CS419 HW1

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Question-1

Proof of convolution being commutative

The definition of convolution is as follows:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)g(x-y)\,dy$$

Now we will perform a change of variables. Let $u = x - y$, which implies $y = x - u$ and $du = -dy$. Now we will rewrite the expression.

$$\begin{aligned} f * g(x) &= \int_{x+\infty}^{x-\infty} f(x-u)g(u)(-du) \\ &= \int_{+\infty}^{-\infty} f(x-u)g(u)\,(-du) \\ &= \int_{-\infty}^{+\infty} f(x-u)g(u)\,du \\ &= \int_{-\infty}^{+\infty} g(u)f(x-u)\,du \\ &= g * f(x) \end{aligned}$$

Since we can turn convolution from one form to another this proves that $f * g = g * f$, which is the commutativity of convolution.

Proof of cross correlation not being commutative

cross correlation for functions

$$(f \star g)(\tau) = \int_{-\infty}^{\infty} f(t) \cdot g(t+\tau)\,dt$$

$$(g \star f)(\tau) = \int_{-\infty}^{\infty} g(t) \cdot f(t+\tau)\,dt$$

we will do a change of variable technique, let $u = t + \tau$ then $t = u - \tau$ and $dt = du$

$$(g \star f)(\tau) = \int_{-\infty}^{\infty} g(u - \tau) \cdot f(u)\, du$$

$$\int_{-\infty}^{\infty} f(u - \tau) \cdot g(t)\, du \neq \int_{-\infty}^{\infty} g(u - \tau) \cdot f(u)\, du$$

then we will compare the 2 integrals

$$\int_{-\infty}^{\infty} f(t) \cdot g(t + \tau)\, dt \neq \int_{-\infty}^{\infty} g(t - \tau) \cdot f(t)\, dt$$

which proves the fact that cross correlation is not commutative

Proof of convolution being associative

we can show what it means to be associative as follows:

$$(f * g) * h = f * (g * h)$$

so if we look at the left side, we can open this as follows:

$$(f * g)(x) * h(x) = \int_{-\infty}^{+\infty} (f * g)(x) h(x - y) dy$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(u) g(y - u) du \right) h(x - y) dy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(u) g(y - u) h(x - y) du) dy$$

$$= \int_{-\infty}^{+\infty} f(u) \left(\int_{-\infty}^{+\infty} (g(y - u) h(x - y) dy) \right) du$$

following the change of variables v = y-u

y = u+v

$dy = dv$, since u is held as constant its derivative is 0 here.

after substituting those variables we get the following equation

$$= \int_{-\infty}^{+\infty} f(u) \left(\int_{-\infty}^{+\infty} g(v) h(x - v - u) dv \right) du$$

the inner integral is the opening of the convolution of (g*h)(x-u)

this would give us $\int_{-\infty}^{+\infty}$ (g(**v**)h(x-u-**v**) dv) which is (g*h)(x-u)

so we can write the following

$$= \int_{-\infty}^{+\infty} f(u) (g * h)(x - u) du = f(x) * (g * h)(x)$$

just like above this is also the opening of convolution f(x)*(g*h)(x)

this proves the point that $(f * g) * h = f * (g * h)$

Question-2

given x' = $xcos(\theta) - ysin(\theta)$
given y' = $xsin(\theta) + ycos(\theta)$
first, we should express the second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} \; and \; \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y}$$

Then later on we should express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ as their derivatives, using the chain rule which is as follows

$$\frac{\partial f}{\partial x} = \cos(\theta)\frac{\partial}{\partial x'} + \sin(\theta)\frac{\partial}{\partial y'}$$
$$\frac{\partial f}{\partial y} = -\sin(\theta)\frac{\partial}{\partial x'} + \cos(\theta)\frac{\partial}{\partial y'}$$

Then we continue with second order derivative of y.

$$\frac{\partial^2 f}{\partial^2 y} = -\frac{\partial}{\partial y}\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y}(-\frac{\partial f}{\partial y'}sin(\theta) + \frac{\partial f}{\partial x'}cos(\theta))$$
$$= -\frac{\partial}{\partial y}\frac{\partial f}{\partial y'}sin(\theta) + \frac{\partial}{\partial y}\frac{\partial f}{\partial x'}cos(\theta)$$

We rewrite the following expression as $\frac{\partial}{\partial y}\frac{\partial f}{\partial y'}$ as $\frac{\partial}{\partial y'}\frac{\partial f}{\partial y}$ and then we rewirte $\frac{\partial}{\partial y}\frac{\partial f}{\partial x'}$ as $\frac{\partial}{\partial x'}\ \frac{\partial f}{\partial y}$
We have the following expression for second order derivative of y:

$$\frac{\partial}{\partial y'}(\frac{\partial f}{\partial y'}(-sin(\theta)) + \frac{\partial f}{\partial x'}cos(\theta)) = -\frac{\partial^2 f}{\partial y'^2}sin(\theta) + \frac{\partial f}{\partial y'\partial x'}cos(\theta)$$

then we also rewrite $\frac{\partial}{\partial x'}\frac{\partial^2 f}{\partial y}$, which is as follows:

$$= \frac{\partial}{\partial x'}(-\frac{\partial f}{\partial y}sin(\theta) + \frac{\partial f}{\partial x'}cos(\theta)) = -\frac{\partial^2 f}{\partial x'\partial y'}sin(\theta) + \frac{\partial^2 f}{\partial x'^2}cos(\theta)$$

