Introduction to Electrical Impedance Tomography *

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1. Introduction. These lectures are intended as an introduction to the mathematical theory of electrical impedance tomography (EIT for short). We do not touch the formidable problems which relate to the practical reconstruction algorithms e.g. relevant optimization algorithms, $\overline{\partial}$–methods and errors caused by the incorrect modeling of the measurement domain. Also, we don not treat the very interesting question of nouniqueness in the EIT.

The reader is assumed to have a basic knowledge of functional analysis and measure theory, and also to be familiar with elementary existence and regularity theory of elliptic divergence form equations. These will be shortly recalled when needed, and relevant references to the literature will be provided.

1.1. Green's formulas. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with C^∞–boundary $\partial\Omega$, i.e. every point $x_0 \in \partial\Omega$ has a neighbourhooud U in $\mathbb{R}^n$ and a function $\rho \in C^\infty(U)$ such that

- $\rho(x) = 0$ if and only if $x \in \partial\Omega \cap U$.
- $\rho(x) > 0$ (resp. $\rho(x) < 0$) if $x \in \Omega^c \cap U$ (resp. $x \in \Omega \cap U$).
- If $\rho(x) = 0$, then $\nabla\rho(x) \neq 0$.

These conditions imply that near a boundary point, possibly after a rotation of coordinate axes, the set $\partial\Omega \cap U$ can be written as $x_n = \psi(x')$ with some C^∞–function ψ where

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$x' = (x_1, \ldots, x_{n_1})$, and the set $\Omega \cap U$ is locally characterized by condition $x_n < \psi(x')$. Let $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ be the outward pointing unit normal at the boundary point x. Then we have a generalization of the fundamental theorem of calculus,

$$\int_\Omega \frac{\partial \varphi(x)}{\partial x_i}\, dx = \int_{\partial\Omega} \nu_i(x)\varphi(x)\, dS(x), \tag{1.1}$$

where $dS(x)$ is the *surface measure* on $\partial\Omega$), i.e. locally, in the set where the boundary is parametrized by $x_n = \psi(x')$ we have $dS(x) = (1 + |\nabla\psi(x')|^2)^{1/2}\, dx'$ where dx' is the $(n-1)$–dimensional Lebesgue–measure on $\mathbb{R}^{n-1}$. If $h = (h_1, \ldots, h_n)$ is a differentiable vector valued function on Ω which is continuous in the closure $\overline{\Omega}$ we may apply this with $\varphi = h_i$, and sum over i to get the *Divergence theorem*,

$$\int_\Omega \nabla \cdot h(x)\, dx = \int_{\partial\Omega} \langle \nu(x), h(x)\rangle\, dS(x). \tag{1.2}$$

Here $\nabla h = \sum_i \partial h_i/\partial x_i$ is the *divergence* of h and $\langle\cdot,\cdot\rangle$ denotes the euclidean inner product.

Assume now that $f,\, g \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Let $h = (\nabla f)g$. Then by the product rule of differentiation

$$\nabla \cdot h = (\Delta f)g + \langle \nabla f, \nabla g\rangle,$$

where $\Delta = \nabla\cdot\nabla = \sum_i \partial^2/\partial x_i^2$ is the *Laplace–operator*. On the other hand

$$\langle \nu(x), (\nabla f)g\rangle = \langle \nu(x), \nabla f\rangle\, g = \frac{\partial f}{\partial \nu} g$$

where $\partial f/\partial\nu$ is *the normal derivative* of f. Inserting these into the Divergence theorem we get the *Green's first identity*,

$$\int_\Omega (\Delta f)g + \langle \nabla f, \nabla g\rangle\, dx = \int_{\partial\Omega} \frac{\partial f}{\partial\nu} g\, dS(x). \tag{1.3}$$

Similarly,

$$\int_\Omega f\Delta g + \langle \nabla f, \nabla g\rangle\, dx = \int_{\partial\Omega} f\frac{\partial g}{\partial\nu}\, dS(x),$$

and subtracting these we get *the Green's theorem*

$$\int_\Omega (\Delta f)g - f\Delta g\, dx = \int_{\partial\Omega} \frac{\partial f}{\partial\nu} g - f\frac{\partial g}{\partial\nu}\, dS(x). \tag{1.4}$$

These three equations, i.e. Divergence theorem, Green's first identity and the Green's theorem will be an important computational set of tools for us.

1.2. Electrostatics. In electrostatics we have three basic ingredients: the electric field E, the electric potential u and the current I. These are relate through the following relations:

- The gradient of the potential is -the electric field, i.e. $E = -\nabla u$.
- The current is proportional to the electric field, i.e Ohm's law: $R(x)I(x) = E(x)$. Here the function $R(x)$ is called the resistivity of the media at x. Often we write this as

$$I(x) = \gamma(x)E(x), \tag{1.5}$$

  where $\gamma(x) = 1/R(x)$ is the *conductivity* at x.
- Outside current sources and sinks, the current flux through the boundary of any bounded domain is equal to zero, i.e "what goes in must come out". We can formulate this mathematically as follows. Let B be any open ball with boundary ∂B that does contain sources or sinks. Let ν be the outward unit normal of ∂B. Then

$$\int_{\partial B} \langle \nu, I \rangle \, dS(x) = 0.$$

  Assume now that the current vector is differentiable. Then by the divergence theorem the above result implies

$$\int_B \nabla \cdot I(x) \, dx = 0$$

  for such balls. Hence we must have

$$\nabla \cdot I = 0$$

  outside sources and sinks.

These three facts imply also that, again outside sources and sinks,

$$0 = \nabla \cdot I = \nabla \cdot \sigma E = -\nabla \cdot \gamma \nabla u. \tag{1.6}$$

This is *the conductivity equation*, which is at the basis of EIT.

1.3. Brief outline of EIT. Somewhat imprecisely, in EIT we want to determine *the unknown* conductivity distribution $\gamma(x)$ inside the domain, for example inside the human torso, from current and voltage measurements done on the surface. In order to do this we must first understand what precisely do we mean for example by the conductivity equation when the conductivty γ is discontinuous. This is typically true in applications. We also need to formulate mathematically the boundary measurements we perform on the boundary. So we start by considering *the direct problem*, known as the Dirichlet problem,

$$\nabla \cdot \gamma \nabla u = 0, \quad \text{in} \quad \Omega,$$

$$u|_{\partial\Omega} = f,$$

where $\Omega \subset \mathbb{R}^n$ is our reconstruction domain. We show that this is uniquely solvable for all measurable bounded and nonvanishing conductivities γ, and once this is done we can define our boundary measurement as the map that takes any Dirichlet boundary value f on the boundary, i.e. the voltage distribution, to the corresponding outflowing current, that is to the term $\sigma \partial u / \partial \nu$. This is known as the Dirichlet–Neumann map. Mathematically we can then formulate the EIT–problem as follows: does the Dirichlet–Neumann map determine the conductivity uniquely, and if it does, then reconstruct γ inside Ω.

This problem was originally considered by Cálderon in 60's, but he published his results only in 1986. He considered a linearized version of this problem, with the linearization done at the constant conductivity $\gamma = 1$, and showed that that this is uniguely solvable. The next step was taken by Bob Kohn and Micheal Vogelius, who showed that the Dirichlet–Neumann map determines the complete Taylor series of a smooth conductivity , and hence assuming that γ is piecewise real–analytic, we obtain a uniqueness result.

Briefly, the contents of this course are:

- The solvability of the direct problem: here we need Sobolev–spaces both on Ω and on its boundary $\partial\Omega$.
- The Dirchlet–Neumann map and the corresponding quadaritc form Q_γ: we need to calculate the derivative dQ_γ of Q_γ with respect to γ.

- We show that the derivative is an injective linear map: this will lead to an important class of harmonic functions, whose generalizations will be crucial in Mikko Salo's course.
- Boundary reconstruction

2. The Dirichlet–problem In this we consider the solvability of the Dirchlet–problem

$$\nabla \cdot \gamma \nabla u = F, \quad \text{in} \quad \Omega,$$

$$u|_{\partial\Omega} = 0.$$

in a bounded C^∞–domain Ω. As alredy pointed out, in practical situations the conductivity γtypically has discontinuities, so it is not clear what we mean by $\nabla \cdot \gamma \nabla u$ in this case. We overcome this problem by solving it weakly.

2.1. Weak derivatives. Let's recall the Hilbert space of (equaivalence classes) of functions which are sqare integrable over Ω:

$$L^2(\Omega) = \{u;\ |u|^2 \text{ integrable and } \textstyle\int_\Omega |u|^2\, dx < \infty\}.$$

This becomes a Hilbert–space, i.e. a complete inner–product space when equipped with the norm induced by the inner product

$$(u, v) = \int_\Omega u\overline{v}\, dx.$$

In order to be able to deal with irreregular coefficients we generalize differentiation to a wider class of functions. Let's assume first that $f \in C^1(\overline{\Omega})$ and $\phi \in C_0^1(\Omega)$. Then by the generalization of the fundamental theorem of calculus (1.1) applied to the product $f\phi$ we get

$$\int_\Omega \frac{\partial f}{\partial x_i} \phi\, dx = -\int_\Omega f \frac{\partial \phi}{\partial x_i}\, dx,$$

which is an integration by parts formula. The integral over $\partial\Omega$ vanishes since $\phi = 0$ in a neighborhood of $\partial\Omega$. We generalize this as follows:

Definition 2.1.1. *A function $u \in L^2(\Omega)$ is said to have a weak derivative (in L^2) with respect to variable x_j if there is a $v \in L^2(\Omega)$ such that*

$$\int_\Omega v\phi\, dx = -\int_\Omega f \frac{\partial \phi}{\partial x_i}\, dx, \text{ for all } \phi \in C_0^\infty(\Omega).$$

We denote $v = \partial u/\partial x_i$, and call it the weak x_j–derivative *of u.*

Every differentiable function obviously has weak x_j–derivatives, and they are equal to ordinary derivatives. It is also easy to see using Lebesgue's theorem that weak derivatives are unique if they exsist. However, one should be somewhat carefull when dealing with them, think for example under what conditions does the product rule of differentiation hold?

Let's introduce the following multi–indice notation: if $\alpha = (\alpha_1, \dots, \alpha_n)$ is a n–tuple of nonnegative integers, we call $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ its length, and define

$$\frac{\partial^\alpha \phi}{\partial x^\alpha} = \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_n} \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

We can now define (integer order) Sobolev–spaces on Ω.

Definition 2.1.2. *Let k be a nonnegative integer. The* Sobolev–space of order k, $H^k(\Omega)$, *consists of all elements of $L^2(\Omega)$ having weak derivatives up to order k in L^2, i.e. all those $u \in L^2(\Omega)$ for which $\partial^\alpha u/\partial x^\alpha$ exists and belongs to $L^2(\Omega)$ for all multi–indices $|\alpha| \leq k$.*

It is easy to see that $H^k(\Omega)$ becomes a Hilbert–space when equipped with the inner product

$$(u, v)_k = \sum_{|\alpha| \leq k} (\frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\alpha v}{\partial x^\alpha}).$$

We will be mostly using the space $H^1(\Omega)$. Note also that H^0 is just L^2.

2.2. Spaces with vanishing Dirichlet–data. Consider the Sobolev–space $H^1(\Omega)$. We want to consider the subspace consisting of all those u for which the restriction to $\partial\Omega$ is equal to zero. Unfortunately, for a general L^2–function the restriction to the boundary does not make sense since the boundary has Lebesgue–measure zero. If the function has more regularity, for example if it is continuous then of course there are no problems: one can prove the so called Sobolev–embedding theorem which says that $H^k(\Omega)$ can be intepreted as a subspace of the space of continuous functions if k is bigger than the integer part of $n/2$. However, for H^1 this doesn't help.

Definition 2.2.1. *Define the subspace of functions with vanishing Dirichlet–data,* $H_0^1(\Omega)$, *as the closure of* $C_0^\infty(\Omega)$ *in the* $(\cdot,\cdot)_1$*–norm.*

Of course, the restriction of the $(\cdot,\cdot)_1$–norm defines a Hilbertian norm in $H_0^1(\Omega)$, but we can actually do better. For this we need an elementary version of the Poincaré's lemma.

