HARMONIC ANALYSIS AND INVERSE PROBLEMS

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Chapter 1

Introduction.

These are the notes of a course that I lectured in The University of Oulu (Finland) during the summer 2002. I completed afterwards some parts and I am still working some other parts that will be soon actualized. I would like to thank Lassi Päivärinta for giving to me the opportunity to enjoy the wonderful environment of that summer and the beautiful trip he organized for me and Juan Antonio Barceló beyond the Arctic Circle.

1.1 Inverse problems

1.2 Brief history.

In construction

Chapter 2

The Helmholtz equation

2.1 Why the Helmholtz equation?

We are going to introduce two inverse problems which in a first treatment reduce to Helmholtz operator $(\Delta + k^2 n(x))$ in the whole space $\mathbb{R}^n$ or to the constant coefficient Hemholtz operator in exterior domains: the scattering problems, for either an inhomogeneous media or an obstacle, and Calderón inverse conductivity problem.

The precise introduction to Calderón inverse conductivity problem can be found in section 6.3 of these notes.

Calderón problem for the case of sufficiently regular isotropic conductivities γI can be reduced by the change $u(x) = \gamma^{-1/2} v(x)$ to the Schödinger equation, see chapter 6.

The study of this inverse problem has been inspirated by the pionering work of Calderón, who proved that the first variational derivative of the operator $\gamma \to \Lambda_\gamma$ at the value $\gamma_0 = 1$ is one to one. To motivate the further development in these notes we give a proof of this fact.

We start with a characterization of the map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

by the sequilinear form on $H^{1/2}(\partial\Omega)$ defined as

$$Q(f, g) := \Lambda_\gamma(f)(\bar{g}) = \int_\Omega \gamma \bigtriangledown u \cdot \overline{\bigtriangledown v},$$

where u and v are weak $W^{1,2}(\Omega)$ solutions of the conductivity initial value problems with Dirichlet datum respectively f and g in the boundary of Ω.

This fact is proved in section 6.3 (it is just an application of Green formula) and it is used as a weak definition of the Dirichlet to Neumann map.

Theorem 2.1. *The differential of the map $\gamma \to \Lambda_\gamma$ at $\gamma = 1$ is one to one.*

To prove Calderón's theorem we take a variation with $h \in C_0^\infty(\Omega)$. Take f and g and the corresponding $W^{1,2}(\Omega)$-solutions u (resp.v) of the Dirichlet boundary value problem for the operator $Div(1+\epsilon h(x))\bigtriangledown$ with datum f (resp. g). We need to evaluate

$$(D\Lambda)_{\gamma=1}(h)(f,g) = \lim_{\epsilon\longrightarrow 0} \frac{Q_{1+\epsilon h(x)}(f,g) - Q_1(h)(f,g)}{\epsilon}.$$

We write $u = u_0 + \delta(u)$ (resp. $v = v_0 + \delta(v)$), where u_0 (resp. v_0) solves $\Delta u_0 = 0$ (resp. $\Delta v_0 = 0$) with datum f (resp g). Then,

$$\Delta\delta(u) = -div(\epsilon h \bigtriangledown u_0) - div(\epsilon h \bigtriangledown \delta(u)),$$

with zero Dirichlet data. From a priori estimates for solution of elliptic homogeneous Dirichtet problems one has:

$$\|\delta(u)\|_{W^{1,2}} \leq C\| - div(\epsilon h \bigtriangledown u_0) - div(\epsilon h \bigtriangledown \delta(u))\|_{W^{-1,2}} \leq C\epsilon\|u_0\|_{W^{1,2}} + C\epsilon\|\delta(u)\|_{W^{1,2}}.$$

Hence for small ϵ,

$$\|\delta(u)\|_{W^{1,2}} \leq C\epsilon\|u_0\|_{W^{1,2}}. \tag{2.1}$$

Now we have

$$Q_{1+\epsilon h(x)}(f,g) = \int_\Omega (1+\epsilon h) \bigtriangledown u \cdot \bigtriangledown \bar{v} dx =$$

$$\int_\Omega (1+\epsilon h) \bigtriangledown (u_0 + \delta(u)) \cdot \bigtriangledown (v_0 \bar{+} \delta(v)) dx = \int_\Omega \bigtriangledown u_0 \cdot \bigtriangledown \bar{v_0} dx$$

$$+\epsilon \left(\int_\Omega h \bigtriangledown u_0 \cdot \bar{\bigtriangledown v_0} + \int_\Omega h \bigtriangledown u_0 \cdot \bigtriangledown \delta(\bar{v}) + \int_\Omega h \bigtriangledown \delta(u) \cdot \bigtriangledown \bar{v_0} + \int_\Omega h \bigtriangledown \delta(u) \cdot \bigtriangledown \delta(\bar{v}) \right)$$

$$+ \int_\Omega \bigtriangledown \delta(u) \cdot \bigtriangledown \delta(\bar{v}) + \int_\Omega \bigtriangledown u_0 \cdot \bigtriangledown \delta(\bar{v}) + \int_\Omega \bigtriangledown \bar{v}_0 \cdot \bigtriangledown \delta(u).$$

By Green formula and the zero data the last two integral vanish, hence

$$\frac{Q_{1+\epsilon h(x)}(f,g) - Q_1(f,g)}{\epsilon}$$

$$= \int_\Omega h \bigtriangledown u_0 \cdot \bar{\bigtriangledown v_0} + \int_\Omega h \bigtriangledown u_0 \cdot \bigtriangledown \delta(\bar{v}) + \int_\Omega h \bigtriangledown \delta(u) \cdot \bigtriangledown \bar{v_0} + \int_\Omega h \bigtriangledown \delta(u) \cdot \bigtriangledown \delta(\bar{v})$$

$$+ \int_\Omega h \bigtriangledown \delta(u) \cdot \bigtriangledown \delta(\bar{v}) + \frac{1}{\epsilon} \int_\Omega \bigtriangledown \delta(u) \cdot \bigtriangledown \delta(\bar{v}).$$

By using Cauchy-Schwarz inequality and (2.1) we have that

$$\lim_{\epsilon\longrightarrow 0} \frac{Q_{1+\epsilon h(x)}(f,g) - Q_1(f,g)}{\epsilon} = \int_\Omega h \bigtriangledown u_0 \cdot \bar{\bigtriangledown v_0}.$$

The above expresion gives a linear operator on h which is the Frechet derivative of the map taking the conductivity to the D-N map. To prove the injectivity, assume that for any f and g

$$(D\Lambda)_{\gamma=1}(h)(f,g) = \int_\Omega h \bigtriangledown u_0 \cdot \bar{\bigtriangledown v_0} = 0,$$

It follows that $h = 0$ as far as the products of gradient of harmonic functions $\bigtriangledown u_0 \cdot \bar{\bigtriangledown} v_0$ form a dense class in, let us say, $L^2(\Omega)$. We make the following choice of harmonic functions (complex exponentials):

$$u_0(x) = e^{z_1 \cdot x}$$

$$v_0(x) = e^{z_2 \cdot x},$$

where the conditions for $z_i \in \mathbb{C}^n$ that $z_i \cdot z_i = 0$ assure they are harmonic functions. Given the vector $\xi \in \mathbb{R}^n$, we choose $z_1 = \eta + i\xi$ and $z_2 = -\eta - i\xi$ so that $\xi \cdot \eta = 0$ and $\|\eta\| = \|\xi\|$. Since the conditions $z_i \cdot z_i = 0$ mean $\xi \cdot \eta = 0$ and $|\eta| = |\xi|$, we obtain

$$0 = \int_\Omega h \bigtriangledown u_0 \cdot \overline{\bigtriangledown v_0} = -2\|\xi\|^2 \int_\Omega h e^{2i\xi \cdot x} dx.$$

From this identity it follows that $h(x) = 0$ on Ω. This proves the injectivity of the differential. □

Any later study of the uniqueness of the inverse problem is influence by Calderón's approach. Namely, after reducing to the Schödinger equation

$$(\Delta + q(x))v = 0,$$

one looks for approximate Calderón's solutions with ansatz $v(x) = e^{z \cdot x} w(z, x))$ where $w \to 1$ as $|z| \to \infty$. By inserting the ansatz in the equation, we need to solve the non homogeneous Fadeev equation

$$(\Delta + 2z \cdot \bigtriangledown)w = -qw. \tag{2.2}$$

This is a motivation to study the perturbations of Laplace operator by lower order operators with constant complex coefficients. This will be achived in chapter .

2.2 Fundamental solutions

The same procedure to find the fundamental solution of the Laplace operator in $\mathbf{R}^n$, can be used in the context of the Helmholtz equation; since the fundamental solution with pole at the origin is singular at this point, we may use polar coordinates to look for solutions out of the origin.

The homogeneous Helmholtz equation

$$(\Delta + k^2)w = 0, \tag{2.3}$$

by taking $u(y) = w(y/k)$, can be reduced to $k = 1$ for u.

Writing the Laplace operator in polar coordinates

$$\Delta = r^{-(n-1)} \partial_r (r^{n-1} \partial_r) + 1/r^2 \Delta_S$$

and assuming that $u(x) = u(|x|)$, is a radial function, we may reduce to solve for $|x| = r \in (0, \infty)$ the modified Bessel ordinary differential equation

$$u_{rr} + \frac{n-1}{r} u_r + u = 0. \tag{2.4}$$

Now the change $u(r) = r^{-(n-2)/2} v(r)$, reduces to Bessel equation of order $\lambda = (n-2)/2$,

$$v_{rr} + \frac{1}{r} v_r + (1 - \frac{\lambda^2}{r^2}) v = 0. \tag{2.5}$$

The usual power series expantion method (Frobenius method) in o.d.e., gives the power series at the origin, when $\lambda \neq 0$:

$$J_\lambda(t) = \sum_{p=0}^{\infty} (-1)^p \frac{1}{p!\Gamma(p+\lambda+1)} \left(\frac{t}{2}\right)^{2p+\lambda} \tag{2.6}$$

The second independent solution has a expression more complicated

$$N_\lambda(t) = \frac{cos\lambda\pi J_\lambda(t) - J_{-\lambda}(t)}{sin\lambda\pi} \tag{2.7}$$

when λ is an integer, the same formula holds understood as a limit as λ approaches the integer.

We use the fundamental set of complex conjugate solutions, known as Hankel functions, $H_\lambda^{(1)}(r) = J_\lambda + iN_\lambda$ and $H_\lambda^{(2)}(r)$.

Lemma 2.1. *The following asymptotic for $r \to 0$ holds, if $\lambda \neq 0$*

$$H_\lambda^{(1)}(r) = -\frac{i}{\pi} \Gamma(\lambda) (\frac{2}{r})^\lambda \tag{2.8}$$

If $\lambda = 0$, then

$$H_0^{(1)}(r) = \frac{2i}{\pi} \log r \tag{2.9}$$

(The case $\lambda = 0$ is special, the second solution, since the inditial equation in Frobenius method has 0 as a double root, has a logarithmic singularity).

In order to normalize the function so obtained, let us define

$$\Phi(x) = c_n k^{(n-2)/2} \frac{H^{(1)}_{(n-2)/2}(k|x|)}{|x|^{(n-2)/2}}, \text{ where } c_n = \frac{1}{2i(2\pi)^{(n-2)/2}}. \tag{2.10}$$

This expresion can be simplified in dimension $n = 3$, since

$$H^{(1)}_{1/2}(r) = -i(\frac{2}{\pi r})^{1/2} e^{ir},$$

to

$$\Phi(x) = \frac{e^{ik|x|}}{4\pi|x|}. \tag{2.11}$$

In dimension $n = 2$, we have

$$\Phi(x) = -\frac{i}{4} H_0^{(1)}(k|x|). \tag{2.12}$$

Lemma 2.2.

$$\begin{aligned} &\frac{d}{dr}H_\lambda^{(1)}(r) = H_{\lambda-1}^{(1)}(r) - \frac{\lambda}{r}H_\lambda^{(1)}(r), \\ &H_{\lambda-1}^{(1)}(r) = r^{-\lambda}\frac{d}{dr}(H_\lambda^{(1)}(r)r^\lambda), \\ &H_{-\lambda}^{(1)}(r) = e^{i\lambda\pi}H_\lambda^{(1)}(r). \end{aligned} \tag{2.13}$$

It can be easily checked, by lemmas 2.1 y 2.2, that for $|x| = r$ we have

Lemma 2.3.

$$\frac{\partial\Phi(x)}{\partial r} = \frac{1}{\omega_{n-1}r^{n-1}} + O(r^{-n+2}) \text{ if } r \to 0, \tag{2.14}$$

where ω_{n-1} is the measure of the unit sphere in $\mathbf{R^n}$.

We will need the following lemma to understand the behavior at infinity of these solutions:

Lemma 2.4.

$$H_\lambda^{(1)}(r) = (\frac{2}{\pi r})^{1/2}e^{i(r-\frac{\lambda\pi}{2}-\frac{\pi}{4})} + O(r^{-3/2}) \text{ if } \lambda \geq 0, \text{ when } r \to \infty. \tag{2.15}$$

Lemma 2.5 (Green formulas)**.** *Let D be a bounded domain with $\mathcal{C}^1$ boundary, and ν its exterior unit normal, let $v \in \mathcal{C}^2(\bar{D})$.*
(1) If $u \in \mathcal{C}^1(\bar{D})$, then

$$\int_D (u\Delta v + \bigtriangledown u \cdot \bigtriangledown v)dx = \int_{\partial D} u\frac{\partial v}{\partial \nu}dS \tag{2.16}$$

(2) If we assume $u \in \mathcal{C}^2(\bar{D})$,then

$$\int_D (u\Delta v - v\Delta u)dx = \int_{\partial D}(u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu})dS. \tag{2.17}$$

Remark: To study some inverse problems, like the obstacle scattering, it is convenient to relax the regularity assumptions in Green formulas, this requires to extend them to domains for which the trace operator on the boundary makes sense (Calderón theorem for Lipschitz domain) and for functions whose traces can be defined as functions in $L^2(\partial D)$, see Lion y Magenes[LiM] or Necas [Ne].

Now we can prove

Proposition 2.1. *Let us take $\Phi(x,y) = \Phi(|x-y|)$. This function is a fundamental solution with pole y of the Helmholtz equation.*

Its behavior at $r = |x| \to \infty$ is

$$\Phi(x,y) = C_n k^{(n-3)/2}e^{ikr}r^{-(n-1)/2}e^{-iky\cdot\frac{x}{r}} + O(r^{-n/2})) \tag{2.18}$$

Furthermore, it satisfies the outgoing Sommerfeld radiation condition:

$$\frac{d}{dr}\Phi(x,y) - ik\Phi(x,y) = o(r^{-\frac{n-1}{2}}), \tag{2.19}$$

uniformly for y in compact sets, when $|x| = r \to \infty$.

Proof

To prove that it is a fundamental solution, we need to see that for $f \in \mathcal{C}_0^\infty$ we have

$$\int \Phi(x,y)(\Delta + k^2)f(x)dx = f(y),$$

It is enough to prove the case $y = 0$. By Green formula

$$\int_{R^n \backslash B(0,\epsilon)} \Phi(x)(\Delta + k^2)f(x)dx = -\int_{S(0,1)} \frac{\partial}{\partial r}(\Phi)(\epsilon)f(\epsilon\theta)(\epsilon)^{n-1}d\sigma(\theta) + C(\epsilon),$$

where, from the asymptotics of Φ as $|x| \to 0$, $C(\epsilon) \to 0$ as $\epsilon \to 0$.

The normalization above, see lemma 2.3, allows us to claim that for $\epsilon \to 0$, this integral converges to $f(0)$.

The asymptotic behavior follows from (2.15), together with

$$|x - y| = |x| - y \cdot x/|x| + O(|x|^{-1})$$

which holds for $|x| \to \infty$, uniformly for y in compact sets.

To obtain the radiation condition, notice that by using lemma 2.2, one has for $s = |x-y|$ y $r = |x|$, $\lambda = (n-2)/2$

$$\begin{aligned}\frac{d}{dr}\Phi(x,y) &= c_n k^{(n-2)/2}\left(sk\frac{d}{ds}H_\lambda^{(1)}(ks) - \lambda H_\lambda^{(1)}(ks)\right)s^{-n/2}\frac{(x-y)\cdot x}{rs}\\ &= c_n k^{(n-2)/2}\left(skH_{\lambda-1}^{(1)}(ks) - k\lambda H_\lambda^{(1)}(ks) - \lambda H_\lambda^{(1)}(ks)\right)s^{-n/2}\frac{(x-y)\cdot x}{rs},\end{aligned}$$

since Hankel functions have a decay as $s^{-1/2}$ and s is asymptotically as r for y on compact sets, the above expresion can be written as

$$c_n k^{(n-2)/2} k H_{\lambda-1}^{(1)}(kr) r^{-(n-2)/2} + O(r^{-n/2}) \text{ if } r \to \infty.$$

Hence

$$\frac{d}{dr}\Phi(x,y) - ik\Phi(x,y) = c_n k^{(n-2)/2}(kH_{\lambda-1}^{(1)}(kr) - e^{i\pi/2}kH_\lambda^{(1)}(kr))r^{-(n-2)/2} + O(r^{-n/2}),$$

now the asymptotic behavior of Hankel functions, lemma 2.4, allows us to cancel the main terms in the difference and to see that this expresion is like $O(r^{-n/2})$.

We can repeat the above calculation for any first order partial derivative of the fundamental solution, just notice the homogeneity of the x_j and y_j partial derivatives of $|x-y|$ and lemmas 2.5 and (2.10), and write,

Proposition 2.2. *Let D be a $\mathcal{C}^1$ domain and consider the normal derivative $F(x,y) = \frac{\partial\Phi(x,\cdot)}{\partial\nu}(y)$; then F is a fundamental solution of Helmholtz operator in $x \neq y$, which at infinity behaves asymptotically as*

$$C_n k^{(n-1)/2} e^{ik|x|}|x|^{-(n-1)/2}\frac{\partial}{\partial\nu_y}e^{-iky\cdot\frac{x}{|x|}} + o(|x|^{-(n-1)/2})). \tag{2.20}$$

Furthetmore F satisfies the outgoing Sommerfeld radiation condition uniformly for $y \in \partial D$.

This proposition is very useful to define single and doble layer potentials on ∂D, in the same way as for the Laplace operator. Notice that the behavior at the pole of the fundamental solution and its normal derivatives are the same as those of the Laplace operator Φ_0, let us remind that

$$\begin{aligned} \Phi_0(x,y) &= C'_n|x-y|^{-(n-2)} \text{if } n > 2. \\ \Phi_0(x,y) &= C'_2 \log|x-y| \text{ if } n = 2. \end{aligned} \tag{2.21}$$

Hence the estimates for layer potentials are the same in both equations.

One of the main differences with the Laplace operator is the asymptotic behavior when $x \to \infty$ of the fundamental solutions. To prove several results, like estimates with dependence control on k, we need to consider the oscillatory character of the fundamental solution of the Helmholtz equation; for these kind of results the Helmholtz equation can be considered of hyperbolic type (it is also known as a reduced wave equation).

2.3 Integral representation formulae

The following representation formulae are similar to those very well known for the Laplace operator.

Theorem 2.2 (Helmholtz representation in bounded domains)**.** *Let D be a bounded $\mathcal{C}^1$ domain and ν its exterior unit normal at $y \in \partial D$, let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$, then if $x \in D$,*

$$\begin{aligned} u(x) = &\int_{\partial D} (\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,\cdot)}{\partial \nu}(y))dS(y) \\ &- \int_D \Phi(x,y)(\Delta + k^2)u(y)dy, \end{aligned} \tag{2.22}$$

We may change the hypothesis and assume $D \in \mathcal{C}^2$ and $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\bar{D})$, in this case we have to assume the existence of the normal derivative as a limit for $x \in \partial D$:

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h \to 0} \nu(x) \cdot \bigtriangledown u(x - h\nu(x)), \tag{2.23}$$

uniformly in ∂D.

For a solution u of the Helmholtz equation we have

$$u(x) = \int_{\partial D} (\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,\cdot)}{\partial \nu}(y))dS(y) \tag{2.24}$$

Proof: We reduce for simplicity to the case $n = 3$, then we have the exact expresion (2.11). The n-dimensional case is similar but using the asymptotic behavior at the origin of the radial derivative, given by lemma 2.3. Let us take the domain $D_\epsilon = D - B(x,\epsilon)$ and its boundary $\partial D \cup S(x,\epsilon)$. Φ is a solution of the Helmholtz equation on D_ϵ, hence the second Green formula allows us to write

$$\begin{aligned}\int_{\partial D\cup S(x,\epsilon)} &(\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,\cdot)}{\partial \nu}(y))dS(y)\\ &= \int_{D_\epsilon}(\Delta u(y) + k^2 u(y))\Phi(x,y)dy.\end{aligned}$$

If $|x-y| = \epsilon$ then $\Phi(x,y) = \frac{e^{ik\epsilon}}{4\pi\epsilon}$ and $\frac{\partial\Phi(x,\cdot)}{\partial\nu}(y) = (1/\epsilon - ik)\frac{e^{ik\epsilon}}{4\pi\epsilon}$. The boundary integral on $S(x,\epsilon)$ is given by

$$\int_{S(x,\epsilon)}\frac{\partial u}{\partial \nu}(y)\frac{e^{ik\epsilon}}{4\pi\epsilon}dS(y) - \int_{S(x,\epsilon)} u(y)(1/\epsilon - ik)\frac{e^{ik\epsilon}}{4\pi\epsilon}dS(y).$$

Take $\epsilon \to 0$, by the mean value theorem this expresion converges to $u(x)$. The limit of the volume integral does exists since Φ has a weak singularity.

The second case in the statement can be reduced to the previous one, by integrating over surfaces parallered to ∂D at distance h in the normal direction and then taking the limit as $h \to 0$. The condition $D \in \mathcal{C}^2$ makes these surfaces to be $\mathcal{C}^1$, for h sufficiently small.

Proposition 2.3 (Corollary)**.** *Let D and u be as in the previous theorem and $(\Delta + k^2)u = 0$ in D, then u is analytic in D.*

Theorem 2.3. *Let Ω be an exterior domain, i.e. the complement of a bounded domain D, as in the previous theorem. Let u be a solution of the Helmholtz equation in Ω such that $u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}^1(\bar\Omega)$ if $D \in \mathcal{C}^1$, or such that $u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}^0(\bar\Omega)$ if $D \in \mathcal{C}^2$ (in this case the normal derivative has to be understood as in the above theorem). Then the following conditions are equivalent:*

1. *(Uniform Sommerfeld radiation condition)*

$$\frac{\partial}{\partial r}u(x) - iku(x) = o(r^{-\frac{n-1}{2}}), \tag{2.25}$$

as $r = |x| \to \infty$ uniformly in $\hat{x} = \frac{x}{r} \in \mathbf{S^{n-1}}$

2. *(L^2-radiation condition)*

$$\lim_{R\to\infty}\int_{S(0,R)}|\frac{\partial u}{\partial r} - iku|^2 dS_R = 0 \tag{2.26}$$

3. *(Exterior representation formula) For every $x \in \Omega$ we have*

$$u(x) = \int_{\partial\Omega}(\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial\Phi(x,\cdot)}{\partial\nu}(y))dS(y) \tag{2.27}$$

Proof: It is clear that (1) implies (2).

Let us prove that (2) implies (3): Fix $x \in \Omega$ and consider the domain $\Omega_R = \Omega \cap B(0,R)$ where R is large enough so that $D \subset B(0,R)$.

The integral representation formula in Ω_R gives

$$u(x) = (\int_{\partial\Omega} + \int_{S(0,R)})(\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,\cdot)}{\partial \nu}(y))dS(y) \tag{2.28}$$

It remains to prove that the integral on the sphere vanishes if $R \to \infty$.

The radiation condition can be written as

$$\int_{S(0,R)} \left(|\frac{\partial u}{\partial \nu}|^2 + k^2|u|^2 \right) dS + 2k \int_{S(0,R)} \Im(u\frac{\partial \bar{u}}{\partial \nu})dS \to 0. \tag{2.29}$$

It can be seen, by using Green formula in an annulus contained in Ω, that the last integral does not depend on R, in fact , by using Green formula on $\Omega \cap B(0,R)$, we can write the radiation condition as

$$\lim_{R\to\infty} \int_{S(0,R)} \left(|\frac{\partial u}{\partial \nu}|^2 + k^2|u|^2 \right) dS = -2k \int_{\partial\Omega} \Im(u\frac{\partial \bar{u}}{\partial \nu})dS, \tag{2.30}$$

where ν is the normal pointing outside Ω.

Let us write $I = I_1 + I_2$, where

$$I_1 = \int_{S(0,R)} (\frac{\partial u}{\partial \nu}(y) - iku(y))\Phi(x,y)dS$$

and

$$I_2 = \int_{S(0,R)} (\frac{\partial \Phi(x,\cdot)}{\partial \nu}(y) - ik\Phi(x,y))u(y)dS.$$

By Cauchy-Schwarz inequality

$$\begin{aligned}(I_2)^2 \leq& \int_{S(0,R)} |u(y)|^2 dS \int_{S(0,R)} |\frac{\partial \Phi(x,y)}{\partial r} - ik\Phi(x,y)|^2 dS(y)\\ =&O(1)o(1) \to 0\end{aligned}$$

when $R \to \infty$, since $\Phi(x,y)$ satisfies the uniform radiation condition.

In a similar way, for the asymptotic behavior of the fundamental solution,

$$(I_1)^2 \leq \int_{S(0,R)} |\frac{\partial u}{\partial r} - iku|^2 dS \int_{S(0,R)} |\Phi(x,y)|^2 dS(y) = o(1)O(1) \to 0.$$

Let us prove now that (3) implies (1):

If u is given by the integral representation formula, we can take derivatives under the integral in x and reduce the radiation condition for u to the same for $\Phi(x,y)$ and $\frac{\partial \Phi(x,\cdot)}{\partial \nu}$ in x (see propositions 2.2 and 2.1).

Corollary 2.1. *If u is a $\mathcal{C}^2$ solution in $\mathbf{R}^n$ of the homogeneous Helmholtz equation, satisfying the radiation condition, then u is identically zero.*

Proof: By Helmholtz integral representation formula in the exterior of $B(0,\epsilon)$ we have for $|x| > c > 0$,

$$u(x) = \int_{S(0,\epsilon)} (\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,\cdot)}{\partial \nu}(y))dS(y),$$

Now take $\epsilon \to 0$, it follows, by the continuity of $\Phi(|x|)$ and its partial derivatives out of the origin that for $|x| > c$ $u(x) = 0$.

Corollary 2.2. *Let u and Ω be as in the previous theorem, and assume that u satisfies SRC, then we can write*

$$u(x) = c_n k^{(n-1)/2} \frac{e^{ik|x|}}{|x|^{(n-1)/2}} u_\infty(\frac{x}{|x|}) + o(|x|^{-(n-1)/2}) \tag{2.31}$$

as $|x| \to \infty$. The function u_∞ is known as the far field pattern or scattering amplitude of u and can be writen as

$$u_\infty(\hat{x}) = \int_{\partial\Omega} (u(y) \frac{\partial e^{-iky\cdot\hat{x}}}{\partial \nu_y} - \frac{\partial u(y)}{\partial \nu} e^{-iky\cdot\hat{x}}) dS(y), \tag{2.32}$$

where $\hat{x} = x/|x| \in \mathbf{S^{n-1}}$.

Proof: It follows easily by Propositions 2.2 and 2.1 applied to the fundamental solutions in the representation formula (2.27).

Remark 1: The scattering amplitudes are the data used in inverse scattering problems, from them one tries to reconstruct the inner structure of any system ruled by any equation which in some exterior domains is the Helmholtz equation. This is the case of the bounded obstacle scattering problem and the inverse problem for a compactly supported inhomogeneous medium. The fundamental solution $\Phi(x, y)$ has far field pattern given by the plane wave $e^{iky\cdot\frac{x}{|x|}}$. The expression (2.32)can be understood as a superposition of plane waves corresponding to the fundamental solution and its normal derivative.

Remark 2: The representation formula (2.27) involves the layer potentials used to solve the Neumann boundary value problem (single layer potential, with density ϕ)

$$\int_{\partial\Omega} \frac{\partial \phi}{\partial \nu}(y)\Phi(x, y) dS(y), \tag{2.33}$$

and the Dirichlet boundary value problem (double layer potential)

$$\int_{\partial\Omega} \phi(y) \frac{\partial \Phi(x, \cdot)}{\partial \nu}(y) dS(y). \tag{2.34}$$

The volume potential, for a function supported on D, is given by

$$\int_D f(y)\Phi(x, y) dy, \tag{2.35}$$

it produces a solution of the non homogeneous Helmholtz equation $(\Delta + k^2)u = f$, with outgoing Sommerfeld radiation condition SRC.

2.4 Rellich uniqueness theorems

The goal is to give conditions at infinity to guarantee uniqueness of the solution of the non homogeneous Helmholtz equation in an exterior domain Ω. By the linearity of the equation we may reduce to study conditions to guarantee that the only solution of the homogeneous equation in Ω is the trivial $u = 0$. It is enough to study the case $k = 1$. Let us start with

Theorem 2.4. *Let u be a $\mathcal{C}^2$ solution of the equation $(\Delta+1)u=0$ in an exterior domain Ω. Choose σ such that $\mathbf{R}^n - B(0,\sigma)\subset\Omega$. Then either there exists $C>0$ and r_0 such that for $r>r_0$ we have*

$$\int_{\sigma<|x|<r}|u(x)|^2dx \geq Cr,$$

or $u=0$ in Ω.

