Module-02, Basic Mathematics and Statistics

Mathematics (calculus)

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Data science and Artificial Intelligence
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Matrix Calculus

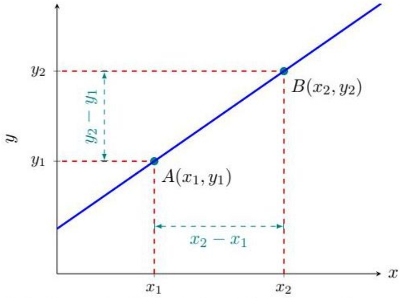
The topic of calculus concerns computing "rate of change" of function as we very their inputs and get different results. It is of vital importance to machine learning. In some case, we use some concepts and notation from matrix algebra.
Before apply into the machine learning model, we need to known about basic and its working.

- Differentiation of vector.
- Differentiation of matrix.

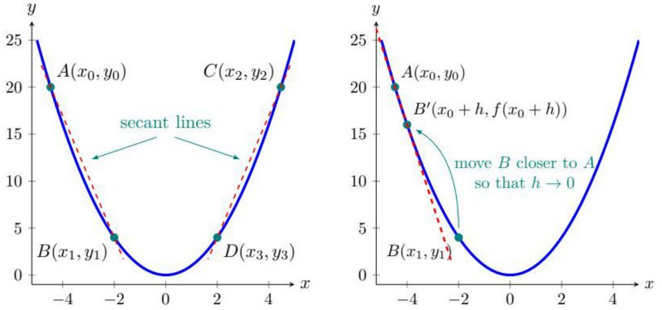
The slope of a line

The slope of a line, and its relationship to the tangent line of a curve is a fundamental concept in calculus.

- The slope of line
  In the calculus the slope of line define, its steepness as a number. This number is calculated by dividing the change in the vertical direction to the change in the horizontal direction when moving from one point on line to another



- We can extend the idea of the slope of a line to the slope of a curve. Consider the left graph of the figure below. If we want to measure the "steepness" of this curve. The average rate of change when moving from point A to pint B is negative as the value of the function is decreasing when x is increasing.



Slope of the curve

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- The average rate of change over the interval $[x_1, x_0 + h]$ represents the rate of change over a very small interval of length h, where h, approaches zero. This is called the slope of the curve at the point x_0. Hence, at any point A$(x_0, f(x_0))$, the slope of the curve is define as:

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$

- The expression of the slope of the curve at a point A is equivalent to the derivative of f(x) at the point x_0. Hence, we can use the derivative to find the slope of the curve.

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The tangent line

- We will define the tangent to curve $f(x)$ at a point $A(x_0, f(x_0))$ as a line satisfies two of the following:
  1. The line passes through A.
  2. The slope of the line is equal to the slope of the curve at the point A.
- Using the above two facts, we can easily determine the equation of the tangent line at a point $(x_0, f(x_0))$.

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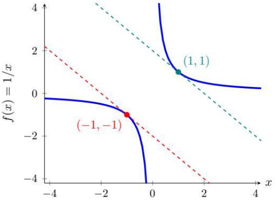
Examples of tangent lines

- The graph of $f(x)$ along with the tangent line at $x = 1$ and $x = -1$ are shown in the figure.

$$f(x) = \frac{1}{x} \tag{2}$$



- Slope of the line at any point is given by the function $f'(x) = -\frac{1}{x^2}$
- Slope of the tangent line to the curve at x =1 is -1, we get y = -x + c
- The tangent line passes through the point (1,1) and we get = -(1) + c $\rightarrow$ c = 2
- The final equation of the tangent line is y = -x + 2

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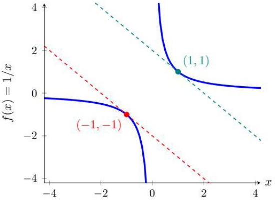

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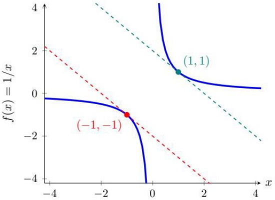

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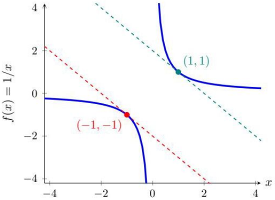

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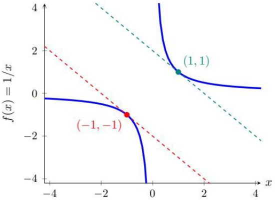

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- Shown below is the curve and the tangent lines at the points x = 2, x = -2, x = 0. At x = 0, the tangent line is parallel to the x-axis as the slope of $f(x)$ at x = 0 is zero.