Proposition 2.2.2. *There exsists a constant* $C(\Omega)$ *depending only on the domain* Ω *such that*

$$\|u\|_{L^2(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^2(\Omega)} \text{ for all } u \in C_0^\infty(\Omega).$$

Proof. Let R be a rectangle with sides parallel to coordinate axes such that $\Omega \subset R$, and extend u by zero as an element of $C_0^\infty(\Omega)$. Choose any j and let $x \in R$. Let $a_j < b_j$ be such that

$$(x_1, \ldots, a_j, \ldots, x_n),\ (x_1, \ldots, b_j, \ldots, x_n) \in \partial R.$$

Then since u vanishes on ∂R we get

$$u(x) = \int_{a_j}^{x} u(x_1, \ldots, t, \ldots, x_n)\, dt,$$

and by Cauchy–Schwartz inequality

$$|u(x)|^2 \leq |x - a_j| \int_{a_j}^{x} |u(x_1, \ldots, t, \ldots, x_n)|^2\, dt \leq |b_j - a_j| \int_{a_j}^{x} |u(x_1, \ldots, t, \ldots, x_n)|^2\, dt.$$

Integrating this over R then gives

$$\|u\|_{L^2(\Omega)}^2 \leq C(\Omega)\|\partial u/\partial x_j\|_{L^2(\Omega)}^2,$$

where $C(\Omega)$ is the diameter of Ω in the j:th coordinate direction. This proves the claim. □

This has the following important corollary:

Corollary 2.2.3. *Assume that $\gamma \in L^\infty(\Omega)$ and that it is bounded away from zero in norm, i.e. there is a positive constant M such that*

$$\frac{1}{M} \leq \gamma(x) \leq M, \text{ for almost all } x \in \Omega. \tag{2.7}$$

Then on the closed subspace $H_0^1(\Omega)$ the conjugate linear pairing

$$(u, v) \mapsto (\nabla u, \gamma \nabla v)$$

defines an inner product $(\cdot,\cdot)_\gamma$, and the norm $\|\cdot\|_\gamma$ determined by it is equivalent with H^1–norm (in $H_0^1(\Omega)$).

Proof. If $u \in C_0^\infty(\Omega)$, then

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq (C(\Omega)^2 + 1)M^2\|u\|^2_\gamma,$$

by the Poincare lemma and the condition (2.7). On the other hand,

$$\|u\|^2_\gamma \leq M^2\|\nabla u\|^2 \leq M^2\|u\|^2_{H^1(\Omega)},$$

and hence the norms are equivalent in $C_0^\infty(\Omega)$. This proves the claim. □

2.3. *Existence of weak solutions.*

Assume for a moment that γ satisfies (2.7) and is in addition differentiable. If $u \in C_0^2(\Omega)$ solves

$$\nabla \cdot \gamma \nabla u = f \in L^2(\Omega),$$

we have by integrating by parts that

$$(\nabla u, \gamma \nabla \phi) = (f, \phi), \text{ for all } \phi \in C_0^\infty(\Omega).$$

This motivates the following definition:

Definition 2.3.1. *A function $u \in H_0^1(\Omega)$ is a weak solution of the Dirichlet–problem*

$$\nabla \cdot \gamma \nabla u = f \in L^2(\Omega), \; u|_{\partial\Omega} = 0, \tag{2.8}$$

if we have

$$(\nabla u, \gamma \nabla \phi) = (f, \phi), \text{ for all } \phi \in H_0^1(\Omega). \tag{2.9}$$

Notice that above in the left hand side we have the inner product $(u, \phi)_\gamma$.

We have an immediate, easy and important result:

THEOREM 2.3.2. *The Dirichlet problem (2.8) has a unique weak solution $u \in H_0^1(\Omega)$.*

Proof. Let $\lambda_f(\phi) = (f, \phi)$ for $\phi \in H_0^1(\Omega)$. This defines a bounded linear functional on $H_0^1(\Omega)$ for any $f \in L^2(\Omega)$, and hence by the Riesz representation theorem there exsists a unique $u \in H_0^1(\Omega)$ such that

$$(u, \phi)_\gamma = \lambda_f(\phi), \text{ for all } \phi \in H_0^1(\Omega),$$

but this is precisely (2.9). □

REMARK 2.3.3. In many instances one can say even more: for example assume that $f \in C^\infty(\overline{\Omega})$ and that the conductivity γ is also a C^∞–function satisfying (2.7). Then the weak solution given by the above theorem belongs actually to $C^\infty(\overline{\Omega})$. The proof of this is not trvial, and we refer the reader for example to [6] and [8].

3. Boundary values in the Dirichlet–problem and the Dirichlet–Neumann–map. So far we have only considered vanishing Dirichlet–data, but for applications to EIT this is not enough: to model our measurements we need non–trivial boundary values. To see what is the correct way of introducing them into the weak formulation, we start by recalling the basic properties of Fourier–transform and then introduce more general Sobolev–spaces.

3.1. Sobolev–spaces for real smoothness indices. For $u \in L^1(\mathbb{R}^n)$ the Fourier–transform $\widehat{u}$ is defined by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi\rangle} \, u(x) \, dx, \ \xi \in \mathbb{R}^n.$$

Sometimes we also denote the Fourier–transform of u by $\mathcal{F}u$. In general, if $u \in L^1$, the Fourier–transform is just a bounded and continuous function. This follows easily from the Dominated Convergence theorem, and the obvious estimate

$$|\widehat{u}(\xi)| \leq \|u\|_{L^1}, \ \xi \in \mathbb{R}^n.$$

If in addition to this we also have that $\widehat{u} \in L^1$, we have an inversion formula:

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\, \widehat{u}(\xi)\, d\xi, \text{ for almost all } x \in \mathbb{R}^n.$$

If $u \in C_0^\infty(\mathbb{R}^n)$, then it is easy to see that $\widehat{u}$ is smooth[1], and $|\xi|^N \widehat{u}(\xi)$ is bounded for all nonegative integers N, hence we may integrate by parts to get

$$\widehat{\frac{\partial u}{\partial x_j}} = i\xi_j \widehat{u}(\xi), \; \xi \in \mathbb{R}^n,$$

and

$$\widehat{x_j u}(\xi) = i\frac{\partial \widehat{u}}{\partial \xi_j}(\xi).$$

hence on the Fourier–side differentiation with respect to x_j corresponds to the multiplication with monomial $-i\xi_j$, and vice versa, multiplication with monomial x_j corresponds in the Fourier–side to differentiation with respect to ξ_j. This will be important for us since it makes it possible to characterize the space $H^k(\mathbb{R}^n)$ in terms of the behaviour of its Fourier–transform. But first we need to recall the following important result: *the map*

$$C_0^\infty(\mathbb{R}^n) \ni u \mapsto \widehat{u} \in L^1 \cap C^\infty$$

has a unique extension to an invertible bounded map $\mathcal{F} : L^2 \to L^2$ *with norm* $(2\pi)^{n/2}$. For a proof see for example [10]. Hence $u \in H^1(\mathbb{R}^n)$ precisely when $\widehat{u} \in L^2$ and $\xi_j \widehat{u} \in L^2$ for all j, i.e. $(1 + |\xi|^2)^{1/2}\, \widehat{u} \in L^2$. Furthermore, the norms

$$\|u\|_1^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$$

and

$$\|u\|_1' = \|(1 + |\xi|^2)^{1/2} \widehat{u}\|_{L^2}$$

are equivalent. Similarly, the space $H^k(\mathbb{R}^n)$ consists of those $u \in L^2$ for which the norm

$$\|u\|_k' = \|(1 + |\xi|^2)^{k/2} \widehat{u}\|_{L^2}$$

is finite. This makes the following definition reasonable.

[1] Infact it has an analytic extension to whole $\mathbb{C}^n$.

Definition 3.1.1. (i) *For any $s \geq 0$ we define*

$$H^s(\mathbb{R}^n) = \{u \in L^2; \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\widehat{u}(\xi)|^2 \, d\xi < \infty\}.$$

This is a Hilbert space when equipped with the innerproduct

$$(u,v)_{H^s} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, d\xi.$$

(ii) *For $s < 0$ we define the space $H^s(\mathbb{R}^n)$ as the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|_{H^s} = \|(1+|\xi|^2)^{s/2}\widehat{u}\|_{L^2}$.*

It is not too difficult to see that C_0^∞ is dense in H^s also for $s \geq 0$, but we will leave this as homework. The definition for $s < 0$ given is not really satisfactory, since the closure can be difficult to characterize. In fact, H^s will then contain elements which cannot anymore be interpreted as functions. They are (tempered) distributions, but in order to avoid this terminology we have given the definition in this form. We can however give the following characterization that makes H^s for $s < 0$ somewhat more concrete. Consider the conjugate linear pairing

$$C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \ni (\phi, \psi) \mapsto \int_{\mathbb{R}^n} \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} \, d\xi. \tag{3.10}$$

Now by the Cauchy–Schwartz–inequality we get for any $s \in \mathbb{R}$ that

$$|\int_{\mathbb{R}^n} \phi(\xi)\overline{\psi(\xi)} \, d\xi| \leq \|(1+|\xi|^2)^{s/2} \, \widehat{\phi}(\xi)\|_{L^2} \, \|(1+|\xi|^2)^{s/2} \, \widehat{\phi}(\xi)\|_{L^2} = \|\phi\|_{H^s}\|\psi\|_{H^{-s}}.$$

Since this conjugate linear pairing is nondegenerate i.e. $\int \phi\overline{\psi} \, dx = 0$ for all $\psi \in C_0^\infty$ implies that $\phi = 0$, we can identify $H^{-s}(\mathbb{R}^n)$with the dual of $H^s(\mathbb{R}^n)$.

Finally we give a version of the Sobolev–embedding theorem:

Proposition 3.1.2. *If $s > n/2 + k$, then $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$ and the embedding is continuous*[2].

Proof. Assume that $2s > n$ and consider the integral $\int |\widehat{u}(\xi)| \, d\xi$. To conclude that u is continuous it is enough to prove that this is finite, for then we can apply the Dominated Convergence theorem to the inverse Fourier–transform

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} \, \widehat{u}(\xi) \, d\xi.$$

[2]So this result says that under the assumption on s it is possible to find a C^k–representative from the equivalence class of L^2–functions determined by any $u \in H^s$.

But we have using Cauchy–Schwartz inequality that

$$\int |\widehat{u}(\xi)|\,d\xi \leq \left(\int (1+|\xi|^2)^{-s}\,d\xi\right)\left(\int (1+|\xi|^2)^s)\,|\widehat{u}(\xi)|^2\,d\xi\right) < \infty,$$

since $2s > n$ and $u \in H^s$. The result for general $k \in \{0,1,2,\ldots\}$ follows from this by considering derivatives for $\widehat{\partial u/\partial x_j} = -i\xi_j\widehat{u}$. □

3.2. The Trace–theorem. In this subsection we show that if $s > 1/2$ elements of $H^s(\mathbb{R}^{n+1})$ have boundary values on the hyperplane $x_{n+1} = 0$. Below $n \geq 1$.

Proposition 3.2.1. *For all $s > 1/2$ the mapping*

$$C_0^\infty(\mathbb{R}^{n+1}) \ni \phi \mapsto \phi|_{x_{n+1}=0} \in C_0^\infty(\mathbb{R}^n)$$

has an extension to a bounded linear map $\mathrm{tr} : H^s(\mathbb{R}^{n+1}) \to H^{s-1/2}(\mathbb{R}^n)$ *called the* trace map.

