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On a problem for a time fractional differential equation on ladder-type metric graph

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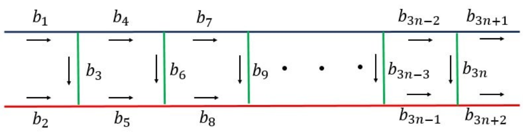
We consider ladder-type graph $\mathbf{\Gamma}$ which obtained by connecting equal finite bonds (see Figure.1). We correspond the bonds b_k, $k = 1, 2, ..., 3n+2$ and $n \in N$. Let us define coordinate x_k on the bond B_k, and $x_k \in (0, L)$. Further we will use x instead of x_k.



$Figure.1$

On the each edges of the over defined graph, we consider fractional differential equations:

$$_CD^{\alpha}_{0t}u^{(k)}(x,t) = au^{(k)}_{xx}(x,t) + bu^{(k)}(x,t) + f^{(k)}(x,t), (x,t) \in B_k \times (0,T) \qquad (1)$$

where a, b are positive constants, $_CD^{\alpha}_{0t}(\cdot)$, $0 < \alpha < 1$ is Caputo time fractional derivative operator [1], $u^{(k)}(\cdot) = \left(u^{(1)}(\cdot), u^{(2)}(\cdot), ..., u^{(3n+2)}(\cdot)\right)^T$, $f^{(k)}(\cdot) = (f^{(1)}(\cdot), f^{(k)}(\cdot), ..., f^{(3n+2)}(\cdot))^T$ and $f^{(k)}(x,t)$ are known functions .

We will study the following problem for equation (1) on $\mathbf{\Gamma}$.

Problem. To find functions $u_k(x,t)$ in the domain $b_k \times (0,T)$, satisfy equation (1), with the following:
initial conditions:

$$u_k(x,t)\Big|_{t=0} = \varphi_k(x), \quad x \in \overline{B_k}, \quad k = \overline{1, 3n+2}, \qquad (2)$$

boundary conditions

$$u_k(x,t)|_{x=0} = 0,\ k = \{1,2\}, \quad u_k(x,t)|_{x=L} = 0,\ k = \{3n+1, 3n+2\}, \quad t \in [0,T] \qquad (3)$$

and vertex (or Kirchhoff) flux conditions

$$\begin{gathered} u_k(x,t)|_{x=L} = u_{k+2}(x,t)|_{x=0} = u_{k+3}(x,t)|_{x=0}, \\ -\frac{\partial}{\partial x}u_k(x,t)|_{x=L} + \frac{\partial}{\partial x}u_{k+2}(x,t)|_{x=0} + \frac{\partial}{\partial x}u_{k+3}(x,t)|_{x=0} = 0, k = \overline{1, 3n-2}, \end{gathered} \qquad (4)$$

$$\begin{gathered} u_k(x,t)|_{x=L} = u_{k+1}(x,t)|_{x=L} = u_{k+3}(x,t)|_{x=0}, \\ -\frac{\partial}{\partial x}u_k(x,t)|_{x=L} - \frac{\partial}{\partial x}u_{k+1}(x,t)|_{x=L} + \frac{\partial}{\partial x}u_{k+3}(x,t)|_{x=0} = 0, k = \overline{2, 3n-1}, \end{gathered} \qquad (5)$$

where $\varphi_k(x)$, $\quad k = \overline{1, 3n+2}$ are given functions, such that

$$\varphi_k(0) = 0,\ k = \{1,2\} \quad \varphi_k(L) = 0,\ k = \{3n+1, 3n+2\}, \quad t \in [0,T]$$

$$\begin{gathered}\varphi_k(L)=\varphi_{k+2}(0)=\varphi_{k+3}(0),\\ -\varphi'_k(L)+\varphi'_{k+2}(0)+\varphi'_{k+3}(0)=0,\quad k=\overline{1,3n-2},\end{gathered}$$

$$\begin{gathered}\varphi_k(L)=\varphi_{k+2}(L)=\varphi_{k+3}(0),\\ -\varphi'_k(L)-\varphi'_{k+2}(L)+\varphi'_{k+3}(0)=0,\quad k=\overline{1,3n-2}.\end{gathered}$$

We consider homogeneous equation of the equation (1). We can separation variables and we will following spectral eigenproblem on the graph $\mathbf{\Gamma}$.

$$X''_k(x)+\lambda^2X_k(x)=0,\quad k=\overline{1,3n+2}, \tag{6}$$

from the conditions (3)-(5), we receive

$$X_k(0)=0,\ k=\{1,2\}\quad X_k(L)=0,\ k=\{3n+1,3n+2\},\quad t\in[0,T] \tag{7}$$

$$\begin{gathered}X_k(L)=X_{k+2}(0)=X_{k+3}(0),\\ -X'_k(L)+X'_{k+2}(0)+X'_{k+3}(0)=0,\quad k=\overline{1,3n-2},\end{gathered} \tag{8}$$

$$\begin{gathered}X_k(L)=X_{k+2}(L)=X_{k+3}(0),\\ -X'_k(L)-X'_{k+2}(L)+X'_{k+3}(0)=0,\quad k=\overline{1,3n-2}.\end{gathered} \tag{9}$$

Similar to (6)-(9) spectral problem was considered in [2].
A solution in terms of (6) on each branch b_k of length L is

$$X_k(x)=c_k\cos\lambda x+d_k\sin\lambda x;\ x\in B_k$$

By using properties of the Mittag-Lefler function, we prove the uniform convergence of the obtained Fourier series. The uniqueness of the solution of the problem is proved using by A-prior estimation (see[3]).

Keywords: *Metric graph, Method of separation of variables, a priori estimates*

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