Proof: Since u is analytic in Ω, u can not vanish identically in the exterior of any ball; hence we may take r_1 such that u does not vanish on $|x|=r_1$ and Ω contains the region $|x|\geq r_1$. Consider the expansion in spherical harmonics of $u(r_1\hat{x})$

$$u(r_1\hat{x})=\sum_{l=0}^{\infty}\sum_{m=1}^{a(l)}u_m^l(r_1)Y_m^l(\hat{x}),$$

where $\{Y_m^l\}_{m=1,\dots,a_l}$ is an orthonormal basis of $L^2(\mathbf{S^{n-1}})$ of spherical polynomials of degree l, i.e. $|x|^lY_m^l(\hat{x})$ is a harmonic polynomial of degree l, remind that

$$\Delta_S Y_m^l=-\frac{l(l+n-2)}{2}Y_m^l, \tag{2.36}$$

where Δ_S is the spherical laplacian.

We also have

$$\int_{S^{n-1}}|u(r_1\hat{x})|^2dS=\sum_{l,m}|u_m^l(r_1)|^2,$$

hence we might choose l and m such that $u_m^l(r_1)\neq 0$. Let us define the radial function given by the projection for $|x|=r\geq r_1$

$$u_m^l(r)=\int_{S^{n-1}}u(r\hat{x})Y_m^l(\hat{x})dS, \tag{2.37}$$

for which we also have

$$\int_{S^{n-1}}|u(r\hat{x})|^2dS\geq |u_m^l(r)|^2,$$

and hence, by taking the average

$$\frac{1}{r}\int_{r_1\leq|x|\leq r}|u(x)|^2dx\geq\frac{1}{r}\int_{r_1}^{r}|u_m^l(s)|^2s^{n-1}ds.$$

The theorem would be proved if we find $C>0$ and r_0 such that for $r>r_0$,

$$\frac{1}{r}\int_{\sigma}^{r}|u_m^l(s)|^2s^{n-1}ds>C. \tag{2.38}$$

Writing the laplacian in polar coordinates

$$\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_S:=\Delta_r+\frac{1}{r^2}\Delta_S, \tag{2.39}$$

since u is a solution of the homogeneous Helmholtz equation, we have

$$\begin{aligned}\Delta_r u_m^l(r) &= \int_{S^{n-1}} \Delta_r u(r\hat{x}) Y_m^l(\hat{x}) dS \\ &= -\frac{1}{r^2}\int_{S^{n-1}} \Delta_S u(r\hat{x}) Y_m^l(\hat{x}) dS - u_m^l(r) \\ &= -\frac{1}{r^2}\int_{S^{n-1}} u(r\hat{x}) \Delta_S Y_m^l(\hat{x}) dS - u_m^l(r) \\ &= \frac{1}{r^2}\frac{l(l+n-2)}{2} u_m^l(r) - u_m^l(r).\end{aligned}$$

where we have used the selfadjointness of Δ_S. This means that $u_m^l(r)$ is a solution of the modified Bessel equation

$$\left(\partial_{rr} - \frac{n-1}{r}\partial_r + (1 - \frac{l(l+n-2)}{2r^2})\right) u_m^l(r) = 0$$

similar to (2.4), which again can be reduced to Bessel equation of order $\mu = l + (n-2)/2$. We may find complex coefficients α and β such that

$$u_m^l(s) = (\alpha J_\mu(s) + \beta Y_\mu(s)) s^{-(n-2)/2},$$

since $u_m^l(r_1) \neq 0$, these coefficients are not zero. By taking real and imaginary parts in this expression, there exist real coefficients $(a, b) \neq (0, 0)$ such that

$$|u_m^l(s)| \geq |a J_\mu(s) + b Y_\mu(s)| s^{-(n-2)/2}.$$

From the asymptotic expansion of Bessel J_μ and Neumann functions Y_μ, obtained by taking real and imaginary parts in lemma 2.2, there exists r_2 such that for any $s > r_2$ it holds

$$|u_m^l(s)| \geq |(a \sin s + b \cos s) s^{-(n-1)/2} + o(s^{-n/2})|.$$

Hence

$$\frac{1}{r}\int_\sigma^r |u_m^l(s)|^2 s^{n-1} ds \geq C \frac{1}{r}\int_{r_2}^r \sin^2(s-\psi)^2 ds + o(1) = O(1).$$

This proves (2.38) and the theorem.

Corollary 2.3 (Rellich first lemma)**.** *Let u be a $\mathcal{C}^2$ solution of the equation $(\Delta + 1)u = 0$ in the exterior domain Ω such that*

$$\lim_{r\to\infty} \int_{|x|=r} |u|^2 dS_r = 0, \tag{2.40}$$

then u vanishes in Ω.

Proof: Define $f(r) = \int_{|x|=r} |u|^2 dS_r$, then $f(r) \to 0$ for $r \to \infty$. From analyticity u does not vanish identically in $|x| \geq r_0$ and from theorem 2.4 there exists $C > 0$ such that for $r > r_0$,

$$\int_\sigma^r f(s) ds \geq Cr.$$

For any $\epsilon > 0$ there exists r_1 such that $f(s) < \epsilon$ if $s > r_1$, then

$$C \leq \frac{1}{r}\int_\sigma^r f(s)ds \leq \frac{1}{r}\int_\sigma^{r_1} f(s)ds + \frac{1}{r}\int_{r_1}^r f(s)ds \leq \frac{C}{r} + \epsilon,$$

which is a contradiction if $r \to \infty$.

Corollary 2.4. *Let Ω be an exterior domain $\mathcal{C}^2$, assume that u is a $\mathcal{C}^2(\Omega) \cup \mathcal{C}(\bar{\Omega})$ outgoing solution of the equation $(\Delta + 1)u = 0$, with normal derivative, pointing out of Ω in the sense of theorem 2.2 such that*

$$\Im \int_{\partial\Omega} u\frac{\partial \bar{u}}{\partial \nu} \geq 0, \tag{2.41}$$

then u vanishes in Ω

Proof: Remark that (see (2.30)) the radiation condition can be writen as

$$\lim_{r\to\infty} \int_{S(0,r)} \left(|\frac{\partial u}{\partial \nu}|^2 + k^2|u|^2 \right) dS_r = -2k \int_{\partial\Omega} \Im(u\frac{\partial \bar{u}}{\partial \nu})dS \leq 0, \tag{2.42}$$

Hence

$$\lim_{r\to\infty} \int_{S(0,r)} |u|^2 dS_r = 0$$

and we can use Rellich first lemma.

Corollary 2.5. *Let $u \in \mathcal{C}^2$ an outgoing solution of the homogeneous Helmholtz equation in an exterior domain Ω, such that its far field pattern u_∞ is zero, then u vanishes in Ω.*

Proof: Since u is an outgoing solution we have (see (2.31))

$$u(x) = c_n k^{(n-1)/2} \frac{e^{ikx}}{|x|^{(n-1)/2}} u_\infty(\frac{x}{|x|}) + o(|x|^{-(n-1)/2}) = o(|x|^{-(n-1)/2}),$$

and hence

$$\lim_{r\to\infty} \int_{S(0,r)} |u|^2 dS_r = 0.$$

Corollary 2.6. *Let u be an outgoing solution of the Schrödinger equation*

$$(\Delta + 1 + q(x))u = 0, \tag{2.43}$$

where q is real valued and compactly supported. Then $u = 0$.

2.5 Remarks and extensions

A motivation to assume the radiation condition is that it asures uniqueness of the non homogeneous Hemholtz equation. Nevertheless we are going to give more reasons in order to justify its use in all the inverse problems in scattering.

- Let us start by justifying the choice of the radial fundamental solutions $\Phi(x)$, with the argument known as the limiting absorption principle. The fundamental solution Φ_ϵ of the operator $\Delta + k^2 + i\epsilon$ satisfies, by taking Fourier transform

$$\hat{\Phi}_\epsilon(\xi) = (-|\xi|^2 + k^2 + i\epsilon)^{-1}.$$

One can prove, see chapter 3 section 1, that the distribution $(-|\xi|^2 + k^2 + i\epsilon)^{-1}$ has a limit if $\epsilon \to 0+$ or $\epsilon \to 0-$, in the sense of tempered distribution; these limits are denoted as $(-|\xi|^2 + k^2 + i0)^{-1}$ and $(-|\xi|^2 + k^2 - i0)^{-1}$. The inverse Fourier transform of the first distribution (see [GS]) is just the outgoing solution $\Phi(x)$, The inverse Fourier transform of the second gives us the fundamental solution Φ_{in} which satisfies the so called incoming Sommerfeld radiation condition

$$\frac{d}{dr}\Phi_{in}(x) + ik\Phi_{in}(x) = o(r^{-\frac{n-1}{2}}).$$

This fact can be expresed in the following "limiting absorption principle":

Theorem 2.5. *Assume $f \in \mathcal{C}_0^\infty$, let u_ϵ be the unique solution of the problem*

$$(\Delta + k^2 + i\epsilon)u_\epsilon = f$$

in $\mathbf{R^n}$. Then $u = \lim_{\epsilon \to 0+} u_\epsilon$ is an outgoing solutions of the Helmholtz equation.

- The understanding of the Helmholtz equation as the equation of the $-k^2$-eigenfunctions of the Laplace operator gives us another motivation of the outgoing condition via the limiting absorption principle by using spectral theory. The spectrum of $-\Delta$ is absolutely continuous and is given by $[0, \infty)$. The so called generalized eigenfunction (generalized because they are not eigenfunctions of the Laplace operator as a selfadjoint unbounded operator on $L^2(\mathbf{R^n})$), given by its spectral resolution are the plane waves

$$u_i(k, x, \hat{\xi}) = e^{ix\cdot\xi},$$

parameterized by $k^2 = |\xi|^2$ in the spectrum $\sigma(\Delta) = \mathbf{R}_+$ by $\xi/k = \hat{\xi} \in \mathbf{S^{n-1}}$. The spectral diagonalization is attained by the unitary operator Fourier transform $\mathcal{F}$. We may find the fundamental solution of the Helmholtz equation as the limit of the distributional Schwartz kernels of the resolvent operators for $\lambda = k^2 + i\epsilon$ out of the spectrum $\mathbf{R}_+$

$$R(\lambda) = (\Delta + \lambda)^{-1} : L^2(\mathbf{R}^n) \to H^2(\mathbf{R}^n).$$

These kernel Φ_ϵ are given by

$$(\mathcal{F}\Phi_\epsilon\mathcal{F})(\eta, \xi) = (-|\xi|^2 + \lambda)^{-1}\delta(\xi - \eta).$$

We are again addressed to the above incoming and outgoing fundamental solutions.

- The method of generalized eigenfuntions has been extended to the Schrödinger operator $\Delta + V$ (see [A]). Following [RS] Chapter XI, we are going to give a heuristic argument to motivate the choice of the outgoing condition in the definition of the scattering solution, in the case of the two body Schrödinger operator $-\Delta + V(x)$. We choose the so called Schrödinger approach to scattering theory: Assume that the operator $L = -\Delta + V(x)$ is selfadjoint in $L^2(\mathbf{R}^n)$. We define the wave operators as the following strong limit

$$W^{\pm} = s - \lim_{t \to \mp\infty} e^{it(-\Delta)} e^{-itL}$$

Assume that the potential V is such that these operators exist and are isometries in $L^2(\mathbf{R}^n)$ (existence and completeness of wave operators, the study of such potentials is the subject of a large literature). Then these operators are intertwining in the sense that

$$W^{+}\Delta = LW^{+}.$$

Assume that in some auxiliary space, this is an estimate, we can make sense out of

$$W^{+}u_i(k, \cdot, \xi) := \phi(k, \cdot).$$

Then $\phi(k, \cdot)$ satisfies the eigenvalue equation for L: $L\phi = k^2\phi$, and we can invert to write

$$u_i = (W^{-})^{*}\phi,$$

which after its definition and the chain rule can be written as

$$\lim_{t \to -\infty} e^{it(-\Delta)} e^{-itL}\phi = \phi - i \lim_{t \to -\infty} \int_0^t e^{is(-\Delta)} V e^{-isL} \phi ds$$

$$= \phi - \lim_{\epsilon \to 0+} i \int_0^{-\infty} e^{is(-\Delta)} V e^{-isk^2} e^{+\epsilon s} \phi ds =$$

$$\phi + \lim_{\epsilon \to 0+} (-\Delta - (k^2 + i\epsilon))^{-1} V\phi,$$

this is the Lippmann-Schwinger integral equation that the outgoing generalized eigenfunctions have to satisfy.

- Let us see a motivation based on the evolution wave equation. Given ψ an outgoing solution in the exterior domain Ω of the Helmholtz equation, the function $u(x, t) = e^{-ikt}\psi(x)$ is a time-harmonic solution of the wave equation. Let $u(x, t)$ be a solution of the exterior problem

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \text{ en } \Omega \times \mathbf{R}$$

Consider B_ρ containing the obstacle $\mathbf{R} - \Omega$, and the domain $\Omega_\rho = \Omega \cap B_\rho$. The energy on Ω_ρ is given by

$$E_\rho = \int_{\Omega_\rho} (|u_t|^2 + |\bigtriangledown u|^2) dx.$$

As a consequence of the divergence theorem and the differential equation we have,

$$\frac{\partial E_\rho}{\partial t} = \int_{\Omega_\rho} (u_t \bar{u}_{tt} + \bar{u}_t u_{tt} + \nabla u \cdot \nabla \bar{u}_t + \nabla \bar{u} \cdot \nabla u_t)$$

$$= \int_{\Omega_\rho} (u_t \bar{u}_{tt} + \bar{u}_t u_{tt} - \Delta u \bar{u}_t - \Delta \bar{u} u_t) + \int_{\partial \Omega_\rho} (\bar{u}_t \frac{\partial}{\partial \nu} u + u_t \frac{\partial}{\partial \nu} \bar{u}) dS$$

$$= \int_{\partial \Omega} (\bar{u}_t \frac{\partial}{\partial \nu} u + u_t \frac{\partial}{\partial \nu} \bar{u}) dS + \int_{S_\rho} (\bar{u}_t \frac{\partial}{\partial \nu} u + u_t \frac{\partial}{\partial \nu} \bar{u}) dS.$$

The last integral is the flow of energy through the sphere of radius ρ. For the time-harmonic solution this flow can be written as

$$2k \int_{S(0,R)} \Im(\psi \frac{\partial \bar{\psi}}{\partial \nu}) dS.$$

Let us write the radiation condition as we did in (2.29). Since, by Rellich uniqueness theorem,

$$\int_{S_\rho} (|\frac{\partial \psi}{\partial \nu}|^2 + k^2 |\psi|^2) dS$$

has a strictly positive limit, we have that, for ρ sufficiently large, the flow through the sphere is strictly negative. As a conclusion the energy of the outgoing time-harmonic solution u flows out of the sphere S_ρ.

The same type of argument proves that for the incoming solutions the energy flows to the inside of the spheres.

- It is natural to wonder if the radiation condition can be substituted by a different one.

  In [AH], where more general equations are studied, one can find different radiation conditions which include the Sommerfeld condition for the Helmholtz equation.

- Let us remark that the outgoing Sommerfeld radiation condition can be understood as a version of the causality principle for solutions of the wave equation in the frequence domain, i.e. if we take k as the Fourier transform variable of t.

2.6 Exercises and further results

- Study if any distributional solution of Helmholtz equation is actually a classical solution (analogous of the Weyl Lemma for Laplace equation)

- Prove Corollary 2.6

- Let us consider the soft obstacle problem

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in the exterior domain } \Omega \\ u = u(x,k,\theta) = u_i + u_s \\ u(x) = 0 \text{ in } \partial\Omega, \end{cases} \tag{2.44}$$

  where u_i is the plane wave $u_i = e^{ik\theta\cdot x}$ and u_s is an outgoing solution. The solution u is known as the scattering solution with incident direction θ and we consider it far field pattern $u_\infty(\tau,\theta,k)$ in the receiver direction $\tau = \frac{x}{|x|}$. Prove, assuming the existence of such a solution the so called reciprocity relation

$$u_\infty(\tau,\theta,k) = u_\infty(-\theta,-\tau,k). \tag{2.45}$$

  (Hints: By using Green formulas and radiation conditions prove an expression for $u_\infty(\omega,\theta) - u_\infty(-\theta,-\omega)$ as a boundary integral on $\partial\Omega$ of products of values of the solutions $u(x,\theta)$ and $u(x,-\omega)$ and their normal derivatives)

  (Assume all the regularity you need)

- The same for Schroedinger equation of Corollary 2.6. (Hint: use the formula in the previous problem for $\mathbb{R}^n \setminus \Omega$ a sufficiently large ball and Green formula in the interior of the ball)

- For the evolution Schrödinger equation, we may find a justification of the outgoing radiation condition similar to the given in the wave evolution. In this case the time-harmonic solution $u(x,t) = e^{-ik^2t}\psi(x)$ for the outgoing solution ψ of Helmholtz equation has a lost of energy through the boundary $\partial B(0,R)$ when the energy on the region $\Omega_R = \Omega \cap B(0,R)$ is given by

$$E(t) = \int_{\Omega_R} |u(x,t)|^2 dx$$

  .

Chapter 3

The spherical harmonics

The use of radial properties of Fourier Transform, spherical harmonics and Hankel transforms are an important source of models in scattering theory (for instance central potentials), in particular point sources arise in this context. The use of Bessel functions and its very well studied properties allow the construction of simple models for more complex systems and to conjecture properties in more general settings.

3.1 The case of the plane: Fourier series.

Let us start with a fast review of the theory of Fourier series, to motivate the tool we are looking for in higher dimension: the spherical harmonics on the sphere S^{n-1}.

Let $f \in L^2(\mathbf{R}^2)$, this in polar coordinates means

$$\int_0^\infty \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta r dr < \infty. \tag{3.1}$$

Hence a.e. $r \in \mathbf{R}$

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

We can expand in Fourier series as

$$f(re^{i\theta}) = \sum_{-\infty}^{+\infty} f_k(r) e^{ik\theta} = .$$

From properties of Fourier series as $N \to \infty$

$$0 \leq \|f(re^{i\theta}) - \sum_{-N}^{+N} f_k(r) e^{ik\theta}\|^2_{L^2(0,2\pi)} = \|f(re^{i\theta})\|^2_{L^2(0,2\pi)} - \sum_{-N}^{+N} |f_k(r)|^2 \to 0$$

Hence

$$\int_0^\infty \int_0^{2\pi} |f(re^{i\theta}) - \sum_{-N}^{+N} f_k(r) e^{ik\theta}|^2 d\theta r dr$$

$$= \int_0^\infty \int_0^{2\pi} |f(re^{i\theta})|^2 - \sum_{-N}^{+N} |f_k(r)|^2 d\theta] r dr$$

$$= \int_0^\infty \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta r dr - \sum_{-N}^{+N} |f_k(r)|^2 r dr.$$

For the monotone converge theorem and Parseval identity, we have

$$\lim \int_0^\infty \sum_{-N}^{+N} |f_k(r)|^2 r dr = \int_0^\infty \sum_{-\infty}^{+\infty} |f_k(r)|^2 r dr = \|f\|_{L^2}^2$$

Hence

$$\|f - \sum_{-N}^{+N} f_k(r) e^{ik\theta}\|_{L^2}^2 \to 0$$

Hence we have a Hilbert sum of spaces

$$L^2(R^2) = \sum_{-\infty}^{\infty} \mathcal{H}_k,$$

where

$$\mathcal{H}_k = \{g(re^{i\theta}) = f_+(r)e^{ik\theta} + f_-(r)e^{-ik\theta} : \int_0^\infty (|f_+(r)|^2 + |f_-(r)|^2) r dr < \infty\}.$$

The characters $e^{ik\theta}$ ($e^{-ik\theta}$) are values of the holomorphic (antiholomorphic) polynomial z^k ($\bar{z}^k$) for $|z| = 1$. Since the real and complex parts of (anti)holomorfic functions are harmonic functions, we can take as a basis (real valued) of the complex span of $e^{\pm ik\theta}$ given by rectriction to S^1 of harmonic polinomial of degree k, i.e. the trigonometric functions $cosk\theta$ and $sink\theta$. This can be generalized to any dimension.

3.2 The spherical harmonics on $\mathbb{R}^n$.

Let us denote $\mathcal{P}_k$ the space of homogeneous polynomials of degree k. The subset of polynomials in $\mathcal{P}_k$ which are harmonic will be denoted by $\mathcal{A}_k$ (solid harmonic polynomials) and it restriction to the unit sphere will be called the spherical harmonic of degree k and denoted by $\mathcal{H}_k$. The dimension of $\mathcal{P}_k$ (it can be measured as the number of choices of $n-1$ positions out of $k+n-1$) is $\frac{(k+n-1)!}{(n-1)!k!}$.

Let us define in $\mathcal{P}_k$ the hermitian product $<P, Q>= P(D)Q$, where for $P(x)$ we denote $P(D)$ the differential polynomial obtained by substituting the monomial x^α by the partial derivative $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ for any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$. Notice that $\{x^\alpha\}$, when α runs the multiindexes $|\alpha| = k$, is an orthogonal basis of $\mathcal{P}_k$ of vectors of lengh $\alpha!$.

Lemma 3.1. *We have the following orthogonal decomposition*

$$\mathcal{P}_k = \mathcal{A}_k + |x|^2 \mathcal{P}_{k-2},$$

Proof: We consider the operator

$$\Delta : \mathcal{P}_k \to \mathcal{P}_{k-2}.$$

Δ is onto: In fact, assume that there exists a $Q \in \mathcal{P}_{k-2}$ so that Q is orthogonal to $\Delta(\mathcal{P}_k)$, i.e. $0 =< \Delta P, Q >$, take $P = |x|^2 Q$, then

$$0 =< \Delta|x|^2Q, Q >= \overline{< Q, \Delta|x|^2Q >} = \overline{Q(D)(\Delta|x|^2Q)} = \overline{\Delta Q(D)(|x|^2Q)} =< |x|^2Q, |x|^2Q >,$$

Hence $Q = 0$.

The proof is finished if we prove that $Ker(\Delta)$ is the orthogonal of $|x|^2\mathcal{P}_{k-2}$. This follows fron the identities

$$< Q, |x|^2P >= \overline{< |x|^2P, Q >} = \overline{\Delta P(D)}Q = \overline{P(D)\Delta}Q = \overline{< P, \Delta Q >} =< \Delta Q, P >$$

Corollary 3.1. *The dimension of $\mathcal{A}_k$ is $a(k,n) = \frac{(k+n-1)!}{(n-1)!k!} - \frac{(k+n-3)!}{(n-3)!k!}$.*

Corollary 3.2. *Any polinomial of degree k on the sphere can be written as a sum of spherical harmonics of degree less than or equal to k*

Weierstrass theorem allows us to prove

Corollary 3.3. *The span of $\cup_{k=0}^{\infty}\mathcal{H}_k$ is a dense subset of*
(a) The continuous functions on S^{n-1} with the L^∞ norm
(b) $L^2(S^{n-1})$.

Proposition 3.1. *Let $\{Y_l^{(k)}\}_{l=1,\dots a(k,n)}$ an orthogonal basis of $\mathcal{H}_k$. Then $\{Y_l^{(k)}\}_{l=1,\dots,a(k,l);k=0,\dots}$ is a Hilbert basis for $L^2(S^{n-1})$.*

Proof: The fact that the solid harmonics of degree k are eigenfunction of the spherical laplacian of eigenvalue $-k(k+n-2)$ is underneath.

Take $u = r^kY^{(k)}(\theta)$ and $v = r^lY^{(l)}(\theta)$, where $\theta = \frac{x}{|x|}$. Then for Green formula

$$0 = \int_{B(0,1)} (u\Delta v - v\Delta u)dx = \int_{S^{n-1}} (u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r})d\sigma$$

$$= \int_{S^{n-1}} (lY^{(k)}(\theta)Y^{(l)}(\theta) - kY^{(l)}(\theta)Y^{(k)}(\theta))d\sigma(\theta)$$

$$= (l-k)\int_{S^{n-1}} Y^{(l)}(\theta)Y^{(k)}(\theta)$$

Corollary 3.4. *Hence we have a Hilbert sum of spaces*

$$L^2(R^n) = \sum_{-\infty}^{\infty} \mathcal{H}_k,$$

where

$$\mathcal{H}_k = \{g(r\theta) = \sum_{l=1}^{a_k} f_l(r)Y_l^{(k)}(\theta) : \sum \int_0^\infty |f_l(r)|^2 r^{n-1}dr < \infty\}.$$

Proof: It follows as in section 3.1.

3.3 The zonal representation

In S^1 the orthogonal projection on $\mathcal{H}_k$, with respect to the basis $cosk\theta$, $sink\theta$ is given by

$$\Pi_k f(\theta) = \int_0^{2\pi} f(\eta) cosk\eta d\eta cosk\theta + \int_0^{2\pi} f(\eta) sink\eta d\eta sink\theta$$

By the addition formula we have

$$\Pi_k f(\theta) = \int_0^{2\pi} f(\eta) cosk(\eta - \theta) d\eta.$$

The fuction $cosk(\eta - \theta)$ is the reproducing kernel of the projection. It is called the zonal harmonic with pole at θ. It has a its maximun at $\eta = \theta$.

In higher dimensions we have similar formulae. We start by choosing an orthonormal basis of real valued spherical harmonics $\{Y_l^{(k)}\}_{l=1,\dots,a(k,n)}$. Fix $\theta \in S^{n-1}$, and the linear form

$$L : \mathcal{H}_k \to \mathbb{C}$$

such that $L(Y) = Y(\theta)$. Then there exists $Z_\theta^{(k)} \in \mathcal{H}_k(\eta)$ such that

$$L(Y) = Z_\theta^{(k)} \cdot Y$$

Then $Z_\theta^{(k)}$ is a reproducing kernel of the orthogonal projection Π_k on $\mathcal{H}_k$ which we will call the harmonic zonal of degree k with pole at θ. Let us denote $a_k = a(k, n)$.

Lemma 3.2. • *(a)*

$$Z_\theta^{(k)}(\eta) = \sum_{m=1}^{a_k} Y_m^{(k)}(\theta) Y_m^{(k)}(\eta).$$

- *(b)* $Z_\theta^{(k)}(\eta) = Z_\eta^{(k)}(\theta)$
- *(c) If ρ is a rotation, then* $Z_{\rho\theta}^{(k)}(\rho\eta) = Z_\theta^{(k)}(\eta)$.
- *(d)* $|Z_\theta^{(k)}(\eta)| \leq Z_\theta^{(k)}(\theta) = \omega_{n-1}^{-1} a_k$.
- *(e)* $\int |Z_\theta^{(k)}(\eta)|^2 d\eta = \omega_{n-1}^{-1} a_k$

Proof: (a) Follows by expanding $Z_\theta^{(k)}$ in the basis $\{Y_l^{(k)}\}_{l=1,\dots,a(k,n)}$.

(b) Follows from the fact that $\{Y_l^{(k)}\}_{l=1,\dots,a(k,n)}$ is real valued. As a consequence the same expresion holds for any orthogonal basis.

To prove (c) let us take $\omega = \rho\theta$ and $Y \in \mathcal{H}_k$. Then

$$\int_{S^{n-1}} Z_{\rho\eta}^{(k)}(\rho\theta) Y(\theta) d\theta = \int_{S^{n-1}} Z_{\rho\eta}^{(k)}(\omega) Y(\rho^{-1}\omega) d\omega$$

$$= Y(\rho^{-1}\rho\eta) = Y(\eta),$$

since $Y(\rho^{-1}\cdot) \in \mathcal{H}_k$, hence

$$Z_{\rho\eta}^{(k)}(\rho\theta) = Z_\eta^{(k)}(\theta).$$

Next, from (c) and the transitivity of S^{n-1} with respect to rotations we get that $Z_\theta^{(k)}(\theta) = \sum |Y_m^{(k)}(\theta)|^2$ is constant. Hence, from (a),

$$Z_\theta^{(k)}(\theta) = \sum_{m=1}^{a_k} |Y_m^{(k)}(\theta)|^2.$$

Then

$$a_k = \int_{S^{n-1}} \sum_{m=1}^{a_k} |Y_m^{(k)}(\theta)|^2 = \omega_{n-1} \sum_{m=1}^{a_k} |Y_m^{(k)}(\theta)|^2 = \omega_{n-1} Z_\theta^{(k)}(\theta).$$

(d) follows by using Cauchy-Schwart in (a). Finally, again by (a) and Plancherel

$$\int_{S^{n-1}} |Z_\theta^{(k)}(\tau)|^2 d\tau = \sum_{m=1}^{a_k} |Y_m^{(k)}(\theta)|^2 = Z_\theta^{(k)}(\theta)$$

Now we look for a more precise description of the zonals. We start by generalizing to higher dimension the following formula for the Poisson kernel of $\mathbb{D}$ as a generating function of the zonals

$$P_r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2rcos\theta} = \frac{1}{\pi}(\frac{1}{2} + \sum_{k=1}^{\infty} r^k cosk\theta).$$

Take an L^2 function f on S^{n-1}, and expand it as a Fourier series (we will call so its expantion in spherical harmonics):

$$f(\theta) = \sum \Pi_k(f)(\theta).$$

By extending as a sum of solid harmonics

$$u(r, \theta) = \sum_{k=0}^{\infty} r^k \Pi_k(f)(\theta)$$

we obtain a harmonic function inside the radius of convergence, formally u solves the dirichlet problem on $B(0,1)$ with data f and by the uniqueness of the Poisson kernel we might write

Theorem 3.1. *The Poisson kernel is the generating function of the zonals, i.e.*

$$P(\eta, x) = \frac{1 - |x|^2}{\omega_{n-1}|x - \eta|^n} = \sum_{k=1}^{\infty} r^k Z_\theta^{(k)}(\eta) \tag{3.2}$$

where $x \in \mathbf{R}^n$, $|x| < 1$ and $\theta = x/|x|$. The convergence is uniform in $|x| < s < r$ for any s.