Proof. So, we have to show that for any $s > 1/2$ there is a constant $C(s)$ such that

$$\|\mathrm{tr}(\phi)\|_{H^{s-1/2}(\mathbb{R}^n} \leq C(s)\|\phi\|_{H^s(\mathbb{R}^{n+1})},\ \ \phi \in C_0^\infty.$$

Let now $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ and denote $\varphi(x) = \mathrm{tr}(\phi)(x) = \phi(x,0)$. Let now $\mathcal{F}_n$ denote the n–dimensional Fourier–transform in x–varibles, $\mathcal{F}_1$ the one–dimensional Fourier–transform in x_{n+1}–variable and finally $\mathcal{F}$ be the $(n+1)$–dimensional Fourier–transform. Then

$$\mathcal{F}\phi(\xi,\xi_{n+1}) = \mathcal{F}_1\{\mathcal{F}_n\phi(\xi,\cdot)\}(\xi_{n+1}),$$

so using the one–dimensional inverse Fourier–transform we get

$$\mathcal{F}_n\phi(\xi,x_{n+1}) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_{n+1}\xi_{n+1}}\,\widehat{\phi}(\xi,\xi_{n+1})\,d\xi_{n+1}.$$

Hence we get

$$\widehat{\varphi}(\xi) = \mathcal{F}_n\phi(\cdot,0)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\phi}(\xi,\xi_{n+1})\,d\xi_{n+1}.$$

Hence we can estimate using Cauchy–Schwarz that

$$|\widehat{\varphi}(\xi)|^2 \leq \frac{1}{4\pi^2}\int_{\mathbb{R}}(1+|\xi|^2+\xi_{n+1}^2)^{-s}\,d\xi_{n+1}\int_{\mathbb{R}}(1+|\xi|^2+\xi_{n+1}^2)^s\,|\widehat{\phi}(\xi,\xi_{n+1})|^2\,d\xi_{n+1}.$$

Let's compute the first integral above:

$$\begin{aligned}\int_{\mathbb{R}} (1+|\xi|^2+\xi_{n+1}^2)^{-s}\, d\xi_{n+1} &= (1+|\xi|^2)^{-s+1/2} \int \left(1+\frac{|\xi_{n+1}|^2}{1+|\xi|^2}\right)^{-s} (1+|\xi|^2)^{-1/2}\, d\xi_{n+1} \\ &= (1+|\xi|^2)^{-s+1/2} \int (1+u^2)^{-s}\, du = C(1+|\xi|^2)^{-s+1/2}.\end{aligned}$$

Combining these we get

$$(1+|\xi|^2)^{s-1/2} |\widehat{\varphi}(\xi)|^2 \le \int_{\mathbb{R}} (1+|\xi|^2+\xi_{n+1}^2)^s\, |\widehat{\phi}(\xi,\xi_{n+1})|^2\, d\xi_{n+1},$$

and integrating this with respect to ξ we get

$$\|\varphi\|_{H^{s-1/2}(\mathbb{R}^n)} \le C\|\phi\|_{H^s(\mathbb{R}^{n+1})},$$

which is the claim. □

One can extend this to the half–space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n;\, x_n > 0\}$ as follows. First, we define $H^s(\mathbb{R}^n_+) = \{u|_{\mathbb{R}^n_+};\, u \in H^s(\mathbb{R}^n)\}$ and equip this with the (quotient) norm

$$\|v\|_{H^s(\mathbb{R}^n_+)} = \inf_{u|_{\mathbb{R}^n_+}=v} \|u\|_{H^s(\mathbb{R}^n)}.$$

It is not too difficult to see that for nonnegative integer values of s this gives on equivalent norm in $H^k(\mathbb{R}^n_+)$. Also one has the following results, whose proof will be left for the first exercise collection:

- Assume that $s > 1/2$ and that $u \in H^s(\mathbb{R}^n)$ is such that $u|_{\{}x_n < 0\} = 0$. Then $tr(u) = 0$. Consequently, there is a well defined map $\mathrm{tr} : H^s(\mathbb{R}^n_+) \to H^{s-1/2}(\mathbb{R}^{n-1})$defined as

  $$\mathrm{tr}(v) = \mathrm{tr}(u),\ \text{where } v \in H^s(\mathbb{R}^n_+) \text{ and } u \in H^s(\mathbb{R}^n)\text{is such that } u|_{\mathbb{R}^n_+} = v.$$

- For $s = 1$ the kernel of tr is precisely $H^1_0(\mathbb{R}^n_+)$.

- For $s = 1$ there is a continuous right inverse E of tr, i.e. a bounded linear map $E : H^{1/2}(\mathbb{R}^{n-1}) \to H^1(\mathbb{R}^n_+)$ such that $\mathrm{tr}(Ef) = f$ for any $f \in H^{1/2}(\mathbb{R}^{n-1})$.

A consequence of this is that one can identify $H^{1/2}(\mathbb{R}^{n-1})$ with the quotient space $H^1(\mathbb{R}^n_+)/H^1_0(\mathbb{R}^n_+)$. Also, if $\Omega \subset \mathbb{R}^n$ is a bounded C^∞–domain, there is a space $H^{1/2}(\partial\Omega)$ which consists of all those elements $f \in L^2(\partial\Omega)$ which are locally in $H^{1/2}(\mathbb{R}^{n-1})$ i.e. if $\phi \in C^\infty(\partial\Omega)$ is a function which is supported in a neighborhood which is diffeomorphic to an open set of $\mathbb{R}^{-1}$ via a map ψ, then $(\phi f) \circ \psi \in H^{1/2}(\mathbb{R}^{n-1})$. Again, we have the following facts:

- Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞–domain. There is a well defined map $\mathrm{tr} : H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ which is an extension of the map $u \mapsto u|_{\partial\Omega}$ (defined for smooth u).
- The kernel of tr is precisely $H_0^1(\Omega)$.
- There is a continuous right inverse E of tr, i.e. a bounded linear map $E : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ such that $\mathrm{tr}(Ef) = f$ for any $f \in H^{1/2}(\partial\Omega)$.

The proof follows almost trivially from the corresponding result for the half–space $\mathbb{R}^n_+$, once one has shown that the space $H^{1/2}(\partial\Omega)$ is well–defined, i.e. that H^s–spaces are invariant under smooth diffeomorphisms. This is easy to prove using the case $s = k$, k a nonnegative integer, and complex interpolation. See for example [5], or the first chapter of [14].

3.3. The Dirichlet–Neumann–map. In this subsection we are finally ready to define the Dirichlet–Neumann–map in a precise way. Let $f \in H^{1/2}(\partial\Omega)$, and consider the following Dirichlet–problem: Find a $u \in H^1(\Omega)$ such that

$$(u, \phi)_\gamma = 0 \;\text{ for all } \phi \in H_0^1(\Omega), \quad \mathrm{tr}(u) = f. \tag{3.11}$$

THEOREM 3.3.1. *Assume that $\gamma \in L^\infty(\Omega)$ satisfies the ellipticity condition* (2.7)*. Then the above problem* (3.11) *has a unique solution $u \in H^1(\Omega)$.*

Proof. Let's first prove the uniqueness. If u solves (3.11) with $f = 0$, then by the results cited in the last subsection we actually have $u \in H_0^1(\Omega)$, and this is then a solution of (2.9) with $f = 0$, and hence by the Theorem (2.3.2) it must vanish.

The existence needs a bit more work: Let $F = Ef \in H^1(\Omega)$be the extension of f. We need to make sense of the term $\nabla \cdot \gamma\nabla F$. This we do as follows. We define it to belong to the dual of $H_0^1(\Omega)$ by identifying it with the linear functional, depending linearly on F,

$$\lambda(F) : H_0^1(\Omega) \ni \phi \mapsto -(\gamma\nabla\phi, \nabla F).$$

Then again by the Riesz–representation theorem there is a unique $U \in H_0^1(\Omega)$ satisfying

$$(\phi, U)_\gamma = \lambda(F)(\phi) \text{ for all } \phi \in H_0^1(\Omega).$$

Then $u = U + F$ will be a weak solution of (3.11). $\square$

To define the Dirichlet–Neumann–map, we need to define the normal derivative of a solution to (3.11) weakly. This can be done as follows. Assuming everything to be smooth, we get from Green's theorem

$$\int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu}\, h\, dS = \int_{\Omega} (\nabla \cdot \gamma \nabla u)\, v \, + \langle \nabla u, \nabla v\rangle\, dx,$$

where $v|_{\partial\Omega} = h$, and if $\nabla \cdot \gamma \nabla u = 0$ in Ω we get

$$\int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu}\, h\, dS = \int_{\Omega} \gamma \langle \nabla u, \nabla v\rangle\, dx. \tag{3.12}$$

Now by the Trace–theorem we can estimate

$$|\int_{\Omega} \gamma \langle \nabla u, \nabla v\rangle\, dx| \leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \leq C_1\|f\|_{H^{1/2}(\partial\Omega)}\|h\|_{H^{1/2}(\partial\Omega)}$$

and hence the bilinear form

$$(f,h) \mapsto \int_{\partial\Omega} \gamma \frac{\partial u}{\partial \nu}\, h\, dS$$

where u is a solution of (3.11) defines a bounded linear functional on $H^{1/2}(\partial\Omega)$, i.e. an element of the dual space $H^{-1/2}(\partial\Omega)$ with norm $\leq C_1\|f\|_{H^{1/2}(\partial\Omega)}$. We call this element the *normal derivative* of u. Note that is essential that u satisfies an equation of the type (3.11); general elements of $H^1(\Omega)$ do not have normal derivatives on $\partial\Omega$.

Definition 3.3.2. *The bounded linear map*

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \ni f \mapsto \gamma \frac{\partial u}{\partial \nu} \in H^{-1/2}(\partial\Omega),$$

where $u \in H^1(\Omega)$ is the unique solution of (3.11), *is called* the Dirichlet–Neumann–map *determined by the conductivity λ.*

As said already the Dirichlet–Neumann map has also the weak definition

$$(\Lambda_\gamma f, h)_{\partial\Omega} = \int_{\Omega} \gamma \langle \nabla u, \nabla v\rangle\, dx,$$

where $v \in H^1(\Omega)$ has trace h. We will be using this representation almost exclusively.

4. Linearization of the map $\gamma \mapsto \Lambda_\gamma$ The mathematical formulation of the EIT is as follows: explain when the map Λ_γ determines the conductivity γ uniquely, and also give a method for reconstructing γ from the map Λ_γ. This is a difficult problem that is still not completely understod. The main difficulty is that this map is *nonlinear*. This may not be completely obvious at the moment, but we will demonstrate this in a moment when we compute its derivative.

4.1. Differentiation of Banach–valued functions. Let $I \subset \mathbb{R}$ be an open interval and $t_0 \in I$. A map $f : I \to \mathbb{R}$ is differentiable at t_0 if there is a real number a such that

$$f(t_0 + h) = f(t_0) + ah + o(h).$$

This condition uniquely determines a, which we denote by $f'(t_0)$. If one wants to generalize this to more general maps a different point of view is useful: a real number $f'(t_0)$ also determines a unique linear map $h \mapsto f'(t_0)h$. One may view this as the *best linear approximation to* f at t_0. Let now E and F be Banach–spaces[3] with norms $\|\cdot\|_E$ and $\|\cdot\|_F$ respectively. Let $U \subset E$ be an open subset and $f : U \to F$ a map. We say that f is differentiable at x_0 if there is a bounded linear map $A \in \mathcal{L}(E, F)$ such that for $h \in E$ small enough in norm we have

$$\frac{\|f(x_0 + h) - f(x_0) - Ah\|_F}{\|h\|_E} \to 0 \text{ as } \|h\|_E \to 0..$$

This determines the linear map A uniquely, and we denote it again by $f'(x_0)$ and call it the derivative of f at x_0. Notice that if f is differentiable at x_0, it is also continuous at x_0.

Example 4.1.1. (i) Let $E = F = L^2(\mathbb{R})$ and consider the mapping $S : f \mapsto f^2$. By the Cauchy–Schwartz theorem this is a continuous function $L^2 \to L^2$, and it is not linear. Fix $f_0 \in L^2$ and let's compute $S'(f_0)$. Now

$$S(f_0 + h) - S(f_0) = (f_0 + h)^2 - f_0^2 = 2f_0h + h^2,$$

and hence $S'(f_0)h = 2f_0h,\ h \in L^2$.

[3]This means that they are normed vector spaces which are complete with respect to this norm.

