Worked Examples from Introductory Physics
Vol. IV: Electric Fields

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Contents

To the Student. Yeah, *You.*

Hi. It's me again. Since you have obviously read all the stuffy pronouncements about the purpose of this problem–solving guide in Volume 1 (the Green Book, in its print version), I won't make them again here.

I will point out that I've got *lots* more work to do on Volume 4, and I'm just making it available so that these chapters (such as they are) may be of some help to you. In fact, the whole set of books is a perpetual work in progress.

However....

Reactions, please!

Please help me with this project: Give me your reaction to this work: Tell me what you liked, what was particularly effective, what was particularly confusing, what you'd like to see more of or less of. I can be reached at `murdock@tntech.edu` or even at x–3044. If this effort is helping you to learn physics, I'll do more of it!

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Chapter 1

Electric Charge; Coulomb's Law

1.1 The Important Stuff

1.1.1 Introduction

During the second semester of your introductory year of physics you will study two special types of forces which occur in nature as a result of the fact that the constituents of matter have electric charge; these forces are the **electric force** and the **magnetic force**. In fact, the study of electromagnetism adds something completely new to the ideas of the mechanics from first semester physics, namely the concept of the electric and magnetic *fields*. These entities are just as real as the masses and forces from first semester and they take center stage when we discuss the phenomenon of electromagnetic radiation, a topic which includes the behavior of visible light.

The entire picture of matter and fields which we will have at the end of this study is known as **classical physics**, but this picture, while complete enough for many fields of engineering, is not a complete statement of the laws of nature (as we now know them). New phenomena which were discovered in the early 20th century demanded revisions in our thinking about the relation of space and time (relativity) and about phenomena on the atomic scale (quantum physics). **Relativity** and **quantum theory** are often known collectively as **modern physics**.

1.1.2 Electric Charge

The phenomenon we recognize as "static electricity" has been known since ancient times. It was later found that there is a physical quantity known as **electric charge** that can be transferred from one object to another. Charged objects can exert forces on other charged objects and also on uncharged objects. Finally, electric charge comes in two types, which we choose to call **positive charge** and **negative charge**.

Substances can be classified in terms of the ease with which charge can move about on their surfaces. **Conductors** are materials in which charges can move about freely; **insulators** are materials in which electric charge is not easily transported.

Electric charge can be measured using the law for the forces between charges (Coulomb's Law). Charge is a *scalar* and is measured in **coulombs** [1]. The coulomb is actually *defined* in terms of electric current (the flow of electrons), which is measured in **amperes**[2]; when the current in a wire is 1 ampere, the amount of charge that flows past a given point in the wire in 1 second is 1 coulomb. Thus,

$$1\,\text{ampere} = 1\,\text{A} = 1\,\tfrac{\text{C}}{\text{s}}\ .$$

As we now know, when charges are transferred by simple interactions (i.e. rubbing), it is a *negative* charge which is transferred, and this charge is in the form of the fundamental particles called **electrons**. The charge of an electron is $1.6022 \times 10^{-19}\,\text{C}$, or, using the definition

$$e = 1.602177 \times 10^{-19}\,\text{C} \tag{1.1}$$

the electron's charge is $-e$. The proton has charge $+e$. The particles found in nature all have charges which are integral multiples of the **elementary charge** e: $q = ne$ where $n = 0, \pm 1, \pm 2 \ldots$. Because of this, we say that charge is **quantized**.

The mass of the electron is

$$m_{\text{e}} = 9.1094 \times 10^{-31}\,\text{kg} \tag{1.2}$$

1.1.3 Coulomb's Law

Coulomb's Law gives the force of attraction or repulsion between two point charges. If two point charges q_1 and q_2 are separated by a distance r then the magnitude of the force of repulsion or attraction between them is

$$F = k\frac{|q_1|\,|q_2|}{r^2} \qquad \text{where} \qquad k = 8.9876 \times 10^9\,\tfrac{\text{N}\cdot\text{m}^2}{\text{C}^2} \tag{1.3}$$

This is the magnitude of the force which *each charge exerts on the other charge* (recall Newton's 3$^{\text{rd}}$ law). The symbol k as used here has to do with electrical forces; it has nothing to do with any spring constants or Boltzmann's constant!

If the charges q_1 and q_2 are of the same sign (both positive or both negative) then the force is mutually *repulsive* and the force on each charge points *away from* the other charge. If the charges are of opposite signs (one positive, one negative) then the force is mutually *attractive* and the force on each charge points *toward* the other one. This is illustrated in Fig. 1.1.

The constant k in Eq. 1.3 is often written as

$$k = \frac{1}{4\pi\epsilon_0} \qquad \text{where} \qquad \epsilon_0 = 8.85419 \times 10^{-12}\,\tfrac{\text{C}^2}{\text{N}\cdot\text{m}^2} \tag{1.4}$$

[1] Named in honor of the... uh... Dutch physicist Jim Coulomb (1766–1812) who did some electrical experiments in... um... Paris. That's it, Paris.

[2] Named in honor of the... uh... German physicist Jim Ampere (1802–1807) who did some electrical experiments in... um... Düsseldorf. That's it, Düsseldorf.

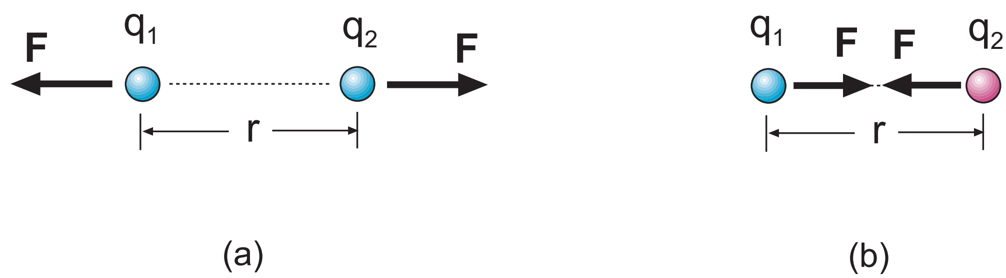

Figure 1.1: (a) Charges q_1 and q_2 have the same sign; electric force is repulsive. (b) Charges q_1 and q_2 have opposite signs; electric force is attractive.

for historical reasons but also because in later applications the constant ϵ_0 is more convenient. ϵ_0 is called the **permittivity constant** [3]

When several points charges are present, the total force on a particular charge q_0 is the *vector sum* of the individual forces gotten from Coulomb's law. (Thus, electric forces have a **superposition** property.) For a continuous distribution of charge we need to divide up the charge distribution into infinitesimal pieces and add up the individual forces with integrals to get the net force.

1.2 Worked Examples

1.2.1 Electric Charge

1. What is the total charge of $75.0\,\mathrm{kg}$ of electrons?

The mass of *one* electron is 9.11×10^{-31} kg, so that a mass $M = 75.0\,\mathrm{kg}$ contains

$$N = \frac{M}{m_{\rm e}} = \frac{(75.0\,\mathrm{kg})}{(9.11 \times 10^{-31}\,\mathrm{kg})} = 8.23 \times 10^{31} \quad \text{electrons}$$

The charge of *one* electron is $-e = -1.60 \times 10^{-19}\,\mathrm{C}$, so that the total charge of N electrons is:

$$Q = N(-e) = (8.23 \times 10^{31})(-1.60 \times 10^{-19}\,\mathrm{C}) = -1.32 \times 10^{13}\,\mathrm{C}$$

2. (a) How many electrons would have to be removed from a penny to leave it with a charge of $+1.0 \times 10^{-7}\,\mathrm{C}$? (b) To what fraction of the electrons in the penny does this correspond? [A penny has a mass of $3.11\,\mathrm{g}$; assume it is made entirely of copper.]

[3] In these notes, k will be used mainly in the first chapter; thereafter, we will make increasing use of ϵ_0!

(a) From Eq. 1.1 we know that as *each* electron is removed the penny picks up a charge of $+1.60 \times 10^{-19}\,\mathrm{C}$. So to be left with the given charge we need to remove N electrons, where N is:

$$N = \frac{q_{\mathrm{Total}}}{q_{\mathrm{e}}} = \frac{(1.0 \times 10^{-7}\,\mathrm{C})}{(1.60 \times 10^{-19}\,\mathrm{C})} = 6.2 \times 10^{11} \ .$$

(b) To answer this part, we will need the total number of electrons in a neutral penny; to find this, we need to find the number of copper atoms in the penny and use the fact that each (neutral) atom contains 29 electrons. To get the *moles* of copper atoms in the penny, divide its mass by the atomic weight of copper:

$$n_{\mathrm{Cu}} = \frac{(3.11\,\mathrm{g})}{(63.54\,\frac{\mathrm{g}}{\mathrm{mol}})} = 4.89 \times 10^{-2}\,\mathrm{mol}$$

The number of copper atoms is

$$N_{\mathrm{Cu}} = n_{\mathrm{Cu}} N_{\mathrm{A}} = (4.89 \times 10^{-2}\,\mathrm{mol})(6.022 \times 10^{23}\,\mathrm{mol}^{-1}) = 2.95 \times 10^{22}$$

and the number of electrons in the penny was (originally) 29 times this number,

$$N_{\mathrm{e}} = 29 N_{\mathrm{Cu}} = 29(2.95 \times 10^{22}) = 8.55 \times 10^{23}$$

so the *fraction* of electrons removed in giving the penny the given electric charge is

$$f = \frac{(6.2 \times 10^{11})}{(8.55 \times 10^{23})} = 7.3 \times 10^{-13}$$

A very small fraction!!

1.2.2 Coulomb's Law

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3. A point charge of $+3.00 \times 10^{-6}\,\mathrm{C}$ is $12.0\,\mathrm{cm}$ distant from a second point charge of $-1.50 \times 10^{-6}\,\mathrm{C}$. Calculate the magnitude of the force on each charge.

Being of opposite signs, the two charges *attract* one another, and the magnitude of this force is given by Coulomb's law (Eq. 1.3),

$$\begin{aligned} F &= k\frac{|q_1 q_2|}{r^2} \\ &= (8.99 \times 10^{9}\,\tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}})\frac{(3.00 \times 10^{-6}\,\mathrm{C})(1.50 \times 10^{-6}\,\mathrm{C})}{(12.0 \times 10^{-2}\,\mathrm{m})^2} = 2.81\,\mathrm{N} \end{aligned}$$

Each charge experiences a force of attraction of magnitude 2.81 N.

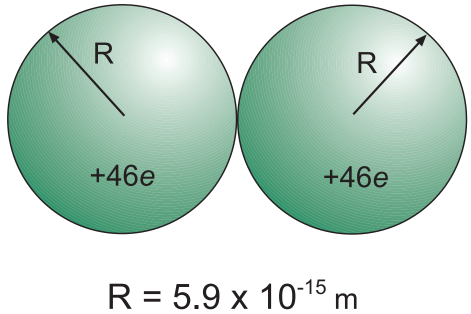

Figure 1.2: Simple picture of a nucleus just after fission. Uniformly charged spheres are "touching".

4. What must be the distance between point charge $q_1 = 26.0\,\mu\mathrm{C}$ and point charge $q_2 = -47.0\,\mu\mathrm{C}$ for the electrostatic force between them to have a magnitude of $5.70\,\mathrm{N}$?

We are given the charges and the magnitude of the (attractive) force between them. We can use Coulomb's law to solve for r, the distance between the charges:

$$F = k\frac{|q_1 q_2|}{r^2} \qquad \Longrightarrow \qquad r^2 = k\frac{|q_1 q_2|}{F}$$

Plug in the given values:

$$r^2 = (8.99 \times 10^9\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}})\frac{(26.0 \times 10^{-6}\,\mathrm{C})(47.0 \times 10^{-6}\,\mathrm{C})}{(5.70\,\mathrm{N})} = 1.93\,\mathrm{m}^2$$

This gives:

$$r = \sqrt{1.93\,\mathrm{m}^2} = 1.39\,\mathrm{m}$$

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5. In fission, a nucleus of uranium–238, which contains 92 protons, divides into two smaller spheres, each having 46 protons and a radius of $5.9 \times 10^{-15}\,\mathrm{m}$. What is the magnitude of the repulsive electric force pushing the two spheres apart?

The basic picture of the nucleus after fission described in this problem is as shown in Fig. 1.2. (Assume that the edges of the spheres are in contact just after the fission.) Now, it is true that Coulomb's law only applies to two *point* masses, but it seems reasonable to take the separation distance r in Coulomb's law to be the distance between the centers of the spheres. (This procedure is *exactly* correct for the gravitational forces between two spherical objects, and because Coulomb's law is another inverse–square force law it turns out to be exactly correct in the latter case as well.)

The charge of each sphere (that is, each nucleus) here is

$$q = +Ze = 46(1.602 \times 10^{-19}\,\mathrm{C}) = 7.369 \times 10^{-18}\,\mathrm{C}\ .$$

The separation of the centers of the spheres is $2R$, so the distance we use in Coulomb's law is

$$r = 2R = 2(5.9 \times 10^{-15}\,\mathrm{m}) = 1.18 \times 10^{-14}\,\mathrm{m}$$

so from Eq. 1.3 the magnitude of the force between the two charged spheres is

$$\begin{aligned} F &= k\frac{|q_1|\,|q_2|}{r^2} \\ &= (8.99 \times 10^9\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}})\frac{(7.369 \times 10^{-18}\,\mathrm{C})(7.369 \times 10^{-18}\,\mathrm{C})}{(1.18 \times 10^{-14}\,\mathrm{m})^2} = 3.5 \times 10^3\,\mathrm{N}\ . \end{aligned}$$

The force between the two fission fragments has magnitude $3.5 \times 10^3\,\mathrm{N}$, and it is a *repulsive* force since the fragments are both positively charged.

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6. Two small positively charged spheres have a combined charge of $5.0 \times 10^{-5}\,\mathrm{C}$. If each sphere is repelled from the other by an electrostatic force of $1.0\,\mathrm{N}$ when the spheres are $2.0\,\mathrm{m}$ apart, what is the charge on each sphere?

We are are not given the values of the individual charges; let them be q_1 and q_2. The condition on the combined charge of the spheres gives us:

$$q_1 + q_2 = 5.0 \times 10^{-5}\,\mathrm{C}\ . \tag{1.5}$$

The next condition concerns the electrostatic force, and so it involves Coulomb's Law. Now, Eq. 1.3 involves the *absolute values* of the charges so we need to be careful with the algebra... but in this case we know that both charges are positive because their sum is positive and they *repel* each other. Thus $|q_1| = q_1$ and $|q_2| = q_2$, and the next condition gives us:

$$F = k\frac{q_1 q_2}{r^2} = 1.0\,\mathrm{N}$$

As we know k and r, this give us the value of the product of the charges:

$$q_1 q_2 = \frac{(1.0\,\mathrm{N})r^2}{k} = \frac{(1.0\,\mathrm{N})(2.0\,\mathrm{m})^2}{\left(8.99 \times 10^9\,\frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)} = 4.449 \times 10^{-10}\,\mathrm{C^2} \tag{1.6}$$

With Eqs. 1.5 and 1.6 we have *two* equations for the *two* unknowns q_1 and q_2. We *can* solve for them; the rest is math! Here's my approach to solving the problem:

From Eq. 1.5 we have:

$$q_2 = 5.0 \times 10^{-5}\,\mathrm{C} - q_1 \tag{1.7}$$

Substitute for q_2 in Eq. 1.6 and get:

$$q_1(5.0 \times 10^{-5}\,\mathrm{C} - q_1) = 4.449 \times 10^{-10}\,\mathrm{C^2}$$

which gives us a quadratic equation for q_1:

$$q_1^2 - (5.0 \times 10^{-5}\,\mathrm{C})q_1 + 4.449 \times 10^{-10}\,\mathrm{C^2} = 0$$

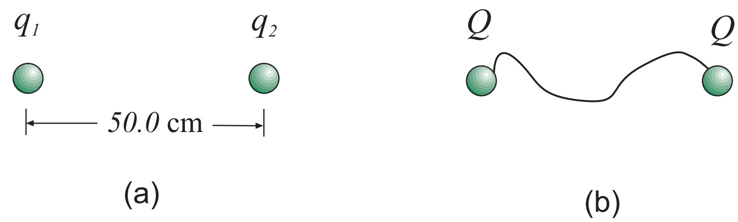

Figure 1.3: (a) Two unknown charges on identical conducting spheres, separated by 50.0 cm, in Example 7. (b) When joined by a wire, the charge evenly divides between the spheres with charge Q on each, such that $q_1 + q_2 = 2Q$.

which we *all* know how to solve. The two possibilities for q_1 are:

$$q_1 = \frac{(5.0 \times 10^{-5}) \pm \sqrt{(5.0 \times 10^{-5})^2 - 4(4.449 \times 10^{-10})}}{2}\,\mathrm{C} = \begin{cases} 3.84 \times 10^{-5}\,\mathrm{C} \\ 1.16 \times 10^{-5}\,\mathrm{C} \end{cases}$$

(Hmm... how do we deal with *two* answers? We'll see...)

Using the two possibilities for q_1 give:

$$q_1 = 3.84 \times 10^{-5}\,\mathrm{C} \quad \Longrightarrow \quad q_2 = 5.0 \times 10^{-5}\,\mathrm{C} - q_1 = 1.16 \times 10^{-5}\,\mathrm{C}$$

$$q_1 = 1.16 \times 10^{-5}\,\mathrm{C} \quad \Longrightarrow \quad q_2 = 5.0 \times 10^{-5}\,\mathrm{C} - q_1 = 3.84 \times 10^{-5}\,\mathrm{C}$$

Actually, these are both the *same* answer, because our numbering of the charges was arbitrary. The answer is that one of the charges is 1.16×10^{-5} C and the other is 3.84×10^{-5} C.

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7. Two identical conducting spheres, fixed in place, attract each other with an electrostatic force of 0.108 N when separated by 50.0 cm, center-to-center. The spheres are then connected by a thin conducting wire. When the wire is removed, the spheres repel each other with an electrostatic force of 0.360 N. What were the initial charges on the spheres?

The initial configuration of the spheres is shown in Fig. 1.3(a). Let the charges on the spheres be q_1 and q_2. If the force of attraction between them has magnitude 0.108 N, then Coulomb's law gives us

$$F = k\frac{|q_1 q_2|}{r^2} = (8.99 \times 10^9\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}})\frac{|q_1 q_2|}{(0.500\,\mathrm{m})^2} = 0.108\,\mathrm{N}$$

from which we get

$$|q_1 q_2| = \frac{(0.108\,\mathrm{N})(0.500\,\mathrm{m})^2}{(8.99 \times 10^9\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}})} = 3.00 \times 10^{-12}\,\mathrm{C^2}$$

But since we are told that the charges *attract* one another, we know that q_1 and q_2 have opposite signs and so their product must be *neagtive*. So we can drop the absolute value sign if we write

$$q_1 q_2 = -3.00 \times 10^{-12}\,\mathrm{C^2} \tag{1.8}$$

Then the two spheres are joined by a wire. The charge is now free to re–distribute itself between the two spheres and since they are identical the total excess charge (that is, $1_1 + q_2$) will be *evenly* divided between the two spheres. If the new charge on each sphere is Q, then

$$Q + Q = 2Q = q_1 + q_2 \tag{1.9}$$

The force of repulsion between the spheres is now 0.0360 N, so that

$$F = k\frac{Q^2}{r^2} = (8.99 \times 10^9 \ \tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}})\frac{Q^2}{(0.500\,\mathrm{m})^2} = 0.0360\,\mathrm{N}$$

which gives

$$Q^2 = \frac{(0.0360\,\mathrm{N})(0.500\,\mathrm{m})^2}{(8.99 \times 10^9 \ \tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}})} = 1.00 \times 10^{-12} C^2$$

We don't know what the sign of Q is, so we can only say:

$$Q = \pm 1.00 \times 10^{-6}\,\mathrm{C} \tag{1.10}$$

Putting 1.10 into 1.9, we get

$$q_1 + q_2 = 2Q = \pm 2.00 \times 10^{-6}\,\mathrm{C} \tag{1.11}$$

and now 1.8 and 1.11 give us two equations for the two unknowns q_1 and q_2, and we're in business!

First, choosing the + sign in 1.11 we have

$$q_2 = 2.00 \times 10^{-6}\,\mathrm{C} - q_1 \tag{1.12}$$

and substituting this into 1.8 we have:

$$q_1(2.00 \times 10^{-6}\,\mathrm{C} - q_1) = -3.00 \times 10^{-12}\,\mathrm{C}^2$$

which we can rewrite as

$$q_1^2 - (2.00 \times 10^{-6}\,\mathrm{C})q_1 - 3.00 \times 10^{-12}\,\mathrm{C}^2 = 0$$

which is a quadratic equation for q_1. When we find the solutions; we get:

$$q_1 = 3.00 \times 10^{-6}\,\mathrm{C} \qquad \text{or} \qquad q_1 = -1.00 \times 10^{-6}\,\mathrm{C}$$

Putting these possibilities into 1.12 we find

$$q_2 = -1.00 \times 10^{-6}\,\mathrm{C} \qquad \text{or} \qquad q_2 = 3.00 \times 10^{-6}\,\mathrm{C}$$

but these really give the same answer: One charge is $-1.00 \times 10^{-6}\,\mathrm{C}$ and the other is $+3.00 \times 10^{-6}\,\mathrm{C}$.

Now make the other choice in 1.11. Then we have

$$q_2 = -2.00 \times 10^{-6}\,\mathrm{C} - q_1 \tag{1.13}$$

Putting this into 1.8 we have:

$$q_1(-2.00 \times 10^{-6}\,\mathrm{C} - q_1) = -3.00 \times 10^{-12}\,\mathrm{C}^2$$

which we can rewrite as

$$q_1^2 + (2.00 \times 10^{-6}\,\mathrm{C})q_1 - 3.00 \times 10^{-12}\,\mathrm{C}^2 = 0$$

which is a different quadratic equation for q_1, and which has the solutions

$$q_1 = -3.00 \times 10^{-6}\,\mathrm{C} \qquad \text{or} \qquad q_1 = 1.00 \times 10^{-6}\,\mathrm{C}$$

Putting these into 1.13 we get

$$q_2 = 1.00 \times 10^{-6}\,\mathrm{C} \qquad \text{or} \qquad q_2 = -3.00 \times 10^{-6}\,\mathrm{C}$$

but these really give the same answer: One charge is $+1.00 \times 10^{-6}\,\mathrm{C}$ and the other is $-3.00 \times 10^{-6}\,\mathrm{C}$.

So in the end we have two distinct possibilities for the initial charges q_1 and q_2 on the spheres. They are

$$-1.00\,\mu\mathrm{C} \qquad \text{and} \qquad +3.00\,\mu\mathrm{C}$$

and

$$+1.00\,\mu\mathrm{C} \qquad \text{and} \qquad -3.00\,\mu\mathrm{C}$$

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8. A certain charge Q is divided into two parts q and $Q - q$, which are then separated by a certain distance. What must q be in terms of Q to maximize the electrostatic repulsion between the two charges?

If the distance between the two (new) charges is r, then the magnitude of the force between them is

$$F = k\frac{(Q-q)q}{r^2} = \frac{k}{r^2}(qQ - q^2)\ .$$

(We know that Q and $Q - q$ both have the same sign so that $Q(Q - q)$ is necessarily a positive number. Force between the charges is repulsive.) To find the value of q which give maximum F, take the derivative of F with respect to q and find where it is zero:

$$\frac{dF}{dq} = \frac{k}{r^2}(Q - 2q) = 0$$

which has the solution

$$(Q - 2q) = 0 \qquad \Longrightarrow \qquad q = \frac{Q}{2}\ .$$

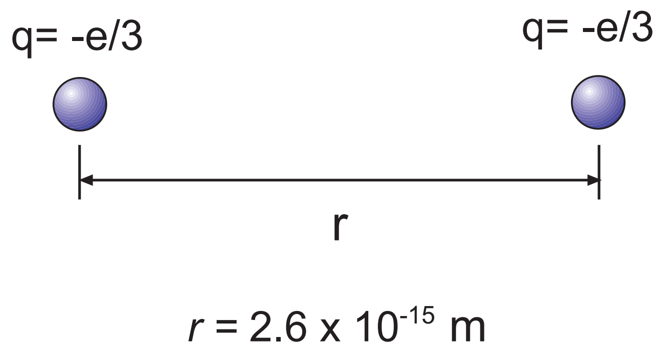

Figure 1.4: Two down quarks, each with charge $-e/3$, separated by 2.6×10^{-15} m, in Example 9.

So the maximum repulsive force is gotten by dividing the original charge Q in half.

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9. A neutron consists of one "up" quark of charge $+\frac{2e}{3}$ and two "down" quarks each having charge $-\frac{e}{3}$. If the down quarks are 2.6×10^{-15} m apart inside the neutron, what is the magnitude of the electrostatic force between them?

We picture the two down quarks as in Fig. 1.4. We use Coulomb's law to find the force between them. (It is *repulsive* since the quarks have the same charge.) The two charges are:

$$q_1 = q_2 = -\frac{e}{3} = -\frac{(1.60 \times 10^{-19}\,\mathrm{C})}{3} = -5.33 \times 10^{-20}\,\mathrm{C}$$

and the separation is $r = 2.6 \times 10^{-15}$ m. The magnitude of the force is

$$F = k\frac{|q_1|\,|q_2|}{r^2} = \left(8.99 \times 10^9\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)\frac{(5.33 \times 10^{-20}\,\mathrm{C})(5.33 \times 10^{-20}\,\mathrm{C})}{(2.6 \times 10^{-15}\,\mathrm{m})^2} = 3.8\,\mathrm{N}$$

The magnitude of the (repulsive) force is 3.8 N.