Proof: Let us take $u(\theta) = \sum_m Y_m$, $Y_m \in H_m$ a finite linear combination of spherical harmonics. Then the function

$$u(x) = \sum_m |x|^m Y_m(\theta)$$

is a harmonic function and hence by Poisson representation formula, since u is continuous we have

$$u(x) = \int_{S^{n-1}} P(\eta, x)u(\eta)d\sigma(\eta)$$

Let us denote $Q(\eta, x) = \sum_{k=1}^{\infty} r^k Z_\theta^{(k)}(\eta)$ We prove the uniform convergence of Q in the range of the statement. From property (d) in lemma 3.2 we have $|Z_\theta^{(k)}(\eta)| \leq \omega_{n-1}a_k$. Hence, since $a_k < c_n k^{n-1}$ the claim follows.

Take now

$$\int_{S^{n-1}} Q(\eta, x)u(\eta)d\sigma(\eta) = \int_{S^{n-1}} \sum_{k=1}^{\infty} r^k Z_\theta^{(k)}(\eta) \sum_m |x|^m Y_m(\eta)$$

$$= \sum_m \int_{S^{n-1}} r^m Z_\theta^{(m)}(\eta)|x|^m Y_m(\eta) = \sum_m |x|^m Y_m(\theta) = u(x).$$

Hence

$$\int_{S^{n-1}} (Q(\eta, x) - P(\eta, x))u(\eta)d\sigma(\eta) = 0$$

Since the u's form a dense class in $L^2(S^{n-1})$, this implies the desired identity since both functions are continuous for $r < 1$.

The above theorem allows us to identify the zonal $Z_\theta^{(k)}(\eta)$ as a polinomial of $t = \theta \cdot \eta$. To achieve this we start by proving that $Z_\theta^{(k)}(\eta)$ depends only on $\theta \cdot \eta$, otherway to say is that it is a function of the geodesic distance between θ and η. This follows easily from property (c) of the previous lemma. In fact, assume η' such that $\theta \cdot \eta = \theta \cdot \eta'$. Hence η and η' are in the same hyperplane with normal θ, and we can find a rotation ρ which lefts invariant θ and takes η to η'. Then

$$Z_\theta^{(k)}(\eta) = Z_{\rho\theta}^{(k)}(\rho\eta) = Z_\theta^{(k)}(\eta')$$

We can write $Z_\theta^{(k)}(\tau) = z^{(k)}(\theta \cdot \eta)$. The generating function formula can be written for $t = \theta \cdot \eta \in [-1, 1]$, as

$$\frac{1 - r^2}{\omega_{n-1}|1 + r^2 - 2tr|^{n/2}} = \sum_{k=1}^{\infty} r^k z^{(k)}(t). \tag{3.3}$$

For $n = 2$ the formula reads as $z^{(k)}(t) = T_k(t)$, where $T_k(t)$ is the Tchebychev polynomial given by

$$T_k(t) = cos(kcos^{-1}t).$$

We obtain similar expresions in higher dimensions by mean of the generating formula for ultraspherical polynomials , see[AB].

$$\frac{1}{(1 + r^2 - 2rt)^\lambda} = \sum_{k=0}^{\infty} r^k C_k^\lambda(t) \tag{3.4}$$

This together with (3.2) gives, for $k = 2, 3....$

$$z^{(k)}(t) = \frac{1}{\omega_{n-1}}(C_k^{n/2}(t) - C_{k-1}^{n/2}(t)).$$

And for $k = 0, 1$

$$z^{(k)}(t) = \frac{1}{\omega_{n-1}} C_k^{n/2}(t).$$

The recurrence formulae for the ultraspherical polynomials

$$kC_k^{\lambda}(t) = 2\lambda(tC_{k-1}^{\lambda+1}(t) - C_{k-2}^{\lambda+1}(t))$$

$$(k+2\lambda)C_k^{\lambda}(t) = 2\lambda(C_k^{\lambda+1}(t) - tC_{k-1}^{\lambda+1}(t))$$

allows us to write

$$z^{(k)}(t) = \frac{2k+n-2}{\omega_{n-1}(n-2)} C_k^{(n-2)/2}(t).$$

To see that the ultraspherical polynomial are, in fact, polynomials, we use the identity

$$2r\lambda \sum_{k=0}^{\infty} r^k C_k^{\lambda+1}(t) = \sum_{k=0}^{\infty} r^k (C_k^{\lambda+1})'(t),$$

which is obtained by taking t-derivative in (3.4), to get the formula $2\lambda C_{k-1}^{\lambda+1}(t) = (C_k^{\lambda+1})'(t)$. This, together with the fact that $C_0^{\lambda}(t) = 1$ for any λ, proves by induction that C_k^{λ} is a polynomial of degree k.

From Weierstrass theorem it follows that the family $\{C_k^{\lambda}\}_{k=1}^{\infty}$ is dense in $\mathcal{C}([-1,1])$. Furthermore we have

Theorem 3.2. *The polynomials $\{C_k^{(n-2)/2}\}_{k=1}^{\infty}$ are orthogonal with respect to the scalar product*

$$(f,g) = \int_{-1}^{1} f(t)g(t)(1-t^2)^{\frac{n-3}{2}} dt.$$

Proof: The orthogonality of $Z_\theta^{(k)}$ and $Z_\theta^{(l)}$, for some fix θ, can be written as

$$0 = \int_{S^{n-1}} Z_\theta^{(k)}(\tau) Z_\theta^{(l)}(\tau) d\sigma(\tau) = \int_{S^{n-1}} C_k^{(n-2)/2}(\theta \cdot \tau) C_l^{(n-2)/2}(\theta \cdot \tau) d\sigma(\tau),$$

which, by taking the integral along the paralels with respect to the pole θ, can be expressed as

$$0 = C \int_0^{\pi} C_k^{(n-2)/2}(cos\alpha) C_l^{(n-2)/2}(cos\alpha) \int_{S^{n-2}_{sin\alpha}} d\sigma_{n-1} d\alpha$$

By taking $cos\alpha = t$ the theorem is proved.

Remark : The same property holds for any $\lambda > 1/2$, when the scalar product has the weight $(1-t^2)^{\lambda - 1/2}$. This follows in general for the family being eigenfuntions of a Sturm-Liouville problem.

Let us go back to Corollary 3.4. We can write

Theorem 3.3. *Let us denote*

$$H_k = span\{f(|x|)p(x),\ such\ that \int |f(r)|^2 r^{n-1+2k} dr < \infty\ and\ p \in A_k\}. \tag{3.5}$$

Then

$$L^2(\mathbf{R}^n) = \sum_{k=0}^{\infty} H_k,$$

in the sense that

- *Each H_k is closed.*
- *The H_k are mutually orthogonal.*
- *Any $f \in L^2(\mathbf{R}^n)$ can be approximated by a finite linear combination of elements in H_k $k = 0, ..., \infty$.*

3.4 The basis of point sources

Our next aim is to use the spherical harmonics to construct a special class of solutions of Helmholtz equation known as point sources.

We look for solutions of the form

$$u(x) = f(|x|)Y(\frac{x}{|x|}),$$

where Y is a spherical harmonics of degree l. The function Y is an eigenfunstion of the spherical laplacean Δ_S. This follows easily by the fact that $r^l Y(\theta)$ where $r = |x|$ and $\theta = \frac{x}{|x|}$ is a harmonic function, and the expression in polar coordenates of the laplacean that

$$\Delta_S Y = -l(l+n-2)Y \tag{3.6}$$

This result was already used in Propotition 3.1. The arguments in section 2.1 show that solutions as above are linear combinations of the functions

$$u_l(x) = j_l(k|x|)Y(\theta) \tag{3.7}$$

and

$$v_l(x) = h_l(k|x|)Y(\theta), \tag{3.8}$$

where we denote the spherical Bessel and Hankel functions as

$$j_l(r) = r^{-(n-2)/2} J_{l+(n-2)/2}(r)$$

and

$$h_l(r) = r^{-(n-2)/2} H^{(1)}_{l+(n-2)/2}(r).$$

We have the following

Proposition 3.2. *The function $u_l(x)$ is an entire solution of Helmholtz equation which satisfies the inequialities*

$$\limsup_{r\to\infty} \frac{1}{r}\int_{B(0,r)} |u_l(x)|^2 dx < \infty. \tag{3.9}$$

$$\liminf_{r\to\infty} \frac{1}{r}\int_{B(0,r)} |u_l(x)|^2 dx > c > 0. \tag{3.10}$$

Solutions of the equation satisfying (3.9) are known as Herglotz wave functions.

The function v_l is an outgoing radiating solution on $\mathbb{R}^n \setminus 0$.

These functions are called point sources or spherical waves.

Proof: As an exercise.

Now we give the expansion of the fundamental solutions of Helmholtz equation in point sources.

Theorem 3.4.

$$\Phi(x,y) = \sum_l \sum_{m=1}^{a_l} ik^{n-2} j_l(k|y|) h_l(k|x|) Y_m^{(l)}(\theta) Y_m^{(l)}(\eta), \tag{3.11}$$

where $Y_m^{(l)}(\theta)$ is an orthonormal basis of spherical harmonics $\eta = y/|y|$, $\theta = \frac{x}{|x|}$ and the series is absolutely and uniformly convergent on compact subsets of $\{(x,y) : |y| < |x|\}$

Proof: Our first aim is to find the Fourier coefficients of the function $\Phi(x, r\eta)$, where $\eta \in S^{n-1}$. Fix r and take x such that $|x| > r$. On one hand, since $u_l(y)$ and $\Phi(x,y)$ are solutions for $|y| < r$, Green formula on $B(0,r)$ yields

$$0 = \int_{|z|=r} (u_l(z)\frac{\partial \Phi(x,z)}{\partial \nu_z} - \Phi(x,z)\frac{\partial u_l}{\partial \nu}(z)) d\sigma(z). \tag{3.12}$$

On the other hand, from the exterior representation formula applied to v_l, we have

$$v_l(x) = \int_{|z|=r} (v_l(z)\frac{\partial \Phi(x,z)}{\partial \nu_z} - \Phi(x,z)\frac{\partial v_l}{\partial \nu}(z)) d\sigma(z). \tag{3.13}$$

For $|z| = r$ we have for $\eta = \frac{z}{|z|}$

$$u_l(z) = j_l(kr)Y(\eta) \text{ and } \frac{\partial v_l}{\partial \nu}(z) = kj_l'(kr)Y(\eta)$$

and

$$v_l(z) = h_l(kr)Y(\eta) \text{ and } \frac{\partial v_l}{\partial \nu}(z) = kh_l'(kr)Y(\eta).$$

Multiplying (3.12) and (3.4) by $h_l(kr)$ and $j_l(kr)$ respectively and adding up we obtain

$$kW(h_l, j_l)(kr)\int_{S^{n-1}} \Phi(x, r\eta)Y(\eta)r^{n-1}d\sigma(\eta) = v_l(x)j_l(kr)$$

Now, since the wronskians

$$W(J_\lambda, H_\lambda^{(1)})(t) = i/t,$$

we have

$$W(h_l, j_l)(kr) = -\frac{i}{(kr)^{n-1}}.$$

Hence we obtain

$$\int_{S^{n-1}} \Phi(x, r\eta) Y_m^{(l)}(\eta) d\sigma(\eta) = ik^{n-2} j_l(kr) h_l(k|x|) Y_m^{(l)}(\theta) \tag{3.14}$$

Then (3.11) follows in $L^2(S^{n-1})$ for fix x, as far as r is fixed and $r < |x|$.

To prove the uniform convergence, we will use the behavior of spherical Bessel functions

$$j_l(t) = \left(\frac{1}{2^{(n-2)/2}\Gamma(l+n/2)} + o(\frac{1}{\Gamma(l+(n-2)/2)}) \right) (\frac{t}{2})^l \tag{3.15}$$

uniformly for t on compacts. Formula (3.15) is obtained directly from (2.6)

We also need the following asymptotic behavior of Hankel functions for large values of the parameter, obtained from (2.7) together with the values of the gamma function for negative values (see [B] for more details),

$$h_l(t) = \left(2^{(n-2)/2}\Gamma(l+(n-2)/2) + o(\Gamma(l+n/2))\right) (\frac{t}{2})^{-l} t^{-(n-2)} \tag{3.16}$$

By Cauchy-Shwartz inequality,

$$\sum_{m=1}^{a_l} |ik^{n-2} j_l(kr) h_l(k|x|) Y_m^{(l)}(\theta) Y_m^{(l)}(\eta)| \le k^{n-2} |j_l(kr) h_l(k|x|)| \sum_{m=1}^{a_l} |Y_m^{(l)}(\theta)|^2$$

$$\le \frac{a_l}{\omega_{n-1}} k^{n-2} |j_l(kr) h_l(k|x|)|.$$

From the above asymptotic formulae we obtain

$$|j_l(kr) h_l(k|x|)| \le C l^{-1} (k|x|)^{-(n-2)} \left(\frac{r}{|x|} \right)^l.$$

This proves the desired uniform convergence.

Corollary 3.5. *(Funk-Hecke formula) Let $Y^{(l)} \in H^l$, then*

$$\int_{S^{n-1}} e^{-ikr\theta\cdot\eta} Y^{(l)}(\eta) d\sigma(\eta) = (\frac{2}{\pi})^{1/2} \frac{i^{-l}}{(rk)^{(n-2)/2}} J_{l+(n-2)/2}(kr) Y^{(l)}(\theta). \tag{3.17}$$

Proof: Left to the reader Recall (3.14),

$$\int_{S^{n-1}} \Phi(x, r\eta) Y^{(l)}(\eta) d\sigma(\eta) = ik^{n-2} j_l(kr) h_l(k|x|) Y^{(l)}(\theta).$$

The formula follows by using the asymptotics (2.18) and (2.15) and then by taking $|x| \to \infty$.

Corollary 3.6. *(Hankel transforms): Let $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, which can be written as $f(x) = f_0(r)Y(x)$, with Y a solid harmonic of degree l. Then $\hat{f}(\xi) = F_0(|\xi|)Y(\xi)$, where*

$$F_0(|\xi|) = (\frac{2}{\pi})^{1/2} \frac{i^{-l}}{|\xi|^{(n-2)/2+l}} \int_0^\infty f_0(r) J_{l+(n-2)/2}(r|\xi|) r^{l+n/2} dr. \tag{3.18}$$

Proof:

We have proved that the space H_k is invariant by the Fourier transform.

We define a family of transforms $\mathcal{F}_l^n$ associated to dimension n and degree l which is rule by formula (3.18). The formula gives a transform on the spaces of functions on $(0, \infty)$ such that

$$\int_0^\infty |f(r)|^2 r^{2l+n-1} dr < \infty$$

We have

Corollary 3.7. *(Dimensional Bochner identity).*

$$\mathcal{F}_l^n = \mathcal{F}_0^{n+2l}$$

3.5 The far field pattern map

The radiating solution of Helmholts equation in exterior domains are uniquely determined by its far feld pattern, as we showed in corollary 2.5. We study now the expresion of the far field patern when we expand the solution in point sources.

Theorem 3.5. *Let u be a radiating solution on $|x| > R_0 > 0$. Then we can write*

$$u(x) = \sum_{l=0}^{\infty} \sum_{m=1}^{a_k} a_m^l Y_m^{(l)}(\theta) h_l(k|x|), \tag{3.19}$$

where $x = |x|\theta$. The series converges in $L^2(S_{R'}^{n-1})$ for any $R > R_0$ and uniformly on compact sets of $|x| > R_0$.

Conversely if the series converges in $L^2(S_{R_0}^{n-1})$ for some $R_0 > 0$, then it converges abssolutely and uniformly on compacts of $|x| > R_0$, and it is a radiating solution of the equation on $|x| > R_0$.

Proof: From Helmholtz representation formula for $R > R_0$,

$$u(x) = \int_{|z|=R} (u(z) \frac{\partial \Phi(x,z)}{\partial \nu_z} - \Phi(x,z) \frac{\partial u}{\partial \nu}(z)) d\sigma(z). \tag{3.20}$$

If we use the expansion of the fundamental solution in point sources we get

$$= \sum_{lm} \int_{S^{n-1}} (u(R\eta) i k^{n-1} h_l(k|x|) j_l'(kR) - \frac{\partial u}{\partial \nu}(R\eta) i k^{n-2} h_l(k|x|) j_l(kR)) Y_m^{(l)}(\theta) Y_m^{(l)}(\eta) R^{n-1} d\sigma(\eta)$$

Hence we have (3.19) with

$$a_m^l(R) = \int_{S^{n-1}} (u(R\eta) i k^{n-1} j_l'(kR) - \frac{\partial u}{\partial \nu}(R\eta) i k^{n-2} j_l(kR)) Y_m^{(l)}(\eta) R^{n-1} d\sigma(\eta). \tag{3.21}$$

The uniform convergence follows by using the asymptotics of spherical Bessel functions, as done in the proof of Theorem 3.11.

Let us assume now that the series converge in $L^2(S^{n-1}_{R_0})$. For Parseval identity we have

$$\sum_{l=0}^{\infty}\sum_{m=1}^{a_l}|a^l_m h_l(kR_0)|^2 < \infty. \tag{3.22}$$

Let us denote $K = \{x : R_0 < R_1 \leq |x| \leq R_2\}$. By Cauchy-Schwartz inequality

$$\left(\sum_{l=0}^{\infty}\sum_{m=1}^{a_l}|a^l_m Y^{(l)}_m(\theta) h_l(k|x|)|\right)^2$$

$$\leq \sum_{l=0}^{\infty}|\frac{h_l(k|x|)}{h_l(kR_0)}|^2 \sum_{m=1}^{a_l}|Y^{(l)}_m(\theta)|^2 \sum_{l=0}^{\infty}\sum_{m=1}^{a_l}|a^l_m Y^{(l)}_m(\theta) h_l(kR_0)|^2 \leq C_{K,R}(\frac{R_0}{|x|})^l.$$

This proves the absolute and uniform convergence on K. Derivation term by term of the series and the fact that it is a sum of point sources show that it is a solution of the equation. (We may prove the same convergence on K for derivatives with respect to $|x|$ in a similar way). The radiation condition follows then from Helmholtz representation formula in the exterior of $|x| > R_0$.

Theorem 3.6. *Let u be as in (3.19) for $|x| > R_0$. Then its far field pattern can be writen as*

$$u_\infty(\theta) = c_n \frac{1}{k}\sum i^{-l-1}\sum_m a^l_m Y^{(l)}_m(\theta). \tag{3.23}$$

The coeficients satisfy, for any $R > R_0$,

$$\sum \frac{\Gamma(l + \frac{n-2}{2})}{(2R)^l}\sum_m |a^l_m|^2 < \infty. \tag{3.24}$$

Proof: left as an exercise

Condition (3.24) follows by (3.22) and using the asymptotics of Hankel funtions as $l \to \infty$

Remark: The above theorems give a description of outgoing solution and its far field pattern in terms of its Fourier coefficients. For solution in the exterior of S^{n-1}_R we can define a far field map from $L^2(S^{n-1}_R)$ to $L^2(S^{n-1})$. Theorem says that the far field fourier series has the very strong convergence condition (3.24). Is then impossible the boundedness of the inverse map in any reasonable space (for instance no Sobolev space of negative order would fit as target space of the inverse map). This means that the inverse map is very unstable, nevertheless the map is injective by the uniqueness of the far field pattern. To be more precise, consider

$$u_l = \frac{1}{l}h_l(k|x|)Y^l(\frac{x}{|x|}).$$

Then

$$u_{l,\infty} = \frac{1}{lki^{n+1}}Y^l(\frac{x}{|x|}).$$

Notice that $\|u_{l,\infty}\|_{L^2(S^{n-1})} \to 0$ and the u_l itself is not uniformly in any negative Sobolev as $l \to \infty$. (The Sobolev norm in S^{n-1} can be calculated by using the Fourier expansion and the fact that the spherical functions Y_m^l are eigenfunctions of the spherical laplacean). The recovery of a near field pattern from the far field is not a well posed problem in the sense of Hadamard.

3.6 Exercises and further results

- Prove (3.7), (3.8) and Proposition 3.2.
- Prove Jacobi-Anger Formula (decomposition of a plane wave in point sources)

$$e^{ikx\cdot\eta} = \sum_{l=0}^{\infty} i^l \frac{(2l+n-2)}{n-2} j_l(k|x|) C_l^{(n-2)/2}(\theta\cdot\eta)$$

- Prove Funk-Hecke expression for zonal multipliers in terms of ultraspherical polynomials:

  A zonal multiplier is an operator

$$T : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}),$$

  given by a funtion $m : [-1,+1] \to \mathbb{C}$ in the following way

$$Tf(\theta) = \int_{\mathbb{S}^{n-1}} m(\theta\cdot\omega) f(\omega) d\sigma(\omega).$$

  Prove that there exists a bounded sequence of complex numbers μ_k such that

$$\Pi_k(Tf)(\theta) = \mu_k \Pi_k(f)$$

  Give an expression for μ_k in terms of the function m and the Gegenbauer polynomials.

- Prove formula (3.23) (Hint: use the expression for the far field pattern as a boundary integral).
- Obtain an expansion in harmonics similar to (3.11) for the fundamental solution of Laplace equation.

Chapter 4

The homogeneous Helmholtz equation

4.1 Herglotz wave functions

The non uniqueness of the $\mathbf{R}^n$-problem

$$(\Delta + k^2)u = f$$

is due to the existence of entire solutions of the homogeneous equation

$$(\Delta + k^2)u = 0$$

the so called generalized eigenfunction, this fact makes Helmholtz equation of hyperbolic type, its Fourier symbol vanishes on the sphere of radius k. A class of solutions of the homogeneous equation are the plane waves parameterized by its frequency k and its direction ω (the direction of its wave front set).

$$\psi_0(k, \omega, x) = e^{ik\omega\cdot x} \tag{4.1}$$

The Fourier transform of this function is a Dirac delta at the point $k\omega$ on the sphere of radius k. The direct scattering problem for the Schrödinger equation is to find out the solution of

$$(\Delta + k^2 + q)u = 0 \tag{4.2}$$

which behaves as a plane wave at infinity modulo a decaying term whose decay is given by the Sommerfeld radiation condition. This solutions are called generalized eigenfunctions of Schrödinger operator, and for short range potentials are studied as perturbations of the plane waves, the generalized eigenfunctions of Laplace operator.

In scattering theory an important role is played by the superposition with a density $g(\omega)$ of plane waves, namely

$$u_i(x) = \int_{S^{n-1}} e^{ik\omega\cdot x} g(\omega) d\sigma(\omega). \tag{4.3}$$

When the density g is a function in $L^2(S^{n-1})$, u_i is called a Herglotz wave function, which is also an entire solution of the homogeneous Helmholtz equation. By expanding the density g as a Fourier series

$$g(\theta) = \sum_{l=0}^{\infty} \sum_{m=1}^{a_l} a_m^l Y_m^{(l)}(\theta).$$

We can also expand the Herglotz wave function

$$u(x) = \sum_{l=0}^{\infty} \sum_{m=1}^{a_l} i^l a_m^l j_l(k|x|) Y_m^{(l)}(\frac{x}{|x|}). \tag{4.4}$$

The function g is related to the assymptotic profile as limit of the spherical component of its principal part in the sense that for $g_1 - g_2 = g$ (orthogonal) defined by

$$g_2(\theta) = \sum_{l=0}^{\infty} \sum_{m=1}^{a_l} (\frac{1+(-1)^l}{2}) a_m^l Y_m^{(l)}(\theta)$$

and

$$g_1(\theta) = -\sum_{l=0}^{\infty} \sum_{m=1}^{a_l} (\frac{1-(-1)^l}{2}) a_m^l Y_m^{(l)}(\theta),$$

one has

$$\big(u(r\theta) - r^{-(n-1)/2}(g_2(\theta) sinkr + i g_1(\theta) coskr)\big) = O(r^{-n/2}) \text{ as } r \to \infty.$$

(compare with proposition 6.7).

For the solvability of inverse problems, the Herglotz wave functions are important, see [CK]. For instance the scattering amplitudes used in inverse problems (either in the acoustic, the Schrödinger or the obstacle inverse problems) are not dense in $L^2(S^{n-1})$, if there exists a solution of an associated problem which is a Herglotz wave function. This density property is crutial from the spectral point of view.

Let us remark that the Herglotz wave functions are just the distributional Fourier transforms of $L^2(S^{n-1})$-densities on the sphere. In the language of Harmonic Analysis they are the range of an operator known as " extension of the Fourier transform", which is defined by

$$E_k(g)(x) = \widehat{g d\sigma}(kx), \tag{4.5}$$

for a function $g \in L^2(S^{n-1})$.

Several characterization of Herglotz wave functions are known, starting with the works of Herglotz, Hartman, Wilcox (see [CK] and the references there).

Theorem 4.1. *An entire solution v of the equation $(\Delta + k^2)v = 0$ is a Herglotz wave function if and only if it satisfies*

$$\sup_R \frac{1}{R} \int_{|x|<R} |v(x)|^2 dx < \infty. \tag{4.6}$$

Furthermore if g is its density, we have

$$\limsup_{R\to\infty} \frac{1}{R}\int_{|x|<R} |v(x)|^2 dx \sim Ck^{n-1}\|g\|^2_{L^2(S^{n-1})} \tag{4.7}$$

The main tools in the original proof are the spherical harmonic expansion of g, Funk-Hecke formulas and uniform estimates in ν of integrals of the square power of Bessel functions J_ν. This proves that (4.7) is actually an equality with some constant C independent on k. These ingredients have been used later to extend the above results and to give different characterizations of Herglotz wave functions, see [Gu], [BRV], [AlFP]. We are going to give a geometric proof of the above theorem that can be extended to manifolds which need not to be spheres. We shall study the Fourier transform of distributions which are measures carried on submanifolds of codimension d in $\mathbf{R}^n$ and whose density is in L^2.

Let us consider the case $k = 1$. If u is a tempered distribution solution of $(\Delta + 1)u = 0$, then its Fourier transform $\hat{u}$ is supported in S^{n-1}, denoting $g = \hat{u}$, we can rewrite the above theorem (except the claim 4.7) as a special case of the two following theorems (see [AH]).

Theorem 4.2. *Let M be a $\mathcal{C}^1$ submanifold of codimension d in $\mathbf{R}^n$. Let us denote by $d\sigma$ the induced measure. Assume that K is a compact subset of M. If $u \in \mathcal{S}'$ and its Fourier transform is supported in K and is given by an $L^2(M)$-function, $\hat{u} = g(\xi)d\sigma(\xi)$ then*

$$\sup_{R>0} \frac{1}{R^d}\int_{|x|\le R} |u(x)|^2 dx \le C\int_M |g(\xi)|^2 d\sigma(\xi), \tag{4.8}$$

where C is independent of u.

Proof: By using a partition of unity we may assume that K is small and we can describe M by the equation $\xi'' = h(\xi')$, where $\xi' = (\xi_1, ..., \xi_{n-d})$ and $\xi'' = (\xi_{n-d+1}, ..., \xi_n)$ and $h \in \mathcal{C}^1$.

Let us write the measure $d\sigma = a(\xi')d\xi'$, for a positive and continuous function a, we have $\hat{u}(\xi) = \hat{u}(\xi', h(\xi'))d\sigma = g(\xi')a(\xi')d\xi'$ and

$$\begin{aligned} u(x) = \hat{u}(e^{ix\cdot\xi}) &= (2\pi)^{-n}\int_{\mathbf{R}^{n-d}} e^{i(x'\cdot\xi' + x''\cdot h(\xi'))} g(\xi')a(\xi')d\xi' \\ &= (2\pi)^{-n}\int_{\mathbf{R}^{n-d}} e^{ix'\cdot\xi'} F(x'', \xi')d\xi', \end{aligned}$$

where $F(x'', \xi') = e^{ix''\cdot h(\xi')}g(\xi')a(\xi')$. By Plancherel formula in x' we have

$$\begin{aligned} \int_{\mathbf{R}^{n-d}} |u(x', x'')|^2 dx' &= \int_{\mathbf{R}^{n-d}} |\widehat{u(\cdot, x'')}(\xi')|^2 d\xi' \\ = \int_{\mathbf{R}^{n-d}} |F(x'', \xi')|^2 d\xi' &= \int_{\mathbf{R}^{n-d}} |\hat{g}(\xi')|^2 a(\xi')^2 d\xi' \\ &\le C\|g\|^2_{L^2(M)}, \end{aligned}$$

Hence

$$\frac{1}{R^d}\int_{B_R}|u(x)|^2dx'dx'' \leq$$
$$\frac{1}{R^d}\int_{[-R,R]^d}\int_{\mathbf{R}^{n-d}}|u(x',x'')|^2dx'dx'' \leq C\|g\|^2_{L^2(M)}$$

As a consequence we obtain one of the estimates in (4.7).

Corollary 4.1. *Assume $g \in L^2(S^{n-1})$ and let us define*

$$u(x) = \int_{S^{n-1}} e^{ik\theta\cdot x}g(\theta)d\sigma(\theta),$$

then

$$\sup_{R\geq 0}\frac{1}{R}\int_{|x|<R}|u(x)|^2dx \leq Ck^{n-1}\|g\|^2_{L^2(S^{n-1})}. \tag{4.9}$$

Proof: Take $v(y) = u(k^{-1}y)$, then v is a temperated distribution such that $\hat{v}$ is supported in S^{n-1}, hence we have from the theorem that

$$\sup_{R\geq 0}\frac{1}{R}\int_{|x|<R}|v(y)|^2dy \leq C\|g\|^2_{L^2(S^{n-1})}.$$

Define the norm

$$|||v|||^2 = \sup_{R\geq 0}\frac{1}{R}\int_{|x|<R}|v(y)|^2dy, \tag{4.10}$$

then we have $|||v||| = k^{n-1}|||u|||$, this proves the corollary.

