MCV4U1 Test on Derivatives II	K&U 22	COM 8	APP 17	TIPS 8	Name: Dorsa Rohani
	22	8	17	8	Date:

Knowledge & Understanding

1. Solve for x and state your solutions in **exact** value. ❻

a) $e^{-3x+8} = 12$	b) $\ln(\ln(7x-5)) = -4$	c) $\ln(3x-2) = -5$
$\ln e^{-3x+8} = \ln 12$ $-3x+8 = \ln 12$ $x = \frac{\ln 12 - 8}{-3}$	$e^{\ln(\ln(7x-5))} = e^{-4}$ $e^{\ln(7x-5)} = e^{e^{-4}}$ $7x-5 = e^{e^{-4}}$ $x = \frac{e^{e^{-4}}+5}{7}$	$e^{\ln(3x-2)} = e^{-5}$ $3x-2 = e^{-5}$ $x = \frac{e^{-5}+2}{3}$

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2. For each of the following functions in the first column, find the <u>simplified</u> expression for $\frac{dy}{dx}$ and place your answer in the second column.
No need to show your work. [12] --------->

Function	$\frac{dy}{dx}$ (simplified)
a) $y = e^{-\sin x}$	$\frac{dy}{dx} = e^{-\sin x}(-\cos x)$
b) $y = \frac{-x}{\cos^2 x}$	$\frac{dy}{dx} = -\frac{\cos x + 2x\sin x}{\cos^3 x}$
c) $y = \pi x^{\pi}$	$\frac{dy}{dx} = \pi^2 x^{\pi - 1}$
d) $y = 3\ln(\sin(x))$	$\frac{dy}{dx} = 3\cot x$
e) $y = \ln x^3$	$\frac{dy}{dx} = \frac{3}{x}$
f) $y = \sin\left(\frac{x}{2}\right)\cos(3x)$	$\frac{dy}{dx} = \frac{1}{2}\cos\frac{x}{2}\cos 3x - 3\sin\left(\frac{x}{2}\right)\sin 3x$
g) $y = \ln(\ln(3x^2))$	$\frac{dy}{dx} = \frac{2}{\ln 3x^2 (x)}$
h) $y = \log_5(x^2-1)$	$\frac{dy}{dx} = \frac{2x}{(x^2-1)\ln 5}$
i) $y = 3^{e^x}$	$\frac{dy}{dx} = e^x \ln 3\,(3^{e^x})$
j) $y = \ln\left(\frac{3}{x}\right)$	$\frac{dy}{dx} = -\frac{1}{x}$
k) $y = [\sin(x)]^x$	$\frac{dy}{dx} = (\sin x)^x[\ln(\sin x) + x\cot x]$
l) $y^2x - 3y^3 + x = 1$	$\frac{dy}{dx} = \frac{-(y^2+1)}{y(2x-9y)}$

12

3. At what value(s) of x does $f(x) = (x+2)^3e^{-x}$ have **horizontal** tangents? ❹

4

$f'(x) = 0$

$f'(x) = [3(x+2)^2]e^{-x} + (x+2)^3 e^{-x}(-1)$

$= 3(x+2)^2e^{-x} - (x+2)^3e^{-x}$

$= e^{-x}(x+2)^2[3-(x+2)]$

$0 = e^{-x}(x+2)^2(-x+1)$

$e^{-x} = 0$ $(x+2)^2 = 0$ $(-x+1) = 0$

no sol. $x = -2$ $x = 1$

$\therefore$ at $x = -2, 1$

COMMUNICATION

Neat and complete solutions are required for full marks. [8]

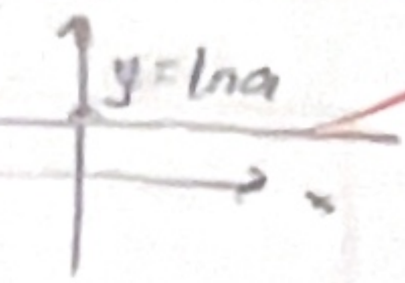
4. Let $f(x) = a^x$, where $0 < a < 1$, and let $g(x) = \frac{f(x)}{f'(x)}$.

a) Predict the **equation** and the **shape** of the graph of g(x) by giving **one example**. ❸

