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HIERARCHICAL OPTIMIZATION: AN INTRODUCTION

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Abstract

Decision problems involving multiple agents invariably lead to conflict and gaming. In recent years, multi-agent systems have been analyzed using approaches that explicitly assign to each agent a unique objective function and set of decision variables; the system is defined by a set of common constraints that affect all agents. The decisions made by each agent in these approaches affect the decisions made by the others and their objectives. When strategies are selected simultaneously, in a noncooperative manner, solutions are defined as equilibrium points [13,51] so that at optimality no player can do better by unilaterally altering his choice. There are other types of noncooperative decision problems, though, where there is a hierarchical ordering of the agents, and one set has the authority to strongly influence the preferences of the other agents. Such situations are analyzed using a concept known as a Stackelberg strategy [13, 14,46]. The hierarchical optimization problem [11, 16, 23] conceptually extends the open-loop Stackelberg model to K players. In this paper, we provide a brief introduction and survey of recent work in the literature, and summarize the contributions of this volume. It should be noted that the survey is not meant to be exhaustive, but rather to place recent papers in context.

1. Problem formulation

Hierarchical optimization was first defined by Bracken and McGill [18, 19] as a generalization of mathematical programming. In this context, the constraint region is implicitly determined by a series of optimization problems which must be solved in a predetermined sequence.

The problem is to find vectors x and v^i ($i = 1, \ldots, m$) to

$$\underset{x \in X}{\text{minimize}}\ f(x)$$

$$\text{subject to } h_i(x) = \min_{v^i \in V^i} g^i(x, v^i) \geq 0, \quad i = 1, \ldots, m. \tag{1}$$

A variation of this problem is to find vectors x, u^i ($i = 1, \ldots, m$) and v^i ($i = 1, \ldots, m$) to

$$\begin{aligned} &\underset{x \in X}{\text{minimize}}\ f(x) \\ &\text{subject to } \bar{h}_i(x) = \max_{u^i \in U^i(x)} \min_{v^i \in V^i} g^i(x, u^i, v^i) \geq 0, \quad i = 1, \ldots, m. \end{aligned} \tag{2}$$

If X, $U^i(\cdot)$ and V^i ($i = 1, \ldots, m$) are convex sets, $f(x)$ is a convex function of x, and $g^i(x, u^i, v^i)$ is concave in x and u^i for every $v^i \in V^i$ ($i = 1, \ldots, m$), then, with several mild restrictions, the mathematical program (2) is convex. If in addition $g^i(x, u^i, v^i)$ is convex in v^i, this program has a saddle point. Bracken and McGill [22] present solution techniques for (2) which employ the sequential unconstrained minimization technique (SUMT) of Fiacco and McCormick [26] for the outer problem. Bracken et al. [20] show that the mathematical program (1) can be transformed into (2) and is thus equivalent to it. Therefore, computational procedures based on SUMT can be used to solve (1) as well.

Recent research on hierarchical optimization problems has generalized the early work. In order to give mathematical formulations of such generalizations, consider a system comprised of K levels, each characterized by individual functions f^i, $i = 1, \ldots, K$, defined over a jointly dependent constraint set S, which are to be maximized by the respective players. Assume that decisions are made sequentially beginning with player 1 who has control over a vector $x^1 \in X^1$, followed by player 2 who has control over a vector $x^2 \in X^2$, down through player K who has control over a vector $x^K \in X^K$, where x^i are nonempty subsets of $\mathbb{R}^n$; $x^i \cap X^j = \varnothing$, $i \neq j = 1, \ldots, K$, $n = n^1 + \ldots + n^K$; and $x = (x^1, \ldots x^K) \in \mathbb{R}^n$. Further assume that S is a compact subset of $\mathbb{R}^n$, x is in S, and each f^i maps S into $\mathbb{R}^1$. By implication, the choice made by a higher-level player may affect the choices available to a lower-level player through S; the strategy selected by any member of the system, however, may influence the outcome realized by any other member through the latter's objective function. The following nested hierarchical optimization problem captures this structure:

P1

$$\begin{aligned} &\underset{x^1 \in X^1}{\text{maximize}}\ f^1 \quad && \text{where } x^2 \text{ solves} \\ &\underset{x^2 \in X^2}{\text{maximize}}\ f^2 \quad && \text{where } x^3 \text{ solves} \\ &\quad\vdots && \\ &\underset{x^{K-1} \in X^{K-1}}{\text{maximize}}\ f^{K-1} \quad && \text{where } x^K \text{ solves} \\ &\underset{x^K \in X^K}{\text{maximize}}\ f^K && \\ &\text{subject to } x = (x^1, \ldots, x^K) \in S. && \end{aligned} \tag{3}$$

When $K = 1$, problem **P1** reduces to a standard nonlinear program; when $K = 2$ and $f^1 = -f^2$, the general "max-min" problem results.

