Mathematics Behind Linear Regression Model

Supervised Algorithm

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Linear Regression

Definition

Linear regression is a fundamental statistical method used to model the relationship between a dependent variable and one or more independent variables. It is widely used in predictive analysis and data modeling.

- Simple linear regression involves a single independent variable. The relationship is modeled with a straight line.

$$y = w_0 + w_1 x \tag{1}$$

- Where,
  y is the dependent variable.
  x is the independent variable.
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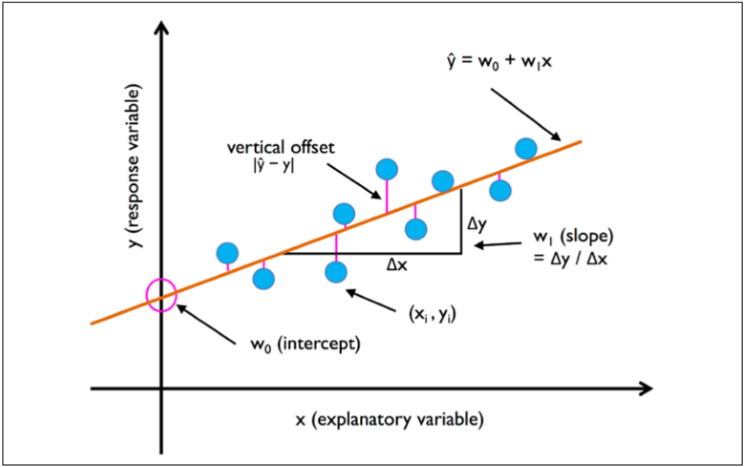
Visual's of Linear regression Model

- Visual's representation of Simple Linear regression Model



Linear Regression Function

- The linear function is denote for i^{th} data points are:

$$\hat{y}_i = w_1 x_i + w_0 \tag{2}$$

- Now our objective, to find a approximated function that best fit the data points over all.
- Error Equations

$$r_i = y_i - \hat{y}_i \tag{3}$$

- The overall error can be calculated in different ways.
- Simply adding of Residual method
- Absolute error method
- Linear least square method

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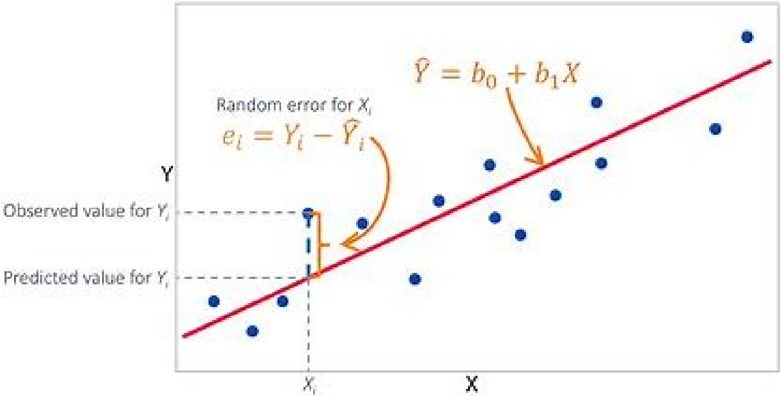
- We have two values actual and approximated value, as shown below:



Visual of error function

Calculating Error Methods: Addition of residuals

One simple way is to add the residuals of all the points

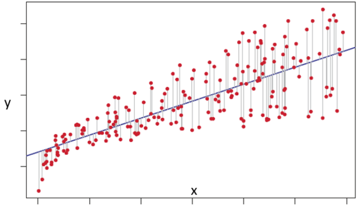
$$E = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} [y_i - (w_1 x_i + w_0)] \tag{4}$$

Residual

Residual –

The difference between residual the *i*th observed response value and the *i*th response value that is predicted by our linear model is known as residual

$$e_i = y_i - \hat{y}_i$$



This is not a good choice to sum all : both +ve and -ve residuals both can be large zero or very close to zero.

Absolute, Linear least square method

- The next possibility: Absolute $||.||$

$$E = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} |y_i - (w_1 x_i + w_0)| \tag{5}$$

Now positive: (total error) but it cannot be used for determining the constants of the function that give the best fit. This is because the measure is not unique.