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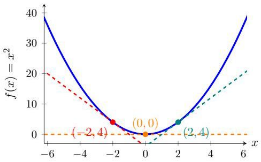

- Slope of the line at any point is given by the function $f'(x) = 2x$
- Slope of the tangent line to the curve at x = 2 is 4,we get y = 4x + c, 4 = 4 ×(2) + c → c = -4
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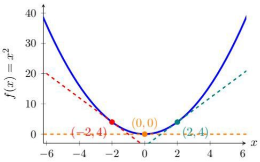

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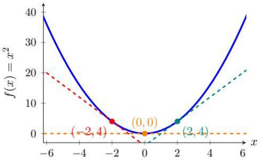

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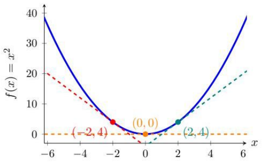

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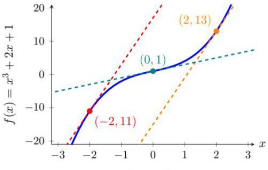
Examples of tangent lines

- This function is shown below, along with its tangent at x = 0, x = 2 and x = -2. Below are the steps to derive an equation of the tangent line at x = 0.

$$f(x) = x^2 + 2x + 1 \tag{4}$$



- Slope of the line at any point is given by the function $f'(x) = 3x^2 + 2$
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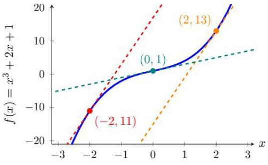

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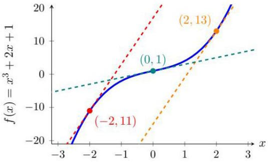

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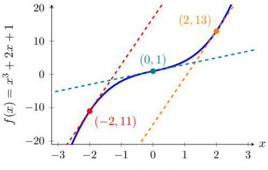

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Vector Differentiation

The simplest form of multi-variable differentiation, vector differentiation generalizes the one-dimensional concept of a derivative to functions with vector-valued inputs or outputs.
We develop the concept of the gradient by generalizing the limit definition of the (single-variable) derivative, which is,

$$\lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t} = f'(x) \tag{5}$$

to functions where the input is a vector.
In the multi variable case, what t $\to$ 0 means is less clear, as there are many directions in which one could approach a point in R^2.

Gradient

This gets the name "gradient" as it represents the set of slopes around a point as one moves one unit in each dimension parallel to the n axes. Let's formally define the gradient.

Definition

The **gradient** vector represents the derivative for a function $f : \mathbb{R}^n \to \mathbb{R}$. If the function is differentiable, the gradient is equal to the $1 \times n$ vector where the ith entry is $\left[\frac{\partial f}{\partial \vec{x}}\right]_i = \frac{\partial f}{\partial x_i}$.

Example

Consider the function $f(\vec{x}) = (x_1^2 + x_2^2)$. Then $\frac{\partial f}{\partial x_1} = 2x_1, \frac{\partial f}{\partial x_2} = 2x_2$, and so $\frac{df}{d\vec{x}} = \nabla f(\vec{x}) = (2x_1, 2x_2)$.

Matrix Differentiation

We introduce matrix differentiation concepts and techniques up to a level commonly used in statistics and data science.

The main constraint for the set of functions that we will differentiable is that the sum of dimensions across the input and output must be at most two. (Why?)

Some examples of possible functions are:

$$
\begin{aligned}
f : \mathbb{R} \to \mathbb{R}^n & \text{ e.g. parameter of symmetric Dirichlet} \\
f : \mathbb{R}^n \to \mathbb{R} & \text{ e.g. vector norm} \\
f : \mathbb{R}^{n \times n} \to \mathbb{R} & \text{ e.g. determinant} \\
f : \mathbb{R}^n \to \mathbb{R}^n & \text{ e.g. linear transformations}
\end{aligned}
$$

Jacobian

Definition

The **Jacobian** matrix represents the derivative for a function $f : \mathbb{R}^n \to \mathbb{R}^m$. It is defined as the $m \times n$ matrix where the term at the ith row and jth column is $\left[\frac{\partial \vec{f}}{\partial \vec{x}}\right]_{\text{ij}} = \frac{\partial f_i}{\partial x_j}$.

Example

Consider the function $\vec{f}(\vec{x}) = (x_1^2 + x_2^2, x_2^3)$. Then,

$$\frac{d\vec{f}}{d\vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 \\ 0 & 3x_2^2 \end{bmatrix}$$

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Great Job
Thank you