As an individual, player K faces a problem parameterized in x^1 through x^{K-1}. Once these variables are selected, he has only to solve a simple optimization problem. Player 1, on the other hand, is required to make the first move and therefore faces a problem implicit in x^2 through x^K. In effect, he must anticipate how each of the succeeding players will react to the control x^1. Finally, player k faces a problem parameterized in x^1 through x^{k-1} and implicit in x^{k+1} through x^K.

There has been little work on the general multi-level hierarchical structure described above. Belonging to this small body of work are the largely methodological papers by Fortuny-Amat [28], Bard [8], and Anandalingam [3]. For an application of multi-level optimization to a conflict resolution problem, see Anandalingam and Apprey [6]. Because much of the hierarchical optimization literature has focussed on the bi-level optimization problem, in the next section we turn our attention to that problem.

2. The bi-level programming problem

Let us consider a two-level hierarchical system where the higher level decision maker (hereafter the "leader") controls decision variables $x \in X$ and the lower level (hereafter the "follower") controls $y \in Y$, respectively. The leader is assumed to select his decision vector first, and the follower select his decision vector after that. In order to formulate the problem, let us define the following:

X = a closed convex set of $\mathbb{R}^{n_1}$,

Y = a closed convex set of $\mathbb{R}^{n_2}$,

$f, F : X \times Y \to \mathbb{R}^1$,

$g : X \times Y \to \mathbb{R}^m$.

Using this notation, the bi-level mathematical programming problem (BLMP) is formulated as:

P2
$$\underset{x \in X}{\text{maximize}}\ F(x, y) \quad \text{where } y \text{ solves} \tag{4}$$

$$\underset{y \in Y}{\text{maximize}}\ f(x, y) \tag{5}$$

$$\text{subject to}\ g(x, y) \le 0, \tag{6}$$

$$x \in X, \quad y \in Y. \tag{7}$$

For problem **P2**, we make the following definition:

DEFINITION 1

The set $\bar{S}(x) = (y : y \in Y, g(x, y) \leq 0)$ is called the Follower's solution set.

DEFINITION 2

The point-to-set mapping $RR : x \to RR(x)$ defined by $RR(x) = (y^* \in Y : f(x, y^*) \geq f(x, y), \forall y \in Y)$ is called the follower's rational reaction set.

Thus, the *feasibility set* of problem **P1** can be denoted alternatively by:

$$S = \{(x, y) : g(x, y) \leq 0, \ x \in X, \ y \in Y\}$$
$$\text{or} \tag{8}$$
$$S = \{(x, y) : x \in X, \ y \in RR(x)\}.$$

Problem **P1** can be rewritten using (8):

P3 $\quad \underset{x,y}{\text{maximize}} \quad F(x, y) \qquad (9)$

$\quad \text{subject to} \quad (x, y) \in S. \qquad (10)$

In the case where all functions are linear, the problem becomes a bi-level linear program which is formulated as follows:

P4 $\quad \underset{x}{\text{maximize}} \quad F(x, y) = ax + by \qquad \text{where } y \text{ solves} \qquad (11)$

$\quad \underset{y}{\text{maximize}} \quad f(x, y) = cx + dy \qquad (12)$

$\quad \text{subject to} \quad g(x, y) = Ax + By - p \leq 0, \qquad (13)$

$\quad x, y \geq 0. \qquad (14)$

Note that once x is given, the follower's objective is simply $\max_y dy$, and cx can be omitted from (12).

3. The bi-level programming literature

The structure of multi-level optimization leads to problem complexities not generally encountered in familiar single-level mathematical programming problems. Bialas and Karwan [17] showed that even a simple two-level resource control problem is nonconvex. In a recent paper, Ben-Ayed and Blair [15] show that the bi-level linear programming problem is NP-hard, making it unlikely that there would be exact algorithms for it.

Early algorithms to solve a special class of *zero-sum* bi-level mathematical programs (BLMP) were based on the branch-and-bound [25,37] and cutting plane techniques [41]. One of the first solutions for the bi-level linear programming problem formulation **P4** was proposed by Candler and Townsley [23]. They observed that once an optimal basis to the inner problem was obtained, changing x might affect its feasibility, but not its optimality. Thus, they proposed a scheme that involved *implicit enumeration* of adjacent bases to test for feasibility and optimality. Bialas and Karwan [16] developed a similar *vertex enumeration* procedure called the "kth best" algorithm; in their approach, the leader solves his problem with respect to both leader and follower decision vectors, and would order all basic feasible solutions in such a way that

$$F(x^k, y^k) \geq F(x^{k+1}, y^{k+1}), \qquad k = 1, 2, \ldots .$$

At any iteration k, given x^k, the follower would solve his problem to obtain $y^{*k} = RR(x^k)$. If $y^{*k} \neq y^k$, then the algorithm proceeds to the next best solution for the leader (x^{k+1}, y^{k+1}) and the follower's computation is repeated. Optimality is reached when $y^{*k} = y^k$.

