Student Information

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Answer 1

We will need the marginal distributions of variables X and Y, and we can compute them by using the Addition Rule to their joint distributions.
$P_X(x) = \sum_y P_{(X,Y)}(x,y)$ and $P_Y(y) = \sum_x P_{(X,Y)}(x,y)$. Therefore:

$P_X(0) = P_{(X,Y)}(0,0) + P_{(X,Y)}(0,2) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12} = 1/4$

$P_X(1) = P_{(X,Y)}(1,0) + P_{(X,Y)}(1,2) = \frac{4}{12} + \frac{2}{12} = \frac{6}{12} = 1/2$

$P_X(2) = P_{(X,Y)}(2,0) + P_{(X,Y)}(2,2) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12} = 1/4$

$P_Y(0) = P_{(X,Y)}(0,0) + P_{(X,Y)}(1,0) + P_{(X,Y)}(2,0) = \frac{1}{12} + \frac{4}{12} + \frac{1}{12} = \frac{6}{12} = 1/2$

$P_Y(2) = P_{(X,Y)}(0,2) + P_{(X,Y)}(1,2) + P_{(X,Y)}(2,2) = \frac{2}{12} + \frac{2}{12} + \frac{2}{12} = \frac{6}{12} = 1/2$

a)

$E(X) = \mu = \sum_x xP(X) = (0)(\frac{1}{4}) + (1)(\frac{1}{2}) + (2)(\frac{1}{4}) = 1$

$Var(X) = E(X - EX)^2 = \sum_x (x-\mu)^2 P(x)$

$= (0-1)^2(\frac{1}{4}) + (1-1)^2(\frac{1}{2}) + (2-1)^2(\frac{1}{4})$

$= (1)(\frac{1}{4}) + (0)(\frac{1}{2}) + (1)(\frac{1}{4}) = 1/2$

b)

$X + Y = Z$ and $P\{X = x \cap Y = y\} = P(x,y)$

$P_Z(0) = P(X + Y = 0) = P(0,0) = 1/12$

$P_Z(1) = P(1,0) = \frac{4}{12} = 1/3$

$P_Z(2) = P(2,0) + P(0,2) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12} = 1/4$

$P_Z(3) = P(1,2) = \frac{2}{12} = 1/6$

$P_Z(4) = P(2,2) = \frac{2}{12} = 1/6$

c)

$Cov(X,Y) = E\{(X - EX)(Y - EY)\} = E(XY) - E(X)E(Y)$

$E(XY) = \sum_x \sum_y xyP(x,y)$

$= (0)(0)P(0,0) + (1)(0)P(1,0) + (2)(0)P(2,0) + (0)(2)P(0,2) + (1)(2)P(1,2) + (2)(2)P(2,2)$

$= 0 + 0 + 0 + 0 + (2)(\frac{2}{12}) + (4)(\frac{2}{12}) = \frac{4+8}{12} = 1$

$E(X) = \sum_x xP_X(x) = (0)P_X(0) + (1)P_X(1) + (2)P_X(2) = (0)(\frac{1}{4}) + (1)(\frac{1}{2}) + (2)(\frac{1}{4}) = 1$

$E(Y) = \sum_y yP_Y(y) = (0)P_Y(0) + (2)P_Y(2) = (0)(\frac{1}{2}) + (2)(\frac{1}{2}) = 1$

$Cov(X,Y) = 1 - (1)(1) = 0$

d)

We know that, if A and B are independent variables, then P(A,B) = P(A).P(B) according to the lemma.

Cov(A,B) = E(A.B) - E(A).E(B)

$E(A.B) = \sum_a \sum_b a.b.P(a,b)$

$E(A).E(B) = \sum_a a.P(a). \sum_b b.P(b)$

If A and B are independent variables then $E(A.B)$ will be equal to $\sum_a \sum_b a.b.P(a,b) = \sum_a a.P(a). \sum_b b.P(b)$. Therefore:

$Cov(A,B) = E(A.B) - E(A).E(B) = \sum_a \sum_b a.b.P(a,b) - \sum_a a.P(a). \sum_b b.P(b)$

$= \sum_a a.P(a). \sum_b b.P(b) - \sum_a a.P(a). \sum_b b.P(b)$

$= 0$

e)

If X and Y are independent variables, then $P(x,y) = P(x).P(y)$ must hold for every value of x and y.

We can observe that $P_{(X,Y)}(0,0) = 1/12 \neq P_X(0).P_Y(0) = \frac{1}{4}.\frac{1}{2} = 1/8$

As we can show a counter-example, we can conclude that variables X and Y aren't independent.