(ii) Consider the map

$$Q : H^1(\Omega) \times H^1(\Omega) \ni (u, v) \mapsto \int_\Omega \gamma \langle \nabla u, \nabla v \rangle \, dx.$$

Now $E = H^1(\Omega) \times H^1(\Omega)$ and $F = \mathbb{C}$. Here $\gamma \in L^\infty(\Omega)$ is arbitrary. This map in not linear, however it is bilinear. Let's compute its derivative at (u, v). Let $h = (h_1, h_2) \in E$. Then

$$Q(u + h_1, v + h_2) - Q(u, v) = Q(h_1, v) + Q(u, h_2) + O(\|h\|^2),$$

so the derivative at (u, v) is the map

$$(h_1, h_2) \mapsto \int_\Omega \gamma(\langle \nabla h_1, \nabla v \rangle + \langle \nabla u, \nabla h_2 \rangle) \, dx.$$

This derivative has all the usual properties: it is linear, i.e.

$$(f + g)'(x_0) = f'(x_0) + g'(x_0), \ (\alpha f)'(x_0) = \alpha f'(x_0)$$

whenever f and g are defined in a neghborhood of x_0, here $\alpha \in \mathbb{C}$. Note also that a linear map is its own derivative: if $A : E \to F$ is bounded and linear, then $A'(x_0)h = Ah$ for all $x_0, \ h \in E$. Also there is a chain rule: Let E, F and G be Banach spaces and assume $U \subset E$and $V \subset F$ are open subsets, $x_0 \in U$, $y_0 \in V$, and $f : U \to F$ differentiable at x_0 and $g : V \to G$ differentiable at $y_0 = f(x_0)$. Then $g \circ f$ is defined in a neighborhood of x_0 and differentiable at x_0 with derivative

$$(g \circ f)'(x_0)h = g'(y_0) f'(x_0)h.$$

Note that this makes sense because $f'(x_0) \in \mathcal{L}(E, F)$, $g'(y_0) \in \mathcal{L}(F, G)$ so the product map $g'(y_0)f'(x_0) \in \mathcal{L}(E, G)$. This has the following important consequence. Assume (with previous notations) that $f : U \to V$is a *diffeomorphism*, i.e. it is differentiable at all points $x \in U$ and it has a differentiable inverse $f^{-1} : V \to U$. Then

$$(f^{-1})'(y_0) f'(x_0) = (f^{-1} \circ f)'(x_0) = \mathrm{id}'_U(x_0) = \mathrm{I}_E$$

and similarly

$$f'(x_0)(f^{-1})'(y_0) = \mathrm{I}_F.$$

Hence the derivatives are invertible linear maps. So, invertibility of the derivative $f'(x_0)$ gives a necessary condition for the local invertibility of a map f at x_0 if the map $x \mapsto f'(x)$ is continuous in neighborhood of x_0. For more on differential calculus in Banach–spaces see [4] or [7],

4.2. Calderón's question. Let's return back to the Dirichlet–Neumann–map

$$(\Lambda_\gamma f, h)_{\partial\Omega} = \int_\Omega \gamma \langle \nabla u, \nabla v \rangle \, dx,$$

where $v \in H^1(\Omega)$ has trace h and $u \in H^1(\Omega)$ solves

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \ u|_{\partial\Omega} = f. \tag{4.13}$$

By the polarization identity Λ_γ is determined by its values on the diagonal

$$(\Lambda_\gamma f, f)_{\partial\Omega} = \int_\Omega \gamma \langle \nabla u, \nabla u \rangle \, dx.$$

Hence it is enough to consider the quadratic form

$$Q_\gamma(f) = \int_\Omega \gamma |\nabla u|^2 \, dx,$$

where u solves (4.13). In Calderon's own words [2]: *The problem is then to decide whether γ is uniquely determined by Q_γ and to calculate γ in terms of Q_γ, if indeed γ is determined by Q_γ.* What Calderon could show was that the derivative of the map $\gamma \mapsto Q_\gamma$ has an injective differential at a constant conductivity. Next we explain his ingenious argument.

4.3. Equation for the remainder. Let γ be a conductivity satisfying the ellipticity condition (2.7), and assume that it is of the form

$$\gamma = 1 + \delta, \tag{4.14}$$

where δ will eventually be small in L^∞–norm. The first task is to expand the solution u of the Dirichlet–problem

$$\nabla \cdot (1 + \delta) \nabla u = 0 \text{ in } \Omega, \ u|_{\partial\Omega} = f. \tag{4.15}$$

in terms of powers of δ. By Theorem 3.3.1, applied with $\gamma = 1$, there is a unique $u_0 \in H^1(\Omega)$ solving

$$\Delta u_0 = 0 \text{ in } \Omega,\ u_0|_{\partial\Omega} = f.$$

Write $u = u_0 + v(\delta)$. Then

$$\nabla \cdot (1+\delta)\nabla(u_0 + v(\delta)) = \Delta v(\delta) + \nabla \cdot \delta \nabla u_0 + \nabla \cdot \delta \nabla v(\delta),$$

and hence u satisfies (4.14) if and only if $v(\delta) \in H_0^1(\Omega)$satisfies

$$\Delta v(\delta) + \nabla \cdot \delta \nabla v(\delta) = -\nabla \cdot \delta \nabla u_0. \tag{4.16}$$

Notice that the condition $v \in H_0^1(\Omega)$ guarantees that u has boundary value f. Now, as already noted in the proof of Theorem 3.3.1, any $F \in H^1(\Omega)$ determines a linear functional $\lambda(F) = \nabla \cdot \delta \nabla F \in H^{-1}(\Omega)$ of $H_0^1(\Omega)$ with norm $\leq \|\delta\|_{L^\infty} \|F\|_{H^1(\Omega)}$. As also noted in the proof of Theorem 3.3.1, there is a bounded linear operator

$$E : H^{-1}(\Omega) \to H_0^1(\Omega),$$

where $E\mu \in H_0^1(\Omega)$ is the unique weak solution of

$$\Delta w = \mu \text{ in } \Omega,\ \ w|_{\partial\Omega} = 0.$$

Hence we can write the equation (4.16) in the equivalent form

$$v(\delta) + E\lambda(v(\delta)) = -E\lambda(u_0). \tag{4.17}$$

This is a linear equation[4] for the unknown $v(\delta) \in H_0^1(\Omega)$ that we can easily solve when $\|\delta\|_{L^\infty}$ is small enough.

4.4. Neumann–series. The inversion of (4.17) is based on the following result, which is often called *Neumann–series* after a 19th–century mathematician Carl Neumann.

Proposition 4.4.1. *Let H be a Hilbert–space and $A : H \to H$ a bounded linear operator with $\|A\| < 1$. Then the operator $I - A : H \to H$is invertible with the inverse given by the norm–convergent series*

$$(I - A)^{-1} = I + A + A^2 + \ldots + A^n + \ldots,$$

[4]Actually, this is a nice Fredholm–type integral equation on $H_0^1(\Omega)$.

and having a norm estimate

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}.$$

Proof. This is based on the summation of a geometric series. Let's start by noticing that

$$(I-A)(I+A+\ldots+A^n) = I - A^{n+1} = (I+A+\ldots+A^n)(I-A). \tag{4.18}$$

Now for the a partial sum $\sum_{k=m}^{n} A^k$ we have an estimate

$$\|\sum_{k=m}^{n} A^k\| \leq \sum_{k=m}^{n} \|A\|^k \leq \|A\|^m(1-\|A\|^{n-m+1})/(1-\|A\|), \tag{4.19}$$

so $\|A\| < 1$ implies that this converges to zero as m, n tend to infinity. So, the series $\sum A^k$ is convergent in $\mathcal{L}(H)$, and letting $n \to \infty$ in (4.18) shows that is defines an inverse of $I - A$. The claim concerning the norm of $(I-A)^{-1}$ follows from (4.19) by choosing $m = 0$ and letting $n \to \infty$. □

Now we can easily solve equation (4.17), at least for δ small enough:

Proposition 4.4.2. *Assume $\|\delta\| < 1/\|E\|$. Then the equation* (4.17) *has a unique solution in $H_0^1(\Omega)$, and it is given by the norm–convergent series*

$$v(\delta) = \sum_{k=0}^{\infty}(-1)^k G^k U_0,$$

where for any $F \in H^1(\Omega)$, $G(F) = E\lambda(F)$, $U = -E\lambda(u_0)$ and $\|G^k\| \leq C\|\delta\|_{L^\infty}^k$.

Proof. We apply the Neumann–series argument with $H = H_0^1(\Omega)$ and $Av = -Gv\,(= -E\lambda(v))$. Then

$$\|Av\|_{H_0^1(\Omega)} \leq \|E\|\|\delta\|_{L^\infty}\|v\|_{H_0^1(\Omega)},$$

where we now equip $H_0^1(\Omega)$ with the norm

$$\|v\|_{H_0^1(\Omega)}^2 = \|\nabla u\|_{L^2(\Omega)}^2.$$

The condition $\|\delta\|_{L^\infty} < 1/\|E\|$ then implies that $\|A\| < 1$, so the claim follows from Neumann-series since $\|G\| \leq C\|\delta\|_{L^\infty}$. □

Notice, that the existence and uniqueness of the solution is not a new result for us, this is already contained in Theorem 3.3.1. What is important for us that it gives an expansion where we can estimate the terms in terms of powers of $\|\delta\|_{L^\infty}$. We will need this when computing the derivative of $\gamma \mapsto Q_\gamma$.

4.5. Computation of the derivative. In this subsection we are finally able to tell the reader how to compute the derivative of $\gamma \mapsto Q_\gamma$ at $\gamma = 1$.

THEOREM 4.5.1. *For any $f \in H^{1/2}(\partial\Omega)$, we have*

$$\frac{dQ_\gamma}{d\gamma}|_{\gamma=1}(\delta)(f) = \int_\Omega \delta |\nabla u_0|^2 \, dx,$$

where $u_0 \in H^1(\Omega)$is the unique solution of

$$\Delta u_0 = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = f.$$

Before the proof we remark, that Q_γ is defined in the open subset U of L^∞ defined by the ellipticity condition (for all $M \geq 1$), and the image is the Banach space of bounded Bilinear forms on $H^{1/2}(\partial\Omega)$. Thus, the derivative at $\gamma = 1$ ought to be a bounded bilinear map from L^∞ to the space of bounded Bilinear forms on $H^{1/2}(\partial\Omega)$, and this is easily seen to be true.

Proof. Let $f \in H^{1/2}(\partial\Omega)$, and let $u = u_0 + v(\delta) \in H^1(\Omega)$be the solution of (4.15). Then in the notations used in previous subsection,

$$\begin{aligned}
(Q_{1+\delta} - Q_1)(f) &= \int_\Omega (1+\delta)|\nabla u|^2 - |\nabla u_0|^2 \, dx \\
&= \int_\Omega \langle \nabla u_0, \nabla \overline{v(\delta)} \rangle + \langle \nabla v(\delta), \overline{\nabla u_0} \rangle + |\nabla v(\delta)|^2 + \delta |\nabla u_0|^2 \\
&+ \delta \left(\langle \nabla u_0, \nabla \overline{v(\delta)} \rangle + \langle \nabla v(\delta), \nabla \overline{u_0} \rangle + |\nabla v(\delta)|^2 \right) dx.
\end{aligned}$$

Since u_0 is a weak solution of $\Delta u_0 = 0$ in Ω and $v(\delta) \in H^1_0(\Omega)$, we have

$$\int_\Omega \langle \nabla u_0, \nabla \overline{v(\delta)} \rangle + \langle \nabla v(\delta), \nabla \overline{u_0} \rangle \, dx = 0.$$

Also,

$$\left| \int_\Omega |\nabla v(\delta)|^2 + \delta \left(\langle \nabla u_0, \nabla \overline{v(\delta)} \rangle + \langle \nabla v(\delta), \nabla \overline{u_0} \rangle + |\nabla v(\delta)|^2 \right) dx \right| \leq C \|\delta\|^2_{L^\infty},$$

since $\|v(\delta)\|_{H^1_0(\Omega)} \leq C\|\delta\|_{L^\infty}$. Hence we have

$$(Q_{1+\delta} - Q_1)(f) = \int_\Omega \delta |\nabla u_0|^2 \, dx + O(\|\delta\|^2_{L^\infty})$$

which proves the claim. □

5. Injectivity of the derivative at constants. In this section we prove that the derivative of $\gamma \mapsto Q_\gamma$is injective at $\gamma = 1$, and hence injective at all constants. Note that this does not really imply anything usefull about the original nonlinear map near $\gamma = 1$, but the method of the proof is extremely clever, and it can be generalized to work also in the nonlinear case.

5.1. A special family of harmonic function. Assume that

$$\int_\Omega \delta |\nabla u_0|^2 \, dx = 0$$

for all u_0 harmonic in Ω. By polarization this means that

$$\int_\Omega \delta \langle \nabla u, \nabla v \rangle \, dx = 0$$

for all u and v harmonic in Ω. We should hence prove that if δ is orthogonal against innerproducts of all gradients of harmonic functions, it must vanish in Ω. A good way to prove this is to test with a large enough family of harmonic functions with simple enough gradients. Let $\zeta \in \mathbb{C}^n$ and consider a function

$$e_\zeta(x) = e^{-i\langle x, \zeta \rangle}.$$

Now

$$\nabla e_\zeta(x) = -i\zeta \, e_\zeta(x), \quad \Delta e_\zeta(x) = -\langle \zeta, \zeta \rangle \, e_\zeta(x),$$

So e_ζ will be harmonic excactly when $\langle \zeta, \zeta \rangle = 0$. Notice that the only real vector ζ such that $\langle \zeta, \zeta \rangle = 0$ is $\zeta = 0$. Actully, writing ζ in terms of its real and imaginary parts $\zeta = \zeta_R + i\zeta_I$ we get

$$\langle \zeta, \zeta \rangle = |\zeta_R|^2 - |\zeta_I|^2 + 2i \langle \zeta_R . \zeta_I \rangle,$$

so $\langle \zeta, \zeta \rangle = 0$ is and only if $|\zeta_R| = |\zeta_I|$ and $\zeta_R \perp \zeta_I$ and thus with a fixed real part ζ_R there is a codimension 2–manifold where the imaginary part must lie. Note that in dimension 2 there are only two possibilities for ζ_I once the real part ζ_R is fixed. This is already an indication of the fact that the EIT–problem is much harder in dimension two.

