface

(1) Implicit equation:

$$\frac{4y^2(x^2+L^2)^2}{(L^2B - Bx^2)^2} + \frac{z^2(x^2+L^2)^2}{T^2(x^2-L^2)^2} = 1.$$

A surface of the 6th order with Agnesi curl, ellipse, Agnesi curl lying in three principal coordinate sections has three planes of symmetry coinciding with the coordinate planes and two conical points (–L; 0; 0) and (L; 0; 0).

The surface is formed by a family of the ellipses lying in the planes x = const which are parallel to the coordinate plane yOz.

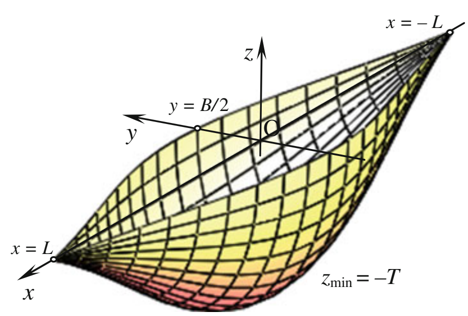


Fig. 1

(2) Parametrical equations (Fig. 1):

$$x = x(u) = uL,$$
$$y = y(u,v) = \frac{1-u^2}{2(1+u^2)}B\cos v,$$
$$z = z(u,v) = \frac{1-u^2}{1+u^2}T\sin v.$$

Additional Literature

Avdon'ev EA, Protod'yakonov SM. Equations and characteristics of some algebraic surfaces of the highest orders. Prikl. Geometriya i Inzhenernaya Grafika. Kiev. 1976; Iss. 21, p. 108-120 (2 refs).

36.5 Algebraic Surfaces of the Seventh Order

Additional Literature

Hill JE. On three septic surfaces. American Journal of Mathematics. 1897; Vol. 19, No. 4, p.289-311.
Krylov IP. Forming and relaying of complex algebraic surfaces through television chanal. PhD Diss. SPb. 2004; 160 p.

■ "Ski Hill"

The surface *"Ski hill"* is an algebraic surface of the seventh order. In outward appearance it resembles the *diagonal cubic surface of Fermat* (see also a Sect. "36.1. Algebraic Surfaces of the Third Order").

The form of the definition of the surface

Explicit equation (Figs. 1, 2 and 3):

$$z = \sqrt[3]{a(x^3 - y^3)^2 y + b(x^2 + y^2)^3 x},$$

where a and b are arbitrary constants.

In Fig. 2, the surface in question with the geometrical parameters:

$$a = 0.75; \quad b = 0.25; \quad -1.5 \le x \le 2; \quad -2 \le y \le 2$$

is presented.

The surface shown in Fig. 3 has

$$a = 0.25; \quad b = 0.75; \quad -1.5 \le x \le 2; \quad -2 \le y \le 2.$$

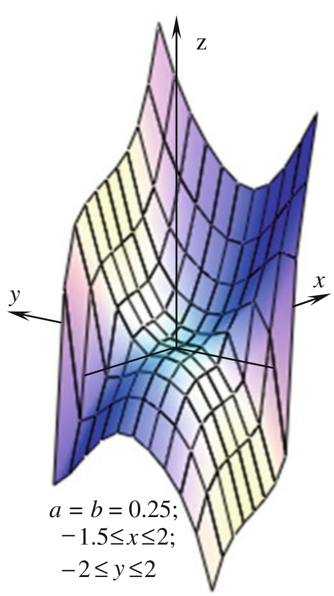


Fig. 1

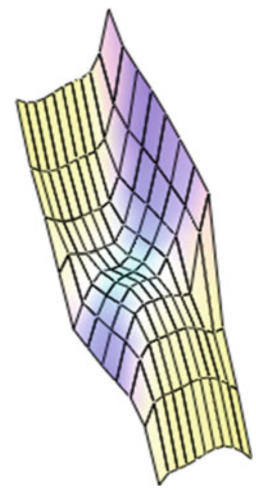


Fig. 2

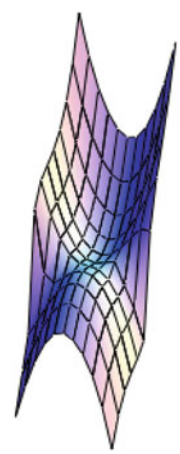


Fig. 3

At the cross-section of the surface by the plane $x = y$, the curve of the seventh order

$$x = u; \quad y = u;$$
$$z^3 = 8bu.^7$$

are placed.

■ The 7th Order Surface with Parabola, Agnesi Curl, Ellipse Lying in Three Principal Coordinate Sections

The surface with a parabola, an Agnesi curl, an ellipse lying at three principal coordinate cross-sections has a parabola

$$y = \frac{B}{2} - \frac{2Bx^2}{L^2}$$

at the cross-section of the surface by the plane xOy, an Agnesi curl

$$z = \frac{2B^2T}{4y^2 + B^2} - T$$

at the cross-section by the plane yOz, and an ellipse

$$\frac{4x^2}{L^2} + \frac{z^2}{T^2} = 1$$

at the cross-section by the plane xOz ($y = 0$).

Here T is the draught of the surface along the axis Oz, B is its maximal width along the axis Oy, L is its length along the axis Ox (Fig. 1).

Having the equations of the main cross-sections of the surface, it is possible to design surfaces with the different requirements given in advance for them. Assuming one and the same main cross-sections, it is possible to design three surfaces essentially differing from each other (see, for example, "The 6th order surface with parabola, an Agnesi curl, an ellipse lying at three principal coordinate cross-sections"). For this, at first, it is necessary to form the continuous frame of the plane curves that are incidental to a family of the planes parallel to one of the coordinate planes.

For practical aims, it is necessary to calculate *the block coefficient* for every obtained surface. *The block coefficient* is a ratio of the area (or the volume) limited by the curve (or the surface) to the area (or the volume) of the rectangle (or the parallelepiped) having one and the same overall dimensions.

Forms of definition of the surface of the 7th order

(1) Implicit equation:

$$\frac{4Bx^2}{L^2(B-2y)} + \frac{\left(4y^2+B^2\right)^2 z^2}{T^2\left(B^2-4y^2\right)^2} = 1,$$

where

$$-L/2 \le x \le L/2; \quad B/2 \le y \le B/2; \quad -T \le z \le 0.$$

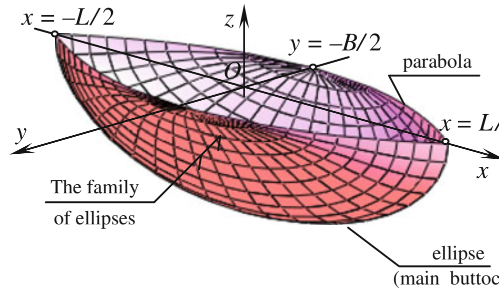


Fig. 1

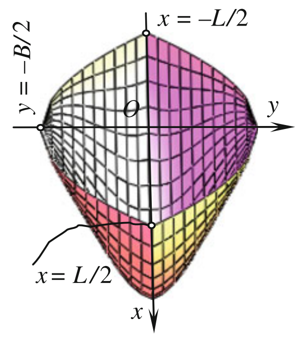


Fig. 2

The surface is formed by a family of the ellipses lying at the planes $y = y_c = \text{const}$ that are parallel to the coordinate plane xOz:

$$\frac{4Bx^2}{L^2(B-2y_c)} + \frac{\left(4y_c^2+B^2\right)^2 z^2}{T^2\left(B^2-4y_c^2\right)^2} = 1.$$

(2) Parametrical equations (Figs. 1 and 2):

$$x = x(y,v) = 0.5L\sqrt{1-\frac{2}{B}y}\cos v, ,$$
$$y = y,$$
$$z = z(y,v) = \frac{T\left(B^2-4y^2\right)}{B^2+4y^2}\sin v.$$

(3) Implicit equation:

$$\left(4y^2+B^2\right)^2(B-2y)\frac{z^2}{T^2} + 4\frac{B}{L^2}x^2\left(B^2-4y^2\right)^2$$
$$-\left(B^2-4y^2\right)^2(B-2y) = 0.$$

The last formula shows, that the surface in question is an algebraic surface of the 7th order.

For this surface, the block coefficient is equal to 2/3 (Avdon'yev and Protod'yakonov 1976).

Additional Literature

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■ The 7th Order Surface with Parabola, Ellipse, Cartesian Folium Lying in Three Principal Coordinate Sections

Surfaces of swimming bodies of the animate nature, in the first place cetaceans and some types of fishes, are interesting for design of ship's surfaces. These bodies possess the perfect form from the point of view of the hydrodynamics.

On the base of analysis of the forms of the outlines of surfaces of the bodies of the animate nature, Avdon'ev confirms, that they may be designed from the arcs of algebraic curves. They give the opportunity to describe the whole profile or its half by one equation. This has the essential significance for providing of the smoothness of the profile. In particular, for the approximation of the external surfaces of the sword-fishes, he suggested to use the 7th order surface with a parabola (cargo waterline)

$$y = \frac{B}{2} - \frac{2B}{L^2}\left(x - \frac{L}{2}\right)^2$$

at the cross-section $z = 0$, the generalized Cartesian folium (*main buttock line*)

$$z = 2.5426\frac{T}{L}x\sqrt{\frac{3(L-x)}{L+3x}}$$

at the cross-section $y = 0$, and an ellipse (*midship frame*)

$$\frac{4y^2}{B^2} + \frac{z^2}{T^2} = 1$$

at the cross-section $x = \frac{L}{\sqrt{3}}$.

Here there are taken the following convention (Fig. 1): B is a maximum width of the surface at the cross-section $x = L/2$ in the direction of the coordinate axis Oy; L is the length of the surface along the axis Ox; $2T$ is a maximum height of the surface at the cross-section сечении $x = L/\sqrt{3}$ along the axis Oz.

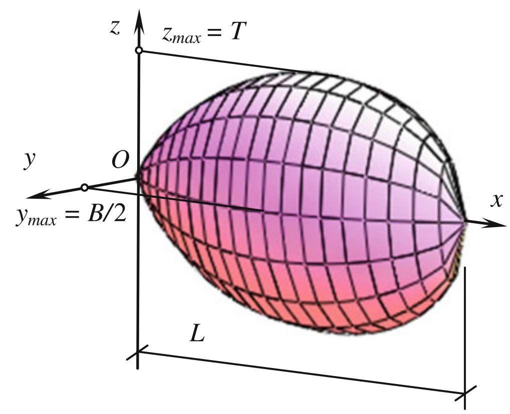

Fig. 1

The form of the definition of the surface of the 7th order

(1) Implicit form of definition (Figs. 1 and 2):

$$\frac{y^2}{\left[\frac{B}{2} - \frac{2B}{L^2}\left(x - \frac{L}{2}\right)^2\right]^2} + \frac{z^2}{\left[2.5426\frac{T}{L}x\sqrt{\frac{3(L-x)}{L+3x}}\right]^2} = 1.$$

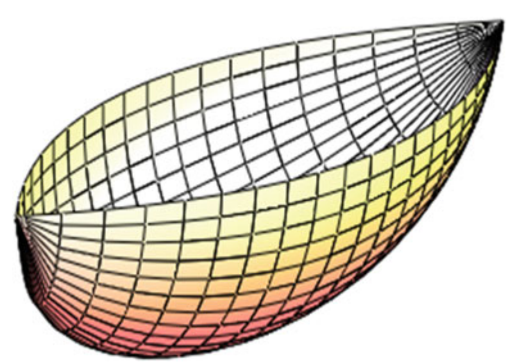

Fig. 2

The variation of the geometrical parameters B, T, L in the equations of the curves and surface makes possible to obtain the given values of the aerohydrodynamic properties (entrance angle, completeness of the form and others).

The surface in question is formed by a family of the ellipses (transverse frame) lying at the parallel planes $x = c = \text{const}$. The family of the ellipses is defined by equations:

$$\frac{y^2}{\left[\frac{B}{2} - \frac{2B}{L^2}\left(c - \frac{L}{2}\right)^2\right]^2} + \frac{z^2}{\left[2.5426\,\frac{Tc}{L}\sqrt{\frac{3(L-c)}{L+3c}}\right]^2} = 1.$$

Additional Literature

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Avdon'ev E.A. Approximation of surfaces of the swimming bodies by algebraic surfaces. Prikladnaya Geometriya i Inzhenernaya Grafika. Kiev. 1975; Iss. 19, p. 99-102 (3 refs).

■ The 7th Order Surface with Parabola, the 4th Order Curve, Parabola Lying in Three Principal Coordinate Sections

The surface with a parabola, the 4th order curve, and a parabola lying in three principle coordinate planes has the parabola

$$\pm y = \frac{B}{2} - \frac{Bx^2}{2L^2}$$

at the cross-section of the surface by the plane $z = T$, a curve of the 4th order

$$\pm y = \frac{\sqrt{3}B}{2T^2} z\sqrt{\frac{4}{3}Tz - z^2}$$

at the cross-section by the plane yOz ($x = 0$), and a parabola

$$z = \frac{T}{L^2}x^2$$

at the cross-section by the plane xOz ($y = 0$).

Here T is the draught of the surface along the axis Oz, B is its maximum width along the axis Oy, $2L$ is its length along the axis Ox.

The origin of the coordinates is disposed at the lowest point of the main buttock where the lowest point of the midship frame is also disposed. The midship frame includes into itself not whole closed curve of the 4th order

$$\pm y = \frac{\sqrt{3}B}{2T^2} z\sqrt{\frac{4}{3}Tz - z^2}$$

laying at the plane yOz, but only the segments in the limits $0 \leq z \leq T$. The coordinate z for whole closed curve of the 4th order changes in the limits of $0 \leq z \leq 4T/3$. At the point $z = T$ (Fig. 1), the curve has the maximum value of y equal to

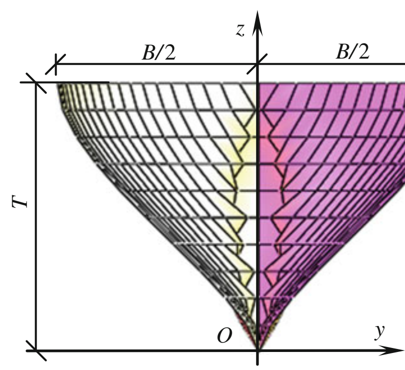

Fig. 1

$$y_{\max}(z = T) = \pm B/2.$$

If the equations of the main cross-sections are available, then it is possible to construct the surfaces with different demands given in advance.

Assuming one and the same main cross-sections, it is possible to design three surfaces essentially differing from each other (see also, for example, "The 6th order surface with parabola, the curve of the 4th order, parabola lying at three principal coordinate cross-sections"). For this, beforehand, it is necessary to form the continuous frame of the plane curves that are incidental to a family of the planes parallel to one of the coordinate planes.

For the practical aims, it is necessary to calculate *the block coefficient* for every obtained surface. *The block coefficient* is a ratio of the area (or the volume) limited by the curve (or the surface) to the area (or the volume) of the rectangle (or the parallelepiped) having one and the same overall dimensions.

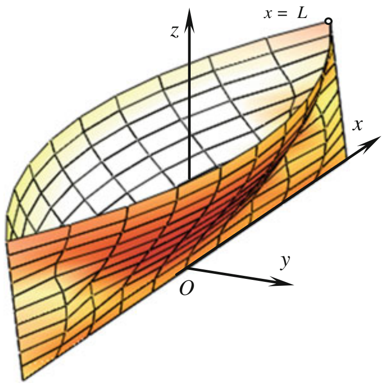

Fig. 2

Forms of definition of the surface of the 7th order

(1) Explicit equation (Figs. 1 and 2):

$$\pm y = \frac{\sqrt{3}B\sqrt{L^2 - x^2}}{2LT^2}\sqrt{\frac{4}{3}Tz - z^2}\sqrt{z^2 - \frac{T}{L^2}x^2 z},$$

where

$$-L \le x \le L; \quad -B/2 \le y \le B/2; \quad 0 \le z \le T.$$

The surface is formed by a family of the curves of the 4th order lying at the planes $x = t$ = const parallel to the coordinate plane yOz:

$$\pm y = \frac{\sqrt{3}B\sqrt{L^2 - t^2}}{2LT^2}\sqrt{\frac{4}{3}Tz - z^2}\sqrt{z^2 - \frac{T}{L^2}t^2 z}.$$

(2) Implicit equation:

$$y^2 = \frac{3B^2(L^2 - x^2)}{4L^2T^4}\left(\frac{4}{3}T - z\right)\left(z - \frac{T}{L^2}x^2\right)z^2.$$

The last formula shows, that the surface in question is an algebraic surface of the 7th order.

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■ The 7th Order Surface with Agnesi Curl, Lame's Curve of the Third Order, Straight Lines Lying in Three Principal Coordinate Sections

The surface with the Agnesi curl, the Lame's curve of the third order, and straight lines lying in three principle coordinate planes has the Agnesi curl

$$\pm y = \frac{L^2B}{x^2 + L^2} - \frac{B}{2}$$

at the cross-section of the surface by the plane xOy (z = 0), the Lame's curve of the third order

$$\frac{4y^2}{B^2} - \frac{z^3}{T^3} = 1$$

at the cross-section by the plane yOz, and three straight lines

$$x = \pm L, \quad y = 0 \quad \text{and} \quad z = -T, \ y = 0$$

at the cross-section by the plane xOz.

Here T is the draught of the surface along the axis Oz, B is the maximum width along the axis Oy, $2L$ is its length along the axis Ox.

Forms of definition of the surface of the 7th order

(1) Explicit form of definition (Fig. 1):

$$\pm y = \frac{L^2B\sqrt{T^3 + z^3}}{T\sqrt{T}(x^2 + L^2)} - \frac{B\sqrt{T^3 + z^3}}{2T\sqrt{T}},$$

where $-L \le x \le L$; $-B/2 \le y \le B/2$; $-T \le z \le 0$.

The surface is formed by a family of the Agnesi curls lying at the planes z = –t = const that are parallel to the coordinate plane yOx:

$$\pm y = \frac{B\sqrt{T^3 - t^3}}{2T\sqrt{T}}\left(\frac{L^2 - x^2}{x^2 + L^2}\right) = \frac{B\sqrt{T^3 - t^3}}{T\sqrt{T}}\left(\frac{L^2}{x^2 + L^2} - \frac{1}{2}\right).$$

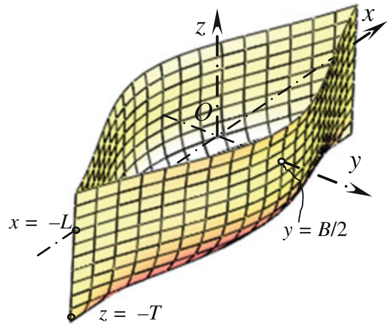

Fig. 1

(2) Implicit form of assignment:

$$B^2(T^3+z^3)(L^2-x^2)^2=4y^2T^3(x^2+L^2)^2.$$

The last formula shows, that the surface in question is an algebraic surface of the 7th order.

Reference

Avdon'ev EA, Protod'yakonov SM. Equations .and characteristics of some algebraic surfaces of the highest orders. Prikl. Geometriya i Inzhenernaya Grafika. Kiev. 1976; Iss. 21, p. 108-120 (2 refs).

36.6 Algebraic Surfaces of the 8th Order

■ The 8th Order Surface with Agnesi Curl, Ellipse, Ellipse Lying in Three Principal Coordinate Sections

The surface has an Agnesi curl

$$y=\frac{L^2B}{4x^2+L^2}-\frac{B}{2}$$

at the cross-section of the surface by the plane xOy ($z = 0$), an ellipse

$$4y^2/B^2+z^2/T^2=0$$

at the cross-section by the plane yOz ($x = 0$), and an ellipse

$$z^2/T^2+4x^2/L^2=1$$

at the cross-section by the plane xOz ($y = 0$). Here T is the draught of the surface, B is its maximum width along the axis Oy, L is its length along the axis Ox.

Forms of definition of the surface

(1) Implicit form of definition (Fig. 1):

$$\frac{y^2}{\left(\frac{L^2B}{4x^2+L^2}-\frac{B}{2}\right)^2}+\frac{z^2}{\frac{T^2(L^2-4x^2)}{L^2}}=1.$$

The volume of the surface may be calculated by a formula:

$$V\;=\;\pi BTL/7.$$

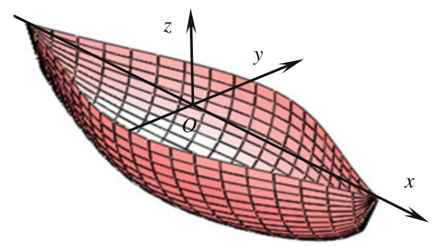

Fig. 1

The surface has three planes of symmetry coinciding with the coordinate planes.

