K&U	App	TIPS
15	20	10
15	19	9½

Test on optimization and related rates

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Instructions: *As part of your solution, make sure to follow the steps below.* **These steps applied to all questions.**

a) *Define all variables used using a diagram (2)*
b) *State any restrictions (the possible values for your independent variable) (1)*
c) *Provide the equation representing the quantity being optimized (3)*
d) *Show your work and state the solution(s) (3)*
e) *Answer the posed question in a concluding statement (1)*

Knowledge & Understanding

1. If $x^3 - y^3 = -9$ and $\frac{dx}{dt} = 3$, find $\frac{dy}{dt}$ when $y = 1$. [3]

$x^3 - (1)^3 = -9$

$x^3 = -9 + 1$

$x^3 = -8$

$x = -2$

$\frac{d}{dt}[x^3 - y^3] = \frac{d}{dt}[-9]$

$3x^2x' - 3y^2y' = 0$

$3(-2)^2(3) - 3(1)^2y' = 0$

$\to y' = 12$

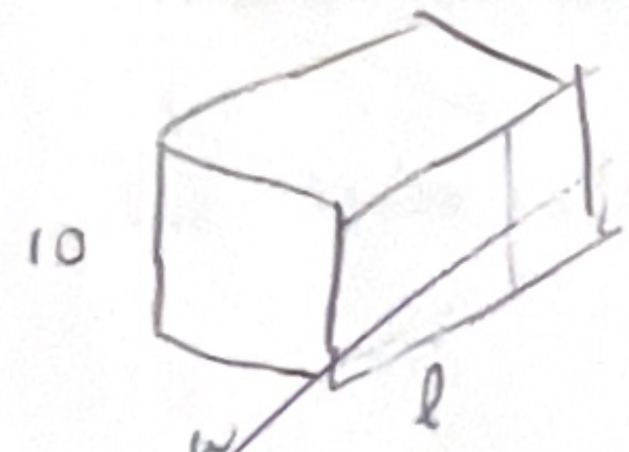
$\therefore \frac{dy}{dt}$ is 12 when $y = 1$

3

Set up the following problems (2 – 3) to the point where you would differentiate but do not differentiate and do not solve (**only steps a to c above**). Include a neat diagram for each problem.

2. A cereal box in the shape of a rectangular prism is required to have a capacity of 1000 cm³, and the **thickness** of the box *must* be 10 cm [to allow for a comfortable grasp by most people]. What dimensions of the box require the **minimum** amount of material? [6]

Let l be length of box (cm), w be width (cm), V be volume (cm³), SA be surface area of box (cm²)



$V = 1000$

$V = 10wl$

$1000 = 10wl$

$100 = wl \to \frac{100}{w} = l$ ①

$w, l > 0$

$SA = 2(10w) + 2wl + 2(10l)$

② $= 20w + 2wl + 20l$

sub ① into ②

$SA = 20w + 2\cancel{w}\left(\frac{100}{\cancel{w}}\right) + 20\left(\frac{100}{w}\right)$

$= 20w + 200 + \frac{2000}{w}$

$= 20w + 200 + 2000w^{-1}$

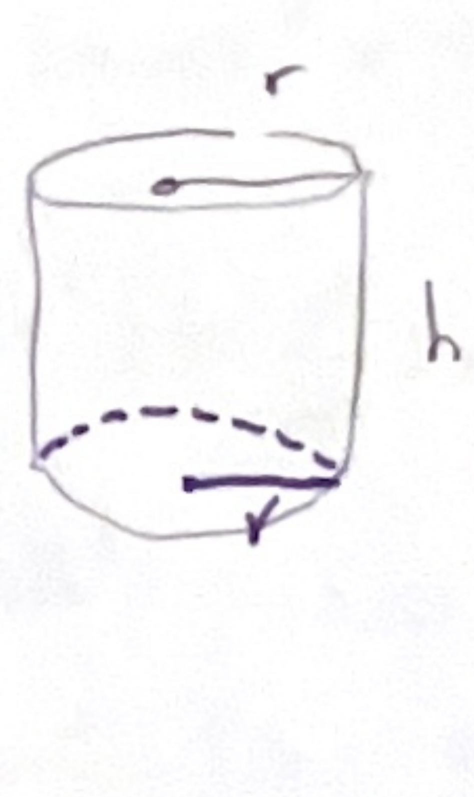
6

3. A manufacturer wishes to produce cylindrical fruit juice cans with a capacity of 250 ml. What dimensions will **minimize** the amount of material required for a can? (1 ml = 1 cm³) [6]

250 cm³

Let r be radius of can (cm), h be height (cm), V be volume (cm³), SA be surface area of can (cm²)