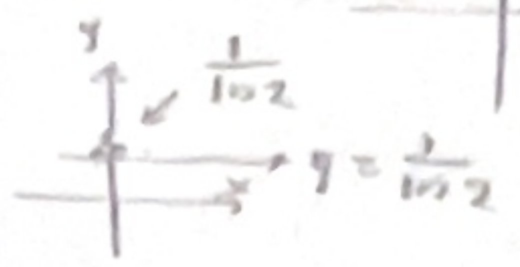
$g(x) = \frac{f(x)}{f'(x)} = \frac{a^x}{a^x(\ln a)} = \frac{1}{\ln a}$

∵ a is a constant and $g(x) = \frac{1}{\ln a}$

∴ g(x) is a straight line



Example: $g(2) = \frac{1}{\ln 2} \approx 0.693...$



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b) For what value of a will $g(x) = e$? ❷

$g(x) = e$

$\frac{1}{\ln a} = e$

$1 = e\ln a$

$\ln a = e^{-1}$

$e^{\ln a} = e^{e^{-1}}$

$a = e^{e^{-1}}$

$= e^{\frac{1}{e}}$

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C8

5. Water in a pot is heated on a stove then removed from the heat. The temperature of the water, ***T*** degrees Celsius, is modelled by the equation $T = 80e^{-0.057t} + 20$ where ***t*** represents time in minutes starting from the time the pot was removed from the stove.

a) What was the temperature of the water after 0.5 hours ? Round to 2 decimal places. ❶

$\frac{1}{2}\text{hours}\left(\frac{60\text{min}}{\text{hour}}\right) = 30\text{min}$ $\quad T = 80e^{-0.057(30)} + 20 \doteq 34.47$ degrees °C

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b) **Determine** $\lim_{t\to\infty} 80e^{-0.057t} + 20$ and **explain** what does this mean about the temperature of pot of water? ❷

$\lim_{t\to\infty} 80e^{-0.057t} + 20$

$= \lim_{t\to\infty} \left(\frac{80}{e^{-0.057t}}\right) + 20$

$= (0) + 20$

$= 20$

∴ the room temp./eventual temperature of the water is 20° degrees after being removed from the stove

Temp will approach, but never touch, 20°C

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Application

1. Find the **equation** of the tangent line to the graph of $f(x) = e^{\frac{-x}{2}}$ that is **perpendicular** to the line $x - 2y - 6 = 0$. Express your answer in **EXACT** value. ❹

$f'(x) = e^{-\frac{1}{2}x}\left(-\frac{1}{2}\right)$

$x - 6 = 2y$

$y = \frac{x-6}{2}$

$y = \frac{1}{2}x - 3$

$m_{\perp} = -2$

$-2 = -\frac{1}{2}e^{-\frac{1}{2}x}$

$4 = e^{-\frac{1}{2}x}$

$\ln(4) = \ln e^{-\frac{1}{2}x}$

$\ln(4) = -\frac{1}{2}x$

$-2\ln 4 = x$

At $x = -2\ln(4)$,

$f(-2\ln(4)) = e^{\frac{-\ln(4)(2)}{2}}$

$= 4$

$(-2\ln(4), 4)$

$y = -2x + b$

$4 = -2(-2\ln(4)) + b$

$4 = 4\ln(4) + b$

$b = 4(1 - \ln(4))$

∴ $y = -2x + 4(1 - \ln 4)$

A4

2. Given $\ln(x-y^2)=y$, determine $\frac{dy}{dx}$ in simplified form. Determine $\frac{dy}{dx}$ when $y=1$ **5**

$\frac{d}{dx}[\ln(x-y^2)] = \frac{d}{dx}[y]$

$\frac{1}{x-y^2}(1-2yy') = y'$

$\frac{1}{x-y^2} - \frac{2yy'}{x-y^2} = y'$

$\frac{1}{x-y^2} = y' + \frac{2yy'}{x-y^2}$

$\frac{1}{x-y^2} = y'\left(1+\frac{2y}{x-y^2}\right)$

$\rightarrow y' = \frac{1}{x-y^2} \cdot \left(\frac{1}{1+\frac{2y}{x-y^2}}\right)$

$= \frac{1}{x-y^2} \cdot \frac{1}{\frac{(x-y^2)+2y}{x-y^2}}$

$= \frac{1}{x-y^2} \cdot \frac{x-y^2}{-y^2+2y+x}$

$\frac{dy}{dx} = \frac{1}{-y^2+2y+x}$

$y=1,$

$\ln(x-1^2)=1$

$x-1^2=e$

$x=e+1$

$\left.\frac{dy}{dx}\right|_{(x=e+1,\,y=1)} = \frac{1}{-(1)^2+2(1)+(e+1)} = \frac{1}{e+2}$