The tangent of the angle of the axis Ox with the tangent line to the Agnesi curl at the points $x=\pm L/2$ can be calculated by a formula

$$\tan\gamma=B/L.$$

With the help of the surface in question, it is possible to construct the surface of a ship hull assuming the line $z = 0$ as a water line. Knowing the general equation of the surface and the expression for the entrance angle (γ) of the waterline, it is possible to construct the surface of the presented type with conditions given in advance for ship hull varying the parameters L, T, B.

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■ The 8th Order Surface with Lame's Curve of the 4th Order, Lame's Curve of the 4th Order, Ellipse Lying in Three Principal Coordinate Sections

The surface has a Lame's curve of the 4th order

$$\frac{16y^4}{B^4} + \frac{x^4}{L^4} = 1$$

at the cross-section of the surface by the plane xOy ($z = 0$), a Lame's curve of the 4th order

$$\frac{16y^4}{B^4} + \frac{z^4}{T^4} = 1$$

at the cross-section by the plane yOz ($x = 0$) and an ellipse

$$\frac{z^2}{T^2} + \frac{x^2}{L^2} = 1$$

at the cross-section by the plane xOz ($y = 0$). Here T is the draught of the surface, B is its maximum width along the axis Oy, $2L$ is its length along the axis Ox.

Forms of definition of the surface

(1) Implicit equation (Fig. 1):

$$\frac{16y^4L^4}{B^4(L^4 - x^4)} + \frac{z^4L^4}{T^4(L^2 - x^2)^2} = 1.$$

The segment of this surface has found the application as a surface of the stern extremity of the ship hull of the river cargo motor vessel designed by a design office ("PKB GURF") in SM UkrSSR (Avdon'ev 1972).

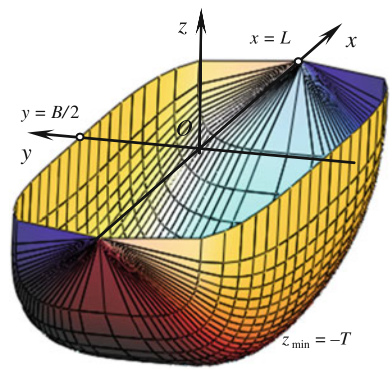

Fig. 1

(2) Parametrical equations (Fig. 1):

$$x = uL,$$
$$y = y(u, v) = \pm B\sqrt[4]{1 - u^4}\sqrt{\cos v}/2,$$
$$z = z(u, v) = T\sqrt{1 - u^2}\sqrt{\sin v}.$$

References

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■ The 8th Order Surface with Parabola, the 4th Order Curve, Parabola Lying in Three Principal Coordinate Sections

The surface has a parabola $y = B(1 - x^2/L^2)/2$ at the cross-section of the surface by the plane xOy, a curve of the fourth order $y = 8Bz\sqrt{z(z-T)}/(3\sqrt{3}T^2)$ at the cross-section by the plane yOz and a parabola $z = 3Tx^2/(4L^2)$ at the cross-section by the plane xOz. The dimensions T, L, and B are shown in Fig. 1.

Forms of definition of the surface

(1) Parametrical equations (Fig. 2):

$$X = X(u, v) = uL,$$

$$Y = Y(u, v) = \pm 8B(1 - u^2)v\sqrt{v(1 - v)}\Big/(3\sqrt{3}),$$
$$Z = Z(u, v) = T(v - 3/4)(1 - u^2),$$

where $u = x/L$; $v = z/T$

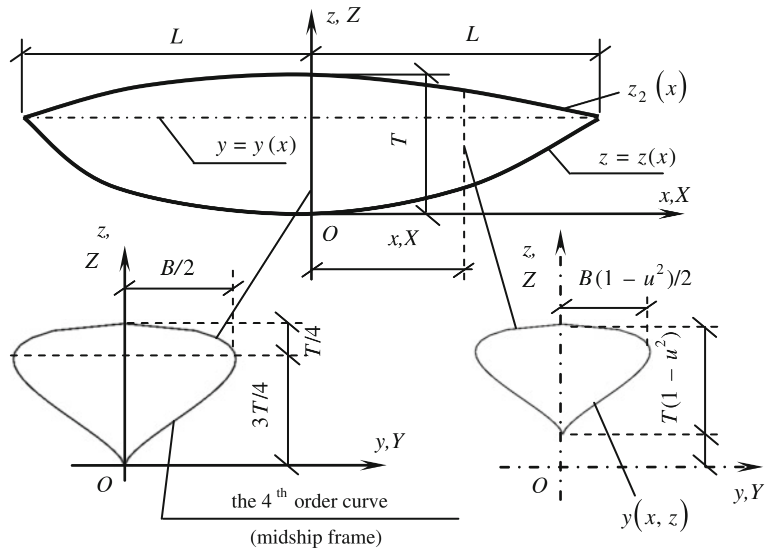
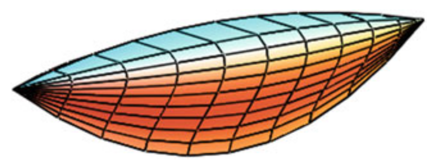


Fig. 1

(2) Implicit equation:

$$y^2 - \frac{64}{27}B^2\left(\frac{z}{T} - \frac{3}{4}\right)^3\left(\frac{1}{4} - \frac{z}{T}\right)\left(1 - \frac{x^2}{L^2}\right)^2 = 0.$$



$-1 \le u \le 1;\ 0 \le v \le 1;$
$L = 2\text{m};\ T = 1\text{m};\ B = 1.2\ \text{m}$

Fig. 2

36.7 Algebraic Surfaces of the 12th Order

■ Surface of the 12th Order with Parabola, the 8th Order Curve, Parabola Lying in Three Principal Coordinate Sections

The surface has a parabola $y = B(1 - x^2/L^2)/2$ at the cross-section of the surface by the plane xOy, a curve of the eighth order

$$y = (CBz/T)\sqrt{(z/T)[1 - (z/T)^p]}$$
$$\times\left[p = 5;\ C = \frac{1}{2}\sqrt{\frac{p+3}{p}\left(\frac{p+3}{3}\right)^{3/p}}\right]$$

at the cross-section by the plane yOz and parabola $z = 3Tx^2/(4L^2)$ at the cross-section by the plane xOz. The dimensions T, L, and B are shown at the page 943 (Fig. 1).

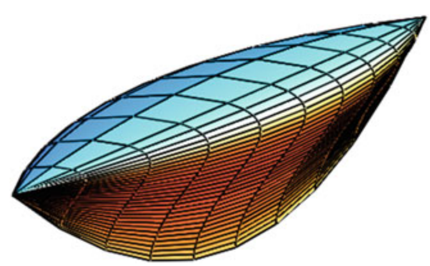


Fig. 1

Forms of definition of the surface

(1) Parametrical equations (Fig. 1):

$$X = X(u, v) = Lu,$$
$$Y = Y(u, v) = CBv\sqrt{v(1 - v^p)}\left(1 - u^2\right),$$
$$Z = Z(u, v) = T(v - v_m)\left(1 - u^2\right),$$

where $v_m = [3/(p+3)]^{\frac{1}{p}}, \quad u = x/L; \quad v = z/T.$

(2) Implicit equation:

$$y^2 - C^2B^2(z/T + v_m)^3[1 - (z/T + v_m)^p](1 - x^2/L^2)^2.$$

For the surface in question, it is necessary to take $p = 5$. Assuming the parameter p equal to any integer, one can obtain algebraic surfaces of other orders.

36.8 Algebraic Surfaces of the 16th Order

A segment of a surface given by analytical equation can be easily pictured with the help of methods of descriptive geometry. Algebraic surfaces of the highest orders formed by a single-parametrical variety of pane curves lying in parallel planes. These plane curves are included into the frame of the surface.

■ Algebraic Surface with a Continuous Net of the Pseudo-Agnesi Curl Passing Through a Parabola and Two Straights

The algebraic surface passing through a parabola

$$z = \frac{c - m}{t^2} y^2 + m,\ x = 0$$

and the straight lines $x = \pm q$ lying at the horizontal plane xOy and carrying on itself a continuous frame of the pseudo-Agnesi curl lying at the planes parallel to the coordinate plane xOz is an algebraic surface of the 16th order.

An equation of the three-parametric set of the pseudo-Agnesi curls has the following form:

$$x^2(z - n + 2R) = 4R^2(n - z),$$

where $n = \frac{c-m}{t^2} y^2 + m$. A radius $R = \frac{q^2 - q\sqrt{q^2 - 4n^2}}{4n}$ is determined from a condition of passing of the pseudo-Agnesi curl through the straight line $x = q$.

Forms of definition of the surface

(1) Implicit equation: (Figs. 1, 2 and 3):

$$x^2(z - n + 2R) = 4R^2(n - z),$$

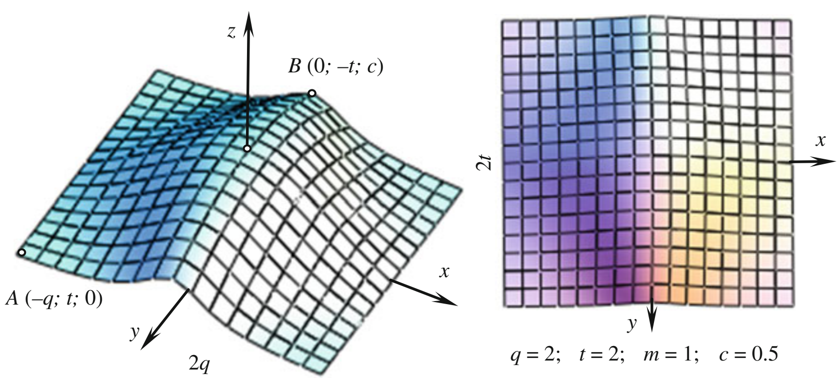


Fig. 1

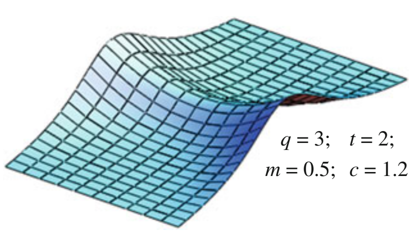


Fig. 2

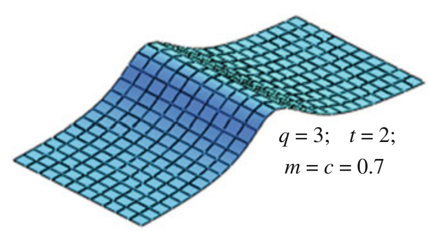


Fig. 3

where the values of the parameters $n = n(y^2)$ and R are given before. The maximum rise of the surface relatively the horizontal plane xOy is equal to m:

$$z_{\max} = z(x = 0; y = 0) = m > c.$$

(2) Implicit equation:

$$\left\{(z-n)\left[2n^2(x^2-q^2)+q^4\right]+q^2nx^2\right\}^2 - q^2(q^2-4n^2)\left[x^2n+(z-n)q^2\right]^2 = 0.$$

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Additional Literature

PS: Additional literature is given at the corresponding pages of the Chap. “36. Algebraic Surfaces of the High Orders”.

37 Polyhedrons and Quasi-polyhedrons

In elementary geometry, a polyhedron (*plural polyhedra* or *polyhedrons*) is a body in three dimensions with flat faces, straight edges and sharp corners or vertices. The word "*polyhedron*" comes from the Classical Greek as poly- ("many") and -hedron (form of "base" or "seat").

A polyhedron is a figure bounded by plane *polygons*. The vertexes and the sides of polygons are the vertexes and the ribs of a polyhedron. They are formed the space net. If vertexes and ribs of a polyhedron are at one side of a plane of any its face, then the polyhedron is called *a convex polyhedron*. In this case, all its faces are convex polygons. *Prisms, pyramids, prismatoids*, and *regular convex polyhedrons*, i.e. Plato's bodies (*tetrahedron, hexahedron, octahedron, dodecahedron*, and *icosahedron*), are of the most interest among polyhedrons. These five regular polyhedrons are also known as *the Platonic solids.* A polyhedron is called *a regular polyhedron* if its faces are regular and identical polygons and the many-sided angles at its vertexes are equal.

A prism is a polyhedron, two bases of which are n-angles and the rest n lateral faces are *parallelograms*. It is known the triangular, quadrangular, and so on prisms due to triangle, quadrangle and n-angle lying in the base. *A pyramid* is a polyhedron, the base of which is a polygon and the rest lateral faces are the triangles with the common vertex. *A prismatoid* is a polyhedron, two faces of which (the bases) lie at the parallel planes bur the rest are the triangles or *the trapeziums*. One side of the triangles and the both bases of the trapeziums are the sides of the bases of a prismatoid. A prismatoid is called *an antiprism*, if its bases are regular polygons with the centers belonging to the common normal to them, but one polygon is turned relative to other polygon about the normal at an angle of $\varphi = 180°/n$, where n is a number of the sides of the polygon).

Tetrahedron consists of four equilateral and identical triangles. Three triangles are connected at every vertex. Tetrahedron is a special case of a pyramid.

Hexahedron is a regular polyhedron with six identical faces (cube) in the form of a square. A cube is a spatial case of a prism. So, a cube is a polyhedron bounded by six polygons meeting at right angles. *Octahedron* is a regular polyhedron consisting of eight equilateral and identical triangles and every four triangles are connected at every vertex. Octahedron is a polyhedron composed of two quadrangular pyramids that are joined by its square bases. *Dodecahedron* is a regular polyhedron with twelve regular and identical five-angle faces, every three of them are connected at every vertex. Assume two parallel fife-angle faces as the bases of a dodecahedron, then the rest ten fife-angle faces will form the lateral surface. *Icosahedron* is a regular polyhedron consisting of twenty equilateral and identical triangles and every five triangles are connected at every vertex.

The cube and the octahedron are a pair of dual polyhedrons, they are obtained from each other if the centroids of the faces of one are taken as the vertices of the other. The

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DOI 10.1007/978-3-319-11773-7_37, © Springer International Publishing Switzerland 2015

dodecahedron and icosahedron are also dual. The centers of faces of a icosahedron are the vertexes of the dodecahedron. The tetrahedron is self-dual.

In 18th century, Swiss mathematician Leonhard Euler showed that for any simple polyhedron, i.e., a polyhedron containing no holes, the sum of the number of vertices V and the number of faces F is equal to the number of edges E plus 2, or $V + F = E + 2$. It was proved also, that three non-convex regular dodecahedrons (Poinsot bodies) exist and that are: (1) small star-shaped dodecahedron; (2) big dodecahedron; and (3) big star-shaped dodecahedron (L.A. Stanislavskaya).

A quasi-polyhedron is called a figure bounded by not plane identical segments of surfaces. The angular points and the boundaries of the segments are the vertexes and the ribs of a quasi-polyhedron. For the basis of the formation of a quasi-polyhedron one may take any polyhedron. For example, one may consider that *the Goursat's surface* is a quasi-polyhedron designed on the cube.

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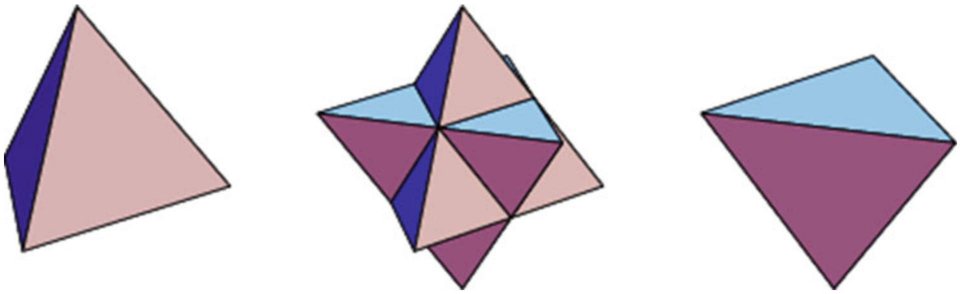
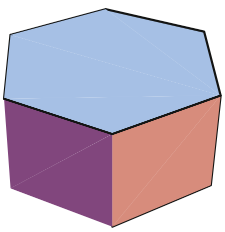
■ The Types of Polyhedrons

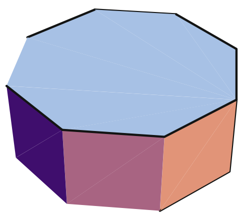
At present time, 75 homogeneous polyhedrons and a large number of their star-shaped forms are known. Here the most known types of the polyhedrons taken in the sites of the Internet are represented. The more detailed information is given in a book of M. Wenninger (1974).



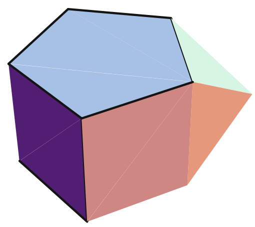
Tetrahedron and polyhedrons designed on its basis



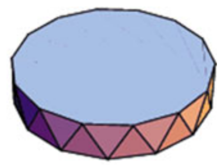
Six-angular prism



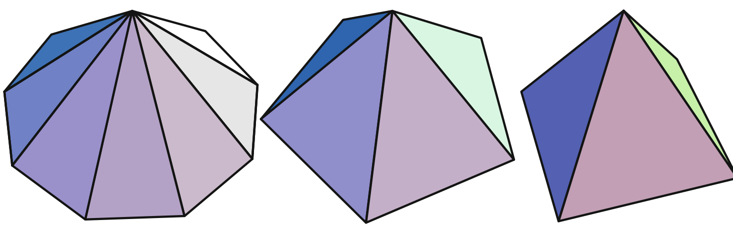
Eight-angular prism



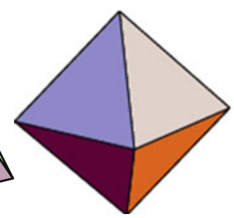
Pentagonal prism with an attached quadrangle pyramid



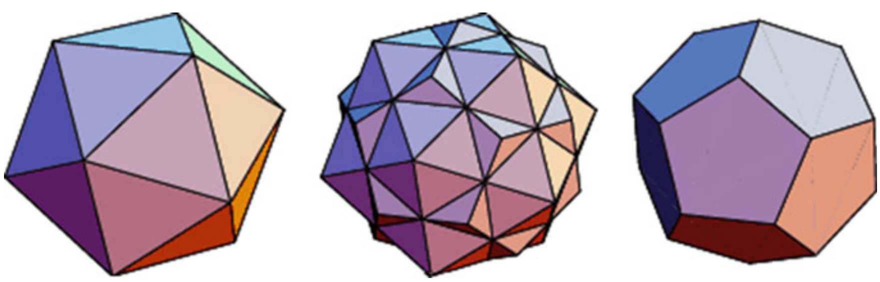
Antiprism



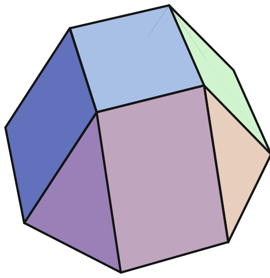
Pyramids

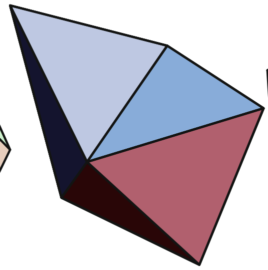


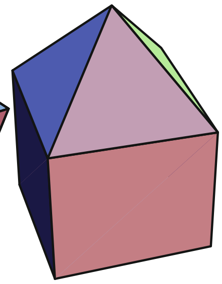
Octahedron

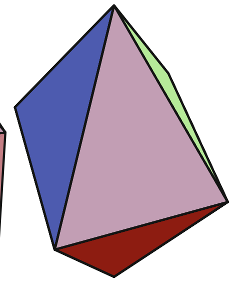


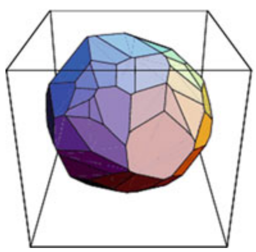
Icosahedron and polyhedrons designed on its basis

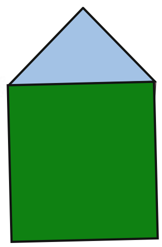

Dome

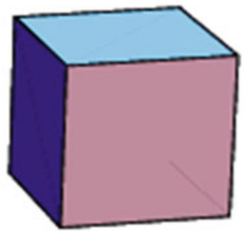

Boat


House


Double square pyramid (Octahedron)

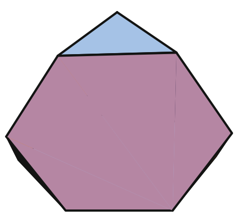

Convex polyhedron

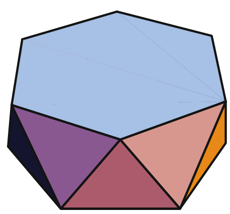

Triangle prism

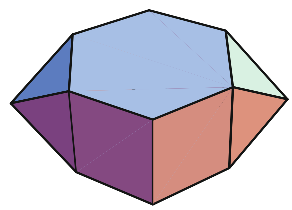

Hexahedron (Cube)

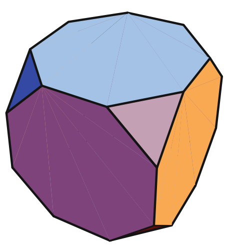

Polyhedron consisting of a cube and an octahedron

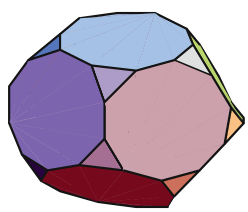
Polyhedrons consisting of the prisms and the pyramids


Truncated double tetrahedron


Hexagonal antiprism


Hexagonal prism with two attached pyramids


Truncated cube


Truncated dodecahedron


Polyhedron formed from a cube and six quadrangle pyramids

PS: The figures of the polyhedrons are taken in Internet sites of W. Weisstein [A Wolfram Web Resource. http://mathworld.wolfram.com/Tetrahedron.htm] and http://mathworld.wolfram.com/Tetradecahedron.html.