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10. The charges and coordinates of two charged particles held fixed in the xy plane are: $q_1 = +3.0\,\mu\mathrm{C}$, $x_1 = 3.5\,\mathrm{cm}$, $y_1 = 0.50\,\mathrm{cm}$, and $q_2 = -4.0\,\mu\mathrm{C}$, $x_2 = -2.0\,\mathrm{cm}$, $y_2 = 1.5\,\mathrm{cm}$. (a) Find the magnitude and direction of the electrostatic force on q_2. (b) Where could you locate a third charge $q_3 = +4.0\,\mu\mathrm{C}$ such that the net electrostatic force on q_2 is zero?

(a) First, make a sketch giving the locations of the charges. This is done in Fig. 1.5. (Clearly, q_2 will be *attracted* to q_1; the force on it will be to the right and downward.)

Find the distance between q_2 and q_1. It is

$$\begin{aligned} r &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-2.0 - 3.5)^2 + (1.5 - 0.50)^2}\,\mathrm{cm} = 5.59\,\mathrm{cm} \end{aligned}$$

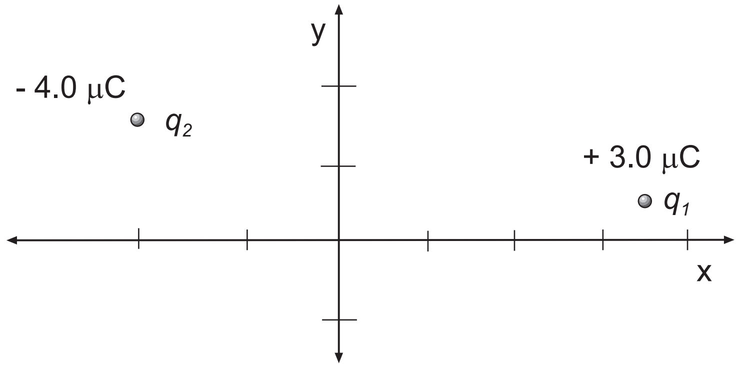

Figure 1.5: Locations of charges in Example 10.

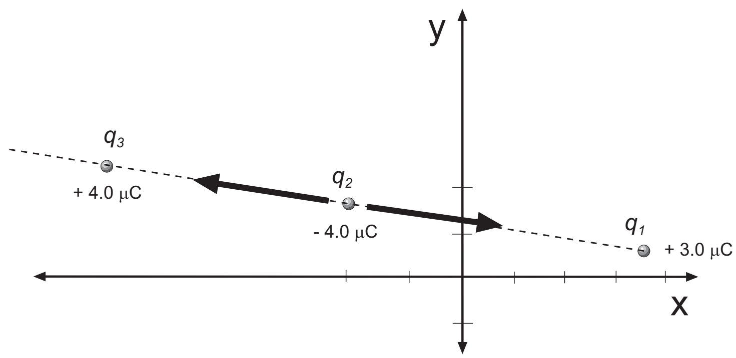

Figure 1.6: Placement of q_3 such as to give zero net force on q_2.

Then by Coulomb's law the force on q_2 has magnitude

$$F = k\frac{|q_1||q_2|}{r^2} = (8.99 \times 10^9\,\tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}})\frac{(3.0 \times 10^{-6}\,\mathrm{C})(4.0 \times 10^{-6}\,\mathrm{C})}{(5.59 \times 10^{-2}\,\mathrm{m})^2} = 35\,\mathrm{N}$$

Since q_2 is attracted to q_1, the *direction* of this force is the same as the vector which points from q_2 to q_1. That vector is

$$\mathbf{r}_{12} = (x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j} = (5.5\,\mathrm{cm})\mathbf{i} + (-1.0\,\mathrm{cm})\mathbf{j}$$

The direction (angle) of this vector is

$$\theta = \tan^{-1}\left(\frac{-1.0}{5.5}\right) = -10.3^\circ$$

(b) The force which the $+4.0\,\mu\mathrm{C}$ charge exerts on q_2 must cancel the force we calculated in part (a) (i.e. the attractive force from q_1). Since this charge will exert an attractive force on q_2, we must place it on the line which joins q_1 and q_2 but on the *other* side of q_2. This is shown in Fig. 1.6.

First, find the distance r' between q_3 and q_2. The force of q_3 on q_2 must also have magnitude 35 N; this allows us to solve for r':

$$F = k\frac{|q_2||q_3|}{r'^2} \qquad \Rightarrow \qquad r'^2 = k\frac{|q_2||q_3|}{F}$$

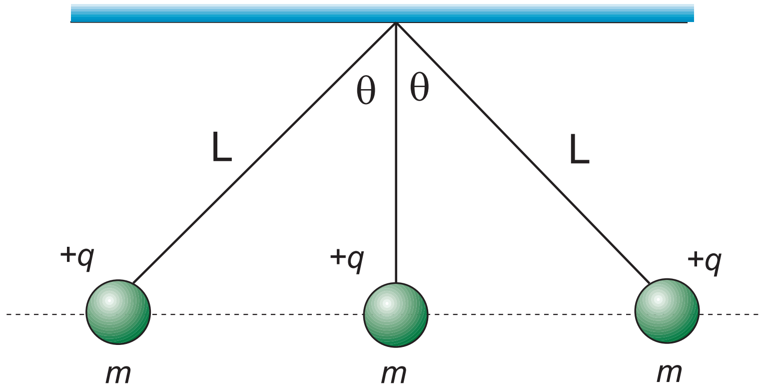

Figure 1.7: Charged masses hang from strings, as described in Example 11.

Plug in the numbers:

$$r'^2 = (8.99 \times 10^9 \, \tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}}) \frac{(4.0 \times 10^{-6}\,\mathrm{C})(4.0 \times 10^{-6}\,\mathrm{C})}{(35\,\mathrm{N})} = 4.1 \times 10^{-3}\,\mathrm{m}$$

$$r' = 6.4 \times 10^{-2}\,\mathrm{m} = 6.4\,\mathrm{cm}$$

This is the distance q_3 from q_2; we also know that being opposite q_1, its direction is

$$\theta' = 180^\circ - 10.3^\circ = 169.7^\circ$$

from q_2. So the *displacement* of q_3 from q_2 is given by:

$$\Delta x = r' \cos\theta' = (6.45\,\mathrm{cm}) \cos 169.7^\circ = -6.35\,\mathrm{cm}$$

$$\Delta y = r' \sin\theta' = (6.45\,\mathrm{cm}) \sin 169.7^\circ = +1.15\,\mathrm{cm}$$

Adding these differences to the coordinates of q_2 we find:

$$x_3 = x_2 + \Delta x = -2.0\,\mathrm{cm} - 6.35\,\mathrm{cm} = -8.35\,\mathrm{cm}$$

$$y_3 = y_2 + \Delta y = +1.5\,\mathrm{cm} + 1.15\,\mathrm{cm} == 2.65\,\mathrm{cm}$$

The charge q_3 should be placed at the point $(-8.35\,\mathrm{cm},\ 2.65\,\mathrm{cm})$.

---

11. Three identical point charges, each of mass $m = 0.100\,\mathrm{kg}$ and charge q hang from three strings, as in Fig. 1.7. If the lengths of the left and right strings are $L = 30.0\,\mathrm{cm}$ and angle $\theta = 45.0^\circ$, determine the value of q.

Make a free–body diagram in order to understand things! Choose the leftmost mass in Fig. 1.7. The forces on this mass are shown in Fig. 1.8. Gravity pulls down with a force mg; the string tension pulls as shown with a force of magnitude T. Both of the other charged masses exert forces of electrostatic repulsion on this mass. The charge in the middle exerts

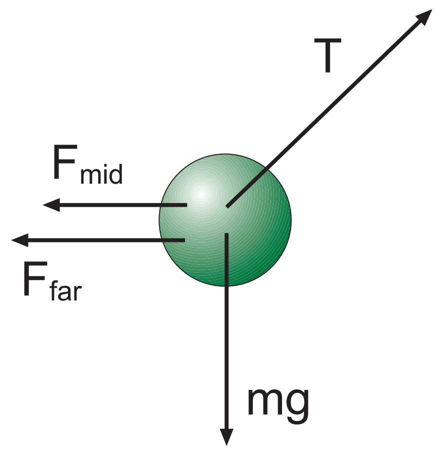

Figure 1.8: Forces acting on the leftmost charged mass in Example 11.

a force of magnitude $F_{\rm mid}$; the rightmost (far) charge exerts a force of magnitude $F_{\rm far}$. Both forces are directed to the left.

We can get expressions for $F_{\rm mid}$ and $F_{\rm far}$ using Coulomb's law. The distance between the left charge and the middle charge is

$$r_1 = (30.0\,{\rm cm})\sin 45.0^\circ = 21.2\,{\rm cm} = 0.212\,{\rm m}$$

and since both charges are $+q$ we have

$$F_{\rm mid} = k\frac{q^2}{(0.212\,{\rm m})^2}\ .$$

Likewise, the distance between the left charge and the rightmost charge is

$$r_2 = 2(30.0\,{\rm cm})\sin 45.0^\circ = 2(0.212\,{\rm m}) = 0.424\,{\rm m}$$

so that we have

$$F_{\rm far} = k\frac{q^2}{(0.424\,{\rm m})^2}\ .$$

The vertical forces on the mass must sum to zero. This gives us:

$$T\sin 45.0^\circ - mg = 0 \qquad \Longrightarrow \qquad T = \frac{mg}{\sin 45.0^\circ} = 1.39\,{\rm N}$$

where we have used the given value of m to evaluate T.

The horizontal forces must also sum to zero, and this gives us:

$$-F_{\rm mid} - F_{\rm far} + T\cos 45.0^\circ = 0$$

Substitute for $F_{\rm mid}$ and $F_{\rm far}$ and get:

$$-k\frac{q^2}{(0.212\,{\rm m})^2} - k\frac{q^2}{(0.424\,{\rm m})^2} + T\cos 45.0^\circ = 0 \qquad (1.14)$$

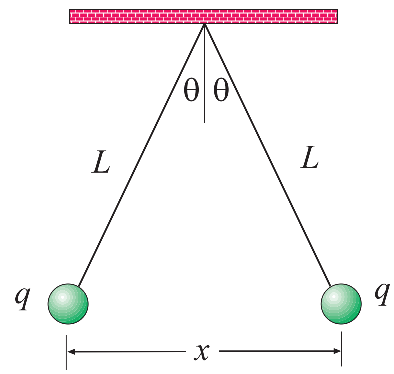

Figure 1.9: Charged masses hang from strings, as described in Example 12.

Since we have already found T, the only unknown in this equation is q. The *physics* part of the problem is done!

A little rearranging of Eq. 1.14 gives us:

$$kq^2\left(\frac{1}{(0.212\,\mathrm{m})^2}+\frac{1}{(0.424\,\mathrm{m})^2}\right)=T\cos 45.0^\circ$$

The sum in the big parenthesis is equal to $27.8\,\mathrm{m}^{-2}$ and with this we can solve for q:

$$\begin{aligned} q^2 &= \frac{T\cos 45.0^\circ}{k(27.8\,\mathrm{m}^{-2})} \\ &= \frac{(1.39\,\mathrm{N})\cos 45.0^\circ}{\left(8.99\times 10^9\,\frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)(27.8\,\mathrm{m}^{-2})} = 3.93\times 10^{-12}\,\mathrm{C}^2 \end{aligned}$$

And then:

$$q = 1.98\times 10^{-6}\,\mathrm{C} = 1.98\,\mu\mathrm{C}$$

---

12. In Fig. 1.9, two tiny conducting balls of identical mass and identical charge q hang from nonconducting threads of length L. Assume that θ is so small that $\tan\theta$ can be replaced by its approximate equal, $\sin\theta$. (a) Show that for equilibrium,

$$x=\left(\frac{q^2L}{2\pi\epsilon_0 mg}\right)^{1/3} ,$$

where x is the separation between the balls. (b) If $L = 120\,\mathrm{cm}$, $m = 10\,\mathrm{g}$ and $x = 5.0\,\mathrm{cm}$, what is q?

(a) We draw a free-body diagram for one of the charge (say, the left one). This is done in Fig. 1.10. The forces acting on the charged ball are the string tension T, the downward force of gravity mg and the force of electrostatic repulsion from the other charged ball, F_{elec}. The direction of this for is to the left because the other ball, having the same charge exerts a

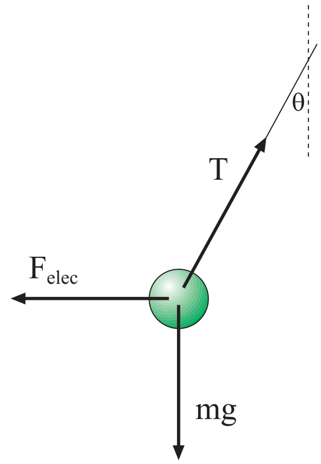

Figure 1.10: The forces acting on one of the charged masses in Example 12.

repulsive force which must point horizontally to the left because of the symmetric position of the other ball.

We do know the magnitude of the force of electrostatic repulsion; from Coulomb's law it is

$$F_{\rm elec} = k\frac{q^2}{x^2}$$

The ball is in static equilibrium, so the forces on the ball sum to zero. The vertical components add to zero, which gives us:

$$T\cos\theta = mg$$

and from the horizontal components we get

$$T\sin\theta = F_{\rm elec} = k\frac{q^2}{x^2}$$

Divide the second of these equations by the first one and get:

$$\frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{kq^2}{mgx^2} \tag{1.15}$$

Now the problem says that the angle θ is so small that we can safely replace $\tan\theta$ by $\sin\theta$ (they are nearly the same for "small" angles). But from the geometry of the problem we can express $\sin\theta$ as:

$$\sin\theta = \frac{x/2}{L} = \frac{x}{2L}$$

Using all of this in Eq. 1.15 we get:

$$\frac{x}{2L} \approx \frac{kq^2}{mgx^2}$$

Now we can solve for x because a little algebra gives:

$$x^3 = \frac{kq^2(2L)}{mg} = \frac{2q^2L}{4\pi\epsilon_0 mg} = \frac{q^2L}{2\pi\epsilon_0 mg} \tag{1.16}$$

which then gives the answer for x,

$$x = \left(\frac{q^2L}{2\pi\epsilon_0 mg}\right)^{1/3}$$

(b) Rearranging Eq. 1.16 we find:

$$q^2 = \frac{2\pi\epsilon_0 mgx^3}{L}$$

and plugging in the given values (in SI units, of course), we get:

$$q^2 = \frac{2\pi(8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}})(10 \times 10^{-3}\,\mathrm{kg})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(1.20\,\mathrm{m})} = 5.68 \times 10^{-16}\,\mathrm{C}^2$$

and then we find q (note the ambiguity in sign!):

$$q = \pm 2.4 \times 10^{-8}\,\mathrm{C} \quad .$$

Chapter 2

Electric Fields

2.1 The Important Stuff

2.1.1 The Electric Field

Suppose we have a point charge q_0 located at $\mathbf{r}$ and a set of *external* charges conspire so as to exert a force $\mathbf{F}$ on this charge. We can define the **electric field** at the point $\mathbf{r}$ by:

$$\mathbf{E} = \frac{\mathbf{F}}{q_0} \tag{2.1}$$

The (vector) value of the $\mathbf{E}$ field depends *only* on the values and locations of the external charges, because from Coulomb's law the force on any "test charge" q_0 is proportional to the value of the charge. However to make this definition really kosher we have to stipulate that the test charge q_0 is "small"; otherwise its presence will significantly influence the locations of the external charges.

Turning Eq. 2.1 around, we can say that if the electric field at some point $\mathbf{r}$ has the value $\mathbf{E}$ then a *small* charge placed at $\mathbf{r}$ will experience a force

$$\mathbf{F} = q_0\mathbf{E} \tag{2.2}$$

The electric field is a *vector*. From Eq. 2.1 we can see that its SI units must be $\frac{\mathrm{N}}{\mathrm{C}}$.

It follows from Coulomb's law that the electric field at point $\mathbf{r}$ due to a charge q located at the origin is given by

$$\mathbf{E} = k\frac{q}{r^2}\hat{\mathbf{r}} \tag{2.3}$$

where $\hat{\mathbf{r}}$ is the unit vector which points in the same direction as $\mathbf{r}$.

2.1.2 Electric Fields from Particular Charge Distributions

• **Electric Dipole**

An **electric dipole** is a pair of charges of opposite sign ($\pm q$) separated by a distance d which is usually meant to be small compared to the distance from the charges at which we

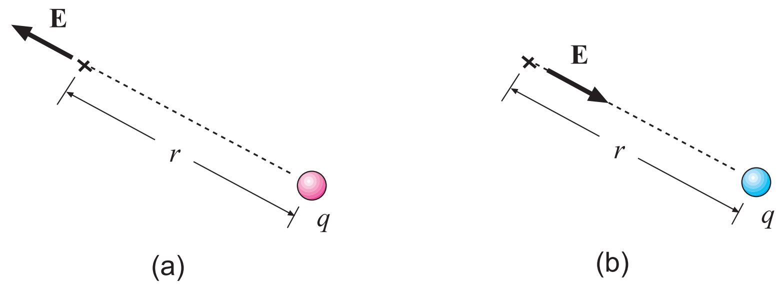

Figure 2.1: The E field due to a point charge q. (a) If the charge q is positive, the E field at some point a distance r away has magnitude $k|q|/r^2$ and points *away* from the charge. (b) If the charge q is negative, the E field has magnitude $k|q|/r^2$ and points *toward* the charge.

want to find the electric field. The product qd turns out to be important; the vector which points from the $-q$ charge to the $+q$ charge and has magnitude qd is known as the **electric dipole moment** for the pair, and is denoted $\mathbf{p}$.

Suppose we form an electric dipole by placing a charge $+q$ at $(0, 0, d/2)$ and a charge $-q$ at $(0, 0, -d/2)$. (So the dipole moment $\mathbf{p}$ has magnitude $p = qd$ and points in the $+\mathbf{k}$ direction.) One can show that when z is much larger than d, the electric field for points on the z axis is

$$E_z = \frac{1}{2\pi\epsilon_0}\frac{p}{z^3} = k\frac{2qd}{z^3} \tag{2.4}$$

- **"Line" of Charge**

A linear charge distribution is characterized by its charger per unit length. **Linear charge density** is usually given the symbol λ; for an arclength ds of the distribution, the electric charge is

$$dq = \lambda ds$$

For a ring of charge with radius R and total charge q, for a point on the axis of the ring a distance z from the center, the magnitude of the electric field (which points along the z axis) is

$$E = \frac{qz}{4\pi\epsilon_0(z^2 + r^2)^{3/2}} \tag{2.5}$$

- **Charged Disk & Infinite Sheet**

A two-dimensional (surface) distribution of charge is characterized by its charge per unit area. **Surface charge density** is usually given the symbol σ; for an area element dA of the distribution, the electric charge is

$$dq = \sigma dA$$

For a disk or radius R and uniform charge density σ on its surface, for a point on the axis of the disk at a distance z away from the center, the magnitude of the electric field (which points along the z axis) is

$$E = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + r^2}}\right) \tag{2.6}$$

The limit $R \longrightarrow \infty$ of Eq. 2.6 gives the magnitude of the E field at a distance z from an infinite sheet of charge with charge density σ. The result is

$$E = \frac{\sigma}{2\epsilon_0} \tag{2.7}$$

2.1.3 Forces on Charges in Electric Fields

An isolated charge q in an electric field experiences a force $\mathbf{F} = q\mathbf{E}$. We note that when q is positive the force points in the same direction as the field, but when q is negative, the force is opposite the field direction!

The potential energy of a point charge in an $\mathbf{E}$ field will be discussed at great length in chapter 4!

When an electric dipole $\mathbf{p}$ is place in a uniform $\mathbf{E}$ field, it experiences no net force, but it *does* experience a torque. The torque is given by:

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \tag{2.8}$$

The potential energy of a dipole also depends on its orientation, and is given by:

$$U = -\mathbf{p} \cdot \mathbf{E} \tag{2.9}$$

2.1.4 Electric Field Lines

Oftentimes it is useful for us to get an *overall visual picture* of the electric field due to a particular distribution of charge. It is useful make a plot where the little arrows representing the direction of the electric field at each point are joined together, forming continuous (directed) "lines". These are the **electric field lines** for the charge distribution.

Such a plot will tell us the basic *direction* of the electric field at all points in space (though we do lose information about the *magnitude* of the field when we join the arrows). One can show that:

• Electric field lines originate on positive charges (they point *away* from the positive charge) and end on negative charges (they point *toward* the negative charge).

• Field lines cannot cross one another.

Whereas a diagram of field lines can contain as many lines as you please, for an accurate representation of the field the number of lines originating from a charge should be *proportional* to the charge.

2.2 Worked Examples

2.2.1 The Electric Field

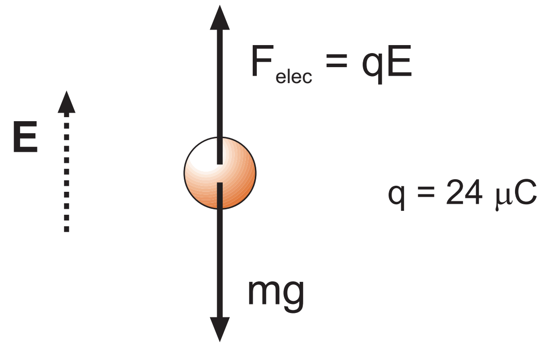

Figure 2.2: Forces acting on the charged mass in Example 1.

1. An object having a net charge of $24\,\mu\mathrm{C}$ is placed in a uniform electric field of $610\,\frac{\mathrm{N}}{\mathrm{C}}$ directed vertically. What is the mass of this object if it "floats" in the field?

The forces acting on the mass are shown in Fig. 2.2. The force of gravity points downward and has magnitude mg (m is the mass of the object) and the electrical force acting on the mass has magnitude $F = |q|E$, where q is the charge of the object and E is the magnitude of the electric field. The object "floats", so the net force is zero. This gives us:

$$|q|E = mg$$

Solve for m:

$$m = \frac{|q|E}{g} = \frac{(24\times10^{-6}\,\mathrm{C})(610\,\frac{\mathrm{N}}{\mathrm{C}})}{(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 1.5\times10^{-3}\,\mathrm{kg}$$

The mass of the object is $1.5\times10^{-3}\,\mathrm{kg} = 1.5\,\mathrm{g}$.

---

2. An electron is released from rest in a uniform electric of magnitude $2.00\times10^4\,\frac{\mathrm{N}}{\mathrm{C}}$. Calculate the acceleration of the electron. (Ignore gravitation.)

The magnitude of the force on a charge q in an electric field is given by $F = |qE|$, where E is the magnitude of the field. The magnitude of the electron's charge is $e = 1.602\times10^{-19}\,\mathrm{C}$, so the magnitude of the force on the electron is

$$F = |qE| = (1.602\times10^{-19}\,\mathrm{C})(2.00\times10^4\,\frac{\mathrm{N}}{\mathrm{C}}) = 3.20\times10^{-15}\,\mathrm{N}$$

Newton's 2^{nd} law relates the magnitudes of the force and acceleration: $F = ma$, so the acceleration of the electron has magnitude

$$a = \frac{F}{m} = \frac{(3.20\times10^{-15}\,\mathrm{N})}{(9.11\times10^{-31}\,\mathrm{kg})} = 3.51\times10^{15}\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

That's the *magnitude* of the electron's acceleration. Since the electron has a negative charge the direction of the *force* on the electron (and also the acceleration) is *opposite* the direction of the electric field.

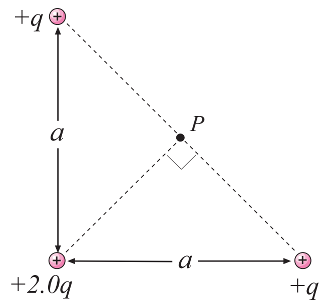

Figure 2.3: Charge configuration for Example 4.

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3. What is the magnitude of a point charge that would create an electric field of $1.00\,\frac{\mathrm{N}}{\mathrm{C}}$ at points $1.00\,\mathrm{m}$ away?

From Eq. 2.3, the magnitude of the E field due to a point charge q at a distance r is given by

$$E = k\frac{|q|}{r^2}$$

Here we are given E and r, so we can solve for $|q|$:

$$\begin{aligned} |q| &= \frac{Er^2}{k} = \frac{(1.00\,\frac{\mathrm{N}}{\mathrm{C}})(1.00\,\mathrm{m})^2}{\left(8.99\times 10^9\,\frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}\right)} \\ &= 1.11\times 10^{-10}\,\mathrm{C} \end{aligned}$$

The *magnitude* of the charge is $1.11\times 10^{-10}\,\mathrm{C}$.

---

4. Calculate the direction and magnitude of the electric field at point P in Fig. 2.3, due to the three point charges.

Since each of the three charges is *positive* they give electric fields at P pointing *away* from the charges. This is shown in Fig. 2.4, where the charges are individually numbered along with their (vector!) E–field contributions.

We note that charges 1 and 2 have the same magnitude and are both at the same distance from P. So the E–field vectors for these charges shown in Fig. 2.4(being in opposite directions) must cancel. So we are left with only the contribution from charge 3.

We know the direction for this vector; it is 45° above the x axis. To find its magnitude we note that the distance of this charge from P is half the length of the square's diagonal, or:

$$r = \tfrac{1}{2}(\sqrt{2}a) = \frac{a}{\sqrt{2}}$$

and so the magnitude is

$$E_3 = k\frac{2q}{r^2} = \frac{2kq}{(a/\sqrt{2})} = \frac{4kq}{a^2} \quad .$$

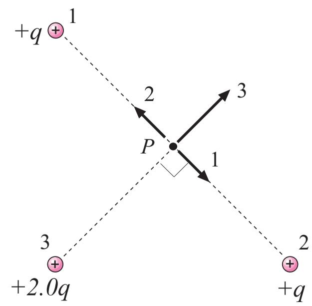

Figure 2.4: Directions for the contributions to the E field at P due to the three positive charges in Example 4.