In the *Kuhn–Tucker approach* [11], the rational reaction set of the follower is replaced by his (Kuhn–Tucker) optimality conditions. The leader takes into account the follower's optimality conditions while solving his own problem; thus, the problem can be written equivalently as

$$\text{P5} \quad \underset{x,\,y,\,w}{\text{maximize}} \quad F(x, y) = ax + by \tag{15}$$

$$\text{subject to} \quad d + w'B = 0, \tag{16}$$

$$w'(Ax + By - p) = 0, \tag{17}$$

$$Ax + By \leq p, \tag{18}$$

$$x, y, w \geq 0, \tag{19}$$

where w is the Lagrangian multiplier associated with eq. (13) and the prime superscript denotes the transpose operation. Note that eq. (17) presents a set of nonlinear equality constraints, making the problem nonconvex *regardless* of what other assumptions are made. Attempts at solving problem **P5** resulting from the Kuhn–Tucker approach include mixed integer programming [28], branch-and-bound techniques [12], grid search [7,8] and parametric complementary pivoting (PCP) [17]. Other attempts at solving bi-level linear programming problems include a penalty function approach [5].

Recent papers by Haurie et al. [38] and Ben-Ayed and Blair [15] have shown that both the grid search method and the PCP method may not converge to optimality.

Simple counter examples are provided for each algorithm, underscoring the difficulty of the hierarchical optimization problem. Ben-Ayed and Blair [15] suggest that branch-and-bound approaches may be more successful. For attempts at using non-traditional, computer intensive search techniques, simulated annealing and the genetic algorithm for solving bi-level linear programming, see Anandalingam et al. [4].

4. Nonlinear hierarchical optimization

The case where $F, f,$ and g in problem **P2** are nonlinear is called the nonlinear hierarchical optimization problem. To date, only a few specialized versions of the nonlinear problem have been addressed with any degree of success. Bard and Falk [11] investigated the BLMP, stated as **P2**, when the follower's problem is a convex program. By assuming that f and g were both smooth and concave in y for x fixed, and that certain regularity conditions held, they were able to reformulate the BLMP as follows:

P6
$$\underset{x, y, u}{\text{maximize}} \quad F(x, y) \tag{20}$$

$$\text{subject to} \quad \nabla_y f(x, y) - u \nabla_y g(x, y) = 0, \tag{21}$$

$$ug(x, y) = 0, \tag{22}$$

$$g(x, y) \le 0, \tag{23}$$

$$u \ge 0, \tag{24}$$

Bard and Falk [11] were able to obtain global solutions under the further assumption of function separability. A more efficient branch-and-bound approach for the strictly convex case is presented in Bard [10]. Other attempts at solving nonlinear hierarchical optimization include the papers by Bracken and McGill [18–21], deSilva [24], and Aiyoshi and Shimizu [2]. Kolstad and Lasdon [40] provide extensive computational results for the sensitivity-based methods.

5. Further generalizations

Hierarchical optimization has been studied in the transport field, where travellers form a multiple lower level and transport operators or system controllers form an upper one. In transportation, variational inequalities are used to model the behavior of the lower-level decision makers, who are said to perform marginal benefit analysis before choosing their optimal route on the network. Applications include transport systems planning [27], signal optimization [44], and network design [1,29,30,36,39,43,47]. In fact, the transportation science literature has led to the definition of what might be called a generalized bi-level mathematical programming (GBLMP) problem. In the GBLMP problem, one of the levels is articulated as a nonlinear complementarity or variational inequality problem. See Friesz [29] for a description of these problems, which have the form:

P7 $$\underset{x \in X}{\text{maximize}} \quad F(x, y) \tag{25}$$

subject to $y \in Y$ such that

$$f(x, y)(y' - y) \geq 0 \qquad \forall y' \in Y. \tag{26}$$

It is well known that under appropriate assumptions, namely symmetry and positive definiteness of the Jacobian $\nabla_y f(x, y)$, that the variational inequality (26) is equivalent to a mathematical programming problem (see, for example, Lemke [42]) and, thus, methods for solving **P7** will also work on **P2**.