■ Astroidal Ellipsoid

Astroidal ellipsoid contains six vertexes and eight non-plane faces.

The form of the definition of the astroidal ellipsoid

(1) Parametrical equation:

$$x = x(u, v) = (a \cos u \cos v)^3,$$
$$y = y(u, v) = (b \sin u \cos v)^3,$$
$$z = z(v) = (c \sin v)^3,$$

where a, b, c are constant geometrical parameters. The surface has three planes of symmetry that are coordinate planes. When

$$a = b = c,$$

the astroidal ellipsoid degenerates into *hyperbolic octahedron*.
In Fig. 1, the surface with

$$a = 1\ \text{m};\ \ b = 1.3\ \text{m};\ \ c = 1.2\ \text{m};$$
$$0 \le u \le 2\pi;\ \ -\pi/2 \le v \le \pi/2$$

is represented.

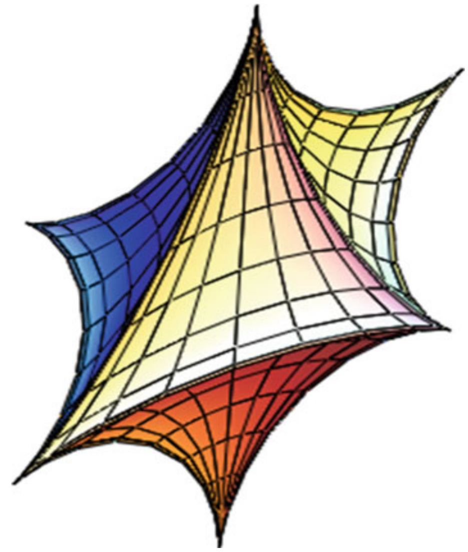


Fig. 1

Additional Literature
Nordstrand T. Astroidal ellipsoid: http://www.uib.no/people/nfyth/asttxt.htm.

■ Hyperbolic Octahedron

Hyperbolic octahedron is a special case of an astroidal ellipsoid when

$$a = b = c.$$

The form of the definition of the hyperbolic octahedron

(1) Parametric form of the definition (Fig. 1):

$$x = x(u, v) = (a \cos u \cos v)^3,$$
$$y = y(u, v) = (a \sin u \cos v)^3,$$
$$z = z(v) = (a \sin v)^3,$$

where a is a constant geometrical parameter; $-\pi/2 \le u \le \pi/2$; $-\pi \le v \le \pi$.

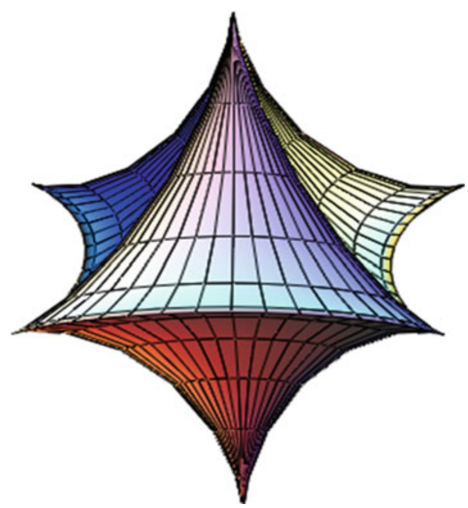


Fig. 1

At the cross-sections of the surface by the coordinate planes, three identical astroids are placed.

Coefficients of the fundamental forms of the surface and its principal curvatures:

$$A = 3a^3 \cos u \sin u \cos^3 v,$$

$$F = \frac{9}{4} a^6 \cos^5 v \sin v \sin(4u),$$

$$B = 3a^3 \cos v \sin v \sqrt{\cos^2 v(\cos^6 u + \sin^6 u) + \sin^2 v};$$

$$A^2B^2 - F^2 = \frac{81}{16} a^{12} \sin^2 2u \sin^2 2v \cos^6 u\left[\cos^2 v \cos^2 u \sin^2 u + \sin^2 v\right] = \frac{81}{16} a^{12} \sin^2 2u \sin^2 2v \cos^6 u \cdot \Psi;$$

$$L = \frac{3a^3 \cos^2 v \sin 2u \sin 2v}{4\sqrt{\cos^2 v \cos^2 u \sin^2 u + \sin^2 v}},$$

$$M = 0$$

$$N = \frac{3a^3 \sin 2u \sin 2v}{4\sqrt{\cos^2 v \cos^2 u \sin^2 u + \sin^2 v}} = \frac{L}{\cos^2 v};$$

$$k_u = \frac{2 \sin v}{3a^3 \cdot \sqrt{\cos^2 v \cos^2 u \sin^2 u + \sin^2 v} \cdot \sin 2u \cos^3 v},$$

$$k_v = \frac{\sin 2u}{3a^3 \sin 2v[\cos^2 v(\cos^6 u + \sin^6 u) + \sin^2 v]\Psi},$$

$$k_{1,2} = \frac{A^2 + B^2 \cos^2 v \pm \sqrt{(A^2 - B^2 \cos^2 v)^2 + 4F^2 \cos^2 v}}{2(A^2B^2 - F^2)} N;$$

$$K = \frac{1}{9a^6 \cos^4 v[\cos^2 v \cos^2 u \sin^2 u + \sin^2 v]^2} > 0.$$

where

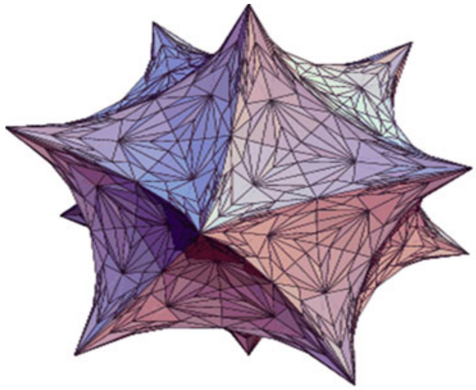
$$\Psi = \sqrt{\cos^2 v \cos^2 u \sin^2 u + \sin^2 v}$$

Additional Literature

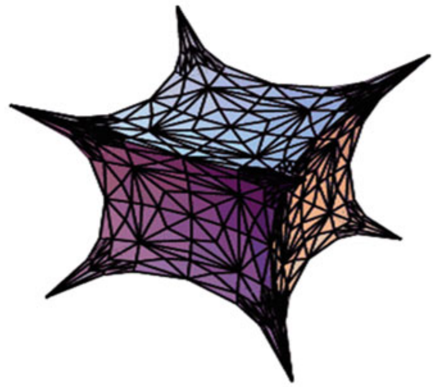
Sabitov IKh. The volume of a polyhedron as a function of the lengths of its edges. Fundam. Prikl. 1996; Mat. 2 (1), p. 305-307.

Mednykh AD, Parker J, Vesnin AYu. On hyperbolic polyhedra arising as convex cores of quasi–Fuchsian punctured torus groups. Bol. Soc. Mat., Mexicana. 2004; (3), 10, p. 357-381.

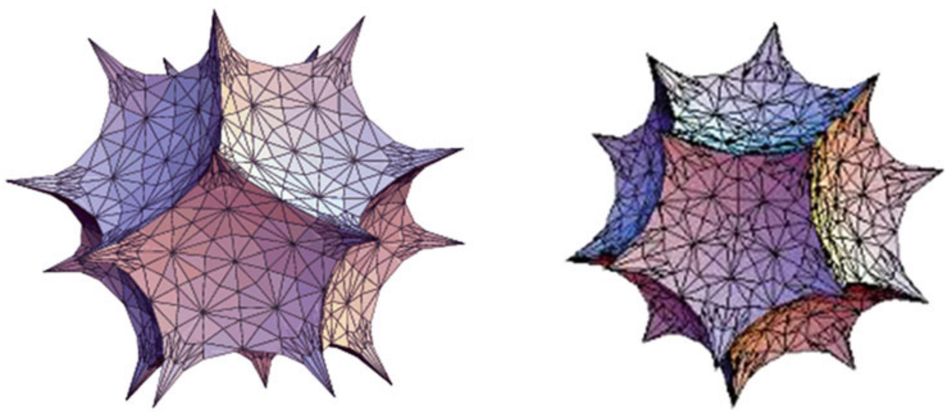
■ The Models of Quasi-polyhedrons Presented in Sites of Internet



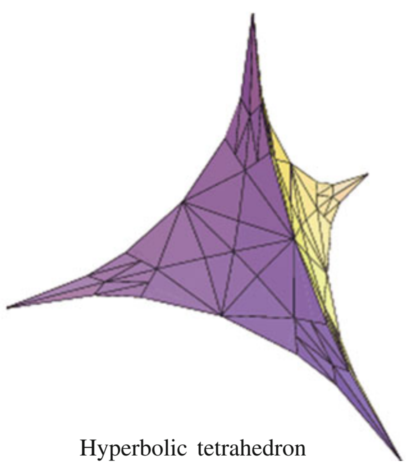
Hyperbolic icosahedron



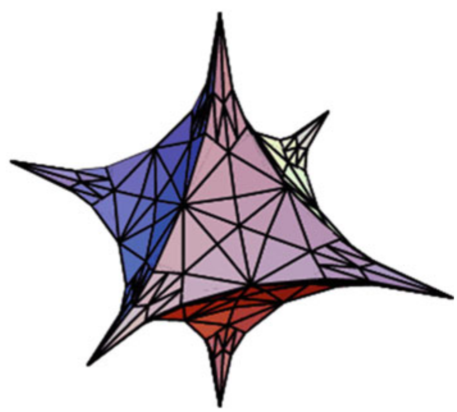
Hyperbolic cube



Hyperbolic dodecahedron



Hyperbolic tetrahedron



Hyperbolic octahedron

These six models of the quasi-polyhedrons are taken in the site: http://mathworld.wolfram.com/HyperbolicPolyhedron.html.

38 Equidistances of Double Systems

A set of points $P_1, P_2, \ldots$ equidistant from the figures $\Phi_1, \Phi_2, \ldots$ in the space R_n (n is a number of the measurements) is called *an equidistance of the system* "$\Phi_1 - \Phi_2 - \cdots$" in R_n. In this definition, *a figure* is any nonempty set of points and the term "equidistance" is not connected with the same name concept in *the plane geometry of Lobachevski* and was introduced as a comfortable abridgement.

The system "$\Phi_1 - \Phi_2 - \cdots - \Phi_m$" is called the double—triple—and so on system according to the number m of the figures forming the system. The double system with $m = 2$ is usually called a system simply.

The equidistances of fifteen double systems formed by points, straight lines, planes, spheres, and by cylindrical surfaces of revolution are known. In five cases, the equidistances were found to be the surfaces of the fourth order and in the rest cases, they were found to be the surfaces of the second order. The detailed description of the equidistances was given in a paper of V.V. Glogovskiy (1955).

Additional Literature

Glogovskiy VV. Equidistances in R_3. Nauchn. Zapiski Lvovskogo Politehn. Instituta. Lvov: LvovPI, 1955; Vol. XXX, "Fiz.-Mat", Iss. 1, p. 72-90.

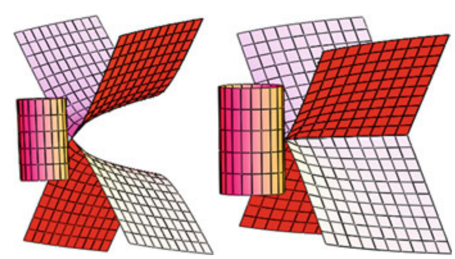
■ Equidistance of the System "Straight—Cylinder"

Equidistance of the system "straight—cylinder" is equidistant from a straight line and a cylinder.

In Fig. 1, the equidistance is presented, when the straight line is perpendicular to the axis of the cylinder,

$$l = 0.$$

If the straight line is parallel to the axis of the cylinder ($l = \infty$), then the equidistance degenerates into *a hyperbolic cylinder*.



$l = 0$; $R = 1$; $1.25 \le y \le 5$;

$0 \le x \le 6$; $0 \le x \le 4.5$

Fig. 1

Forms of definition of the surface

(1) Implicit equation (V.V. Glogovskiy 1955):

$$\sqrt{x^2 + y^2} - \sqrt{(lx - z)^2 \big/ (1 + l^2) + (y - a)^2} - R = 0,$$

where a, l, and R are constants.

(2) Explicit equation:

$$z = lx \pm \sqrt{1 + l^2}\sqrt{x^2 + y^2 - 2R\sqrt{x^2 + y^2} + R^2 - (y - a)^2}.$$

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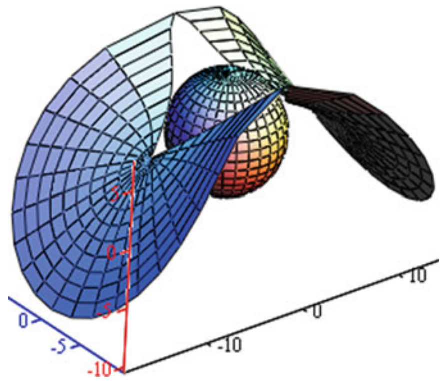
■ Equidistance of the System “Straight—Sphere”

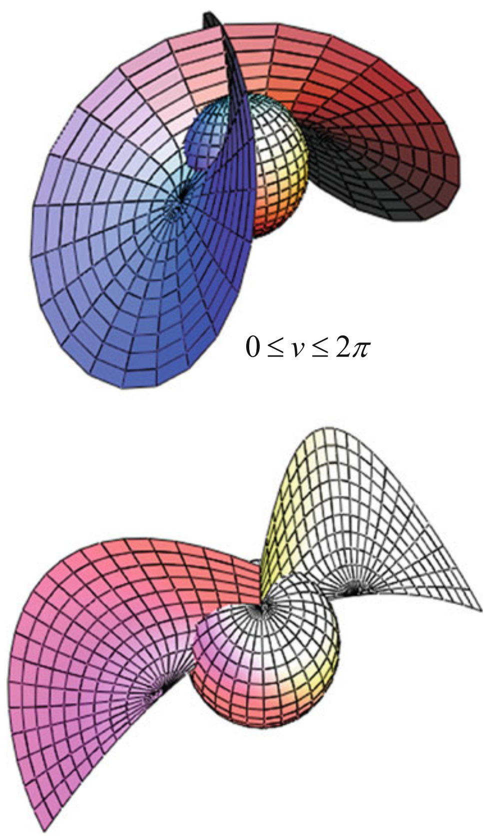
Equidistances of the system “*straight—sphere*” is equidistant from a straight line and a sphere. Here, three cases are possible:

- a straight line and a sphere do not intersect each other (Fig. 1),
- a straight line touches a sphere (Fig. 2),
- a straight line intersects a sphere (Fig. 3).



$a = 3.7;\ R = 5;\ 0 \le r \le 10;\ -1.247\pi \le v \le 0.247\pi$

Fig. 1



$a = 2.5;\ \ R = 5;\ \ 0 \le r \le 10;\ 0 \le v \le \pi$

Fig. 2

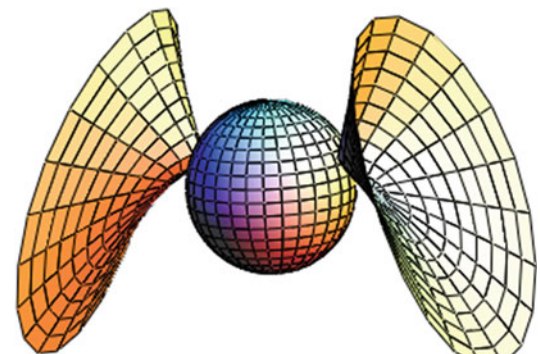

$a = 1.5;\ R = 5;\ 0 \le r \le 10;\ 0 \le v \le 2\pi$

Fig. 3

Forms of definition of the surface

(1) Implicit equation:

$$\sqrt{x^2 + y^2 + (z + a)^2} - \sqrt{y^2 + (z - a)^2} - R = 0,$$

where a and R are constants.

(2) Parametrical form of the definition at the cylindrical system of coordinates (Figs. 1, 2 and 3):

$$x = x(r, v) = \pm\sqrt{R^2 + 2R\sqrt{r^2\cos^2 v + (r\sin v - a)^2} - 4ar\sin v},$$
$$y = y(r, v) = r\cos v,$$
$$z = z(r, v) = r\sin v.$$

Three images in the figures correspond to the cases, when the straight line does not cross the sphere ($2a > R$, Fig. 1), touches it ($2a = R$, Fig. 2), and intersects the sphere ($2a < R$, Fig. 3).

(3) Implicit equation:

$$4R^2y^2 = x^4 + 2x^2(4az - R^2) + (4a^2 - R^2)(4z^2 - R^2).$$

So the equidistance in question represents an algebraic surface of the fourth order. The coordinate planes zOx and zOy are planes of symmetry.

The cross sections of the surface by the planes $x = x_c =$ const perpendicular to the straight line are curves of the second order and the cross sections $y = y_c =$ const and $z = z_c =$ const are curves of the fourth order.

(4) Explicit equation:

$$x = \pm\sqrt{R^2 + 2R\sqrt{y^2 + (z - a)} - 4az}.$$

Additional Literature

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■ Equidistance of the System “Point—Cylinder”

Equidistance of the system “point—cylinder” is equidistant from a point and a cylinder. Here, three cases are possible:

- a point and a straight line (a cylinder degenerates into the straight line, i.e., $R = 0$, Fig. 1);
- a point and a cylinder, where $0 < R < \infty$, Fig. 2;
- a point and a plane (a cylinder degenerates into the plane, i.e., $R = \infty$).

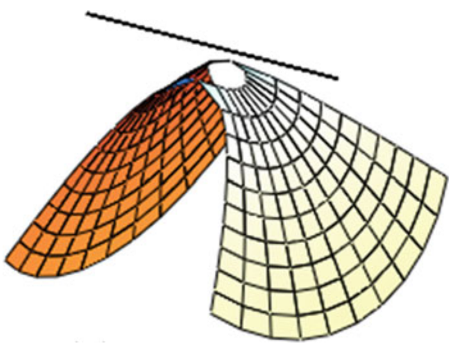
A paraboloid of revolution is equidistance of the system “point—cylinder” when we have the second ultimate case, i.e., when $R = \infty$.

Forms of definition of the surface

(1) Implicit equation:

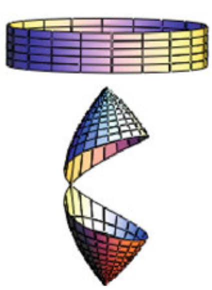
$$\sqrt{x^2 + y^2} - \sqrt{y^2 + z^2 + (x-a)^2} - R = 0,$$

where a and R are constants.



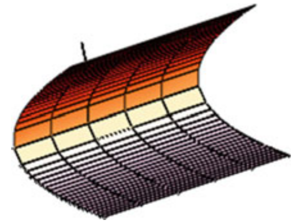
$a = 1;\ R = 0;\ 0.7072 \le r \le 5;$
$-\pi/4 \le v \le \pi/4$

Fig. 1 .



$a = 0.8;\ R = 5;$
$0 \le r \le 2.1;$
$-\pi \le v \le \pi$

Fig. 2 .



$a = 1;\ R = 0;\ 0.5 \le x \le 5;$
$-5 \le y \le 5$

Fig. 3 .

(2) Implicit form of assignment:

$$z^4 - 2R^2(2x^2 - 2y^2 + z^2) + 2a(a-2x)(z^2 - R^2)$$
$$+ a^2(a-2x)^2 + R^4 = 0.$$

Equidistance of the system “point—cylinder” is *an algebraic surface of the fourth order.*

The cross sections of the equidistance in the question by the planes $z = z_c =$ const that are perpendicular to the axis of the cylinder and by the planes of the pencil

$$y = kx$$

passing through the axis of the cylinder are the curves of the second order.

The cross sections $y = y_c =$ const and $x = x_c =$ const are the curves of the fourth order.

(3) Parametrical equations (Fig. 1):

$$x = x(r, v) = r\cos v,$$
$$y = y(r, v) = r\sin v.$$
$$z = z(r, v) = \pm\sqrt{R^2 + 2r\cos v - 2rR - a^2}$$

In Fig. 1, the equidistance of the system “point–straight line” which is the ultimate case of the equidistance “point—cylinder” with $R = 0$ is shown.