$V = 250\ cm^3$



$r, h > 0$

$V = \pi r^2 h$

$250 = \pi r^2 h$

① $h = \frac{250}{\pi r^2}$

② $SA = 2\pi r^2 + 2\pi rh$

sub ① into ②

$SA = 2\pi r^2 + 2\cancel{\pi}\cancel{r}\left(\frac{250}{\cancel{\pi} r^{\cancel{2}}}\right)$

$= 2\pi r^2 + \frac{500}{r}$

$= 2\pi r^2 + 500r^{-1}$

6

PART B: Application

1. Water is flowing out of a **cylindrical** storage tank at a rate of 3 m^3/min. If the tank has a radius of 2 m, how fast is the water level falling? Exact solution with proper unit. [4]

Let r be radius of tank, h be height of water, V be volume of water

$\frac{dV}{dt} = -3$

$\frac{dh}{dt} = ?$

$V, h \geq 0$

$V = \pi r^2 h$

$\frac{dV}{dt} = \frac{d}{dt}[\pi r^2 h]$

$V' = \pi r^2 h'$

$-3 = \pi(2)^2 h'$

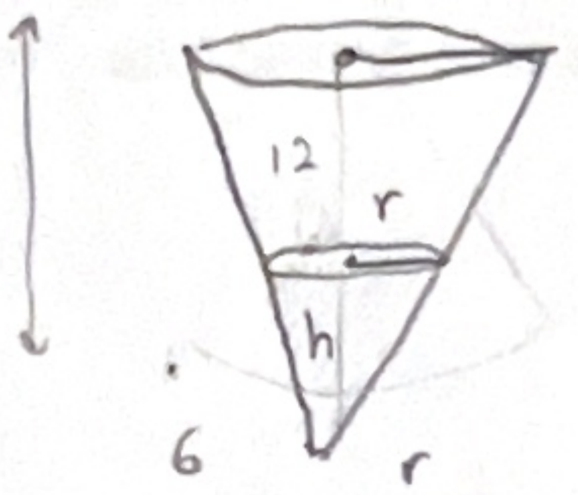
$h' = \frac{-3}{4\pi}$ m/min

∴ the water level is falling at a rate of $\frac{3}{4\pi} \approx 0.238$ m/min

4

2. A conical tank is 12 m high. Its dimeter across the top is 12 m. Water is being drained from the tank at the rate of 5 m^3/min. Determine the **exact** rate at which the water level is decreasing when h is 4 m. [6]

Let V be volume of water, h be height of water, r be radius of water



$\frac{dV}{dt} = -5$

$\frac{dh}{dt} = ?$ when $h = 4$

$V, r, h \geq 0$

$0 \leq r \leq 6$

$0 \leq h \leq 12$

② $V = \frac{1}{3}\pi r^2 h$

$\frac{6}{12} = \frac{r}{h}$

① $r = \frac{1}{2}h$

sub ① into ②

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$= \frac{\pi}{3}\left(\frac{h^2}{4}\right)h$

$= \frac{\pi}{12}h^3$

$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{\pi}{12}h^3\right]$

$V' = \frac{\pi}{4}h^2 h'$

$-5 = \frac{\pi}{4}(4)^2 h'$

$h' = -\frac{5}{4\pi}$ m/min

∴ the water level is decreasing by $\frac{5}{4\pi}$ m/min at a height of 4 m

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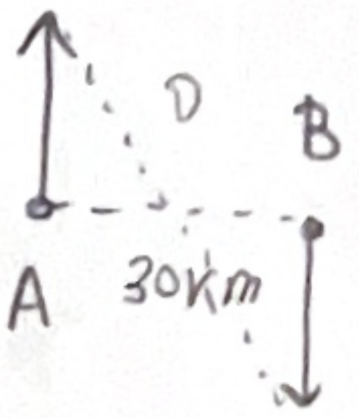
3. At 2:00 p.m. ship A is 30 km west of ship B. Ship A is sailing north at 50 km/hr and ship B is sailing south at 30 km/hr. How fast is the distance between them changing at 4:00 pm? [10]

Let D be distance between ships A & B (km), t be time (h)

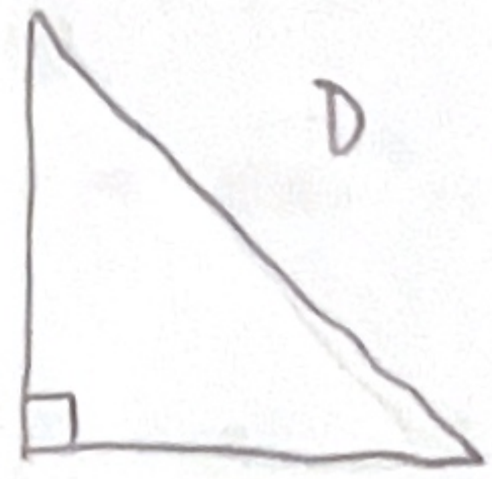
$t \geq 0$

$D \geq 0$

at 2pm:



50t + 30t



10

$\frac{dD}{dt} = ?$ at $t = 2$

$D^2 = (80t)^2 + 30^2$

$D^2 = 6400t^2 + 900$

$\frac{d}{dt}[D^2] = \frac{d}{dt}[6400t^2 + 900]$

$2DD' = 12800t$

$2(10\sqrt{265})D' = 12800(2)$

$D' = \frac{1280}{\sqrt{265}}$

at $t = 2$

$D^2 = (80 \cdot 2)^2 + 30^2$

$D = \pm 10\sqrt{265}$

reject

$D' \approx 78.63$ km/h

∴ distance between ships A & B is changing at approx. 78.6 km/h ($\frac{1280}{\sqrt{265}}$ km/h) at 4pm

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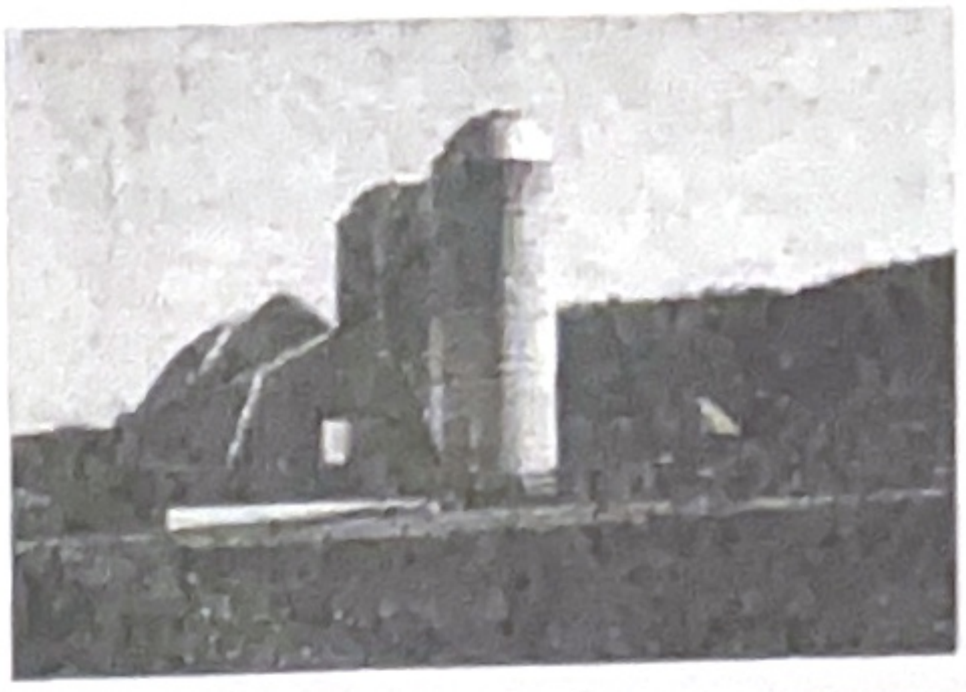
Thinking Inquiry TIPS: /10

Instruction: For the following 2 questions please choose **ONLY** one question to solve. Please check mark ✓your selection.