5.2. The injectivity. In this section we show that $dQ_\gamma/d\gamma|_{\gamma=1}$ is an injective linear map. To this end, fix a *real* vector $\xi \in \mathbb{R}^n$, and let $\zeta,\ \zeta^* \in \mathbb{C}^n$ be such that

$$\langle \zeta, \zeta \rangle = \langle \zeta^*, \zeta^* \rangle = 0,\ \ \zeta + \zeta^* = \xi. \tag{5.20}$$

This is always possible. We may for example take

$$\zeta_R = \zeta_R^* = \xi/2,\ \ \zeta_I = -\zeta_I^* = |\xi|\omega/2,$$

where ω is any unit vector orthogonal to ξ. Note that in dimension two this exhausts all the possibilities we have, in higher dimensions there is still room left to ask for more. Let's compute the inner–product of the gradients:

$$\langle \nabla e_\zeta, \nabla e_{\zeta^*} \rangle = -\langle \zeta, \zeta^* \rangle e^{-i\langle \zeta+\zeta^*, x\rangle} = -\langle \zeta, \xi - \zeta \rangle e^{-i\langle \xi, x\rangle}.$$

Now

$$\langle \zeta, \xi - \zeta \rangle = \langle \zeta, \zeta \rangle - \langle \xi/2 + i|\xi|\omega/2, \xi \rangle = -|\xi|^2/2,$$

So we see that

$$\int_\Omega e^{-i\langle \xi, x\rangle} \frac{|\xi|^2}{2} \delta(x)\, dx = 0 \text{ for all } \xi \in \mathbb{R}.$$

Intepreting now $\delta \in L^2(\mathbb{R}^n)$ by extending it as zero to $\mathbb{R}^n \setminus \Omega$ we get from the above equations that

$$|\xi|^2\, \widehat{\delta}(\xi) = 0 \text{ for all } \xi \in \mathbb{R},$$

and hence $\widehat{\delta} = 0$ almost everywhere, i.e. $\delta = 0$. This proves the injectivity of the derivative at $\gamma = 1$. Note that the derivative at a general conductivity γ_0 is just as easy to compute:

$$\frac{dQ_\gamma}{d\gamma}|_{\gamma=\gamma_0}(\delta)(f) = \int_\Omega \delta |\nabla u_0|^2\, dx,$$

where $u_0 \in H^1(\Omega)$is the unique solution of

$$\nabla \cdot \gamma_0 \nabla u_0 = 0 \text{ in } \Omega,\ \ \ u|_{\partial\Omega} = f.$$

The difficulty in proving the injectivity of $dQ_\gamma/d\gamma$ at γ_0 is in finding suitable solutions to replace e_ζ. The existence of solutions generalizing these is due to J.Sylvester and G. Uhlmann [13]. Their results will be explained in the second part of the course.

6. Boundary reconstruction. This section deals with the following question: assume we know the Dirichlet–Neumann–map corresponding to the conductivity γ. Is there a way of determining the values of γ and its derivatives on the boundary $\partial\Omega$ directly. The proof we present here follows that given in [9], but with some simplifications because we assume more regularity on $\partial\Omega$ and γ than needed. There is also a very strong pointwise boundary recovery result due to [1], but we use the Nachman–proof because it seems somewhat more easier to adapt to our smooth case.

6.1. Motivation. Below we try to convince the reader that this is an important question, evnthough it is a long way from being a solution to the inverse problem of EIT.

Let's consider first a method called *Layer stripping.* Assume that our domain is the unit disk $\mathbb{D} = \{|z| < 1\}$ of the complex plane. Let γ be a unknown conductivity distribution in $\mathbb{D}$. Briefly, the idea in Layer Stripping is the following: let's for the moment believe that there is way to conveniently recover the $\gamma|_{\partial\mathbb{D}}$ from the Dirichlet–Neumann–map Λ_γ. Use this value as an approximation in a thin layer near the boundary, and *propagate* (we'll explain in moment how) $\Lambda_\gamma$from $\partial\mathbb{D}$ to the boundary $\partial D(r)$ of a smaller disk $D(r)$ of radius $r < 1$. Denote the Dirichlet–Neumann–map on $\partial D(r)$ by $\Lambda_\gamma(r)$. Again compute an (approximative) value of γ near $\partial D(r)$, and march inward to the next layer.

The propagation of Λ_γ is done using the so–called operator valued Riccati-equation

$$\partial_r\Lambda_\gamma(r) + \Lambda_\gamma(r)\gamma^{-1}\Lambda_\gamma(r) + r^{-1}\Lambda_\gamma(r) + r^{-2}P(\gamma, \nabla_\theta) = 0,$$

where we have used the polar coordinates (r, θ) and $P(\gamma, \nabla_\theta)$ is an explicit second order differential operator in the angular variables θ. This method was introduced by E. Somersalo, M. Cheney, D. Isaacson and E. Isaacson in [11]. However, its does not give a good reconstruction far away from the boundary, and also it's convergence properties are poorly understod. However, it is known to converge to the original conductivity in the two–dimensional radially symmetric case. This is due to J. Sylvester [12].

Another reason is the following. Let u satisfy

$$\nabla \cdot \gamma\nabla u = 0 \ \text{ in } \Omega,$$

where $\gamma \in L^\infty(\Omega)$ with a positive lower bound and has two derivatives in $L^\infty(\Omega)$. Then it is a staightforward computation to see (Exercise collection no.2) that $v = \gamma^{1/2}\, u$ satisfies the zero–energy Schrödinger–equation

$$-\Delta v + q(x)\, v = 0 \text{ in } \Omega,$$

where

$$q(x) = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}.$$

Let's consider the Dirichlet–problem

$$-\Delta v + q(x) v = 0 \text{ in } \Omega,\ v|_{\partial\Omega} = f \in H^{1/2}(\partial\Omega).$$

Assuming that this problem has a unique solution $v \in H^1(\Omega)$, i.e. that zero is not an Dirichlet–eigenvalue of $-\Delta + q(x)$ in Ω, we may again define the Dirichlet–Neumann map of $-\Delta + q(x)$ by

$$\Lambda_q f = \frac{\partial v}{\partial \nu} \in H^{-1/2}(\partial\Omega).$$

It is easy to see that this corresponds to the weak definition given by

$$(\Lambda_q f, g) = \int_\Omega q v \overline{w} + \langle \nabla v, \nabla \overline{w} \rangle \, dx$$

where $w \in H^1(\Omega)$ has trace $g \in H^{1/2}(\partial\Omega)$ on $\partial\Omega$. It is now easy to see that maps Λ_γ and Λ_q are connected by

$$\Lambda_q = \gamma^{-1/2}(\Lambda_\gamma + \frac{1}{2}\frac{\partial \gamma}{\partial \nu})\gamma^{-1/2},$$

and hence we know Λ_q once we know γ and $\partial\gamma/\partial\nu$ on the boundary $\partial\Omega$, and vice versa. This makes possible to consider the somewhat easier problem of recovering the potential q from Λ_q. Also, as we'll see later it makes it possible to propagate the Dirichlet–Neumann–map so that we can assume that the conductivity is equal to one in a neighborhood of the boundary.

6.2. A useful boundary identity. In this section we derive the indentity from which we will recover the boundary value of γ. To do this we will need the Neumann–Didirchlet–map of the Laplacian. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a C^2–domain. Consider the the Neumann–problem

$$\Delta w = 0 \text{ in } \Omega,\ \partial w/\partial \nu = h \in H^{-1/2}(\Omega). \tag{6.21}$$

It is easy to see that the above problem has atmost one H^1–solution, modulo constants. Namely, if w solves (6.21) with $h = 0$, we get from Green's formulae

$$-\int_\Omega |\nabla w|^2\, dx = \int_\Omega w\Delta w\, dx + \int -\partial\Omega w \frac{\partial w}{\partial \nu}\, dS = 0,$$

Hence $\nabla w = 0$ in Ω, and thus w is a constant. The existence is also easy to prove, either using weak methods as was the case for the Dirichlet–problem (see for example [8]), or using boundary integral equation methods (see [3]). Note that to guarantee the existence one must have $h \in H_0^{-1/2}(\partial\Omega) = H^{-1/2}(\partial\Omega)/\{1\}$, i.e. h is orthogonal to all constants. Let now

$$R : H_0^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)/\text{constants},\ h \mapsto [w|_{\partial\Omega}],$$

where w is the solution of (6.21) and $[w]$ denotes its equivalence class modulo constants. This is a bounded linear map. The following lemma is useful.

Lemma 6.2.1. *The map R is symmetric, i.e. for all g, $h \in H_0^{-1/2}(\partial\Omega)$ we have*

$$(Rg, h)_{\partial\Omega} = (g, Rh)_{\partial\Omega}.$$

Note that this makes sense since the Neumann–data is orthogonal to all constants, it doesn't matter which representative for Rg or Rh we choose.

Proof. Let v and w be $H^1(\Omega)$–solutions of (6.21) with Neumann–data g and h respectively. Then

$$(Rg, h)_{\partial\Omega} = \int_\Omega \langle \nabla v, \nabla \overline{w}\rangle\, dx = (g, Rh)_{\partial\Omega}$$

since both v and w are weak solutions. □

Now we can formulate the boundary identity that we need:

Proposition 6.2.2. *Let $\Omega \subset \mathbb{R}^n$be a bounded C^2–domain, $n \geq 2$, and assume $\gamma \in C^1(\overline{\Omega})$be a positive function. Then for any $f \in H^{1/2}(\Omega)$ and $h \in H_0^{-1/2}(\Omega)$ we have*

$$((\gamma - R\Lambda_\gamma)f, h)_{\partial\Omega} = \int_\Omega u\langle \nabla\overline{w}, \nabla\gamma\rangle\, dx,$$

where $u \in H^1(\Omega)$ solves $\nabla \cdot \gamma\nabla u = 0$ in Ω and has boundary value f, and w solves (6.21) above.

Proof. (From [9].) This is a straightforward application of Green's formulae: since w solves (6.21) weakly, we have

$$(v|_{\partial\Omega}, h)_{\partial\Omega} = \int_\Omega \langle \nabla v, \nabla \overline{w}\rangle \, dx \tag{6.22}$$

for any $v \in H^1(\Omega)$. Also, $Rh = w_{\partial\Omega}$ modulo constants, so applying the weak definition of Λ_γ we get

$$(\Lambda_\gamma f, Rh)_{\partial\Omega} = \int_\Omega \gamma \langle u, \nabla \overline{w}\rangle \, dx. \tag{6.23}$$

Notice that since $\Lambda_\gamma f$ is a Neumann–boundary value, we have $(\Lambda_\gamma f, 1)_{\partial\Omega} = 0$, so it doesn't matter that w above is only defined up to a constant. Now let's apply (6.22) with $v = \gamma u$, note that since $\gamma \in C^1(\overline{\Omega})$, multiplication by γ defines a bounded linear map $H^1(\Omega) \to H^1(\Omega)$. We then get

$$(\gamma f|_{\partial\Omega}, h)_{\partial\Omega} = \int_\Omega \gamma \langle \nabla u, \nabla \overline{w}\rangle \; + u\langle \nabla \gamma, \nabla \overline{w}\rangle \, dx. \tag{6.24}$$

Now we can use the symmetry of R to write (6.23) in the form

$$(R\Lambda_\gamma f, h)_{\partial\Omega} = \int_\Omega \gamma \langle u, \nabla \overline{w}\rangle \, dx.$$

Subtracting this from (6.22) with $v = \gamma u$ we get

$$(\gamma f - R\Lambda_\gamma f, h)_{\partial\Omega} = \int_\Omega u\langle \nabla \overline{w}, \nabla \gamma\rangle \, dx,$$

which is the claim. □

The essential point in this formula is that the right hand–side does not depend anymore on the derivatives of u. So, somewhat loosely speaking, R approximately diagonalizes Λ_γ, and the diagonal term is just $\gamma|_{\partial\Omega}$.

