ELECTRIC FLUX, GAUSS'S LAW AND ELECTRIC POTENTIAL

LECTURE NOTE ON PHY102

BY

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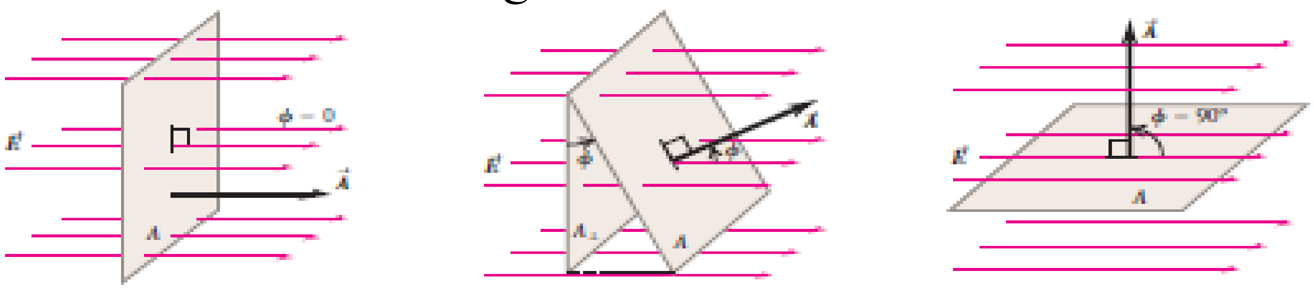
INTRODUCTION

- Apart from Coulomb's Law of Electrostatics, a simple way of estimating the electric field due to a charge is the *Gauss' Law*. Gauss's Law relates the electric field at points on an imaginary closed surface (Gaussian surface) and the net charge enclosed by the surface.
- Coulomb's Law illustrates the interaction of electric field created by point charges using the Faraday's concept of electric field lines. In Gauss' Law, this interaction can be illustrated using the concept of electric flux.

ELECTRIC FLUX (ϕ)

Electric flux is defined as the number of electric field lines that passes through a given surface. Let us consider a surface (say a square) immersed in a uniform electric field as shown in the figure below:



These electric field lines that flow through the defined surface is called the **flux**. The magnitude of the flux, ϕ is directly proportional to the number of field lines that passes through the given surface. Hence,

$$\phi = E.A = EA\cos\theta \ldots\ldots\ldots\ldots(1)$$

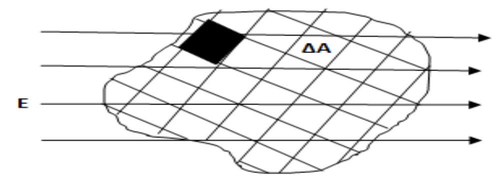
Where θ is the angle between the field and the normal to the surface. If the field direction and the surface are perpendicular, then the field direction and the normal to the surface are parallel. Then equation (1) becomes:

$$\phi = EA\cos 0 = EA \ldots\ldots\ldots\ldots(2)$$

If the field direction and the surface are parallel, then the field direction and the normal to the surface are perpendicular. Then equation (1) becomes:

$$\phi = EA\cos 90 = 0 \ldots\ldots\ldots\ldots(3)$$

If the electric field is not uniform and the surface is not flat, we divide the surface into small elemental area ΔA in such a way that the electric field at ΔA is uniform as shown in the figure below. Then, the electric flux through the entire surface will be the sum of the 'fluxes' through each surface.



As the area vector approaches a differential limit dA, the equation becomes:

$$\phi = \oint E.dA$$

NOTE:

1. There can be no excess charge at any point within a solid conductor; any excess charge must reside on the conductor's surface.
2. The total flux through a surface depends on the amount of charge Q enclosed in the surface.
3. When a conductor is in equilibrium, the electric field everywhere inside the conductor is zero (electrostatic conduction) i.e. no net charge inside the conductor

4. For enclosed surface, enclosing no charge:

$$\phi = \oint E.dA = 0$$

That is, when a region contains no charge, any field lines caused by charges outside the region that enters on one side must leave again on the other side.