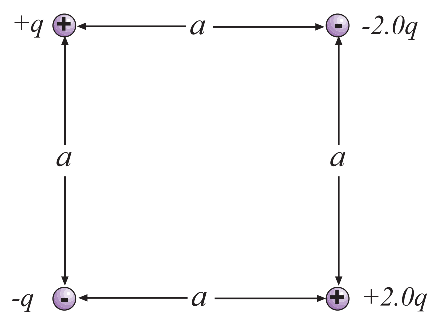

Figure 2.5: Charge configuration for Example 5.

So the electric field at P has magnitude

$$E_{\text{net}} = \frac{4kq}{a^2} = \frac{4q}{(4\pi\epsilon_0)a^2} = \frac{q}{\pi\epsilon_0 a^2}$$

and points at an angle of 45°.

---

5. What are the magnitude and direction of the electric field at the center of the square of Fig. 2.5 if $q = 1.0 \times 10^{-8}\,\mathrm{C}$ and $a = 5.0\,\mathrm{cm}$?

The center of the square is equidistant from all the charges. This distance r is half the diagonal of the square, hence

$$r = \tfrac{1}{2}(\sqrt{2}a) = \frac{a}{\sqrt{2}} = \frac{(5.0\,\mathrm{cm})}{\sqrt{2}} = 3.55 \times 10^{-2}\,\mathrm{m}$$

Then we can find the *magnitudes* of the contributions to the **E** field from each of the charges. The charges of magnitude q have contributions of magnitude

$$E_{1.0q} = k\frac{q}{r^2} = (8.99 \times 10^9\,\tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}})\frac{(1.0 \times 10^{-8}\,\mathrm{C})}{(3.55 \times 10^{-2}\,\mathrm{m})^2} = 7.13 \times 10^4\,\tfrac{\mathrm{N}}{\mathrm{C}}$$

The charges of magnitude $2.0q$ contribute with fields of twice this magnitude, namely

$$E_{2.0q} = 2E_{1.0q} = 1.43 \times 10^5\,\tfrac{\mathrm{N}}{\mathrm{C}}$$

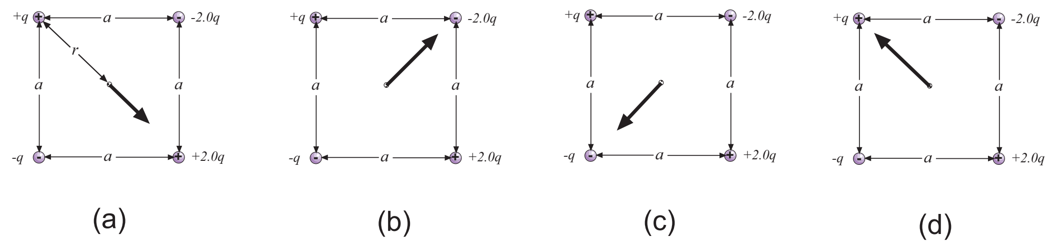

Figure 2.6: Directions of $\mathbf{E}$ field at the center of the square due to three of the corner charges. (a) Upper left charge is at distance $r = a/\sqrt{2}$ from the center (as are the other charges). $\mathbf{E}$ field due to this charge points *away from* charge, in -45° direction. (b) $\mathbf{E}$ field due to upper right charge points *toward* charge, in $+45^\circ$ direction. (c) $\mathbf{E}$ field due to lower left charge points *toward* charge, in $+225^\circ$ direction. (d) $\mathbf{E}$ field due to lower left charge points *away from* charge, in $+135^\circ$ direction.

The *directions* of the contributions to the total $\mathbf{E}$ field are shown in Fig. 2.6(a)–(d). The $\mathbf{E}$ field due to the upper left charge points *away from* charge, which is in -45° direction (as measured from the $+x$ axis, as usual. The $\mathbf{E}$ field due to upper right charge points *toward* the charge, in $+45^\circ$ direction. The $\mathbf{E}$ field due to lower left charge points *toward* that charge, in $180^\circ + 45^\circ = +225^\circ$ direction. Finally, $\mathbf{E}$ field due to lower right charge points *away from* charge, in $180^\circ - 45^\circ = +135^\circ$ direction.

So we now have the magnitudes and directions of four vectors. Can we add them together? Sure we can!

$$\begin{aligned}
\mathbf{E}_{\rm Total} &= (7.13 \times 10^4\,\tfrac{\rm N}{\rm C})(\cos(-45^\circ)\mathbf{i} + \sin(-45^\circ)\mathbf{j}) \\
&+ (1.43 \times 10^5\,\tfrac{\rm N}{\rm C})(\cos(+45^\circ)\mathbf{i} + \sin(45^\circ)\mathbf{j}) \\
&+ (7.13 \times 10^4\,\tfrac{\rm N}{\rm C})(\cos(225^\circ)\mathbf{i} + \sin(225^\circ)\mathbf{j}) \\
&+ (1.43 \times 10^5\,\tfrac{\rm N}{\rm C})(\cos(+135^\circ)\mathbf{i} + \sin(135^\circ)\mathbf{j})
\end{aligned}$$

(I know, this is the clumsy way of doing it, but I'll get to that.) The sum gives:

$$\mathbf{E}_{\rm Total} = 0.0\mathbf{i} + (1.02 \times 10^5\,\tfrac{\rm N}{\rm C})\mathbf{j}$$

So the magnitude of $\mathbf{E}_{\rm Total}$ is $1.02 \times 10^5\,\frac{\rm N}{\rm C}$ and it points in the $+y$ direction.

This particular problem can be made easier by noting the cancellation of the $\mathbf{E}$'s contributed by the charges on opposite corners of the square. For example, a $+q$ charge in the upper left and a $+2.0q$ charge in the lower right is equivalent to a *single* charge $+q$ in the lower right (as far as this problem is concerned).

2.2.2 Electric Fields from Particular Charge Distributions

---

6. *Electric Quadrupole* **Fig. 2.7 shows an electric quadrupole. It consists of two dipole moments that are equal in magnitude but opposite in direction. Show that the value of E on the axis of the quadrupole for points a distance z from its center (assume $z \gg d$) is given by**

$$E = \frac{3Q}{4\pi\epsilon_0 z^4}\ ,$$

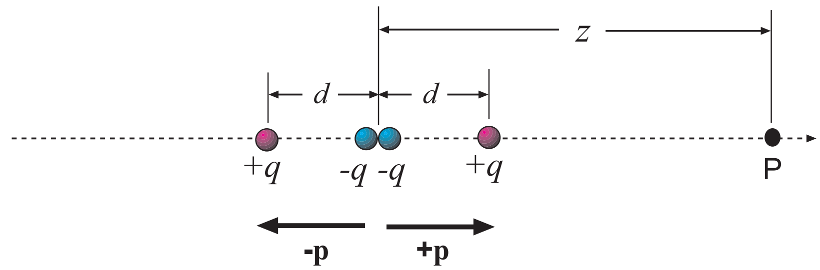

Figure 2.7: Charges forming electric quadrupole in Example 6.

in which Q (defined by $Q \equiv 2qd^2$) is known as the *quadrupole moment* of the charge distribution.

We note that as the problem is given we really have *three* separate charges in this configuration: A charge $-2q$ at the origin, a charge $+q$ at $z = -d$ and a charge $+q$ at $z = +d$. (Again, see Fig. 2.7.)

We are assuming that q is positive; for now let us also assume that the point P (for which we want the electric field) is located on the z axis at some positive value of z, as indicated in Fig. 2.7. We will now find the contribution to the electric field at P for each of the three charges.

The center charge ($-2q$) lies at a distance z from the point P. So then the *magnitude* of the E field due to this charge is $k\frac{2q}{z^2}$, but since the charge is *negative* the field points *toward* the charge, which in this case in the $-z$ direction. So then the contribution E_z (at point P) by the center charge is

$$E_z^{(\mathrm{center})} = -k\frac{2q}{z^2}$$

The charge on the left ($+q$) lies at a distance $z + d$ from the point P. So then the magnitude of the E field due to this charge is $k\frac{q}{(z+d)^2}$. Since this charge is positive, this field points *away from* the charge, namely in the $+z$ direction. So the contribution to E_z by the left charge is

$$E_z^{(\mathrm{left})} = +k\frac{q}{(z+d)^2}$$

The charge on the right ($+q$) lies at a distance $z - d$ from the point P. The magnitude of the field due to this charge is $k\frac{q}{(z-d)^2}$ and this charge is also positive so we get a contribution

$$E_z^{(\mathrm{right})} = +k\frac{q}{(z-d)^2}$$

The field at point P is the sum of the three contributions. We factor out kq from each term to get:

$$E_z = kq\left[-\frac{2}{z^2} + \frac{1}{(z+d)^2} + \frac{1}{(z-d)^2}\right] \ .$$

At this point we are really done with the *physics* of the problem; the rest of the work is doing the *mathematical* steps to get a simpler (approximate) expression for E_z.

We can remove a factor of z^2 from the denominator of each term. We choose z because it is much larger than d and later this will allow us to use the binomial expansion. We get:

$$\begin{aligned} E_z &= kq\left[-\frac{2}{z^2}+\frac{1}{z^2\left(1+\frac{d}{z}\right)^2}+\frac{1}{z^2\left(1-\frac{d}{z}\right)^2}\right] \\ &= \frac{kq}{z^2}\left[-2+\frac{1}{\left(1+\frac{d}{z}\right)^2}+\frac{1}{\left(1-\frac{d}{z}\right)^2}\right] \\ &= \frac{kq}{z^2}\left[-2+\left(1+\frac{d}{z}\right)^{-2}+\left(1-\frac{d}{z}\right)^{-2}\right] \qquad (2.10) \end{aligned}$$

At this point the expression for E_z is exact, but it is useful to get an approximate expression for the case where z is much larger than the size of the quadrupole: $z \gg d$. For this we can make use of the binomial expansion (another name for the Taylor expansion of $(1+x)^n$ about $x = 0$):

$$(1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\cdots \qquad \text{valid for } x^2 < 1$$

The formula is especially useful when x is small in absolute value; then the first several terms of the expansion will give a good approximation. We will use the binomial expansion to simplify our last expression by associating $\pm\frac{d}{z}$ with x (because it *is* small), using $n = -2$, and just to be safe, we'll use the first three terms. This give us the approximations:

$$\left(1+\frac{d}{z}\right)^{-2} \approx 1-2\left(\frac{d}{z}\right)+\frac{6}{2}\left(\frac{d}{z}\right)^2 = 1-\frac{2d}{z}+3\left(\frac{d}{z}\right)^2$$

$$\left(1-\frac{d}{z}\right)^{-2} \approx 1+2\left(\frac{d}{z}\right)+\frac{6}{2}\left(\frac{d}{z}\right)^2 = 1+\frac{2d}{z}+3\left(\frac{d}{z}\right)^2$$

Now putting these results into Eq. 2.10 we find:

$$\begin{aligned} E_z &\approx \frac{kq}{z^2}\left[-2+1-\frac{2d}{z}+3\left(\frac{d}{z}\right)^2+1+\frac{2d}{z}+3\left(\frac{d}{z}\right)^2\right] \\ &= \frac{kq}{z^2}\left[6\left(\frac{d}{z}\right)^2\right] \\ &= \frac{6kqd^2}{z^4} \end{aligned}$$

Using the definition $Q \equiv 2qd^2$, and also $k = \frac{1}{4\pi\epsilon_0}$ we can also write this result as

$$E_z = \frac{3kQ}{z^4} = \frac{3Q}{4\pi\epsilon_0 z^4}\ .$$

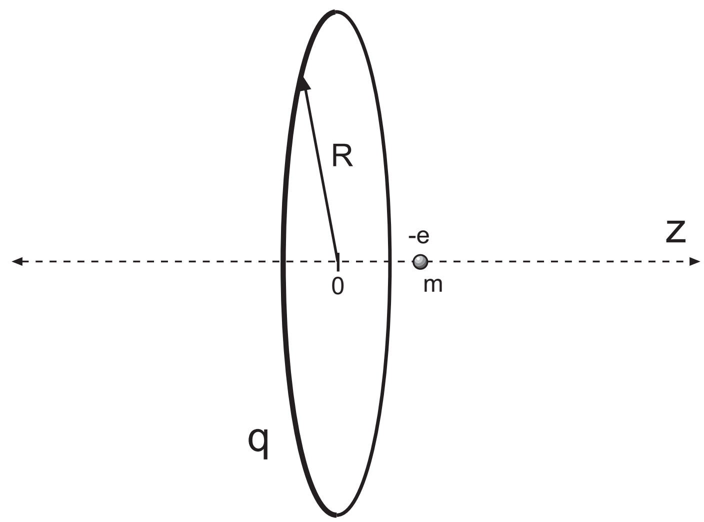

Figure 2.8: Electron oscillates on z axis through center of charged ring or radius R and total charge q, as in Example 7.

---

7. An electron is constrained to the central axis of the ring of charge with radius R and total charge q. Show that the electrostatic force exerted on the electron can cause it to oscillate through the center of the ring with an angular frequency

$$\omega = \sqrt{\frac{eq}{4\pi\epsilon_0 m R^3}} ,$$

where m is the electron's mass.

A picture of this problem is given in Fig. 2.8; included is the electron which oscillates through the center. For this to happen, the charge q must be positive so that the electron is always attracted back to the ring (i.e. the force is *restoring*.)

From Eq. 2.5 we have the magnitude of the E field for points on the axis of the ring (which we will call the z axis, with its origin at the center of the ring):

$$E = \frac{q|z|}{4\pi\epsilon_0(z^2 + R^2)^{3/2}}$$

Note, we need an absolute value sign on the coordinate z to get a *positive* magnitude. Now, the E field points along the $\pm z$ axis; when z is positive it goes in the $+z$ direction and when z is negative it goes in the $-z$ direction. So the z *component* of the electric field which the electron "sees" at coordinate z is in fact

$$E_z = \frac{qz}{4\pi\epsilon_0(z^2 + R^2)^{3/2}}$$

From Eq. 2.2 we get force on the electron as it moves on the z axis:

$$F_z = (-e)E_z = \frac{-eqz}{4\pi\epsilon_0(z^2 + R^2)^{3/2}} \tag{2.11}$$

This expression for the force on the electron is rather messy; it is a *restoring* force since its direction is opposite the displacement of the electron from the center, but it is not *linear*, that is, it is not simply proportional to z. (Our work on harmonic motion assumed a *linear* restoring force.) We will assume that the oscillations are *small* in the sense that the maximum value of $|z|$ is much smaller than R. If that is true, then in the denominator of Eq. 2.11 we can approximate:

$$(z^2 + R^2)^{3/2} \approx \left(R^2\right)^{3/2} = R^3$$

Making this replacement in Eq. 2.11, we find:

$$F_z \approx \frac{-eqz}{4\pi\epsilon_0 R^3} = -\left(\frac{eq}{4\pi\epsilon_0 R^3}\right) z$$

This is a force "law" which just like the Hooke's Law, $F_z = -kz$ with the role of the spring constant k being played by

$$k \Leftrightarrow \frac{eq}{4\pi\epsilon_0 R^3} \tag{2.12}$$

By analogy with the harmonic motion of a mass m on the end of a spring of force constant k, where the result for the angular frequency was $\omega = \sqrt{\frac{k}{m}}$, using Eq. 2.12 we find that angular frequency for the electron's motion is

$$\omega = \sqrt{\frac{eq}{4\pi\epsilon_0 R^3 m}}$$

---

8. A thin nonconducting rod of finite length L has a charge q spread uniformly along it. Show that the magnitude E of the electric field at point P on the perpendicular bisector of the rod (see Fig. 2.9) is given by

$$E = \frac{q}{2\pi\epsilon_0 y} \frac{1}{(L^2 + 4y^2)^{1/2}} \; .$$

We first set up a coordinate system with which to do our calculation. Let the origin be at the center of the rod and let the x axis extend along the rod. In this system, the point P is located at $(0, y)$ and the ends of the rod are at $(-L/2, 0)$ and $(+L/2, 0)$.

If the charge q is spread uniformly over the rod, then it has a linear charge density of

$$\lambda = \frac{q}{L} \; .$$

Then if we take a section of the rod of length dx, it will contain a charge λdx.

Next, we consider how a *tiny bit* of the rod will contribute to the electric field at P. This is shown in Fig. 2.10. We consider a piece of the rod of length dx, centered at the coordinate

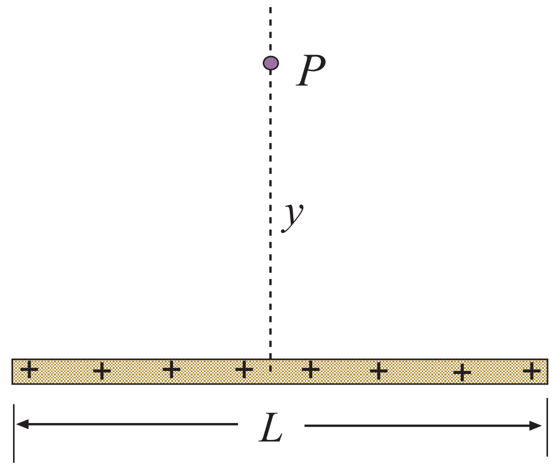

Figure 2.9: Charged rod and geometry for Example 8.

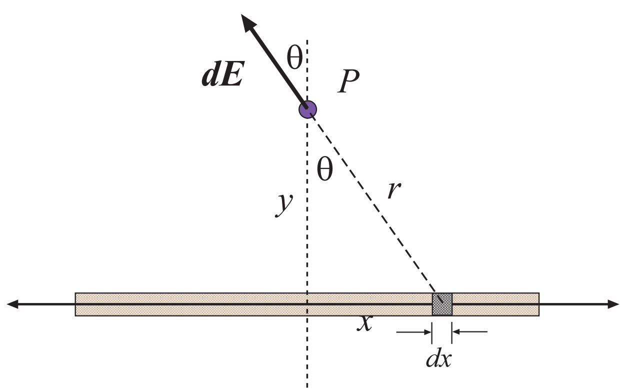

Figure 2.10: An element of the rod of length dx located at x gives a contribution to the electric field at point P.

x. This element is small enough that we can treat it as a point charge...and we *know* how to find the electric field due to a point charge.

The distance of this small piece from the point P is

$$r = \sqrt{x^2 + y^2}$$

and the amount of electric charge contained in the piece is

$$dq = \lambda dx$$

Therefore the electric charge in this little bit of the rod gives an electric field of magnitude

$$dE = k\frac{dq}{r^2} = \frac{k\lambda dx}{(x^2 + y^2)}$$

This little bit of electric field $d\mathbf{E}$ points at an angle θ away from the y axis, as shown in Fig. 2.10.

Eventually we will have to *add up* all the little bits of electric field $d\mathbf{E}$ due to each little bit of the rod. In doing so we will be adding *vectors* and so we will need the components of the $d\mathbf{E}$'s. As can be seen from Fig. 2.10, the components are:

$$dE_x = dE \sin\theta = -\frac{k\lambda dx}{(x^2 + y^2)} \sin\theta \qquad \text{and} \qquad dE_y = \cos\theta \, dE = \frac{k\lambda dx}{(x^2 + y^2)} \cos\theta \tag{2.13}$$

Also, some basic trigonometry gives us:

$$\sin\theta = \frac{x}{r} = \frac{x}{(x^2 + y^2)^{1/2}} \qquad \cos\theta = \frac{y}{r} = \frac{y}{(x^2 + y^2)^{1/2}}$$

Using these in Eqs. 2.13 gives:

$$dE_x = -\frac{kx\lambda dx}{(x^2 + y^2)^{3/2}} \qquad dE_y = \frac{ky\lambda dx}{(x^2 + y^2)^{3/2}} \tag{2.14}$$

The next step is to add up all the individual dE_x's and dE_y's. The result for the sum of the dE_x's is easy: It must be zero! By considering *all* of the little bits of the rod, we can see that dE_x will be positive just as often as it is negative, and when the sum is taken the result is zero. (We sometimes say that this result follows "from symmetry".) The same is not true for the dE_y's; they are always positive and we *will* have to do some work to add them up.

Eq. 2.14 gives the contribution to E_y arising from an element of length dx centered on x. The bits of the rod extend from $x = -L/2$ to $x = +L/2$, so to get the sum of *all* the little bits we do the integral:

$$E_y = \int_{\text{rod}} dE_y = \int_{-L/2}^{+L/2} \frac{ky\lambda dx}{(x^2 + y^2)^{3/2}} \tag{2.15}$$

Eq. 2.15 gives the result for the E field at P (which is what the problem asks for) so we are now done with the *physics* of the problem. All that remains is some *mathematics* to work out the integral in Eq. 2.15.

First, since k, λ and y are constants as far as the integral is concerned, they can be taken outside the integral sign:

$$E_y = ky\lambda \int_{-L/2}^{+L/2} \frac{1}{(x^2+y^2)^{3/2}} dx$$

The integral here is not difficult; it can be looked up in a table or evaluated by computer. We find:

$$E_y = k\lambda y \frac{x}{y^2\sqrt{x^2+y^2}} \Bigg|_{-\frac{L}{2}}^{+\frac{L}{2}}$$

Evaluate!

$$\begin{aligned} E_y &= k\lambda y \left\{ \frac{L/2}{y^2\left(\frac{L^2}{4}+y^2\right)^{1/2}} - \frac{(-L/2)}{y^2\left(\frac{L^2}{4}+y^2\right)^{1/2}} \right\} \\ &= k\lambda y \frac{L}{y^2\left(\frac{L^2}{4}+y^2\right)^{1/2}} \end{aligned}$$

We can cancel a factor of y; also, make the replacements $k = \frac{1}{4\pi\epsilon_0}$ and $\lambda = q/L$. This gives:

$$\begin{aligned} E_y &= \frac{1}{4\pi\epsilon_0}\frac{q}{L}\frac{L}{y\left(\frac{L^2}{4}+y^2\right)^{1/2}} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{y\left(\frac{L^2}{4}+y^2\right)^{1/2}} \end{aligned}$$

We're getting close! We can make the expression look a little neater by pulling a factor of $\frac{1}{4}$ out of the parentheses in the denominator. Use:

$$\left(\frac{L^2}{4}+y^2\right)^{1/2} = \frac{1}{2}\left(L^2+4y^2\right)^{1/2}$$

in our last expression and finally get:

$$\begin{aligned} E_y &= \frac{q}{4\pi\epsilon_0}\frac{2}{y\left(L^2+4y^2\right)^{1/2}} \\ &= \frac{q}{2\pi\epsilon_0 y}\frac{1}{\left(L^2+4y^2\right)^{1/2}} \end{aligned}$$

Since at point P, $\mathbf{E}$ has no x component, the magnitude of the E field is

$$E = \frac{q}{2\pi\epsilon_0 y}\frac{1}{\left(L^2+4y^2\right)^{1/2}}$$

(and the field points in the $+y$ direction for positive q.)

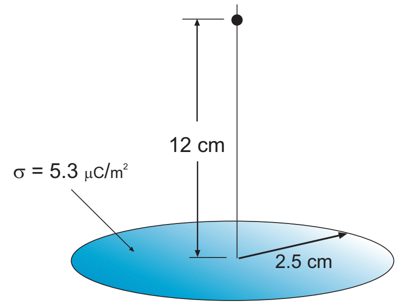

Figure 2.11: Geometry for Example 9.

9. A disk of radius $2.5\,\mathrm{cm}$ has a surface charge density of $5.3\,\frac{\mu\mathrm{C}}{\mathrm{m}^2}$ on its upper face. What is the magnitude of the electric field produced by the disk at a point on its central axis at distance $z = 12\,\mathrm{cm}$ from the disk?

The geometry for this problem is shown in Fig. 2.11. Here we *do* have a formula we can use, Eq. 2.6. With $r = 2.5 \times 10^{-2}\,\mathrm{m}$, $\sigma = 5.3\,\frac{\mu\mathrm{C}}{\mathrm{m}^2}$ and $z = 12 \times 10^{-2}\,\mathrm{m}$ we find:

$$\begin{aligned} E &= \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+r^2}}\right) \\ &= \frac{(5.3\,\frac{\mu\mathrm{C}}{\mathrm{m}^2})}{2\left(8.85\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m}^2}\right)}\left(1 - \frac{(12\times10^{-2}\,\mathrm{m})}{\sqrt{(12\times10^{-2}\,\mathrm{m})^2+(2.5\times10^{-2}\,\mathrm{m})^2}}\right) \\ &= 629\,\frac{\mathrm{N}}{\mathrm{C}} \end{aligned}$$

At the given point, the E field has magnitude $629\,\frac{\mathrm{N}}{\mathrm{C}}$ and points away from the disk.

2.2.3 Forces on Charges in Electric Fields

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10. An electron and a proton are each placed at rest in an electric field of $520\,\mathrm{N/C}$. Calculate the speed of each particle $48\,\mathrm{ns}$ after being released.

Consider the electron. From $\mathbf{F} = q\mathbf{E}$, and the fact that the *magnitude* of the electron's charge is $1.60 \times 10^{-19}\,\mathrm{C}$, the *magnitude* of the force on the electron is

$$F = |q|E = (1.60 \times 10^{-19}\,\mathrm{C})(520\,\mathrm{N/C}) = 8.32 \times 10^{-17}\,\mathrm{N}$$

and since the mass of the electron is $m_{\mathrm{e}} = 9.11 \times 10^{-31}\,\mathrm{kg}$, from Newton's 2$^{\mathrm{nd}}$ Law, the *magnitude* of its acceleration is

$$a = \frac{F}{m_{\mathrm{e}}} = \frac{(8.32 \times 10^{-17}\,\mathrm{N})}{(9.11 \times 10^{-31}\,\mathrm{kg})} = 9.13 \times 10^{13}\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

Since the electron starts from rest ($\mathbf{v}_0 = 0$), we have $\mathbf{v} = \mathbf{a}t$ and so the *magnitude* of its velocity 48 ns after being released is

$$v = at = (9.13 \times 10^{13}\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(48 \times 10^{-9}\,\mathrm{s}) = 4.4 \times 10^6\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

So the final *speed* of the electron is $4.4 \times 10^6\,\frac{\mathrm{m}}{\mathrm{s}}$.