6.3. A family of oscillatory functions. The proof is based on the following idea: let

$$P(x, D) = \sum_{|\alpha| \le m} p_\alpha(x) D^\alpha, \quad D = -i\nabla, \; p_\alpha \in C^\infty,$$

be a linear differential operator (with smooth coefficients) of order m. Its *principal part* is the formed by all the derivatives of exact order m i.e.

$$P_m(D) = \sum_{|\alpha| = m} p_\alpha(x) D^\alpha.$$

Is there a simple way of recovering the principal part of the operator from its action on some special class of functions? The answer is yes. Let $\lambda > 0$, $\xi \in \mathbb{R}^n$ and note that

$$P(x,D)(e^{i\lambda\langle x,\xi\rangle}) = \sum_{|\alpha|\le m} p_\alpha(x)\lambda^{|\alpha|}\xi^\alpha e^{i\lambda\langle x,\xi\rangle},$$

and hence

$$\lambda^{-m}e^{-i\lambda\langle x,\xi\rangle}P(x,D)(e^{i\lambda\langle x,\xi\rangle}) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha + \text{terms of order } \lambda^{-1}.$$

So, the *principal symbol* $p_m(x,\xi)$ of $P(x,D)$ can be recovered from

$$p_m(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-m}e^{-i\lambda\langle x,\xi\rangle}P(x,D)(e^{i\lambda\langle x,\xi\rangle}).$$

This determines the principal part $P_m(x,D)$ of the operator using the inverse Fourier–transformation. Now, Λ_γ is not a differential operator on $\partial\Omega$. If both γ and $\partial\Omega$ are smooth, it is actually a pseudodifferential operator, but even for these a suitable generalization of the above argument holds.

Fix now $x_0 = (x_0', x_{0,n}) \in \partial\Omega$, and let $B \subset \mathbb{R}^{n-1}$ be an open neighbourhood of x_0', and I an open interval containing $x_{0,n}$ such that there is $\phi \in C^\infty(B)$ so that

$$\Omega \cap U = \{(x', x_n) \in U; x_n < \phi(x')\}, \quad U = B \times I.$$

Let $h \in L^2(\partial\Omega)$ have support in $U \times \partial\Omega$, and for any $\eta \in \mathbb{R}^{n-1}$ define

$$h_\eta(x) = h(x)e^{-i\langle x,\eta\rangle} - \frac{1}{|\partial\Omega|}\int_{U\cap\partial\Omega} h(y)e^{-i\langle y,\eta\rangle}\,dS(y).$$

Here $|\partial\Omega|$ is the measure of $\partial\Omega$. Note that

$$\int_{p\Omega} h_\eta(x)\,dS(x) = 0,$$

since h vanishes outside $U \times \partial\Omega$. These are the oscillatory functions that we use to probe Λ_γ at the point x_0. We need them to have mean zero so that they can be used as Neumann–data.

6.4. A maximum principle. In the boundary recontruction result we will need a maximum principle for the operator $\nabla \cdot \gamma \nabla$. It will be more convenient to consider an operator of more general form. So let $A = (a_{ij}) : \overline{\Omega} \to \mathbb{R}^{n \times n}$ be a C^1–function, i.e. all the component functions $a_{ij} \in C^1(\overline{\Omega})$. Define

$$L_A u := \sum_{i,j} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_i}, \ \ u \in C^2(\Omega). \tag{6.25}$$

We assume below that L_A is *elliptic* in the sense that the matrix $A(x)$ defines a positive definite quadratic form at every point of Ω, i.e. there is a positive constant c such that

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n, \, x \in \Omega. \tag{6.26}$$

PROPOSITION 6.4.1. *If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $L_A u \geq \ (\leq) \, 0$, then u cannot have a local maxima (minima) at any point of Ω.*

Proof. Assume at first that $x_0 \in \Omega$ is a local minima and that $L_A u > 0$. By composing with a suitable affine transformation (Homework), we may assume that $x_0 = 0$ and that $A(0)$ is diagonal at x_0. We can now write

$$L_A u = \sum_{i,j} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_1(x, \nabla)u,$$

where $b_1(x, \nabla)$is a first order linear differential operator with continuous coefficients. Since at the local minima $\nabla u(0) = 0$, we have

$$\sum_{i,j} a_{i,j}(0) \frac{\partial^2 u(x_0)}{\partial x_i \partial x_j} > 0.$$

Thus, if $\lambda_i > 0$ are the eigenvalues of $A(0)$, we have

$$\sum_i \lambda_i \frac{\partial^2 u(0)}{\partial x_i^2} > 0$$

but at a local maxima we must have

$$\frac{\partial^2 u(0)}{\partial x_i^2} \leq 0 \text{ for all } i,$$

so this is a contradiction. Assume now that we only have $L_A u \geq 0$. Still working in coordinates where 0 is a local maximun and $A(0)$ is diagonal, let for any $M > 0$,

$$r_M = e^{M(x_1 + \ldots + x_n)}.$$

Then

$$\begin{aligned} L_A r_M(x) &= e^{M(x_1+\ldots+x_n)}\left(M^2\sum_{i,j} a_{ij}(x) + M\sum_{i,j}\frac{\partial a_{ij}(x)}{\partial x_i}\right) \\ &\geq Me^{M(x_1+\ldots+x_n)}(nM - \sum_{i,j}\|\frac{\partial a_{ij}}{\partial x_i}\|_{L^\infty}) > 0 \end{aligned} \tag{6.27}$$

when M is large enough. Then for any $\varepsilon > 0$ we know that

$$u(0) + \varepsilon \leq \sup_{\partial\Omega} |u(x)| + \varepsilon e^{Mn\,\mathrm{diam}\,(\Omega)}$$

so by letting $\varepsilon \to 0$ we get that

$$u(0) \leq \sup_{\partial\Omega} |u(x)|.$$

The proof of the other claim is similar with obvious changes. □

6.5. A mini–mini course on compact operators. In the reconstruction proof below we will need some basic properties of compact operators. Let H_1 and H_2 be separable Hilbert–spaces. A bounded operator $F : H_1 \to H_2$ is called a *finite rank operator*, if the subspace (of H_2)

$$\mathrm{Im}\,(F) = \{y; y = F(x) \text{ for some } x \in H_1\}$$

is finite dimensional. Recall that a bounded and closed subset of any finite dimensional Hilbert–space is always compact, but that in an infinite dimensional space this is not enough to guarantee compactness. However, if F is a finite rank operator, and $E \subset H_1$ is closed and bounded, then the closure of $F(E)$ will be comapct, since it is a closed and bounded subset of a finite dimensional subspace[5].

Motivated by this we say that a bounded linear map $A : H_1 \to H_2$ is *compact* if the image $A(B_1)$ of the closed unit ball B_1 of H_1 is precompact, i.e. the closure $\overline{A(B_1)}$ is a compact subset of H_2. In the Hilbert–space setting there is another characterization for compactness: operator $A : H_1 \to H_2$ is compact if and only if there is a sequence (F_j) of finite rank operators $F_j : H_1 \to H_2$ such that $\|A - F_j\| \to 0$ as $j \to \infty$. This is not true for compact operators between Banach–spaces: the above definition works equally well if

[5]In a Hilbert–space finite dimensional subspaces are always closed.

H_1 and H_2 are repalced by a pair of Banach-spaces, but the approximation by finite rank operators may fail.

Denote the set of compact operators $H_1 \to H_2$ by $\mathcal{K}(H_1, H_2)$. The important thing about compactness is that $\mathcal{K}(H_1, H_2)$ is a closed subspace of $\mathcal{L}(H_1, H_2)$. And that if one multiplies any bounded operator either from the left or from the right with a compact operator, the product will be a compact operator. This is easy to see using the approximation by finite rank operators. To prove the fact that $\mathcal{K}(H_1, H_2)$ is closed subspace it is useful to know the following result: an operator $A : H_1 \to H_2$ is compact if and only if for any bounded sequence (x_i) of H_1, the sequnce (Ax_i) contains a convergent subsequence.

To sum up, the basic properties are as follows:

- By definition, a bounded linear operator $A : H_1 \to H_2$ is *compact* if the image $A(B_1)$ of the closed unit ball B_1 of H_1 is precompact, i.e. the closure $\overline{A(B_1)}$ is a compact subset of H_2.

- This is equivalent with the following: for any bounded sequence (x_i) of H_1, the sequnce (Ax_i) contains a subsequence which is convergent in H_2.

- For operators acting between Hilbert–spaces, this is also equivalent with the property that there is a sequence of finite–rank operators $A_i : H_1 \to H_2$ such that $\|A - A_i\| \to 0$ as $j \to \infty$.

- If H is a third Hilbert space, $L_1 : H \to H_1$ and $L_2 : H_2 \to H$ are bounded and linear and $A : H_1 \to H_2$ is compact, then also $AL_1 : H \to H_2$ and $L_1 A : H_1 \to H$ are compact.

- The set of compact operators $\mathcal{K}(H_1, H_2)$ forms a closed linear subspace of $\mathcal{L}(H_1, H_2)$.

- In the case when $H = H_1 = H_2$, the set $\mathcal{K}(H)$ of compact operators $H \to H$ is a closed two–sided ideal of the set $\mathcal{L}(H)$ of bounded linear operators $H \to H$.

Next we give two typical examples of compact operators:

Proposition 6.5.1. *Let* $I = [0,1]$, *and* $H = L^2(I)$. *Let* $a : I \times I \to \mathbb{C}$ *be* a continuous *function and*

$$Af(t) = \int_0^1 a(t,s)f(s)\,ds,\ \ f \in L^2(I)\ \textit{and}\ t \in I.$$

Then A *is compact.*

Proof. Let n be a postive integer, and divide I into intervals

$$I_k^n = [k/n, (k+1)/n],\ \ k = 0, \dots, n-1,$$

and similarly divide $I \times I$ into squares

$$Q_{k,l}^n = I_k^n \times I_l^n.$$

Choose points $(x_k^n, y_l^n) \in Q_{k,l}^n$ and define the piecewise constant function

$$a_n(t,s) = \sum_{k,l} a(x_k^n, y_l^n)\chi_{Q_{k,l}^n}(s,t),$$

where $\chi_{Q_{k,l}^n}$ is the characteristic function of the square $Q_{k,l}^n$. Since a is uniformly continuous, for any $\varepsilon > 0$ there is $n(\varepsilon)$ such that $n > n(\varepsilon)$ implies that

$$|a(t,s) - a_n(t,s)| \leq \varepsilon \text{ for all } (t,s) \in Q.$$

Let

$$A_nf(t) = \int_0^1 a_n(t,s)f(s)\,ds,\ \ f \in L^2(I) \text{ and } t \in I.$$

Thus if $n > n(\varepsilon)$ then

$$\|A - A_n\| \leq \varepsilon.$$

On the other hand,

$$\chi_{Q_{k,l}^n} = \chi_{I_k^n}\chi_{I_l^n}$$

and thus

$$A_nf(t) = \sum_{k,l} a(x_k^n, y_l^n)\chi_{I_k^n}(t)\int \chi_{I_{k,l}^n}(s)f(s)\,ds,$$

so the image of $A_n$is contained in span $(I_1^n, \dots, I_n^n)$, i.e. it is a finte rank operator. Thus A is compact. $\square$

Proposition 6.5.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded(smooth) domain, and assume $\delta > 0$. Then the inclusion $H_0^{s+\delta}(\Omega) \subset H_0^s(\Omega)$ is compact if $s \geq 0$.*

Proof. In the proof we use the following result that is proved in the "Functional Analysis" course: a bounded sequence (f_i) of $L^2(\Omega)$ contains a weakly convergent subsequence $(f_{i(n)})$, i.e. for all $g \in L^2(\Omega)$ the inner products $(f_{i(n)}, g)$ converge in $\mathbb{C}$. Let now (f_i) be a bounded sequence in $H^{s+\delta}(\Omega)$, i.e. for all i,

$$\|f_i\|_{s+\delta} \leq 1.$$

Since f_i are supprted in Ω we have for all $\xi \in \mathbb{R}^n$ that

$$|\widehat{f}_i(\xi)| \leq \int_\Omega |f_i| \, dx \leq C\|f_i\|_{L^2} \leq C.$$

Hence the set $|f_i(\xi)|$ is bounded both in ξ and i. This is where we use the fact that Ω is bounded. We may also assume that the sequence (f_i) is weakly convergent in $L^2(\Omega)$, by passing to a subsequence if necessary. Since $e^{-i\langle \cdot,\xi\rangle} \in L^2(\Omega)$ and they form a bounded set with respect to ξ in the $L^2(\Omega)$ norm, we have

$$\|f_k - f_l\|_s^2 \leq \int_{|\xi|\leq r} + \int_{|\xi|>r} (1+|\xi|^2)^s |\widehat{f}_k(\xi) - \widehat{f}_l(\xi)|^2 \, d\xi,$$

where the the difference $\widehat{(f_k - f_l)}(\xi) = (f_k - f_l, e^{-i\langle \cdot,\xi\rangle})$ converges to zero for all $\xi \in \mathbb{R}^n$. Hence by Fatou's lemma the integral over the sphere $\{|\xi| \leq r\}$ converges to zero for any fixed $r > 1$. On the other hand, since (f_i) was a bounded sequence in $H^{s+\delta}$–norm we have

$$\int_{|\xi|>r} (1+|\xi|^2)^s |\widehat{f}_k(\xi) - \widehat{f}_l(\xi)|^2 \, d\xi \leq r^{-2\delta} \int_{|\xi|>r} (1+|\xi|^2)^{s+\delta} |\widehat{f}_k(\xi) - \widehat{f}_l(\xi)|^2 \, d\xi,$$

and by taking r large enough this can be made arbitrarily small for all k, l. Thus (f_i) converges to zero in H^s–norm. □

The above result also holds for negative values of s, and what is more important for us, also for the imbedding $H^{s+\delta}(\Omega) \subset H^s(\Omega)$.