Theorem 4.3. *Assume $u \in L^2_{loc}\cap\mathcal{S}'$ such that*

$$\limsup_{R\to\infty}\frac{1}{R^d}\int_{|x|<R}|u(x)|^2dx < \infty.$$

Let Ω be an open set in $\mathbf{R}^n$ such that $\hat{u}$ restricted to Ω, $g = \hat{u}_{|\Omega}$, is compactly supported in a $\mathcal{C}^\infty$-submanifold M of codimension d, then

$$g \in L^2(M),$$

and furthermore

$$\int_M |g|^2d\sigma \leq C\limsup_{R\to\infty}\frac{1}{R^d}\int_{|x|<R}|u(x)|^2dx. \tag{4.11}$$

Proof: Let us start with the following claim on the mollification of resolution ϵ of $\hat{u}$

Lemma 4.1. *Let $u \in L^2_{loc}\cap\mathcal{S}'$ and $\chi \in \mathcal{C}_0^\infty$ supported on $B(0,1)$, and denote $g_\epsilon = \hat{u}(\cdot)\star\epsilon^{-n}\chi(\cdot/\epsilon)$. Then, for fixed $d > 0$, we have*

$$\|g_\epsilon\|^2_{L^2} \leq C\epsilon^{-d}K_d(\epsilon)C_d(\chi),$$

where $K_d(\epsilon) = \sup_{R\epsilon\geq 1}\frac{1}{R^d}\int_{|x|<R}|u(x)|^2dx$ and $C_d(\chi)$ only depends on χ.

The lemma follows from the following inequalities (Plancherel and taking the sup out of the integral and then out of the sum):

$$\|g_\epsilon\|_{L^2}^2 = \|u(\cdot)\hat{\chi}(\epsilon(\cdot))\|_{L^2}^2 =$$

$$\sum_{j=1}^{\infty} \int_{2^{j-1}\leq|\epsilon x|\leq 2^j} |u(x)\widehat{\chi}(\epsilon x)|^2 dx + \int_{|\epsilon x|\leq 1} |u(x)\widehat{\chi}(\epsilon x)|^2 dx \leq$$

$$\sum_{j=1}^{\infty} (\sup_{2^{j-1}\leq|\epsilon x|\leq 2^j} \widehat{\chi}(\epsilon x))^2 \int_{2^{j-1}\leq|\epsilon x|\leq 2^j} |u(x)|^2 dx + (\sup_{|\epsilon x|\leq 1} \chi(\epsilon x))^2 \int_{|\epsilon x|\leq 1} |u(x)|^2 dx$$

$$\leq \epsilon^{-d} (\sup_{j=1,2,\ldots} (\frac{2^j}{\epsilon})^{-d} \int_{2^{j-1}\leq|\epsilon x|\leq 2^j} |u(x)|^2 dx \cdot (\sum_{j=1}^{\infty} \sup_{2^{j-1}\leq|y|\leq 2^j} |\widehat{\chi}(y)|^2 2^{jd})$$

$$+ \sup_{|y|\leq 1} |\widehat{\chi}(y)|^2 \epsilon^d \int_{|\epsilon x|\leq 1} |u(x)|^2 dx)$$

$$\leq \epsilon^{-d} \sup_{\epsilon R\geq 1} (R)^{-d} \int_{B(0,R)} |u(x)|^2 dx C_d(\chi).$$

Going back to the proof of the theorem, let us see that g is a density L^2 on M. Since g is supported on M, then g_ϵ is supported on $M_\epsilon = \{x \in \mathbf{R}^n : d(x, M) \leq \epsilon\}\}$; since $u \in \mathcal{S}'$ then $g_\epsilon \to g$ en $\mathcal{S}'$. Let us see the action of g on test functions $\psi \in \mathcal{C}_0^\infty$:

$$|g(\psi)| = |\lim_{\epsilon\to 0}(g_\epsilon)(\psi)| = \lim_{\epsilon\to 0} |\int_{M_\epsilon} (\hat{u} \star \chi_\epsilon(x))\psi(x) dx|$$

$$\leq \lim_{\epsilon\to 0} (\int |g_\epsilon|^2 dx)^{1/2} (\int_{M_\epsilon} |\psi(x)|^2 dx)^{1/2}$$

$$\leq \lim_{\epsilon\to 0} (\epsilon^{-d} \int_{M_\epsilon} |\psi(x)|^2 dx)^{1/2} (K_d(\epsilon) C_d)^{1/2},$$

and hence, considering that

$$\epsilon^{-d} \int_{M_\epsilon} |\psi(x)|^2 dx \to \int_M |\psi(x)|^2 d\sigma(x),$$

we have

$$|g(\psi)| \leq \limsup_{\epsilon\to 0} K(\epsilon)^{1/2} \|\psi\|_{L^2(M)} C_d^{1/2},$$

this means that g is a function in $L^2(M)$ such that

$$\int_M |g(\theta)|^2 d\sigma(\theta) \leq C_d \limsup_{\epsilon\to 0} K(\epsilon),$$

this ends the proof of the theorem.

As a consequence we obtain the following characterization of the Helglotz wave functions.

Corollary 4.2. *A solution of the homogeneous Helmholtz equation is a Herglotz wave function if and only if (4.6) holds*

Notice that theorem 4.1 gives us an isometry from $L^2(S^{n-1})$ to the space of solutions satisfying (4.6) though the norm (4.10). Corollary 4.2 just proves the equivalence of the two norms.

Theorem 4.2 and theorem 4.3 can be understood as "the dual trace theorem at the end point". To see how this can be done, let us start with the dual version of the above.

Let us define the Besov space

$$B_s = \{v \in L^2_{loc} : \|v\|_{B_s} = \sum_{j=0}^{\infty} R^s_{j+1} (\int_{\Omega_j} |v|^2 dx)^{1/2} < \infty\}, \tag{4.12}$$

where $R_j = 2^{j-1}$ if $j \geq 1$, $R_0 = 0$ and $\Omega_j = \{x : R_j \leq |x| \leq R_{j+1}\}$.

Notice that the elements of the dual B^*_s are the functions $v \in L^2_{loc}$ such that

$$\|v\|^2_{B^*_s} = \sup_{j=1,2\ldots} R^{2s}_j \int_{\Omega_j} |v|^2 dx \leq \infty \tag{4.13}$$

The norm defined above is equivalent to the one defined in (4.10) when the supremun there is taken over $R > 1$.

Corollary 4.3. *Let M be a $\mathcal{C}^1$-submanifold in $\mathbf{R}^n$ and K a compact contained in M. Then the operator given by the restriction of the Fourier transform to K, defined for $v \in \mathcal{S}$ as*

$$T(v) = \hat{v}_{|K} \in L^2_K(d\sigma) \tag{4.14}$$

can be extended by continuity to an onto map from $B_{d/2}$ to $L^2_K(d\sigma)$.

Proof: Let us prove the continuity of T. Write

$$\|T(v)\|_{L^2_K(d\sigma)} = sup\{< Tv, \phi >: \|\phi\|_{L^2_K(d\sigma)} = 1\}$$

We have

$$< Tv, \phi >= \int_K \hat{v}\phi d\sigma = \int_{R^n} v(x)\widehat{\phi d\sigma}(x)dx \leq \|\widehat{\phi d\sigma}\|_{B^*_{d/2}} \|v\|_{B_{d/2}} \leq C\|v\|_{B_{d/2}},$$

where we used (4.8).

The adjoint of T, defined for $\psi \in L^2_K(d\sigma)$ by

$$T^*(\psi) = \widehat{(\psi d\sigma)} \in B^*_{d/2} \tag{4.15}$$

is one to one from 4.11 and has closed range, hence T is onto, see [Rud] Thm 4.15.

This corollary is a trace theorem at the end point, which means that it gives a substitute of the Sobolev space $W^{d/2,2}(\mathbf{R}^n)$ in order to obtain traces in L^2 when restricted to a submanifold of codimension d. To compare with, let us recall the classical trace theorem in Sobolev spaces.

Theorem 4.4. *Let M be a $\mathcal{C}^\infty$ manifold of codimension d and $\alpha > d/2$, then there exists a bounded operator*

$$\tau : W^{\alpha,2}(\mathbf{R}^n) \to W^{\alpha-d/2,2}(M),$$

such that for $\psi \in \mathcal{C}_0^\infty$, $\tau(\psi) = \psi_{|M}$. This operator is called the trace operator on M and $\tau(f)$ the trace of f on M which we also denote by $f_{|M}$.

We are going to prove this theorem in the special case $M = \mathbf{R}^{n-d}$, the case for general M can be reduced to this from the invariance of $W^{s,2}$ by a $\mathcal{C}^N$- change of coordinates, where $N \geq s$, and by partitions of unity.

Theorem 4.5. *Let $u \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ and define $\tau u \in \mathcal{C}_0^\infty(\mathbf{R}^{n-d})$, for $x' \in \mathbf{R}^{n-d}$, by $\tau u(x') = u(x', 0)$. Then τ can be extended by continuity to a map $W^{\alpha,2}(\mathbf{R}^n) \to W^{\alpha-d/2,2}(\mathbf{R}^{n-d})$ if $\alpha > d/2$. Furthermore, this map is onto.*

Proof: From the Fourier inversion theorem in $\mathbf{R}^d$ we may write

$$\widehat{(\tau u)}(\xi') = \int_{\mathbf{R}^d} \hat{u}(\xi', \xi'') d\xi'',$$

hence

$$|\widehat{(\tau u)}(\xi')|^2 \leq \int_{\mathbf{R}^d} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^\alpha d\xi'' \int_{\mathbf{R}^d} (1 + |\xi|^2)^{-\alpha} d\xi''.$$

If we use that for $\alpha > d/2$,

$$\int_{\mathbf{R}^d} (1 + |\xi|^2)^{-\alpha} d\xi'' = C(\alpha, d)(1 + |\xi'|^2)^{-\alpha + d/2}, \tag{4.16}$$

where

$$C(\alpha, d) = \int_0^\infty \frac{s^{d-1}}{(1 + s^2)^\alpha} ds < \infty$$

for $\alpha > d/2$. Then we have

$$|\widehat{(\tau u)}(\xi')|^2 (1 + |\xi'|^2)^{\alpha - d/2} \leq \int_{\mathbf{R}^d} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^\alpha d\xi''.$$

Integrating in ξ' we obtain the continuity.

In order to see that τ is onto, let $g \in W^{\alpha-d/2,2}(\mathbf{R}^{n-d})$. Now define

$$\psi(\xi) = \frac{\hat{g}(\xi')(1 + |\xi'|^2)^{\alpha - d/2}}{(1 + |\xi|^2)^\alpha}$$

one can see that $\hat{\psi} = u \in W^{\alpha,2}(\mathbf{R}^n)$ and that $\widehat{\tau u} = C(\alpha, d)\hat{g}$ again from Fourier inversion theorem and (4.16).

Let us explain in which sense Agmon-Hormander theorem is a dual version of the above theorem for $\alpha = d/2$. Let us write $\alpha = d/2 + \epsilon$. Then the following a priori estimate, for every $\epsilon \geq 0$, holds

$$\|\tau(g)\|_{L^2(M)} \leq C\|g\|_{W^{d/2+\epsilon,2}(R^n)} = \|\hat{g}\|_{L^2((1+|\xi|^2)^{d/2+\epsilon} d\xi)}.$$

If we take $\hat{g} = f$, we obtain that

$$\|\tau(\hat{f})\|_{L^2(M)} \leq C\|f\|_{L^2((1+|x|^2)^{d/2+\epsilon} dx)}.$$

This means that the restriction operator

$$Tf = \hat{f}_{|M} : L^2(1+|x|^2)^{d/2+\epsilon} \to L^2(M) \tag{4.17}$$

Hence the adjoint operator

$$T^*(\psi) = \widehat{(\psi d\sigma)} \tag{4.18}$$

satisfies

$$\|T^*(\psi)\|_{L^2((1+|\xi|^2)^{-d/2-\epsilon}d\xi)} \leq C\|\psi\|_{L^2(M)}. \tag{4.19}$$

This inequality can also be written as

$$\sup_{R\geq 1} \frac{1}{R^{d/2+\epsilon}} \int_{B(0,R)} |T^*(\psi)(\xi)|^2 d\xi \leq C\|\psi\|^2_{L^2(M)}. \tag{4.20}$$

Estimate (4.8) is the above inequality for $\epsilon = 0$.

From the point of view of traces of functions, it is false that the traces of a function f such that $\hat{f} \in L^2(< x >^{d/2})$ exists as a function in $L^2(M)$, but the claim is true if we substitute by the assumption $\hat{f} \in B_{d/2}$ and, in some sense, this is the sharp class of functions whose traces are in $L^2(M)$.

From theorem 4.2 and corollary 4.3 we obtain

Theorem 4.6.

$$T^*T(f) = \widehat{d\sigma} \star f : B_{d/2} \to B^*_{d/2}. \tag{4.21}$$

In the case $M = \mathbf{S}^{n-1}$ we will see that this operator is related to the imaginary part of the resolvent of the Laplace operator. Formula (4.21) gives a factorization of this imaginary part when considered as an operator from some space to its dual, by inserting the intermediate space $L^2(\mathbf{S}^{n-1})$. In $L^p(\mathbf{R}^n)$ this factorization is known as "P. Tomas' argument", see [T]. Let us define

$$I_k f(x) = \frac{1}{k}(\widehat{d\sigma_k} * f)(x), \tag{4.22}$$

where $d\sigma_k$ is the measure on the sphere of radius k. Consider the norm defined in (4.10) and which is the dual of

$$\|u\|_{\tilde{B}_{1/2}} = \sum_{-\infty}^{\infty} \left(R_j \int_{\Omega_j} |u(x)|^2 dx \right)^{1/2}, \tag{4.23}$$

(compare with (4.12))

This norm behaves well with respect to dilations as oposite to ((4.12)), it was used by Kenig, Ponce and Vega [] to study well posedness of non linear dispersive equations.

We have

Corollary 4.4. *There exists a constant $C > 0$ uniform in k such that*

$$|||I_k f||| \leq Ck^{-1}\|f\|_{\tilde{B}_{1/2}} \tag{4.24}$$

Proof: We reduce to the case $k = 1$, noticing that for $u_k(x) = u(x/k)$, we have

$$|||f_k||| = k^{(n-1)/2}|||f|||,$$

$$\|f_k\|_{\tilde{B}_{1/2}} = k^{(n+1)/2}\|f\|_{\tilde{B}_{1/2}}$$

and

$$(I_k f)(x/k) = k^{-2} I_1(f_k)(x).$$

The case $k = 1$ can be proved from (4.9) duality and P.Tomas' argument as was done to prove Theorem 4.6.

4.2 The Restriction of the Fourier Transform

Given $f \in L^p(\mathbf{R}^n)$ and a submanifold M in $\mathbf{R}^n$, when does it make sense to restrict $\hat{f}$ to M in the sense of this restriction being a function in $L^t(M)$? A lot has been written about this question, see the remarks below. We are going to study the case $M = \mathbf{S}^{n-1}$, starting with $t = 2$. It is important to remark that in these theorems (this is not the case of corollary (4.3)) the positive curvature of the sphere plays a fundamental role. We start will the dual theorem

Theorem 4.7 (Extension theorem)**.** *Let ψ be an $L^2(\mathbf{S}^{n-1})$ density, then its extension $T^*\psi = \widehat{\psi d\sigma}$ is in $L^q(\mathbf{R}^n)$, for $q \geq \frac{2(n+1)}{n-1}$, i.e. if q satisfies the relation*

$$1/2 - 1/q \geq 1/(n+1). \tag{4.25}$$

Furthermore we have the estimate

$$\|T^*\psi\|_{L^q(\mathbf{R}^n)} \leq C\|\psi\|_{L^2(S^{n-1})}. \tag{4.26}$$

As a consequence we have:

Corollary 4.5 (Restriction theorem [T])**.** *Let $f \in L^p(\mathbf{R}^n)$ where $p \leq \frac{2(n+1)}{n+3}$ we can define $Tf = \hat{f}_{|S^{n-1}}$ as a L^2 density and it holds*

$$\|Tf\|_{L^2(\mathbf{S}^{n-1})} \leq C\|f\|_{L^p} \tag{4.27}$$

Remark 1: By Cauchy-Schwarz we can write for ω_n the measure of the sphere:

$$|\widehat{\psi d\sigma}(\xi)| = \int_{\mathbf{S}^{n-1}} e^{ix\cdot\xi}\psi(x)d\sigma(x)| \leq \|\psi\|^{1/2}_{L^2(S^{n-1})}\omega_n^{1/2}.$$

Then by interpolation it suffices to prove the theorem at the end point $q = \frac{2(n+1)}{n-1}$.

Remark 2: The range of q is sharp. This can be proved by the following homogeneity argument in the corollary : Take a non negative function $\phi \in \mathcal{C}_0^\infty$ and construct $(\phi_\delta)(\xi', \xi_n) = \phi(\frac{\xi'}{\delta}, \frac{\xi_n - e_n}{\delta^2})$. Then

$$\widehat{\phi_\delta}(x) = e^{i\delta^2 x_n}\delta^{n+1}\phi(\delta x', \delta^2 x_n)$$

and hence $\|T\widehat{\phi_\delta}\|_{L^t(S^{n-1})} = \|\phi_\delta\|_{L^t(S^{n-1})} \geq C\delta^{(n-1)/t}$ and $\|\widehat{\phi_\delta}\|_{L^p} \leq C\delta^{n+1-(n+1)/p}$, where $p' = q$. Assume that

$$\|T\psi\|_{L^t(\mathbf{S}^{n-1})} \leq C\|\psi\|_{L^p}, \tag{4.28}$$

and take $\psi = \widehat{\phi_\delta}$, then if $\delta \to 0$, we obtain the necessary condition $(n+1)/q = (n+1)(1-\frac{1}{p}) < (n-1)/t$.

There is another necessary condition that comes from the evaluation of $T^*(1)$ in term of Bessel function given by Funk-Ecke formula. That is the constrain $q > 2n/(n-1)$. The sufficiency of $(n+1)/q < (n-1)/t$, together with $q > 2n/(n-1)$ to have inequality (4.28) is known as the "Restriction conjecture", an open question in classical Fourier Analysis. Notice that in the particular case $t = 2$ we obtain the range of the Corollary.

To prove the extension theorem, we need the following lemma, which gives a bound for the mollification with resolution ϵ of a measure on the sphere with density f.

Lemma 4.2. *Let $\chi \in \mathcal{S}$, denote $d\sigma_\epsilon = f(\cdot)d\sigma(\cdot) \star \epsilon^{-n}\chi(\cdot/\epsilon)$, where $f \in L^\infty(\mathbf{S}^{n-1})$ then*

$$sup_x|d\sigma_\epsilon(x)| \leq C\epsilon^{-1}$$

Proof: We make a reduction to the case where χ is compactly supported, we will refer to this reduction as a " Schwartz tails argument": Take a $\mathcal{C}_0^\infty$ partition of unity in $\mathbf{R}^n$ such that

$$\sum_{j=0}^{\infty} \psi_j(x) = 1,$$

where ψ_0 is supported in $B(0,1)$ and $\psi_j = \psi(2^{-j}x)$ for $j > 0$, and ψ is supported in the annulus $1/2 \leq |x| \leq 2$. We adapt the partition to the resolution ϵ, which means to take

$$\sum_{j=0}^{\infty} \psi_j(\epsilon^{-1}x) = 1.$$

Now write

$$d\sigma_\epsilon(x) = d\sigma(\cdot) \star \epsilon^{-n} \sum \psi_j(\epsilon^{-1}(\cdot))\chi(\cdot/\epsilon),$$

We are going to estimate this sum term by term. Notice that the jth term , $j > 0$ is an integral on the sphere of radius 1 centered at x of a function supported on the annulus $2^{j-1}\epsilon \leq |y| \leq 2^j\epsilon$. In this annulus, since χ is rapidly decreasing, we have:

$$|\chi(\epsilon^{-1}x)| \leq \frac{C_N}{(1+2^j)^N},$$

Hence

$$|d\sigma(\cdot) \star \epsilon^{-n}\psi_j(\epsilon^{-1}(\cdot))\chi(\cdot/\epsilon)(x)| \leq C_N \frac{(2^j\epsilon)^{n-1}}{(1+2^j)^N}\epsilon^{-n}.$$

By taking N big enough, the sum in j converges bounded by $C\epsilon^{-1}$. To end the proof of the lemma, notice that the term $j = 0$, satisfies trivially the inequality .

Proof of the extension theorem.

We give only the proof of the weak type inequality $L^2 \to L^{\frac{2(n+1)}{n-1},\infty}$ at the end point, where the target space denotes the weak Lorentz space.

The strong inequality at the end point can be proved by using Stein interpolation theorem for an analytic family of operators, see [St] and the original P. Tomas' proof. The proof we give of the weak type is due to J. Bourgain [Bo]. We will show that for ϕ a function in $L^2(S^{n-1})$, we have

$$|\{x : |(\phi d\sigma)\check{}(x)| \geq \lambda\}| \leq C(\frac{\|\phi\|_{L^2}}{\lambda})^{\frac{2(n+1)}{n-1}}.$$

We may assume $\|\phi\|_{L^2} = 1$. From remark 1, we have that

$$|\{x : |(\phi d\sigma)\check{}(x)| \geq \omega_n^{1/2}\}| = 0,$$

which allows us to reduce to the case of small λ.

Let $A = \{x \in B(0, R) : \Re\widehat{\phi d\sigma}(x) \geq \lambda\}$, where R is fixed. A similar argument can estimate the cases $\Re\widehat{\phi d\sigma}(x) \leq -\lambda$ ad the similar cases for the imaginary part. Split $\mathbf{R}^n = \cup Q_\beta$ where Q_β are cubes with disjoint interiors and side length $\rho = \lambda^{-\alpha}$, with an exponent α to be chosen later and $\beta \in \mathbf{Z}^n$.

Let $A_\beta = A \cap Q_\beta$, we have

$$\lambda|A| \leq |\int_{R^n} \chi_A \widehat{\phi d\sigma}(x)dx| = |\int_{S^{n-1}} \phi\widehat{\chi_A} d\sigma| \leq$$

$$\|\phi\|_{L^2(S^{n-1})}(\int_{S^{n-1}} |\sum_\beta \widehat{\chi_{A_\beta}}|^2 d\sigma)^{1/2},$$

where $\chi_{A_\beta} = \chi_\beta$, vanishes except for a finite number of β's, we have

$$\lambda^2|A|^2 \leq \int_{S^{n-1}} |\sum_\beta \widehat{\chi_\beta}|^2 d\sigma = \sum_{\beta,\gamma} \int_{S^{n-1}} \widehat{\chi_\beta}\bar{\widehat{\chi_\gamma}} d\sigma.$$

Fix an integer $s > 0$, to be chosen later, we may write

$$\lambda^2|A|^2 \leq \sum_{|\beta-\gamma|>s} \int_{S^{n-1}} \widehat{\chi_\beta}\bar{\widehat{\chi_\gamma}} d\sigma + \sum_{|\beta-\gamma|\leq s} \int_{S^{n-1}} \widehat{\chi_\beta}\bar{\widehat{\chi_\gamma}} d\sigma = \Sigma_1 + \Sigma_2.$$

We start by estimating

$$\Sigma_2 \leq \sum_{|\beta-\gamma|\leq s} \int_{S^{n-1}} (|\widehat{\chi_\beta}|^2 + |\widehat{\chi_\gamma}|^2) d\sigma \leq C_n(s) \sum_\beta \int_{S^{n-1}} (|\widehat{\chi_\beta}|^2 d\sigma,$$

where $C_n(s)$ is the number of multiindices at distance of β less than s.

Let us write $(|\widehat{\chi_\beta}|^2)\check{} = \chi_\beta * \chi_\beta$ and notice that this function is supported in a cube $Q^*_{2\beta}$ centered at x_β of side length 2ρ. Take $\psi \in \mathcal{C}_0^\infty$ such that $\psi = 1$ in $B(0, 2\sqrt{n})$, then $\psi(\frac{x-x_\beta}{\rho}) = 1$ for $x \in Q^*_{2\beta}$, hence

$$\int_{S^{n-1}} |\widehat{\chi_\beta}|^2 d\sigma = d\sigma(|\widehat{\chi_\beta}|^2) = \widehat{d\sigma}((|\widehat{\chi_\beta}|^2)\check{}) = \widehat{d\sigma}\left(\psi(\rho^{-1}((\cdot) - x_\beta))(|\widehat{\chi_\beta}|^2)\check{}(\cdot)\right) =$$

$$= \left(\psi(\rho^{-1}((\cdot)-x_\beta))\widehat{d\sigma}\right)\left(\left(|\widehat{\chi_\beta}|^2\right)\check{}(\cdot)\right) = (\psi(\rho^{-1}((\cdot)-x_\beta))\check{}*d\sigma)(|\widehat{\chi_\beta}|^2) =$$

$$= (e^{i\rho(\cdot)x_\beta}\rho^n\widehat{\psi}(\rho(\cdot))*d\sigma)(|\widehat{\chi_\beta}|^2) = \int_{\mathbf{R}^n} d\sigma_{1/\rho}(\xi)|\widehat{\chi_\beta}(\xi)|^2 d\xi$$

$$\leq C\rho\int_{\mathbf{R}^n}|\widehat{\chi_\beta}(\xi)|^2 d\xi = C\rho\int_{\mathbf{R}^n}|\chi_\beta(x)|^2 dx = C\rho|A_\beta|,$$

where we have used lemma 4.2. Hence we have proved that

$$\Sigma_2 \leq C_n(s)\rho|A|.$$

To estimate Σ_1 notice that

$$\widehat{d\sigma}(\xi) = C|\xi|^{-\frac{n-2}{2}}J_{\frac{n-2}{2}}(\xi),$$

which for large $|\xi|$ behaves like $C|\xi|^{-\frac{n-1}{2}}$. Since $dist(A_\beta, A_\gamma) \geq C\rho s$, we have

$$\int_{S^{n-1}}\widehat{\chi_\beta}\bar{\widehat{\chi_\gamma}}d\sigma = \int_{\mathbf{R}^n}\chi_\beta(\xi)\int_{\mathbf{R}^n}\chi_\gamma(\eta)\widehat{d\sigma}(\xi-\eta)d\eta d\xi$$

$$\leq (C\rho s)^{-\frac{n-1}{2}}|A_\beta||A_\gamma|,$$

and hence

$$\Sigma_1 \leq (C\rho s)^{-\frac{n-1}{2}}\sum_{\beta,\gamma}|A_\beta||A_\gamma| = (C\rho s)^{-\frac{n-1}{2}}|A|^2.$$

Putting the two terms together we have proved

$$\lambda^2|A|^2 \leq C_n(s)\rho|A| + C(s\rho)^{-\frac{n-1}{2}}|A|^2.$$

The choice $\rho = \lambda^{-\frac{4}{n-1}}$ and s large enough allows the last term to be absorbed by the left hand side, hence we have

$$\lambda^2|A|^2 \leq C(n)\rho|A|,$$

uniformly on R. This proves the inequality.

The extension theorem is just an estimate of the L^q-norm of Herglotz wave functions, this follows inmediately from (4.5).

Corollary 4.6. *If v is a Herglotz wave function corresponding to the eigenvalue k^2 with density g, then for*

$$1/2 - 1/q \geq 1/(n+1)$$

it holds

$$\|v\|_{L^q} \leq Ck^{-n/q}\|g\|_{L^2(S^{n-1})}. \tag{4.29}$$

We also have an analogous of theorem 4.4

Corollary 4.7. *Let $k > 0$, and consider as in (4.22)*

$$I_k f(x) = \frac{1}{k}(\widehat{d\sigma_k}*f)(x).$$

Then, for

$$\frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$$

and $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$\|I_k f\|_{L^q} \leq C k^{n(\frac{1}{p}-\frac{1}{q})-2} \|f\|_{L^p} \tag{4.30}$$

Proof: Write

$$I_k f(x) = k^{-1} \int \widehat{d\sigma_k}(x-y) f(y) dy = k^{-1} \int_{S_k^{n-1}} \widehat{f}(\theta) e^{ix\cdot\theta} d\sigma_k(\theta)$$

$$= k^{n-2} \int_{S^{n-1}} \widehat{f}(k\theta) e^{ikx\cdot\theta} d\sigma(\theta) = k^{-2} \int_{S^{n-1}} g(\theta) e^{ikx\cdot\theta} d\sigma(\theta),$$

where $g(\theta) = \widehat{f(\frac{(\cdot)}{k})}(\theta)$. This means $I_k f(x) = k^{-2} v(x)$, where v is a Herglotz wave function with density g. Hence

$$\|I_k f\|_q \leq k^{-2} k^{-n/q} \|g\|_2,$$

for $1/2 - 1/q \geq 1/(n+1)$. But if $\frac{1}{p} - \frac{1}{2} \geq \frac{1}{n+1}$, the restriction theorem implies

$$\|g\|_2 \leq C \|f(\frac{(\cdot)}{k})\|_p \leq k^{n/p} \|f\|_p,$$

this proves the corollary.

Remark: The exponent of k in Corollary 4.7 is zero for p and q in the Sobolev gap for the Laplacian $2/n = 1/p - 1/q$.

4.3 The transmission problem

4.4 Exercises and further results

- Prove (4.4) and study the convergence of the series (4.4).
- The evolution Schroedinger equation. Estimates of the extension operator can be seen as estimates of the initial value problem for the evolution Schrödinger equation, namely the solution of the problem

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = 0, (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = u_0. \end{cases} \tag{4.31}$$

Let us denote the solution $u(x,t) = e^{it\Delta} u_0$. Then

$$\|D_x^\gamma e^{it\Delta} u_0\|_{L_x^q(L_t^2)} \leq C \|u_0\|_{L^2}. \tag{4.32}$$

Where D^γ denotes the fractional derivative

$$D_x^\gamma f(x) = \int_{R^n} e^{ix\cdot\xi} |\xi|^\gamma \hat{f}(\xi) d\xi,$$

γ satisfies $\gamma = 1 - n(1//2 - 1/q)$, q is in the range $1/(n+1) \leq 1/2 - 1/q \leq 1/2$ and $n \geq 2$. Also it holds

$$\|D_x^{1/2} e^{it\Delta} u_0\|_{B^*_{1/2}(L^2_t)} \leq C \|u_0\|_{L^2}. \tag{4.33}$$

Prove these estimates. Hints: To obtain (4.32)

– (a) Write

$$D_x^\gamma e^{it\Delta} u_0(x) = \int_{R^n} e^{ix\cdot\xi} e^{it|\xi|^2} |\xi|^\gamma \hat{u}_0(\xi) d\xi$$

Then write the integral in polar coordenates (r, θ) and change variable $s = r^2$.