We do a similar calculation for the proton; the only difference is its larger mass. Since the *magnitude* of the proton's charge is the same as that of the electron, the magnitude of the force will be the same:

$$F = 8.32 \times 10^{-17}\,\mathrm{N}$$

But as the proton mass is 1.67×10^{-27} kg, its acceleration has magnitude

$$a = \frac{F}{m} = \frac{(8.32 \times 10^{-17}\,\mathrm{N})}{(1.67 \times 10^{-27}\,\mathrm{kg})} = 4.98 \times 10^{10}\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

And then the magnitude of the velocity 48 ns after being released is

$$v = at = (4.98 \times 10^{10}\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(48 \times 10^{-9}\,\mathrm{s}) = 2.4 \times 10^3\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

So the proton's final speed is $2.4 \times 10^3\,\frac{\mathrm{m}}{\mathrm{s}}$.

---

11. Beams of high–speed protons can be produced in "guns" using electric fields to accelerate the protons. (a) What acceleration would a proton experience if the gun's electric field were $2.00 \times 10^4\,\frac{\mathrm{N}}{\mathrm{C}}$? (b) What speed would the proton attain if the field accelerated the proton through a distance of $1.00\,\mathrm{cm}$?

(a) The proton has charge $+e = 1.60 \times 10^{-19}$ C so in the given (uniform) electric field, the force on the protons has magnitude

$$F = |q|E = eE = (1.60 \times 10^{-19}\,\mathrm{C})(2.00 \times 10^4\,\tfrac{\mathrm{N}}{\mathrm{C}}) = 3.20 \times 10^{-15}\,\mathrm{N}$$

Then we use Newton's second law to get the magnitude of the protons' acceleration. Using the mass of the proton, $m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg,

$$a = \frac{F}{m_{\mathrm{p}}} = \frac{(3.20 \times 10^{-15}\,\mathrm{N})}{(1.67 \times 10^{-27}\,\mathrm{kg})} = 1.92 \times 10^{12}\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

(b) The protons start from rest (this is assumed) and move in one dimension, accelerating with $a_x = 1.92 \times 10^{12}\,\frac{\mathrm{m}}{\mathrm{s}^2}$. If they move through a displacement $x - x_0 = 1.00\,\mathrm{cm}$, then one of our favorite equations of one–dimensional kinematics gives us:

$$v_x^2 = v_{x0}^2 + 2a_x(x - x_0) = 0^2 + 2(1.92 \times 10^{12}\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.00 \times 10^{-2}\,\mathrm{m}) = 3.83 \times 10^{10}\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

so that the final velocity (and speed) is

$$v_x = 1.96 \times 10^5\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

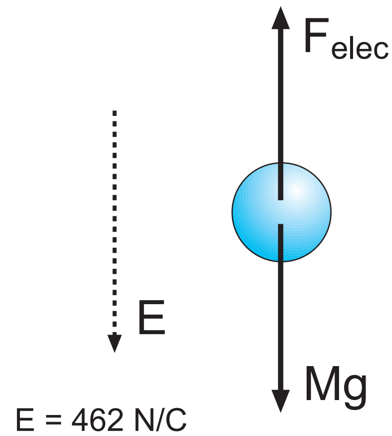

Figure 2.12: Forces acting on the water drop in Example 12.

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12. A spherical water drop $1.20\,\mu$m in diameter is suspended in calm air owing to a downward–directed atmospheric electric field $E = 462\,\frac{\rm N}{\rm C}$. (a) What is the weight of the drop? (b) How many excess electrons does it have?

(a) Not much physics for this part. The volume of the drop is

$$V = \tfrac{4}{3}\pi R^3 = \tfrac{4}{3}\pi(1.20 \times 10^{-6})\,{\rm m}^3 = 9.05 \times 10^{-19}\,{\rm m}^3$$

Assuming that the density of the drop is the usual density of water, $\rho = 1.00 \times 10^3\,\frac{\rm kg}{\rm m^3}$, we can get the mass of the drop from $M = \rho V$. Then the *weight* of the drop is

$$W = Mg = \rho V g = (1.00 \times 10^3\,\tfrac{\rm kg}{m^3})(9.05 \times 10^{-19}\,{\rm m}^3)(9.80\,\tfrac{\rm m}{\rm s^2}) = 8.87 \times 10^{-15}\,{\rm N}$$

(b) The forces acting on the water drop are as shown in Fig. 2.12, namely gravity with magnitude Mg directed downward and the electric force with magnitude $F_{\rm elec} = |q|E$, directed upward. Here, E is the magnitude of the electric field and $|q|$ is the magnitude of the charge on the drop. The net force on the drop is zero, and so this allows us to solve for $|q|$:

$$F_{\rm elec} = |q|E = Mg \qquad \Longrightarrow \qquad |q| = \frac{Mg}{E}$$

Plug in the weight $W = Mg$ found in part (a) and the given value of the E:

$$|q| = \frac{Mg}{E} = \frac{(8.87 \times 10^{-15}\,{\rm N})}{(462\,\frac{\rm N}{\rm C})} = 1.92 \times 10^{-17}\,{\rm C}$$

We know that the drop has a *negative* charge (electric field points down, but the electric *force* points *up*) so that the charge on the drop is

$$q = -1.92 \times 10^{-17}\,{\rm C}\ .$$

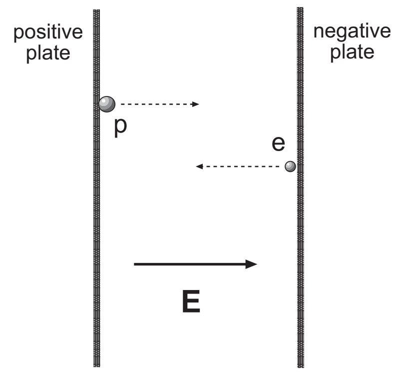

Figure 2.13: Proton and electron are released at the same time and move in opposite directions in a uniform electric field, as given in Example 13.

Figure 2.14: Coordinates for the motion of proton and electron in Example 13.

This negative charge comes from an accumulation of electrons on the water drop. The charge of *one* electron is $q_e = -1.60 \times 10^{-19}$ C. So the number of electrons on the drop must be

$$N = \frac{(-1.92 \times 10^{-17}\,\mathrm{C})}{(-1.60 \times 10^{-19}\,\mathrm{C})} = 120$$

The drop has 120 excess electrons.

---

13. Two large parallel copper plates are 5.0 cm apart and have a uniform electric field between them as depicted in Fig. 2.13. An electron is released from the negative plate at the same time that a proton is released from the positive plate. Neglect the force of the particles on each other and find their distance from the positive plate when they pass each other. (Does it surprise you that you need not know the electric field to solve this problem?

We organize our work by setting up coordinates; we suppose that the proton and electron both move along the x axis, as shown in Fig. 2.14 (though we ignore the force they exert on each other). If the distance between the plates is $D = 5.0\,\mathrm{cm}$ then the proton starts off at $x = 0$ and the electron starts off at $x = D$. The proton will accelerate in the $+x$ direction and the electron will accelerate in the $-x$ direction.

The charge of the proton is $+e$. The electric field between the plates points in the $+x$ direction and has magnitude E. Then the force on the proton has magnitude eE and by

Newton's second law the magnitude of the proton's acceleration is $\frac{eE}{m_\mathrm{p}}$, where m_p is the mass of the proton. Then the $x-$acceleration of the proton is

$$a_{x,\mathrm{p}} = \frac{eE}{m_\mathrm{p}}$$

and since the proton starts from rest, its position is given by

$$x_\mathrm{p} = \tfrac{1}{2}\frac{eE}{m_\mathrm{p}}t^2 \tag{2.16}$$

The charge of the electron is $-e$. Then the force on the electron will have magnitude eE and point in the $-x$ direction. The magnitude of the electron's acceleration is $\frac{eE}{m_\mathrm{e}}$, where m_e is the mass of the electron, but the electron's acceleration points in the $-x$ direction. Then the $x-$acceleration of the electron is

$$a_{x,\mathrm{e}} = -\frac{eE}{m_\mathrm{p}}$$

and since the electron starts from rest and is initially at $x = D$, its position is given by

$$x_\mathrm{e} = D - \tfrac{1}{2}\frac{eE}{m_\mathrm{e}}t^2 \tag{2.17}$$

Clearly, there is some time at which x_p and x_e are equal. This happens when

$$\tfrac{1}{2}\frac{eE}{m_\mathrm{p}}t^2 = D - \tfrac{1}{2}\frac{eE}{m_\mathrm{e}}t^2$$

A little bit of algebra will allow us to solve for t. Regroup some terms:

$$\tfrac{1}{2}\frac{eE}{m_\mathrm{p}}t^2 + \tfrac{1}{2}\frac{eE}{m_\mathrm{e}}t^2 = D$$

$$\tfrac{1}{2}eEt^2\left(\frac{1}{m_\mathrm{p}} + \frac{1}{m_\mathrm{e}}\right) = D$$

$$t^2 = \frac{2D}{eE}\left(\frac{1}{m_\mathrm{p}} + \frac{1}{m_\mathrm{e}}\right)^{-1} \tag{2.18}$$

Taking the square root to get t is not necessary because we want to plug t back into either one of the equations for the coordinates to find the value of x at which the meeting occurred, and both of those equations contain t^2. Putting 2.18 into 2.16 we find:

$$\begin{aligned} x_\mathrm{meet} &= \tfrac{1}{2}\frac{eE}{m_\mathrm{p}}t^2_\mathrm{meet} \\ &= \tfrac{1}{2}\frac{eE}{m_\mathrm{p}}\frac{2D}{eE}\left(\frac{1}{m_\mathrm{p}} + \frac{1}{m_\mathrm{e}}\right)^{-1} \end{aligned}$$

Then we keep doing algebra to get a beautiful, simple form!

$$\begin{aligned} x_{\rm meet} &= \frac{D}{m_{\rm p}}\left(\frac{1}{m_{\rm p}}+\frac{1}{m_{\rm e}}\right)^{-1} \\ &= \frac{D}{m_{\rm p}}\left(\frac{m_{\rm p}m_{\rm e}}{(m_{\rm e}+m_{\rm p})}\right) \\ &= \frac{Dm_{\rm e}}{(m_{\rm e}+m_{\rm p})} \end{aligned} \tag{2.19}$$

Now just plug numbers into Eq. 2.19. We are given D, and we also need:

$$m_{\rm p} = 1.67\times 10^{-27}\,\mathrm{kg} \qquad m_{\rm e} = 9.11\times 10^{-31}\,\mathrm{kg}$$

These give:

$$\begin{aligned} D &= \frac{(5.0\times 10^{-2}\,\mathrm{m})(9.11\times 10^{-31}\,\mathrm{kg})}{(9.11\times 10^{-31}\,\mathrm{kg}+1.67\times 10^{-27}\,\mathrm{kg})} \\ &= 2.73\times 10^{-5}\,\mathrm{m} = 27\,\mu\mathrm{m} \end{aligned}$$

As for the concluding question...

We note that the final answer did not depend on the value of the electric field strength E (a good thing, since it wasn't given). We can think about why this happened: Since at the meeting place both particles had been travelling for the same amount of time, the distances travelled by each are proportional to their accelerations. If we change *both* accelerations by the same factor, the meeting point will occur at the *same* place because the ratio of distances travelled by each particle is the same. So applying the same scale factor to both $a_{\rm p}$ and $a_{\rm e}$ does not change the answer. But changing the value of the electric field *does* apply the same scale factor to both accelerations; since this will not change the answer, we don't expect to see E show up in the expression for $x_{\rm meet}$.

(That was long–winded... do you have a simpler way to see this?)

---

14. The electrons in a particle beam each have a kinetic energy of 1.60×10^{-17} J. What are the magnitude and direction of the electric field that will stop these electrons in a distance of $10.0\,\mathrm{cm}$?

Let's think about the direction first. The acceleration of the electrons must be *directly opposite* their initial (beam) velocities in order for them to come to a halt. So the force on them is also opposite the beam direction. From the *vector* equation $\mathbf{F} = q\mathbf{E}$ we see that if the charge q is negative —as it is for an electron— then the force and electric field have opposite directions. So the electric field must point in the *same* direction as the initial velocities of the electrons (the beam direction). See Fig. 2.15.

It is easiest to use the work–energy theorem to solve the problem. Recall:

$$W_{\rm net} = \Delta K$$

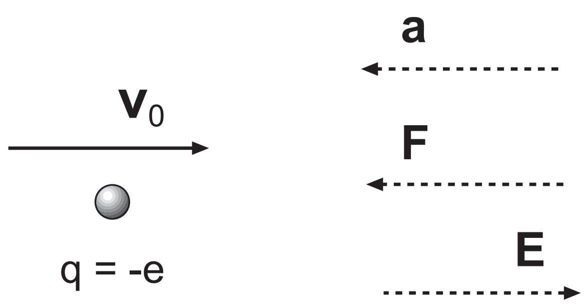

Figure 2.15: Directions for the acceleration, force and (uniform) electric field for Example 14.

As the electron slows to a halt, its change in kinetic energy is

$$\Delta K = K_f - K_i = 0 - (1.60 \times 10^{-17}\,\mathrm{J}) = -1.60 \times 10^{-17}\,\mathrm{J}$$

Suppose the electric force on the electron has magnitude F. The electron moves a distance $s = 10.0\,\mathrm{cm}$ *opposite* the direction of the force so that the work done is

$$W = -Fd = -F(10.0 \times 10^{-2}\,\mathrm{m})$$

which is also the *net* work done. The work–energy theorem says that these are equal, so:

$$-F(10.0 \times 10^{-2}\,\mathrm{m}) = -1.60 \times 10^{-17}\,\mathrm{J}$$

Solve for F:

$$F = \frac{(1.60 \times 10^{-17}\,\mathrm{J})}{(10.0 \times 10^{-2}\,\mathrm{m})} = 1.60 \times 10^{-16}\,\mathrm{N}$$

Now since we know the charge of an electron we can find the magnitude of the electric field. Here we have $E = F/|q| = F/e$, so the magnitude of the E field is

$$E = \frac{F}{e} = \frac{(1.60 \times 10^{-16}\,\mathrm{N})}{(1.60 \times 10^{-19}\,\mathrm{C})} = 1.00 \times 10^{3}\,\tfrac{\mathrm{N}}{\mathrm{C}}$$

We have now found both the magnitude *and* direction of the E field.

Chapter 3

Gauss'(s) Law

3.1 The Important Stuff

3.1.1 Introduction; Grammar

This chapter concerns an important mathematical result which relates the electric *field* in a certain region of space with the electric *charges* found in that same region. It is useful for finding the value of electric field in situations where the charged objects are highly symmetrical. It is also valuable as an alternate mathematical expression of the inverse–square nature of the electric field from a point charge (Eq. 2.3).

Alas, physics textbooks can't seem to agree on the *name* for this law, discovered by Gauss. Some call it **Gauss' Law**. Others call it **Gauss's Law**. Do we need the extra "s" after the apostrophe or not? *Physicists do not yet know the answer to this question!!!!*

3.1.2 Electric Flux

The concept of **electric flux** involves a *surface* and the (vector) values of the electric field at all points of the surface. To introduce the way that flux is calculated, we start with a simple case. We will consider a *flat* surface of area A and an electric field which is *constant* (that is, has the same vector value) over the surface.

The surface is characterized by the "area vector" $\mathbf{A}$. This is a vector which points perpendicularly (normal) to the surface and has magnitude A. The surface and its area vector along with the uniform electric field are shown in Fig. 3.1.

Actually, there's a little problem here: There are really *two* choices for the vector $\mathbf{A}$. (It could have been chosen to point in the opposite direction; it would still be normal to the surface and have the same magnitude.) However in every problem where we use electric flux, it will be made clear which choice is made for the "normal" direction.

Now, for this simple case, the electric flux Φ is given by

$$\Phi = \mathbf{E} \cdot \mathbf{A} = EA\cos\theta$$

where theta is the angle between $\mathbf{E}$ and $\mathbf{A}$.

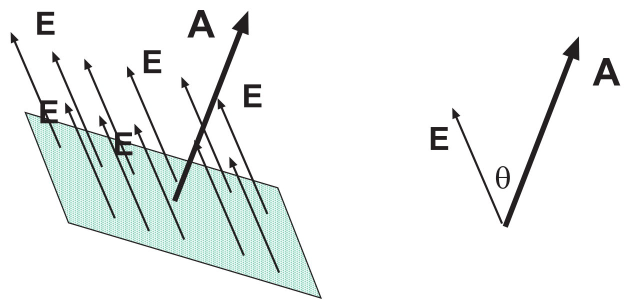

Figure 3.1: Electric field **E** is uniform over a flat surface whose area vector is **A**.

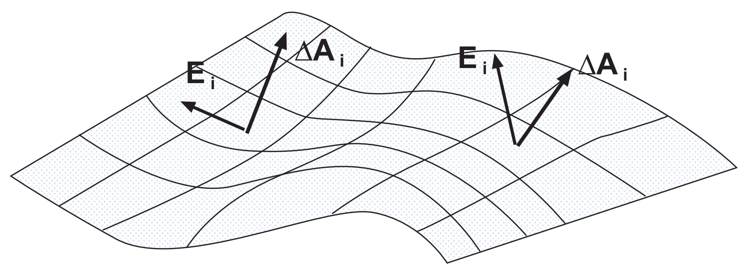

Figure 3.2: How flux is calculated (conceptually) for a general surface. Divide up the big surface into small squares; for each square find the area vector $\Delta\mathbf{A}_i$ and average electric field $\mathbf{E}_i$. Take $\Delta\mathbf{A}_i \cdot \mathbf{E}_i$ and add up the results for all the little squares.

We see that electric flux is a *scalar* and has units of $\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{C}}$.

In general, a surface is *not* flat, and the electric field will not be uniform (either in magnitude or direction) over the surface. In practice one must use the machinery of advanced calculus to find the flux for the general case, but it is not hard to get the basic idea of the process: We divide up the surface into little sections (say, squares) which are all small enough so that it is a good approximation to *treat* them as flat *and* small enough so that the electric field $\mathbf{E}$ is reasonably constant. Suppose the ith little square has area vector $\Delta\mathbf{A}_i$ and the value of the electric field on that square is close to $\mathbf{E}_i$. Then the electric flux for the little square is found as before,

$$\Delta\Phi_i = \Delta\mathbf{A}_i \cdot \mathbf{E}_i$$

and the electric flux for the whole surface is *roughly* equal to the sum of all the individual contributions:

$$\Phi \approx \sum_i \Delta\mathbf{A}_i \cdot \mathbf{E}_i$$

The procedure is illustrated in Fig. 3.2.

The procedure outlined above gets closer and closer to the real value of the electric flux Φ when we make the little squares more numerous and smaller. A similar procedure in beginning calculus gives an integral (for one variable). Here, we arrive at a **surface integral** and the *proper* way to write out definition of the electric flux over the surface S is

$$\Phi = \int_S \mathbf{E} \cdot d\mathbf{A} \tag{3.1}$$

3.1.3 Gaussian Surfaces

For now at least, we are only interested in finding the flux for a special class of surfaces, ones which we call **Gaussian surfaces**. Such a surface is a *closed* surface... that is, it encloses a particular volume of space and doesn't have any holes in it. In principle it have any shape at all, but in our problem–solving we will have the most use for surfaces which have a high degree of symmetry, for example spheres and cylinders.

When we find the electric flux on a *closed* surface, it is agreed that the unit normal for all the little surface elements $d\mathbf{A}$ points *outward*.

There is a special notation for a surface integral done over a *closed* surface; the integral sign will usually have a circle superimposed on it. Thus for a *Gaussian* surface S, the electric flux is written

$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} \tag{3.2}$$

We will be considering Gaussian surfaces constructed around different configurations of charges, configurations for which we are interested in finding the electric field. We get an interesting result for the electric flux for a Gaussian surface when it *encloses some electric charge*... and also when it doesn't!

3.1.4 Gauss'(s) Law

Suppose we choose a *closed* surface S in some environment where there are charges and electric fields. We can (in principle, at least) compute the electric flux Φ on S. We can also find the total electric charge enclosed by the surface S, which we will call $q_{\rm enc}$. **Gauss'(s) Law** tells us:

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\rm enc}}{\epsilon_0} \tag{3.3}$$

3.1.5 Applying Gauss'(s) Law

Gauss'(s) Law is used to find the electric field for charge distributions which have a symmetry which we can exploit in calculating both sides of the equation: $\oint \mathbf{E} \cdot d\mathbf{A}$ and $q_{\rm enc}/\epsilon_0$.

• **Point Charge**

Of course, we already know how to get the magnitude and direction of the electric field due to a point charge q. Here we show how this result follows from Gauss'(s) Law. (The purpose here is to give a patient discussion of how we get a *known* result so that we can use Gauss'(s) Law to obtain *new* results.

We imagine a spherical surface of radius r centered on q, as shown in Fig. 3.3. The spherical shape takes advantage of the fact that a single point gives no preferred direction in space. When we are done with the calculation, we will know the electric field for any point a distance r away from the charge.

Having drawn the proper surface, we have to use a little "common sense" for determining the direction of the electric field. From symmetry we can see the the the E field *must* point radially. Imagine looking at the point charge from any direction. It doesn't look any different!

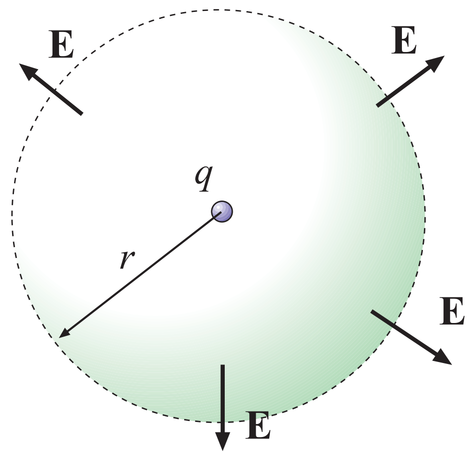

Figure 3.3: Gaussian surface of radius r centered on a point charge q. Symmetry dictates that the E field must point in the radial direction so that for points on the surface it is (locally) perpendicular to the surface.

But if the electric field's direction were anything but radial (straight out from the charge) we *could* distinguish the direction from which we were observing the charge.

Furthermore at a given distance r from the charge, the magnitude of the E field must be the same, although for all we know right now it could depend upon r. So we conclude that at all points of our (spherical) surface the E field is radial everywhere and its magnitude is the same. This fact is indicated in Fig. 3.3.

With this very reasonable assumption about $\mathbf{E}$ we can evaluate $\oint \mathbf{E}\cdot d\mathbf{A}$ *without explicitly doing any integration.* We note that every where on the surface the vector $\mathbf{E}$ is *parallel* to the area vector $d\mathbf{A}$, so that $\mathbf{E}\cdot d\mathbf{A} = E\,dA$. Since the magnitude of E is constant over the surface it can be taken outside the integral sign:

$$\oint \mathbf{E}\cdot d\mathbf{A} = \oint E dA = E \oint dA \ .$$

But the expression $\oint dA$ just tells us to add up all the area elements of the surface, giving us the total area of the spherical surface, which is $4\pi r^2$. So we find:

$$\oint \mathbf{E}\cdot d\mathbf{A} = E(4\pi r^2) \ .$$

Now the charge enclosed by the Gaussian surface is simply q, that is:

$$q_{\rm enc} = q \ .$$

Putting these facts into Gauss'(s) Law (Eq. 3.3) we have:

$$E(4\pi r^2) = \frac{q}{\epsilon_0} \qquad \Longrightarrow \qquad E = \frac{q}{4\pi\epsilon_0 r^2} \ ,$$

which we know is the correct answer for the electric field due to a point charge q.

• **Spherically–Symmetric Distributions of Charge**

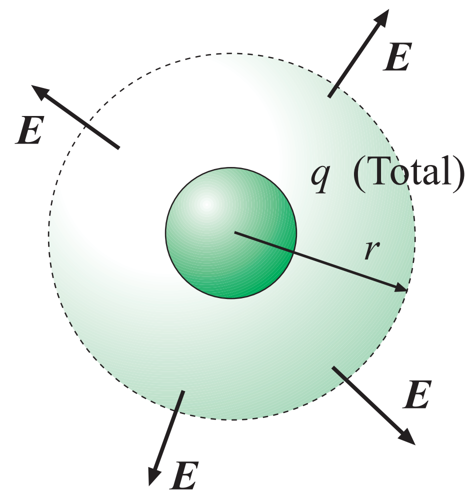

Figure 3.4: Gaussian surface of radius r centered on *spherically symmetric* charge distribution with total charge q. E field points radially outward on the surface.

Using Gauss'(s) Law and a spherical Gaussian surface, we can find the electric field outside of *any* spherically symmetric distribution of charge. Suppose we have a ball with total charge q, where the charge density only depends on the distance from the center of the ball. (That is to say, is has spherical symmetry.) We can draw a Gaussian surface of radius r (r being large than the radius of the ball) and use the same arguments as for the point charge to find the electric field.

We again argue that since the system looks the same regardless of the direction from which we view it, the E field on the spherical surface must point in the radial direction. (See Fig. 3.4.) So for the surface integral in Gauss'(s) Law, we get *exactly* the same thing we had before:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint E dA = E(4\pi r^2) \ .$$

As for the charge enclosed, since the *total* charge is given as q, Gauss'(s) Law gives us:

$$E(4\pi r^2) = \frac{q}{\epsilon_0}$$

so as before we find that the magnitude of the electric field at a distance r from the center of the ball of charge is

$$E = \frac{q}{4\pi\epsilon_0 r^2}$$

From a mathematical point of view, this result is quite interesting. It is the same as the field due to a point charge (as long as r is bigger than the ball's radius). The exact nature of the distribution of charge does not matter, just so long as it is spherically symmetric and its total is q. If you were to try to calculate the electric field *explicitly* by doing a integral over the volume elements of the sphere it would be a lot of work! Using Gauss'(s) Law the calculation is very easy.

As a further example involving spherical symmetry, we consider a *hollow* spherically-symmetric charge distribution. We can find the value of the electric field *inside* all of charge.