6.6. Reconstruction of γ. We start by giving the first boundary reconstruction theorem.

Theorem 6.6.1. *Assume $\gamma \in C^\infty(\overline{\Omega})$ has a positive lower bound and that $\Omega \subset \mathbb{R}^n$ is a bounded C^∞–domain. Let h_η be the zero–mean functions constructed in subsection* 6.3*, with*

U a cylindrical neighbourhood of x_0, and let $f \in C^\infty(\partial\Omega)$. Then $\gamma_{\partial\Omega\cap U}$ can be determined form the formula

$$(\gamma f, h) = \lim_{|\eta|\to\infty,\, \eta\in\mathbb{R}^{n-1}} (R\Lambda_\gamma e^{-i\langle\cdot,\eta\rangle} f, h_\eta).$$

Proof. Let $f_\eta = e^{-i\langle\cdot,\eta\rangle} f$. Then by Proposition 6.2.2 we can estimate

$$((\gamma - R\Lambda_\gamma) f_\eta, h)_{\partial\Omega} \leq \|u_\eta\|_{L^\infty(\Omega)} \|\nabla\|_{L^2(\Omega)} \|\nabla\gamma\|_{L^2(\Omega)},$$

where u_η satisfies $\nabla \cdot \gamma\nabla u_\eta = 0$ in Ω and $u_\eta|_{\partial\Omega} = f_\eta$.

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Calderón problem

Lecture notes, Spring 2008

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Contents

CHAPTER 1

Introduction

Electrical Impedance Tomography (EIT) is an imaging method with potential applications in medical imaging and nondestructive testing. The method is based on the following important inverse problem.

> **Calderón problem:** Is it possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary?

In this course we will prove a fundamental uniqueness result due to Sylvester and Uhlmann, which states that the conductivity is determined by the boundary measurements. We will also consider stable dependence of the conductivity on boundary measurements, and the case where measurements are only made on part of the boundary. In addition, we will discuss useful techniques in partial differential equations, Fourier analysis, and inverse problems.

Let us begin by recalling the mathematical model of EIT, see [**5**] for details. The purpose is to determine the electrical conductivity $\gamma(x)$ at each point $x \in \Omega$, where $\Omega \subseteq \mathbf{R}^n$ represents the body which is imaged (in practice $n = 3$). We assume that Ω is a bounded open subset of $\mathbf{R}^n$ with C^∞ boundary, and that γ is a positive C^2 function in $\overline{\Omega}$.

Under the assumption of no sources or sinks of current in Ω, a voltage potential f at the boundary $\partial\Omega$ induces a voltage potential u in Ω, which solves the Dirichlet problem for the conductivity equation,

$$\left\{ \begin{array}{rl} \nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{array} \right. \tag{1.1}$$

Since γ is positive, there is a unique weak solution $u \in H^1(\Omega)$ for any boundary value $f \in H^{1/2}(\partial\Omega)$. One can define the Dirichlet to Neumann map (DN map) formally as

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

This is the current flowing through the boundary. More precisely, the DN map is defined weakly as

$$(\Lambda_\gamma f, g)_{\partial\Omega} = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx, \quad f, g \in H^{1/2}(\partial\Omega),$$

where u is the solution of (1.1), and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. The pairing on the boundary is integration with respect to the surface measure,

$$(f, g)_{\partial\Omega} = \int_{\partial\Omega} f g \, dS.$$

With this definition Λ_γ is a bounded linear map from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$.

The Calderón problem (also called the inverse conductivity problem) is to determine the conductivity function γ from the knowledge of the map Λ_γ. That is, if the measured current $\Lambda_\gamma f$ is known for all boundary voltages $f \in H^{1/2}(\partial\Omega)$, one would like to determine the conductivity γ. There are several aspects of this inverse problem which are interesting both for mathematical theory and practical applications.

1. **Uniqueness.** If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, show that $\gamma_1 = \gamma_2$.
2. **Reconstruction.** Given the boundary measurements Λ_γ, find a procedure to reconstruct the conductivity γ.
3. **Stability.** If Λ_{γ_1} is close to Λ_{γ_2}, show that γ_1 and γ_2 are close (in a suitable sense).
4. **Partial data.** If Γ is a subset of $\partial\Omega$ and if $\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma$ for all boundary voltages f, show that $\gamma_1 = \gamma_2$.

Starting from the work of Calderón in 1980, the inverse conductivity problem has been studied intensively. In the case where $n \geq 3$ and all conductivities are in $C^2(\overline{\Omega})$, the following positive results are an example of what can be proved.

Theorem. *(Sylvester-Uhlmann 1987) If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in Ω.*

Theorem. *(Nachman 1988) There is a convergent algorithm for reconstructing γ from Λ_γ.*

Theorem. *(Alessandrini 1988) Let $\gamma_j \in H^s(\Omega)$ for $s > \frac{n}{2} + 2$, and assume that $\|\gamma_j\|_{H^s(\Omega)} \leq M$ and $1/M \leq \gamma_j \leq M$ $(j = 1, 2)$. Then*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)})$$

where $\omega(t) = C|\log t|^{-\sigma}$ *for small* $t > 0$*, with* $C = C(\Omega, M, n, s) > 0$*,* $\sigma = \sigma(n, s) \in (0, 1)$*.*

THEOREM. *(Kenig-Sjöstrand-Uhlmann 2007) Assume that* Ω *is convex and* Γ *is any open subset of* $\partial\Omega$*. If* $\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma$ *for all* $f \in H^{1/2}(\partial\Omega)$*, and if* $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$*, then* $\gamma_1 = \gamma_2$ *in* Ω*.*

During this course we will discuss the methods involved in these results. The main tool will be the construction of special solutions, called *complex geometrical optics* solutions, to the conductivity equation and related equations. This will involve Fourier analysis, and we will begin the course with a discussion of n-dimensional Fourier series.

References. This course a continuation of the class "Impedanssitomografian perusteet" (Principles of EIT) given by Petri Ola [**5**]. From [**5**] we will mainly use the solvability of the Dirichlet problem for elliptic equations, the definition of the DN map, and results which state that the boundary values of the conductivity can be recovered from the DN map. Chapter 2 on multiple Fourier series is classical, an excellent reference is Zygmund [**8**]. In Chapter 3 we prove the uniqueness result of Sylvester-Uhlmann [**6**]. The proof of the main estimate, Theorem 3.7, follows Hähner [**4**]. The stability question is taken up in Chapter 4, where the main results are due to Alessandrini [**1**]. The treatment also benefited from Feldman-Uhlmann [**3**]. In the final Chapter 5 we introduce Carleman estimates and prove the result in Bukhgeim-Uhlmann [**2**], which states that it is enough to measure currents on roughly half of the boundary to determine the conductivity.

CHAPTER 2

Multiple Fourier series

2.1. Fourier series in L^2

Joseph Fourier laid the foundations of the mathematical field now known as Fourier analysis in his 1822 treatise on heat flow. The basic question is to represent periodic functions as sums of elementary pieces. If the period of f is 2π and the elementary pieces are sine and cosine functions, then the desired representation would be

$$f(x) = \sum_{k=0}^{\infty} (a_k \cos(kx) + b_k \sin(kx)).$$

Since $e^{ikx} = \cos(kx) + i\sin(kx)$, we may alternatively consider the series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}. \tag{2.1}$$

We will need to represent functions of n variables as Fourier series. If f is a function in $\mathbf{R}^n$ which is 2π-periodic in each variable, then a natural analog of (2.1) would be

$$f(x) = \sum_{k \in \mathbf{Z}^n} c_k e^{ik\cdot x}.$$

This is the form of Fourier series which we will study. Note that the terms on the right-hand side are 2π-periodic in each variable.

There are many subtle issues related to various modes of convergence for the series above. However, we will mostly just need the case of convergence in L^2 norm for Fourier series of L^2 functions, and in this case no problems arise. Consider the cube $Q = [-\pi, \pi]^n$, and define an inner product on $L^2(Q)$ by

$$(f, g) = (2\pi)^{-n} \int_Q f\bar{g}\,dx, \quad f, g \in L^2(Q).$$

With this inner product, $L^2(Q)$ is a separable infinite-dimensional Hilbert space. The space of functions which are locally square integrable and

2π-periodic in each variable may be identified with $L^2(Q)$. Therefore, we will consider Fourier series of functions in $L^2(Q)$.

LEMMA 2.1. *The set $\{e^{ik\cdot x}\}$ is an orthonormal subset of $L^2(Q)$.*

PROOF. A direct computation: if $k, l \in \mathbf{Z}^n$ then

$$\begin{aligned}
(e^{ik\cdot x}, e^{il\cdot x}) &= (2\pi)^{-n} \int_Q e^{i(k-l)\cdot x}\,dx \\
&= (2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{i(k_1-l_1)x_1} \cdots e^{i(k_n-l_n)x_n}\,dx_n \cdots dx_1 \\
&= \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}
\end{aligned}$$

□

We recall a Hilbert space fact: if $\{e_j\}_{j=1}^{\infty}$ is an orthonormal subset of a separable Hilbert space H, then the following are equivalent:

(1) $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis, in the sense that any $f \in H$ may be written as the series

$$f = \sum_{j=1}^{\infty} (f, e_j) e_j$$

with convergence in H,

(2) for any $f \in H$ one has

$$\|f\|^2 = \sum_{j=1}^{\infty} |(f, e_j)|^2,$$

(3) if $f \in H$ and $(f, e_j) = 0$ for all j, then $f \equiv 0$.

If the condition (3) is satisfied, the orthonormal system $\{e_j\}$ is called *complete*. The main point is that $\{e^{ik\cdot x}\}_{k\in\mathbf{Z}^n}$ is complete in $L^2(Q)$.

LEMMA 2.2. *If $f \in L^2(Q)$ satisfies $(f, e^{ik\cdot x}) = 0$ for all $k \in \mathbf{Z}^n$, then $f \equiv 0$.*

The proof is given below. The main result on Fourier series of L^2 functions is now immediate. Below we denote by $\ell^2(\mathbf{Z}^n)$ the space of complex sequences $c = (c_k)_{k\in\mathbf{Z}^n}$ with norm

$$\|c\|_{\ell^2(\mathbf{Z}^n)} = \Big(\sum_{k\in\mathbf{Z}^n} |c_k|^2 \Big)^{1/2}.$$

THEOREM 2.3. *If $f \in L^2(Q)$, then one has the Fourier series*

$$f(x) = \sum_{k \in \mathbf{Z}^n} \hat{f}(k) e^{ik\cdot x}$$

with convergence in $L^2(Q)$, where the Fourier coefficients are given by

$$\hat{f}(k) = (f, e^{ik\cdot x}) = (2\pi)^{-n} \int_Q f(x) e^{-ik\cdot x}\, dx.$$

One has the Plancherel formula

$$\|f\|^2_{L^2(Q)} = \sum_{k \in \mathbf{Z}^n} |\hat{f}(k)|^2.$$

Conversely, if $c = (c_k) \in \ell^2(\mathbf{Z}^n)$, then the series

$$f(x) = \sum_{k \in \mathbf{Z}^n} c_k e^{ik\cdot x}$$

converges in $L^2(Q)$ to a function f satisfying $\hat{f}(k) = c_k$.