this is equal to

$$\frac{\partial^2 f}{\partial y^2} = (-\frac{\partial^2 f}{\partial y'^2}sin(\theta) + \frac{\partial f}{\partial y'\partial x'}cos(\theta)) * (-sin(\theta)) + (\frac{\partial^2 f}{\partial x'\partial y'}sin(\theta) + \frac{\partial^2 f}{\partial x'^2}cos(\theta))cos(\theta)$$

we distribute sine and cosine functions accordingly which will give us the following result:

$$\frac{\partial^2 f}{\partial y^2} = \sin^2(\theta)\frac{\partial^2}{\partial x'^2} - 2\sin(\theta)\cos(\theta)\frac{\partial^2}{\partial x'\partial y'} + \cos^2(\theta)\frac{\partial^2}{\partial y'^2}$$

we do the same thing for the $\frac{\partial^2 f}{\partial x^2}$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} cos(\theta) + \frac{\partial f}{\partial y'} sin(\theta)$$

then we want to expand for the second order derivative

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y'} sin(\theta) + \frac{\partial f}{\partial x'} cos(\theta)) = (\frac{\partial}{\partial x}\frac{\partial f}{\partial y'} sin(\theta) + \frac{\partial}{\partial x}\frac{\partial f}{\partial x'} cos(\theta))$$

we again rewrite the expression $\frac{\partial}{\partial x}\frac{\partial f}{\partial y'}$ as $\frac{\partial}{\partial y'}\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial x}\frac{\partial f}{\partial x'}$ as $\frac{\partial}{\partial x'}\frac{\partial f}{\partial x}$

$$\frac{\partial}{\partial y'}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x'}(\frac{\partial f}{\partial x'} cos(\theta) + \frac{\partial f}{\partial y'} sin(\theta)) = \frac{\partial^2 f}{\partial x'^2} cos(\theta) + \frac{\partial^2 f}{\partial y' 2} sin(\theta)$$

$$\frac{\partial}{\partial x'}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x'}(\frac{\partial f}{\partial x'} cos(\theta) + \frac{\partial f}{\partial y'} sin(\theta)) = \frac{\partial^2 f}{\partial x'^2} cos(\theta) + \frac{\partial^2 f}{\partial x' \partial y'} sin(\theta)$$

this expressions when subbed in to $\frac{\partial^2 f}{\partial x^2}$ will give the following result

$$\frac{\partial^2 f}{\partial x^2} = (\frac{\partial^2 f}{\partial y'^2} cos^2(\theta) + 2sin(\theta)cos(\theta)\frac{\partial^2 f}{\partial y' \partial x'} + \frac{\partial^2 f}{\partial x'^2} cos^2(\theta))$$

all we have to do now is basically add the second derivative to find out the laplacian

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (\cos^2(\theta) + \sin^2(\theta))\frac{\partial^2}{\partial x'^2} + (\sin^2(\theta) + \cos^2(\theta))\frac{\partial^2}{\partial y'^2} + 2\cos(\theta)\sin(\theta)\left(\frac{\partial^2}{\partial x' \partial y'} - \frac{\partial^2}{\partial y' \partial x'}\right)$$

hence this proves laplacian operator is rotation invariant

$$\frac{\partial^f 2}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$$

Question-3

Given:

$$h(x,y) = 3f(x,y) + 2f(x-1,y) + 2f(x+1,y) - 17f(x,y-1) + 99f(x,y+1)$$

Step 1: Check for Additivity

A filter is additive if:

$$h(f_1(x,y) + f_2(x,y)) = h(f_1(x,y)) + h(f_2(x,y))$$

Applying h to $f_1(x,y) + f_2(x,y)$:

$$\begin{aligned}
h(f_1(x,y) + f_2(x,y)) &= 3(f_1(x,y) + f_2(x,y)) + 2(f_1(x-1,y) + f_2(x-1,y)) \\
&\quad + 2(f_1(x+1,y) + f_2(x+1,y)) - 17(f_1(x,y-1) + f_2(x,y-1)) \\
&\quad + 99(f_1(x,y+1) + f_2(x,y+1)) \\
&= (3f_1(x,y) + 2f_1(x-1,y) + 2f_1(x+1,y) - 17f_1(x,y-1) + 99f_1(x,y+1)) \\
&\quad + (3f_2(x,y) + 2f_2(x-1,y) + 2f_2(x+1,y) - 17f_2(x,y-1) + 99f_2(x,y+1)) \\
&= h(f_1(x,y)) + h(f_2(x,y))
\end{aligned}$$

Step 2: Check for Homogeneity

A filter is homogeneous if:

$$h(af(x,y)) = ah(f(x,y))$$

Applying h to $a \cdot f(x,y)$:

$$\begin{aligned}
h(a \cdot f(x,y)) &= 3(a \cdot f(x,y)) + 2(a \cdot f(x-1,y)) + 2(a \cdot f(x+1,y)) \\
&\quad - 17(a \cdot f(x,y-1)) + 99(a \cdot f(x,y+1)) \\
&= a(3f(x,y) + 2f(x-1,y) + 2f(x+1,y) - 17f(x,y-1) + 99f(x,y+1)) \\
&= a \cdot h(f(x,y))
\end{aligned}$$

Since h satisfies both additivity and homogeneity, it is a linear filter.

Convolution Mask

The matrix given by the coordinates with corresponding coefficients is:

$$\begin{bmatrix} 0 & -17 & 0 \\ 2 & 3 & 2 \\ 0 & 99 & 0 \end{bmatrix}$$