5. Electric field lines can begin or end inside a region of space only when there is charge in that region.

Example 1:

A positive point charge 4.0μC is surrounded by a sphere with radius 0.26m centred on the charge. Find the electric flux through the sphere due to this charge.

Solution

Q = 4 μC = 4 x 10^{-6}C, r = 0.26m, k = 9 x 10^9Nm²/C²

$$E = \frac{kQ}{r^2} = \frac{9\times10^9\times4\times10^{-6}}{(0.26)^2} = 5.33\times10^5\,N/C$$

$$\phi = \oint E.dA = E\oint dA$$

But $\oint dA = 4\pi r^2$

$$\phi = E\oint dA = E.4\pi r^2 = 5.33\times10^5\times4\pi(0.26)^2 = 4.52\times10^5\,Nm^2/C$$

GAUSS' LAW

The precise relationship between the flux through a closed (Gaussian) surface and the net charge Q within that surface is given by Gauss' Law. This law states that:

The total electric flux through a closed surface is equal to the total net charge, q inside the surface divided by the permittivity, ε_0

$$\phi = \frac{q_{enclosed}}{\varepsilon_0}$$

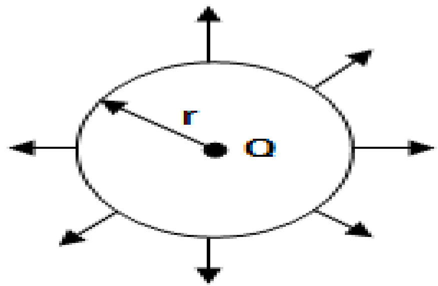
We can also write the equation above as:

$$\phi = \oint E.dA = \frac{q_{enclosed}}{\varepsilon_0}$$

To apply Gauss' Law, usually a hypothetical closed surface is chosen. Although Gaussian surface of any shape can be taken, the problem is often simplified if a Gaussian surface that mimics symmetry of charge distribution is chosen.

APPLYING GAUSS LAW TO A POINT CHARGE

Lets enclose the point charge Q in a spherical Gaussian surface of radius r as shown below



The total flux ϕ, through the surface is given as:

$$\phi = \oint EdA = \frac{q_{enclosed}}{\varepsilon_0}$$

But $\oint dA$ = surface area of a sphere = $4\pi r^2$

Also, $E = \dfrac{kQ}{r^2}$

Therefore,

$$\phi = \oint EdA = \left(\frac{kQ}{r^2}\right)\left(4\pi r^2\right) = 4\pi kQ$$

This means that the flux is independent of the radius of the sphere. It only depends on the charge enclosed by the sphere.

Example 2

What is the electric flux through a sphere that has a radius of 1.00m and carries a charge of +1.00μC at its centre?