In Fig. 2, the fragment of the equidistance with the cylinder is given.

(4) Explicit equation:

$$z = \pm\sqrt{R^2 + 2ax - 2R\sqrt{x^2 + y^2} - a^2}.$$

When we have an ultimate case with $R = 0$, the equidistance degenerates into *a parabolic cylinder* (Fig. 3).

Additional Literature

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Liu Zhaoshgne Song Zuozong, Yao Xinghui. Slant Equidistant Surface Of Cylinder And Its Properties. Natural Science Journal of Harbin Normal University. 2000; 03.

■ Equidistance of the System “Straight—Torus”

Assume a torus with a radius r of the generating circles and a straight line l that is parallel to the axis of the torus. A straight line l of the system coincides with the coordinate axis Ox (Fig. 1). The line of the centers of the generating circles of the torus has a form of a circle with a radius R. *Equidistance of the system “straight—torus”* is formed by the points disposed at equal distances from the straight line and the torus.

Forms of definition of the surface

(1) Implicit equation (N.S. Gumen, O.V. Smerichko):

$$x^2 - 2cz + c^2 + (R \pm r)^2 = \pm 2(R \pm r)\sqrt{y^2 + (z - c)^2}.$$

(2) Implicit equation:

$$x^2 - 2cz_1 - a = \pm 2(R \pm r)\sqrt{y^2 + z_1^2},$$

where

$$a = c^2 - (R \pm r)^2.$$

The second variant of an implicit form of the definition of the surface was obtained from the first variant of the definition by the substitution $z = z_1 + c$, i.e., parallel translation of the coordinate system $Oxyz$ along the positive direction of the axis Oz (Fig. 1).

(3) Implicit equation:

$$(x^2 - a)^2 \big/ 4 + az_1^2 - cz_1(x^2 - a) = (R \pm r)^2 y^2.$$

The equation shows that the equidistance “straight—torus” is *an algebraic surface of the fourth order.*

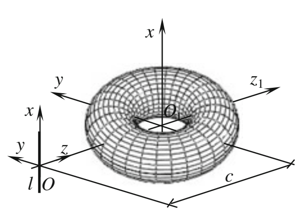

Fig. 1 .

(4) Parametrical equations (N.S. Gumen, O.V. Smerichko):

$$\begin{aligned} x &= x(\rho, \varphi) \\ &= \pm\sqrt{2p\rho[(R + pr) + pc\cos\varphi] + a},\ p = \pm 1, \\ y &= y(\rho, \varphi) = \rho \sin\varphi, \\ z &= z(\rho, \varphi) = \rho\cos\varphi, \end{aligned}$$

where $0 \le \varphi \le 2\pi$. At the cross sections of the surface by the planes $x = x_c =$ const, the curves

$$\rho = \frac{t}{1 + \varepsilon\cos\varphi} \text{ lie,}$$
$$\text{where } t = \pm\frac{x_c^2 - a}{2(R \pm r)},\ \varepsilon = \pm\frac{c}{R \pm r}.$$

Designing the equidistance “straight—torus”, we can discover three cases:

1. Let $c > R \pm r$, then $a > 0; \varepsilon > 1$. An equation of the curve is defined two hyperbolas, the focal parameters p of which are changed depending on the current coordinate x_c. The continuous family of these hyperbolas forms two surfaces of the two-sheeted hyperbolic paraboloids of the fourth order with the spaces mutually contacting at two points (Fig. 2). The directing curves for the hyperbolas are parabolas. At the boundary $x = \pm\sqrt{a}$, the hyperbolas degenerate into pairs of the intersecting straight lines.
2. Let $c < R \pm r$, then $a < 0; \varepsilon < 1$. The equation of the curve determines a continuous family of the ellipses. When $c = 0$, the ellipses degenerate into the circles with the centers on the axis x. The formed surface is a

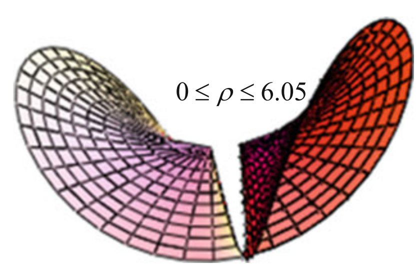

Fig. 2 $R = 5$; $r = 1$; $c = 6.1$; $p = 1$

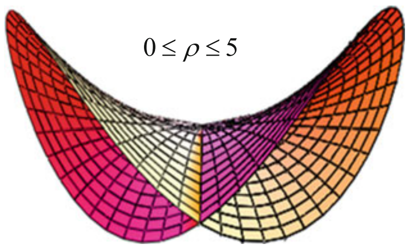

Fig. 3 $R = 5$; $r = 1$; $c = 6$; $p = 1$

one-sheeted elliptic paraboloid of the fourth order. When $c = 0$, we have a circular paraboloid. The directing curves along which the ellipse slides by its vertexes are parabolas with the same equations that the directing parabolas of the surface of the first case have.

3. Let $c = R \pm r$, then $a = 0; \varepsilon = 1$. The equation of the curve determines the continuous family of parabolas, the focal parameters of which are changed depending on the current coordinate x by the formula $p = 0.5x^2/(R \pm r)$. A continuous family of these parabolas forms two surfaces of the parabolic paraboloids of the fourth order, and a straight line z (Fig. 3) is also among them. The vertexes of the generatrix parabolas move along the directrix parabolas $z = 0.5x^2/(R \pm r)$.

Reference

Gumen NS, Smerichko OV. Paraboloids of the 4th order as the geometrical places of the points equidistant from a torus and a straight line parallel to its axis. Prikl. Geom. i Inzhen. Grafika. Kiev. 1991; Iss. 51, p. 46-52

Brief Information on the Surfaces Not Included in the Basic Content of the Encyclopedia

Brief information on some classes of the surfaces which were not picked out into the special section in the encyclopedia is presented at the part "Surfaces", where rather known groups of the surfaces are given.

At this section, the less known surfaces are noted. For some reason or other, the authors could not look through some primary sources and that is why these surfaces were not included in the basic contents of the encyclopedia. In the basis contents of the book, the authors did not include the surfaces that are very interesting with mathematical point of view but having pure cognitive interest and imagined with difficultly in real engineering and architectural structures.

Non-orientable surfaces may be represented as kinematics surfaces with ruled or curvilinear generatrixes and may be given on a picture. *The surfaces suggested by L.S. Pontryagin* are studied in a paper [1] as a two-metric ruled manifolds. A generatrix of this surface is a straight line moving at the space under corresponding law. Togino Kadzuto [2] finds the conditions, under which conjugation of several *Kuen surfaces* (special algebraic surfaces) gives a C^2-regular surface. In a work [3], an equation of *a Bress hyperboloid* for a case of general motion is obtained. The possible cases were investigated and it was proved, that in the general case, geometric locus of points with zero tangent accelerations is the one-sheeted hyperboloid, but at a special case, these are cone, hyperbolic paraboloid, circular cylinder, hyperbolic cylinder, two planes, straight line, or locus can be absent at all.

The special Euler cones are mentioned in an article [4]. *Surfaces with the constant equiaffine invariants* were studded by N.M. Onischuk [5]. This group of surfaces contains the second order surfaces and some more 6 surfaces.

Bergmann Horst [6] considered *inscribe and circumscribeellipsoids of Steiner* for *n*-metric simplex satisfying the demand of the extreme volume. They generalize the problem solved by Steiner for the plane case.

In a paper of G. Brauner [7], ruled surfaces at the three-dimensional Euclidian space permitting the conform mapping different from the isometry and similarity keeping their straight generatrixes are considered. It is proved, that only cylinders, cones and *ortoid ruled surfaces* with a constant distribution parameter possess this property. Other properties of these surfaces are considered as well.

It is known, that *the Plücker conoid* carries two-parametrical family of ellipses. The straight lines, perpendicular to the planes of these ellipses and passing through their centers, form the right congruence which is *an algebraic congruence of the4th order of the 2nd class*. This congruence attracted attention of D. Palman [8] who studied its properties. Taking into account, that on the Plücker conoid, ∞^2 of conic cross-sections are disposed, O. Bottema [9] examined the congruence of the normals to the planes of these conic cross-sections passed through their centers and prescribed a number of the properties of a congruence of straight lines, the order of which is equal to four.

In the site [10], a ruled surface "Swallow Surface", cyclic surfaces "Horn", "Feder I", «Frdre II» with generatrix circles of the variable radius lying at the planes of pencil.

The variety of the self-intersecting surfaces such as "Twisted Fano", "Twisted Plane" (Fig. 1), "Stiletto", "Lemniscape", "Slipper", "Fano Planes", "Pseudocatenoid", "Triaxial Teardrop", there are shown in the site [11].

The exotic surfaces "Jeener's klein surface", "Bonan–Jeener's klein surface 1" (Fig. 2), "Bent horns surface", "Triangular trefoil", "Triple corkscrew", "Tori link" consisting of the repeating segments can be seen in the site [12].

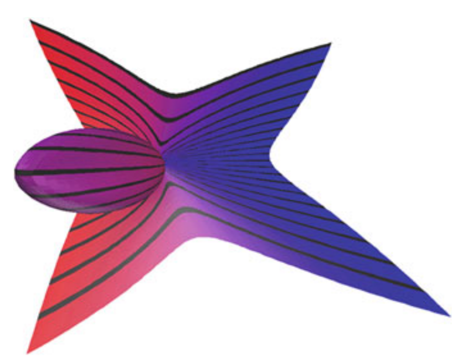

Fig. 1

DOI 10.1007/978-3-319-11773-7,

Fig. 2

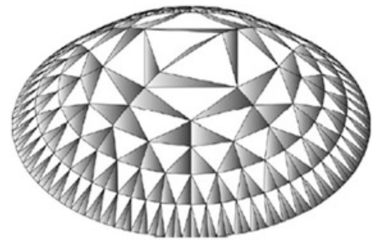

Fig. 3

Several non-orientable surfaces of constant negative Gaussian curvature ("Kink Surface", "Kuen Surface", "Breather Surface", Two-Soliton and Three-Soliton Surfaces), a number of algebraic surfaces given in implicit form (Cayley Cubic, Pretzel Surface, Pilz Surface, Orthocircles) one can see in the site [13].

Having appeared at the end of 70th years, the concepts "fractal" and "fractal geometry" after the middle of the 80th came into use of mathematicians and programmers. The word "fractal" comes from the Latin word "*fractus*" that means "fractional" and "*frahgere*" i.e. "break". A fractal surface is a surface consisting of the self-similar segments. Fractals may describe many physical phenomenon and natural formation such as mountains, clouds, trees, landscapes with the good exactness. Firstly, the fractal nature of our world was noticed by Benoît B. Mandelbrot. The main salient feature of the fractals is their continuous self-similarity. *Fractal surface* consists of polygon or bi-polynomial surfaces given by chance. In the machine graphics, fractals are constructed by simple and quick iteration algorithms [14]. In Fig. 3 [15], an elliptic paraboloid designed with the help of methods of fractal geometry is shown. In the encyclopedia, these surfaces are not described.

These are surfaces which are used usually for the acquaintance with topologic objects and with theorems or they are used for the illustrations of topological works and ideas. Such surfaces as limaçon-shaped dunce hat, Duns egg, Cayley cubic, Whitney bottle, surfaces of Morin, Seifert, Haken and some others are described in papers and books dealing with topology (see, for example, a book of Francis [16]).

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Classification of All Surfaces Presented in the Encyclopedia

(class—subclass—group—subgroup—surfaces)

1. Ruled Surfaces

1.1. Ruled Surfaces of Zero Gaussian Curvature

1.1.1. Torse Surfaces (Torses)

The literature on geometry and strength analysis of shells in the form of developable surfaces

■ Open (evolvent) helicoid
■ Monge ruled surface with the circular cylindrical directing surface
■ Developable conic helicoid
■ Developable helicoid with a cuspidal edge on the paraboloid of revolution
■ Parabolic torse
■ Torse with an edge of regression on the ellipsoid of revolution
■ Torse with an edge of regression on one sheet hyperboloid of revolution
■ Torse with an edge of regression given as $x = e^{-t}\cos t$; $y = e^{-t}\sin t$, $z = e^{-t}$
■ Torse with an edge of regression given as $x = v - v^3/3$; $y = v^2$; $z = a(v + v^3/3)$
■ Torse with an edge of regression in the form of a line of intersection of two cylinders with the perpendicular axes
■ Torse with an edge of regression in the form of hyperbolic helical line
■ Torse with a given line of curvature in the form of the second order parabola
■ Torse with generating straight lines lying in the normal planes of a spherical curve
Developable helical surface with slope angles of straight generators from 0^o till 90^o (it is presented in the Class "9. Helix-shaped surfaces")
Ruled conic limaçon of revolution (it is presented in the Subclass "4.2. Monge surfaces with a conic directrix surface")

Developable surfaces with two plane directrix curves

■ Torse with two parabolas with intersecting axes
■ Torse with two parabolas lying in intersecting planes but with parallel axes
■ Torse with two ellipses lying in parallel planes and with parallel axes
■ Torse with two parabolas having one common axis and lying in intersecting planes
■ Torse with two parabolas of the second and forth order placed in parallel planes and with parallel axes
■ Torse with parabola and circle in parallel planes
■ Torse with parabola and ellipse in parallel planes
■ Torse with hyperbola and parabola in parallel planes
■ Torse with two ellipses placed in mutually perpendicular planes
Torse with two parabolas lying in mutually perpendicular planes and with the apexes on the same coordinate axis (synonym is parabolic torse)

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The third and fourth order developable surfaces

■ Torse with an edge of regression in the form of the line of intersection of circular cylinder and circular cone
Parabolic torse (it is presented in the Group “1.1.1. Torse surfaces (torses)”)

Surfaces of constant slope

■ Torse of constant slope with directrix parabola
■ Torse of constant slope with directrix catenary
■ Torse of constant slope with an edge of regression on one sheet hyperboloid of revolution
■ Torse of constant slope with directrix ellipse
Torse of constant slope with directrix curve in the form of evolvent of the circle (synonyms are open (evolvent) helicoid and Monge’s ruled surface with circular cylindrical directing surface)
Conical helicoid of constant slope (synonym is developable conic helicoids)
Surface of constant slope with an edge of regression on a paraboloid of revolution (synonym is developable helicoid with a cuspidal edge on the paraboloid of revolution)
Surface of constant slope with directrix circle (synonym is conical surface of revolution)
Surface of constant slope with an edge of regression given as $x = v–v^3/3$; $y = v^2$; $z = a(v + v^3/3)$ (synonym is torse with an edge of regression given as $x = v–v^3/3$; $y = v^2$; $z = a(v + v^3/3)$)
Torse with an edge of regression on the ellipsoid of revolution (it is presented in the Group “1.1.1. Torse surfaces (torses)”

1.1.2. Cylindrical Surfaces

■ Cylindrical surfaces presented in the encyclopedia
■ Cylindrical helical strip
■ Elliptical cylinder
■ Parabolic cylinder
■ Hyperbolic cylinder
■ Right cylindrical surface with directrix semi-cubical parabola
■ Right cylinder with a directrix catenary
■ Right astroidal cylindrical surface
■ Evolvent cylindrical surface
■ Lemniscate cylinder
■ Right cylinder with a directrix logarithmic spiral
■ Right cylinder with a directrix hyperbolic spiral
■ Right cylinder with a directrix spiral of Fermat
■ Right cylinder with a directrix spiral of Archimedes
■ Surface of cylindrical flexible hopper-type bin for the keeping of dry materials
■ Cylindrical surface “Eight”
■ Cylindrical-and-conical spiral strip
■ Oblique circular cylinder
■ Oblique elliptical cylinder
■ Oblique parabolic cylinder
■ Oblique hyperbolic cylinder
Cylindrical surface of revolution (it is presented in the Class “35. The second order surfaces”
Right wave-shaped cylindrical surface (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)
Elliptical cylinder with two directrix circlers in mutually perpendicular planes (synonym is cylindroid with two directrix circles lying in mutually perpendicular planes)
Epicycloidal cylinder (it is presented in the Group “34.1.2. Rotational surfaces with axoids “cylinder–cylinder””)
Hypocycloidal cylinder (it is presented in the Group “34.1.2. Rotational surfaces with axoids “cylinder–cylinder””)
Cylindrical helical strip of variable pitch (it is presented in the Subclass “7.2. Helical surfaces of variable pitch”)
Sinusoidal cylindrical surface (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)
Cylindrical-and spherical spiral-shaped strip (it is presented in the Class “9. Spiral-shaped surfaces”)
Cylindrical helical strip with given slope angles of tangents at the beginning and at the end of generatrix cylindrical helical line of centers of variable pitch (it is presented in the Subclass “7.2. Helical surfaces of variable pitch”)

1.1.3. Conical Surfaces

■ Key conical surfaces presented in the encyclopedia

The literature on geometry and strength analysis of shells in the form of conical surfaces

■ Elliptical conical surface
■ Oblique circular conical surface
■ Oblique elliptical conical surface
■ Conical surface with a directrix Agnesi curve
■ Conical surface with a directrix curve in the form of Cartesian folium
■ Evolvent conical surface
■ Hyperbolic conical surface
■ Right conical surface with a plane director curve in the form of circular sinusoid
■ Spiral conical strip
■ Conical surface with a directrix curve on a sphere
■ Helical cone

Conical surface of revolution (it is presented in the Class “35. The second order surfaces”)
Parabolic conical surface (it is presented in the Class “35. The second order surfaces”)
Cubic cone (it is presented in the Subclass“36.1. Algebraic surfaces oh the third order”)
Right conical sinusoidal wave-shaped surface (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)
Waving conical surface in lines of principle curvatures with inner vertex (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)
Rotational surface with axoids “cone–cone” generated by a straight line coming through the common vertex of the axoids (it is presented in the Group “34.1.3. Rotational surfaces with axoids “cone–cone””)
Rotational surface with axoids “plane–cone” generated by a straight line coming through the vertex of a mobile cone (it is presented in the Group “34.1.4. Rotational surfaces with axoids “plane–cone»”)
Ruled conic limaçon of revolution (it is presented in the Subclass “4.2. Monge surfaces with a conic directrix surface”)
Honeycomb conical surface (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)

1.2. Ruled Surfaces of Negative Gaussian Curvature

The literature on geometry and analysis of shells in the form of ruled surfaces of negative total curvature

■ Oblique helicoid
■ Ruled surface with straight generatrixes passing through a logarithmic spiral and intersecting the fixed axis under constant angle
■ Spherical helicoid

Right helicoid (it is presented in the Class “19. Minimal Surfaces”)
Hyperboloid of one sheet (it is presented in the Class “35. Surfaces of the second order”)
One-sheet hyperboloid of revolution (it is presented in the Class “2. Surfaces of revolution”)
Convolute helicoid (it is presented in the Group “7.1.1. Ruled helical surfaces”)
Helical surface generated by binormals of a cylindrical helix (it is presented in the Group “7.1.1. Ruled helical surfaces”)
Ruled surface of the trajectory of movement of straight generatrix of an evolvent helicoid in the process of its parabolic bending (it is presented in the Class “8. Spiral surfaces”)
Spiral surface with straight generatrixes in the planes of pencil (it is presented in the Class “8. Spiral surfaces”)
Ruled rotational surfaces of negative total curvature (they are presented in the Subclass “34.1. Rotational surfaces”)
Edlinger’s surfaces (they are presented in the Class “29. Edlinger’s surfaces”)
Ruled spiroidal surfaces of negative Gaussian curvature (they are presented in the Subclass “34.2. Spiroidal Surfaces”)
Spiroidal ruled surfaces of Rachkovskaya–Kharabaev (they are presented in the Group “34.2.5. Spiroidal ruled surfaces of Rachkovskaya – Kharabaev”)
Helix-shaped preliminarily twisted strip (it is presented in the Class “10. Helix-shaped surfaces”)
Ruled surface of Skidan (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)

1.2.1. Catalan Surfaces

■ Whitney umbrella (Cartan umbrella)
■ Pseudo-developable helicoid
■ Ruled rotor cylindroid
■ Saddle in the drum
Hyperbolic paraboloid (it is presented in the Class "35. Surfaces of the second order")
Elliptic helicoid (it is presented in the Class "10. Helix-shaped surfaces")
Cayley surface (it is presented in the Class "3. Translation surfaces")
Cubic surface $x^3 + xy + z = 0$ (it is presented in the Subclass "36.1. The third order algebraic surfaces")
Spiral ruled surface with straight generatrix perpendicular to an axis of a directrix conic spiral and to the tangent of the same spiral (it is presented in the Class "8. Spiral surfaces")
Right helicoidal surface with variable pitch (it is presented in the Subclass "7.2. Helical surfaces of variable pitch")
Pseudo-developable helix-shaped surface with variable pitch (it is presented in the Class "10. Helix-shaped surfaces")
Right spherical helicoid (synonym is spherical helicoid)
Right waving helicoid (it is presented in the Class "25. Wave-shaped, waving, corrugated surfaces")