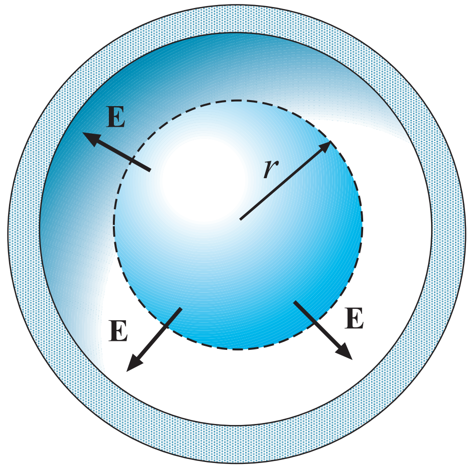

Figure 3.5: Gaussian surface of radius r centered in the interior of a *spherically symmetric* charge distribution with total charge q. $\mathbf{E}$ must point in the radial direction everywhere on the surface, but in fact $\mathbf{E}$ is zero.

To do this we once again draw a spherical Gaussian surface, this time of radius r, where r is smaller than the inner radius of the hollow ball.

What can we say about the electric field on *this* Gaussian surface? Symmetry tells us exactly the same thing as before: The electric field (if there is one!) must point in the radial direction because of the symmetry of the problem, and it must have the same magnitude everywhere on the surface. This is shown in Fig. 3.5. So again we have

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint E dA = E(4\pi r^2) \ .$$

But this time the Gaussian surface *encloses no charge at all*. So Gauss'(s) Law gives

$$E(4\pi r^2) = \frac{q_{\mathrm{enc}}}{\epsilon_0} = 0$$

so that $E = 0$ anytime we are inside the hollow sphere of charge. This result comes about very simply using Gauss'(s) Law but it is rather challenging to show it by doing all the integrals by hand.

• Other Geometries

We can use Gauss'(s) Law to find the electric field around other charge distributions which have some type of symmetry, but we need to chose Gaussian surfaces of different shapes in order to take advantage of the symmetry.

If a charge distribution has symmetry about an axis (that is, *cylindrical* symmetry, like a long line of charge) then it is most useful to choose a *cylindrical* Gaussian surface, as shown in Fig. 3.6.

Using a cylindrical Gaussian surface, one can show that for a line of charge with a (positive) linear charge density λ, the electric field $\mathbf{E}$ at a distance r from the points radially outward and has magnitude

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \tag{3.4}$$

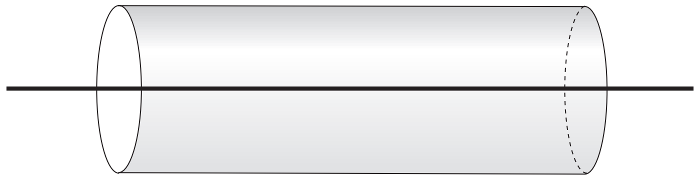

Figure 3.6: Gaussian surface for a line charge or more generally a distribution with *cylindrical* symmetry.

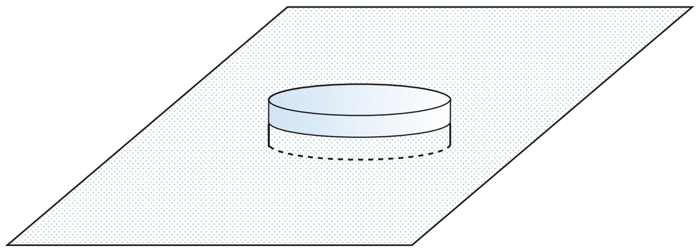

Figure 3.7: Gaussian surface for a sheet of charge (or, more generally, a charge distribution with planar symmetry).

If the charge density λ is negative, the E field points radially inward with a magnitude given by 3.4 with λ being the magnitude of the charge density.

If a charge distribution has planar symmetry that is, it stretches out uniformly and forever in the x and y directions) then it turns out to be quite useful to choose a Gaussian surface shaped like a "pillbox", that is, a cylindrical shape of very small thickness. Such a construction is shown in Fig. 3.7.

Using such a "pillbox" Gaussian surface, one can show that for a plane of charge with a (positive) charge density (charge per unit area) σ, the electric field $\mathbf{E}$ points outward from the sheet and has magnitude

$$E = \frac{\sigma}{2\epsilon_0} \tag{3.5}$$

If the charge density is *negative* the electric field points inward toward the sheet and has a magnitude given by 3.5 with σ being the magnitude of the charge density. This is the same result as Eq. 2.7.

Note that the magnitude of the E field in 3.5 does not depend on the distance from the sheet of charge.

3.1.6 Electric Fields and Conductors

For the *electrostatic* conditions that we are considering all through Vol. 4, the electric field is *zero* inside any conductor. Using Gauss'(s) law it follows that if a conductor carries any net charge, the charge will reside on the surface(s) of the conductor.

Also using Gauss'(s) law one can show that the electric field just outside a conducting surface is perpendicular to the surface and is given by

$$E = \frac{\sigma}{\epsilon_0} \tag{3.6}$$

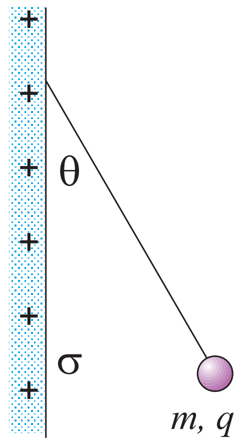

Figure 3.8: Example 1.

where σ is the surface charge density at the chosen location on the conductor and where we mean that if σ is positive the E field points outward and if it is negative the E field points toward the surface. Note that Eq. 3.6 differs from Eq. 3.5; the reasons are subtle! Be careful in choosing which one to use!

3.2 Worked Examples

3.2.1 Applying Gauss'(s) Law

1. In Fig. 3.8, a small nonconducting ball of mass $m = 1.0\,\mathrm{mg}$ and charge $q = 2.0 \times 10^{-8}\,\mathrm{C}$ (distributed uniformly through its volume) hangs from an insulating thread that makes an angle $\theta = 30^\circ$ with a vertical, uniformly charged nonconducting sheet (shown in cross section). Considering the gravitational force on the ball and assuming that the sheet extends far vertically and into and out of the page, calculate the surface charge density σ of the sheet.

Draw a free–body diagram for the sphere! This is done in Fig. 3.9. The forces on the ball are gravity, mg downward, the tension in the string T and the force of electrostatic repulsion (F_{elec}, straight out from the sheet), arising from the sheet of positive charges. We know that the electrostatic force must point straight out from the sheet because the electric field arising from the charge points straight out, so the force exerted on the ball must point straight out as well. (We can assume the ball acts like a point charge with the charge concentrated at its center.)

First, find F_{elec}. The ball is in static equilibrium, so that the vertical and horizontal forces sum to zero. This gives us the equations:

$$T\cos\theta = mg \qquad\qquad T\sin\theta = F_{\mathrm{elec}}$$

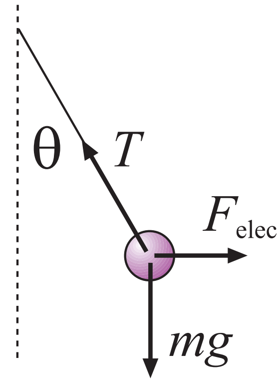

Figure 3.9: Free–body diagram for the small ball in Example 1.

Divide the second of these by the first to cancel out T and give:

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{F_{\text{elec}}}{mg} \qquad \Longrightarrow \qquad F_{\text{elec}} = mg\tan\theta$$

Plug in the numbers (note: $1.0\,\text{mg} = 1.0\times10^{-6}\,\text{kg}$) and get

$$F_{\text{elec}} = (1.0\times10^{-6}\,\text{kg})(9.80\,\tfrac{\text{m}}{\text{s}^2})\tan30^\circ = 5.7\times10^{-6}\,\text{N}$$

From $F_{\text{elec}} = |q|E$ we can get the magnitude of the electric field:

$$E = F_{\text{elec}}/|q| = (5.7\times10^{-6}\,\text{N})/(2.0\times10^{-8}\,\text{C}) = 2.8\times10^{2}\,\tfrac{\text{N}}{\text{C}}$$

This is the magnitude of an E field on one side of an infinite sheet of charge so that from Eq. 3.5 we can find the charge density of the sheet:

$$\sigma = 2\epsilon_0 E = 2(8.85\times10^{-12}\,\tfrac{\text{C}^2}{\text{N}\cdot\text{m}^2})(2.8\times10^{2}\,\tfrac{\text{N}}{\text{C}}) = 5.0\times10^{-9}\,\tfrac{\text{C}}{\text{m}^2} = 5.0\,\tfrac{\text{nC}}{\text{m}^2}$$

Since the E field points away from the sheet, this is the correct sign for the charge density; the charge density of the sheet is $+5.0\,\frac{\text{nC}}{\text{m}^2}$.

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2. In a 1911 paper, Ernest Rutherford said: "In order to form some idea of the forces required to deflect an α particle through a large angle, consider an atom [as] containing a point positive charge Ze at its center and surrounded by a distribution of negative electricity $-Ze$ uniformly distributed within a sphere of radius R. The electric field E at a distance r from the center for a point *inside* the atom [is]

$$E = \frac{Ze}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right) \ ."$$

Verify this equation.

Rutherford's model of the atom is shown in Fig. 3.10(a). The charge density of the

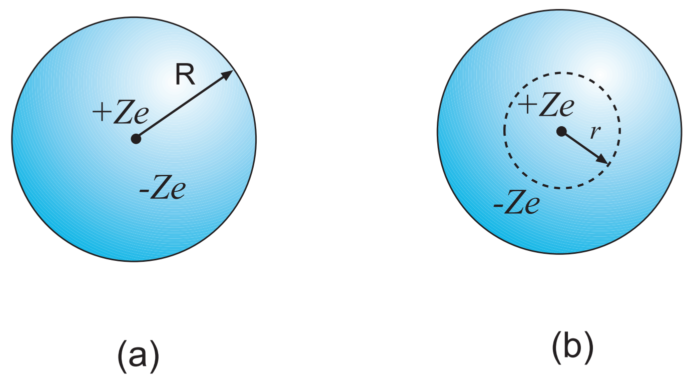

Figure 3.10: (a) Rutherford's atomic model. Point charge $+Ze$ is at the center, with a ball of uniform charge density of radius R and total charge $-Ze$ surrounding it. (b) Spherical Gaussian surface of radius r.

distribution of "negative electricity" is found by dividing the total charge $-Ze$ by the volume of the ball:

$$\rho_{-Ze} = \frac{-Ze}{\frac{4}{3}\pi R^3} = -\frac{3Ze}{4\pi R^3}$$

To find the electric field at a distance r from the center (where $r < R$), we will assume from the spherical symmetry of the problem that the $\mathbf{E}$ field points *radially*, and its magnitude depends on the distance from the center, r. Then a spherical Gaussian surface will be useful, and such a surface is shown in Fig. 3.10(b). The surface has radius r and is centered on the point charge $+Ze$. Since the $\mathbf{E}$ field is (assumed) radial, the surface integral of $\mathbf{E}$ will give a simple result, and it won't be too hard to find the charge enclosed by this surface. Then Gauss'(s) Law will give us E.

What is the charge enclosed by the surface in Fig. 3.10(b)? It encloses the point charge $+Ze$ but it also encloses *some* of the continuous charge distribution. How much? The volume of our surface is $\frac{4}{3}\pi r^3$ and multiplying this volume by the charge density found above gives the amount of the charge from the ball of negative charge which is enclosed. Thus the *total* charge enclosed by the surface is

$$\begin{aligned} q_{\rm enc} &= +Ze + \left(\tfrac{4}{3}\pi r^3\right)\left(\frac{-3Ze}{4\pi R^3}\right) = +Ze - Ze\left(\frac{r^3}{R^3}\right) \\ &= Ze\left(1 - \frac{r^3}{R^3}\right) \end{aligned}$$

(Notice that when $r = R$ the total charge enclosed is *zero*, as it should be.)

Now, the surface integral of $\mathbf{E}$ is just the (common) magnitude of $\mathbf{E}$ on the surface multiplied by its area, $4\pi r^2$. Putting *all* of this together, Gauss'(s) Law gives us:

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{q_{\rm enc}}{\epsilon_0} \qquad \Longrightarrow \qquad E(4\pi r^2) = \frac{Ze}{\epsilon_0}\left(1 - \frac{r^3}{R^3}\right)$$

Divide through by $4\pi r^2$ to get E, the radial component of the $\mathbf{E}$ field inside the "atom"

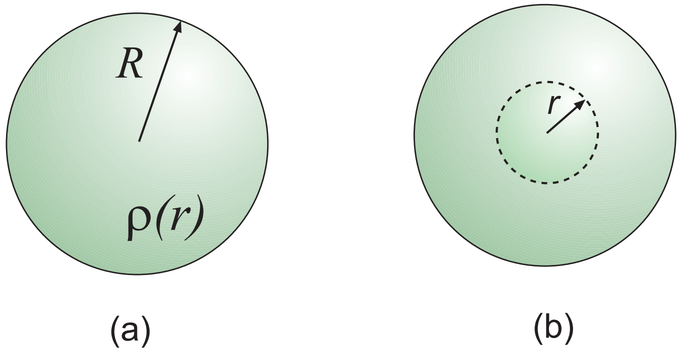

Figure 3.11: (a) Ball of charge, radius R. The charge density depends on the distance r. (b) Spherical Gaussian surface of radius r drawn inside the sphere.

(which is also the magnitude of the **E** field):

$$E = \frac{Ze}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right)$$

---

3. A solid nonconducting sphere of radius R has a nonuniform charge distribution of volume charge density $\rho = \rho_s r/R$, where ρ_s is a constant and r is the distance from the center of the sphere. Show (a) that the total charge on the sphere is $Q = \pi\rho_s R^3$ and (b) that

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^4}r^2$$

gives the magnitude of the electric field inside the sphere.

(a) The ball of charge with nonuniform density $\rho(r)$ is drawn in Fig. 3.11(a). To get the total charge, integrate the charge density $\rho(\mathbf{r})$ over the volume of the sphere. (We must do an integral since the density is not uniform.) When integrating functions like $\rho(r)$ which depend only on the distance r over a spherical volume, we multiply $\rho(r)$ by the volume of the spherical shell element $4\pi r^2 dr$ and sum up from $r = 0$ to $r = R$:

$$Q = \int_{\text{sphere}} \rho(r)d\tau = \int_0^R \rho(r)4\pi r^2 dr$$

Substitute the given expression for $\rho(r)$ and get:

$$\begin{aligned} Q &= \int_0^R \frac{\rho_s r}{R}(4\pi)r^2 dr = \frac{4\pi\rho_s}{R}\int_0^R r^3 dr = \frac{4\pi\rho_s}{R}\frac{r^4}{4}\bigg|_0^R \\ &= \frac{4\pi\rho_s}{R}\frac{R^4}{4} = \pi\rho_s R^3 \end{aligned}$$

(b) To find the E field inside the sphere: Assume that the E field points in the radial direction (from the spherical symmetry of the problem). Imagine a spherical Gaussian surface of radius

r centered on the center of the charge distribution, as drawn in Fig. 3.11(b). Then the surface integral of $\mathbf{E}$ will have a simple expression and if we can calculate the charge enclosed by this surface, we can find E using Gauss'(s) Law.

To get the enclosed charge, perform an integral as in part (a) but this time only integrate out to the radius r. This gives us:

$$\begin{aligned} q_{\rm enc} &= \int_0^r \rho(r')(4\pi r'^2)dr' = \int_0^r \frac{\rho_s r'}{R}(4\pi r'^2)dr' \\ &= \frac{4\pi\rho_s}{R}\int_0^r r'^3 dr' = \frac{4\pi\rho_s}{R}\frac{r^4}{4} = \frac{\pi\rho_s r^4}{R} \end{aligned}$$

As usual (yawn) the surface integral of the E field over the spherical Gaussian surface is

$$\oint \mathbf{E}\cdot d\mathbf{A} = E(4\pi r^2)$$

Putting all of this into Gauss'(s) Law, we find:

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{q_{\rm enc}}{\epsilon_0} \qquad \Longrightarrow \qquad E(4\pi r^2) = \frac{\pi\rho_s r^4}{\epsilon_0 R}$$

Divide out the $4\pi r^2$ and get:

$$E = \frac{\rho_s r^2}{4\epsilon_0 R}$$

This answer is correct, but we would like to express E in terms of the *total charge* of the sphere (instead of ρ_s). In part (a) we found that we can write:

$$Q = \pi\rho_s R^3 \qquad \Longrightarrow \qquad \rho_s = \frac{Q}{\pi R^3}$$

so then our answer for E is

$$E = \frac{r^2}{4\epsilon_0 r}\left(\frac{Q}{\pi R^3}\right) = \frac{Qr^2}{4\pi\epsilon_0 R^4}$$

This is the radial component of the E field as well as its magnitude.

3.2.2 Electric Fields and Conductors

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4. An isolated conductor of arbitrary shape has a net charge of $+10\times10^{-6}$ C. Inside the conductor is a cavity within which is a point charge $q = +3.0\times10^{-6}$ C. What is the charge (a) on the cavity wall and (b) on the outer surface of the conductor?

(a) The system is shown in Fig. 3.12(a). Consider any Gaussian surface which lies within the material of the conductor and encloses the cavity, as shown in Fig. 3.12(b). Since $\mathbf{E} = 0$

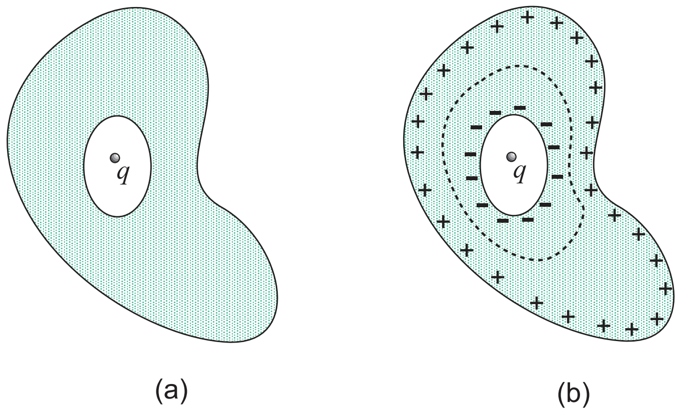

Figure 3.12: (a) Conductor carrying a net charge has a cavity inside of it. Cavity contains a charge $q = 3.0 \times 10^{-6}\,\mathrm{C}$. (b) Charges in the conductor arrange themselves; negative charges collect on the inner surface; the Gaussian surface shown must enclose a net charge of zero! An even larger number of positive charges collect on the outer surface since the conductor must have a net positive charge.

everywhere inside the conductor the integral $\oint \mathbf{E} \cdot d\mathbf{A}$ on this surface gives zero and then by Gauss'(s) Law the total charge enclosed is zero. The surface encloses the charge $q = +3.0 \times 10^{-6}\,\mathrm{C}$ and also the charge which accumulates on the inner surface of the conductor. This implies that the charge which collects on the inner surface is $q_{\mathrm{inner}} = -3.0 \times 10^{-6}\,\mathrm{C}$.

(b) The rest of the free charge on the conductor accumulates on the outer surface. But the total must come out to $+10 \times 10^{-6}\,\mathrm{C}$, as advertised! Thus:

$$q_{\mathrm{outer}} + q_{\mathrm{inner}} = +10 \times 10^{-6}\,\mathrm{C} \ .$$

Then:

$$\begin{aligned} q_{\mathrm{outer}} &= +10 \times 10^{-6}\,\mathrm{C} - q_{\mathrm{inner}} \\ &= +10 \times 10^{-6}\,\mathrm{C} - (-3.0 \times 10^{-6}\,\mathrm{C}) = 13 \times 10^{-6}\,\mathrm{C} \end{aligned}$$

The charge on the outer surface is $+13 \times 10^{-6}\,\mathrm{C}$.

Chapter 4

The Electric Potential

4.1 The Important Stuff

4.1.1 Electrical Potential Energy

A charge q moving in a constant electric field $\mathbf{E}$ experiences a force $\mathbf{F} = q\mathbf{E}$ from that field. Also, as we know from our study of work and energy, the work done on the charge by the field as it moves from point $\mathbf{r}_1$ to $\mathbf{r}_2$ is

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{s}$$

where we mean that we are summing up all the tiny elements of work $dW = \mathbf{F} \cdot d\mathbf{s}$ along the length of the path. When $\mathbf{F}$ is the electrostatic force, the work done is

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} q\mathbf{E} \cdot d\mathbf{s} = q \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{s} \tag{4.1}$$

In Fig. 4.1, a charge is shown being moved from $\mathbf{r}_1$ to $\mathbf{r}_2$ along two different paths, with $d\mathbf{s}$ and $\mathbf{E}$ shown for a bit of each of the paths.

Now it turns out that from the mathematical form of the electrostatic force, the work done by the force does not depend on the path taken to get from $\mathbf{r}_1$ to $\mathbf{r}_2$. As a result we say

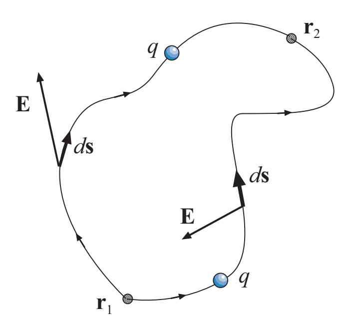

Figure 4.1: Charge is moved from $\mathbf{r}_1$ to $\mathbf{r}_2$ along two separate paths. Work done by the electric force involves the summing up $\mathbf{E} \cdot d\mathbf{s}$ along the path.

that the electric force is **conservative** and it allows us to calculate an **electric potential energy**, which as usual we will denote by U. As before, only the changes in the potential have any real meaning, and the change in potential energy is the negative of the work done by the electric force:

$$\Delta U = -W = -q\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\mathbf{s} \tag{4.2}$$

We usually want to discuss the potential energy of a charge *at a particular point*, that is, we would like a function $U(\mathbf{r})$, but for this we need to make a definition for the potential energy *at a particular point*. Usually we will make the choice that the potential energy is zero when the charge is infinitely far away: $U_\infty = 0$.

4.1.2 Electric Potential

Recall how we developed the concept of the electric field $\mathbf{E}$: The force on a charge q_0 is always proportional to q_0, so by dividing the charge out of $\mathbf{F}$ we get something which can conveniently give the force on *any* charge. Likewise, if we divide out the charge q from Eq. 4.2 we get a function which we can use to get the change in potential energy for any charge (simply by multiplying by the charge). This new function is called the **electric potential**, V:

$$\Delta V = \frac{\Delta U}{q}$$

where ΔU is the change in potential energy of a charge q. Then Eq. 4.2 gives us the difference in electrical potential between points $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\Delta V = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\mathbf{s} \tag{4.3}$$

The electric potential is a *scalar*. Recalling that it was defined by dividing *potential energy* by *charge* we see that its units are $\frac{\mathrm{J}}{\mathrm{C}}$ (joules per coulomb). The electric potential is of such great importance that we call this combination of units a **volt**[1]. Thus:

$$1\,\mathrm{volt} = 1\,\mathrm{V} = 1\tfrac{\mathrm{J}}{\mathrm{C}} \tag{4.4}$$

Of course, it is then true that a joule is equal to a coulomb-volt=C · V. In general, multiplying a charge times a potential difference gives an *energy*. It often happens that we are multiplying an elementary charge (e) (or some multiple thereof) and a potential difference in volts. It is then convenient to use the unit of energy given by the product of e and a volt; this unit is called the **electron-volt**:

$$1\,\mathrm{eV} = (e)\cdot(1\,\mathrm{V}) = 1.60\times 10^{-19}\,\mathrm{C}\cdot(1\,\mathrm{V}) = 1.60\times 10^{-19}\,\mathrm{J} \tag{4.5}$$

Equation 4.3 can only give us the *differences* in the value of the electric potential between two points $\mathbf{r}_1$ and $\mathbf{r}_2$. To arrive at a function $V(\mathbf{r})$ defined at all points we need to specify

[1] Named in honor of the. . . uh. . . French physicist Jim Volt (1813–1743) who did some electrical experiments in. . . um. . . Bologna. That's it, Bologna.

a point at which the potential V is *zero*. Often we will choose this point to be "infinity" (∞) that is, as we get very far away from the set of charges which give the electric field, the potential V becomes very small in absolute value. However this "reference point" can be chosen anywhere and for each problem we need to be sure *where* it is understood that $V = 0$ before we can sensibly talk about the function $V(\mathbf{r})$. Then in Eq. 4.3 equal to this reference point and calculate an potential *function* $V(\mathbf{r})$ for all other points. So we can write:

$$V(\mathbf{r}) = -\int_{\mathbf{r}_{\rm ref}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{s} \tag{4.6}$$

4.1.3 Equipotential Surfaces

For a given configuration of charges, a set of points where the electric potential $V(\mathbf{r})$ has a given value is called an **equipotential surface**. It takes no work to move a charged particle from one point on such a surface to another point on the surface, for then we have $\Delta V = 0$.

From the relations between $\mathbf{E}(\mathbf{r})$ and $V(\mathbf{r})$ it follows that the field lines are perpendicular to the equipotential surfaces everywhere.

4.1.4 Finding E from V

The definition of V an *integral* involving the $\mathbf{E}$ field implies that the electric field comes from V by taking *derivatives*:

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z} \tag{4.7}$$

These relations can be written as one equation using the notation for the gradient:

$$\mathbf{E} = -\nabla V \tag{4.8}$$

4.1.5 Potential of a Point Charge and Groups of Points Charges

Using Eq. 4.3, one can show that if we specify that the electrical potential is zero at "infinity", then the potential due to a point charge q is

$$V(\mathbf{r}) = k\frac{q}{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{4.9}$$

where r is the distance of the charge from the point of interest. Furthermore, for a set of point charges q_1, q_2, q_3, ... the electrical potential is

$$V(\mathbf{r}) = \sum_i k\frac{q_i}{r_i} = \sum_i \frac{1}{4\pi\epsilon_0}\frac{q_i}{r_i} \tag{4.10}$$

where r_i is the distance of each charge from the point of interest.

