6 Graphs and their properties

6.1 Establish Lemma 6.1.

Solution. It holds that:

$$[A^{n_o}]_{\ell,k} = \sum_{\{j_1,\ldots,j_{n_o-1}\}\in\mathcal{P}^{n_o}_{\ell,k}} a_{\ell j_1} a_{j_1 j_2} \cdots a_{j_{n_o-1}k}$$

Here, $\mathcal{P}^{n_o}_{\ell,k}$ denotes the set of agents $j_1, \ldots, j_{n_o-1}$ on the a path of length n_o from agent ℓ to agents k. For a connected graph, we know that for any pair ℓ, k, there exists some $n_o^{\ell,k}$, such that at least one path of non-zero weights exists. In light of the fact that all weights are non-negative, this ensures that $[A^{n_o^{\ell,k}}]_{\ell,k} > 0$. In general, however, this power $n_o^{\ell,k}$ may be different for different pairs of agents. We now aim to show that if there exists at least one agent with a self-loop, then there exists a common n_o, such that $[A^{n_o}]_{\ell,k} > 0$ for all ℓ, k. We make the argument by construction. Let us denote by ℓ^s the agent with the self-loop. Since the graph is connected, we can always construct a path from agent ℓ to k by taking a detour first from ℓ to ℓ^s, and then from ℓ^s to k. Let us denote the length of this path hy $n_o^{\ell k,s}$. Let $n_o^s = \max_{k,\ell} n_o^{\ell k,s}$. We can then take a path from any node ℓ to any other node k of length exactly n_o^s by taking an appropriate number of self-loops at node ℓ^s. It then follows that:

$$[A^{n_o^s}]_{\ell,k} > 0$$

6.2 Establish (6.7).

Solution. We have:

$$\begin{aligned}
\lim_{i\to\infty} A^i &= \left(V_\epsilon J V_\epsilon^{-1}\right)^i \\
&= V_\epsilon \left(\lim_{i\to\infty} J^i\right) V_\epsilon^{-1} \\
&= \begin{bmatrix} p & V_R \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lim_{i\to\infty} J_\epsilon^i \end{bmatrix} \begin{bmatrix} \mathbb{1}^\mathsf{T} \\ V_L^\mathsf{T} \end{bmatrix} \\
&= \begin{bmatrix} p & V_R \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{1}^\mathsf{T} \\ V_L^\mathsf{T} \end{bmatrix} \\
&= p\mathbb{1}^\mathsf{T}
\end{aligned}$$

6.3 Show that the Laplacian and Metropolis rules in Table 6.1 are doubly-stochastic.

Solution. We begin with the Laplacian. First, for symmetry, we have:

$$A^\mathsf{T} = (I - \beta L)^\mathsf{T} = I - \beta L^\mathsf{T} = I - \beta L = A$$

For stochasticity:

$$A\mathbb{1} = \left(I - \beta L^\mathsf{T}\right) \mathbb{1} = \mathbb{1} - \beta L \mathbb{1} = \mathbb{1}$$

For the Metropolis rule, symmetry follows directly from:

$$a_{\ell k} = \frac{1}{\max\{n_k, n_\ell\}} = \frac{1}{\max\{n_\ell, n_k\}} = a_{k\ell}$$

For stochasticity, note that:

$$\sum_{k=1}^{K} a_{\ell k} = a_{kk} + \sum_{\ell \in \mathcal{N}_k \setminus k} a_{\ell k} = 1 - \sum_{\ell \in \mathcal{N}_k \setminus k} a_{\ell k} + \sum_{\ell \in \mathcal{N}_k \setminus k} a_{\ell k} = 1$$

7 Averaging over Graphs

7.1 Establish relation (7.46).

Solution. We can reformulate:

$$
\begin{aligned}
&\eta \sum_{k=1}^{K} \sum_{\ell \in \mathcal{N}_k} c_{\ell k} \|w_k - w_\ell\|^2 \\
&= \eta \sum_{k=1}^{K} \sum_{l > k} c_{\ell k} \left(\|w_k\|^2 - 2 w_\ell^\mathsf{T} w_k + \|w_\ell\|^2 \right) \\
&= \eta \sum_{k=1}^{K} \sum_{\ell=1}^{K} c_{\ell k} \left(\|w_\ell\|^2 - w_\ell^\mathsf{T} w_k \right) \\
&= \eta \sum_{k=1}^{K} \sum_{\ell=1}^{K} c_{\ell k} \|w_\ell\|^2 - \eta \sum_{k=1}^{K} \sum_{\ell=1}^{K} c_{\ell k} w_\ell^\mathsf{T} w_k \\
&= \eta \sum_{\ell=1}^{K} \left(\sum_{k=1}^{K} c_{\ell k} \right) \|w_\ell\|^2 - \eta \sum_{k=1}^{K} \left(\sum_{\ell=1}^{K} c_{\ell k} w_\ell^\mathsf{T} \right) w_k \\
&= \eta\, \mathcal{w}^\mathsf{T} \left(\mathrm{diag}\{C\mathbb{1}\} \otimes I_M \right) \mathcal{w} - \eta\, \mathcal{w}^\mathsf{T} \left(C \otimes I_M \right) \mathcal{w} \\
&= \eta\, \mathcal{w}^\mathsf{T} \left(\left(\mathrm{diag}\{C\mathbb{1}\} - C \right) \otimes I_M \right) \mathcal{w} \\
&= \eta\, \mathcal{w}^\mathsf{T} \mathcal{L}\, \mathcal{w}
\end{aligned} \tag{7.1}
$$

7.2 We saw at several points throughout this chapter that the mixing rate of an matrix A, quantified through its second-largest singular value $\sigma_2(A)$ plays a key role in quantifying the rate at which averages can be computed. In this problem, we will verify Theorem 7.1 in code. To this end, generate a random collection of signals $\{g_k\}_{k=1}^K$ using a statistical model of your choice. For the graph, we generate Erdos-Renyi graphs with varying edge probability p_{edge}. Any pair of agents ℓ and k is linked with probability p_{edge}, independently of all other edges. Construct the adjacency matrix A following the Metropolis rule of Chapter 6. For different choices of p_{edge}, compute the associated mixing rate $\sigma_2(A)$. Subsequently implement the static consensus algorithm (7.16), plot the evolution of $\sum_{k=1}^K \|\overline{g} - w_{k,i}\|^2$ and discuss the relationship between edge probablities p_{edge}, $\sigma_2(A)$ and the rate of convergence.

Solution. The solution is provided as a Jupyter notebook in the separate file `Problem_7_2.ipynb`.