Conoids

Right helical conoid (synonym is right helicoid)
■ Parabolic conoid
■ Conoid with a directrix circle
■ Conoid with a directrix catenary
■ Right sinusoidal conoid
■ Right conoid with a directrix parabola the axis of which is parallel to the axis of conoid
■ Evolvent conoid
■ Plücker conoid
■ Wallis's conical edge
■ Zindler's conoid
■ Continuous topographic ruled surface with distributing ellipse

Cylindroids

■ Cylindroid with two directrix ellipses
■ Cylindroid with two directrix circles lying in mutually perpendicular planes
■ Ball's cylindroid
■ Cylindroid with a parabola and a sinusoid lying on the parallel ends
■ Frezier's cylindroid

1.2.2. Twice Oblique Cylindroids

■ Twice oblique trochoid cylindroid

2. Surfaces of Revolution

■ Surfaces of revolution presented in the encyclopedia
■ One-sheet hyperboloid of revolution
■ Fairing of cycloidal type
■ Pseudo-sphere
■ Paraboloid of revolution
■ Circular torus
■ Elliptic torus
■ Surface of revolution of a curve $z = b\exp(-a^2x^2)$ around the z axis
■ Two-sheeted hyperboloid of revolution

■ Surface of conjugation of two coaxial cylinders of different diameters
■ Surface of revolution «Wellenkugel»
■ Surface of conjugation of coaxial cylinder and cone
■ Surface formed by rotation of a meridian in the form of semi-cubical parabola
■ Surface of revolution of a hyperbola $z = b/x$ around the Oz axis
■ Parabolic humming-top
■ Surface of revolution of an astroid
■ Astroidal torus
■ Surface of revolution of the Agnesi curl
■ Deformed sphere
■ Surface of revolution of a parabola
■ Parabolic-and- logarithmic surface of revolution
■ Hyperbolic-and- logarithmic surface of revolution
■ Bullet nose
■ The fourth order paraboloid of revolution
■ Surface of revolution with damping circular waves
■ "Kiss surface"
■ Soucoupoid
■ Globoid (toroid)
■ Surface of revolution of a usual cycloid
■ Pseudo-catenoid
■ Surface of revolution «Pear»
■ Surface of revolution of a general sinusoid
■ Corrugated surface of revolution of a general sinusoid
■ Surface of revolution of a parabola of arbitrary position
■ Surface of revolution of a biquadrate parabola
■ Ellipsoid of revolution
■ Ding–Dong surface
■ "Eight surface"
■ Surface of revolution "Egg" of the fourth order
■ Surface of revolution "Egg" of the third order
■ Piriform Surface
■ "Drop"

Kappa Surface (it is presented in the Class "6. Continuous topographic and topographic surfaces")
Spherical surface (sphere) (it is presented in the Subclass "23.1. Surfaces of constant positive Gaussian curvature")
Cylindrical surface of revolution (it is presented in the Class "35. The second order surfaces")
Conical surface of revolution (it is presented in the Class "35. The second order surfaces")
Catenoid (it is presented in the Class "19. Minimal surfaces")
Tractroid (antisphere, tractrisoid) (synonym is pseudo- sphere)
Surface of revolution of an inclined sinusoid (synonym is helical surface with sinusoidal generatrix)
Surface of revolution given by a harmonic function $z = \ln[x^2 + y^2]^{1/2}$ (it is presented in the Class "31. Surfaces given by harmonic functions ")
Cycloidal torus (synonym is helical surface with generatrix cycloid)
Surface of revolution of the evolvent of a circle (synonym is helical surface with the generatrix evolvent of a circle)
Surface of revolution of a hyperbola of arbitrary position (it is presented in "Helical surface with generatrix curve in the form of hyperbola")
Surface of revolution of the Agnesi curl (it is presented in the Subclass "36.4. Algebraic surfaces of the sixth order")
Cyclic surfaces of revolution (it is presented in the Class "17. Cyclic surfaces")

The literature on analysis of shells in the form of surfaces of revolution

2.1. Middle Surfaces of Bottoms of Shells of Revolution Made by Winding of One Family of Threads Along the Lines of Limit Deviation

2.2. Middle Surfaces of Bottoms of Shells of Revolution Made by Plane Winding of Threads

2.3. Middle Surface of Bottom of Shell of Revolution Made by Winding of Threads Along Geodesic Lines

2.4. Middle Surfaces of Shells of Revolution with Given Properties

■ Surfaces of revolution with geometrically optimal rise
■ Middle surface of non-bending shell of revolution under uniform pressure
Surface of conjugation of coaxial cylinder and cone (it is presented in the Class "2. Surfaces of revolution")

2.5. Surfaces of Revolution with Extreme Properties

Circular torus (it is presented in the Class "2. Surfaces of revolution")
■ Surface of catenoidal type
■ "PenKa"

2.6. The Surfaces of Delaunay

Catenoid (it is presented in the Class "19. Minimal surfaces")
Spherical surface (sphere) (it is presented in the Subclass "23.1. Surfaces of the constant positive Gaussian curvature")
Cylindrical surface of revolution (it is presented in the Class "2. Surfaces of revolution")

2.6.1 Nodoid and Unduloid Surfaces of Revolution

■ Nodoid surface connecting two circular cones

3. Translation Surfaces

■ Translation surfaces presented in the encyclopedia

3.1. Surfaces of Right Translation

Cylindrical surfaces (it is presented in "Cylindrical surfaces presented in the encyclopedia")
Paraboloid of revolution (it is presented in the Class "2. Surfaces of revolution")
Elliptic paraboloid (it is presented in the Class "35. The second order surfaces")
Hyperbolic paraboloid (it is presented in the Class "35. The second order surfaces")
Surface of translation of circle along sinusoid (it is presented in the Subclass "17.3. Cyclic surfaces with a plane of parallelism")

Surface of translation of circle along ellipse (it is presented in the Subclass “17.3. Cyclic surfaces with a plane of parallelism”)
The first Scherk’s minimal surface (it is presented in the Class “19. Minimal surfaces”)
■ Circular surface of translation
■ Cayley surface
■ Surface of translation of catenary along catenary
■ Surface of translation of circle along parabola
■ Surface of translation of sinusoid along parabola
■ Elliptic surface of translation
■ Shoe surface
■ Surface of translation of parabola along hyperbola
■ Surface of translation of sinusoid along sinusoid
■ Cycloidal surface of translation
The literature on geometry and analysis of shells in the form of parallel translation surfaces

3.2. Translation Surfaces with Congruent Generatrix and Directrix Curves

Paraboloid of revolution (it is presented in the Class “2. Surfaces of revolution”)
Biquadratic surface of translation (synonym is circular surface of translation)
Bicatenary surface of translation (synonym is surface of translation of catenary along catenary)
Bi-elliptical surface of translation (synonym is elliptic surface of translation)
Bicosine surface of translation (it is presented in the Class “5. Surfaces of congruent sections”)
Bicycloidal surface of translation (synonym is cycloidal surface of translation)
Bisemicubic surface of translation (it is presented in the Class “5. Surfaces of congruent sections”)
Literature on geometry of translation surfaces with congruent generatrix and directrix curves (it is given in the Class “5. Surfaces of congruent cross sections”)

3.3. Surfaces of Oblique Translation

■ Volkov’s diagonal circular surface of translation
■ Oblique parabolic surface of translation
Oblique cosine surface of translation (synonym is bicosine surface of translation)

3.4. Velaroidal Surfaces

■ Sinusoidal velaroid
■ Surface of velaroidal type on annulus plan with two families of sinusoids
■ Parabolic velaroid
■ Elliptical velaroid

4. Carved Surfaces

■ Carved surfaces presented in the encyclopedia
Tubular surfaces (they are presented in the Subclass “17.2. Normal cyclic surfaces”)

4.1 Monge Surfaces with a Circular Cylindrical Directrix Surface

Monge ruled surfaces with a circular cylindrical directing surface (they are presented in the Group "1.1.1. Torse surfaces (torses)")
Monge surface with a cylindrical directrix surface and a circular meridian (synonym is a tubular surface with a plane line of centers in the form of the evolvent of a circle)
■ Monge surface with a cylindrical directrix surface and a parabolic meridian
■ Monge surface with a cylindrical directrix surface and with a sinusoid as meridian
■ Monge surface with a cylindrical directrix surface and a hyperbolic meridian
■ Monge surface with a cylindrical directrix surface and meridian in the form of a cycloid
■ Monge surface with a cylindrical directrix surface and meridian in the form of catenary

4.2. Monge Surfaces with a Conic Directrix Surface

■ Ruled conic limaçon of revolution

4.3. Carved Surfaces of General Type

■ Carved sinusoidal surface
■ Carved surface with directrix cubic parabola and generating ellipse
■ Carved surface with directrix ellipse and generatrix parabola
■ Carved surface with directrix cycloid and generatrix ellipse
■ Carved surface with directrix sinusoid and generatrix cycloid
■ Carved surface with directrix sinusoid and generatrix parabola
■ Carved surface with directrix sinusoid and generatrix ellipse
■ Carved surface with directrix ellipse and generatrix sinusoid
The literature on geometry and application of shells in the form of carved surfaces

5. Surfaces of Congruent Sections

Kinematical surfaces of general type (they are presented in the Class "34. Kinematical surfaces of general type")
Monge surfaces (they are presented in the Class "4. Carved surfaces")
Helical surfaces (they are presented in the Class "7. Helical surfaces")
Translation surfaces with congruent generatrix and directrix curves (they are presented in the Class "3. Translation surfaces)
Tubular surfaces (they are presented in the Subclass "17.2. Normal cyclic surfaces")
Cyclic surfaces of revolution (they are presented in the Class "17. Cyclic surfaces")
Right circular surface on a cylinder (it is presented in the Subclass "17.3. Cyclic surfaces with a plane of parallelism")
Sinusoidal helicoid (it is presented in the Class "25. Wave-shaped, waving, and corrugated surfaces")
■ Bicosine surface of translation
■ Bisemicubic surface of translation
■ Twisted surface with congruent ellipses in parallel planes
The literature on geometry, application, and analysis of shells in the form of surfaces of congruent section

6. Continuous Topographic and Topographic Surfaces

Surfaces of revolution (they are presented in the Class “2. Surfaces of revolution”)
Surfaces of right translation (they are presented in the Class “3. Translation surfaces”)
Cyclic surfaces with a plane of parallelism (they are presented in the Class “17. Cyclic surfaces”)
Catalan surfaces (they are presented in the Subclass “1.2. Ruled surfaces of negative Gaussian curvature”)
Continuous topographic ruled surface with distributing ellipse (it is presented in the Group “1.2.1. Catalan surfaces”)
■ Topographic surface with the given elliptical cross sections
■ “Trash Can”
■ “Paper Bag”
■ Kappa surface
■ Continuous topographic surface of Cassini

6.1. Aerodynamic Surfaces Given by Algebraic Plane Curves

■ Aerodynamic surface given by a continuous framework of elliptical ribs
■ Aerodynamic surfaces given by a continuous framework of water-lines in the form of generalized Agnesi curls
■ The 6^{th} order surface with Cartesian folium, ellipse, Cartesian folium lying in three principal coordinate sections
■ Surface of the 5^{th} order with parabola, ellipse, parabola lying in three principal coordinate sections
■ Aerodynamic surfaces with a continuous framework of plane curves presented in the encyclopedia
Aerodynamic surfaces with a continuous framework of plane curves (they are presented in the Class “36. Algebraic surfaces of the high orders”)

7. Helical Surfaces

7.1. Ordinary Helical Surfaces

■ Ordinary helical surfaces presented in the encyclopedia

7.1.1. Ruled Helical Surfaces

Right helicoid (it is presented in the Class “19. Minimal surfaces”)
Oblique helicoid (it is presented in the Subclass “1.2. Ruled surfaces of negative Gaussian curvature”)
Open (evolvent) helicoid (it is presented in the Group “1.1.1. Torse surfaces (torses)”)
Pseudo-open helicoid (it is presented in the Subclass “1.2. Ruled surfaces of negative Gaussian curvature”)
Cylindrical helical strip (it is presented in the Group “1.1.2. Cylindrical surfaces”)
Helical surface formed by the principle normals of helix (synonym is right helicoid)
Helical surface formed by the tangents of helix (synonym is evolvent helicoid)
■ Convolute helicoid
■ Helical surface generated by binormals of a cylindrical helix

7.1.2. Circular Helical Surfaces

■ Circular helical surface with generatrix circles in the planes of pencil
■ Tubular helical surface
■ Right circular helical surface
■ Circular helical surface with generating circles lying in the osculating plane of a helix of centers of the circles
“The Saint Elias surface” (it is presented in the Subclass “17.4. Cyclic surfaces with circles in planes of pencil”)
Surface of a helical pole (it is presented in the Subclass “17.3. Cyclic surfaces with a plane of parallelism””)
Helix-shaped preliminarily twisted surface of circular cross section (it is presented in the Subclass “10.1. Helix-shaped preliminarily twisted surfaces with plane generatrix curve” and in the Group “7.1.2. Tubular helical surface”)
Helical twisted surface with circles in the planes of pencil (it is presented in the Subclass “10.2. Helix-shaped twisted surfaces with plane generating curves in the planes of pencil”)

7.1.3. Ordinary Helical Surfaces with Arbitrary Plane Generatrix Curves

■ Dini's helicoid
■ Helical surface with parabolic generatrix of arbitrary position
■ Helical surface with sinusoidal generatrix
■ Helical surface with generatrix ellipse
■ Helical sinusoidal strip
■ Helical surface with generatrix curve in the form of evolvent of the circle
■ Astroidal helicoid
■ Helical surface with generatrix cycloid
■ Helical surface with generatrix curve in the form of hyperbola

7.2. Helical Surfaces of Variable Pitch

■ Helical surfaces of variable pitch presented in the encyclopedia
■ Cyclic helical surface with given slope angles of tangents at the beginning and at the end of a cylindrical helical line of centers of variable pitch
■ Cylindrical helical strip of variable pitch
■ Right helicoidal surface with variable pitch
■ Cyclic helical surface with a line of centers of variable pitch
■ Cylindrical helical strip with given slope angles of tangents at the beginning and at the end of the directrix cylindrical helical line of centers of variable pitch

The literature on application and analysis of shells in the form of helical surfaces

8. Spiral Surfaces

■ Spiral surfaces presented in the encyclopedia

The literature on geometry, application, and analysis of shells in the form of spiral and spiral-shaped surfaces

Cylindrical-and-conical spiral strip (it is presented in the Group "1.1.2. Cylindrical surfaces")
Spiral conical strip (it is presented in the Group "1.1.3. Conical Surfaces")
Developable conic helicoid (it is presented in the Group "1.1.1. Torse surfaces (torses)")
Torse with an edge of regression given as $x = e^{-t}\cos t$; $y = e^{-t}\sin t$, $z = e^{-t}$ (it is presented in the Group "1.1.1. Torse surfaces (torses)")
Right circular spiral surface (it is presented in the Subclass "17.3. Cyclic surfaces with a plane of parallelism")
Circular spiral surface with a generatrix circle of constant radius lying in planes of a pencil (it is presented in the Subclass "17.4. Cyclic surfaces with circles in planes of a pencil")
Tubular spiral surface (it is presented in the Group "17.2.1. Tubular surfaces")
Cyclic surface with generatrix circles in planes of a pencil and with a plane line of centers in the form of a logarithmic spiral (it is presented in the Subclass "17.4. Cyclic surfaces with circles in planes of a pencil")
Right cylinder with a directrix logarithmic spiral (it is presented in the Group "1.1.2. Cylindrical surfaces")
Ruled surface with straight generatrixes passing through a logarithmic
spiral and intersecting the fixed axis under constant angle (it is presented in the Subclass "1.2. Ruled surfaces of negative Gaussian curvature")

■ Spiral ruled surface with straight generatrix perpendicular to an axis of a directrix conic spiral and to the tangent of the same spiral
■ Spiral surface with straight generatrixes in the planes of pencil
■ Spiral surface with parabolic generatrix of arbitrary position
■ Spiral surface with a generatrix ellipse
■ Spiral surface with a hyperbolic generatrix
■ Spiral surface with generatrix in the form of cycloid

■ Spiral surface with a sinusoidal generatrix
■ Spiral surface with generatrix in the form of the evolvent of a circle
■ Spiral surface with directrix logarithmic spiral and with parabolic generatrix

9. Spiral-Shaped Surfaces

■ Spiral-shaped surfaces presented in the encyclopedia
The literature on geometry, the application, and analysis of shells in the form of spiral and spiral-shaped surfaces (the literature is presented in the Class “8. Spiral Surfaces”)
Developable helicoid with a cuspidal edge on the paraboloid of revolution (it is presented in the Group “1.1.1. Torse surfaces (torses)”)
Cyclic surface with a generating circle in the planes of pencil and with a plane line of centers in the form of spiral of Archimedes (it is presented in the Group “17.4.1. Cyclic surfaces with circles in the planes of pencil and with a plane center-to-center line”)
Cyclic surface with generatrix circles of variable radius and with a plane line of centers constructed about a circular cylinder (it is presented in the Group “17.4.1. Cyclic surfaces with circles in the planes of pencil and with a plane center-to-center line)
Cyclic surface in a cylinder (it is presented in the Class “10. Helix-Shaped Surfaces”)
Right circular spiral-shaped surface with a generatrix circle of variable radius (it is presented in the Subclass “17.3. Cyclic surfaces with a plane of parallelism”)
Spiral-shaped surface with generatrix sinusoids and with a directrix line of constant pitch on a circular cone (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)
Normal cyclic surface with generatrix circles of variable radius and with a plane center-to-center line in the form of a logarithmic spiral (it is presented in the Group “17.2.2. Normal cyclic surfaces with generatrix circles of variable radius”)
Conical surface with a directrix curve on a sphere (in is presented in the Group “1.1.3. Conical surfaces”)
Tubular surface on the sphere (it is presented in the Group “17.2.1. Tubular surfaces”)
Surfaces with a spherical directrix curve (they are presented in the Class “21. Surfaces with a spherical directrix curve)
■ Tubular surface with a line of centers on one-sheet hyperboloid of revolution
■ Ruled surface of the trajectory of movement of straight generatrix of an evolvent helicoid in the process of its parabolic bending
■ Snail surface
■ Steinbach screw
■ Spiral-shaped surface with parabolic generatrixes and directrix line of constant pitch on a circular cone
■ Hyperbolic helicoid
■ “Cornucopia”
■ Cylindrical-and-spherical spiral-shaped strip
■ Transcendental surface with two families of plane isotropic coordinate lines
■ Spiral-shaped surface with elliptical generatrixes and with a line of centers of constant pitch on a circular cone
■ “Seashell”

9.1. Spiral-Shaped Cyclic Surfaces with Circles of Variable Radius in the Planes of Pencil

■ Spiral screw
■ Spiral-shaped surface “Shell without vertex”
■ Spiral-shaped surface “Shell with vertex”

10. Helix-Shaped Surface

■ Helix-shaped surfaces presented in the encyclopedia
Normal cyclic helix-shaped surface consisting of identical elements (it is presented in the Class "17. Cyclic Surfaces")
Hyperbolic helicoid (it is presented in the Class "9. Spiral-shaped surfaces")
Right waving helicoid (it is presented in the Class "25. Wave-shaped, waving, and corrugated surfaces)
Tubular helix-shaped surface with a line of centers of variable pitch (it is presented in the Subclass "17.2. Normal Cyclic Surfaces")
Spiroidal surfaces with axoids "cylinder - cylinder" (they are presented in the Subclass "34.2. Spiroidal Surfaces")
■ Elliptic helicoid
■ Helix-shaped twisted strip with straight generatrixes in the planes of pencil
■ Helix-shaped preliminarily twisted strip
■ Helix-shaped surface with variable elliptical cross section
■ Helix-shaped surface with generatrix circle of variable radius lying on a plane
■ Cyclic surface in a cylinder
■ Quasi-helical cyclic surface with boundary circles given in advance
■ Developable helix-shaped surface with slope angles of straight generators changing from 0^o till 90^o
■ Pseudo-developable helix-shaped surface with variable pitch
■ Rotational oblique helicoid

10.1. Helix-Shaped Preliminarily Twisted Surfaces with Plane Generatrix Curve

■ Helix-shaped preliminarily twisted surface of elliptical cross section
■ Helix-shaped preliminarily twisted surface of circular cross section

10.2. Helix-Shaped Twisted Surfaces with Plane Generating Curves in the Planes of Pencil

■ Helix-shaped twisted surface of elliptical cross section in the planes of pencil
■ Helical twisted surface with circles in the planes of pencil