Using Eq. 4.10, one can show that the potential due to an electric dipole with magnitude p at the origin (pointing upward along the z axis) is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} \tag{4.11}$$

Here, r and θ have the usual meaning in spherical coordinates.

4.1.6 Potential Due to a Continuous Charge Distribution

To get the electrical potential due to a continuous distribution of charge (with $V = 0$ at infinity assumed), add up the contributions to the potential; the potential due to a charge dq at distance r is $dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r}$ so that we must do the integral

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\mathbf{r})d\tau}{r} \tag{4.12}$$

In the last expression we are using the charge density $\rho(\mathbf{r})$ of the distribution to get the element of charge dq for the volume element $d\tau$.

4.1.7 Potential Energy of a System of Charges

The potential energy of a pair of point charges (i.e. the work W needed to bring point charges q_1 and q_2 from infinite separation to a separation r) is

$$U = W = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r} \tag{4.13}$$

For a larger set of charges the potential energy is given by the sum

$$U = U_{12} + U_{23} + U_{13} + \ldots = \frac{1}{4\pi\epsilon_0}\sum_{\text{pairs } ij}\frac{q_iq_j}{r_{ij}} \tag{4.14}$$

Here r_{ij} is the distance between charges q_i and q_j. Each pair is only counted *once* in the sum.

4.2 Worked Examples

4.2.1 Electric Potential

1. The electric potential difference between the ground and a cloud in a particular thunderstorm is 1.2×10^9 V. What is the magnitude of the change in energy (in multiples of the electron-volt) of an electron that moves between the ground and the cloud?

The *magnitude* of the change in potential as the electron moves between ground and cloud (we don't care which way) is $|\Delta V| = 1.2 \times 10^9\,\mathrm{V}$. Multiplying by the magnitude of the electron's charge gives the *magnitude* of the change in potential energy. Note that lumping "e" and "V" together gives the eV (electron-volt), a unit of energy:

$$|\Delta U| = |q\Delta V| = e(1.2 \times 10^9\,\mathrm{V}) = 1.2 \times 10^9\,\mathrm{eV} = 1.2\,\mathrm{GeV}$$

---

2. An infinite nonconducting sheet has a surface charge density $\sigma = 0.10\mu\mathrm{C/m^2}$ on one side. How far apart are equipotential surfaces whose potentials differ by $50\,\mathrm{V}$?

In Chapter 3, we encountered the formula for the electric field due a nonconducting sheet of charge. From Eq. 3.5, we had: $E_z = \sigma/(2\epsilon_0)$, where σ is the charge density of the sheet, which lies in the xy plane. So the plane of charge in this problem gives rise to an E field:

$$\begin{aligned} E_z &= \frac{\sigma}{2\epsilon_0} \\ &= \frac{(0.10 \times 10^{-6}\,\frac{\mathrm{C}}{\mathrm{m^2}}}{2(8.85 \times 10^{-12}\,\frac{\mathrm{C^2}}{\mathrm{N\cdot m^2}})} = 5.64 \times 10^3\,\tfrac{\mathrm{N}}{\mathrm{C}} \end{aligned}$$

Here the E field is *uniform* and also $E_x = E_y = 0$.

Now, from Eq. 4.7 we have

$$\frac{\partial V}{\partial z} = -E_z = -5.64 \times 10^3\,\tfrac{\mathrm{N}}{\mathrm{C}} \ .$$

and when the rate of change of some quantity (in this case, with respect to the z coordinate) is constant we can write the relation in terms of *finite* changes, that is, with "Δ"s:

$$\frac{\Delta V}{\Delta z} = -E_z = -5.64 \times 10^3\,\tfrac{\mathrm{N}}{\mathrm{C}}$$

and from this result we can find the change in z corresponding to any change in V. If we are interested in $\Delta V = 50\,\mathrm{V}$, then

$$\Delta z = -\frac{\Delta V}{E_z} = -\frac{(50\,\mathrm{V})}{(5.64 \times 10^3\,\frac{\mathrm{N}}{\mathrm{C}})} = -8.8 \times 10^{-3}\,\mathrm{m} = -8.8\,\mathrm{mm}$$

i.e. to get a change in potential of $+50\,\mathrm{V}$ we need a change in z coordinate of $-8.8\,\mathrm{mm}$.

Since the potential only depends on the distance from the plane, the equipotential surfaces are *planes*. The distance between planes whose potential differs by $50\,\mathrm{V}$ is $8.8\,\mathrm{mm}$.

---

3. Two large, parallel conducting plates are $12\,\mathrm{cm}$ apart and have charges of equal magnitude and opposite sign on their facing surfaces. An electrostatic force of $3.9 \times 10^{-15}\,\mathrm{N}$ acts on an electron placed anywhere between the two plates.

(Neglect fringing.) (a) Find the electric field at the position of the electron. (b) What is the potential difference between the plates?

(a) We are given the *magnitude* of the electric force on an electron (whose charge is $-e$). Then the magnitude of the E field must be:

$$E = \frac{F}{|q|} = \frac{F}{e} = \frac{(3.9 \times 10^{-15}\,\mathrm{N})}{(1.60 \times 10^{-19}\,\mathrm{C})} = 2.4 \times 10^4\,\tfrac{\mathrm{N}}{\mathrm{C}} = 2.4 \times 10^4\,\tfrac{\mathrm{V}}{\mathrm{m}}$$

(b) The **E** field in the region between two large oppositely–charged plates is *uniform* so in that case, we can write

$$E_x = -\frac{\Delta V}{\Delta x}$$

(where the **E** field points in the x direction, i.e. perpendicular to the plates), and the potential difference between the plates has magnitude

$$|\Delta V| = |E_x \Delta x| = (2.4 \times 10^4\,\tfrac{\mathrm{V}}{\mathrm{m}})(0.12\,\mathrm{m}) = 2.9 \times 10^3\,\mathrm{V}$$

---

4. The electric field inside a nonconducting sphere of radius R with charge spread uniformly throughout its volume, is radially directed and has magnitude

$$E(r) = \frac{qr}{4\pi\epsilon_0 R^3} \quad .$$

Here q (positive or negative) is the total charge within the sphere, and r is the distance from the sphere's center. (a) Taking $V = 0$ at the center of the sphere, find the electric potential $V(r)$ inside the sphere. (b) What is the difference in electric potential between a point on the surface and the sphere's center? (c) If q is positive, which of those two points is at the higher potential?

(a) We will use Eq. 4.6 to calculate $V(r)$ using $r = 0$ as the reference point: $V(0) = 0$. The electric field has only a radial component $E_r(r)$ so that we will evaluate:

$$V(\mathbf{r}) = -\int_{\mathbf{r}_{\mathrm{ref}}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{s} = -\int_0^r E_r(r')dr'$$

Using the given expression for $E_r(r')$ (which one can *derive* using Gauss'(s) law) we get:

$$\begin{aligned} V(r) &= -\int_0^r \frac{qr'}{4\pi\epsilon_0 R^3} dr = -\frac{q}{4\pi\epsilon_0 R^3}\int_0^r r'\,dr' \\ &= -\frac{q}{4\pi\epsilon_0 R^3}\frac{r^2}{2} \\ &= -\frac{qr^2}{8\pi\epsilon_0 R^3} \end{aligned}$$

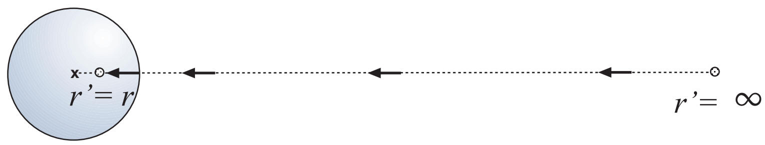

Figure 4.2: Path of integration for Example 5. Integration goes from $r' = \infty$ to $r' = r$.

(b) Using the result of part (a), the difference between values of $V(r)$ on the sphere's surface and at its center is

$$V(R) - V(0) = -\frac{qR^2}{8\pi\epsilon_0 R^3} = -\frac{q}{8\pi\epsilon_0 R}$$

(c) For q positive, the answer to part (b) is a *negative* number, so the center of the sphere must be at a higher potential.

---

5. A charge q is distributed uniformly throughout a spherical volume of radius R. (a) Setting $V = 0$ at infinity, show that the potential at a distance r from the center, where $r < R$, is given by

$$V = \frac{q(3R^2 - r^2)}{8\pi\epsilon_0 R^3} \quad .$$

(b) Why does this result differ from that of the previous example? (c) What is the potential difference between a point of the surface and the sphere's center? (d) Why doesn't this result differ from that of the previous example?

(a) We find the function $V(r)$ just as we did the last example, but this time the reference point (the place where $V = 0$) is at $r = \infty$. So we will evaluate:

$$V(\mathbf{r}) = -\int_{\mathbf{r}_{\rm ref}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{s} = -\int_{\infty}^{r} E_r(r')\, dr' \quad . \tag{4.15}$$

The integration path is shown in Fig. 4.2. We note that the integration (from $r' = \infty$ to $r' = r$ with $r < R$) is over values of r both outside and inside the sphere.

Just as before, the E field for points *inside* the sphere is

$$E_{r,\,\rm in}(r) = \frac{qr}{4\pi\epsilon_0 R^3} \quad , \tag{4.16}$$

but now we will also need the value of the E field *outside* the sphere. By Gauss'(s) law the external E field is that same as that due to a point charge q at distance r, so:

$$E_{r,\,\rm out}(r) = \frac{q}{4\pi\epsilon_0 r^2} \quad . \tag{4.17}$$

Because $E_r(r)$ has two different forms for the interior and exterior of the sphere, we will have to split up the integral in Eq. 4.15 into two parts. When we go from ∞ to R we need to use Eq. 4.17 for $E_r(r')$. When we go from R to r we need to use Eq. 4.16 for $E_r(r')$. So from Eq. 4.15 we now have

$$\begin{aligned} V(r) &= -\int_{\infty}^{R} E_{r,\,\mathrm{out}}(r')\,dr' - \int_{R}^{r} E_{r,\,\mathrm{in}}(r')\,dr' \\ &= -\int_{\infty}^{R} \left(\frac{q}{4\pi\epsilon_0 r'^2}\right)\,dr' - \int_{R}^{r} \left(\frac{qr'}{4\pi\epsilon_0 R^3}\right)\,dr' \\ &= -\frac{q}{4\pi\epsilon_0}\left\{\int_{\infty}^{R}\frac{dr'}{r'^2} + \int_{R}^{r}\frac{r'}{R^3}\,dr'\right\} \end{aligned}$$

Now do the individual integrals and we're done:

$$\begin{aligned} V(r) &= -\frac{q}{4\pi\epsilon_0}\left\{-\frac{1}{r'}\bigg|_{\infty}^{R} + \frac{r'^2}{2R^3}\bigg|_{R}^{r}\right\} \\ &= -\frac{q}{4\pi\epsilon_0}\left\{-\frac{1}{R} + \frac{(r^2-R^2)}{2R^3}\right\} \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{2R^2}{2R^3} + \frac{(R^2-r^2)}{2R^3}\right) \\ &= \frac{q(3R^2-r^2)}{8\pi\epsilon_0 R^3} \end{aligned}$$

(b) The difference between this result and that of the previous example is due to the different choice of reference point. There is no problem here since it is only the *differences* in electrical potential that have any meaning in physics.

(c) using the result of part (a), we calculate:

$$\begin{aligned} V(R) - V(0) &= \frac{q(2R^2)}{8\pi\epsilon_0 R^3} - \frac{q(3R^2)}{8\pi\epsilon_0 R^3} \\ &= -\frac{qR^2}{8\pi\epsilon_0 R^3} = -\frac{q}{8\pi\epsilon_0 R} \end{aligned}$$

This is the *same* as the corresponding result in the previous example.

(d) *Differences* in the electrical potential will *not* depend on the choice of the reference point, the answer *should* be the same as in the previous example... if $V(r)$ is calculated correctly!

---

6. What are (a) the charge and (b) the charge density on the surface of a conducting sphere of radius $0.15\,\mathrm{m}$ whose potential is $200\,\mathrm{V}$ (with $V = 0$ at infinity)?

(a) We are given the radius R of the conducting sphere; we are asked to find its charge Q.

From our work with Gauss'(s) law we know that the electric field outside the sphere is the same as that of a point charge Q at the sphere's center. Then if we were to use Eq. 4.6

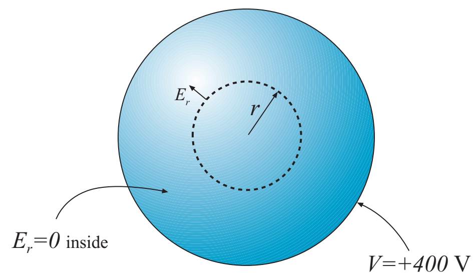

Figure 4.3: Conducting charged sphere, has potential of 400 V, from Example 7.

with the condition $V = 0$ at infinity (which *is* outside the sphere!), we would get the *same* result for V as we would for a point charge Q at the origin and $V = 0$ at infinity, namely:

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \qquad \text{(outside sphere)}$$

This equation holds for $r \geq R$.

Then at the sphere's surface ($r = R$) we have:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$$

Solve for Q and plug in the numbers:

$$\begin{aligned} Q &= 4\pi\epsilon_0 V R \\ &= 4\pi(8.85 \times 10^{-12}\,\tfrac{\mathrm{C}^2}{\mathrm{N\cdot m^2}})(200\,\mathrm{V})(0.15\,\mathrm{m}) \\ &= 3.3 \times 10^{-9}\,\mathrm{C} \end{aligned}$$

The charge on the sphere is 3.3×10^{-9} C.

(b) The charge found in (a) resides on the surface of the conducting sphere. To get the charge density, divide the charge by the surface are of the sphere:

$$\sigma = \frac{Q}{4\pi R^2} = \frac{(3.3 \times 10^{-9}\,\mathrm{C})}{4\pi(0.15\,\mathrm{m})2} = 1.2 \times 10^{-8}\,\frac{\mathrm{C}}{\mathrm{m}^2}$$

The charge density on the sphere's surface is $1.2 \times 10^{-8}\,\frac{\mathrm{C}}{\mathrm{m}^2}$.

---

7. An empty hollow metal sphere has a potential of $+400\,\mathrm{V}$ with respect to ground (defined to be at $V = 0$) and has a charge of $5.0 \times 10^9\,\mathrm{C}$. Find the electric potential at the center of the sphere.

The problem is diagrammed in Fig. 4.3. From considering a spherical Gaussian surface drawn inside the sphere, we see that the electric field E_r must be *zero* everywhere in side the sphere because such a surface will enclose no charge. But for spherical geometries, E_r and V are related by

$$E_r = -\frac{dV}{dr}$$

Figure 4.4: Conducting charged sphere, has potential of 400 V (with $V = 0$ at $r = \infty$), from Example 8.

so that with $E_r = 0$, V must be constant throughout the interior of the spherical conductor. Since the value of V on the sphere itself is +400 V, V then must also equal +400 V at the center.

So $V = +400\,\mathrm{V}$ at the center of the sphere. (There was no calculating to do on this problem!)

---

8. What is the excess charge on a conducting sphere of radius $R = 0.15\,\mathrm{m}$ if the potential of the sphere is $1500\,\mathrm{V}$ and $V = 0$ at infinity?

The problem is diagrammed in Fig. 4.4. If the sphere has net charge Q then from Gauss' law the radial component of the electric field for points outside the sphere is

$$E_r = k\frac{Q}{r^2}$$

Using Eq. 4.6 with $r = \infty$ as the reference point, the potential at distance R from the sphere's center is:

$$\begin{aligned} V(R) &= -\int_\infty^r E_r dr = -\int_\infty^r \frac{kQ}{r^2} dr \\ &= \left.\frac{kQ}{r}\right|_\infty^R = \frac{kQ}{R} - 0 \\ &= \frac{kQ}{R} \end{aligned}$$

(Note that the integration takes place over values of r *outside* the sphere so that the expression for E_r is the correct one. E_r is zero *inside* the sphere.)

We are given that $V(R) = 400\,\mathrm{V}$, so from $kQ/R = 400\,\mathrm{V}$ we solve for Q and get:

$$Q = \frac{R(400\,\mathrm{V})}{k} = \frac{(0.15\,\mathrm{m})(400\,\mathrm{V})}{(8.99\times 10^9\,\frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}})} = 2.5\times 10^{-8}\,\mathrm{C}$$

---

9. The electric potential at points in an xy plane is given by

$$V = (2.0\,\tfrac{\mathrm{V}}{\mathrm{m^2}})x^2 - (3.0\,\tfrac{\mathrm{V}}{\mathrm{m^2}})y^2\ .$$

What are the magnitude and direction of the electric field at the point $(3.0\,\mathrm{m}, 2.0\,\mathrm{m})$**?**

Equations 4.7 show how to get the components of the **E** field if we have the electric potential V as a function of x and y. Taking partial derivatives, we find:

$$E_x = -\frac{\partial V}{\partial x} = -(4.0\,\tfrac{\mathrm{V}}{\mathrm{m}^2})x \qquad \text{and} \qquad E_y = -\frac{\partial V}{\partial y} = +(6.0\,\tfrac{\mathrm{V}}{\mathrm{m}^2})y \ .$$

Plugging in the given values of $x = 3.0\,\mathrm{m}$ and $y = 2.0\,\mathrm{m}$ we get:

$$E_x = -12\,\tfrac{\mathrm{V}}{\mathrm{m}} \qquad \text{and} \qquad E_y = -12\,\tfrac{\mathrm{V}}{\mathrm{m}}$$

So the magnitude of the **E** field at the given is

$$E = \sqrt{(12.0)^2 + (12.0)^2}\,\tfrac{\mathrm{V}}{\mathrm{m}} = 17\,\tfrac{\mathrm{V}}{\mathrm{m}}$$

and its direction is given by

$$\theta = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}(1.0) = 135^\circ$$

where for θ we have made the proper choice so that it lies in the second quadrant.

4.2.2 Potential Energy of a System of Charges

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10. (a) What is the electric potential energy of two electrons separated by $2.00\,\mathrm{nm}$**? (b) If the separation increases, does the potential energy increase or decrease?**

Since the charge of an electron is $-e$, using Eq. 4.13 we find:

$$\begin{aligned} U &= \frac{1}{4\pi\epsilon_0}\frac{(-e)(-e)}{r} \\ &= \frac{1}{4\pi(8.85\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}})}\frac{(1.60\times10^{-19}\,\mathrm{C})^2}{(2.00\times10^{-9}\,\mathrm{m})} \\ &= 1.15\times10^{-19}\,\mathrm{J} \end{aligned}$$

As the charges are both positive, the potential energy is a positive number and is inversely proportional to r. So the potential energy *decreases* as r increases.

---

11. Derive an expression for the work required to set up the four-charge configuration of Fig. 4.5, assuming the charges are initially infinitely far apart.

The work required to set up these charges is the same as the potential energy of a set of point charges, given in Eq. 4.14. (That is, sum the potential energies $k\frac{q_iq_j}{r_{ij}}$ over all pairs of

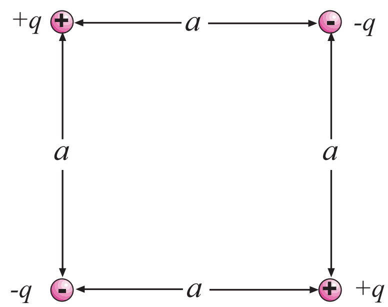

Figure 4.5: Charge configuration for Example 11.

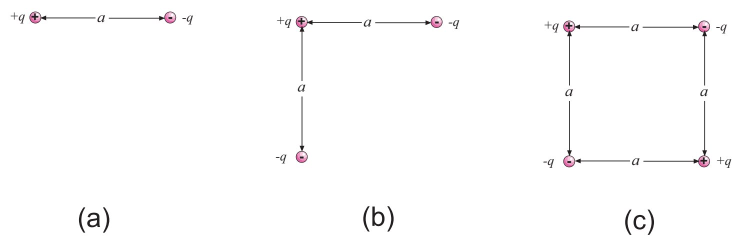

Figure 4.6: (a) Second charge is brought in from ∞ and put in place. (b) Third charge is brought in. (c) Last charge is brought in.

charges.) We can arrive at the same answer and understand that formula a little better if we assemble the system one charge at a time.

Begin with the charge in the upper left corner of Fig. 4.5. Moving this charge from infinity to the desired location requires *no* work because it is never near any other charge. We can write: $W_1 = 0$.

Now bring up the charge in the upper right corner ($-q$). Now we have the configuration shown in Fig. 4.6(a). While being put into place it has experienced a force from the first charge and the work required of the external agency is the change in potential energy of this charge, namely

$$W_2 = \frac{1}{4\pi\epsilon_0}\frac{(+q)(-q)}{a} = -\frac{q^2}{4\pi\epsilon_0 a}$$

Now bring the charge in the lower left corner ($-q$), as shown in 4.6(b). When put into place it is a distance a from the first charge and $\sqrt{2}a$ from the second charge. The work required for this step is the potential energy of the third charge in this configuration, namely:

$$W_3 = \frac{1}{4\pi\epsilon_0}\frac{(+q)(-q)}{a} + \frac{1}{4\pi\epsilon_0}\frac{(-q)(-q)}{\sqrt{2}a} = \frac{q^2}{4\pi\epsilon_0 a}\left(-1 + \frac{1}{\sqrt{2}}\right)$$

Finally, bring in the fourth charge ($+q$) to give the configuration in Fig. 4.6(c). The last charge is now a distance a from *two* $-q$ charges and a distance $\sqrt{2}a$ from the other $+q$ charge. So the work required for this step is

$$W_4 = 2\frac{1}{4\pi\epsilon_0}\frac{(+q)(-q)}{a} + \frac{1}{4\pi\epsilon_0}\frac{(+q)(+q)}{\sqrt{2}a}$$

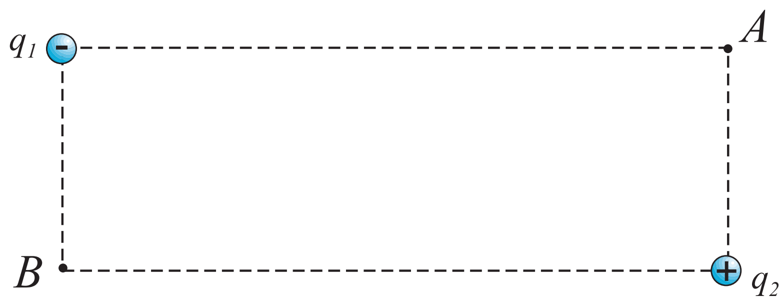

Figure 4.7: Charge configuration for Example 12.

$$= \frac{q^2}{4\pi\epsilon_0 a}\left(-2+\frac{1}{\sqrt{2}}\right)$$

So now add up all the W's to get the total work done:

$$\begin{aligned} W_{\rm Total} &= W_1 + W_2 + W_3 + W_4 \\ &= \frac{q^2}{4\pi\epsilon_0 a}\left(-1-1+\frac{1}{\sqrt{2}}-2+\frac{1}{\sqrt{2}}\right) \\ &= \frac{q^2}{4\pi\epsilon_0 a}\left(-4+\frac{2}{\sqrt{2}}\right) \end{aligned}$$

This is a nice analytic answer; if we combine all the numerical factors (including the 4π) we get:

$$W_{\rm Total} = \frac{(-0.21)q^2}{\epsilon_0 a}$$

This is the same result as we'd get by using Eq. 4.14.

---

12. In the rectangle of Fig. 4.7, the sides have lengths $5.0\,\mathrm{cm}$ and $15\,\mathrm{cm}$, $q_1 = -5.0\,\mu\mathrm{C}$ and $q_2 = +2.0\,\mu\mathrm{C}$. With $V = 0$ at infinity, what are the electric potentials (a) at corner A and (b) corner B? (c) How much work is required to move a third charge $q_3 = +3.0\,\mu\mathrm{C}$ from B to A along a diagonal of the rectangle? (d) Does this work increase or decrease the electric energy of the three–charge system? Is more, less or the same work required if q_3 is moved along paths that are (e) inside the rectangle but not on the diagonal and (f) outside the rectangle?

(a) To find the electric potential due to a group of point charges, use Eq. 4.10. Since point A is 15 cm away from the $-5.0\,\mu\mathrm{C}$ charge and 5.0 cm away from the $+2.0\,\mu\mathrm{C}$ charge, we get:

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1}+\frac{q_2}{r_2}\right] \\ &= (8.99\times10^9\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}})\left[\frac{(-5.0\times10^{-6}\,\mathrm{C})}{(15\times10^{-2}\,\mathrm{m})}+\frac{(+2.0\times10^{-6}\,\mathrm{C})}{(5.0\times10^{-2}\,\mathrm{m})}\right] = 6.0\times10^4\,\mathrm{V} \end{aligned}$$

(b) Perform the same calculation as in part (a). The charges q_1 and q_2 are at different distances from point B so we get a different answer:

$$V = (8.99 \times 10^9\, \tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}}) \left[\frac{(-5.0 \times 10^{-6}\,\mathrm{C})}{(5.0 \times 10^{-2}\,\mathrm{m})} + \frac{(+2.0 \times 10^{-6}\,\mathrm{C})}{(15 \times 10^{-2}\,\mathrm{m})} \right] = -7.8 \times 10^5\,\mathrm{V}$$

(c) Using the results of part (a) and (b), calculate the change in potential V as we move from point B to point A:

$$\Delta V = V_A - V_B = 6.0 \times 10^4\,\mathrm{V} - (-7.8 \times 10^5\,\mathrm{V}) = 8.4 \times 10^5\,\mathrm{V}$$

The change in potential energy for a $+3.0\,\mu\mathrm{C}$ charge to move from B to A is

$$\Delta U = q\Delta V = (3.0 \times 10^{-6}\,\mathrm{C})(8.4 \times 10^5\,\mathrm{V}) = 2.5\,\mathrm{J}$$

(d) Since a positive amount of work is done by the outside agency in moving the charge from B to A, the electric energy of the system has *increased*. We can see that this must be the case because the $+3.0\,\mu\mathrm{C}$ charge has been moved closer to another positive charge and farther away from a negative charge.