Answer 2

The probability for a pen to be broken is 0.2, and it is our probability of success in this question. Then according to complement rule our probability of failure is $q = 1-p = 1-0.2 = 0.8$ (Probability for a pen to be NOT broken).

a)

This is the Binomial Distribution.

$P(x) = \binom{n}{x}p^x q^{n-x}$ is the probability mass function for this distribution. (x is the number of successes and n is the number of trials)

As we need to find at least 3 pens to be broken we are trying to find $P\{X \geq 3\}$.

$P\{X \geq 3\} = 1 - P\{X \leq 2\}$ according to the complement rule. Also $P\{X \leq 2\} = F(2)$ so,

$P\{X \geq 3\} = 1 - F(2)$

We have parameters n = 12, p = 0.2 and x = 2. From Table A2, from our textbook we can find the value of F(2).

F(2) = 0.558 according to the table. Therefore:

$P\{X \geq 3\} = 1 - 0.558 = 0.442$

b)

This is the Negative Binomial Distribution.

$P(x) = \binom{x-1}{k-1}.p^k.q^{x-k}$ is the probability mass function for this distribution.(x is number of trials and k is the number of successes)

$P(5) = \binom{5-1}{2-1}.p^2.q^{5-2} = \binom{4}{1}.p^2.q^3$

$= \frac{4!}{1!.3!}(0.2)^2(0.8)^3 \approx 0.0819$

c)

This is the Negative Binomial Distribution. To find the average we must compute the expected value.(Expectation)

$E(X) = \frac{k}{p}$ is the expectation for this distribution, which means it is the expected number of trials to obtain the k number of successes.(k is the number of successes and p is the probability of success. In our case p is the probability for a pen to be broken.)

$E(X) = \frac{4}{0.2} = 20$

Answer 3

a)

This is the Exponential Distribution.

$F_T(t) = 1 - e^{-\lambda t}$ is the cumulative distribution function and $E(T) = \frac{1}{\lambda}$ is the expectation for this distribution. (where λ is the frequency parameter)

We are trying to find the possibility that the time required to get the first call is greater than 2 hours. So we try to find $P_T\{T > 2\}$.

$P_T\{T > 2\} = 1 - P_T\{T \leq 2\}$ according to complement rule.

$E(T) = \frac{1}{\lambda} = 4$ hrs, so $\lambda = 0.25\ hrs^{-1}$.

$F_T(2) = P_T\{T \leq 2\} = 1 - e^{-\lambda t} = 1 - e^{-0.5}$

$P_T\{T > 2\} = 1 - 1 + e^{-0.5} = e^{-0.5} \approx 0.607$

b)

This is the Gamma Distribution. In this question we are going to use the relation between the Gamma distribution and the Poisson distribution. We will use a $Gamma(\alpha, \lambda)$ variable T and a $Poisson(\lambda t)$ variable X. We have already found that $\lambda = 0.25$ and as we are dealing with the 3rd rare event our $\alpha = 3$. We try to find the value of $P_T\{T \geq 10\}$.

$P_T\{T \geq 10\} = P_X\{X \leq 3\}$

$P_X\{X \leq 3\} = F_X(3)$, we can look at the Table A3 from our textbook for the Poisson Distribution. (x = 3, t = 10 and $\lambda = 0.25$, so $\lambda t = 2.5$. We use this value when we look at the table.)

$F_X(3) = P_T\{T \geq 10\} = 0.758$

c)

This is the Gamma Distribution. We will deal with $Gamma(\alpha, \lambda)$ variable T where $\alpha = 3$ and $\lambda = 0.25$. We are trying to find $P_T\{T \geq 16 \mid T \geq 10\}$.

$P_T\{T \geq 16 \mid T \geq 10\} = \frac{P_T\{T \geq 16 \cap T \geq 10\}}{P_T\{T \geq 10\}} = \frac{P_T\{T \geq 16\}}{P_T\{T \geq 10\}}$

$P_T\{T \geq 16\} = P_X\{X \leq 3\} = F_X(3)$

We can compute the $F_X(3)$ by looking at the Table A3 for the Poisson Distribution, where parameters $x = 3$, $t = 16$ and $\lambda = 0.25$, so $\lambda t = 4$. We will use this value when we look at the table.

$F_X(3) = P_T\{T \geq 16\} = 0.433$. We have already computed the value of $P_T\{T \geq 10\} = 0.758$ in 3B.

$P_T\{T \geq 16 \mid T \geq 10\} = \frac{P_T\{T \geq 16\}}{P_T\{T \geq 10\}} = \frac{0.433}{0.758} \approx 0.571$