11. Blutel Surfaces

Quadrics (they are presented in the Class "35. The Second Order Surfaces")
Dupin's cyclides (they are presented in the Group "17.1.2. Dupin's Cyclides")

12. Veronese Surfaces

Steiner surfaces (synonyms are the Roman surface, cross cap, Steiner surfaces of the first and second types)

13. Tzitzéica Surfaces

■ Tzitzéica surface of the second order with the centroaffine invariant $I = -a^2$
■ Tzitzéica surface of the third order with the centroaffine invariant $I = 1/27$
■ Tzitzéica surface of the second order with the centroaffine invariant $I = a^2$
■ Tzitzéica surface of the third order with the centroaffine invariant $I = -4/27$

14. Peterson Surfaces

Monge surfaces with a circular cylindrical directrix surface (they are presented in the Subclass "4.1. Monge Surfaces With a Circular Cylindrical Directrix Surface")
Surfaces of translation of the 2nd order curves (they are presented in the Class "3. Translation surfaces")
Surfaces of revolution of the 2nd order curves (they are presented in the Class "2. Surfaces of Revolution")
Minimal surfaces of Peterson (they are presented in the Class "19. Minimal surfaces")
■ Bendings of a three-axial ellipsoid

15. Surfaces of Bézier

■ Bicubic surface of Bézier
The literature on application and geometry of surfaces of Bézier

16. Quasi-Ellipsoidal Surfaces

■ Quasi-ellipsoidal surface with three values of semi-axes

16.1. Quasi-Ellipsoidal Surfaces with Six Values of Semi-Axes

■ Quasi-ellipsoidal surface with concave segments between ribs
■ Ruled quasi-ellipsoidal surface
■ Quasi-ellipsoidal surface with convex segments between ribs

16.2. Quasi-Ellipsoidal Surfaces with Cylindrical Insertions

■ Quasi-ellipsoidal surface with convex segments and a cylindrical insertion

17. Cyclic Surfaces

■ Classification of cyclic surfaces
Literature on geometry and analysis of shells in the form of cyclic surfaces
Circular helical surface with generating circles lying in the osculating plane of a helix of centers of the circles (it is presented in the Group "7.1.2. Circular helical surfaces")
Regular cyclic rotational surfaces (they are presented in "Cyclic rotational surface with axoids "cylinder–plane"")

17.1. Canal Surfaces

Tubular surfaces (they are presented in the Subclass "17.2. Normal cyclic surfaces")

17.1.1. Canal Surfaces of Joachimsthal

Virich's cyclic surface (it is presented in the Class "32. Surfaces of Joachimsthal")

Canal surfaces of Joachimsthal in the lines of principle curvature

■ Epitrochoidal surface
■ Joachimsthal cosine canal surfaces of the 1st type
■ Joachimsthal cosine canal surfaces of the 2nd type
■ Joachimsthal cosine canal surfaces of the 3d type

17.1.2. Dupin Cyclides

■ Dupin cyclides of the first type (of the forth order)
■ Dupin cyclide of the second type (of the third order)
Circular torus is Dupin cyclide of the third type but of the forth order (it is presented in the Class "2. Surfaces of revolution")

17.2. Normal Cyclic Surfaces

17.2.1. Tubular Surfaces

■ Tubular surfaces presented in the encyclopedia
Tubular helical surface (it is presented in the Group "7.1.2. Circular Helical Surfaces")
Circular torus (it is presented in the Class "2. Surfaces of revolution")
Wave-shaped torus on the sphere (it is presented in the Class "25. Wave-shaped, waving, and corrugated surfaces")
Helix-shaped preliminarily twisted surface of circular cross section (it is presented in the Class "10. Helix-shaped surfaces")
Tubular loxodrome (it is presented in the Class "21. Surfaces with spherical director curve")
Tubular surface with a center-to-center line on the one- sheet hyperboloid of revolution (it is presented in the Class "9. Spiral-shaped surfaces")
Tubular surface winding the sphere (it is presented in the Class "21. Surfaces with spherical director curve")
■ Tubular spiral surface
■ Tubular surface with a plane line of centers in the form of the evolvent of a circle
■ Tubular surface with a plane line of centers in the form of a cycloid
■ Tubular surface with a plane parabolic line of centers
■ Tubular surface with a plane hyperbolic centerline
■ Tubular surface with a plane elliptical line of centers
■ Tubular surface with a plane sinusoidal centerline
■ Tubular surface on the sphere
■ Tubular helix-shaped surface with a center-to-center line of variable lead

17.2.2. Normal Cyclic Surfaces with Generatrix Circle of Variable Radius

■ Normal cyclic surfaces with generatrix circle of variable radius presented in the encyclopedia
■ Normal cyclic surface with plane circular line of centers and with a generator circle of variable radius
■ Normal cyclic helix-shaped surface consisting of identical elements
■ Normal cyclic surface with generatrix circles of variable radius and with a plane center-to-center line in the form of a logarithmic spiral
■ Normal cyclic surface with generatrix circle of variable radius and with a plane center-to-center line in the form of a conical spiral
■ Connecting canal for two cylindrical surfaces with parallel axes
■ Normal cyclic surface with an elliptical line of centers and with a generatrix circle of variable radius (the first type)
■ Normal cyclic surface with an elliptical line of centers and with a generatrix circle of variable radius (the second type)

17.3. Cyclic Surfaces with a Plane of Parallelism

Right circular helical surface (it is presented in the Group “7.1.2. Circular helical surface”)
Oblique circular conical surface (it is presented in the Group “1.1.3. Conical surface”)
Oblique circular cylinder (it is presented in the Group “1.1.2. Cylindrical surfaces”)
Circular surface of translation (it is presented in the Subclass “3.1. Surfaces of right translation”)
Surface of translation of circle along parabola (it is presented in the Subclass “3.1. Surfaces of right translation”)
Volkov’s diagonal circular surface of translation (it is presented in the Subclass “3.3. Surfaces of oblique translation”)
■ Right circular spiral surface
■ Surface of translation of a circle along a sinusoid
■ Surface of translation of a circle along an elliptical centerline
■ Surface of a helical pole
■ Right circular spiral-shaped surface with a generatrix circle of variable radius…
■ Right circular surface on a cylinder

17.4. Cyclic Surfaces with Circles in Planes of Pencil

■ Cyclic surfaces with circles in planes of pencil presented in the encyclopedia
Circular helical surface with generatrix circles in the planes of pencil (it is presented in the Group “7.1.2. Circular Helical Surfaces”)
Spiral-shaped cyclic surfaces with circles of variable radius in the planes of pencil (they are presented in the Class “9. Spiral-Shaped Surfaces”)
Helix-shaped surface with generatrix circle of variable radius lying on a plane (it is presented in the Class “10. Helix-Shaped Surfaces”)
Helical twisted surface with circles in the planes of pencil (it is presented in the Class “10. Helix-Shaped Surfaces”)
Wave-shaped torus (synonym is cyclic surface with circles in the planes of pencil and with a waving line of centers on a cylinder)
Cyclic surface in a cylinder (it is presented in the Class “10. Helix-shaped surfaces”)
Quasi-helical cyclic surface with boundary circles given in advance (it is presented in the Class “10. Helix-shaped surfaces”)
Cyclic helical surface with a line of centers of variable pitch (it is presented in the Subclass “7.2. Helical surfaces of variable pitch”)
Cyclic helical surface with given slope angles of tangents at the beginning and at the end of a cylindrical helical line of centers of variable pitch (it is presented in the Subclass “7.2. Helical surfaces of variable pitch”)
■ Circular spiral surface with a generatrix circle of constant radius lying in planes of a pencil
■ “The Saint Elias surface”
■ Cyclic surface with circles in the planes of pencil and with a waving line of centers on a cylinder
■ Cyclic surface with circles in the planes of meridians of the sphere and with a center-to-center line on the same sphere
■ “Clover knot”
■ “Hornlet”

17.4.1. Cyclic Surfaces with Circles in the Planes of Pencil and with a Plane Center-to-Center Line

Circular torus (it is presented in the Class “2. Surfaces of revolution”)
Preliminarily twisted circular torus (it is presented in “Helix-shaped preliminarily twisted surface of circular cross section)
Epitrochoidal surface (it is presented in the Group “17.1.1. Canal surfaces of Joachimsthal”)
Dupin cyclides (they are presented in the Group “17.1.2. Dupin cyclides”)
Canal surfaces of Joachimsthal (they are presented in the Group “17.1.1. Canal surfaces of Joachimsthal”)
“Cornucopia” (it is presented in the Class “9. Spiral-shaped surfaces”)

- Cyclic surface with a generatrix circle in the planes of pencil and with a plane line of centers in the form of a logarithmic spiral
- Cyclic surface with generatrix circles of variable radius and with a plane line of centers constructed about a circular cylinder
- Cyclic surface with a generating circle in the planes of pencil and with a plane line of centers in the form of spiral of Archimedes
- Cyclic surface with a generatrix circle in the planes of pencil and with a plane centerline in the form of the 2^{nd} order curve
- Cyclic surface with circles in planes of pencil and with a straight line of centers
- Cyclic surface with circles in planes of pencil with a straight directrix and a fixed straight of pencil that are lying on different sides of a plane center-to-center line
- Cyclic surface with circles in planes of pencil, with a straight directrix and a fixed straight of pencil that are lying on the same side of a plane center-to-center line
- Cyclic surface with circles of variable radius in planes of pencil and with three straight parallel directrixes

17.5. Cyclic Surfaces of Revolution

Circular torus (it is presented in the Class "2. Surfaces of revolution")
Spherical surface (it is presented in the Class "23. Surfaces of the constant positive Gaussian curvature")

- Cyclic surface of revolution with the rotation axis parallel to the planes with generatrix circles
- Cyclic surfaces of revolution with the rotation axis intersecting the planes with generatrix circles at the constant angle

18. One-Sided Surfaces

Steiner surface of the first type (it is presented in "The Roman surface")
The literature on geometry of one-sided surfaces

- One-sided ruled surface (Möbius strip)
- Cross cap
- The Roman surface
- The Klein surface (The Klein bottle)
- The Boy surface
- The lemniscate one-sided surface

19. Minimal Surfaces

The literature on geometry of minimal surfaces and analysis of shells having the form of these surfaces

- Catenoid
- Right helicoid
- Scherk's minimal surface (the first one)
- Enneper's surface
- Schwarz surface
- Neovius' surface
- Catalan's surface
- Bour's minimal surface
- Costa minimal surface
- Gyroid
- Henneberg minimal surface

■ Trinoid
■ Lichtenfels minimal surface
■ The second Scherk's minimal surface
■ Richmond's minimal surface
Hoffman's minimal surface (it is presented at the end of "Minimal surfaces presented in Internet sites")
Transcendental affine minimal surface (it is presented in the Class "20. Affine minimal surfaces")
Chen–Gackstatter surfaces (they are presented at the end of "Minimal surfaces presented in Internet sites")
Lopez Minimal Surface (it is presented at the end of "Minimal surfaces presented in Internet sites")
Oliveira's minimal surface (it is presented at the end of "Minimal surfaces presented in Internet sites")

19.1. Minimal Surfaces Pulled Over a Rigid Support Contour and Given by Point Frame

19.2. Minimal Surfaces with Free Boundaries

■ Minimal surfaces presented in Internet sites

19.3. Complete Minimal Surfaces

Catenoid (it is presented in the Class "19. Minimal surfaces")
Enneper's surface (it is presented in the Class "19. Minimal surfaces")
Right helicoid (it is presented in the Class "19. Minimal surfaces")
Scherk's minimal surface (it is presented in the Class "19. Minimal surfaces")
Richmond's minimal surface (it is presented in the Class "19. Minimal surfaces")

19.4. Minimal Surfaces of Peterson

Catenoid (it is presented in the Class "19. Minimal surfaces")

19.5. Minimal Surfaces of Thomsen

Enneper's surface (it is presented in the Class "19. Minimal surfaces")
■ Minimal surface of Thomsen permitting transition to Enneper's surface

19.6. The Chen-Gackstatter Surfaces

19.7. Algebraic Minimal Surfaces

Enneper's surface (it is presented in the Class "19. Minimal surfaces")

19.8. Embedded Triply-Periodic Minimal Surfaces

■ Several examples of embedded triply-periodic minimal surfaces

20. Affine Minimal Surfaces

Enneper's surface (it is presented in the Class "19. Minimal surfaces")
Elliptical paraboloid (it is presented in the Class "35. The 2nd order surfaces")
Minimal surfaces of Thomsen (they are presented in the subclass "19.5. Minimal surfaces of Thomsen")
Ttranscendental surface with two families of plane isotropic coordinate lines (it is presented in the Class "9. Spiral-shaped surfaces")
■ Transcendental affine minimal surface

21. Surfaces with Spherical Director Curve

■ Surfaces with a spherical director curve presented in the encyclopedia
Spherical helicoid (it is presented in the Subclass "1.2. Ruled surfaces of negative Gaussian curvature")
Right spherical helicoid (synonym is spherical helicoid)
Conical surface with a directrix curve on a sphere (it is presented in the Group "1.1.3. Conical surfaces")
Cylindrical-and-spherical spiral-shaped strip (it is presented in the Class "9. Spiral-shaped surfaces")
Tubular surface on the sphere (it is presented in the Group "17.2.1. Tubular surfaces")
Cyclic surface with circles in the planes of meridians of the sphere and with a center-to-center line on the same sphere (it is presented in the Subclass "17.4. Cyclic surfaces with circles in planes of pencil")
Wave-shaped torus on the sphere (it is presented in the Class "25. Wave-shaped, waving, and corrugated surfaces")
Torse with generating straight lines lying in the normal planes of a spherical curve (it is presented in the Group "1.1.1. Torse surfaces (torses)")
■ Surface with a spherical director curve and with a parabolic generatrix in the planes of meridians of the sphere
■ Surface with a spherical director curve and with an elliptical generatrix in the planes of meridians of the sphere
■ Tubular loxodrome
■ Tubular surface winding the sphere

22. Weingarten Surfaces

Surfaces of revolution (they are presented in the Class "2. Surfaces of revolution")
Surfaces of the constant Gaussian curvature (they are presented in the Class "23. Surfaces of the constant Gaussian curvature")
Surfaces of the constant mean curvature (they are presented in the Class "24. Surfaces of the constant mean curvature")

23. Surfaces of the Constant Gaussian Curvature

23.1. Surfaces of the Constant Positive Gaussian Curvature

■ Spherical surface (sphere)
■ Rembs' surface
■ Sievert's surface

23.2. Surfaces of the Constant Negative Gaussian Curvature

Pseudosphere (it is presented in the Class "2. Surfaces of revolution")
Dini's helicoid (it is presented in the Group "7.1.3. Ordinary helical surfaces with arbitrary plane generatrix curves")
■ Kuen surface
The literature on geometry and analysis of shells in the form of surfaces of the constant negative Gaussian curvature

Surfaces of the constant zero Gaussian curvature

Ruled surfaces of zero Gaussian curvature (they are presented in the Class “1. Ruled surfaces”)

24. Surfaces of the Constant Mean Curvature

Spherical surface (sphere) (it is presented in the Class “23. Surfaces of the constant positive Gaussian curvature”)
Cylindrical surface of revolution (it is presented in the Class “35. The second order surfaces”)
Minimal surfaces (they are presented in the Class “19. Minimal surfaces”)
Nodoid and unduloid surfaces of revolution (they are presented in the Group “2.6.1. Nodoid and unduloid surfaces of revolution”)
The literature on geometry of surfaces of the constant mean curvature

25. Wave-Shaped, Waving, and Corrugated Surfaces

The literature on geometry, analysis, and the application of shells in the form of wave-shaped, waving, and corrugated surfaces
■ Wave-shaped, waving, and corrugated surfaces presented in the encyclopedia…
Right conical surface with a plane director curve in the form of circular sinusoid (it is presented in the Group “1.1.3. Conical surfaces”)
Right sinusoidal conoid (it is presented in the Group “1.2.1. Catalan Surfaces”)
Surface of translation of sinusoid along parabola (it is presented in the Class “3.Translation surfaces”)
Surface of translation of sinusoid along sinusoid (it is presented in the Class “3. Translation surfaces”)
Surface of revolution of a general sinusoid (it is presented in the Class “2. Surfaces of revolution”)
Corrugated surface of revolution of a general sinusoid (it is presented in the Class “2. Surfaces of revolution”)
Carved sinusoidal surface (it is presented in the Subclass “4.3. Carved surfaces of general type”)
Carved surface with directrix sinusoid and generatrix ellipse (it is presented in the Subclass “4.3. Carved surfaces of general type”))
Carved surface with directrix ellipse and generatrix sinusoid (it is presented in the Subclass “4.3. Carved surfaces of general type”)
Carved surface with directrix sinusoid and generatrix cycloid (it is presented in the Subclass “4.3. Carved surfaces of general type”)
Cyclic surface with circles in the planes of pencil and with a waving line of centers on a cylinder (it is presented in the Subclass “17.4. Cyclic surfaces with circles in planes of pencil”)
Spiral surface with a sinusoidal generatrix (it is presented in the Class “8. Spiral surfaces”)
Wave-shaped torus (synonym is cyclic surface with circles in the planes of pencil and with a waving line of centers on the cylinder)
Right circular surface on a cylinder (it is presented in the Subclass “17.4. Cyclic surfaces with a plane of parallelism“)
Spiroidal surfaces with axoides “cylinder–cylinder” (they are presented in the Subclass “34.2. Spiroidal surfaces”)
■ Sinusoidal helicoid
■ Waving sinusoidal velaroid
■ Right conical sinusoidal wave-shaped surface
■ Waving conical surface in lines of principle curvatures with inner vertex
■ Honeycomb conical surface
■ Right wave-shaped cylindrical surface
■ Skidan’s ruled surface
■ Right waving helicoid

- Wave-shaped torus on the sphere
- Sinusoidal cylindrical surface
- Spiral-shaped surface with generatrix sinusoids and with a directrix line of constant pitch on a circular cone
- Waving surface with the pseudo Agnesi curls of cylindrical type
- Waving surface with the pseudo Agnesi curls on a circular plan
- Corrugated paraboloid of revolution
- Corrugated sphere
- Ruled rotational surface with axoids “plane–cylinder”
- Parabolic rotational surface with axoids “plane–cylinder”
- Sphere with cycloidal crimps
- Waving surface with cubical parabolas
- Waving surface with semi-cubical parabolas

25.1. Waving Chains with Elliptical Cross Sections Limited by Surfaces of the 2nd Order

- Waving chain with elliptical cross sections limited by an elliptical cylinder
- Waving chain with elliptical cross sections limited by an elliptical cone
- Waving chain with elliptical cross sections limited by an elliptical paraboloid

26. Surfaces of Umbrella Type

- Surfaces of umbrella type presented in the encyclopedia

The literature on geometry, the application, and analysis of umbrella shells and shells of umbrella type

- Surface of umbrella type with parabolic generatrixes and with the opening at the vertex
- Surface formed by parabolas but with radial waves damping in the central point
- Waving ellipsoidal surface
- Paraboloid of revolution with radial waves
- Surface of umbrella type with parabolic generatrixes and with a circular opening at the vertex

Skidan’s ruled surface (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)
Corrugated paraboloid of revolution (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)
Waving surface with the pseudo Agnesi curls on a circular plan (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)
Waving surface with cubical parabolas (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)
Sphere with cycloidal crimps (it is presented in the Class “25. Wave-shaped, waving, and corrugated surfaces”)

26.1. Surfaces of Umbrella Type with the Central Plane Point

- Surface formed by cubic parabolas but with radial waves damping in the central point
- Crossed trough
- Umbrella surface formed by biquadratic parabolas and with astroidal level line
- Surface of umbrella type formed by cubic parabolas on a cycloidal plan

26.2. Surfaces of Umbrella Type with Singular Central Point

- Surface of umbrella type with a sinusoidal generatrix
- Surface of umbrella type formed by semi-cubic parabolas on a cycloidal plan

27. Special Profiles of Cylindrical Products

■ Triangular profile of cylindrical fragment of a shaft for the profile detachable joint
■ Superellipses
■ Two types of aerodynamic cylindrical profiles
■ Generalized superellipses
■ Compound profiles formed by a curve and its mirror reflection
■ Profiles of products of trochoidal rotary machines

28. Bonnet Surfaces

Cylindrical surface of revolution (it is presented in the Class "35. The second order surfaces")
Right cylinder with a directrix logarithmic spiral (it is presented in the Group "1.1.2. Cylindrical surfaces")

29. Edlinger's Surfaces

One-sheet hyperboloid of revolution (it is presented in the Class "2. Surfaces of revolution")

30. Coons Surfaces

■ Coons surfaces on a curvilinear quadrangular plane

31. Surfaces Given by Harmonic Functions

Right helicoid ($z = c\mathrm{arctg}(y/x)$) (it is presented in the Subclass "1.2. Ruled surfaces of negative Gaussian curvature")
■ Surface of revolution given by the harmonic function $z = \ln[x^2 + y^2]^{1/2}$
■ Harmonic surface of right translation of a sinusoid with changing amplitude
■ Harmonic surface of right translation