(e) The force which a point charge (or set of point charges) exerts on a another charge is a *conservative* force. So the work which it does (or likewise the work *required* of some outside force) as the charge moves from one point to another is *independent of the path taken*. Therefore we would require the same amount of work if the path taken was some other path inside the rectangle.

(f) Since the work done is independent of the path taken, we require the same amount of work even if the path from A to B goes outside the rectangle.

---

13. Two tiny metal spheres A and B of mass $m_A = 5.00\,\mathrm{g}$ and $m_B = 10.0\,\mathrm{g}$ have equal positive charges $q = 5.00\,\mu\mathrm{C}$. The spheres are connected by a massless nonconducting string of length $d = 1.00\,\mathrm{m}$, which is much greater than the radii of the spheres. (a) What is the electric potential energy of the system? (b) Suppose you cut the string. At that instant, what is the acceleration of each sphere? (c) A long time after you cut the string, what is the speed of each sphere?

(a) The initial configuration of the charges in shown in Fig. 4.8(a). The electrostatic potential energy of this system (i.e. the work needed to bring the charges together from far away is

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} = (8.99 \times 10^9\, \tfrac{\mathrm{N \cdot m^2}}{\mathrm{C^2}}) \frac{(5.00 \times 10^{-6}\,\mathrm{C})^2}{(1.00\,\mathrm{m})} = 0.225\,\mathrm{J}$$

We are justified in using formulae for *point charges* because the problem states that the sizes of the spheres are small compared to the length of the string (1.00 m).

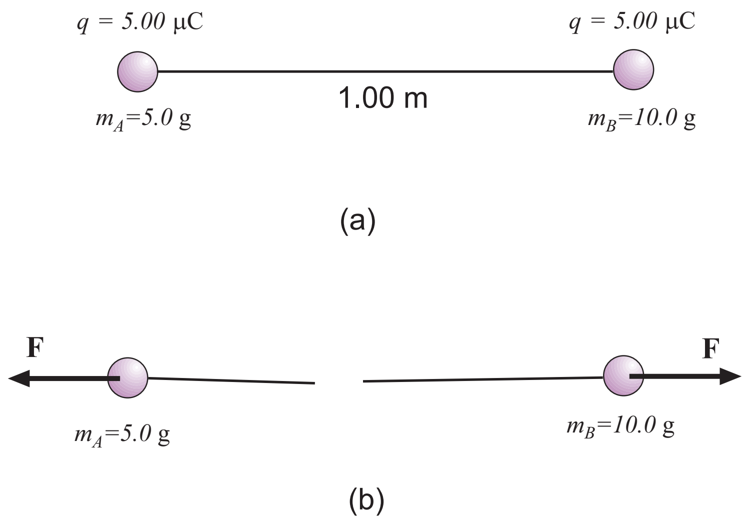

Figure 4.8: (a) Charged spheres attached to a string, in Example 13. The electrostatic repulsion is balanced by the string tension. (b) After string is cut there is a mutual force of electrical repulsion **F**. Magnitude of the *force* on each charge is the same but their *accelerations* are different!

(b) From Coulomb's law, the magnitude of the mutual force of repulsion of the two charges is

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} = (8.99\times 10^9\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{C^2}})\frac{(5.00\times 10^{-6}\,\mathrm{C})^2}{(1.00\,\mathrm{m})^2} = 0.225\,\mathrm{N}$$

but since the masses of the spheres are different their *accelerations* have different magnitudes. From Newton's 2$^{\mathrm{nd}}$ law, the accelerations of the masses are:

$$a_1 = \frac{F}{m_1} = \frac{(0.225\,\mathrm{N})}{(5.00\times 10^{-3}\,\mathrm{kg})} = 45.0\,\tfrac{\mathrm{m}}{\mathrm{s^2}}$$

$$a_2 = \frac{F}{m_2} = \frac{(0.225\,\mathrm{N})}{(10.0\times 10^{-3}\,\mathrm{kg})} = 22.2\,\tfrac{\mathrm{m}}{\mathrm{s^2}}$$

Of course, the accelerations are in opposite *directions*.

(c) From the time that the string breaks to the time that we can say that the spheres are "very far apart", the only force that each one experiences is the force of electrical repulsion (arising from the other sphere). This is a *conservative force* so that total mechanical energy is conserved. It is also true that there are no *external forces* being exerted on the two–sphere system. Then we know that the total (vector) momentum of the system is also conserved.

First, let's deal with the condition of energy conservation. The total energy right after the string is cut is just the potential energy found in part (a) since the spheres are not yet in motion. So $E_{\mathrm{init}} = 0.225\,\mathrm{J}$.

When the spheres are a long ways apart, there is no electrical potential energy, but they are in motion with respective speeds v_A and v_B so there is kinetic energy at "large" separation. Then energy conservation tells us:

$$\tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_B^2 = 0.225\,\mathrm{J} \tag{4.18}$$

Momentum conservation gives us the other equation that we need. If mass B has x–velocity v_B then mass A has x–velocity $-v_A$ (it moves in the other direction. The system begins and ends with a total momentum of *zero* so then:

$$-m_A v_A + m_B v_B = 0 \qquad \Longrightarrow \qquad v_B = \frac{m_A}{m_B} v_A$$

Substitute this result into 4.18 and get:

$$\tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B \left(\frac{m_A^2}{m_B^2} \right) v_A^2 = 0.225\,\mathrm{J}$$

Factor out v_A^2 on the left side and plug in some numbers:

$$\tfrac{1}{2}\left(m_A + \frac{m_A^2}{m_B} \right) v_A^2 = \tfrac{1}{2}\left(5.00\,\mathrm{g} + \frac{(5.00\,\mathrm{g})^2}{(10.0\,\mathrm{g})} \right) v_A^2 = (3.75 \times 10^{-3}\,\mathrm{kg}) v_A^2 = 0.225\,\mathrm{J}$$

So then we get the final speed of A:

$$v_A^2 = \frac{0.225\,\mathrm{J}}{3.75 \times 10^{-3}\,\mathrm{kg}} = 60.0\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2} \qquad \Longrightarrow v_A = 7.75\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

and the speed of B:

$$v_B = \frac{m_A}{m_B} v_A = \left(\frac{5.00\,\mathrm{g}}{10.0\,\mathrm{g}} \right) 7.75\,\tfrac{\mathrm{m}}{\mathrm{s}} = 3.87\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

---

14. Two electrons are fixed $2.0\,\mathrm{cm}$ **apart. Another electron is shot from infinity and stops midway between the two. What is its initial speed?**

The problem is diagrammed in Fig. 4.9(a) and (b). Since the electrostatic force is a conservative force, we know that *energy is conserved* between configurations (a) and (b). In picture (a) there is energy stored in the repulsion of the pair of electrons as well as the kinetic energy of the third electron. (Initially the third electron is too far away to "feel" the first two electrons.) In picture (b) there is no kinetic energy but the electrical potential energy has increased due to the repulsion between the third electron and the first two. If we can calculate the change in potential energy ΔU then by using energy conservation, $\Delta U + \Delta K = 0$ we can find the initial speed of the electron.

The potential energy of a set of point charges (with $V = 0$ at ∞) is given in Eq. 4.14. When the third electron comes from infinity and stops at the midpoint, the increase in potential energy the contribution given by the third electron as it "sees" its new neighbors. With $r = 1.0\,\mathrm{cm}$, this increase is

$$\Delta U = \frac{1}{4\pi\epsilon_0}\frac{(-e)(-e)}{r} + \frac{1}{4\pi\epsilon_0}\frac{(-e)(-e)}{r} = \frac{e^2}{2\pi\epsilon_0 r}$$

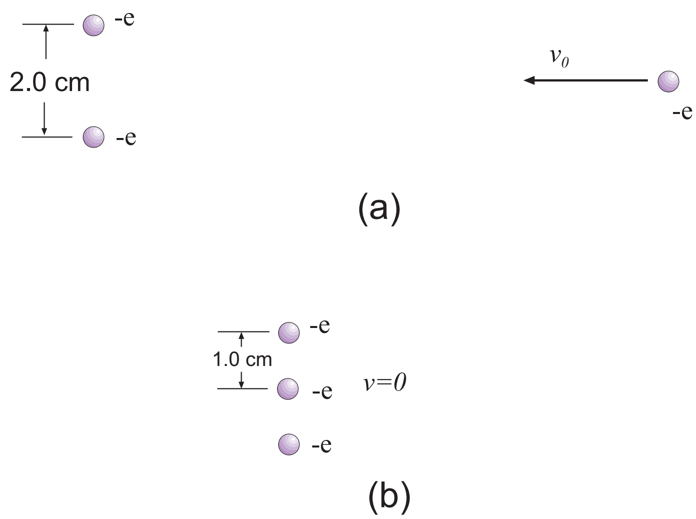

Figure 4.9: (a) Electron flies in from ∞ with speed v_0. (b) Electron comes to rest midway between the other two electrons.

The change in kinetic energy is $\Delta K = -\frac{1}{2}mv_0^2$. Then energy conservation gives:

$$\Delta K = -\Delta U \qquad \Longrightarrow \qquad -\tfrac{1}{2}mv_0^2 = -\frac{e^2}{2\pi\epsilon_0 r}$$

Solve for v_0:

$$\begin{aligned} v_0^2 &= \frac{e^2}{\pi\epsilon_0 m r} \\ &= \frac{(1.60 \times 10^{-19}\,\mathrm{C})^2}{\pi(8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N \cdot m^2}})(9.11 \times 10^{-31}\,\mathrm{kg})(1.0 \times 10^{-2}\,\mathrm{m})} = 1.01 \times 10^5\,\frac{\mathrm{m}^2}{\mathrm{s}^2} \end{aligned}$$

which gives

$$v_0 = 3.18 \times 10^2\,\frac{\mathrm{m}}{\mathrm{s}}$$

Chapter 5

Capacitance and Dielectrics

5.1 The Important Stuff

5.1.1 Capacitance

Electrical energy can be stored by putting opposite charges $\pm q$ on a pair of isolated conductors. Being conductors, the respective surfaces of these two objects are all at the same potential so that it makes sense to speak of a potential difference V *between the two conductors*, though one should really write ΔV for this. (Also, we will usually just talk about "the charge q" of the conductor pair though we really mean $\pm q$.)

Such a device is called a **capacitor**. The general case is shown in Fig. 5.1(a). A particular geometry known as the *parallel plate capacitor* is shown in Fig. 5.1(b).

It so happens that if we don't change the configuration of the two conductors, the charge q is proportional to the potential difference V. The proportionality constant C is called the **capacitance** of the device. Thus:

$$q = CV \tag{5.1}$$

The SI unit of capacitance is then $1\,\frac{\mathrm{C}}{\mathrm{V}}$, a combination which is called the **farad**[1]. Thus:

$$1\,\mathrm{farad} = 1\,\mathrm{F} = 1\,\frac{\mathrm{C}}{\mathrm{V}} \tag{5.2}$$

The permittivity constant can be expressed in terms of this new unit as:

$$\epsilon_0 = 8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N{\cdot}m^2}} = 8.85 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}} \tag{5.3}$$

5.1.2 Calculating Capacitance

For various simple geometries for the pair of conductors we can find expressions for the capacitance.

- **Parallel-Plate Capacitor**

[1]Named in honor of the...uh...Austrian physicist Jim Farad (1602–1796) who did some electrical experiments in...um...Berlin. That's it, Berlin.

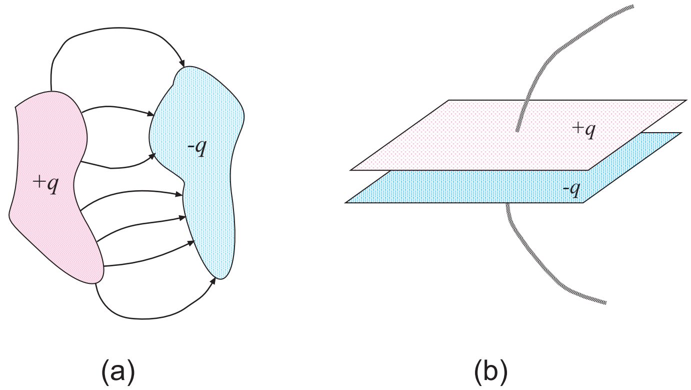

Figure 5.1: (a) Two isolated conductors carrying charges $\pm q$: A capacitor! (b) A more common configuration of conductors for a capacitor: Two isolated parallel conducting sheets of area A, separated by (small) distance d.

The most common geometry we encounter is one where the two conductors are parallel plates (as in Fig. 5.1(b), with the stipulation that the dimensions of the plates are "large" compared to their separation to minimize the "fringing effect".

For a parallel-plate capacitor with plates of area A separated by distance d, the capacitance is given by

$$C = \frac{\epsilon_0 A}{d} \tag{5.4}$$

- **Cylindrical Capacitor**

In this geometry there are two coaxial cylinders where the radius of the inner conductor is a and the inner radius of the outer conductor is b. The length of the cylinders is L; we stipulate that L is large compared to b.

For this geometry the capacitance is given by

$$C = 2\pi\epsilon_0 \frac{L}{\ln(b/a)} \tag{5.5}$$

- **Spherical Capacitor**

In this geometry there are two concentric spheres where the radius of the inner sphere is a and the inner radius of the outer sphere is b. For this geometry the capacitance is given by:

$$C = 4\pi\epsilon_0 \frac{ab}{b-a} \tag{5.6}$$

5.1.3 Capacitors in Parallel and in Series

- **Parallel Combination:** Fig. 5.2 shows a configuration where three capacitors are combined in **parallel** across the terminals of a battery. The battery gives a constant potential

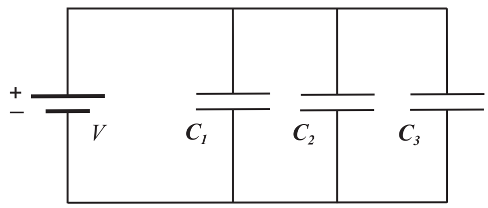

Figure 5.2: Three capacitors are combined in *parallel* across a potential difference V (produced by a battery).

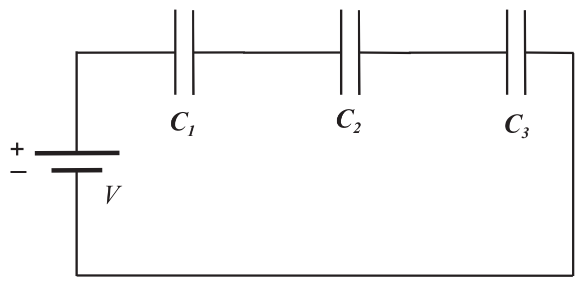

Figure 5.3: Three capacitors are combined in *series* across a potential difference V (produced by a battery).

difference V across the plates of *each* of the capacitors. The charges q_1, q_2 and q_3 which collect on the plates of the respective capacitors are *not* the same, but will be found from

$$q_1 = c_1 V \qquad q_2 = C_2 V \qquad q_3 = C_3 V \ .$$

The total charge on the plates, $q = q_1 + q_2 + q_3$ is related to the potential difference V by $q = C_{\rm equiv} V$, where $C_{\rm equiv}$ is the equivalent capacitance of the combination. In general, the equivalent capacitance for a set of capacitors which are in parallel is given by

$$C_{\rm equiv} = \sum_i C_i \qquad\qquad \textbf{Parallel} \tag{5.7}$$

• **Series Combination:** Fig. 5.3 shows a configuration where three capacitors are combined in **series** across the terminals of a battery. Here the charges which collect on the respective capacitor plates *are* the same (q) but the potential differences across the capacitors are *different*. These potential differences can be found from

$$V_1 = \frac{q}{C_1} \qquad V_2 = \frac{q}{C_2} \qquad V_3 = \frac{q}{C_3}$$

where the individual potential differences add up to give the total: $V_1 + V_2 + V_3 = V$. In general, the effective capacitance for a set of capacitors which are in series is

$$\frac{1}{C_{\rm equiv}} = \sum_i \frac{1}{C_i} \qquad\qquad \textbf{Series} \tag{5.8}$$

5.1.4 Energy Stored in a Capacitor

When we consider the work required to charge up a capacitor by moving a charge $-q$ from on plate to another we arrive at the potential energy U of the charges, which we can view as the energy stored in the electric field between the plates of the capacitor. This energy is:

$$U = \frac{q^2}{2C} = \tfrac{1}{2}CV^2 \tag{5.9}$$

If we associate the energy in Eq. 5.9 with the region where there *is* any electric field, the interior of the capacitor (the field is effectively zero outside) then we arrive at an energy per unit volume for the electric field, i.e. an **energy density**, u. It is:

$$u = \tfrac{1}{2}\epsilon_0 E^2 \tag{5.10}$$

This result also holds for *any* electric field, regardless of its source.

5.1.5 Capacitors and Dielectrics

If we fill the region between the plates of a capacitor with an insulating material the capacitance will be increased by some numerical factor κ:

$$C = \kappa C_{\rm air} \quad . \tag{5.11}$$

The number κ (which is unitless) is called the **dielectric constant** of the insulating material.

5.2 Worked Examples

5.2.1 Capacitance

---

1. Show that the two sets of units given for ϵ_0 in Eq. 5.3 are in fact the same.

Start with the new units for ϵ_0, $\frac{\rm F}{\rm m}$. From Eq. 5.2 we substitute $1\,{\rm F} = 1\,\frac{\rm C}{\rm V}$ so that

$$1\,\frac{\rm F}{\rm m} = 1\,\frac{\rm C/V}{\rm m} = 1\,\frac{\rm C}{\rm V\cdot m}$$

Now use the definition of the volt from Eq. 4.4: $1\,{\rm V} = 1\,{\rm J/C} = 1\,{\rm N\cdot m/C}$ to get

$$1\,\frac{\rm F}{\rm m} = 1\,\frac{\rm C}{\frac{\rm N\cdot m}{\rm C}\cdot{\rm m}} = 1\,\frac{\rm C^2}{\rm N\cdot m^2}$$

So we arrive at the original units of ϵ_0 given in Eq. 1.4.

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2. The capacitor shown in Fig. 5.4 has capacitance $25\,\mu{\rm F}$ and is initially un-

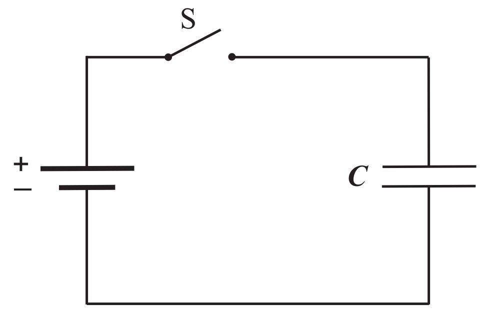

Figure 5.4: Battery and capacitor for Example 2.

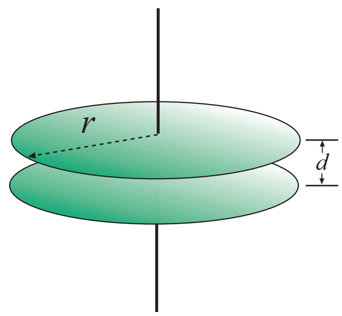

Figure 5.5: Capacitor described in Example 3.

charged. The battery provides a potential difference of $120\,\mathrm{V}$**. After switch S is closed, how much charge will pass through it?**

When the switch is closed, then the charge q which collects on the capacitor plates is given by $q = CV$. Plugging in the given values for the capacitance C and the potential difference V, we find:

$$\begin{aligned} q &= CV = (25 \times 10^{-6}\,\mathrm{F})(120\,\mathrm{V}) \\ &= 3.0 \times 10^{-3}\,\mathrm{C} = 3.0\,\mathrm{mC} \end{aligned}$$

This is the amount of charge which has been exchanged between the top and bottom plates of the capacitor. So $3.0\,\mathrm{mC}$ of charge has passed through the switch.

5.2.2 Calculating Capacitance

---

3. A parallel–plate capacitor has circular plates of $8.2\,\mathrm{cm}$ **radius and** $1.3\,\mathrm{mm}$ **separation. (a) Calculate the capacitance. (b) What charge will appear on the plates if a potential difference of** $120\,\mathrm{V}$ **is applied?**

(a) The capacitor is illustrated in Fig. 5.5. The area of the plates is $A = \pi r^2$ so that with

$r = 8.2\,\mathrm{cm}$ and $d = 1.3\,\mathrm{mm}$ and using Eq. 5.4 we get:

$$\begin{aligned} C &= \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 \pi r^2}{d} \\ &= \frac{(8.85 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}})\pi(8.2 \times 10^{-2}\,\mathrm{m})^2}{(1.3 \times 10^{-3}\,\mathrm{m})} \\ &= 1.4 \times 10^{-10}\,\mathrm{F} = 140\,\mathrm{pF} \end{aligned}$$

(b) When a potential of $120\,\mathrm{V}$ is applied to the plates of the capacitor the charge which appears on the plates is

$$q = CV = (1.4 \times 10^{-10}\,\mathrm{F})(120\,\mathrm{V}) = 1.7 \times 10^{-8}\,\mathrm{C} = 17\,\mathrm{nC}$$

---

4. You have two flat metal plates, each of area $1.00\,\mathrm{m}^2$, with which to construct a parallel-plate capacitor. If the capacitance of the device is to be $1.00\,\mathrm{F}$, what must be the separation between the plates? Could this capacitor actually be constructed?

In Eq. 5.4 (formula for C for a parallel-plate capacitor) we have C and A. We can solve for the separation d:

$$C = \frac{\epsilon_0 A}{d} \qquad \Longrightarrow \qquad d = \frac{\epsilon_0 A}{C}$$

Plug in the numbers:

$$d = \frac{(8.85 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}})(1.00\,\mathrm{m}^2)}{(1.00\,\mathrm{F})} = 8.85 \times 10^{-12}\,\mathrm{m}$$

This is an *extremely* tiny length if we are thinking about making an actual device, because the typical "size" of an atom is on the order of $1.0 \times 10^{-10}\,\mathrm{m}$. Our separation d is ten times *smaller* than that, so the atoms in the plates would not be truly separated! So a suitable capacitor could *not* be constructed.

---

5. A $2.0-\mu\mathrm{F}$ spherical capacitor is composed of two metal spheres, one having a radius twice as large as the other. If the region between the spheres is a vacuum, determine the volume of this region.

The capacitance of a ("air–filled") spherical capacitor is

$$C = 4\pi\epsilon_0 \frac{ab}{(b-a)} \quad .$$

where a and b are the radii of the concentric spherical plates. Here we are given that $b = 2a$, so we then have:

$$C = 4\pi\epsilon_0 \frac{2a^2}{a} = 8\pi\epsilon_0 a$$

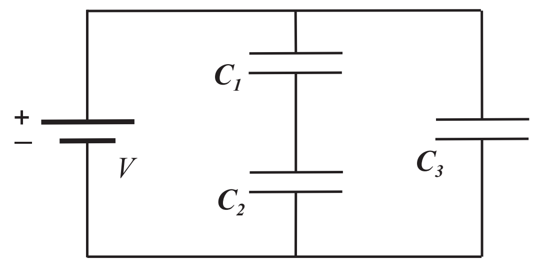

Figure 5.6: Configuration of capacitors for Example 6.

We are given the value of C so we can solve for a:

$$a = \frac{C}{8\pi\epsilon_0} = \frac{(2.0\times 10^{-6}\,\mathrm{F})}{8\pi(8.85\times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}})} = 9.0\times 10^{3}\,\mathrm{m} \qquad (!)$$

so that $b = 2a = 1.8\times 10^4\,\mathrm{m}$.

Then the volume of the enclosed region between the two plates is:

$$\begin{aligned} V_{\mathrm{enc}} &= \tfrac{4}{3}\pi b^3 - \tfrac{4}{3}\pi a^3 = \tfrac{4}{3}\pi((2a)^3 - a^3) = \tfrac{4}{3}\pi(7a^3) \\ &= 2.1\times 10^{13}\,\mathrm{m}^3 \end{aligned}$$

5.2.3 Capacitors in Parallel and in Series

---

6. In Fig. 5.6, find the equivalent capacitance of the combination. Assume that $C_1 = 10.0\,\mu\mathrm{F}$, $C_2 = 5.00\,\mu\mathrm{F}$, and $C_3 = 4.00\,\mu\mathrm{F}$

The configuration given in the figure is that of a *series* combination of two capacitors (C_1 and C_2) combined in *parallel* with a single capacitor (C_3). We can use the reduction formulae Eq. 5.8 and Eq. 5.7 to give a *single* equivalent capacitance.

First combine the series capacitors with Eq. 5.8. The equivalent capacitance is:

$$\frac{1}{C_{\mathrm{equiv}}} = \frac{1}{10.0\,\mu\mathrm{F}} + \frac{1}{5.00\,\mu\mathrm{F}} = 0.300\,\mu\mathrm{F}^{-1} \qquad \Longrightarrow \qquad C_{\mathrm{equiv}} = 3.33\,\mu\mathrm{F}$$

After this reduction, the configuration is as shown in Fig. 5.7(a). Now we have two capacitors in parallel. By Eq. 5.7 the equivalent capacitance is just the *sum* of the two values:

$$C_{\mathrm{equiv}} = 3.33\,\mu\mathrm{F} + 4.00\,\mu\mathrm{F} = 7.33\,\mu\mathrm{F}$$

The final equivalent capacitance is shown in Fig. 5.7(b).

The equivalent capacitance of the combination is $7.33\,\mu\mathrm{F}$.

---

7. How many $1.00\,\mu\mathrm{F}$ capacitors must be connected in parallel to store a charge of $1.00\,\mathrm{C}$ with a potential of $110\,\mathrm{V}$ across the capacitors?