– (b) With the above expression write $\|D_x^\gamma e^{it\Delta} u_0\|^2_{L^q_x(L^2_t)}$ and use Plancherel identity in t and then Minkowski integral inequality (notice that $q > 2$). End the proof by using the extension estimate for spheres of the appropriate radius.

Estimate (4.33) follows in a similar way.

- Look for an example to disprove the trace theorem at the endpoint in the Sobolev version.

Chapter 5

Estimates for the resolvent

As we showed in chapter 1, the outgoing solution of the equation in $\mathbf{R}^n$

$$(\Delta + k^2)u = f \tag{5.1}$$

is the function

$$u(x) = \int \Phi(x-y)f(y)dy,$$

where Φ is defined in (2.10). In the Fourier transform side, as was observed in section 1.4, the outgoing fundamental solution $\Phi(x)$ is given by

$$\hat{\Phi}(\xi) = (-|\xi|^2 + k^2 + i0)^{-1}, \tag{5.2}$$

This distribution can be expressed in terms of the homogeneous distributions of degree -1, principal value (p.v.) and Dirac delta (see Gelfand-Shilov Vol 1, pg 209-236). We can obtain the expression from the one variable formula

$$\lim_{\epsilon \to 0+} (t + i\epsilon)^{-1} = pv\frac{1}{t} + i\pi\delta,$$

which can be extended to the $\mathbf{R}^n$-function $t = H(\xi)$ as far as we can take locally H as a coordinate function in a local patch of a neighborhood in $\mathbf{R}^n$ at any point ξ_0 for which $H(\xi_0) = 0$. We have

Proposition 5.1. *Let $H : \mathbf{R}^n \to \mathbf{R}$ such that $| \bigtriangledown H(\xi)| \neq 0$ at any point where $H(\xi) = 0$, then we can define the distribution limit*

$$(H(\xi) + i0)^{-1} = \lim_{\epsilon \to 0+} (H(\xi) + i\epsilon)^{-1}. \tag{5.3}$$

And

$$(H(\xi) + i0)^{-1} = pv\frac{1}{H(\xi)} + i\pi\delta(H) \tag{5.4}$$

in the sense of the tempered distributions.

The distribution $\delta(H)$ is defined as

$$\delta(H)(\psi) = \int_{H(\xi)=0} \psi(\xi)\omega(\xi),$$

where ω is any $(n-1)$-form such that $\omega \wedge dH = d\xi$. It is easily seen, from the change of variable formula, that this integral does not depends on the choice of the form ω. The existence of such ω can be proved by using local coordinates in $\mathbf{R}^n$ adapted to the manifold $H(\xi)=0$. Also it is easily seen that for α any function which does not vanish at the points ξ with $H(\xi)=0$, then

$$\delta(\alpha H)=\alpha^{-1}\delta(H).$$

We can choose an orthonormal moving frame on the tangent plane to $H(\xi)=0$, namely $\omega_1,...,\omega_{n-1}$, for this frame we have $\omega_1\wedge...\wedge\omega_{n-1}\wedge\frac{dH}{|\bigtriangledown H|}=d\xi$, it follows that

$\delta(|\bigtriangledown H|^{-1}H)$ is the measure $d\sigma$ induced by $\mathbf{R}^n$ on the hypersurface $H(\xi)=0$ and hence

$$\delta(H)=|\bigtriangledown H|^{-1}d\sigma.$$

We summarize the above in the following

Lemma 5.1. *Let $H(\xi)=-|\xi|^2+k^2$, then*

$$(H(\xi)+i0)^{-1}=pv\frac{1}{H(\xi)}+\frac{i\pi}{2k}d\sigma \tag{5.5}$$

The operator I_k defined in (4.22) is going to model some of the estimates we want to prove for the resolvent, since

$$\begin{aligned}R_+(k^2)(f)(x)&=(\Delta+k^2+i0)^{-1}(f)(x)\\&=p.v.\int_{\mathbf{R}^n}e^{ix\cdot\xi}\frac{\hat{f}(\xi)}{-|\xi|^2+k^2}d\xi+\frac{i\pi}{2k}\widehat{d\sigma}*f(x).\end{aligned}\tag{5.6}$$

5.1 Selfdual Besov estimates

The most celebrated and widely used is the estimate due to Agmon in his study of Schrödinger operator with short range potentials. We start with the extension of the estimate (4.23), which holds for the imaginary part operator $I_kf=(i\pi/2k)\widehat{d\sigma}*f$, to the complete resolvent. This is an improvement due to Agmon and Hörmander for $B_{1/2}$ of the mentioned estimate of Agmon. The following is a dilation invariant version due to Kenig, Ponce and Vega in their study of nonlinear Schrödinger equations. Notice that we extend the sum in estimate (4.12) of chapter 1 to $j\in\mathbf{Z}$, as was done in corollary 4.4.

We state also some estimates for derivatives of solution of Helmholtz equation. The key point is that we can substitute some power of k in the estimates by some derivatives in the left hand side of the inequalities. Since in the characteristic variety we have $|\xi|=k$, we have to control the symbol far away of this set, which means to estimate Riezs potentials type operators.

Theorem 5.1 ([KPV]). *There exists $C > 0$, independent of k such that*

$$\sup_{R\geq 0}\left(\frac{1}{R}\int_{|x|<R}|R_+(k^2)f(y)|^2dy\right)^{1/2}\leq Ck^{-1}\sum_{j=-\infty}^{\infty}(R_j\int_{\Omega_j}|f|^2dx)^{1/2}, \tag{5.7}$$

and

$$\sup_{R\geq 0}\left(\frac{1}{R}\int_{|x|<R}||D|R_+(k^2)f(y)|^2dy\right)^{1/2}\leq C\sum_{j=-\infty}^{\infty}(R_j\int_{\Omega_j}|f|^2dx)^{1/2}. \tag{5.8}$$

Where $|D|$ denotes the Fourier multiplier operator

$$\widehat{|D|f}(\xi) = |\xi|\hat{f}(\xi)$$

Remark: From the above estimate there is a gain of one derivative with respect to the left hand side in the solution of the equation

$$(\Delta + k^2)u = f$$

assuming the radiation condition. The point in the above estimate is that the a priori contant C is independent of k.

As a corollary we obtain the analogous of the trace theorem for the complete resolvent, this is Agmon estimate:

Corollary 5.1 ([A]). *Let $\omega(x) =< x >^{-(1+\epsilon)/2}$, $\epsilon > 0$, then*

$$\|R_+(k^2)f\|_{L^2(w(x)dx)} \leq Ck^{-1}\|f\|_{L^2(w(x)^{-1}dx)} \tag{5.9}$$

The estimate with derivatives is the following, which is written for the particular case of Agmon weights

Corollary 5.2. *Let $k > 0$ and α a multiindex with $|\alpha| \leq 1$, then there exists a constant C such that*

$$\|D^\alpha R_+(k^2)f\|_{L^2(w(x)dx)} \leq Ck^{\alpha-1}\|f\|_{L^2(w(x)^{-1}dx)} \tag{5.10}$$

Proof of Theorem 5.8: By homogeneity we might reduce to the case $k = 1$. Only the first term in (5.6)

$$Kf(x) = p.v.\int_{\mathbf{R}^n} e^{ix\cdot\xi}\frac{\hat{f}(\xi)}{-|\xi|^2+1}d\xi$$

need to be estimated. We can make the following localization

Claim 1: There exists $C > 0$ such that for any $r, s > 0$

$$\frac{1}{r}\int_{B(0,r)}|K(g\chi_s)(x)|^2dx + \frac{1}{r}\int_{B(0,r)}||D|K(g\chi_s)(x)|^2dx \leq Cs\int|g(x)\chi_s|^2dx, \tag{5.11}$$

where

$$\Omega_s = B(0,2s)\setminus B(0,s),$$

and χ_s is a C^∞ function supported on Ω_s

Estimates (5.7) and (5.8) follow from the claim by splitting $g = \sum_{j\in\mathbb{Z}} g\chi_{2^j}$, for a partition of the unity, where $\chi_{2^j}(x) = \varphi(2^j|x|)$ for a basic function $\varphi(t)$ supported on $[1/2, 2]$ such that $\sum_{j\in\mathbb{Z}} \varphi(2^j t) = 1$ for $t \in (0, \infty)$ and $0 \le \varphi(t) \le 1$. In fact from Minkowsky inequality and the finite overlapping of the partition of unity,

$$\left(\frac{1}{r}\int_{B(0,r)} |K(g\sum_j \chi_{2^j})(x)|^2 dx\right)^{1/2} \le \sum_j \left(\frac{1}{r}\int_{B(0,r)} |K(g\chi_{2^j})(x)|^2 dx\right)^{1/2}$$

$$\le C\sum_j \left(2^j \int |g(x)\chi_{2^j}|^2 dx\right)^{1/2} \le 16C\sum_j \left(2^j \int_{\Omega_j} |g(x)\chi_{2^j}|^2 dx\right)^{1/2}.$$

To prove the claim, take $\psi \in C_0^\infty$ such that $\psi = 1$ on a $\delta/2$- neighborhood of S^{n-1} and supported on the δ- neighborhood of S^{n-1} for δ sufficiently small depending on the dimension and take

$$m(\xi) = p.v.\frac{1}{|\xi|^2 - 1} = p.v.\frac{\psi}{|\xi|^2 - 1} + \frac{1-\psi}{|\xi|^2 - 1} = m_1(\xi) + m_2(\xi) \qquad (5.12)$$

Let us denote the Fourier multiplier operator corresponding to m_i by $m_i(D)$, $i = 1, 2$. To estimate $m_1(D)$ we need another decomposition, since we can split for some $R < 1$

$$A = supp\psi \subset \cup_{j=1,\dots n}\{\xi : R < |\xi_j| < 1 + \delta\},$$

We chose δ small enough to assure that for $\xi \in supp\psi \cap \{\xi : R < |\xi_j| < 1 + \delta\}$ we have $|\xi|^2 - |\xi_j|^2 < 1 - c_n$ with some $c_n > 0$, we can write

$$m_1(\xi) = \sum_{j=\pm1,\dots,\pm n} m_1(\xi)\phi_j(\xi) = \sum_{j=\pm1,\dots\pm n} n_j(\xi),$$

where $\phi_j(\xi) = \phi(\xi_j)$, for a C_0^∞ function ϕ supported on $\{t \in \mathbb{R} : R/2 < t < 1 + 2\delta\}$ or in $\{t \in \mathbb{R} : -R/2 > t > -1 - 2\delta\}$ according to the sign of j. By rotating it suffices to estimate the Fourier multiplier operator $n_1(D)$ given by

$$n_1(\xi) = m_1(\xi)\phi(\xi_1).$$

Take $(x_1, x') = x \in \mathbb{R}^n$, $x_1 \in \mathbb{R}$, then

$$\frac{1}{r}\int_{B(0,r)} |n_1(D)(g\chi_s)(x)|^2 dx \le \frac{1}{r}\int_{|x_1|<r}\int_{\mathbb{R}^{n-1}} |n_1(D)(g\chi_s)(x_1, x')|^2 dx' dx_1.$$

Since

$$n_1(D)(g\chi_s)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} n_1(\xi)\widehat{(g\chi_s)}(\xi_1, \xi')e^{ix\cdot\xi} d\xi_1 d\xi' = (h_{x_1})\check{}(x'),$$

where

$$h_{x_1}(\xi') = \int_{\mathbb{R}} n_1(\xi_1, \xi')\widehat{(g\chi_s)}(\xi_1, \xi')e^{ix_1\xi_1} d\xi_1,$$

by Plancherel in the x' variable we can write

$$\frac{1}{r}\int_{B(0,r)}|n_1(D)(g\chi_s)(x)|^2dx \leq \frac{1}{r}\int_{|x_1|<r}\int_{\mathbb{R}^{n-1}}|h_{x_1}(\xi')|^2d\xi'dx_1 \tag{5.13}$$

Write

$$\begin{aligned}h_{x_1}(\xi') &= \int_{\mathbb{R}} n_1(\xi_1,\xi')\int_{\mathbb{R}}((g\chi_s)(y_1,(\cdot)'))\hat{}(\xi')e^{-i\xi_1 y_1}dy_1 e^{ix_1\xi_1}d\xi_1,\\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} n_1(\xi_1,\xi')e^{i(x_1-y_1)\xi_1}d\xi_1\right)((g\chi_s)(y_1,(\cdot)'))\hat{}(\xi')dy_1\\ &= \int_{\mathbb{R}} a(\xi',x_1,y_1)\,((g\chi_s)(y_1,(\cdot)'))\hat{}(\xi')dy_1,\end{aligned} \tag{5.14}$$

where we denote

$$a(\xi',x_1,y_1) = \int_{\mathbb{R}} n_1(\xi_1,\xi')e^{i(x_1-y_1)\xi_1}d\xi_1, \tag{5.15}$$

we have

Claim 2: Let $|x_1|<r$, $|y_1|<s$, then $|a(\xi',x_1,y_1)|<C$

Then by Claim 2 we have that (5.13) is bounded by

$$\frac{1}{r}\int_{|x_1|<r}\int_{\mathbb{R}^{n-1}}|\int_{\mathbb{R}} a(\xi',x_1,y_1)\,((g\chi_s)(y_1,(\cdot)'))\hat{}(\xi')dy_1|^2d\xi'dx_1 \tag{5.16}$$

$$\leq C^2\int_{\mathbb{R}^{n-1}}|\int_{\mathbb{R}}((g\chi_s)(y_1,(\cdot)'))\hat{}(\xi')dy_1|^2d\xi', \tag{5.17}$$

which by Cauchy-Schwartz can be bounded by

$$\leq C^2\int_{\mathbb{R}^{n-1}} s\int_{\mathbb{R}}|\,((g\chi_s)(y_1,(\cdot)'))\hat{}(\xi')|^2dy_1d\xi'. \tag{5.18}$$

By using Plancherel in ξ', we have

$$\frac{1}{r}\int_{B(0,r)}|n_1(D)(g\chi_s)(x)|^2dx \leq C^2 s\int_{\mathbb{R}^n}|(g\chi_s)(y_1,y')|^2dy. \tag{5.19}$$

The estimate for

$$\frac{1}{r}\int_{B(0,r)}||D|n_1(D)(g\chi_s)(x)|^2dx$$

can be obtained in the same way.

Proof of Claim 2: We can express, since on the support of n_1 we have $1-|\xi'|^2 > c_n > 0$,

$$n_1(\xi) = p.v.\frac{h(\xi)}{\xi_1-(1-|\xi'|)},$$

where $h(\xi) = \frac{\psi(\xi)\phi(\xi_1)}{\xi_1+(1-|\xi'|^2)^{1/2}}$ is a C_0^∞ function, then

$$a(\xi',x_1,y_1) = p.v.\int_{\mathbb{R}}\frac{h(\xi)}{\xi_1-(1-|\xi'|^2)^{1/2}}e^{i(x_1-y_1)\xi_1}d\xi_1$$

$$= \left(p.v. \frac{h((\cdot)_1, \xi')}{(\cdot)_1 - (1 - |\xi'|^2)^{1/2}} \right)\hat{}(y_1 - x_1)$$

$$= \left(p.v. \frac{h((1-|\xi'|^2)^{1/2}, \xi')}{(\cdot)_1 - (1 - |\xi'|^2)^{1/2}} \right)\hat{}(y_1 - x_1) + \int_{\mathbb{R}} \frac{h(\xi_1, \xi') - h((1-|\xi'|^2)^{1/2}, \xi')}{\xi_1 - (1-|\xi'|^2)^{1/2}} e^{i(x_1 - y_1)\xi_1} d\xi_1,$$

since the function

$$\left|\frac{h(\xi_1, \xi') - h((1-|\xi'|^2)^{1/2}, \xi')}{\xi_1 - (1-|\xi'|^2)^{1/2}}\right| \leq sup \left|\frac{\partial h(\xi_1, \xi')}{\partial \xi_1}\right|$$

and it is compactly supported, hence

$$\left| \int_{\mathbb{R}} \frac{h(\xi_1, \xi') - h((1-|\xi'|^2)^{1/2}, \xi')}{\xi_1 - (1-|\xi'|^2)^{1/2}} e^{i(x_1 - y_1)\xi_1} d\xi_1 \right| \leq D sup \left|\frac{\partial h(\xi', \xi_1)}{\partial \xi_1}\right|$$

where D is the diameter of the support of n_1. By using the Hilbert transform

$$\left(p.v. \frac{h((1-|\xi'|^2)^{1/2}, \xi')}{(\cdot)_1 - (1 - |\xi'|^2)^{1/2}} \right)\hat{}(y_1 - x_1) = h((1-|\xi'|^2)^{1/2}, \xi') e^{i(1-|\xi'|)(x_1 - y_1))} sign(x_1 - y_1),$$

this proves the claim.

To estimate $m_2(D)$, we can write

$$m_2(\xi) = \frac{1-\psi}{|\xi|^2 - 1} = \omega(\xi)(\lambda^2 + |\xi|^2)^{-1}, \tag{5.20}$$

and

$$|\xi| m_2(\xi) = \frac{|\xi|(1-\psi)}{|\xi|^2 - 1} = \omega(\xi)(\lambda^2 + |\xi|^2)^{-1/2},$$

where ω are bounded functions and in $C^\infty(\mathbb{R}^n \setminus \{0\})$ and $\lambda > 0$. Then, using Holder inequality with $1/p = 1/2 - 1/(2n)$, we have

$$\frac{1}{r} \int_{B(0,r)} ||D| m_2(D)(g\chi_{\Omega_s})(x)|^2 dx = \left(\int_{\mathbb{R}^n} (\lambda^2 + |D|^2)^{-1/2} \omega(D)(g\chi_{\Omega_s})(x)|^p dx \right)^{2/p}$$

From L^p- estimates of Bessel potential of order 1, this can be estimated by

$$C(\lambda) \left(\int_{\mathbb{R}^n} |\omega(D)(g\chi_{\Omega_s})(x)|^{p'} dx \right)^{2/p'},$$

where $1/p - 1/p' = 1/n$. Finally, by using Hörmander multiplier theorem, the Fourier multiplier operator $\omega(D)$ are $L^{p'}$-bounded which together with Hölder inequality give us the desired estimate

$$\leq Cs \int_{\mathbb{R}^n} |g\chi_{\Omega_s}(x)|^2 dx.$$

The operator given by (5.20) can by estimated in the same way but with the help of the Bessel potential of order 2.

5.2 Selfdual L^p estimates.

We start by proving the analogous of Corollary 4.7

Theorem 5.2. *Let $k > 0$ and $\frac{2}{n} \geq \frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$ and $n > 2$ or $n = 2$ and $1 > \frac{1}{p} - \frac{1}{q} \geq \frac{2}{3}$ where $\frac{1}{p} + \frac{1}{q} = 1$, then there exists a constant C independent of k and f such that*

$$\|R_+(k^2)f\|_{L^q} \leq Ck^{n(\frac{1}{p}-\frac{1}{q})-2}\|f\|_{L^p}$$

Remarks:

- The restriction $\frac{2}{n} \geq \frac{1}{p} - \frac{1}{q}$ needs to be added, since the Fourier multiplier $(-|\xi|^2 + k^2)^{-1}$ behaves as a Bessel potential of order 2 when $|\xi| \to \infty$.
- The whole range (p, q) follows by Stein interpolation theorem of analytic families, see [KRS]. We are going to use real interpolation and to prove only the range $\frac{2}{n} \geq \frac{1}{p} - \frac{1}{q} > \frac{2}{n+1}$.

The proof requires the following analogous of Lemma 4.2 on the mollification of p.v. distribution:

Lemma 5.2. *Let $\chi \in \mathcal{S}$, and*

$$R_\epsilon(\xi) = \frac{1}{-|(\cdot)|^2 + 1 + i0} * \epsilon^{-n}\chi(\cdot/\epsilon)(\xi),$$

then

$$|R_\epsilon(\xi)| \leq C\epsilon^{-1}$$

Proof: After lemma 4.2 it remains to prove the same estimate for

$$P_\epsilon(\xi) = pv\frac{1}{-|(\cdot)|^2 + 1} * \epsilon^{-n}\chi(\cdot/\epsilon)(\xi).$$

Let us denote $\chi_\epsilon = (\epsilon)^{-n}\chi((\cdot)/\epsilon)$, then

$$\begin{aligned} P_\epsilon(\xi) &= -pv\left(\int_{1-\epsilon\leq|\eta|\leq 1+\epsilon} + \int_{1-\epsilon>|\eta|} + \int_{|\eta|>1+\epsilon}\right)\chi_\epsilon(\xi-\eta)\frac{1}{|\eta|^2-1}d\eta \\ &= I_1 + I_2 + I_3. \end{aligned}$$

I_2 and I_3 can be easily bounded by $\epsilon^{-1}\|\chi\|_{L^1}$. Let us write

$$I_1 = \lim_{\delta\to 0}\int_{\delta\leq|1-|\eta||\leq\epsilon}\chi_\epsilon(\xi-\eta)\frac{1}{|\eta|^2-1}d\eta$$

$$= \lim_{\delta\to 0}(\int_{1-\epsilon}^{1-\delta} + \int_{1+\delta}^{1+\epsilon})\int_{S^{n-1}}\chi_\epsilon(\xi - r\theta)\frac{1}{r^2-1}r^{n-1}d\sigma(\theta),$$

Changing $r = 2 - s$ in the second integral we obtain

$$I_1 = \lim_{\delta\to 0}\int_{1-\epsilon}^{1-\delta}F(r,\xi)(r-1)^{-1}dr,$$

where

$$\begin{aligned}F(r,\xi) = &\int_{S^{n-1}} \chi_\epsilon(\xi - r\theta)\frac{r^{n-1}}{(r+1)}d\sigma(\theta)\\ &- \int_{S^{n-1}} \chi_\epsilon(\xi-(2-r)\theta)\frac{(2-r)^{n-1}}{(3-r)}d\sigma(\theta)\end{aligned} \tag{5.21}$$

If we observe that $F(1,\xi) = 0$, we may write by the mean value theorem

$$\int_{1-\epsilon}^{1-\delta} F(r,\xi)(r-1)^{-1}dr \le \epsilon \sup_{1-\epsilon\le r\le 1} |\frac{\partial F}{\partial r}(r,\xi)|.$$

The radial derivative of the first integral in the definition of F, (5.21), is given by

$$\frac{\partial}{\partial r}(\frac{r^{n-1}}{r+1})\int_{S^{n-1}} \chi_\epsilon(\xi - r\theta)d\sigma(\theta) + \frac{r^{n-1}}{r+1}\int_{S^{n-1}} \theta\cdot\bigtriangledown\chi_\epsilon(\xi-r\theta)d\sigma(\theta).$$

The second of these integrals can be written as

$$\epsilon^{-1}\sum_{i=1}^{n}\frac{r^{n-1}}{r+1}\int_{S^{n-1}}\theta_i(\frac{\partial}{\partial x_i}\chi)_\epsilon(\xi-r\theta)d\sigma(\theta),$$

hence both integrals can be understood as mollifications with resolution ϵ of the measures $\theta_i d\sigma(\theta)$, which, from lemma 4.2, are bounded by $C(\chi)\epsilon^{-1}$. This gives the desired estimate for the first integral in (5.21) and the second integral can be treated in the same way. We have

$$|\frac{\partial F}{\partial r}(r,\xi)| \le C\epsilon^{-2},$$

hence we obtain he desired estimate $|I_1(\xi)| \le C\epsilon^{-1}$

Proof of Theorem 5.2 : By homogeneity we can reduce to the case $k = 1$. The operator R_+ is convolution with the outgoing fundamental solution Φ. Take a partition of unity $\sum_{j=0}^{\infty}\phi_j(x) = 1$, such that $supp\phi_0 \subset B(0,1)$ and $supp\phi_j \subset \{|x| \in (2^{j-1}, 2^{j+1})\}$, where $\phi_j(x) = \phi(2^{-j}x)$ for a fixed function ϕ. Let us denote $\Phi_j = \phi_j\Phi$ and the operators $K_j f = \Phi_j * f$. From estimates of the Hankel functions $H^{(1)}_{(n-1)/2}$ at the origin we have

$$\Phi_0(x) \le |x|^{-(n-2)}\chi_{B(0,2)} \text{ if } n > 2 \tag{5.22}$$

$$\Phi_0(x) \le log|x|\chi_{B(0,2)} \text{ if } n = 2. \tag{5.23}$$

From estimates of the Bessel potential of order 2 these operators are bounded $L^p \to L^q$, for the ranges

$$\frac{2}{n} \ge \frac{1}{p} - \frac{1}{q} \ge 0 \text{ if } n > 2,$$

and

$$1 > \frac{1}{p} - \frac{1}{q} \ge 0 \text{ if } n = 2,$$

Let us now estimate K_j if $j > 1$.

From Lemma 5.2 we have on one hand

$$\|\widehat{\Phi_j}\|_\infty = \|(-|\cdot|^2 + 1 + i0)^{-1} * \hat{\phi}_j\|_\infty \le C2^j \tag{5.24}$$

hence

$$\|K_j\|_{L^2\to L^2} \le C2^j. \tag{5.25}$$

On the other hand, from estimates of the fundamental solutions, we have

$$|\Phi_j(x)| \le 2^{-j(n-1)/2}, \tag{5.26}$$

this gives

$$\|K_j\|_{L^1\to L^\infty} \le C2^{-j(n-1)/2}. \tag{5.27}$$

Interpolating estimates (5.25) and (5.27) we obtain the self dual estimate

$$\|K_j\|_{L^p\to L^q} \le C(2^j)^{2/q-(n-1)(1-2/q)/2}.$$

The sums of the norms is finite in the desired (p,q)-range.

5.3 Uniform Sobolev estimates

From the estimates of the resolvent given in the above section, we are going to obtain an a priori estimates for lower order perturbations of the laplacian, which include estimates for the Faddeev operator, the key operator in the inverse conductivity problem. Let us start with the zero order perturbations of the laplacian. We have the following

Theorem 5.3. *Let $z \in \mathbf{C}$, p and q in the range of theorem 5.2 and $u \in \mathcal{C}_0^\infty$ then there exists a constant C independent of z such that*

$$\|u\|_q \le C|z|^{(1/p-1/q)n/2-1}\|(\Delta+z)u\|_p \tag{5.28}$$

Let us remark that if, the two derivatives Sobolev gap, $1/p - 1/q = 2/n$ holds the estimate is uniform in z and then it can be understood as a uniform version of the Sobolev estimate for any zero order perturbation of the laplacian. Also observe that in two dimensions there is no such uniform estimate.
Proof: We use Phragmén-Lindelöf maximun principle:

Proposition 5.2. *Let $F(z)$ analytic in the open half complex plane $\{Imz > 0\} = \mathbf{C}_+$ and continuous in the closure. Assume that $|F(z)| \le L$ in $\partial\mathbf{C}_+$ and that for any $\epsilon > 0$ there exists C such that $|F(z)| \le Ce^{\epsilon|z|}$ as $|z| \to \infty$ uniformly on the argument of z. Then $|F(z)| \le L$ for any $z \in \mathbf{C}_+$.*

Let us assume that $\hat{u}$ and $\hat{v}$ are compactly supported functions on $\mathbf{R}^n$, and consider in $\mathbf{C}_+$ the analytic function

$$F(z) = z^{-(1/p-1/q)n/2+1}\int v(\Delta+z)^{-1}u$$

$$= z^{-(1/p-1/q)n/2+1}\int(-|\xi|^2+z)^{-1}\hat{v}(\xi)\hat{u}(\xi)d\xi,$$

where we use the principal determination of $\log z$. F is continuous on the closure of $\mathbf{C}_+$, (this follow from the convergence of the distribution $(-|\xi|^2+z)^{-1} \to (-|\xi|^2 + \Re z + i0)^{-1}$).

From the estimate for the outgoing resolvent $F(z)$ is bounded by $C\|u\|_p\|v\|_p$ in the part $\mathbb{R}_+$ of the boundary $\Im z = 0$. In $\mathbb{R}_-$ the same estimate holds from estimates of Bessel potentials of order 2. Finally $F(0) = 0$ except when $1/p - 1/q = 2/n$, but then the same bound follows from estimates of the Riesz potential of order two.

To bound F at infinity we consider $|z|$ sufficiently large so that for $\xi \in \operatorname{supp} \hat{u}$ we have

$$(-|\xi|^2 + |z|)^{-1} \leq \frac{2}{|z|},$$

and hence we obtain

$$|F(z)| \leq C|z|^{-(1/p-1/q)n/2} \int |\hat{v}(\xi)\hat{u}(\xi)| d\xi \leq Ce^{\epsilon|z|}.$$

From the maximun principle we have

$$\int (\Delta + z)^{-1} u(x)v(x)dx \leq C|z|^{(1/p-1/q)n/2-1}\|u\|_p\|v\|_p,$$

which, by density, proves the desired estimate.