32. Surfaces of Joachimsthal

Canal surfaces of Joachimsthal (they are presented in the Group "17.1.1. Canal surfaces of Joachimsthal")
Surfaces of revolution (they are presented in the Class "2. Surfaces of revolution")
■ Virich cyclic surface
The literature on geometry of Joachimsthal surfaces

33. Saddle Surfaces

Saddle in the drum (it is presented in the Group "1.2.1. Catalan surfaces")
Ruled saddle surfaces (synonym is ruled surfaces of negative Gaussian curvature)
Surfaces of the constant negative Gaussian curvature (they are presented in the Subclass "23.2. Surfaces of the constant negative Gaussian curvature")
Minimal surfaces (they are presented in the Class "19. Minimal surfaces ")

- Saddle surface of the ε class
- Peano saddle
- Narrowing saddle surface of Rosendorn
- Flat saddle in the drum
- Monkey saddle
- Saddle surface with zero rotation of horn

34. Kinematical Surfaces of General Type.

The literature on geometry of rotational and spiroidal surfaces
Surfaces of right translation (they are presented in the Class "3. Translation surfaces")

34.1. Rotational Surfaces

Surfaces of revolution (they are presented in the Class "2. Surfaces of Revolution")
Oblique helicoid (it is presented in the Subclass "1.2. Ruled surfaces of negative Gaussian curvature")
Convolute helicoid (it is presented in the Group "7.1.1. Ruled helical surfaces")
Rotational surfaces with axoids "plane–cylinder" (they are presented in the Class "25. Wave-shaped, waving, and corrugated surfaces")
Monge surfaces with a circular cylindrical directrix surface (they are given in the Class "4. Carved surfaces")
Rotational surfaces with axoids "cone–plane» (synonym is Monge surface with a conic directrix surface)

34.1.1. Rotational Surfaces with Axoids "Cylinder–Plane"

- Ruled rotational surface with axoids "cylinder–plane"
- Cyclic rotational surface with axoids "cylinder–plane"

34.1.2. Rotational Surfaces with Axoids "Cylinder–Cylinder"

- Ruled rotational surface of Lusta
- Epicycloidal cylinder
- Hypocycloidal cylinder
- Rotational surface with axoids "cylinder–cylinder" formed by a straight not intersecting the axis of a mobile cylinder in the process of external rolling
- Rotational surface with axoids "cylinder–cylinder" formed by a straight not intersecting the axis of a mobile cylinder in the process of internal rolling
- Rotational surface with axoids "cylinder–cylinder" formed by a straight intersecting the axis of a mobile cylinder in the process of external rolling
- Rotational surface with axoids "cylinder–cylinder" formed by a straight intersecting the axis of a mobile cylinder in the process of internal rolling

34.1.3. Rotational Surfaces with Axoids "Cone–Cone"

- Rotational surface with axoids "cone–cone" generated by a straight line coming through the common vertex of the axoids (external rolling)
- Rotational surface with axoids "cone–cone" generated by a straight line coming through the common vertex of the axoids (internal rolling)
- Rotational e surface with axoids "cone–cone" formed by a straight intersecting the axis of a mobile cone in the process of external rolling
- Rotational surface with axoids "cone–cone" formed by a straight parallel to the axis of a mobile cone in the process of external rolling
- Rotational surface with axoids "cone–cone" formed by a parabola in the process of external rolling (the first type)
- Rotational surface with axoids "cone–cone" formed by a parabola in the process of external rolling (the second type)

34.1.4. Rotational Surfaces with Axoids “Plane–Cone”

- Rotational surface with axoids “plane - cone” generated by a straight line coming through the vertex of a mobile cone
- Rotational surface with axoids “plane–cone” formed by a straight intersecting the axis of a mobile cone
- Rotational surface with axoids “plane–cone” formed by a straight parallel to the axis of a mobile cone

34.2. Spiroidal Surfaces

Ordinary helical surfaces (they are presented in the Subclass “7. 1. Ordinary helical surfaces”)

34.2.1. Spiroidal Surfaces with Axoids “Cylinder–Plane”

- Regular ruled cylindrical spiroidal surface
- Regular cyclic cylindrical spiroidal surface
- Cylindrical helical limaçon with parabolic generatrix

34.2.2. Spiroidal Surfaces with Axoids “Cylinder–Cylinder”

- Spiroidal surfaces with axoids “cylinder–cylinder” formed by a straight not intersecting the axis of a mobile cylinder in the process of external rolling
- Spiroidal surface with axoids “cylinder–cylinder” formed by a straight not intersecting the axis of a mobile cylinder in the process of internal rolling
- Spiroidal surface with axoids “cylinder–cylinder” formed by a straight intersecting the axis of a mobile cylinder in the process of external rolling
- Spiroidal surface with axoids “cylinder–cylinder” formed by a straight intersecting the axis of a mobile cylinder in the process of internal rolling

34.2.3. Spiroidal Surfaces with Axoids “Plane–Cone”

- Spiroidal surface with axoids “plane–cone” formed by a straight passing through the vertex of a mobile cone…

34.2.4. Spiroidal Surfaces with Axoids “Plane–Cylinder”

- Spiroidal surface with axoids “plane–cylinder» formed by a straight parallel to the axis of a rolling cylinder
- Spiroidal surface with axoids “plane–cylinder» formed by a straight intersecting the axis of a rolling cylinder
- Parabolic spiroidal surface with axoids “plane–cylinder”

34.2.5. Spiroidal Ruled Surfaces of Rachkovskaya–Kharabaev

- Spiroidal ruled surface of Rachkovskaya–Kharabaev with axoids “evolvent helicoid–right circular cone”
- Spiroidal ruled surface of Rachkovskaya–Kharabaev with axoids “developable conical helicoid–right circular cone”
- Spiroidal ruled surface of Rachkovskaya–Kharabaev with axoids “evolvent helicoid–right elliptical cone”
- Kinematic ruled surface on the base of one-sheet hyperboloidal surfaces of revolution as fixed and loose axoids (one axoid is located outside another)
- Kinematic ruled surface on the base of one-sheet hyperboloidal surfaces of revolution as fixed and loose axoids (one axoid is located in the interior of another)

35. The Second Order Surfaces

- Quadratic surfaces presented in the encyclopedia
- Hyperbolic paraboloid
- Parabolic conic surface
- Cylindrical surface of revolution
- Conical surface of revolution
- One-sheet hyperboloid
- Ellipsoid

■ Elliptic paraboloid
■ Hyperboloid of two sheets
Spherical surface (sphere) (it is presented in the Subclass "23.1. Surfaces of the constant positive Gaussian curvature")
Two-sheeted hyperboloid of revolution (it is presented in the Class "2. Surfaces of Revolution")
Paraboloid of revolution (it is presented in the Class "2. Surfaces of Revolution")
Elliptical conical surface (it is presented in the Group "1.1.3. Conical surfaces")
Oblique circular conic surface (it is presented in the Group "1.1.3. Conical surfaces")
The Morse cone (synonym is conical surface of revolution)
Oblique elliptical conical surface (it is presented in the Group "1.1.3. Conical surfaces")
Parabolic conical surface (it is presented in the Group "1.1.3. Conical surfaces")
Hyperbolic conical surface (it is presented in the Group "1.1.3. Conical surfaces")
Elliptical cylinder (it is presented in the Group "1.1.2. Cylindrical surfaces")
Hyperbolic cylinder (it is presented in the Group "1.1.2. Cylindrical surfaces")
Parabolic cylinder (it is presented in the Group "1.1.2. Cylindrical surfaces")
Oblique circular cylinder (it is presented in the Group "1.1.2. Cylindrical surfaces")
Oblique elliptical cylinder (it is presented in the Group "1.1.2. Cylindrical surfaces")
Oblique hyperbolic cylinder (it is presented in the Group "1.1.2. Cylindrical surfaces")
Oblique parabolic cylinder (it is presented in the Group "1.1.2. Cylindrical surfaces")
■ Quadrics

36. Algebraic Surfaces of the High Orders

■ Algebraic surfaces of the high orders presented in the encyclopedia

36.1. Algebraic Surfaces of the 3th Order

■ "Handkerchief surface"
■ Cubic surface $x^3 + xy + z = 0$
■ Diagonal cubic surface of Fermat
■ Cubic cone
■ The cubic surface with 24 straight lines

36.2. Algebraic Quartic Surface

■ Euler surface of the forth order
■ Surface of circles of Feuerbach
■ Nordstrand's weird surface
■ Menn's surface
■ Środa's parabolic surface
■ Overlimited parabolic velaroid
■ "Chair"
■ Goursat's surface
■ Tooth surface
■ Tanglecube
■ Kummer surface
■ Quartic surface with parabola, ellipse, parabola in three principal coordinate sections
■ Quartic surface with the 4th order curve, ellipse, the 4th order curve in three principal coordinate sections
■ Pillow Shape
■ Additional information on surfaces of the 4th order

36.2.1. The 4th Order Surfaces With Multiply Lines

■ Quartic surface with a triple straight line
■ Quartic surface with a double conic and with a double straight line
■ Quartic surface with two double straight lines

Quartics with a double conic

Dupin cyclides (they are presented in the Group “17.1.2. Cyclic surfaces”)
■ Cyclides with a triple point

Quartics with three double straight lines

■ Ruled surfaces of the 4th order with three double lines
■ Steiner surfaces of the first and second types

Quartics with a double straight line

■ Quartic surface with a double straight line and a triple point

36.3. Algebraic Quintic Surfaces

■ Peninsula surface
■ Parabolic surface of conoidal type

36.4. Algebraic Surfaces of the Sixth Order

■ Sine surface
■ Heart surface
■ Hunt‘s surface
■ The 6th order surface with the pseudo Agnesi curl and two parabolas lying in parallel planes
■ The 6th order surface with parabola, the 4th order curve, parabola lying in three principal coordinate sections
■ Surface of revolution of an Agnesi curl (the 2nd variant)
■ Algebraic surface of the 6th order with two nets of translation
■ The 6th order surface with Agnesi curl, ellipse, Agnesi curl lying in three principal coordinate sections

36.5. Algebraic Surfaces of the 7th Order

■ “Ski hill”
■ The 7th order surface with parabola, Agnesi curl, ellipse lying in three principal coordinate sections
■ The 7th order surface with parabola, ellipse, Cartesian folium lying in three principal coordinate sections
■ The 7th order surface with parabola, the 4th order curve, parabola lying in three principal coordinate sections
■ The 7th order surface with Agnesi curl, Lame’s curve of the third order, straight lines lying in three principal coordinate sections

36.6. Algebraic Surfaces of the 8th Order

■ The 8th order surface with Agnesi curl, ellipse, ellipse lying in three principal coordinate sections
■ The 8th order surface with Lame’s curve of the 4th order, Lame’s curve of the 4th order, ellipse lying in three principal coordinate sections
■ The 8th order surface with parabola, the 4th order curve, parabola lying in three principal coordinate sections

36.7. Algebraic Surfaces of the 9th Order

Enneper's surface (it is preented in the Class "19. Minimal surfaces")

36.8. Algebraic Surfaces of the 10th Order

Aerodynamic surfaces given by a continuous framework of elliptical ribs (it is presented in the Subclass "6.1. Aerodynamic surfaces given by algebraic plane curves")

36.9. Algebraic Surfaces of the 12th Order

■ Surface of the 12th order with parabola, the 8th order curve, parabola lying in three principal coordinate sections

36.10. Algebraic Surfaces of the 16th Order

■ Algebraic surface with a continuous net of the pseudo Agnesi curl passing through a parabola and two straights
The literature on geometry of algebraic surfaces

37. Polyhedrons and Quasi-Polyhedrons

■ The types of polyhedrons
■ Astroidal ellipsoid
■ Hyperbolic octahedron
■ The models of quasi-polyhedrons presented in sites of Internet

38. Equidistances of Double Systems

■ Equidistance of the system "straight–cylinder"
■ Equidistance of the system "straight–sphere"
■ Equidistance of the system "point–cylinder"
■ Equidistance of the system "straight–torus"

The Surfaces Not Included in the Classification

Pontryagin's surfaces
Kuen surfaces
Bress hyperboloid
Euler special cones
Surfaces with the constant equi-affine invariants
Steiner ellipsoid
Ortoid ruled surfaces
Algebraic congruence of the 4th order of the 2nd class
Fractal surfaces

Russian, English, French, and German Dictionary of Surfaces and Curves

See Table 1

Table 1

Russian	English	French	German
Абелева минимальная поверхность	Abelian minimal surfaces		Abelsche Minimalflächen
Алгебраическая геометрия	Algebraic geometry	géométrie *f* algébrique *f*	algebraische Geometrie *f*
алгебраическая поверхность	Algebraic surface	surface *f* algébrique	algebraische Fläche *f*
аналитическая поверхность	Analytical surface	surface *f* analitique	Analytische Fläche *f*
асимптота	Asymptote of the curve	asymptote *f*	Asymptote *f*
асимптотическая линия	Asymptotic line	ligne *f* asymptotique	Schmiegtangentkurve *f*
астроида	Astroid	astroïde *f*	Astroide *f*
аффинно-минимальная поверхность	Affine minimal surface	surface minimale affine	Affinminimalfläche *f*
аффинно-минимальные поверхности Томсена	Affine minimal surfaces of Thomsen	surface minimale affine de Thomsen	Die Affinminimalflächen von G. Thomsen
Башмачная поверхность	Shoe surface	surface souliève	Schuhfläche *f*
бинормаль	Binormal	binormale *f*	Binormale *f*
Богемский купол	Bohemian dome	Dôme Bohemian	
бутылка Клейна	Klein bottle	la bouteille de Klein	
бутылка Уитни	Whitney bottle	la bouteille de Whitney	
Вектор	Vector	vecteur *m*	Vektor *m*
велароидальная поверхность	Velaroidal surface	surface vélaroidale	
Верзиера (локон Аньези)	Agnesi curl	boucle Agnesi	
вершина конуса	Vertex of a cone	sommet *m* d'un cône	Spitze *f*, Scheitel *m*
винт Архимеда	Archimedes' screw	vis d' Archimède	Archimedische Schnecke
винтовая линия	Helix, coil	helice *f*	Schraubenlinie *f*
винтовая поверхность	Helical surface	surface *f* hélicoïdale	Schraubfläche
винтовая трубчатая поверхность	Helical tubular surface	surface *f* hélicoïdale tubulaire	
винтовое движение	Screw motion	mouvement *m* hélicoïdal	schraubenförmige Bewegung
винтовой коноид	Helical conoid	conoïde hélicoïdal	

(continued)

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Table 1 (continued)

Russian	English	French	German
винтовой цилиндроид	Helical cylindroid	cylindroïde hélicoïdal	
винт Штейнбаха	Steinbach screw	vis de Steinbách	
волновая поверхность Френнела	Fresnel’s elasticity surface	surface ondulatoire de Fresnel	
вторая основная квадратичная форма поверхности	The second fundamental form of a surface	seconde forme *f* quadratique fondamentale d’une surface	zweite Fundamentalform *f* einer Fläche
вытянутый эллипсоид вращения	Prolate spheroid	ellipsoïde *m* de revolution allongé	langgestrecktes Rotationsellipsoid *n*
Гауссова кривизна поверхности	The Gaussian curvature of surface	courbure *f* de Gauss	Gaußsche Krümmung *f*
гексаэдр	Hexahedron	hexaèdre *m*	Hexaeder *n*
геликоид Дини	Dini’s surface	hélicoïde Dini	Dinische Helicoid *n*
географическая система криволинейных координат	Geographic system of curvilinear coordinates	système géographique des coordonnées curvilignes	
геодезическая кривизна	Geodesic curvature	courbure *f* géodesique	geodätische Krümmung *f*
геодезическая линия	Geodesic line	ligne géodésique	geodätische Linie *f*
гипар	Hypar	hypare	
гипербола	Hyperbola	hyperbole *f*	Hyperbel *m*
гиперболические координаты	Hyperbolic coordinates	coordonnèes *f pl* hyperboliques	hyperbolische Koordinaten *f pl*
гиперболический геликоид	Hyperbolic helicoid	hélicoïde hyperbolique	hyperbolisches Helikoid
гиперболический параболоид (гипар)	Hyperbolic paraboloid	paraboloïde *m* hyperbolique	hyperbolisches Paraboloid *n*
гиперболический торс	Hyperbolic developable surface	surface éveloppable hyperbolique	
гиперболический цилиндр	Hyperbolic cylinder	cylindre *m* hyperbolique	hyperbolischer Zylinder
гипоциклоида	Hypocycloid	hypocycloïde *f*	Hypozykloide *f*
гипоциклоидный цилиндр	Hypocycloidal cylinder		hypozykloid Zylinder
главная нормаль	Principal normal	normale principale	Hauptnormale *f*
главная кривизна	Principal curvature	courbure *f* principale	Hauptkrümmung *f*
главные кривизны поверхности	Principal curvatures of surface	surface d’incurvation principale	Hauptkrümmung *f*
гладкая кривая	Smooth curve	courbe *f* lisse	glatte Kurve *f*
глобоид	Globoid	globoïde	Globoid
горловая окружность	Waist circle		Kehlekreis *f*
горловой эллипс	Waist ellipse		Kehleellipse *f*
гофр	Crimp		
Дважды линейчатая поверхность	Doubly ruled Surface	surface double règlèe	
двуполостный гиперболоид	Hyperboloid of two sheets	hyperboloïde *m* deux nappes	zweischaliges Hyperboloid *n*
двуполостный гиперболоид вращения	Two-sheeted hyperboloid of revolution	hyperboloïde à deux nappes de rotation	zweischaliges Drehhyperboloid *n*
декартовы координаты	Cartesian coordinates	coordonnées *f pl* cartésiennes	kartesische Koordinaten *f pl*
дифференциальная геометрия	Differential geometry	géométrie *f* différentielle	Differentialgeometrie *f*
длина дуги	Arc length	longueur *f* d’un arc	Bogenlänge *f*
додекаэдр	Dodecahedron	didécaèdre *m*	Dodekaeder *n*

(continued)

Table 1 (continued)

Russian	English	French	German
Закрытый тор	Spindle torus; horn torus	tore fermé	geschlossener Torus *m*
замкнутая пространственная кривая	Closed space curve	courbe fermée gauche	räumliche abgeschlossene Kurve
зонтик Картана	Cartan umbrella	parapluie de Cartan	
зонтик Уитни	Whitney umbrella	parapluie de Whitney	
зубная поверхность	Tooth surface		
Изгибание	Bending	déformation *f*	Verbiegung *f*
изометрия	Isometry, isometric projection	isométrie *f*	isometrische Abbildung
икосаэдр	Icosahedron	icosaèdre *m*	Ikosaeder *n*
Каноническое уравнение поверхности	The support function of the surface	equation canonique de surface	kanonische Gleichung einer Oberfäche
каналовая поверхность Иоахимсталя	Canal surface of Joachimsthal	surface canale de Joachimsthal	
каплевидная поверхность	Drop shaped surface	surface de la goute	
касательная плоскость	Tangent plane	plan *m* tangent	Tangentialebene *f*
катеноид	Catenoid	caténoïde *f*	Katenoide *f*
квадратичная поверхность	Quadric surface	surface quadratique	quadratische Fläche *f*
класс	Class	classe *f*	Klasse *f*
конгруэнтность	Congruence	congruence *f*	Kongruenz *f*
коника	Conic	conique	
коническая винтовая линия	Conical helix, conic spiral	spirale *f* conique	konische Schraubenlinie
коническая кромка Уоллиса	Wallis's conical edge	bord conique de Wallis	
коническая поверхность вращения	Conic surface of revolution	surface conique de revolution	Drehkegelfläche *f*
коническая спираль	Conical spiral	spirale conique	
конический геликоид одинакового ската	Conic helicoid of constant slope	hélicoïde conique de pente constante	
коноид	Conoid	conoïde *m*	Konoid *n*, Konoidfläche *f*
коноид Плюккера	Plücker conoid	conoïde Plücker	Plückersches Konoid
коноид Циндлера	Zindler's conoid	conoïde de Zindler	Zindlersches Konoid
конус	Cone	cône *m*	Kegel *m*
косой геликоид	Oblique helicoid	hélicoïde *m* oblique	
косой гипар	Oblique hypar	hypare oblique	
коэффициенты первой квадратичной формы поверхности	Coefficients of the first fundamental form of surface	coefficients de première forme quadratique de surface	
коэффициенты второй квадратичной формы поверхности	Coefficients of the second fundamental form of surface	coefficients de deuxième forme quadratique de surface	
коэффициенты Ламе в теории поверхностей	*Lame's* coefficients in the surface theory	coefficient de Lamé à la théorie des surfaces	
кратные линии	Multiple lines	lignes multiples	
крестообразный желоб	Crossed trough	canal croix	
кривая второго порядка (коника)	Quadratic curve, conic	courbe de deuxiéme orde, conique *f*	Kurve zweiter Ordnung, Kegelschnitt
кривая Ламе	Lame's curve	courbe de Lamé	Lamesche Kurve *f*
кривая 3-го порядка	Cubic (curve)	courbe d'ordre trois, cubique *f*	Kurve dritter Ordnung, Kubik *f*