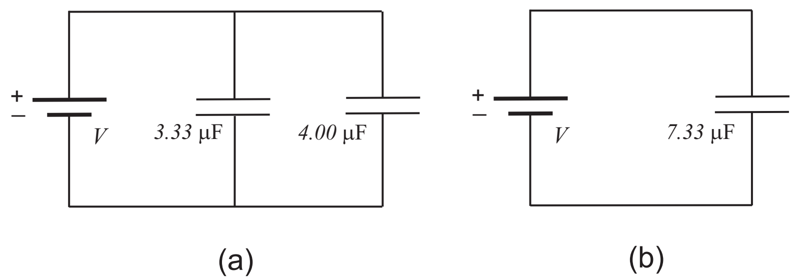

Figure 5.7: (a) Series capacitors in previous figure have been combined as a single equivalent capacitor. (b) Parallel combination in (a) has been combined to give a single equivalent capacitor.

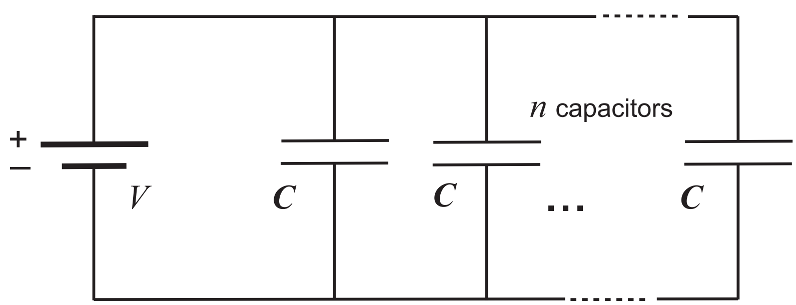

Figure 5.8: n capacitors in parallel, for Example 7.

In this problem we imagine a configuration like that shown in Fig. 5.8, where we have n capacitors with $C = 1.00\,\mu\mathrm{F}$ connected in parallel across a potential difference of $V = 110\,\mathrm{V}$. Since parallel capacitors simply *add* to give the equivalent capacitance (see Eq. 5.7) we have $C_{\mathrm{equiv}} = nC$, and the potential difference across the combination is related to the *total* charge q_{tot} on the plates by $q_{\mathrm{tot}} = C_{\mathrm{equiv}}V = nCV$. We then use this to solve for n:

$$n = \frac{q_{\mathrm{tot}}}{CV} = \frac{(1.00\,\mathrm{C})}{(1.00\times10^{-6}\,\mathrm{F})(110\,\mathrm{V})} = 9.09\times10^{3} \ .$$

So one would need to hook up $n = 9090$ capacitors (!) to store the 1.00 C of charge.

---

8. Each of the uncharged capacitors in Fig. 5.9 has a capacitance of $25.0\,\mu\mathrm{F}$. A potential difference of $4200\,\mathrm{V}$ is established when the switch is closed. How many coulombs of charge then pass through the meter A?

The (total) charge which passes through the (current) meter A is the total charge which collects on the plates of the three capacitors. We note that for each capacitor the potential difference across the plates (after the switch is closed) is 4200 V. So the charge on each capacitor is

$$q = CV = (25.0\times10^{-6}\,\mathrm{F})(4200\,\mathrm{V}) = 0.105\,\mathrm{C}$$

and the total charge is

$$q_{\mathrm{Total}} = 3(0.105\,\mathrm{C}) = 0.315\,\mathrm{C} \ .$$

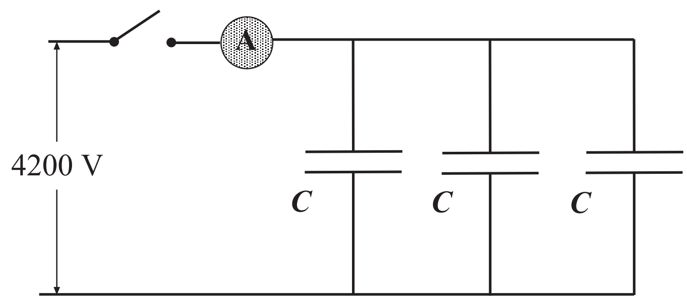

Figure 5.9: Configuration of capacitors for Example 8.

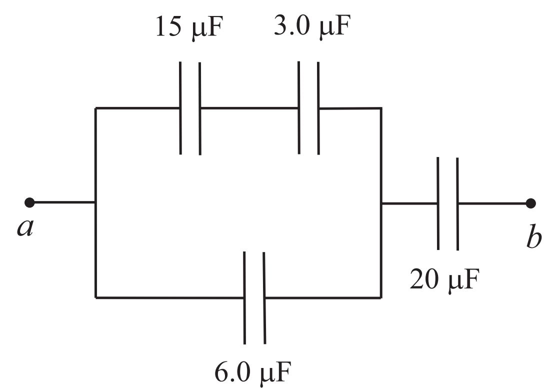

Figure 5.10: Combination of capacitors for Example 9.

So this is the amount of charge which passes through meter A.

We could also note that the equivalent capacitance of the three parallel capacitors is

$$C_{\rm equiv} = 3(25.0\,\mu{\rm F}) = 75.0\,\mu{\rm F}$$

and with 4200 V across the leads of the equivalent capacitance the total charge which collects on the plates is

$$q_{\rm Total} = C_{\rm equiv}V = (75.0 \times 10^{-6}\,{\rm F})(4200\,{\rm V}) = 0.315\,{\rm C} \ .$$

---

9. Four capacitors are connected as shown in Fig. 5.10. (a) Find the equivalent capacitance between points a and b. (b) Calculate the charge on each capacitor if $V_{ab} = 15\,{\rm V}$.

(a) To get the equivalent capacitance of the set of capacitors between a and b: First note that the $15\,\mu$F and $3.0\,\mu$F capacitors are in series so they combine as:

$$\frac{1}{C_{\rm equiv}} = \frac{1}{15\,\mu{\rm F}} + \frac{1}{3.0\,\mu{\rm F}} = 0.40\,\mu{\rm F}^{-1} \qquad \Longrightarrow \qquad C_{\rm equiv} = 2.5\,\mu{\rm F}$$

After this reduction, the configuration is as shown in Fig. 5.11(a). The reduced circuit now

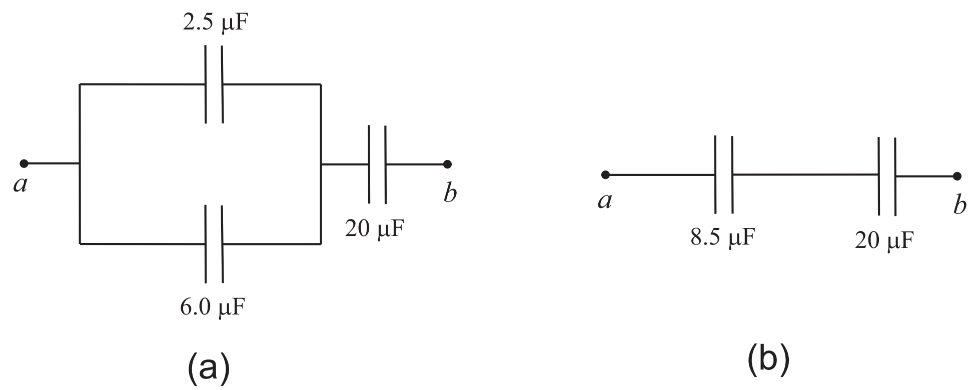

Figure 5.11: (a) After "reduction" of the series pair. (b) After combining two parallel capacitances.

has $2.5\,\mu\mathrm{F}$ and $6.0\,\mu\mathrm{F}$ capacitors in parallel which combine as:

$$C_{\mathrm{equiv}} = 2.5\,\mu\mathrm{F} + 6.0\,\mu\mathrm{F} = 8.5\,\mu\mathrm{F}$$

which gives us the combination shown in 5.11(b).

Finally, the $8.5\,\mu\mathrm{F}$ and $20\,\mu\mathrm{F}$ capacitors in series reduce to:

$$\frac{1}{C_{\mathrm{equiv}}} = \frac{1}{8.5\,\mu\mathrm{F}} + \frac{1}{20\,\mu\mathrm{F}} = 0.168\,\mu\mathrm{F}^{-1} \qquad \Longrightarrow \qquad C_{\mathrm{equiv}} = 5.96\,\mu\mathrm{F}$$

so the equivalent capacitance between points a and b is $5.96\,\mu\mathrm{F}$.

(b) Since the equivalent capacitance between a and b is $5.96\,\mu\mathrm{F}$, the charge which collects on *either end* of the combination is

$$Q = C_{\mathrm{equiv}} V_{ab} = (5.96 \times 10^{-6}\,\mathrm{F})(15\,\mathrm{V} = 8.95 \times 10^{-5}\,\mathrm{C}$$

This is the same as the charge on the far end of the $20\,\mu\mathrm{F}$ capacitor (and thus on either plate of that capacitor) , so we have the charge on that capacitor:

$$Q_{20\,\mu\mathrm{F}} = 8.95 \times 10^{-5}\,\mathrm{C}$$

Now we can find the potential difference across the $20\,\mu\mathrm{F}$ capacitor:

$$V_{20\,\mu\mathrm{F}} = \frac{Q_{20\,\mu\mathrm{F}}}{C_{20\,\mu\mathrm{F}}} = \frac{(8.95 \times 10^{-5}\,\mathrm{C})}{(20 \times 10^{-6}\,\mathrm{F})} = 4.47\,\mathrm{V}$$

With this value, we can find the potential difference between points a and c (see Fig. 5.12):

$$V_{ac} = 15.0\,\mathrm{V} - 4.47\,\mathrm{V} = 10.5\,\mathrm{V}$$

This is now the potential difference across the $6.0\,\mu\mathrm{F}$ capacitor, so we can find its charge:

$$Q_{6.0\,\mu\mathrm{F}} = (6.0 \times 10^{-6}\,\mathrm{F})(10.5\,\mathrm{V}) = 6.32 \times 10^{-5}\,\mathrm{C}$$

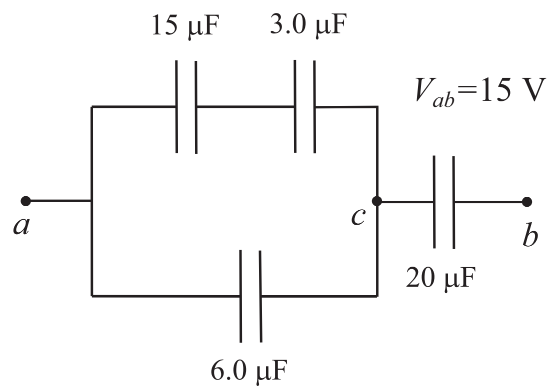

Figure 5.12: Point c comes just before the $20\,\mu$F capacitor. Find V_{ac} by subtracting $V_{20\,\mu\mathrm{F}}$ from 15 V

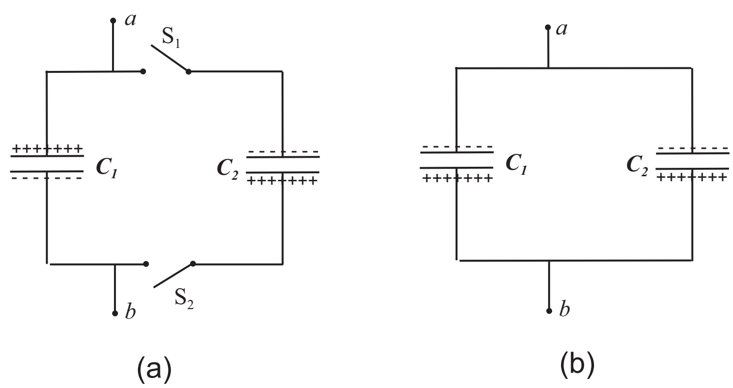

Figure 5.13: Capacitor configuration for Example 10. (a) before switches are closed. (b) After switches are closed, charges redistribute on the plates of C_1 and C_2.

Finally, we note that the potential difference across the $15\,\mu$F — $3.0\,\mu$F series pair is also 10.5 V. Now, the equivalent capacitance of this pair was $2.50\,\mu$F, so that the charge which collects on each end of this combination is

$$Q = C_{\mathrm{equiv}} V = (2.5 \times 10^{-6}\,\mathrm{F})(10.5\,\mathrm{V}) = 2.63 \times 10^{-5}\,\mathrm{C}\ .$$

But this is the same as the charge on the outer plates of the two capacitors, and that means that both capacitors have the same charge, namely:

$$Q_{15\,\mu\mathrm{F}} = Q_{3.0\,\mu\mathrm{F}} = 2.63 \times 10^{-5}\,\mathrm{C}$$

We now have the charges on all four of the capacitors.

---

10. In Fig. 5.13(a), the capacitances are $C_1 = 1.0\,\mu$F and $C_2 = 3.0\,\mu$F and both capacitors are charged to a potential difference of $V = 100\,$V but with opposite polarity as shown. Switches S_1 and S_2 are now closed. (a) What is now the potential difference between a and b? What are now the charges on capacitors (b) 1 and (c) 2?

Let's first find the charges which the capacitors had before the switch was closed. For C_1 the magnitude of its charge was

$$Q_1 = C_1 V = (1.0\,\mu\mathrm{F})(100\,\mathrm{V}) = 1.00 \times 10^{-4}\,\mathrm{C}$$

What we mean here is that the *upper* plate of C_1 had a charge of 1.00×10^{-4} C, because the polarity matters here! So the lower plate of C_1 had a charge of -1.00×10^{-4} C.

For C_2, the *magnitude* of its charge was

$$Q_2 = C_2 V = (3.0\,\mu\mathrm{F})(100\,\mathrm{V}) = 3.00 \times 10^{-4}\,\mathrm{C}$$

but here we mean that the upper plate of C_2 had a charge of -3.00×10^{-4} C because of the polarity indicated in Fig. 5.13(a). So its lower plate had a charge of $+3.00 \times 10^{-4}$ C.

We note that the *total* charge on the upper plates is

$$Q_{1,\,\mathrm{upper}} + Q_{1,\,\mathrm{upper}} = -2.00 \times 10^{-4}\,\mathrm{C}$$

and the total charge on the lower plates is $+2.00 \times 10^{-4}$ C.

Now when the switches are closed the charges on the upper plates will redistribute themselves on the upper plates of C_1 and C_2. Lets call these new charges (on the *upper* plates) Q'_1 and Q'_2. We note that since the total charge on the upper plates was *negative* then it is a net negative charge which shifts around on the upper plates and Q'_1 and Q'_2 are both negative, as indicated in Fig. 5.13(b). By conservation of charge, the total is still equal to -2.00×10^{-4} C:

$$Q'_1 + Q'_2 = -2.00 \times 10^{-4}\,\mathrm{C}$$

Though we don't yet know the new potential difference across each capacitor, we do know that it is the *same* for both. Actually, we know that b must be at the higher potential; we will let the potential change in going from a to b be called V'. Now, the potential for each capacitor is found from $V = Q/C$; actually because of the polarities here (the Q's being negative) we need a minus sign, but the fact that the potential differences are the *same* across both capacitors gives:

$$V' = \frac{-Q'_1}{C_1} = \frac{-Q'_2}{C_2} \qquad \Longrightarrow \qquad Q'_2 = \frac{C_2}{C_1} Q'_1 = \left(\frac{3.0\,\mu\mathrm{F}}{1.0\,\mu\mathrm{F}}\right) Q'_1 = 3.0 Q'_1$$

Substituting this result into the previous one gives

$$Q'_1 + 3.0 Q'_1 = -2.0 \times 10^{-4}\,\mathrm{C} \qquad \Longrightarrow \qquad Q'_1 = \frac{-2.0 \times 10^{-4}\,\mathrm{C}}{4.0} = -5.0 \times 10^{-5}\,\mathrm{C}$$

Having solved for one of the unknowns, we're nearly finished!

The change in potential as we go from a to b is then:

$$V' = \frac{-Q'_1}{C_1} = \frac{+5.0 \times 10^{-5}\,\mathrm{C}}{1.0 \times 10^{-6}\,\mathrm{F}} = 50\,\mathrm{V}$$

(b) The *magnitude* of the new charge on capacitor 1 is $|Q'_1| = 5.0 \times 10^{-5}$ C

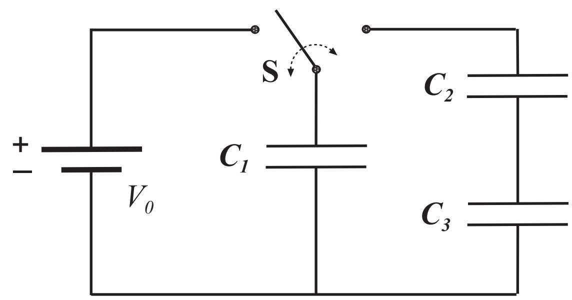

Figure 5.14: Configuration of capacitors and potential difference with switch for Example 11.

(c) Using $Q_2' = 3.0Q_1'$, the *magnitude* of the new charge on the second capacitor is

$$|Q_2'| = 3.0|Q_1'| = 3.0(5.0 \times 10^{-5}\ \mathrm{C}) = 1.5 \times 10^{-4}\ \mathrm{C} \ .$$

---

11. When switch S is thrown to the left in Fig. 5.14, the plates of capacitor 1 acquire a potential difference V_0. Capacitors 2 and 3 are initially uncharged. The switch is now thrown to the right. What are the final charges q_1, q_2 and q_3 on the capacitors?

Initially the only capacitor with a charge is C_1, with a charge given by:

$$q_{1,\,\mathrm{init}} = C_1 V_0 \tag{5.12}$$

since the potential across its plates is V_0.

Now consider what happens when the switch is thrown to the right and the capacitors have charges q_1, q_2 and q_3. Since C_2 and C_3 are joined in series, their charges will be equal, so $q_2 = q_3$ and we only need to find q_2. Also, note that the upper plate of C_1 is only connected to the upper plate of C_2 so that q_1 and q_2 must add up to give the original charge on C_1:

$$q_1 + q_2 = q_{1,\,\mathrm{init}} \tag{5.13}$$

Finally, we note that the potential difference across C_1 is equal to the potential difference across the C_2-C_3 series combination. The equivalent capacitance of the C_2-C_3 combination is:

$$\frac{1}{C_{\mathrm{equiv}}} = \frac{1}{C_2} + \frac{1}{C_3} = \frac{C_2 + C_3}{C_2 C_3} \qquad \Longrightarrow \qquad C_{\mathrm{equiv}} = \frac{C_2 C_3}{C_2 + C_3}$$

The potential across C_1 is q_1/C_1, and the potential across the series pair is q_2/C_{equiv}. So equating the potential differences gives

$$\frac{q_1}{C_1} = \frac{q_2}{C_{\mathrm{equiv}}} = \left(\frac{C_2 + C_3}{C_2 C_3}\right) q_2 \tag{5.14}$$

And that's all the equations we need; we can now solve for q_1 and q_2. Eq. 5.14 gives

$$q_2 = \frac{1}{C_1}\frac{C_2 C_3}{C_2 + C_3} q_1 \tag{5.15}$$

and then substitute this and also Eq. 5.12 into Eq. 5.13. We get:

$$q_1 + \frac{1}{C_1}\frac{C_2C_3}{C_2+C_3}q_1 = C_1V_0$$

Factor out q_1 on the left:

$$\left(1 + \frac{1}{C_1}\frac{C_2C_3}{C_2+C_3}\right)q_1 \;=\; \left(\frac{C_1(C_2+C_3)+C_2C_3}{C_1(C_2+C_3)}\right)q_1 \;=\; C_1V_0$$

Now we can isolate q_1:

$$q_1 = \frac{C_1^2(C_2+C_3)}{C_1C_2+C_1C_3+C_2C_3}V_0$$

Then go back and use Eq. 5.15 to get q_2:

$$\begin{aligned} q_2 &= \frac{1}{C_1}\frac{C_2C_3}{(C_2+C_3)}q_1 \;=\; \frac{1}{C_1}\frac{C_2C_3}{(C_2+C_3)}\frac{C_1^2(C_2+C_3)}{(C_1C_2+C_1C_3+C_2C_3)}V_0 \\ &= \frac{C_1C_2C_3}{C_1C_2+C_1C_3+C_2C_3}V_0 \end{aligned}$$

Finally, we recall that $q_3 = q_2$. This gives us expressions for all three charges in terms of the initial parameters.

5.2.4 Energy Stored in a Capacitor

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12. How much energy is stored in one cubic meter of air due to the "fair weather" electric field of magnitude $150\,\mathrm{V/m}$**?**

From Eq. 5.10 we have the energy density of an electric field. (As noted there, the *source* of the electric field is irrelevant.) We get:

$$\begin{aligned} u &= \tfrac{1}{2}\epsilon_0E^2 \\ &= \tfrac{1}{2}(8.85\times10^{-12}\,\tfrac{\mathrm{C^2}}{\mathrm{N\cdot m^2}})(150\tfrac{\mathrm{V}}{\mathrm{m}})^2 \;=\; 9.96\times10^{-8}\,\tfrac{\mathrm{J}}{\mathrm{m^3}} \end{aligned}$$

So in one cubic meter, 9.96×10^{-8} J of energy are stored.

---

13. What capacitance is required to store an energy of $10\,\mathrm{kW\cdot h}$ **at a potential difference of** $1000\,\mathrm{V}$**?**

First, convert the given energy to some sensible units!

$$E \;=\; 10\,\mathrm{kW\cdot h} = 10\times10^3\,\tfrac{\mathrm{J}}{\mathrm{s}}\cdot(1\,\mathrm{h})\left(\frac{3600\,\mathrm{s}}{1\,\mathrm{h}}\right) \;=\; 3.60\times10^7\,\mathrm{J}$$

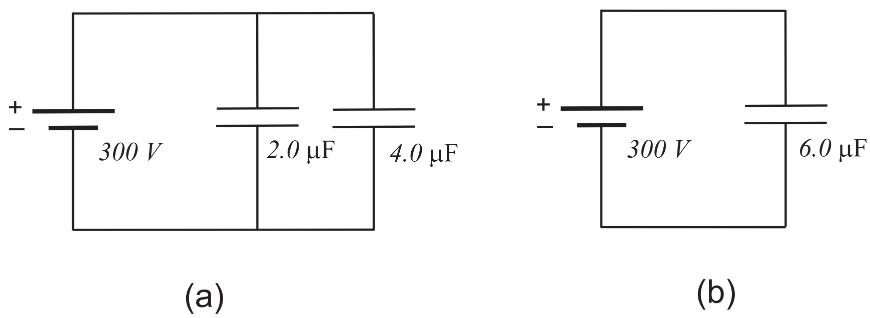

Figure 5.15: (a) Capacitor configuration for Example 14. (b) Equivalent capacitor.

Then use Eq. 5.9 for the energy stored in a capacitor:

$$E = \tfrac{1}{2}CV^2 \qquad \Longrightarrow \qquad C = \frac{2E}{V^2}$$

Plug in the numbers:

$$C \;=\; \frac{2(3.60 \times 10^7\,\mathrm{J})}{(1000\,\mathrm{V})^2} \;=\; 72\,\mathrm{F}$$

A capacitance of $72\,\mathrm{F}$ (big!) is needed.

---

14. Two capacitors, of 2.0 and $4.0\,\mu\mathrm{F}$ capacitance, are connected in parallel across a $300\,\mathrm{V}$ potential difference. Calculate the total energy stored in the capacitors.

The capacitors and potential difference are diagrammed in Fig. 5.15(a). For the purpose of finding the *total* energy in the capacitors we can replace the two parallel capacitors with a single equivalent capacitor of value $6.0\,\mu\mathrm{F}$ (the original two were in *parallel*, so we *sum* the values). This is because the charge which collects on the equivalent capacitor *is* the sum of charges on the plates of the original two capacitors.

Then the energy stored is

$$E = \tfrac{1}{2}CV^2 \;=\; \tfrac{1}{2}(6.0 \times 10^{-6}\,\mathrm{F})(300\,\mathrm{V})^2 \;=\; 0.27\,\mathrm{J}$$

Appendix A: Useful Numbers

Conversion Factors

Length	cm	meter	km	in	ft	mi
1 cm =	1	10^{-2}	10^{-5}	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 m =	100	1	10^{-3}	39.37	3.281	6.214×10^{-4}
1 km =	10^5	1000	1	3.937×10^4	3281	06214
1 in =	2.540	2.540×10^{-2}	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 ft =	30.48	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mi =	1.609×10^5	1609	1.609	6.336×10^4	5280	1

Mass	g	kg	slug	u
1 g =	1	0.001	6.852×10^{-2}	6.022×10^{26}
1 kg =	1000	1	6.852×10^{-5}	6.022×10^{23}
1 slug =	1.459×10^4	14.59	1	8.786×10^{27}
1 u =	1.661×10^{-24}	1.661×10^{-27}	1.138×10^{-28}	1

An object with a *weight* of 1 lb has a *mass* of 0.4536 kg.

Constants:

$$
\begin{aligned}
e &= 1.6022 \times 10^{-19}\,\mathrm{C} = 4.8032 \times 10^{-10}\,\mathrm{esu} \\
\epsilon_0 &= 8.85419 \times 10^{-12}\,\tfrac{\mathrm{C}^2}{\mathrm{N\cdot m^2}} \\
k &= 1/(4\pi\epsilon_0) = 8.9876 \times 10^9\,\tfrac{\mathrm{N\cdot m^2}}{\mathrm{C}^2} \\
\mu_0 &= 4\pi \times 10^{-7}\,\tfrac{\mathrm{N}}{\mathrm{A}^2} = 1.2566 \times 10^{-6}\,\tfrac{\mathrm{N}}{\mathrm{A}^2} \\
m_{\mathrm{electron}} &= 9.1094 \times 10^{-31}\,\mathrm{kg} \\
m_{\mathrm{proton}} &= 1.6726 \times 10^{-27}\,\mathrm{kg} \\
c &= 2.9979 \times 10^8\,\tfrac{\mathrm{m}}{\mathrm{s}} \\
N_A &= 6.0221 \times 10^{23}\,\mathrm{mol}^{-1}
\end{aligned}
$$