Now we extend the theorem to first order perturbations. We start by making some simplifications using the group of invariance of the Laplace operator. Let us assume that we have the following a priori estimate for $a \in \mathbf{C}^n$ and $b \in \mathbf{C}$

$$\|u\|_q \leq C(a,b)\|(\Delta + a\cdot\bigtriangledown + b)u\|_p. \tag{5.29}$$

The change $u(x) = e^{ic\cdot x}v(x)$ for $c = -\Im a/2$ reduces this inequality to

$$\|v\|_q \leq C(a,b)\|(\Delta - i\Re a\cdot\bigtriangledown + \tilde{b})v\|_p, \tag{5.30}$$

where $\tilde{b} = b - i\Im a \cdot \Re a/2 + |\Im a|^2/4$. We may then reduce to prove (5.29) for the case $a \in \mathbf{R}^n$. By rotation invariance we may assume that $\Re a \cdot \bigtriangledown = |\Re a|\frac{\partial}{\partial x_1}$. Hence we may reduce to prove

$$\|v\|_q \leq C_1(\epsilon,\tilde{b})\|(\Delta + \epsilon\frac{\partial}{\partial x_1} + \tilde{b})v\|_p, \tag{5.31}$$

and we will take $C(a,b) = C_1(\epsilon,\tilde{b})$ for $\epsilon = |\Re a| \neq 0$ and $\tilde{b}$ as above. Finally we use the dilation behavior of the estimate depending on p and q. Take $v(x) = v(y/\lambda) = v_\lambda(y)$, then $\|v\|_q = \lambda^{-n/q}\|v_\lambda\|_q$. And

$$((\Delta_x + \epsilon\frac{\partial}{\partial x_1} + \tilde{b})v)_\lambda = \lambda^2(\Delta_y + \epsilon\lambda^{-1}\frac{\partial}{\partial y_1} + \lambda^{-2}\tilde{b})(v_\lambda).$$

Hence, assuming that $\Re\tilde{b} \neq 0$, the choice $\lambda = |\Re\tilde{b}|^{1/2}$ reduces the estimate to

$$\|v_\lambda\|_q \leq \lambda^{2+n(1/q-1/p)}C(a,b)\|(\Delta + \epsilon\lambda^{-1}\frac{\partial}{\partial y_1} \pm 1 + i\Im\tilde{b}\lambda^{-2}))v_\lambda\|_p. \tag{5.32}$$

The key point is that we can prove the above estimate with constant independent of all the parameters, namely:

Proposition 5.3. *There exists $C_2 > 0$ such that for any real numbers ϵ and β and any $u \in \mathcal{C}_0^\infty$ it holds*

$$\|u\|_q \leq C_2 \|(\Delta + \epsilon(\frac{\partial}{\partial y_1} + i\beta) \pm 1)u\|_p. \tag{5.33}$$

From this proposition and the above reductions we obtain by taking

$$\lambda^{2+n(1/q-1/p)} C(a,b) = C_2$$

with

$$\lambda = |\Re \tilde{b}|^{1/2} = |\Re b + |\Im a|^2/4|^{1/2}$$

Theorem 5.4. *([KRS]) Let $a \in \mathbf{C}^n$ and $b \in \mathbf{C}$ such that $\Re b + |\Im a|^2/4 \neq 0$, then for any $u \in \mathcal{C}_0^\infty$, there exists C independent of a and b such that*

$$\|u\|_q \leq C|\Re b + |\Im a|^2/4|^{(1/p-1/q)n/2-1} \|(\Delta + a \cdot \bigtriangledown + b)u\|_p. \tag{5.34}$$

Proof of Proposition 5.3 We may assume $\epsilon \neq 0$, otherwise the estimate was already proved. By taking Fourier transform, the claim is reduced to prove that the Fourier multiplier

$$(Tf)\check{}(\xi) = m(\xi)\hat{f}(\xi),$$

where

$$m(\xi) = (-|\xi|^2 \pm 1 + i\epsilon(\xi_1 + \beta))^{-1}, \tag{5.35}$$

is bounded from L^p to L^q. We observe that the multiplier is a locally integrable function. Let us concentrate in the case $m(\xi) = (-|\xi|^2 + 1 + i\epsilon(\xi_1 + \beta))^{-1}$, being the other easier. We are going to use one dimensional Littlewood-Paley theory: consider $\chi = \chi_{[1.2]}$,

$$\chi_k(\xi_1) = \chi(2^k(\xi_1 + \beta)),$$

and $m_k(\xi) = m(\xi)\chi_k(\xi_1)$.If we denote $m_k(D)$ the corresponding Fourier multiplier operator, it will suffice if we prove the following estimate with C independent of ϵ, β and k:

$$\|(m_k(D)f)\|_q \leq C\|f\|_p. \tag{5.36}$$

In fact we have, by Littlewood-Paley

$$\|Tf\|_q \leq \| \left(\sum_{-\infty}^{+\infty} |m_k(D)f|^2 \right)^{1/2} \|_q = \| \sum_{-\infty}^{+\infty} |m_k(D)f|^2 \|_{q/2}^{1/2},$$

Since $q/2 > 1$, we may write from Minkowski inequality

$$\leq \left(\sum_{-\infty}^{+\infty} \| |m_k(D)f|^2 \|_{q/2} \right)^{1/2} = \left(\sum_{-\infty}^{+\infty} \|m_k(D)f\|_q^2 \right)^{1/2}$$

$$\leq C \left(\sum_{-\infty}^{+\infty} \|\chi_k(D)f\|_p^2 \right)^{1/2} = C \left(\sum_{-\infty}^{+\infty} (\int |\chi_k(D)f|^p dx)^{2/p} \right)^{1/2}$$

$$= C \left(\| \{ \int |\chi_k(D) f(x)|^p dx \} \|_{l^{2/p}} \right)^{1/p},$$

where $\|\{a_k\}\|_{l^{2/p}}$ denotes the $l^{p/2}$-norm of the sequence $\{a_k\}$. Since $2/p > 1$, by Minkowski this can be bounded by

$$C \left(\int \| \{ |\chi_k(D) f(x)|^p \} \|_{l^{2/p}} dx \right)^{1/p} = C \left(\int (\sum_{-\infty}^{+\infty} |\chi_k(D) f(x)|^2)^{p/2} dx \right)^{1/p}$$

$$= C \| \left(\sum_{-\infty}^{+\infty} |\chi_k(D) f|^2 \right)^{1/2} \|_p,$$

again Littlewood-Paley gives

$$\leq C \|f\|_p.$$

It remains to prove (5.36). We freeze the variable ξ_1 at the value 2^{-k} and substitute m_k by

$$\tilde{m}_k = \frac{\chi_k(\xi_1)}{-|\xi|^2 + 1 + i\epsilon 2^{-k}}.$$

We can apply the estimates for the resolvent (5.28) with $z = 1 + i\epsilon 2^{-k}$, the Hilbert transform and write

$$\|\tilde{m}_k(D)\|_q \leq C \|\chi_k(D) f\|_p \leq C \|f\|_p.$$

It remains to prove the boundedness of the Fourier multiplier operator T_k given by the difference

$$m_k - \tilde{m}_k = \frac{\chi_k(\xi_1)(i\epsilon(\xi_1 + \beta - 2^{-k})}{(-|\xi|^2 + 1 + i\epsilon 2^{-k})(-|\xi|^2 + 1 + i\epsilon(\xi_1 + \beta))}.$$

Using polar coordinates $\xi = \rho\omega$ and Minkowski inequality

$$\|T_k f\|_q$$

$$= \| \int_0^\infty \int_{S_\rho^{n-1}} \frac{\chi_k(\xi_1) i\epsilon(\xi_1 + \beta - 2^{-k}) \hat{f}(\xi)}{(-\rho^2 + 1 + i\epsilon 2^{-k})(-\rho^2 + 1 + i\epsilon(\xi_1 + \beta))} e^{ix\cdot\xi} d\sigma_\rho(\xi) d\rho \|_q$$

$$\leq \int_0^\infty \| \int_{S_\rho^{n-1}} \frac{\chi_k(\xi_1) i\epsilon(\xi_1 + \beta - 2^{-k}) \hat{f}(\xi)}{(-\rho^2 + 1 + i\epsilon 2^{-k})(-\rho^2 + 1 + i\epsilon(\xi_1 + \beta))} e^{ix\cdot\xi} d\sigma_\rho(\xi) \|_q d\rho,$$

from Corollary 4.7, me may write

$$\leq \int_0^\infty \| \left(\frac{\chi_k(D_1) i\epsilon(D_1 + \beta - 2^{-k})}{(-\rho^2 + 1 + i\epsilon 2^{-k})(-\rho^2 + 1 + i\epsilon(D_1 + \beta))} f \right) \|_p \rho^{-1+n(1/p-1/q)} d\rho. \quad (5.37)$$

Let us remark that the Fourier multiplier operator given by the function

$$n_k(\xi_1) = \frac{\chi_k(\xi_1) i\epsilon(\xi_1 + \beta - 2^{-k})}{(-\rho^2 + 1 + i\epsilon 2^{-k})(-\rho^2 + 1 + i\epsilon(\xi_1 + \beta))}$$

is acting only in the variable ξ_1, in order to prove its L^p-boundedness we need to check (see [D], page 64) that its multiplier function defines a measure dn_k of bounded variation. But the total variation is given by

$$\int |dn_k| \leq \frac{\epsilon 2^{-k}}{(\rho^2-1)^2+(\epsilon 2^{-k})^2},$$

hence

$$\|T_k f\|_q \leq \int_0^\infty \rho^{n-1-2n/q} \frac{\epsilon 2^{-k}}{(\rho^2-1)^2+(\epsilon 2^{-k})^2} d\rho \|f\|_p.$$

The last integral is uniformly bounded with respect to all the parameters on the desired range $2/(n+1) \leq 1/p - 1/q = (1-2/q) \leq 2/n$, which makes the exponent of ρ between 0 and 1.

As a special case of the above theorem we obtain mapping properties of the Faddeev operator, which is used in the inverse conductivity problem, see (2.2),

Corollary 5.3. *Let $\rho \in \mathbf{C}^n$ such that $\rho \cdot \rho = 0$. Assume that $\frac{2}{n} \geq \frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$ if $n > 2$ and $1 > \frac{1}{p} - \frac{1}{q} \geq \frac{2}{3}$ if $n = 2$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a constant C independent of ρ and f such that*

$$\|f\|_{L^q} \leq C|\rho|^{n(\frac{1}{p}-\frac{1}{q})-2} \|(\Delta + \rho \cdot \bigtriangledown) f\|_{L^p} \tag{5.38}$$

Just remark that the condition $\rho \cdot \rho = 0$ reads $|\Re\rho| = |\Im\rho|$, $\Re\rho \cdot \Im\rho = 0$.

5.4 Non selfdual estimates:

The estimates in the previous section can be extended to a range of (p,q) out the duality line $\frac{1}{p} + \frac{1}{q} = 1$, see [KRS] for the parameters independent case. Nevertheless these extensions are very close to such duality line. In this section we are going to prove an estimate for the resolvent which is halfway between the analogous of theorem 4.6 for the resolvent and the selfdual estimate (5.2). From the point of view of applications to inverse problems this is an inprovement, since, for big k's, the exponent of k is better than in the mentioned extensions.

Theorem 5.5 ([RuV]). *Let p be such that $\frac{1}{n+1} \leq \frac{1}{p} - \frac{1}{2} \leq \frac{1}{n}$. Then there exists $C > 0$ independent of k such that*

$$\sup_{R,x_0} \left(\frac{1}{R} \int_{B(x_0,R)} |R_+(k^2)(f)(x)|^2 dx \right)^{1/2} \leq C k^{-3/2+n(\frac{1}{p}-\frac{1}{2})} \|f\|_{L^p(\mathbf{R}^n)}. \tag{5.39}$$

Proof: Since the estimate is invariant by translations, dilations and rotations, we may assume, without loss of generality, that $k = 1$ and $x_0 = 0$. From formula (5.6) we have

$$\begin{aligned} R_+(1)(f)(x) &= \int e^{ix\cdot\xi} pv \frac{\hat{f}(\xi)}{-|\xi|^2+1} d\xi + \frac{i\pi}{2} \widehat{d\sigma} * f(x) \\ &= R_+(f)(x) + R_0(f)(x) \end{aligned} \tag{5.40}$$

The last term $R_0(f)(x)$ can be written, as in formula (4.21) in the previous chapter,

$$R_0(f)(x) = \frac{i\pi}{2}\widehat{d\sigma} * f(x) = T^*T(f).$$

For this operator the estimate follows from Corollary 4.5 and Corollary 4.1 in the previous chapter.
Hence only the principal value part of $R_+(f)(x)$ remains to be bounded.
Take a cutoff function $\phi \in \mathcal{C}_0^\infty([-1/2, 1/2])$, $\phi(t) = 1$ for $t \in [-1/4, 1/4]$ and define f_i as

$$\hat{f}_1(\xi) = \phi(|\xi|)\hat{f}(\xi),$$
$$\hat{f}_2(\xi) = (1 - \phi(|\xi|/8))\hat{f}(\xi),$$
$$\hat{f}_3(\xi) = \hat{f} - \hat{f}_1 - \hat{f}_2.$$

Let us write $R_+(1)(f)(x) = \sum_{i=1}^3 R_+(f_i)$. We will prove the estimate for each R_j.

It is easy to see that for $i = 1, 2$, $R_+(f_i)$ is pointwise majorized by $J^2(f_i)$, the Bessel potential of order two. From classical estimates for Bessel potentials, we have

$$\left(\frac{1}{R}\int_{B(0,R)} |R_+(f_i)(x)|^2 dx\right)^{1/2} =$$

$$\left(\frac{2}{R}\int_0^\infty |\{x \in B(0, R) : |R_+(f_i)(x)| > \lambda\}|\lambda d\lambda\right)^{1/2}$$
$$\leq \left(\frac{2}{R}\int_0^\infty |\{x \in B(0, R) : |J^2(f_i)(x)| > c\lambda\}\|\lambda d\lambda\right)^{1/2}$$
$$\leq \left(2\frac{1}{R}\int_0^\infty \min(R^n, \frac{c\|f_i\|_p^q}{\lambda^q})\lambda d\lambda\right)^{1/2},$$

where $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}$. The choice $1/2 = 1/q + 1/(2n)$ makes the above bounded by $C\|f_i\|_p$.

Consider now $R_+(f_3)$. By a partition of unity and rotation invariance we may assume

$$supp\hat{f} \subset \{\xi = (\xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{n-1} : |\xi'| \leq |\xi_1|\}.$$

Notice that in this case $R_+(f_3)$ can be written as

$$R_+(f_3) = \int_{\mathbf{R}^{n-1}} e^{ix'\cdot\xi'} \int_{\mathbf{R}} e^{ix_1\xi_1}\phi(\xi_1) pv\frac{1}{-|\xi|^2 + 1}\hat{f}_3(\xi)d\xi_1 d\xi',$$

where ϕ is a function in $\mathcal{C}_0^\infty$ supported on $[-4, -1/2n] \cup [1/2n, 4]$. Taking the Fourier transform and using Plancherel identity with respect to the x'-variable, we have

$$\frac{1}{R}\int_{B(0,R)} |R_+(f_3)(x)|^2 dx \leq$$

$$\frac{1}{R}\int_{-R}^R \int_{\mathbf{R}^{n-1}} |\int_{\mathbf{R}} e^{ix_1\xi_1}\phi(\xi_1) pv\frac{1}{-|\xi|^2 + 1}\hat{f}_3(\xi)d\xi_1|^2 d\xi' dx_1.$$

Define

$$T_{x_1} f(\xi') = \int_{\mathbf{R}} e^{ix_1\xi_1} \phi(\xi_1) pv \frac{1}{-|\xi|^2 + 1} \hat{f}(\xi) d\xi_1. \tag{5.41}$$

It will be sufficient to prove that

$$\|T_{x_1} f(\xi')\|_{L^2(\mathbf{R}^{n-1})} \leq C \|f\|_{L_p(\mathbf{R}^n)}, \tag{5.42}$$

with C independent of x_1. Write

$$\begin{aligned} &\int_{\mathbf{R}^{n-1}} |T_{x_1} f(\xi')|^2 d\xi' = \int_{\mathbf{R}^{n-1}} T_{x_1} f(\xi') \bar{T}_{x_1} f(\xi') d\xi' \\ &= \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}} f(y_1, y') \left(\int_{\mathbf{R}} \left(\bar{f}(z_1, \cdot) * \bar{K}_{x_1}(z_1, \cdot) * K_{x_1}(y_1, \cdot) \right) (y') dz_1 \right) dy_1 dy', \end{aligned} \tag{5.43}$$

Where

$$\hat{K}(\xi) = \phi(\xi_1) pv \frac{1}{-|\xi|^2 + 1} \tag{5.44}$$

and $K_{x_1}(s, x') = K(s - x_1, x')$.
Define for x_1, y_1 and z_1 fixed the following operator, $y' \in \mathbf{R}^{n-1}$,

$$(S_{x_1}(y_1, z_1) g)(y') = (g(\cdot) * \bar{K}_{x_1}(z_1, \cdot) * K_{x_1}(y_1, \cdot))(y').$$

Claim. $S_{x_1}(y_1, z_1)$ is bounded from $L^p(\mathbf{R}^{n-1})$ to $L^{p'}(\mathbf{R}^{n-1})$ with operator norm bounded by

$$C(1 + |y_1 - z_1|)^{-(n-1)(1/p - 1/2)}$$

with C independent of x_1, y_1 and z_1, and $1 \leq p \leq 2$.
Assume that the claim is proved, then by Hölder inequality

$$\int_{\mathbf{R}^{n-1}} |T_{x_1} f(\xi')|^2 d\xi' \leq \|f\|_{L_p(\mathbf{R}^n)} \| \int_{\mathbf{R}} S_{x_1}(\cdot, z_1) \bar{f}(z_1, \cdot) dz_1 \|_{L_{p'}(\mathbf{R}^n)}.$$

Then Minkowski integral inequality and the claim give

$$\begin{aligned} &\| \int_{\mathbf{R}} S_{x_1}(\cdot, z_1) \bar{f}(z_1, \cdot) dz_1 \|_{L_{p'}(\mathbf{R}^n)} \leq \\ &\left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} \|S_{x_1}(\cdot, z_1) \bar{f}(z_1, \cdot)\|_{L_{p'}(\mathbf{R}^n)} dz_1 \right)^{p'} dy_1 \right)^{1/p'} \leq \\ &C \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} (1 + |y_1 - z_1|)^{-(n-1)(1/p - 1/2)} \|f(z_1, \cdot)\|_{L_p(\mathbf{R}^{n-1})} dz_1 \right)^{p'} dy_1 \right)^{1/p'} \\ &\leq C \|f\|_{L_p(\mathbf{R}^n)} \end{aligned}$$

with $1 - (n-1)(1/p - 1/2) \leq 1/p - 1/p'$, where the last inequality is a consequence of the fractional integration theorem.

To prove the claim we shall use the Riesz-Thorin interpolation theorem. On one hand, we have

$$\|S_{x_1}(y_1, z_1)\|_{L^2 \to L^2} \leq \sup_{\xi'} |K_{x_1}(z_1, \cdot)\check{}(\xi')| |K_{x_1}(y_1, \cdot)\check{}(\xi')|.$$

From (5.44) and writing $\eta_1 = \xi_1 - (1-|\xi'|^2)^{1/2}$,

$$
\begin{aligned}
K_{x_1}(s,\cdot)\hat{\ }(\xi') &= \int e^{i\xi_1(s-x_1)}\phi(\xi_1)pv\frac{1}{-|\xi|^2+1}d\xi_1 \\
&= e^{i(s-x_1)(1-|\xi'|^2)^{1/2}}\int e^{-\eta_1 x_1}e^{i\eta_1 s}\tilde{\phi}(\eta_1,\xi')pv\frac{1}{\eta_1}d\eta_1
\end{aligned}
\tag{5.45}
$$

where

$$
\tilde{\phi}(\eta_1,\xi') = \frac{\phi(\eta_1+(1-|\xi'|^2)^{1/2})}{\eta_1+2(1-|\xi'|^2)^{1/2}}.
$$

From the support properties of ϕ this function is in the Schwartz class $\mathcal{S}(\mathbf{R})$ uniformly in ξ' hence

$$
|K_{x_1}(s,\cdot)\hat{\ }(\xi')| \le C((pv\frac{1}{(\cdot)})\hat{\ }*(\phi((\cdot),\xi')\hat{\ })(x_1-s) \le C\|(\phi(\eta_1,\xi')\hat{\ })(\cdot)\|_{L^1(\mathbf{R})} \le C
$$

uniformly in x_1, s and ξ'. Hence

$$
\|S_{x_1}(y_1,z_1)\|_{L^2\to L^2} \le C \tag{5.46}
$$

On the other hand

$$
\|S_{x_1}(y_1,z_1)\|_{L^1\to L^\infty} \le \sup_{y'}|(\bar{K}_{x_1}(z_1,\cdot)*K_{x_1}(y_1,\cdot))(y')|.
$$

But from (5.45) we have

$$
|(\bar{K}_{x_1}(z_1,\cdot)*K_{x_1}(y_1,\cdot))(y')| = |\int e^{i(z_1-y_1)(1-|\xi'|^2)^{1/2}}e^{i\xi'\cdot y'}\gamma(z_1,\xi')\gamma(y_1,\xi')d\xi'|
$$

where

$$
\gamma(s,\xi') = \int e^{i\eta_1(s-x_1)}\tilde{\phi}(\eta_1,\xi')pv\frac{1}{\eta_1}d\eta_1.
$$

Notice that $1-|\xi'| > C > 0$ if $\xi' \in supp\gamma(t,\cdot)$ and denote $\Phi(\xi') = (z_1-y_1)(1-|\xi'|^2)^{1/2}$. Since $\det(\partial^2_{ij}\Phi) \ge C|z_1-y_1|^{n-1}$, the stationary phase lemma allows one to write

$$
\|S_{x_1}(y_1,z_1)\|_{L^1\to L^\infty} \le C(1+|z_1-y_1|)^{-(n-1)/2} \tag{5.47}
$$

The Riesz-Thorin interpolation theorem together with (5.46) and (5.47) prove the claim and the theorem.

5.5 Some extensions

In this section we state some estimates for the solution of Helmholtz equation which are extensions of the previously stated.

A local version without weights of Theorem 5.1 can be proved by using the scheme of theorem 5.2.

There is also a version of corollary 5.2 which we will need in the applications. It allows us to go up to $|\alpha| = 2$, but for derivatives of order greater than one the constant is local in $k \in \mathbf{R}_+$.

Theorem 5.6 ([A])**.** *Let $k \in K$ a compact in $\mathbf{R}_+$. Then there exists a constant only depending on K, such that*

$$\|R_+(k^2)f\|_{W^{2,2}(w(x)dx)} \leq C\|f\|_{L^2(w(x)^{-1}dx)} \tag{5.48}$$

Where

$$f \in W^{2,2}(w(x)dx) \text{ if and only if } D^\alpha f \in L^2(w(x)dx) \text{ for any } |\alpha| \leq 2 \tag{5.49}$$

One can extend the results in section 5.1 in a weighted version more appropriate to treat Schrödinger equations.
Let us start with some definitions. We denote by $\mathcal{X}_r$ the class of non negative weight functions such that

$$V = V(|x|) \text{ is a radial function,}$$

and

$$\|V\|_X := \sup_{\mu \in \mathbf{R}} \int_\mu^\infty \frac{V(r)r}{(r^2 - \mu^2)^{1/2}} dr < \infty. \tag{5.50}$$

Theorem 5.7 ([BRV])**.** *Let W_1 and W_2 in $\mathcal{X}_r$, then there exists a constants $C > 0$ such that*

$$\int_{\mathbf{R}^n} |R_+(k^2)(x)|^2 W_1(x)dx \leq \frac{C}{k^2}\|W_1\|_X\|W_2\|_X \int_{\mathbf{R}^n} |f(x)|^2 W_2(x)^{-1}dx. \tag{5.51}$$

$$\int_{\mathbf{R}^n} |\nabla R_+(k^2)(x)|^2 W_1(x)dx \leq C\|W_1\|_X\|W_2\|_X \int_{\mathbf{R}^n} |f(x)|^2 (W_2(x))^{-1}dx. \tag{5.52}$$

To see that theorem 5.1 and the corollaries can be obtained from theorem 5.7 take

$$f = \sum_{-\infty}^{+\infty} f\chi_{\Omega_j} := \sum_{-\infty}^{+\infty} f_j,$$

$W_1 = R^{-1}\chi_{B(0,R)}$ and $W_2 = {R_j}^{-1}\chi_{\Omega_j} + (1 + |x|^2)(1 - \chi_{\Omega_j})$. Then we have for any $R > 0$

$$\frac{1}{R}\int_{B(0,R)} |R_+(k^2)f_j(x)|^2 dx$$
$$\leq \frac{C}{k^2}\|W_1\|_X\|W_2\|_X \int_{\mathbf{R}^n} |f_j(x)|^2 |W_2(x)|^{-1}dx \leq Ck^{-2}R_j \int_{\Omega_j} |f(x)|^2 dx.$$

Hence, by taking square root, adding up and using Minkowski inequality

$$\left(\frac{1}{R}\int_{B(0,R)} |R_+(k^2)f(x)|^2 dx\right)^{1/2} = \left(\frac{1}{R}\int_{B(0,R)} |\sum R_+(k^2)f_j(x)|^2 dx\right)^{1/2}$$

$$\leq \sum \left(\frac{1}{R}\int_{B(0,R)} |R_+(k^2)f_j(x)|^2 dx\right)^{1/2} \leq Ck^{-1}\sum\left(R_j\int_{\Omega_j} |f(x)|^2 dx\right)^{1/2}.$$

We have also a version with derivatives of theorems 5.2 and 5.5.

Theorem 5.8 ([RuV]). *Let $k > 0$ and $\frac{2}{n} \geq \frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$ and $n > 2$ or $n = 2$ and $1 > \frac{1}{p} - \frac{1}{q} \geq \frac{2}{3}$ where $\frac{1}{p} + \frac{1}{q} = 1$, let $\alpha = 2 - n(\frac{1}{p} - \frac{1}{q})$. Then there exists a constant C independent of k and f such that*

$$\|D^\alpha R_+(k^2)f\|_{L^q} \leq C\|f\|_{L^p} \tag{5.53}$$

Proof: By homogeneity we may reduce to the case $k = 1$. Let us split $f = f_1 + f_2$, where $\hat{f}_1 = \phi\hat{f}$ and $\hat{f}_2 = 1 - \hat{f}_1$, and $\phi \in \mathcal{C}_0^\infty$ is supported in $\{|\xi| \leq 4\}$.

Then from theorem 5.2

$$\|D^\alpha R_+(k^2)f_1\|_{L^q} \leq C\|Mf\|_p,$$

where $Mf = \widehat{\phi(\xi)|\xi|^\alpha} * f$, which is a convolution with an L^1 kernel and hence bounded in L^p.

Now since $D^\alpha R_+(k^2)f_2(x) \leq \int f(y)|x-y|^{\alpha-2-n}dy$, we may use the mapping properties of the Riesz potential of order $2-\alpha$ to estimate this part.

In a similar fashion we can prove

Theorem 5.9 ([RuV]). *Let p such that $\frac{1}{n+1} \leq \frac{1}{p} - \frac{1}{2} \leq \frac{1}{n}$ and $0 \leq \alpha \leq 3/2 - n(1/p - 1/2)$. Then there exists $C > 0$ independent of k such that*

$$\sup_{R,x_0}\left(\frac{1}{R}\int_{B(x_0,R)}|D^\alpha R_+(k^2)(f)(x)|^2dx\right)^{1/2} \leq Ck^{\alpha-3/2+n(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^p}. \tag{5.54}$$

- *Remark 1:* In the estimates with derivatives one sees the hyperbolic character of these estimates: The dependence on k gives them that character. All the estimates involving more than one derivative have a bad dependence on k, the idea is that from the wave equation one never can gain more than one derivative.

- *Remark 2:* Condition (5.50) is a special case of the boundedness of the X-rays transform of V if we take off the assumption of radiality:

$$|||V|||_X = \sup_{x\in R^n, \omega\in S^{n-1}}\int_0^\infty V(x - t\omega)dt < \infty \tag{5.55}$$

  An open question is if theorem 5.7 is true for the class of non negative weights V with the above condition. It has been proved this condition to be necessary, see [BRV]. The same estimate for the imaginary part of the resolvent is also an open question and is related to some conjecture in harmonic analysis, see [CaS].

Extensions for Morrey-Campanato weights (to be completed) Exercise. Complete the proof of the theorem en 2D

Chapter 6

Applications

6.1 Unique continuation

The first consequence of the estimates in the previous chapter is a weak unique continuation principle for solution of the Schrödinger equation with rough potential V. The theorem, actually a stronger version of this principle, is due to Jerison and Kenig; they proved that any solution of the inequality

$$|\Delta u(x)| \leq |V(x)u(x)| \tag{6.1}$$

which vanishes up to infinity order at a point must be identically zero . We prove:

Theorem 6.1. *Let u a function in the Sobolev space $W^{2,p}_{loc}$, for p such that $2/p-1 = 1/r$ satisfying (6.1) in a domain Ω with $V \in L^r_{loc}$, where $r = n/2$ for $n \geq 3$ and $r > 1$ if $n = 2$. Then if u vanishes in an open subdomain of Ω it must vanish identically in Ω.*

Remark 1 : The Theorem can be proved for V in the weak Lorentz space $L^{n/2,\infty}$ with sufficiently small norm. This result has been proved to be sharp [KN].

Remark 2: There are also results for differential inequalities

$$|\Delta u(x)| \leq |V(x)u(x)| + |W(x) \cdot \nabla u(x)|$$

The best result is due to T. Wolff [W]

The proof of this theorem is based on the following Carleman estimate.