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3. Determine for what value(s) of x on the interval $-\pi \le x \le 2\pi$ where the slope of the tangent to the function $y=-\cot(x)$ is equal to 2 . Use **exact** radian measure. **4**

$y = -\frac{1}{\tan x}$

$= -\tan x^{-1}$

$y' = \tan x^{-2}(\sec^2 x)$

$= \frac{\sec^2 x}{\tan^2 x}$

$= \frac{1}{\cos^2 x}\left(\frac{\cos^2 x}{\sin^2 x}\right)$

$= \frac{1}{\sin^2 x}$

$y' = \frac{1}{\sin^2 x}$

$2 = \frac{1}{\sin^2 x}$

$\sqrt{\sin^2 x} = \sqrt{\frac{1}{2}}$

$\sin x = \pm\frac{1}{\sqrt{2}}$

$x = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$x = \pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

$x = \pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{5\pi}{4}$

$x = 2\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{7\pi}{4}$

$x = -\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

$x = -\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = -\frac{3\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, -\frac{\pi}{4}, -\frac{3\pi}{4}$

4

4. Show that $y=\frac{1-\cos(x)}{\sin(x)}$ has **no horizontal** tangent within its domain. **4**

$y'=0$

$y' = \frac{(1-\cos x)'\sin x - (1-\cos x)\sin x'}{\sin^2 x}$

$= \frac{\sin^2 x - (1-\cos x)\cos x}{\sin^2 x}$

$= \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}$

$= \frac{-\cos x + 1}{\sin^2 x}$

$\rightarrow = \frac{1-\cos x}{\sin^2 x}\left(\frac{1+\cos x}{1+\cos x}\right)$

$= \frac{1-\cos^2 x}{\sin^2 x(1+\cos x)}$

$= \frac{\sin^2 x}{\sin^2 x(1+\cos x)}$

$= \frac{1}{1+\cos x}$

$0 = \frac{1}{1+\cos x} \leftarrow$ no solutions/DNE

$\therefore$ no horiz. tangent

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Thinking & Problem Solving

6. At what **point(s)** on the curve $y = 2^{-\sqrt{3}\sin x+\cos x}$, $-2\pi \le x \le 2\pi$ is the tangent line horizontal? Express the coordinates of the points in **exact** values where: (4)

$y' = 0$

$y' = 2^{-\sqrt{3}\sin x+\cos x}(\ln 2)(-\sqrt{3}\sin x+\cos x)'$

$y'=0 \to 0 = 2^{-\sqrt{3}\sin x+\cos x}(\ln 2)(-\sqrt{3}\cos x-\sin x)$

$2^{-\sqrt{3}\sin x+\cos x} = 0$ no sol.

$\ln 2 = 0$ no sol.

$-\sqrt{3}\cos x - \sin x = 0$

$\sqrt{3} = \frac{\sin x}{-\cos x}$

$-\sqrt{3} = \tan x$

$\to x = \pi - \tan^{-1}(\sqrt{3}) = \frac{2\pi}{3}$

$x = 2\pi - \tan^{-1}(\sqrt{3}) = \frac{5\pi}{3}$

$x = -2\pi + \tan^{-1}(\sqrt{3}) = -\frac{4\pi}{3}$

$x = -\pi + \tan^{-1}(\sqrt{3}) = -\frac{1\pi}{3}$

$\therefore (\frac{2\pi}{3}, \frac{1}{4}), (\frac{5\pi}{3}, 4), (-\frac{4\pi}{3}, \frac{1}{4}), (-\frac{\pi}{3}, 4)$

4

7. State the <u>**exact**</u> value of the $\lim\limits_{h\to 0}\frac{5e^{-1+h}-5e^{-1}}{h}$? What is the **graphical** significance of this quantity or what is the **geometrical representation** of this limit? (4)

1.839

It is the slope of the tangent line (derivative) of the function $y = 5e^x$ at $x = -1$

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$= \lim\limits_{h\to 0}\frac{5e^{-1}\cdot e^{h} - 5e^{-1}}{h}$

$= \lim\limits_{h\to 0}\frac{5e^{-1}(e^{h}-1)}{h}$

$= 5e^{-1}\lim\limits_{h\to 0}\frac{e^{h}-1}{h}$

recall: $\lim\limits_{h\to 0}\left(\frac{b^{h}-1}{h}\right) = \ln b$

$= 5e^{-1}(\ln e)$

$= 5e^{-1}(1)$

$= 5e^{-1}$

$= \frac{5}{e}$ //

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have a great day!