Solution

$\phi = 4\pi kQ$

$\phi = 4\pi \times 9 \times 10^{9} \times 10^{-6}$

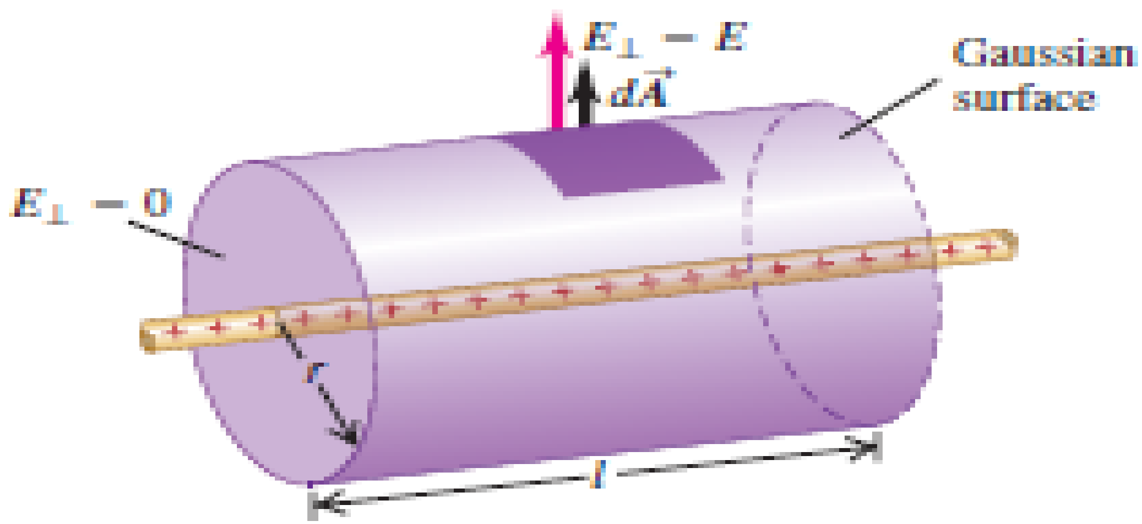
$\phi = 1.13 \times 10^{5}\ Nm^2/C^2$

GAUSS LAW AND CHARGE DISTRIBUTION

To use Gauss law to determine the electric field strength due to distribution of charge over a surface, we first construct a Gaussian surface over the charge distribution to pass through the point of interest and evaluate the flux at the surface of the Gaussian.

To Determine the Electric Field of an Infinite Line of Charge

We consider uniform distribution of positive charges (of total charge Q) along an infinite long thin wire of charge per unit length λ (charge/length). Then construct a Gaussian surface to surround it and evaluate the flux on a small segment of area dA and sum over the whole volume of the surface.



From Gauss Law,

$$\phi = \oint E dA = \frac{q}{\varepsilon_0}$$

$\oint dA =$ Curved surface area of a cylinder, $2\pi rl$

Therefore, $$\phi = E\oint dA = 2\pi r l E = \frac{q}{\varepsilon_0}$$

$$E = \frac{q}{2\pi r l \varepsilon_0} = \frac{\lambda}{2\pi r \varepsilon_0}$$

Where $q/l = \lambda$ (linear charge density)

Example 3

Evaluate the magnitude of electric field strength at a point 3.0cm from an infinite line of charge of linear charge density 18.0μC/cm, situated in a medium of relative permittivity 1.5

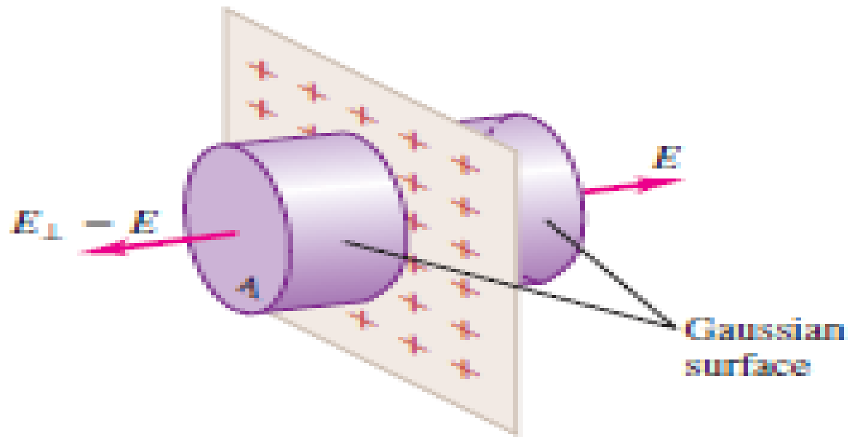
Solution

$$E = \frac{\lambda}{2\pi r \varepsilon} = \frac{\lambda}{2\pi r \varepsilon_0 \varepsilon_r}$$

$$E = \frac{18\times10^{-4}}{2\times3.142\times3\times10^{-2}\times8.85\times10^{-12}\times1.5} = 7.19\times10^{8}\,N/C$$

To Determine the Electric Field of an Infinite Plane Sheet of Charge

We consider a charge Q distributed uniformly over a very large non-conducting plane sheet of surface charge density, σ. To determine the field, we construct a Gaussian surface in the form of a closed cylinder over a small segment of the sheet with the axis of the cylinder perpendicular to the sheet and the electric field E directly perpendicular to the plane on both sides.