Question 1: ☐ **Question 2:** ☑

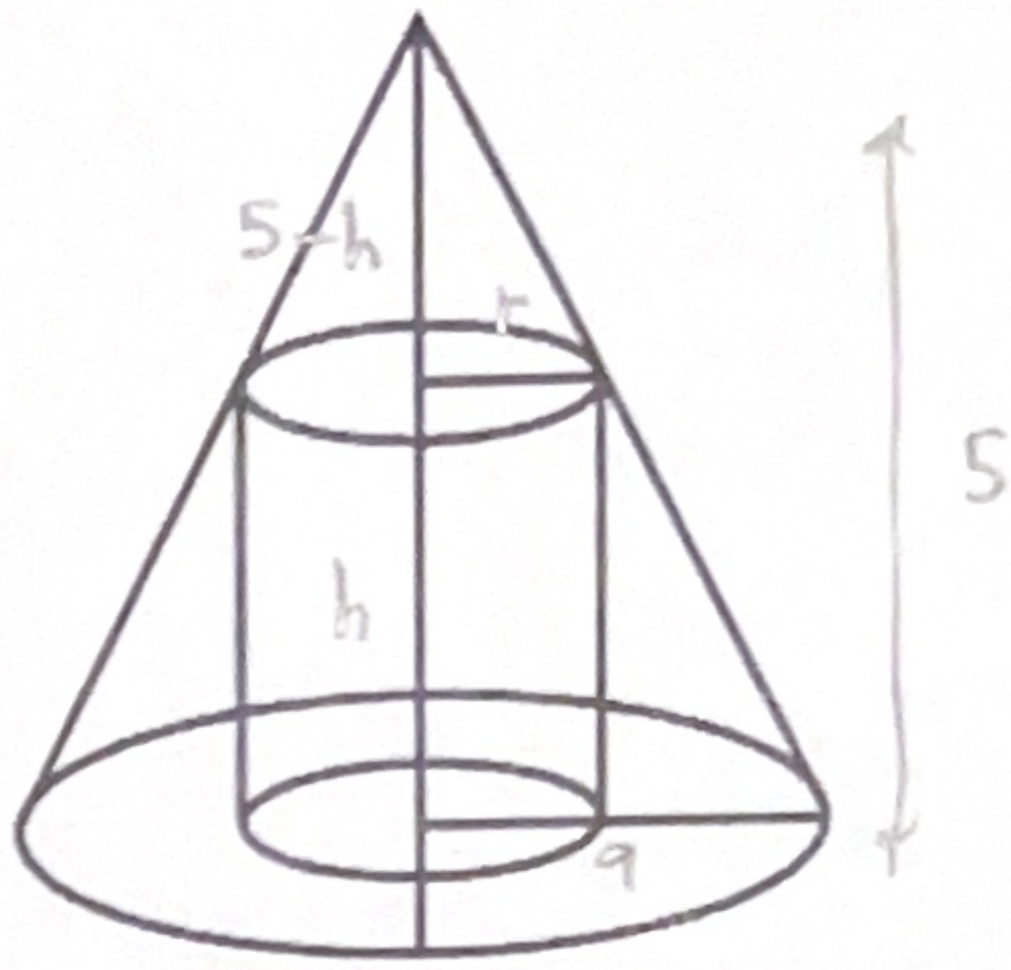
Question 1

1. Suppose you wish to build a grain silo with volume V = 10000 m^3 of made up of a steel cylinder and a hemispherical roof. The steel sheets covering the surface of the silo are quite expensive, so you wish to minimize the surface area of your silo. What **height** and **radius** should the silo have with a circular floor to keep out gophers.



Or Question 2

2. Determine the largest volume of a cylinder inscribed in a cone with height of 5 cm and base radius of 9 cm.



let r be radius of cylinder (cm)

let h be height " " (cm)

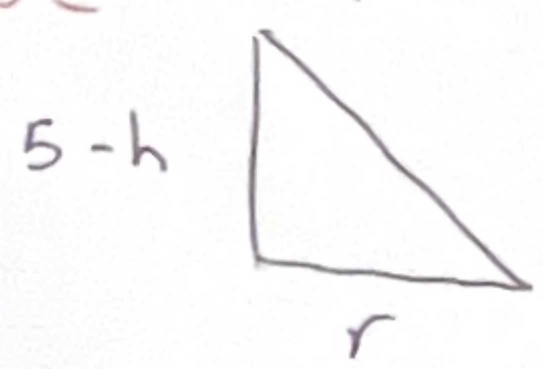
let V be volume of cylinder (cm^3)

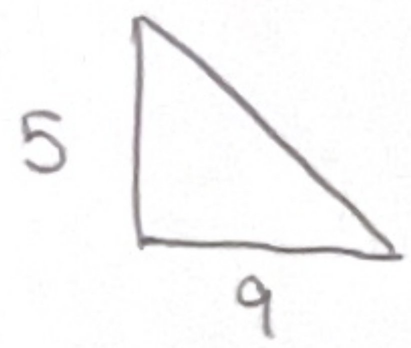
$r, h > 0$

$0 < h < 5$

$0 < r < 9$

$9\frac{1}{2}$





$$\frac{5-h}{r} = \frac{5}{9}$$

$$5-h = \frac{5r}{9}$$

① $h = 5 - \frac{5r}{9}$

② $V_{cylinder} = \pi r^2 h$

sub ① into ②

$$= \pi r^2 \left(5 - \frac{5r}{9}\right)$$

$$= 5\pi r^2 - \frac{5}{9}\pi r^3$$

$$V' = 10\pi r - \frac{5}{3}\pi r^2$$

$V' = 0$

$$0 = \pi r\left(10 - \frac{5}{3}r\right)$$

$r = 0$ reject $r = 6$ cm

check:

$$V'' = 10\pi - \frac{10}{3}\pi r$$

at $r = 6$,

$V'' < 0$

is max at $r = 6$

$$h = 5 - \frac{5r}{9}$$

$$= 5 - \frac{5(6)}{9}$$

$$= \frac{5}{3} \text{ cm}$$

$$V = \pi(6)^2\left(\frac{5}{3}\right)$$

$$= 60\pi \text{ cm}^3$$

∴ largest volume is 60π cm^3 ≈ 188.5 cm^3 with a radius of 6cm that can be inscribed in a cone with height 5cm & radius 9cm