Proposition 6.1. *Let $\rho \in \mathbf{R}$ and $v \in \mathbf{S}^{n-1}$, then there exists a $C > 0$ independent of ρ and v, such that for any $u \in \mathcal{C}_0^\infty$ if with p as above and q its dual exponent*

$$\|e^{\rho v \cdot x} u\|_q \leq C|\rho|^{(1/p-1/q)n-2} \|e^{\rho v \cdot x} \Delta u\|_p \tag{6.2}$$

Proof: Take $u = e^{-\rho v \cdot x}\tilde{u}$, then the estimate reduces to prove

$$\|\tilde{u}\|_q \leq C|\rho|^{(1/p-1/q)n-2} \|(\Delta + 2\rho v \cdot \nabla + \rho^2)\tilde{u}\|_p.$$

which follows from theorem 5.4

Proof of theorem 6.1: We may reduce to the following

Claim: Suppose $u \in W^{2,p}_{loc}$ satisfies $|\Delta u(x)| \leq |V(x)u(x)|$ in a neighborhood of $\mathbf{S}^{n-1}$, where $V \in L^r_{loc}$. Then if u vanishes on one side of the sphere $\mathbf{S}^{n-1}$ it vanishes in a neighborhood of the sphere.

To obtain the theorem from the claim, consider the invariance by dilations and rotations of the statements and asume that $x_0 = 0$ is a point in the open set where u vanishes. Assume $d = dist(x_0, suppu \cap \Omega) < \infty$; by rescaling we may assume that $d = 1$, but from the claim u must vanish in a bigger ball, hence $d = \infty$.

To prove the claim let us assume first that $u = 0$ on the outside of $B(0,1)$. By dilation and rotation invariance assume that $u = 0$ outside of $B = B(-e_n, 1)$ where e_n is the n-th vector in the canonical basis of $\mathbf{R}^n$. It will be enough to prove that $u = 0$ in a neighborhood of the origin. Take $\eta \in \mathcal{C}_0^\infty([-2\delta, 2\delta])$, $\eta = 1$ on $[-\delta, \delta]$ such that

$$\|V\|_{L^r(A_1)} \leq \epsilon. \tag{6.3}$$

Then, from the Carleman estimate with $v = e_n$, denoting $\eta = \eta(x_n)$, $A_1 = B \cap \{x_n \geq -2\delta\}$ and $A_2 = B \cap \{-2\delta \leq x_n \leq -\delta\}$, we have

$$\|e^{\rho x_n}\eta u\|_{L^q} \leq C|\rho|^{(1/p-1/q)n-2}\|e^{\rho x_n}\Delta(\eta u)\|_{L^p}$$

$$\leq C|\rho|^{(1/p-1/q)n-2}(\|e^{\rho x_n}u\Delta\eta\|_{L^p(A_2)} + \|e^{\rho x_n} \bigtriangledown \eta \cdot \bigtriangledown u\|_{L^p(A_2)} + \|e^{\rho x_n}\eta\Delta u\|_{L^p(A_1)})$$

$$\leq C|\rho|^{(1/p-1/q)n-2}(e^{-\rho\delta}\|u\|_{W^{1,p}(A_1)} + +C\|e^{\rho x_n}\eta V u\|_{L^p(A_1)}),$$

From Hölder

$$\leq C|\rho|^{(1/p-1/q)n-2}e^{-\rho\delta}\|u\|_{W^{1,p}(A_1)} + C\epsilon|\rho|^{(1/p-1/q)n-2}\|e^{\rho x_n}\eta u\|_{L^q(A_1)}$$

take $C\epsilon < 1/2$ (actually condition (6.3) is not needed in dimension $n = 2$, it is enough to take ρ sufficiently large), then

$$\|e^{\rho x_n}\eta u\|_{L^q} \leq 2C|\rho|^{(1/p-1/q)n-2}e^{-\rho\delta}\|u\|_{W^{1,p}(A_1)}.$$

If we restrict to $A_3 == B \cap \{x_n \geq -\delta/2\}$ we have

$$e^{-\rho\delta/2}\|u\|_{L^q(A_3)} \leq 2C|\rho|^{(1/p-1/q)n-2}e^{-\rho\delta}\|u\|_{W^{1,p}(A_1)}.$$

Hence by taking $\rho \to \infty$ we prove that

$$\|u\|_{L^q(A_3)} = 0.$$

Now let us assume that $u = 0$ in the interior of $B(0,1)$. By using the Kelvin transform we may reduce to the above case. In fact take

$$u_1(x) = u(x/|x|^2)|x|^{-(n-2)}.$$

Now notice that if u satisfies $\Delta u = W$, then $\Delta u_1 = W(x/|x|^2)|x|^{-n}$. Since $|W(x)| \leq |u(x)V(x)|$ we have

$$|\Delta u_1(x)| \leq |u(x/|x|^2)V(x/|x|^2)||x|^{-n} = |V_1(x)u_1(x)|,$$

where $V_1(x) = |x|^{-2}V(x/|x|^2)$. It follows that u_1 satisfies the conditions of the claim and then vanishes in the exterior of $B(0,1)$, hence u_1 vanishes in a neighborhood of $\mathbf{S}^{n-1}$. The same is true, then for u.

6.2 The inverse boundary value problem for the Schrödinger equation

Let Ω be a bounded domain, which we assume smooth, we consider the Schrödinger hamiltonian $\Delta+V$ with the electrostatic potential q, which we assume in $L^r(\Omega)$. We try to recover q from boundary measurements. These measurements are given by the so called Dirichlet to Neumann map (D-N map). For a given boundary Dirichlet datum $f \in W^{1/2,2}(\partial\Omega)$ its image by the D-N map is defined as

$$\Lambda_q(f) = \frac{\partial}{\partial\nu}u \tag{6.4}$$

where u is the solution of the problem

$$\begin{cases} (\Delta+q)u = 0 \ \text{ in } \Omega \\ u_{|\partial\Omega} = f. \end{cases} \tag{6.5}$$

In quantum mechanic the natural inverse problem, proposed by Fadeev, [F], is to recover the potential q from far field or scattering measurements. Actually it has been proved that the far field measurements, corresponding to a fix energy, are equivalent to the D-N map boundary measurements for a sufficiently large sphere, [].

Since 0 could be an eigenvalue of the Dirichlet operator, the uniqueness of the above problem can not in general be proved and hence the D-N not need to be a map. In order to avoid this constrain, we are going to substitute the D-N map by the so called "Cauchy data set" of q which is defined by

$$\mathcal{C}_q = \{(u_{|\partial\Omega}, \frac{\partial}{\partial\nu}u) : u \in W^{1,2}(\bar\Omega), (\Delta+q)u = 0\} \tag{6.6}$$

Remark that in the case that the D-N map exists, $\mathcal{C}_q$ is a graph. In general we may claim

Proposition 6.2. *Let $q \in L^r$, with $r \geq n/2$, then*

$$\mathcal{C}_q \subset W^{1/2,2}(\partial\Omega) \times W^{-1/2,2}(\partial\Omega) \tag{6.7}$$

Proof: the problem is that, a priori, with the only assumption $u \in W^{1,2}(\bar\Omega)$, we can not claim that the trace of ∇u is in $W^{-1/2,2}(\partial\Omega)$, but, by using the equation, we are going to prove that $\frac{\partial}{\partial\nu}u \in W^{-1/2,2}(\partial\Omega)$.

Assume that u is a weak $W^{1,2}$ solution of $(\Delta+q)u = 0$. This means that for any $\psi \in \mathcal{C}_0^\infty(\Omega)$ one has

$$-\int_\Omega \nabla u \cdot \nabla\psi + \int_\Omega qu\psi = 0 \tag{6.8}$$

Assume that u is a smooth solution of the equation and take a test funtion $\phi \in W^{1/2,2}(\partial\Omega)$, by the trace theorem we can extend ϕ to a $W^{1,2}(\Omega)$ function $\tilde{\phi}$ such that

$$\|\tilde{\phi}\|_{W^{1,2}(\Omega)} \leq C\|\phi\|_{W^{1/2,2}(\partial\Omega)}.$$

By the Green formula

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial}{\partial\nu} u\phi d\sigma &= \int_\Omega (\bigtriangledown u \cdot \bigtriangledown\tilde{\phi}) + \int_\Omega \Delta u\tilde{\phi} \\ &= \int_\Omega (\bigtriangledown u \cdot \bigtriangledown\tilde{\phi}) - \int_\Omega (qu\tilde{\phi})\end{aligned} \tag{6.9}$$

Hence by Hölder inequality

$$|\int_{\partial\Omega} \frac{\partial}{\partial\nu} u\phi d\sigma| \leq \| \bigtriangledown u\|_2 \| \bigtriangledown \tilde{\phi}\|_2 + \|q\|_r \|u\|_{p'} \|\tilde{\phi}\|_{p'},$$

where $1/r + 1/p' + 1/p' = 1$ and $1/p - 1/p' = 1/r$.

Since we have $1/r = 1 - 2/p' \leq 2/n$, then $1/2 - 1/p' \leq 1/n$ and we can use the Sobolev embedding,

$$\|\tilde{\phi}\|_{p'} \leq C\|\tilde{\phi}\|_{W^{1,2}(\Omega)},$$

which together with the trace estimate give us:

$$|\int_{\partial\Omega} \frac{\partial}{\partial\nu} u\phi d\sigma| \leq C\| u\|_{W^{1,2}} \|\phi\|_{W^{1/2,2}(\partial\Omega)}.$$

This proves that the normal derivative can be defined, from (6.9) by density, as an element of $W^{-1/2,2}(\partial\Omega)$, with the only assumption $u \in W^{1,2}(\Omega)$.

It remains to prove that this definition does not depend on the choice of the extension $\tilde{\phi}$ of ϕ. This follows by density from (6.8).

Now we can state the main result of this section, the uniquenes of the inverse boundary value problem. This is a L^r-version of Sylvester and Uhlmann's pioneering result, see [SyU], due to Jerison and Kenig and Chanillo, see [Ch].

Theorem 6.2. *Let q_1 and q_2 be functions in $L^r(\Omega)$, $r > n/2$, $n \geq 3$. Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ then $q_1 = q_2$.*

The proof is based on the existence of Calderón approximated solutions :

Proposition 6.3 (Sylvester-Uhlmann solutions)**.** *Let $\rho \in \mathbf{C}^n$ such that $\rho \cdot \rho = 0$ and $q = q_1\chi_\Omega$ with $q_1 \in L^r$, $r \geq n/2$ and $\|q_1\|_{n/2} \leq \epsilon(n)$ if $r = n/2$. Then for $|\rho|$ sufficiently large there exists a $W^{1.2}_{loc}$ solution u of $(\Delta + q)u = 0$ in $\mathbf{R}^n$ which can be written as*

$$u(x) = e^{\rho\cdot x}(1 + \psi(\rho, x)),$$

where for $r > n/2$

$$\|\psi(\rho, \cdot)\|_{p'} \to 0 \text{ as } |\rho| \to \infty. \tag{6.10}$$

Proof: Insert $u = e^{\rho\cdot x}(1+\psi(\rho,x))$ in the equation, then we are reduced to find a solution of the Faddeev equation

$$(\Delta + 2\rho\cdot\bigtriangledown)\psi = q + q\psi, \tag{6.11}$$

satisfying (6.10). If we take Fourier transform in (6.11), we are reduced to find a solution of the integral equation

$$\psi = K_\rho(q) + K_\rho(q\psi), \tag{6.12}$$

where $\widehat{K_\rho(f)}(\xi) = (-|\xi|^2 - 2i\rho\cdot\xi)^{-1}\hat{f}(\xi)$.
The mapping properties of K_ρ are given in (5.38). We write $T_\rho(f) = K_\rho(qf)$. Then we have to solve the Fredholm equation

$$(I - T_\rho)(\psi) = K_\rho(q).$$

By using Corollary 5.3 and Hölder inequality we have

$$\|T_\rho(f)\|_{p'} \leq C\|qf\|_p \leq C\|q\|_r\|f\|_{p'},$$

from the assumptions on q we have that T_ρ is bounded in $L^{p'}$ with norm less than 1. Hence we can write

$$\psi = (I - T_\rho)^{-1}K_\rho(q)$$

and

$$\|\psi\|_{p'} \leq \|K_\rho(q)\|_{p'} \leq |\rho|^{n/r-2}\|q\|_p.$$

Since $p < r$ this gives condition (6.10) if $r > n/2$. The fact that $u \in W^{1,2}$ follows from a priori estimates for the Laplace operator since $\Delta u = qu \in L^p$ and is compactly supported.

The above proposition allows us to generate Cauchy data, now we state the other ingredient in the proof of theorem 6.2.

Proposition 6.4. *Let $q_i \in L^r(\Omega)$, $i = 1,2$ and $r \geq n/2$ such that $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, and assume that u_i are $W^{1,2}(\bar{\Omega})$-solutions of $(\Delta + q_i)u_i = 0$ in Ω. Then*

$$\int_\Omega (q_1 - q_2)u_1u_2 = 0. \tag{6.13}$$

Proof: Let $(f_i, g_i) \in \mathcal{C}_{q_i}$ be the Cauchy data generated by u_i. From the fact that the Cauchy data for both potentials coincide, there exists a function $v_1 \in W^{1,2}(\bar{\Omega})$ satisfying $(\Delta + q_1)v_1 = 0$ with Cauchy data (f_2, g_2). Hence

$$0 = g_2(f_1) - g_2(f_1) = \frac{\partial}{\partial\nu}v_1(f_1) - \frac{\partial}{\partial\nu}u_2(f_1)$$

From (6.9) we have

$$\frac{\partial}{\partial\nu}v_1(f_1) = +\int_\Omega \bigtriangledown v_1\cdot\bigtriangledown\tilde{f}_1 - \int_\Omega q_1v_1\tilde{f}_1$$

We may choose the extension $\tilde{f}_1 = u_1$, hence

$$\frac{\partial}{\partial\nu}v_1(f_1) = \int_\Omega \nabla v_1 \cdot \nabla u_1 - \int_\Omega q_1 v_1 u_1$$
$$= \frac{\partial}{\partial\nu}u_1(f_2)$$

since v_1 is an extension of f_2. Then we have

$$0 = \frac{\partial}{\partial\nu}u_1(f_2) - \frac{\partial}{\partial\nu}u_2(f_1)$$
$$= + \int_\Omega \nabla u_1 \cdot \nabla u_2 - \int_\Omega q_1 u_1 u_2 - \int_\Omega \nabla u_2 \cdot \nabla u_1 + \int_\Omega q_2 u_2 u_1$$

This proves the proposition.

Proof of theorem 6.2:

Fix $\xi \in \mathbf{R}^n$ and take the two complex vectors $\rho_1 = l + i(\xi + m)$ and $\rho_2 = -l + i(\xi - m)$, where l, m and ξ are vectors in $\mathbf{R}^n$ orthogonal to each other and $|l| = |\xi + m|$. From proposition 6.3 we can take for $i = 1, 2$ a solution of $(\Delta + q_i)u_i = 0$ of the form $u_i = e^{\rho_i \cdot x}(1 + \psi_i(\rho, x))$. Then from proposition 6.5 we have

$$0 = \int_\Omega (q_1 - q_2)u_1 u_2 = \int_\Omega (q_1 - q_2)e^{\rho_1 \cdot x}(1 + \psi_1(\rho, x))e^{\rho_2 \cdot x}(1 + \psi_2(\rho, x))dx$$
$$= \int_\Omega (q_1 - q_2)e^{2i\xi\cdot x}dx + \int_\Omega (q_1 - q_2)e^{2i\xi\cdot x}\psi_1(\rho_1, x)(1 + \psi_2(\rho_2, x))$$
$$+ \int_\Omega (q_1 - q_2)e^{2i\xi\cdot x}\psi_2(\rho_2, x)(1 + \psi_1(\rho_1, x)).$$

But

$$\left|\int_\Omega q_i e^{2i\xi\cdot x}\psi_2(\rho_2, x)(1 + \psi_1(\rho_1, x))\right| \le \|q_i\|_r \|\psi_2\|_{p'} \|1 + \psi_2\|_{L^{p'}(\Omega)}$$

tends to zero as $|\rho| = c|l|$ tends to ∞, hence we obtain $\hat{q}_1 - \hat{q}_2 = 0$.

Remarks:

- The proof can be extended for potentials in $L^{n/2}$. In this case one proves that in proposition 6.3 $\|\psi(\rho, \cdot)\|_{p'}$ tends to zero in the weak sense, which is what we need to apply proposition 6.4 There is also an extension for potential in Morrey spaces, see [Ch], which is based on the uniform Sobolev estimates of [ChS] and [ChiR], this result contains potential in the Lorentz space $L^{n/2,\infty}$ with small norm.

- The two dimensional case for the Schrodinger equation was solved by Bukhgeim, [Bu] in the case of $q \in L\infty(\Omega)$. For the special case of potential coming from conductivities with two derivatives it was proved by Nachman, [N1] and with just one derivative by Brown and Uhlmann, [BrU]. They use the scattering transform of the potential. Let us remark that, in this case, the inverse problem is formally well determined and one needs to control all the solutions of Faddeev equation, even for $|\rho| = 0$.

6.3 The Calderón inverse conductivity problem

The problem of electrical impedance tomography deals with the reconstruction of the symmetric conductivity matrix $\gamma = (\gamma^{jk})_{n\times n}$, in the potential equation on a bounded domain Ω, with the ellipticity condition ($c > 0$)

$$\xi_j \gamma^{jk} \xi_k > c|\xi|^2 > 0,$$

$$\operatorname{div} \gamma \nabla u = 0, \tag{6.14}$$

from boundary measurements given by yhe Dirichlet to Neumann (voltage-current) map at the boundary. One considers the Dirichlet boundary value problem

$$\begin{cases} \operatorname{div} \gamma \nabla u = 0 \ \text{ in } \Omega \\ u_{|\partial\Omega} = f. \end{cases} \tag{6.15}$$

The D-N map is given by

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu} \tag{6.16}$$

where the last denotes the normal derivative at a boundary point. We have to understand this map in the weak sense as

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \tag{6.17}$$

defined via the Green formula as

$$\Lambda_\gamma(f)(\phi) = \int_\Omega \gamma \nabla u \cdot \nabla \bar{\tilde{\phi}} \tag{6.18}$$

for a $W^{1,2}$-extension $\tilde{\phi}$ to Ω of the test function $\phi \in H^{1/2}(\partial\Omega)$. In a similar fashion as we did for Schrödinger equation we can prove, via Green formulas and trace theorems, the a priori estimate

$$|\Lambda_\gamma(f)(\phi)| \le C \|f\|_{H^{1/2}(\partial\Omega)} \|\phi\|_{H^{1/2}(\partial\Omega)},$$

which can be used as the mapping boundedness of Λ_γ. This weak definition does not depends on the choice of the extension, in particular if we choose an extension v which is a solution of the conductivity equation (6.14) we obtain that the map Λ is selfadjoint:

$$\Lambda_\gamma(f)(\phi) = \int_\Omega \gamma \nabla u \cdot \nabla \bar{v}.$$

We are going to study the isotropic case $\gamma^{jk} = \gamma(x)\delta^{jk}$ for a real function γ (in the general case there are some obstructions to uniqueness of the inverse problem). In this case the problem can be reduced, assumed that the conductivity is sufficiently smooth, to the inverse boundary value problem for the Schrödinger equation. Let us point out that the reduction of the electrical impedance tomography problem, proposed by Calderón, to the analogous problem for the Schrödinger equation requires the recovery of the conductivity and its normal derivative at the boundary of the domain; this has been attained by Sylvester and Uhlmann, see [SyU2] and by Alessandrini, see [Al] and Brown, see [Br], for less regular domains and conductivities. We are going to study the different steps of this reduction.

Reduction to the Schrödinger equation. Assume $\gamma \in \mathcal{C}^2(\bar{\Omega})$. We carry over the change of function in equation (6.14)

$$v = \gamma^{1/2}u.$$

Lemma 6.1.

$$\operatorname{div}(\gamma \nabla u) = \gamma^{1/2}(\Delta + q)(\gamma^{1/2}u), \tag{6.19}$$

where

$$q = -\frac{\Delta(\gamma^{1/2})}{\gamma^{1/2}}. \tag{6.20}$$

Proof: A straight forward calculation gives,

$$\begin{aligned} \operatorname{div}(\gamma\nabla(\gamma^{1/2}v)) &= div(\gamma(-1/2\nabla\gamma)\gamma^{-3/2}v) + \operatorname{div}(\gamma^{1/2}\nabla v) \\ &= \operatorname{div}(-1/2\gamma^{-1/2}(\nabla\gamma)v) + 1/2\gamma^{-1/2}\nabla\gamma\cdot\nabla v + \gamma^{1/2}\Delta v \\ &= -1/2\operatorname{div}(\gamma^{-1/2}(\nabla\gamma))v + \gamma^{1/2}\Delta v \\ &= -\Delta(\gamma^{1/2}) + \gamma^{1/2}\Delta v. \end{aligned}$$

We have proved that v is solution of the Schrödinger equation

$$(\Delta + q)v = 0. \tag{6.21}$$

We will reduce the conductivity inverse problem for (6.15) to the inverse boundary value problem for (6.5). To achive this we need the following

Lemma 6.2. *Let (f, g) Cauchy data for (6.5). Then*

$$g = \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2}f) + 1/2\gamma^{-1}\frac{\partial\gamma}{\partial\nu}f \tag{6.22}$$

Proof: Let ω a solution of (6.5) which generates (f, g), take $u = \gamma^{-1/2}\omega$, then u is a solution of (6.15) with Dirichlet datum $\gamma^{-1/2}f$. We perform the following calculation with the weak definition of the normal derivatives induced by both equations:

$$\frac{\partial\omega}{\partial\nu} = \gamma^{1/2}\frac{\partial u}{\partial\nu} + \frac{\partial\gamma^{1/2}}{\partial\nu}u$$

this, by using (6.16), gives the desired identity.

Corollary 6.1. *If γ_1, γ_2 such that*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

$$\gamma_1 = \gamma_2 \ \text{ and } \frac{\partial\gamma_1}{\partial\nu} = \frac{\partial\gamma_2}{\partial\nu} \ \text{ on } \partial\Omega,$$

then the Cauchy data sets $\mathcal{C}_{q_1} = \mathcal{C}_{q_1}$.

Proposition 6.5. *let*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

and

$$\gamma_1 = \gamma_2 \ \ and\ \frac{\partial \gamma_1}{\partial \nu} = \frac{\partial \gamma_2}{\partial \nu} \ \ on\ \partial\Omega.$$

Then $\gamma_1 = \gamma_2$.

Proof: From the Corollary and Theorem 6.2, it is enough to prove that if $q_1 = q_2$, then $\gamma_1 = \gamma_2$. Since for $i = 1, 2$

$$q_1 = q_2 = q = -\frac{\Delta(\gamma_i^{1/2})}{\gamma_i^{1/2}}$$

We have two solutions $v_i = \gamma_i^{1/2}$ of the equation $\Delta v + qv = 0$ with the same boundary data v and $\frac{\partial v}{\partial \nu}$ on $\partial\Omega$. From theorem on unique continuation from the boundary $v_1 = v_2$ (notice that for the Schrödinger equation one boundary data does not suffices). We can avoid to refer to unique continuation results, by reducing again to a conductivity equation. Take $u = log\frac{\gamma_1}{\gamma_2}$, then

$$div((\gamma_1\gamma_2)^{1/2}\nabla u) = 2(\gamma_1\gamma_2)^{1/2}(q_1 - q_2) = 0,$$

since $u = 0$ on $\partial\Omega$, by uniqueness $u = 0$.

Theorem 6.3 ([SyU], [Ch])**.** . *Assume* $\gamma \in W^{2,r}(\bar{\Omega})$, *where* $r > n/2$, *then if*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then $\gamma_1 = \gamma_2$.

To prove the theorem we only need to prove that the DtoN map Λ_{γ_1} determines the boundary values of γ, this will be the subject of the next section.

Boundary determination of the conductivity

Remarks on the 2d case (to be completed).

Exercises: Prove (6.22) by using the definition of the map (6.18) and (6.9).

6.4 The Scattering Problem

The third application of the estimates is to the inverse scattering problem. We shall firstly state results concerning the direct problem and introduce some properties of the scattering solution we will need to pose the inverse problem. We may use the same model to treat the direct problem for two different equations, the Schrödinger and the acoustic.

We consider the direct problem of scattering for the Hamiltonian $H = \Delta + V(x)$. In the case of Schrödinger or potential scattering we assume that $V(x) = q(x) \in L^p$

for some p. The scattering solution of wave number k is the solution of the problem

$$\begin{aligned}&(\Delta + k^2)u = V(x)u\\&u = u_i + u_s\\&u_s \text{ satisfies the outgoing Sommerfeld radiation condition (S.R.C)}\end{aligned} \tag{6.23}$$

In the case of the acoustic equation $V = k^2(1 - n(x))$ depends itself on the wave number k, n is the refraction index of the medium which can be complex $n(x) = n_1(x) + in_2(x)/k$, $n_2 \geq 0$ and $n_1(x) > 0$.

Both equations can be treated in the same way, in order to prove existence of solutions and to study the asymptotic behavior at ∞, if one assumes that V is a compactly supported function. For some other properties they have to be treated very differently. In fact, for hyperbolic stationary estimates, see the remark at the end of chapter 3, we have to control the behavior on the parameter k and the two equations are very different from this point of view.

The incident wave u_i is an entire solution of the homogeneous Helmholtz equation; this is the case of plane waves

$$u_i(x) = u_0(k, \theta, x) = e^{ik\theta\cdot x}, \tag{6.24}$$

The scattering solution $u = u(k, \theta, x)$ is a solution of the Helmholtz equation in the exterior of the object $D = suppV$, and so is the scattered solution u_s; since it satisfies the S.R.C., it has the following asymptotics as $|x| \to \infty$ (see chapter 1):

$$u_s(x) = c_n k^{(n-1)/2} \frac{e^{ik|x|}}{|x|^{(n-1)/2}} u_\infty(k, \theta, \frac{x}{|x|}) + o(|x|^{-(n-1)/2}) \tag{6.25}$$

The function $u_\infty(k, \theta, \omega)$, is known as the scattering amplitude or far field pattern, it represents the measurements in the inverse scattering problem.

The way of constructing the scattering solutions is by mean of the so called Lippmann-Schwinger integral equation. We are going to state this equation, to sketch the proof of existence of the solution and to get an expression for the far field pattern which will be useful for our purposes.

Let us remark the equation satisfied by u_s,

$$(\Delta + k^2)u_s = Vu_i + Vu_s \tag{6.26}$$

If we apply the resolvent $R_+ = R_+(k^2)$, since u_s is outgoing, we obtain

$$u_s = R_+(Vu).$$

This is the Lippmann-Schwinger integral equation

$$u_s(x) = \int_{\mathbf{R}^n} \Phi_k(x, y)V(y)(u_i(y) + u_s(y))dy \tag{6.27}$$

Denote $< x >= (1 + |x|^2)^{1/2}$.

Theorem 6.4. *Let $V \in L^r$ compactly supported with $r > n/2$, and $k > 0$. Then there exists a unique solution u_s of the Lippmann-Schwinger integral equation such that: $u_s \in W^{s,p'}(< x >^{-\beta}\, dx)$, where $s < 2 - n/r$ and $1/p - 1/p' = 1/r$, and β is some exponent depending on r such that $\beta < 1/2$ if $r < \infty$.*

Proof: We use a factorization similar to the one used in the proof of Proposition 6.3. Write the Lippmann-Schwinger equation as

$$u_s = R_+(Vu_i) + R_+(Vu_s) = R_+(Vu_i) + T_k(u_s)$$

We use Fredholm theorem. To prove that T_k is compact in $W = W^{s,p'}(< x >^{-\beta}\, dx)$ we use the following estimate, which is obtained by interpolation of theorem 5.6 and theorem 5.2 in chapter 3

Theorem 6.5. *Let $0 \leq 1/p - 1/p' = 1/r \leq 2/n$ if $n > 2$ or $0 \leq 1/p - 1/p' < 1$ for $n = 2$. Let $k > 0$ and $0 \leq s \leq 2 - n/r$. Then there exists a β depending on r and $\beta < 1/2$ if $r < \infty$, such that*

$$\|R_+(k^2)f\|_{W^{s,p'}(<x>^{-\beta}dx)} \leq C(k)\|f\|_{L^p(<x>^{\beta}dx)} \tag{6.28}$$

The compactness of T_k follows from this theorem, Hölder inequality and Rellich compactness theorem. Now to obtain the existence we need to prove that the only solution in $W^{s,p'}(< x >^{-\beta}\, dx)$ of the integral equation $u = T_k(u)$ is $u = 0$. To prove this, notice that u is a solution of the equation $(\Delta + k^2)u = Vu$ and since V is compactly supported we can use Rellich uniqueness theorem Corollary 2.4, from which we deduce that u has to be compactly supported and hence the unique continuation principle implies that $u = 0$.

To justify the use of Rellich uniqueness theorem we have to check (2.41) in chapter 1. By the Green formula in a ball containing the support of V

$$\int_{\partial B} u\frac{\partial \bar{u}}{\partial \nu} = \int_B (\nabla u \cdot \nabla \bar{u} + Vu\bar{u} - k^2 u\bar{u})dx,$$

where it is easy to see that the gradient integral makes sense (from elliptic estimates u is locally in $W^{2,p}$) hence

$$\Im \int_{\partial\Omega} u\frac{\partial \bar{u}}{\partial \nu} = \int_B \Im V|u|^2 \geq 0$$

from the conditions on the potential at the beginning of this section.

In the case of the Schrödinger equation, i.e. $V(x) = q(x)$, we have a good decay of the solution, appropriated for some partial recovery of q in the inverse problem. The key point is the negative power of k in the estimate:

Theorem 6.6. *Let $V = q(x) \in L^r$ compactly supported with $r > n/2$, and $k > 0$, assume that $0 \leq t \leq 1 - n/2r$ and $1/p - 1/p' = 1/r$. Then there is a $\beta > 0$ such that the solution u given by theorem 6.5 satisfies*

$$\|D^t u_s\|_{L^{p'}(<x>^{-\beta}dx)} \leq Ck^{t+n/(2r)-1}\|q\|_{L^p(<x>^{-\beta}dx)} \tag{6.29}$$

The proof follows the lines of Proposition 6.3, just changing the Faddeev kernel by the resolvent kernel and using the following estimate which can be obtained by interpolation of corollary 5.2 and theorem 5.2 in chapter 3.