There is no flux through the curved part of the Gaussian; all the flux is through the two flat ends. From Gauss Law, we have:

$$\phi = \oint E dA = \frac{q}{\varepsilon_0}$$

Since there are two circular surface, the equation can be written as:

$$\phi = 2\oint EdA = 2EA = \frac{q}{\varepsilon_0} \quad \text{where } \oint dA = A$$

Hence, $E = \frac{q}{2A\varepsilon_0} = \frac{\sigma}{2\varepsilon_0}$

Where q/A = σ (surface charge density)

Applications of Gauss Law

1. To solve complex electrostatic problems involving unique symmetries like cylindrical, spherical or planar symmetry.
2. Used to simplify the evaluation of electric field which may be complex as a result of tough integration.

PROBLEMS

1. The nucleus of an atom has a charge of 3e, where e is the electronic charge. Find the flux through a sphere of radius 1Å (1Å = 10^{-10}m)
2. A physicist surrounded a point charge of 5.0μC with a sphere of radius 0.26m centred on the charge. Calculate the electric flux through the sphere.
3. What will be the magnitude of the electric field strength at a point 5.0cm from an infinite line of charge of linear charge density 21.5μC/cm situated in a medium of relative permittivity 2.5?
4. A negative point charge 4.8 μC is surrounded by a sphere with radius 0.55m centred on the charge. Find the magnitude of the electric field and electric flux through the sphere due to this charge.

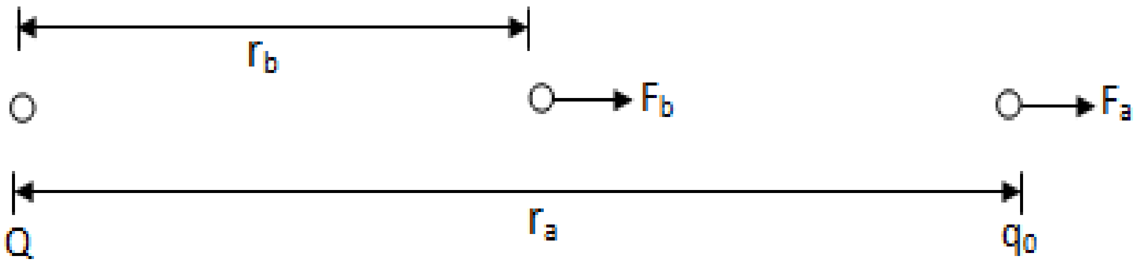
ELECTRIC POTENTIAL

INTRODUCTION

The work done on a charged particle when it moves in an electric field under the influence of the field can de defined in terms of the Electric Potential Energy. Since electric force is a conservative force, the work done on the charged particle in the field will depend on the position.

ELECTRIC POTENTIAL ENERGY, U

In a region where there is an existing reference charge Q, a test charge q_0 introduced into the region at a distance r_a will experience a conservative force F_a exerted on it by the field.



If we now choose to move the test charge q_0 to a point r_b closer to the reference charge Q, it will experience a greater force F_b directed away from Q. Since the force involved is conservative, work is done on the charge. This work done by the force can be expressed in terms of the Electric Potential Energy, U.

When the charged particle is moved from a point where the potential energy is U_a to a point where the energy is U_b, the work done on the charge is equal to the change in potential energy ΔU experienced by the particle between the two points i.e.

$$W_{a\to b} = U_a - U_b = -(U_b - U_a) = -\Delta U$$

If the field is uniform, then the force can be defined as: $F = q_0 E$

Thus, we can re-write the equation above as:

$$W_{a\to b} = -q_0 E(r_b - r_a)$$

Electric Potential Energy, U can de defined as the work that must be done to move a charged particle between two points within the field (i.e. from the reference position r_a to a new position r_b).

The work done dW is defined as:

$$U = \int dW = \int F dr = \int_{r_a}^{r_b} F.dr$$

But the electrostatic force, $F = \dfrac{kQq_0}{r^2}$

Substituting into equation (*)

$$U = \int_{r_a}^{r_b} \frac{kQq_0}{r^2} dr = kQq_0 \left(\frac{1}{r_a} - \frac{1}{r_b} \right)$$

In the presence of several point charges, the net potential energy follows the superposition law i.e.

$$U = kq_0 \left[\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \ldots + \frac{Q_n}{r_n} \right] = kq_0 \sum_{i=1}^{n} \frac{Q_i}{r_i}$$