- Now we apply square on the all over error E is:

$$E = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} [y_i - (w_1 x_i + w_0)]^2 \tag{6}$$

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Calculate w_1 and w_0

- $$w_1 = \frac{nS_{xy} - S_x S_y}{nS_{xx} - (S_x)^2} \tag{7}$$
  $$w_0 = \frac{S_{xx} S_y - S_{xy} S_x}{nS_{xx} - (S_x)^2} \tag{8}$$
- We have to calculate w_0 and w_1 using below formulas Where:

  w_1 : Coefficients for the respective independent variables
  w_0 : Intercept
  S_x : Sum of x
  S_y : Sum of y
  S_{xy} : Sum of x and y product
  S_{xx} : Sum of x product itself

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Example for Linear Regression

Example

We want to predict the atmospheric pressure on given temperature and in between temperature the dataset are given below,

X_i	T(c^0)	0^0	30^0	70^0	100^0
y_i	P(atm)	0.94	1.05	1.17	1.28

$$S_x = \sum_{i=1}^{4} x_i = 0 + 30 + 70 + 100 = 200 \tag{9}$$

$$S_y = \sum_{i=1}^{4} y_i = 0.94 + 1.05 + 1.17 + 1.28 = 4.44 \tag{10}$$

Example For Linear Regression

- Similarly we have,

$$S_{xx} = \sum_{i=1}^{4} x_i^2 = 15800 \tag{11}$$

$$S_{xy} = \sum_{i=1}^{4} (x_i y_i) = 241.4 \tag{12}$$

- Putting these values in parameters equations we get,

$$w_1 = \frac{4(241.4) - 200 \times 4.44}{4(15800) - (200)^2} = 0.003345 \tag{13}$$

$$w_0 = 0.9428 \tag{14}$$

- The Linear Equation is

$$\hat{y}_i = 0.003345 x_i + 0.9428. \tag{15}$$

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Example for Linear Regression

Data Overview

x_i	1	2	3	4	5
y_i	2	3	4	5	6

Calculating Means

$$\bar{x} = \frac{1+2+3+4+5}{5} = 3$$
$$\bar{y} = \frac{2+3+5+4+6}{5} = 4$$

Calculating Slope (w_1)

$$w_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$$

Calculation

$$
\begin{aligned}
&= \frac{(1-3)(2-4)+(2-3)(3-4)+(0)(5-4)+(4-3)(0)+(5-3)(2)}{(1-3)^2+(2-3)^2+(3-3)^2+(4-3)^2+(5-3)^2} \\
&= \frac{(-2)(-2)+(-1)(-1)+(0)(1)+(1)(0)+(2)(2)}{(-2)^2+(-1)^2+(0)^2+(1)^2+(2)^2} \\
&= \frac{4+1+0+0+4}{4+1+0+1+4} \\
&= \frac{9}{10} = 0.9
\end{aligned}
$$

Example for Linear Regression

Calculating Intercept (w_0)

$$\begin{aligned} w_0 &= \bar{y} - w_1\bar{x} \\ &= 4 - 0.9 \times 3 \\ &= 4 - 2.7 \\ &= 1.3 \end{aligned}$$

Regression Equation

$$\hat{y} = 1.3 + 0.9x \tag{16}$$

Verification

$$\begin{aligned} \hat{y}_1 &= 1.3 + 0.9 \times 1 = 2.2 \\ \hat{y}_2 &= 1.3 + 0.9 \times 2 = 3.1 \\ \hat{y}_3 &= 1.3 + 0.9 \times 3 = 4.0 \\ \hat{y}_4 &= 1.3 + 0.9 \times 4 = 4.9 \\ \hat{y}_5 &= 1.3 + 0.9 \times 5 = 5.8 \end{aligned}$$

Example for Linear Regression

Sum of Squared Residuals (SSR)

$$
\begin{aligned}
e_1 &= 2 - 2.2 = -0.2 \\
e_2 &= 3 - 3.1 = -0.1 \\
e_3 &= 5 - 4.0 = 1.0 \\
e_4 &= 4 - 4.9 = -0.9 \\
e_5 &= 6 - 5.8 = 0.2 \\
SSR &= (-0.2)^2 + (-0.1)^2 + (1.0)^2 + (-0.9)^2 + (0.2)^2 \\
&= 0.04 + 0.01 + 1.00 + 0.81 + 0.04 \\
&= 1.90
\end{aligned}
$$

This show model fit

Model Evaluation

1. R-squared ($\mathbf{R}^2$)

The Total Sum of Squares (SST) is:

$$\text{SST} = \sum_{i=1}^{n} (y_i - \bar{y})^2 = 10$$

$$R^2 = 1 - \frac{\text{SSR}}{\text{SST}} = 1 - \frac{1.90}{10} = 0.81$$

2. Mean Squared Error (MSE)

$$\text{MSE} = \frac{\text{SSR}}{n} = \frac{1.90}{5} = 0.38$$

3. Root Mean Squared Error (RMSE)

$$\text{RMSE} = \sqrt{\text{MSE}} = \sqrt{0.38} \approx 0.62$$

Conclusion

The linear regression model fits the data well:

- $R^2 = 0.81$ indicates that 81% of the variance in y is explained by x.
- MSE $= 0.38$ and RMSE $= 0.62$ indicate the average magnitude of prediction errors.

Questions and Answers

Search for Resource of Workshop