Proposition 6.6. *let $k > 0$, and $0 \leq t \leq 1 - n/2r$ and $1/p - 1/p' = 1/r$. Then there exists $C > 0$ such that*

$$\|D^t R_+(k^2) f\|_{L^{p'}(<x>^{-\beta}dx)} \leq C k^{t+n/2r-1} \|f\|_{L^p(<x>^{\beta}dx)} \tag{6.30}$$

Let us remark the Lipmann-Schwinger integral equation

$$u_s(k,\omega,x) = R_+(k^2)(Vu)(x) = \int_{\mathbf{R}^n} \Phi_k(x-y)V(y)u(k,\omega,y)dy, \tag{6.31}$$

We are going to give an expression for the far field pattern, which can be directly obtained from (6.31) by using the asymptotics of the fundamental solution and the compactness of the support of V. It is interesting to state it in two steps, the first is a scattering interpretation of the Fourier transform

Proposition 6.7. *Let v be an outgoing solution of the inhomogeneous Helmholts equation with a source f.*

$$\Delta v + k^2 v = f,$$

where $f \in \mathcal{C}_0^\infty$. Then v can be written when $|x| \to \infty$ as

$$v(x) = C k^{(n-3)/2} \frac{e^{ik|x|}}{|x|^{(n-1)/2}} v_\infty(k, x/|x|) + o(|x|^{-(n-1)/2})$$

and the far field pattern is given by

$$v_\infty(k, x/|x|) = \hat{f}(kx/|x|) \tag{6.32}$$

The proof follows from the volume potential formula (2.35) in chapter 1 and the asymptotics of the fundamental solution $\Phi_k(x,y)$ as $|x| \to \infty$.
Notice that the compactness of the support is essential to obtain the above proposition.

If we apply proposition 6.7 to the Schrödinger equation by taking $f = V(x)u(k,\omega,x)$ we obtain

Proposition 6.8. *If V is compactly supported, then the far field pattern of the scattering solution is given by*

$$u_\infty(k,\theta,\frac{x}{|x|}) = C \int_{\mathbf{R}^n} e^{-ikx/|x| \cdot y} V(y)u(k,\theta,y)dy. \tag{6.33}$$

For the case of non compactly supported potentials this is used as a definition of the scattering amplitude or far field pattern, see [EsR1].

Now we can pose

<u>The inverse scattering problem</u>:

Can one recover the potential $V(x)$ from the knowledge of the scattering amplitude $u_\infty(k,\omega,\theta)$? Notice that $\omega \in \mathbf{S}^{n-1}$ is the direction of the incident plane wave u_i, θ is the direction of the measurement and k is the wave number, k^2 is the energy of the incident wave.

The problem with complete data is overdetermined, we have $2n-1$ parameters and n unknowns. Hence one assumes partial knowledge of the scattering amplitude, the most celebrated problems are:

- Fixed energy data: One assumes $u_\infty(k,\omega,\theta)$ for $k=k_0$ known, the problem is formally well determined only in dimension two.
- Fixed angle data: One assumes $u_\infty(k,\omega,\theta)$ known for $\theta=\theta_0$, the problem is formally well determined in any dimension.
- Backscattering data: One assumes $u_\infty(k,\omega,\theta)$ data with $\theta=-\omega$. The problem is formally well determined.

The fixed energy scattering for Schrödinger equation can be reduced to the boundary inverse Dirichlet to Neumann problem which can be solved by the complex exponentials functions of Sylvester and Uhlmann (see Isakov [I]). For the direct approach by using Faddeev equation see the notes of Päivärinta []. Just remark that these two problems have the same degree of overdeterminacy.

We state the following reciprocity relation for the scattering amplitude, which follows by the representation formula in exterior domains, see [CK]:

Proposition 6.9. *Let u_∞ be the scattering amplitudes of the problem $(\Delta+k^2)u = V(x)u$, where V is real, then*

$$u_\infty(k,\omega,\theta) = u_\infty(k,-\theta,-\omega) \tag{6.34}$$

From now on we consider the case of Schrödinger scattering, $V=q(x)$. Since the potential is real, we can see that the change from k to $-k$ in the scattering problem preserves the equation and conjugates the radiation condition and the incident plane wave. Hence we may extend

$$u_\infty(-k,\theta,\omega) = \overline{u_\infty(k,\theta,\omega)}$$

The scattering amplitude can be expanded in the so called Born series, which is obtained by inserting the Lippmann-Schwinger equation in the expression (6.33):

$$\begin{aligned} u_\infty(k,\theta,\omega) = & \int_{\mathbf{R}^n} e^{-ik(\omega-\theta)\cdot y} q(y)dy \\ & + \sum_{j=1}^{m} \int e^{-ik\omega\cdot y}(qR_+(k^2))^j (q(\cdot)u_i(k,\theta,\cdot))(y)dy \\ & + \int e^{-ik\omega\cdot y}(qR_+(k^2))^m (q(\cdot)u_s(k,\theta,\cdot))(y)dy. \end{aligned} \tag{6.35}$$

Notice that the first term on the right can be expressed as
$\hat{q}(k(\omega - \theta))$, hence we define the Born approximated potential according to the partial scattering data as

- (a) Born fixed angle approximation at θ_0

$$\widehat{q_{\theta_0}}(k(\omega - \theta_0)) = u_\infty(k, \theta_0, \omega) \tag{6.36}$$

  where $k \in \mathbf{R}$ and $\omega \in \mathbf{S}^{n-1}$

- (b) Born backscattering approximation

$$\hat{q}_b(\xi) = u_\infty(|\xi|/2, -\theta, \theta) \tag{6.37}$$

  where $\theta = \xi/|\xi|$.

The first term in the Born expansion is the similar expression for the Fourier transform of the actual potential q. The so called **Diffraction Tomography** substitutes the potential q by its Born approximation. This is the basis of the **Ultrasound Tomography in Medicine**. The Born approximation can be calculated from band limited data obtained by experiments for energy k^2, these data are reduced to the so called Ewald spheres $\{\xi = k(\omega - \theta_0) : \omega \in \mathbf{S}^{n-1}\}$.

Let us start by stating the recovery of the actual potential from an overdetermined set of scattering amplitudes. This is based on the following expression

Proposition 6.10. *Let q be a compactly supported potential in L^r, $r > n/2$. Assume that θ_0 is orthogonal to ξ , and $\theta_n \to \theta_0$. Then*

$$\lim_{n\to\infty} \widehat{(q_{\theta_n})}(\xi) = \hat{q}(\xi) \tag{6.38}$$

Proof: If we use (6.35) with $m = 1$ we have

$$\hat{q}(k(\omega - \theta)) = u_\infty(k, \theta, \omega) + \int e^{-ik(\omega-\theta)\cdot y} q(y) e^{-ik\theta\cdot y} u_s(k, \theta, y) dy$$

Now the take $\xi = k(\omega - \theta)$ this gives $k = \frac{|\xi|}{2\hat{\xi}\cdot\theta}$, where $\hat{\xi} = \xi/|\xi|$ hence

$$|\widehat{q - q_\theta}(\xi)| = |\int e^{-i\xi\cdot y} q(y) e^{-i|\xi|/(2\hat{\xi}\cdot\theta)\theta\cdot y} u_s(|\xi|/(2\hat{\xi}\cdot\theta), \theta, y) dy|$$

$$\leq C\|q\|_{L^p} \|u_s(|\xi|/(2\hat{\xi}\cdot\theta), \theta, (\cdot))\|_{L^{p'}(<x>^{-\beta})},$$

from (6.29) we have

$$|\widehat{q - q_\theta}(\xi)| \leq C\|q\|^2_{L^p} ||\xi|/(2\hat{\xi}\cdot\theta)|^{n/2r-1}$$

if we take $\theta = \theta_n \to \theta_0$ this tends to 0.

Theorem 6.7. *Let q be a compactly supported potential in L^r, $r > n/2$. Then the knowledge of the fixed angle Born approximation for all incident angles in a maximal half circle S_1 of an $n-1$ sphere centered at the origin, determines uniquely q.*

Proof: Let $\xi \in \mathbf{R}^n$ and consider the normal hyperplane $\Pi = \{\xi \cdot \theta = 0\}$, assume that q_θ is known for θ on the 1-dimensional circle $S^{n-1} \cap \Pi_2$, where Π_2 is a two plane through the origin, then there exists a vector $\theta_0 \in S^{n-1}$ in the line $\Pi \cap \Pi_2$. Then from the proposition

$$\hat{q}(\xi) = \lim_{n \to \infty} \widehat{q_{\theta_n}}(\xi) \tag{6.39}$$

where $\theta_n \to \theta_0$. Hence we have determined the Fourier transform $\hat{q}(\xi)$ for any ξ, non orthogonal to the circle, by (6.39).

Notice that there is one degree of overdeterminacy in the above theorem.

There is an inconvenient in the above theorem, beside the overdeterminacy: We need to measure $\widehat{q_\theta}(\xi)$ for θ approaching to a point θ_0 orthogonal to ξ in order to recover the Fourier transform $\hat{q}(\xi)$, to reach this point we need incident plane waves with energy $k^2 = \frac{|\xi|}{2\hat{\xi}\cdot\theta}$ increasing to ∞. Another way to say : the fixed angle scattering amplitude at points orthogonal to the fixed incident vector is obtained by experiments with plane waves whose energy tends to ∞. This is not practical in applications, since the model for high energy increases the absorption factor of the media and the Helmholtz equation model is getting far from the actual one. For medical tomography this is an obvious inconvenient. In applications to actual recovery of potential sometimes only some parameters of the potential are needed, this means partial recovery of q. We shall see next that from fixed angle scattering only an incident direction suffices to recover the main singularities of q. This has been also proved for backscattering in dimension 2, see [OPS].

There are several results on recovery of singularities by taking spherical means of the whole scattering amplitude, see [PSo] [PSSo]. Generic uniqueness and local uniqueness is also known, see [St] for fixed angle scattering and [EsR1], [EsR2], [Pr] and [L] for backscattering.

Theorem 6.8. *Assume that q is a compactly supported real valued function in $W^{s_0,p}(\mathbf{R}^n)$, for $0 \le s_0 < \frac{n}{2}$, $n(\frac{1}{p} - \frac{2}{n+1}) \le s_0 \le \frac{n}{p}$, $n \ge 2$ and $p \ge 2$. Then $q - q_\theta \in W^{s,2}$ for any $s < s_1(s_0,p)$ modulo a function in $\mathcal{C}^\infty$ where*

$$s_1 = s_0 + \frac{1}{2} + \frac{n-1}{2}\left(\frac{s_0}{n} - \frac{1}{p}\right).$$

From theorem 6.8 we obtain an improvement from q to $q - q_\theta$ in the scale of Hilbertian Sobolev Spaces, we state it in dimension $n = 2, 3$.

Corollary 6.2 ($n = 3$)**.** *If q is compactly supported real valued function in $W^{s_0,2}$ for $0 \le s_0 < 3/2$, then $q - q_\theta \in W^{s,2}$ modulo a function in $\mathcal{C}^\infty$ for any $s < s_1$, given by*

$$s_1 = \frac{4s_0}{3}.$$

Corollary 6.3 ($n = 2$)**.** *If q is compactly supported and in $W^{s_0,2}$, $0 \le s_0 < 1$, then $q - q_\theta \in W^{s,2}$ modulo a function in $\mathcal{C}^\infty$ for any $s < s_1$, given by*

$$s_1 = \frac{5s_0}{4} + \frac{1}{4}.$$

To prove the above theorem we need some extensions of the estimates in chapter 3 and a theorem on products of functions of Sobolev spaces (actually we need a version with weights $< x >^\delta$, but since we are dealing with compactly supported potential we do not go into the details) .

We will denote

$$R_{k,\theta}(f)(x) = e^{-ikx\cdot\theta} R_+(k^2)(f(\cdot)e^{ik\theta\cdot(\cdot)})(x). \tag{6.40}$$

We state then:

Proposition 6.11. *Given r and t such that $0 \leq \frac{1}{t} - \frac{1}{2} \leq \frac{1}{n+1}$ and $0 \leq \frac{1}{2} - \frac{1}{r} \leq \frac{1}{n+1}$, and $s \geq 0$, and for an appropriate choice of δ and δ', there exists $C > 0$, independent of $k > 0$ and $\theta \in \mathbf{S}^{n-1}$, such that*

$$\|R_{k,\theta} f\|_{W^{s,r}(<x>^{-\delta})} \leq C k^{-1+\frac{n-1}{2}(\frac{1}{t}-\frac{1}{r})} \|f\|_{W^{s,t}(<x>^{\delta'})}.$$

Proof of proposition 6.11:

Since

$$(R_{k,\theta} f)\hat{}(\xi) = \frac{\hat{f}(\xi)}{-|\xi - k\theta|^2 + k^2 + i0},$$

the commutativity of fractional derivatives and $R_{k,\theta}$ allows one to reduce to the case $s = 0$. But in L^p, we can take away the exponentials in (6.40) and reduce to prove

$$\|R_+ f\|_{L^r(<x>^{-\delta})} \leq C k^{-1+\frac{(n-1)}{2}(\frac{1}{t}-\frac{1}{r})} \|f\|_{L^t(<x>^{\delta'} dx)}. \tag{6.41}$$

Now, (6.41) follows from interpolation of theorem 5.2 in the case $1/p - 1/p' = 2/(n+1)$, $p' = q$ and the three following inequalities:

First Corollary 5.1

$$\|R_+ f\|_{L^2(<x>^{-\delta})} \leq C k^{-1} \|f\|_{L^2(<x>^{\delta} dx)},$$

second

$$\|R_+ f\|_{L^r} \leq C k^{-\frac{1}{2}-\frac{1}{n+1}} \|f\|_{L^2(<x>^{\delta} dx)}, \tag{6.42}$$

where $\frac{1}{2} - \frac{1}{r} = \frac{1}{n+1}$ and finally

$$\|R_+ f\|_{L^2(<x>^{-\delta} dx)} \leq C k^{-\frac{1}{2}-\frac{1}{n+1}} \|f\|_{L^t}, \tag{6.43}$$

where $\frac{1}{t} - \frac{1}{2} = \frac{1}{n+1}$.

The estimate (6.43) follows from the non selfdual estimate for the resolvent, theorem 5.5, since

$$C \sup_{R,x_0} \left(\frac{1}{R} \int_{B(x_0,R)} |R_+(f)(x)|^2 dx \right)^{1/2} \geq \|R_+(f)\|_{L^2(<x>^{-\delta} dx)}$$

for $\delta > 1/2$.

Finally (6.42) is the dual of (6.43) for the incoming limit $R_-(k^2)$.

Let us define for fixed $\theta_0 \in \mathbf{S}^{n-1}$ and $\theta \in \mathbf{S}^{n-1}$,

$$S_k(f)(\theta) = \int_{\mathbf{R}^n} e^{-ikx\cdot(\theta-\theta_0)} f(x) dx. \tag{6.44}$$

Lemma 6.3. *For $\omega(\theta) = |\theta - \theta_0|^s$, we have for some $\delta > 0$*

$$\|S_k(f)\omega\|_{L^2(\mathbf{S}^{n-1})} \le Ck^{-\frac{n-1}{2}-s}\|D^s f\|_{L^2(<x>^\delta dx)}. \tag{6.45}$$

Lemma 6.4. *Let $\frac{1}{n+1} \le \frac{1}{t} - \frac{1}{2} \le \frac{1}{2}$, then*

$$\|S_k(f)\omega\|_{L^2(\mathbf{S}^{n-1})} \le Ck^{n(\frac{1}{t}-1)-s}\|D^s f\|_{L^t(\mathbf{R}^n)}. \tag{6.46}$$

Proof of lemma 6.3 The following identity allows one to reduce to the case $k = 1$

$$S_k f(\theta) = k^{-n-s}|\theta - \theta_0|^{-s} S_1\left((D^s f)(\frac{\cdot}{k})\right)(\theta).$$

To see this, write

$$\begin{aligned} S_k f(\theta) = \hat{f}(k(\theta-\theta_0)) &= |\theta-\theta_0|^{-s}k^{-s}(D^s f)\hat{}(k(\theta-\theta_0)) \\ &= |\theta-\theta_0|^{-s}k^{-s-n}((D^s f)(\frac{\cdot}{k}))\hat{}(\theta-\theta_0) \\ &= |\theta-\theta_0|^{-s}k^{-s-n}S_1((D^s f)(\frac{\cdot}{k}))(\theta). \end{aligned}$$

Now

$$\begin{aligned} \|S_k(f)\omega\|_{L^2(\mathbf{S}^{n-1})} &= k^{-n-s}\|S_1((D^s f)(\frac{\cdot}{k}))\|_{L^2(\mathbf{S}^{n-1})} \\ &\le Ck^{-n-s}\|(D^s f)(\frac{\cdot}{k})\|, \end{aligned}$$

where the last inequality follows from the restriction theorem 4.21 in chapter 2 with the homogeneous norm defined in (4.23):

$$\|f\|_{\tilde{B}_{1/2}} = \sum_{j=-\infty}^{\infty}(R_j\int_{\Omega_j}|f|^2dx)^{1/2} \tag{6.47}$$

instead of $B_{1/2}$ (see formula (5.7)):

$$\|\hat{g}_{|\mathbf{S}^{n-1}}\|_{L^2(\mathbf{S}^{n-1})} \le C\|g\|_{\tilde{B}_{1/2}},$$

for $g = (D^s f)(\frac{\cdot}{k})e^{i(\cdot)\cdot\theta_0}$

Hence, since $\|g(k(\cdot)\|_{\tilde{B}_{1/2}} = k^{(n+1)/2}\|g\|_{\tilde{B}_{1/2}}$, and the fact that

$$\|g\|_{\tilde{B}_{1/2}} \le \|g\|_{L^2(<x>^\delta dx)}$$

we obtain the desired estimate.

Proof of lemma 6.4: As above we reduce to the case $k = 1$, by using the previous argument, the estimate in this case is Stein-Tomas theorem for the restriction of the Fourier transform to the unit sphere Corollary 4.5 in chapter 1.

The following statement is obtained by interpolation of lemma 6.3 and lemma 6.4 in the case $\frac{1}{n+1} = \frac{1}{t} - \frac{1}{2}$.

Proposition 6.12. *Given t such that $0 \le \frac{1}{t} - \frac{1}{2} \le \frac{1}{n+1}$, and $s \ge 0$, there exists $\delta(t) > 0$ such that, for $\omega(\theta) = |\theta - \theta_0|^s$, one has*

$$\|S_k(f)\omega\|_{L^2(\mathbf{S}^{n-1})} \le Ck^{\frac{n-1}{2}(\frac{1}{t}-\frac{3}{2})-s}\|f\|_{\dot{W}^{s,t}(<x>^\delta dx)}. \tag{6.48}$$

The next is a theorem due to Zolesio, see [Gr],

Proposition 6.13. *Let $s_1, s_2, s_3 \geq 0$, $s_3 \leq s_1$, $s_3 \leq s_2$ and p_1, p_2 and p such that*

$$s_1 + s_2 - s_3 \geq n(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0 \text{ and } p_j > p \ , \ j = 1, 2$$

then

$$\|uv\|_{W^{s_3,p}(\mathbf{R}^n)} \leq \|u\|_{W^{s_1,p_1}(\mathbf{R}^n)} \|v\|_{W^{s_2,p_2}(\mathbf{R}^n)}.$$

Proof of theorem 6.8 Let us take a cutoff $\chi \in \mathcal{C}^\infty(\mathbf{R})$ such that $\chi = 1$ for $t > 2$ and $\chi = 0$ for $t < 1$ and write

$$(q - q_{\theta_0})\widehat{\ }(\xi) = (1 - \chi(|\xi|))(q - q_{\theta_0})\widehat{\ }(\xi) + \chi(|\xi|)(q - q_{\theta_0})\widehat{\ }(\xi). \tag{6.49}$$

We have to control the second term in the above decomposition, since the first one gives a smooth function. Take $\xi = k(\theta - \theta_0)$, recall the definition of the fixed angle Born approximation (6.36), the Born expansion for the scattering amplitude (6.35) and write

$$\chi(|\xi|)\widehat{(q - q_\theta)}(\xi) = \sum_{j=1}^{m} \widehat{Q_j(q)}(k, \theta) + \widehat{q_m^R}(\xi), \tag{6.50}$$

where

$$\widehat{Q_j(q)}(\xi) = \chi(|\xi|) \int_{\mathbf{R}^n} e^{-ik(\theta-\theta_0)\cdot y} (qR_{k,\theta_0})^j (q(\cdot)) dy, \tag{6.51}$$

The following proposition controls the terms $\tilde{Q}_j(q)$ in the above sum. It states that the Born series is a sort of asymptotic expansion from the point of view of regularity.

Proposition 6.14. *Let q be a compactly supported function in $W^{s_0,p}(\mathbf{R}^n)$, $0 \leq s_0 < n/2$, $n(\frac{1}{p} - \frac{2}{n+1}) \leq s_0 \leq \frac{n}{p}$, and $p \geq 2$. Then $Q_j(q) \in W^{s,2}$ for $s < s_j$, given by*

$$s_j = s_0 - \frac{1}{2} + j\left(1 - \frac{(n-1)}{2}(\frac{1}{p} - \frac{s_0}{n})\right).$$

Proof of proposition 6.14

Make the change of variable in the integral (6.51), reduced to the halfspace $\xi \cdot \theta_0 < 0$, given by $\xi = k(\theta - \theta_0)$,$\theta \in \mathbf{S}^{n-1}$, $|\xi| = k|\theta - \theta_0|$ and $|\xi|^{n-1} d|\xi| d\sigma_{n-1}(\tilde{\xi}) = k^{n-1}|\theta - \theta_0|^2 dk d\sigma_{n-1}(\theta)$, where $\tilde{\xi} \in \mathbf{S}^{n-1}$. Then

$$\|(Q_j(q))\|^2_{W^{s,2}} \leq 2 \int_{1/2}^{\infty} k^{n-1} k^{2s} \int_{\mathbf{S}^{n-1}} \chi(k|\theta - \theta_0|)^2 |\theta - \theta_0|^{2s+2} \times$$
$$| \int_{\mathbf{R}^n} e^{-ik(\theta-\theta_0)\cdot y} (qR_{k,\theta_0})^j (q) dy|^2 d\sigma dk \tag{6.52}$$
$$\leq \int_{1/2}^{\infty} k^{n-1+2s} \|S_k (qR_{k,\theta_0})^j (q) \chi(k|\theta - \theta_0|) |\theta - \theta_0|^{s+1}\|^2_{L^2(\mathbf{S}^{n-1})} dk.$$

From propositions 6.11, 6.12 and 6.13, there exists some δ´s, which may change at each occurrence, such that

$$\begin{aligned}
&\|S_k(qR_{k,\theta_0})^j(q)\chi(k|\theta-\theta_0|)|\theta-\theta_0|^{s+1}\|_{L^2(\mathbf{S}^{n-1})} \leq \\
&C\|S_k\|_{\dot W^{s_0,t}(<x>^\delta)\to L^2(\mathbf{S}^{n-1}\omega_1 d\sigma)}\|q\|_{W^{s_0,t_1}}\times \\
&\prod_{l=1}^{j}\left(\|q\|_{W^{s_0,p_l}((<x>^\delta)}\|R_{k,\theta_0}\|_{W^{s_0,t_l}(<x>^\delta)\to W^{s_0,r_l}(<x>^{-\delta})}\right).
\end{aligned} \tag{6.53}$$

Where $\omega_1(\theta)=|\theta-\theta_0|^{2s+2}$ and p_l, r_l has to satisfy the following requirements, for $t_{j+1}=t$:

$$\begin{cases}
0\leq \frac{1}{p_l}+\frac{1}{r_l}-\frac{1}{t_{l+1}}\leq\frac{s_0}{n}\ ,\ p_l>t_{l+1}, r_l>t_{l+1},\\
0\leq\frac{1}{t_l}-\frac{1}{2}\leq\frac{1}{n+1},\\
0\leq\frac{1}{2}-\frac{1}{r_l}\leq\frac{1}{n+1},\\
0\leq\frac{1}{t}-\frac{1}{2}\leq\frac{1}{n+1}.
\end{cases} \tag{6.54}$$

For exponents r_l and t_l under these constrains and with the appropriate weight exponents δ, in propositions 6.11, 6.12 and 6.13, we can bound (6.53) by

$$C^j(\|q\|_{W^{s_0,p}(<x>^\delta)})^{j+1}k^\alpha, \tag{6.55}$$

where α is given by

$$-j+\frac{n-1}{2}\left(\frac{1}{t_1}-\sum_{l=1}^{j}(\frac{1}{r_l}-\frac{1}{t_{l+1}})-\frac{3}{2}\right)-s_0$$

Now we choose $p=p_l$, $\frac{1}{p}+\frac{1}{r_l}-\frac{1}{t_{l+1}}=\frac{s_0}{n}$, and the condition $n(\frac{1}{p}-\frac{2}{n+1})\leq s_0\leq\frac{n}{p}$ guaranties the existence of r_l and t_l with the above requirements and with α given by

$$j\left(-1-\frac{n-1}{2}(\frac{s_0}{n}-\frac{1}{p})\right)+\frac{n-1}{2}(\frac{1}{t_1}-\frac{3}{2})-s_0.$$

Now the choice of $t_1=2$ makes the integral in (6.52) convergent for the desired values $s<s_j$ in the statement of the proposition.

To end with the second term in (6.50), we may use the estimates (6.55) in the proof of proposition 6.14 and take $\int_{k_0}^{\infty}$ in (6.52) by using an appropriate cutoff function in (6.49). Then we may write if $s<s_1$ for some $\beta>0$,

$$\|Q_j(q)\|_{W^{s,2}}\leq k_0^{-j\beta}C^j(\|q\|_{W^{s_0,p}})^{j+1},$$

and, by taking k_0 big enough, we make the sum in (6.50) a convergent series in the above Sobolev norm.

We are going to give, for future use, the following estimate for the remainder term $\tilde q_m^R$ in (6.50), which only involves L^p condition on q. By taking m big enough, this estimate will also control the Sobolev norm of the remainder term in (6.50).

Proposition 6.15. *Let q be a compactly supported function in L^p for $p > n/2$ and $p \geq 2$. Let $\tilde{q}_m^R$ be defined in (6.50), then $q_m^R \in W^{s,2}$ for any $s < s_m^R = (m+1/2)(2 - n/p) - n/(2p)$.*

Proof :

$$\|q_m^R\|^2_{W^{s,2}}$$
$$\leq C \int_{1/2}^{\infty} k^{n-1+2s} \|S_k(qR_{k,\theta_0})^m (qu_i e^{-ik\theta_0\cdot(\cdot)})\|^2_{L^2(\mathbf{S}^{n-1})} dk.$$

Now take r such that $\frac{1}{r'} - \frac{1}{r} = \frac{1}{p}$ (i.e. $r' = \frac{2p}{p+1}$) .

From theorem 5.2 in chapter 3, Stein-Tomas restriction theorem and Hölder inequality,

$$\|S_k(qR_{k,\theta_0})^m (qu_s e^{-ik\theta_0\cdot(\cdot)})\|_{L^2(\mathbf{S}^{n-1})} \leq C k^{n(\frac{p+1}{2p}-1)+m(\frac{n}{p}-2)} \|q\|^{m+1}_{L^p} \|u_s\|_{L^r}.$$

From theorem 6.6, we have, for k big enough,

$$\|u_s\|_{L^r} \leq C \|R_+(k^2)(qu_i)\|_{L^r} \leq C k^{n/(2p)-1} \|q\|_{L^{r'}},$$

which gives the desired value s_m^R. This ends with the proof of proposition 6.15.

Remarks

- One can obtain the following local result by using the above technics, mainly the observation leading to the convergence of the series (6.50) at the end of the proof of theorem 6.8, together with the control of the zero energy case which is refered in the section 2 in [Ru]: Assume that q is compactly supported then there exists an $\epsilon > 0$ and a δ such that if $\|q\|_{W^{s,2}(<x>^\delta)} \leq \epsilon$, for positive $s \geq \frac{n(n-3)}{2(n-1)}$, then the fixed angle scattering amplitude determines uniquely the potential. For fixed angle scattering Stefanov, [St] obtained a similar result but assuming more regularity, he supposes q compactly supported and $\|q\|_{W^{4,\infty}} \leq \epsilon$.

- In the case of backscattering data, the local results have been obtained for differents a priori spaces. Let us mention Prosser [Pr], for the so called Friedrich class, which is the set of functions whose Fourier transform $\hat{q}$ is in the closure of $\mathcal{C}_0^\infty$ with the α-Lipschitz norm (Hölder norm)

$$\| < x >^N \hat{q}\|_{\Lambda^\alpha}$$

for $0 < \alpha \leq 1$ and $N > 1$, also Stefanov [St], for q compactly supported and $\|q\|_{W^{4,\infty}} \leq \epsilon$ and finally Lagergren [L] in the case $n = 3$ for the closure of $\mathcal{C}_0^\infty$ with with the seminorms

$$\|q\| = \sum_{|\alpha|=1} \|q^{(\alpha)}\|_{L^1}$$

$$\|q\|_K = \int_K |q(x)| dx$$

where K runs the compact sets in $\mathbf{R}^3$.

- If one considers the Born approximation constructed from backscattering data, it can be proved that it contains the main singularities up to 1/2 of the actual potential, see [**?**] [**?**] and [**?**]:

  Theorem 6.9. *Assume that* $n \in \{2,3\}$, q *is a compactly supported function in* $W^{\alpha\,,2}(\mathbb{R}^n)$ *and* $\alpha \geq 0$*. Then* $q - q_B \in W^{\beta\,,2}(\mathbb{R}^n) + C^{\infty}(\mathbb{R}^n)$ *, for any* $\beta \in \mathbb{R}$ *such that* $0 \leq \beta < \alpha + \frac{1}{2}$ *.*

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