The interesting and challenging aspect of GBLMP occurs when (26) does not have a standard mathematical programming equivalent. In this case, as (26) makes clear, the outer or higher-level problem may be viewed as a mathematical program with infinitely many constraints. Marcotte [44] has proposed an exact constraint accumulation method for such problems. Marcotte [45] has also proposed and conducted worst case analyses of various heuristics which involve the approximate solution of the inner problem. Suwansirikul et al. [47] have suggested a heuristic which decomposes the problem into a set of 1 leader–1 follower subproblems. DeSilva [24], Kolstad and Lasdon [40], and Friesz et al. [33] have advocated the use of nonlinear sensitivity analysis to develop local expressions for the inner problem's decision variables and derivatives in terms of the outer problem's decision variables. These sensitivity analysis methods, although lacking formal convergence proofs, have proven to be quite effective on medium size problems. Moreover, they are particularly amenable to parallel computation and are expected to perform quite well on large problems when the associated software is fully vectorized. See in particular the detailed numerical tests made by Friesz et al. [33] of alternative implementations of sensitivity analysis based methods for GBLMP.

Simulated annealing has been recently employed by Friesz et al. [34,35] to solve versions of GNLMP. Although simulated annealing is extremely computationally demanding, it has identified solutions to medium-scale network design problems which are 5 to 10 percent better than the best local solutions previously determined with other methods, suggesting that global optima have been found.

6. Applications

Among the earliest applications of hiearchical optimization is the work by Bracken and McGill [21,22]. They formulate and solve hierarchical optimization models motivated by a wide variety of defense problems such as (1) strategic offensive and defensive force structure design, (2) strategic bomber force structure and basing, (3) strategic defense to achieve specified post-attack production capabilities, (4) general models of weapon mix and targeting to achieve desired attrition over time, and (5) allocation of tactical aircraft to missions [21]. Other applications of the Bracken and McGill work include analysis of competitive economies through a model which enables firms to maximize profit subject to constraints on maintenance of market share in the face of collusive action by competitors and subject to resource constraints [22].

Recently, hierarchical optimization techniques and multi-level mathematical programming have been applied to many systems in which independent agents of unequal influence make decisions in an interrelated fashion. Problems are found in such areas as energy planning [24], government regulation [9,54], equipment scheduling [52], decentralized control [8,53], and conflict resolution [6]. As mentioned before, transportation network design has been an especially active application area. Furthermore, in recent years, hierarchical optimization has been extensively applied to the study of imperfectly competitive spatial economies and equilibrium facility location [31,32,49,50].

7. This volume

This volume of the *Annals of Operations Research* extends the present literature from the perspectives of both theory and applications in a number of important areas.

The first paper by Blair generalizes the work by Ben-Ayed and Blair [15] and shows that multi-level mathematical programs are NP-hard. Tobin, and Leleno and Sherali examine the hierarchical optimization problems inherent in multi-firm systems. Tobin uses the theory of sensitivity analysis to provide theoretical results on the existence and uniqueness of Stackelberg–Cournot–Nash equilibria; these results yield an efficient algorithm for finding the equilibrium. Leleno and Sherali provide theoretical results for a network of oligopolies involved in two-stage production. They also provide existence and uniqueness results for equilibria, and provide mechanisms for performing sensitivity analysis.

The next five papers provide new algorithms for a number of different hierarchical optimization problems. Ishizuka and Aiyoshi derive theoretical properties of a double penalty function method for bi-level linear programming. Júdice and Faustino provide a sequential method that extends the parametric complementary pivoting algorithm and corrects its flaws. They also provide extensive compuational results. DeSilva and McCormick provide new results for nonlinear bi-level programs (NLBLP) and extend deSilva's [24] dissertation that formed the basis of much work in NLBLP. Al-Khayyal, Horst and Pardalos provide branch-and-bound and piecewise linear approximation methods for solving NLBLP. Edmunds and Bard provide a new algorithm for solving integer nonlinear bi-level programs.

The next four papers focus on transportation network design problems or generalized hierarchical optimization. Marcotte and Marquis provide results from the implementation of a new heuristic for the continuous network design problem. Miller, Friesz and Tobin provide a heuristic for the problem of locating facilities in a network where the firms interact with each other in a hierarchical fashion and there is spatial competition. Suh and Kim provide the definitive comparative review of applications of nonlinear bi-level programming to equilibrium network design problems. The paper by Ben-Ayed, Blair, Boyce and LeBlanc gives details of constructing a real-world bi-level programming model. It reports on the results of a study at the University of Illinois on designing a highway system in Tunisia.

Three other applications of hierarchical optimization are reported in this volume. In one, Hobbs and Nelson report on how they used bi-level nonlinear programs to examine various issues that affect electric utility planning and provide advice on energy conservation. Desai examines the organization of independent and integrated channel structures in marketing using a hierarchical optimization model. Reyniers examines supplier–customer interactions in quality control using a Stackelberg equilibrium approach and derives optimal strategies.

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