For static charge distribution in a field, the total potential energy U is the sum of the potential energy of interaction for each pair of charge i.e.

$$U = k \sum_{i<j} \frac{Q_i Q_j}{r_{ij}}$$

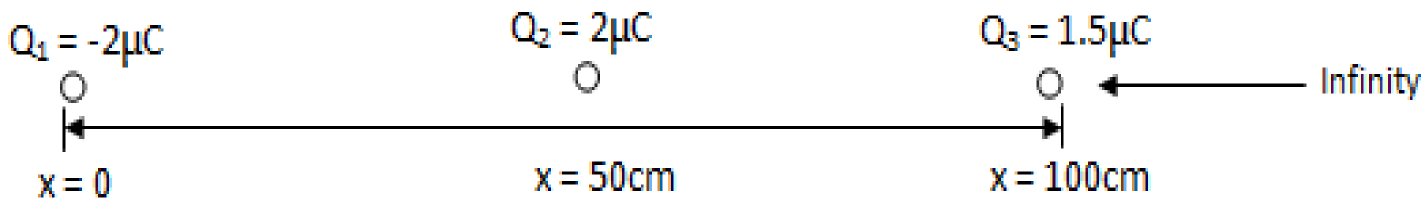
Example 1:

Two point charges Q1=-2μC is located at x = 0 and Q2 = +2μC is located at x = 50cm. Find

i. The work done by external force to bring a charge Q3 = 1.5μC from infinity to x = 100cm

ii. The total potential energy of the system of the three charges

Solution



The work done by the external force to bring Q_3 from infinity to $x = 100\text{cm}$ is the total work done on Q_3 in the field caused by Q_1 and Q_2. Thus,

$$W = U = kQ_3\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2}\right)$$

$$U = 9\times10^{9}\times1.5\times10^{-6}\left(\frac{-2\times10^{-6}}{1} + \frac{2\times10^{-6}}{0.5}\right) = 2.7\times10^{-2}\,J$$

The potential energy of the collection of the three charges is:

$$U = k\sum_{i<j}\frac{Q_iQ_j}{r_{ij}} = k\left[\frac{Q_1Q_2}{r_{12}} + \frac{Q_1Q_3}{r_{13}} + \frac{Q_2Q_3}{r_{23}}\right]$$

$$U = 9\times10^{9}\left[\frac{(-2\times2)}{0.5} + \frac{(-2\times1.5)}{1} + \frac{(2\times1.5)}{0.5}\right]\times10^{-12} = -4.5\times10^{-2}\,J$$

The negative sign means that work is done by external force to bring the 3 charges to the location from infinity.

ELECTRIC POTENTIAL, V

The work done in moving a test charge q_0 from infinity ($r=\infty$) to a position r, within the electric field generated by the presence of another charge Q is defined by the equation below:

$$U = kQq_0\left(\frac{1}{r}\right)$$

The potential energy per unit charge at a point in an electric field is called the Electric Potential (V).

$$V = \frac{U}{q_0}$$

Substituting into the first equation,

$$V = \frac{kQ}{r}$$

Electric potential, V is a scalar quantity measured in Volts. Also, V = Ed

Example 2

An electron is placed between two parallel plate conductors separated by a distance of 50cm. If the final velocity of the electron is 10^8m/s, calculate:

a. The work done on the electron

b. The potential difference and the electric field between the two plates

Solution

m = 9.1 x 10^{-31}kg, v = 10^{8}m/s, q = 1.6 x 10^{-19}C, d = 50cm = 50 x 10^{-2}m

a. The work done, W equals the kinetic energy of the electron, since the work done is by a conservative force

$$W = \frac{1}{2}mv^2 = \frac{1}{2}\times 9.1\times 10^{-31}\times \left(10^{8}\right)^2 = 4.55\times 10^{-15}\,J$$

b. The potential difference, V

$$V = \frac{W}{q} = \frac{4.55\times 10^{-15}}{1.6\times 10^{-19}} = 2.84\times 10^{4}\,V$$

The electric field, E

$$E = \frac{V}{d} = \frac{2.84\times 10^{4}}{50\times 10^{-2}} = 5.68\times 10^{4}\,N/C$$