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Higher order accuracy finite-difference schemes for hyperbolo-parabolic equations

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In this work, compact difference schemes of 4+2 and 4+4 approximation orders are considered and studied for the hyperbolic-parabolic equations, including cases with constant coefficients and quasilinear equation. By *compact* schemes, we mean difference schemes of higher order of approximation written on stencils standard for a given equation [1, 2].

This study uses the notation from [3, 4].

In the domain $\overline{Q}_T = \{(x,t):\ 0 \le x \le l,\ 0 \le t \le T\}$, $\overline{Q}_T = \overline{\Omega} \times [0,T]$, $\overline{\Omega} = \Omega \cup \Gamma$, $\Omega = \{0 < x < l\}$, it is required to find continuous function $u(x,t)$, satisfying following initial boundary value problem

$$\frac{\partial}{\partial t}\left(\rho_1 \frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}(\rho_2 u) = Lu + f(x,t), \quad x \in \Omega, \quad t \in (0,T], \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \quad \frac{\partial u}{\partial t}(x,0) = \overline{u}_0(x), \quad x \in \Omega, \tag{2}$$

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad t \in (0,T]. \tag{3}$$

Here $Lu = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)$, $\quad \rho_m \ge 0, \quad m = 1,2, \quad k > 0.$

In the case of constant coefficients ρ_1, ρ_2, k, we approximate the differential problem (1)-(3) by the weighted finite difference scheme of approximation order 4+2:

$$\rho_1 y_{\overline{t}t} + \rho_1 \frac{h^2}{12} y_{\overline{t}t\overline{x}x} + \rho_2 y_{\overset{\circ}{t}} + \rho_2 \frac{h^2}{12} y_{\overset{\circ}{t}\overline{x}x} = k y^{(\sigma,\sigma)}_{\overline{x}x} + \varphi, \quad (x,t) \in \omega_h \times \omega_\tau, \tag{4}$$

$$y(x,0) = u_0(x), \quad x \in \overline{\omega}_h, \quad y_t(x,0) = u_1(x), \quad x \in \omega_h, \tag{5}$$

$$y(0,t) = \mu_1(t), \quad y(l,t) = \mu_2(t), \quad t \in \omega_\tau, \tag{6}$$

where

$$v^{(\sigma,\sigma)} = v + \sigma\tau^2 v_{\overline{t}t}, \quad 0 \le \sigma \le 1, \quad \varphi = f + \frac{h^2}{12} f_{\overline{x}x},$$

$$u_1(x) = \overline{u}_0 + \frac{\tau}{2\rho_1}\left[k\frac{\partial^2 u}{\partial x^2}(x,0) - \rho_2 \overline{u}_0 + f(x,0)\right].$$

Using the method of energy inequalities we obtain a priori estimates for the stability of the difference solution.

Theorem 1. *Let the condition*

$$\sigma \ge \frac{1}{2} + \rho_1 \frac{h^2}{12\tau^2}, \quad \tau \ge \sqrt{\frac{\rho_1}{6}} h$$

be satisfied. Then the difference scheme (4)-(6) is ρ-stable with respect to the initial data and the right-hand side, and its solution satisfies the a priori estimate

$$Q^{n+1} \leq Q^1 + \frac{\tau}{\rho_2}\sum_{k=1}^{n} ||\varphi^k||^2.$$

Remark 1. The following difference scheme approximate the differential equation (1) with approximation order 4+4:

$$\left(\rho_1 + \frac{\rho_2^2\tau^2}{12\rho_1}\right) y_{\bar{t}t} + \rho_1\frac{h^2}{12}y_{\bar{t}t\bar{x}x} + \rho_2 y_{\overset{\circ}{t}} + \left(\rho_2\frac{h^2}{12} - k\frac{\rho_2\tau^2}{12\rho_1}\right) y_{\overset{\circ}{t}\bar{x}x} = k y_{\bar{x}x}^{(1/12,1/12)} + \varphi.$$

Here

$$\varphi = f + \frac{h^2}{12}f_{\bar{x}x} + \frac{\tau^2}{12}\left(f_{\bar{t}t} + \frac{\rho_2}{\rho_1}f_{\overset{\circ}{t}}\right).$$

Remark 2. For the quasilinear hyperbolic-parabolic equation

$$\rho_1\frac{\partial^2 u}{\partial t^2} + \rho_2\frac{\partial u}{\partial t} = L\phi(u) + f(x,t),$$

with the conditions $\phi_u' = k(u) \geq k_1 > 0, \quad \rho_m = const \geq 0, \quad m = 1,2$, we write the weighted compact difference schemes, of 4+2 and 4+4 order of accuracy respectively:

$$\rho_1 y_{\bar{t}t} + \rho_1\frac{h^2}{12}y_{\bar{t}t\bar{x}x} + \rho_2 y_{\overset{\circ}{t}} + \rho_2\frac{h^2}{12}y_{\overset{\circ}{t}\bar{x}x} = (\phi(y))_{\bar{x}x}^{(\sigma,\sigma)} + f + \frac{h^2}{12}f_{\bar{x}x}$$

and

$$\left(\rho_1 + \frac{\rho_2^2\tau^2}{12\rho_1}\right) y_{\bar{t}t} + \rho_1\frac{h^2}{12}y_{\bar{t}t\bar{x}x} + \rho_2 y_{\overset{\circ}{t}} + \rho_2\frac{h^2}{12}y_{\overset{\circ}{t}\bar{x}x} = (\phi(y))_{\bar{x}x}^{(1/12,1/12)} + \frac{\rho_2\tau^2}{12\rho_1}(\phi(y))_{\overset{\circ}{t}\bar{x}x} + \varphi,$$

$$\varphi = f + \frac{h^2}{12}f_{\bar{x}x} + \frac{\tau^2}{12}\left(f_{\bar{t}t} + \frac{\rho_2}{\rho_1}f_{\overset{\circ}{t}}\right).$$

Keywords compact difference schemes, hyperbolo-parabolic equation, quasilinear equation, priori estimates, stability, higher order accuracy finite-difference scheme.

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