(continued)

Table 1 (continued)

Russian	English	French	German
кривизна кривой	Curvature of the curve	courbure *f* de la ligne	Krümmung der Kurve
криволинейные координаты на поверхности	Curvilinear coordinates on the surface	coordonnées *f pl* curvilignes de la surface	krummlinige Koordinaten *f pl*
круговая поверхность переноса	Circular translation surface	surface de translation curculaire	
круговой конус	Circular cone	cône *m* à base circulaire	Kreiskegel *m*
круговой тор	Circular torus	tore circulaire	Kreistorus *m*
круговой цилиндр	Circular cylinder	cylindre *m* de revolution	Kreiszylider *m*, Drehzylinder *m*
куб	Cube	cube *m*	Würfel *m*
кубическая кривая (кубика)	Cubic curve (cubic)	cubique *f*	Kubik *f*
кручение кривой	Torsion of the curve	courbe de torsion	Schmiegung *f* einer Raumkurve
кубический торс	Cubic developable surface	surface développable cubique	
кубоид	Cuboid	cuboïde *m*	
Лемниската	Lemniscate	lemniscate *f*	Lemniskate *f*
Лимон	Lemon surface		Die Lemon Surface
линейчатая винтовая поверхность	Ruled helical surface		Regelschraubfläche *f*
линейчатая поверхность	Ruled surface, scroll surface, scroll	surface *f* réglée	Regelfläche *f*
линия кривизны	Line of curvature	ligne *f* de courbure	Krümmungslinie *f*
линия откоса	Sloping line, line of slope		
линия центров окружностей	The line of centers of circles, center-to-center line	ligne des centres des cercles	die Linie der Kreismittelpunkt
Лист Декарта	Cartesian folium (leaf)		
лист Мебиуса	Möbius strip	ruban *m* de Moebius	Möbiussches Band *n*
логарифмическая спираль	Logarithmic spiral	spirale *f* logarithmique	logarithmische Spirale *f*
локон Аньези	Agnesi curl	boucle Agnesi	
локсодрома	Loxodrome	loxodromie *f*	Loxodrome *f*
Меридиан	Meridian	méridien *m*	Meridian *m*
Меркаторова проекция	The Mercator parameterization	projection de Mercator	
минимальная поверхность	Minimal surface	surface *f* minimale	Minimalfläche *f*
минимальная поверхность Бора	Bour’s minimal surface	surface minimale de Bour	Bohrsche Minimalfläche
минимальная поверхность Косты	Costa minimal surface	surface minimale de Costa	
Минимальная поверхность Томсена	Thomsen’s minimal surface	surface minimale de Thomsen	Thomsensche Minimalfläche *f*; Minimalfläche *f* von G. Thomsen
минимальные поверхности со свободными границами	Minimal surfaces with free boundaries		
многогранник	Polyhedron	polyèdre *m*	Polyeder *n*, Vielflach *n*
морская ракушка	seashell	coquille *f*	
Наклонная коническая поверхность	Oblique conic surface	surface conique inclinée	
наклонный цилиндр	Oblique cylinder	cylindre incliné	
направляющая кривая	Directrix curve	courbe guide	Leitkurve *f*
направляющие косинусы	Directional cosines		
направляющий конус	Director cone		

(continued)

Table 1 (continued)

Russian	English	French	German
начертательная геометрия	Descriptive geometry	géométrie *f* descriptive *f*	darstellende Geometrie *f*
непрерывный каркас прямолинейных образующих торса	A continuous net of rectilinear generatrixes of developable surface	carcasse continue des génératrices rectilignes de surface développable	
нормальная плоскость пространственной кривой	Normal plane of space curve	plan normal de courbe spatiale	Normalebene *f* der raumlichen Kurve
нормальная циклическая поверхность	Normal cyclic surface	surface normale cylindrique	
Обезьянье седло	Monkey saddle	celle de singe	Affenstattel
обобщенный	Generalized	généralisé	verallgemeinert
образующая окружность	Generating circle	cercle génératrice	erzeugende des Kreislinie
обычная циклоида	Ordinary cycloid	cycloïde ordinnaire	
однопараметрическое семейство плоскостей	Single-parametric System of planes	famille des plans à un parametre	
однополостный гиперболоид	Hyperboloid of one sheet	hyperboloïde *m* à une nappe	einschaliges Hyperboloid *n*
однополостный гиперболоид вращения	One-sheet hyperboloid of revolution	hyperboloïde de revolution à une nappe	einschaliges Drehhyperboloid
окружность	Circle	cercle *m*	Kreislinie *f*
окружности Вилларсо		cercles de Villarceau	
октаэдр	Octahedron	octaèdre *m*	Oktaeder *n*
омбилическая поверхность	Umbilical surface	surface ombilique	Nabelfläche *f*
основание цилиндра	Base of cylinder	cylindre de base	
особая точка	Singular point	point *m* singulier	singulärer Punkt *m*
ось	Axis	axe *m*	Achse *f*
ось абсцисс	*x*-axis, abscissa axis	axe d’abscisses	Abszissenache *f*
ось аппликат	*z*-axis, applicate axis	axe de cotes, axe des *Z*	Applikatenachse *f*
ось вращения	Rotation axis	axe de rotation	
ось ординат	*y*-axis, ordinate axis	axe d’ordonnées	Ordinatenachse *f*
ось симметрии	Axis of symmetry	axe *m* de symétrie	Symmetrieachse *f*
открытый тор	Ring torus	tore ouvert	
Парабола	Parabola	parabole *f*	Parabel *f*
парабола Нейля	Neil’s parabola	parabole *f* de Neil	Neilsche Parabel *f*
параболический велароид	Parabolic velaroidal surface	surface vélaroïdale parabolique	
параболический коноид	Parabolic conoid	conoïde parabolique	
параболоид вращения	Paraboloid of revolution	paraboloïde de révolution	Rotationsparaboloid *n*
параболическая точка	Parabolic point	point parabolique	parabolischer Punkt
параболический торс	Parabolic developable surface	surface développable parabolique	
параболический цилиндр	Parabolic cylinder	cylindre *m* parabolique	parabolischer Zylinder *m*
параллель	Parallel	parallele	Parallele *f*
параметрические уравнения поверхности	Parametric equations of surface	equations paramétrique de surface	
пенка	Skin	plau *f*	
первая основная квадратичная форма поверхности	The first fundamental form of a surface	première forme quadratique fondamentale d’ une surface	erste Fundamentalform *f* einer Fläche
перекрученная лента	Twisting ribbon	ruban forcé	überdrehendes Band

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Table 1 (continued)

Russian	English	French	German
Пиренейская поверхность	Peninsula surface	surface Peninsula	
плоскость	Plane	plan *m*	Ebene *f*
площадь поверхности	The surface area	l'aire de la surface	
плужный отвал	Mouldboard	versoir de charrue	Haldenpflug *m*
поверхности Безье	Surfaces of Bézier	surfaces de Bézier	Bézier-Flächen
поверхности Блютеля	Blutel's surfaces	surfaces de Blutel	Die Blutelschen Kegelschnittflächen
поверхности Бонне	Bonnet surfaces	surfaces d'Ossian Bonnet	
поверхности Вейнгартена	Weingarten surfaces	surfaces de Weingarten	*W*-Flächen
поверхности Веронезе	Veronese surfaces	surfaces de Veronese	
поверхности Каталана	Catalan's surfaces	surfaces de Catalan	Katalanischen Flächen
Поверхности Кунса	Coons surfaces		
поверхности Петерсона	Peterson surfaces	surfaces de Peterson	Petersonischen Flächen
поверхности Хакена	Haken surfaces	surfaces de Haken	Hackenischen Flächen
поверхности Цицейки	Tzitzéica's surfaces	surfaces de Tzitzéica	
поверхности Эдлингера	Edlinger's surfaces	surfaces d'Edlinger	der Edlinger-Flächen
поверхность Боя	Boy surface	surface *f* de Boy	Boyische Fläche
поверхность вращения	Surface of revolution	surface *f* de révolution	Rotationsfläche *f*
поверхность вращения астроиды	Surface of revolution of astroid	surface de revolution des astroïdes	Rotationsfläche der Astroide
поверхность вращения «Воронка»	Funnel		Der Funnel
поверхность вращения полукубической параболы	The surface of revolution of semicubical parabola	surface semi-cubique des paraboles de revolution	Rotationsfläche der semikubische Parabel
поверхность вращения циклоиды	Surface of revolution of cycloid	surface de revolution cycloïde	Rotationsfläche der Zykloide
поверхность второго порядка	Quadratic surface	surface quadratique	zweiten Grades Fläche *f*
поверхность Гурса	Goursat's surface	surface de Goursat	Gurssche Fläche
поверхность Дарбази	Darbazi surface	surface de Darbazi	Darbazische Fläche
поверхность дважды косогоклина	Surface of a double oblique wedge	surface d'un coin double-oblique	
поверхность дважды косого цилиндра	Surface of a double oblique cylinder	surface d'un cylindre double-oblique	
поверхность Зейферта	Seifert surface	surface de Seifert	
поверхность Куммера	Kummer surface	surface de Kummer	
поверхность Куена	Kuen surface	surface de Kuen	
поверхность Кэли	Cayley surface	surface de Cayley	Cayleyfläche
поверхность косого клина	Surface of an oblique wedge	surface d'un coin oblique	
Поверхность «Кресло»	"Chair"	surface "Chaise"	
Поверхность кругов Фейербаха	Surface of circles of Feuerbach		
поверхность Мебиуса	Möbius surface	surface de Moebius	
поверхность Морэна	Morin's surface	surface de Morin	
поверхность косого цилиндра	Surface of an oblique cylinder	surface d'un cylindre oblique	
поверхность Мэнна	Menn's surface	surface de Menn	

(continued)

Table 1 (continued)

Russian	English	French	German
поверхность одинакового ската	Surface of constant slope	surface de pente constante	
поверхность «Падающая капля»	“Kiss surface”		
поверхность «Платок»	“Handkerchief surface”		
поверхность прямого переноса	Translation surface	surface de translation	
поверхность пятого порядка	Quintic surface		
поверхность Рембса	Rembs’ surface	surface de Rembs	
поверхность «Рог изобилия»	“Cornucopia”	“Corne d’Abondance”	
Поверхность святого Ильи	“The Saint Elias surface”		
Поверхность «Сердце»	Heart surface	surface «Coeur»	
поверхность Сиверта	Sievert’s surface	surface de Sievert	
поверхность Ханта	Hunt’s surface	surface de Hunt	
поверхность четвертого порядка	Quartic surface; the 4th order surface	surface quatrique	
поверхность Шерка	Scherk’s minimal surface	surface minimale de Scherk	Die Scherksche Minimalfläche
поверхность Штейнера	Steiner surface	surface de Steiner	Steiner Fläche *f*
поверхность Эннепера	Enneper’s surface	surface d’Enneper	Die Ennepersche Minimalfläche
полиномиальная функция	Polynomial function	fonction polynomiale	Polynom *m*
полная минимальная поверхность	Complete minimal surface		
полукубическая парабола	Semicubical parabola (cuspidal cubic)	parabole *f* semi-cubique	semikubische Parabel *f*
полуоси	Semi-axes	demi-axes	
полярная поверхность	Polar surface	surface polaire	Polfläche *f*
полярные координаты	Polar coordinates	coordonèes *f pl* polaires	polare Koordinaten *f pl*
правильный гиперболоид	Regular hyperboloid	hyperboloïde régulier	
предметная плоскость	Object plane	plan object	Grundebene
призма	Prism	prisme *m*	Prisma *n*
призма усеченная	Truncated prism	tronc *m* de prisme, prisme *m* tronqué	Prismenstumpf *m*
пролет оболочки	Span	travée	Feld
прямой геликоид	Right helicoid	helicoïde droit	
прямой гипар	Right hypar	hypare droit	
прямой цилиндроид	Right cylindroid	cylindroïde droit	
прямолинейная образующая	Rectilinear generatrix	generatrice rectiligne	
прямоугольные координаты	Orthogonal coordinates	coordonnées *f pl* rectangulaires	rechtwinklige Koordinaten *f pl*
псевдосфера	Pseudosphere, tractroid, tractrisoid	pseudo-sphère *f*	Pseudosphäre *f*
Радиус	Radius	rayon *m*	Radius *m*
радиус-вектор	Radius-vector	rayon *m* vecteur	Radiusvektor *m*
развертывающаяся поверхность	Developable surface	surface *f* dévelopable	abwickelbare Fläche *f*
развертывающийся геликоид	Open helicoid	hélicoïde ouvert	abwickelbare Helykoid
развертывающийся конический геликоид	Developable conic helicoid	hélicoïde développable conique	abwickelbare Kegelhelikoid

(continued)

Table 1 (continued)

Russian	English	French	German
развертка	Development	developpement *m*	entwickelte Fläche *f*
ребро возврата	Cuspidal edge	l'arête de rebroussement	Gratlinie *f*, Striktionslinie
регулярная поверхность	Regular surface	surface reguliére	
резная линейчатая поверхность Монжа	Monge ruled surface	surface regleé de Monge	
резные поверхности Монжа двойной кривизны	Monge surface of double curvature	surfaçes moulures	Gesimsflächen
римская поверхность	Roman surface	surface romaine	
рулетта	Roulette	roulette *f*	Rollkurve *f*
Седло Пеано	Peano saddle	celle de Peano	
сеть переноса	Translation net		
сжатый эллипсоид вращения	Oblate spheroid	ellipsoïde *m* de revolution aplati	zusammengedrücktes Rotationsellip-soid *n*
синусовая поверхность	Sine surface	surface sunicoidale	
синусоида	Sine curve, sinusoid	sinusoïde *f*	Sinuskurve *f*
скрещенный колпак Штейнера	Steiner cross cap		
соприкасающаяся плоскость	Osculating plane	plan *m* osculateur	Schmiegungsebene *f*
сопряженные криволинейные координаты	Conjugate curvilinear coordinates	coordonnées curvilignes conjuguées	
сотовая поверхность	Honeycomb surface		
спираль Архимеда	Spiral of Archimedes	spirale *f* d'Archimède	Archimedische Spirale *f*
спиральная поверхность	Spiral surface		Spiralfläche
спираль Ферма	Spiral of Fermat		Fermat Spirale *f*
сплющенный эллипсоид вращения	Oblate ellipsoid of revolution	ellipsoïde *m* de révolution aplati	zusammengedrücktes Rotationsellipsoid *n*
спрямляющая плоскость	Rectifying plane	plan *m* rectifiant	rektifizierende Ebene *f*
срединная поверхность оболочки	Middle surface of a shell	surface moyenne d'une enveloppe	Schalenmittelfläche *f*
средняя кривизна поверхности	The mean curvature of surface	courbure *f* moyenne	mittlere Krümmung *f* der Fläche
стрела подъема	Rise	fléche, montée d'enveloppe	Pleilhöhe *f*
стрикционная линия	Striction curve		
сфера	Sphere	sphère *f*	Sphäre *f*
сферические координаты	Spherical coordinates	coordonnées *f pl* spherique	Kugelkoordinaten *fpl*
сфероид	Spheroid	sphéroïde *m*	Sphäroid, Rotationsellipsoid
Тетраэдр	Tetrahedron	tétraèdre *m*	Tetraeder *n*
тор	Torus	tore *m*	Torus *m*
тороид	Toroid	toroïde *m*	Toroid *n*
торс	Torse	surface développable	Torse
торс 4-го порядка	The forth order torse surface	surface developpable de 4-ème ordre	
точка возврата первого рода	The first type cuspidal point	point *m* de rebroussement	Rückkehrpunkt *m*, erster Art
точка на бесконечности	A point at infinity	point *m* à l'infini	Fernpunkt *m*
точка пинча	Pinch point	point de pinch	
трактриса	Tractrix	tractrice *f*	Traktrix *f*

(continued)

Table 1 (continued)

Russian	English	French	German
трактроид	Tractroid	tractroïde *m*	
трансцендентная	Transcendental		
трехосный эллипсоид	Triaxial ellipsoid	ellipsoïde triaxiale	
триноид	Trinoid	trinoïde *m*	
трохоида	Trochoid	trocoïde *f*	Trochoide *f*
трубчатая поверхность	Tubular surface, tube	surface tubulaire	Röhrenfläche *f*
Угол	Angle	angle *m*	Winkel *m*
улитка Паскаля	Limaçon of Pascal	limaçon *m* de Pascal	Pascalsche Schnecke *f*
ундулоид	Unduloid, onduloid		Unduloid
уравнение в неявной форме	Implicit equation	èquation *f* implicite	implizite Gleichung *f*
уравнение в явной форме	Explicit equation	èquation *f* explicite	explizite Gleichung *f*
усеченный конус	Truncated cone; conical frustum	tronc *m* de cône	Kegelstupf *m*
условие единственности торсовой поверхности	Condition of uniqueness of the developable surface	condition d'unicité de surface développable	
условие торсовости	Necessary condition for the developable surface	condition nécessaire pour la surface développable	
Фигура	Figure, configuration	figure *f*	Figur *f*
формула	Formula	formule *f*	Formel *f*
функция	Function	fonction *f*	Funktion *f*
Хорда	Chord	corde *f*	Sehne *f*
Цепная линия	Catenary	caténaire *f*	Kettenlinie *f*
циклида с тройной точкой	Cyclide with a triple point	cyclide avec le triple poïnt	
циклиды Дюпена	Dupin's cyclides	cyclide de Dupin	
циклическая поверхность	Cyclic surface	surface cyclique	
циклоида	Cycloid	cycloide *f*	Zykloide *f*
цилиндрическая винтовая линия	Circular helix, Cylindrical helix	hélice *f* cylindrique	gemeine Schraubenlinie *f*
цилиндрическая винтовая полоса	Cylindrical helical strip	barre cylindrique hélicale	
цилиндрическая линейчатая ротативная улитка	Cylindrical ruled rotational limaçon	limaçon cylindique réglé rotational	
цилиндр	Cylinder	cylindre *m*	Zylinder *m*
цилиндрическая поверхность	Cylindrical surface	surface *f* cylindrique	Zylinderfläche *f*
цилиндрическая поверхность вращения	Cylindrical surface of revolution	surface de revolution cylindrique	
цилиндрические координаты	Cylindrical coordinates	coordonnèes *f pl* cylindriques	zylindrische Koordinaten *f pl*
цилиндроид	Cylindroid	cylinroïde *n*	Zylindroid *n*
Чебышевская сеть на поверхности	Chebyshev's net on the surface		
Шаг винтовой линии	The lead of a helix		
Шаг резьбы	The pitch of a screw		
шар	Ball, solid sphere	boule *f*	Kugel *m*
шутовской колпак	Dunce hat	bonnet de bouffon	
Эвольвента окружности	Evolvent of the circle	évolvente *f* circulaire	Evolvente *f* ………
эвольвентный геликоид	Evolvent helicoid	hélicoïde d'évolvente	
эллипс	Ellipse	ellipse *f*	Ellipse *f*

(continued)

Table 1 (continued)

Russian	English	French	German
эллипсоид вращения	Spheroid (ellipsoid of revolution)	ellipsoïde *m* de revolution	Rotationsellipsoid *n*, Drehellipsoid
эллиптический велароид	Elliptic velaroidal surface	surface elliptique vélaroidale	
эллиптический геликоид	Elliptic helicoid	hélicoïde elliptique	
эллиптический конус	Elliptic cone	cône elliptique	
эллиптический параболоид	Elliptic paraboloid	paraboloïde *m* elliptique	elliptisches Paraboloid *n*
эллиптический цилиндр	Elliptic cylinder	cylindre *m* elliptique	elliptischer Zylinder *m*
эпитрохоида		épitrohoïde	
эпитрохоидальная поверхность		surface épitrohoïde	
эпициклоида	Epicycloid	épicycloïde *f*	Epizykloide *f*
эпициклоидный цилиндр	Epicycloidal cylinder		epizykloid Zylinder
Яйцо Дунса	Duns egg	oeuf de Duns	

PS The encyclopedia contains a large number of additional literature and references written in Russian language. That is why, the first column in the Dictionary is given for Russian terms. It must help to read the presented Russian original sources

Name Index of the Basic Text

The names and initials of the scientists used in the basic text of the encyclopedia are given in *the Name Index of the Basic Text*. Sometimes these names go down in the names of surfaces, constants, or coefficients. After the names of scientists at the same line, the pages of the encyclopedia are pointed out where these names were used.

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Name Index of the Authors of Additional Literature

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