PROOF. The facts on the Fourier series of $f \in L^2(Q)$ follow directly from the discussion above, since $\{e^{ik\cdot x}\}_{k \in \mathbf{Z}^n}$ is a complete orthonormal system in $L^2(Q)$. For the converse, if $(c_k) \in \ell^2(\mathbf{Z}^n)$, then

$$\| \sum_{\substack{k \in \mathbf{Z}^n \\ M \leq |k| \leq N}} c_k e^{ik\cdot x} \|^2_{L^2(Q)} = \sum_{\substack{k \in \mathbf{Z}^n \\ M \leq |k| \leq N}} |c_k|^2$$

by orthogonality. Since the right hand side can be made arbitrarily small by choosing M and N large, we see that $f_N = \sum_{k \in \mathbf{Z}^n, |k| \leq N} c_k e^{ik\cdot x}$ is a Cauchy sequence in $L^2(Q)$, and converges to $f \in L^2(Q)$. One obtains $\hat{f}(k) = (f, e^{ik\cdot x}) = c_k$ again by orthogonality. $\square$

It remains to prove Lemma 2.2. We begin with the most familiar case, $n = 1$. The partial sums of the Fourier series of a function $f \in L^2([-\pi, \pi])$, extended as a 2π-periodic function into $\mathbf{R}$, are given by

$$\begin{aligned} S_m f(x) = \sum_{k=-m}^{m} \hat{f}(k) e^{ikx} &= \sum_{k=-m}^{m} \Big(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky}\, dy \Big) e^{ikx} \\ &= \int_{-\pi}^{\pi} D_m(x-y) f(y)\, dy \end{aligned}$$

where $D_m(x)$ is the *Dirichlet kernel*

$$D_m(x) = \frac{1}{2\pi}\sum_{k=-m}^{m} e^{ikx} = \frac{1}{2\pi}e^{-imx}(1+e^{ix}+\ldots+e^{i2mx})$$
$$= \frac{1}{2\pi}e^{-imx}\frac{e^{i(2m+1)x}-1}{e^{ix}-1} = \frac{1}{2\pi}\frac{e^{i(m+\frac{1}{2})x}-e^{-i(m+\frac{1}{2})x}}{e^{i\frac{1}{2}x}-e^{-i\frac{1}{2}x}} = \frac{1}{2\pi}\frac{\sin((m+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}.$$

Thus, definining the *convolution* of two 2π-periodic locally integrable functions by

$$f * g(x) = \int_{-\pi}^{\pi} f(x-y)g(y)\,dy,$$

we see that $S_m f(x) = (D_m * f)(x)$. The Dirichlet kernel acts in a similar way as an approximate identity, but is problematic because it takes both positive and negative values.

Definition. A sequence $(K_N(x))_{N=1}^{\infty}$ of 2π-periodic continuous functions on the real line is called an *approximate identity* if

(1) $K_N \geq 0$ for all N,
(2) $\int_{-\pi}^{\pi} K_N(x)\,dx = 1$ for all N, and
(3) for all $\delta > 0$ one has

$$\lim_{N\to\infty} \sup_{\delta\leq|x|\leq\pi} K_N(x) \to 0.$$

Thus, an approximate identity (K_N) for large N resembles a Dirac delta at 0, extended in a 2π-periodic way. It is possible to approximate L^p functions by convolving them against an approximate identity.

Lemma 2.4. *Let (K_N) be an approximate identity. If $f \in L^p([-\pi,\pi])$ where $1 \leq p < \infty$, or if f is a continuous 2π-periodic function and $p = \infty$, then*

$$\|K_N * f - f\|_{L^p([-\pi,\pi])} \to 0 \quad \text{as } N \to \infty.$$

Proof. Since K_N has integral 1, we have

$$(K_N * f)(x) - f(x) = \int_{-\pi}^{\pi} K_N(y)[f(x-y)-f(x)]\,dy$$

Let first f be continuous and $p = \infty$. To estimate the L^∞ norm of $K_N * f - f$, we fix $\varepsilon > 0$ and compute

$$|(K_N * f)(x) - f(x)| \leq \int_{-\pi}^{\pi} K_N(y)|f(x-y) - f(x)|\,dy$$

$$\leq \int_{|y|\leq\delta} K_N(y)|f(x-y)-f(x)|\,dy + \int_{\delta\leq|y|\leq\pi} K_N(y)|f(x-y)-f(x)|\,dy.$$

Here $\delta > 0$ is chosen so that

$$|f(x-y) - f(x)| < \frac{\varepsilon}{2} \quad \text{whenever } x \in \mathbf{R} \text{ and } |y| \leq \delta.$$

This is possible because f is uniformly continuous. Further, we use the definition of an approximate identity and choose N_0 so that

$$\sup_{\delta\leq|x|\leq\pi} K_N(x) < \frac{\varepsilon}{4\pi\|f\|_{L^\infty}}, \quad \text{for } N \geq N_0.$$

With these choices, we obtain

$$|(K_N * f)(x) - f(x)| \leq \frac{\varepsilon}{2}\int_{|y|\leq\delta} K_N(y)\,dy + 4\pi\|f\|_{L^\infty} \sup_{\delta\leq|x|\leq\pi} K_N(x) < \varepsilon$$

whenever $N \geq N_0$. The result is proved in the case $p = \infty$.

Let now $f \in L^p([-\pi,\pi])$ and $1 \leq p < \infty$. We need the integral form of Minkowski's inequality,

$$\Big(\int_X \Big|\int_Y F(x,y)\,d\nu(y)\Big|^p\,d\mu(x)\Big)^{1/p} \leq \int_Y \Big(\int_X |F(x,y)|^p\,d\mu(x)\Big)^{1/p}\,d\nu(y),$$

which is valid on σ-finite measure spaces (X,μ) and (Y,ν), cf. the usual Minkowski inequality $\|\sum_y F(\,\cdot\,,y)\|_{L^p} \leq \sum_y \|F(\,\cdot\,,y)\|_{L^p}$. Using this, we obtain

$$\|K_N * f - f\|_{L^p([-\pi,\pi])} \leq \int_{-\pi}^{\pi} K_N(y)\|f(\,\cdot\,-y) - f\|_{L^p([-\pi,\pi])}\,dy$$

$$= \int_{\delta\leq|y|\leq\pi} K_N(y)\|f(\,\cdot\,-y)-f\|_{L^p}\,dy + \int_{|y|\leq\delta} K_N(y)\|f(\,\cdot\,-y)-f\|_{L^p}\,dy$$

$$\leq 4\pi\|f\|_{L^p} \sup_{\delta\leq|x|\leq\pi} K_N(x) + \sup_{|y|\leq\delta}\|f(\,\cdot\,-y) - f\|_{L^p}.$$

Now, for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|f(\,\cdot\,-y) - f\|_{L^p([-\pi,\pi])} < \frac{\varepsilon}{2} \quad \text{whenever } |y| \leq \delta.$$

Thus the second term can be made arbitrarily small by choosing δ sufficiently small, and then the first term is also small if N is large. This shows the result. $\square$

Now, if the Dirichlet kernels (D_m) were an approximate identity, by Lemma 2.4 one would have $S_m f \to f$ in L^2 for any $f \in L^2$. This would in particular imply that any $f \in L^2$ which satisfies $(f, e^{ikx}) = \hat{f}(k) = 0$ for all $k \in \mathbf{Z}^n$, would be the zero function.

However, D_m is not an approximate identity because it takes negative values. One does get an approximate identity if a different summation method used: instead of the partial sums $S_m f$ consider the *Cesàro sums*

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{m=0}^{N} S_m f(x).$$

This can be written in convolution form as

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{m=0}^{N} (D_m * f)(x) = (F_N * f)(x)$$

where F_N is the *Fejér kernel*,

$$\begin{aligned}
F_N(x) &= \frac{1}{2\pi(N+1)} \sum_{m=0}^{N} \frac{e^{i(m+\frac{1}{2})x} - e^{-i(m+\frac{1}{2})x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\
&= \frac{1}{2\pi(N+1)} \frac{e^{i\frac{1}{2}x} \frac{e^{i(N+1)x}-1}{e^{ix}-1} - e^{-i\frac{1}{2}x} \frac{e^{-i(N+1)x}-1}{e^{-ix}-1}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\
&= \frac{1}{2\pi(N+1)} \frac{e^{i(N+1)x} - 1 + e^{-i(N+1)x} - 1}{(e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x})^2} \\
&= \frac{1}{2\pi(N+1)} \frac{\sin^2(\frac{N+1}{2}x)}{\sin^2(\frac{1}{2}x)}.
\end{aligned}$$

Clearly this is nonnegative, and in fact F_N is an approximate identity (exercise). It follows from Lemma 2.4 that Cesàro sums of the Fourier series an L^p function always converge in the L^p norm if $1 \leq p < \infty$.

Lemma 2.5. *If $f \in L^p([-\pi, \pi])$ where $1 \leq p < \infty$, or if f is a continuous 2π-periodic function and $p = \infty$, then*

$$\|\sigma_N f - f\|_{L^p([-\pi,\pi])} \to 0 \quad \textit{as } N \to \infty.$$

Proof of Lemma 2.2. We begin with the case $n = 1$. If $f \in L^2([-\pi, \pi])$ and $(f, e^{ikx}) = 0$ for all $k \in \mathbf{Z}$, then $S_m f = 0$ and also $\sigma_N f = 0$ for all N. By Lemma 2.5 it follows that $f \equiv 0$.

Now let $n \geq 2$, and assume that $f \in L^2(Q)$ and $(f, e^{ik\cdot x}) = 0$ for all $k \in \mathbf{Z}^n$. Since $e^{ik\cdot x} = e^{ik_1x_1} \cdots e^{ik_nx_n}$, we have

$$\int_{-\pi}^{\pi} h(x_1; k_2, \ldots, k_n) e^{-ik_1x_1}\, dx_1 = 0$$

for all $k_1 \in \mathbf{Z}$, where

$$h(x_1; k_2, \ldots, k_n) = \int_{[-\pi,\pi]^{n-1}} f(x_1, x_2, \ldots, x_n) e^{-i(k_2x_2 + \ldots + k_nx_n)}\, dx_2 \cdots dx_n.$$

Now $h(\,\cdot\,; k_2, \ldots, k_n)$ is in $L^2([-\pi, \pi])$ by the Cauchy-Schwarz inequality. By the completeness of the system $\{e^{ik_1x_1}\}$ in one dimension, we obtain that $h(\,\cdot\,; k_2, \ldots, k_n) = 0$ for all $k_2, \ldots, k_n \in \mathbf{Z}$. Applying the same argument in the other variables shows that $f \equiv 0$. $\square$

We have now proved the main facts on Fourier series of L^2 functions.

2.2. Sobolev spaces

In this section we wish to consider Sobolev spaces of periodic functions. In fact, most of the results given here will not be used in their present form, but they provide good motivation for the developments in Chapter 3.

Let $T^n = \mathbf{R}^n / 2\pi\mathbf{Z}^n$ be the n-dimensional torus. Note that $L^2(Q)$ above may be identified with $L^2(T^n)$. However, $C(Q)$ is different from $C(T^n)$; in fact $C(T^n)$ (resp. $C^\infty(T^n)$) can be identified with the continuous (resp. C^∞) 2π-periodic functions in $\mathbf{R}^n$.

First we need to define weak derivatives of periodic functions. This is similar to the nonperiodic case, except that here the test function space will be $C^\infty(T^n)$. We use the notation

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \qquad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

Definition. Let $f \in L^2(T^n)$. We say that $D^\alpha f \in L^2(T^n)$ if there is a function $v \in L^2(T^n)$ which satisfies

$$\int_{T^n} f D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_{T^n} v\varphi\, dx$$

for all $\varphi \in C^\infty(T^n)$. In this case we define $D^\alpha f = v$.

As in the nonperiodic case, weak derivatives are unique, and for smooth functions the definition coincides with the usual derivative.

This uses the fact that there are no boundary terms arising from integration by parts, because of periodicity.

DEFINITION. If $m \geq 0$ is an integer, we denote by $H^m(T^n)$ the space of functions $f \in L^2(T^n)$ such that $D^\alpha f \in L^2(T^n)$ for all $\alpha \in \mathbf{N}^n$ satisfying $|\alpha| \leq m$.

We equip $H^m(T^n)$ with the inner product

$$(f, g)_{H^m(T^n)} = \sum_{|\alpha| \leq m} (D^\alpha f, D^\alpha g).$$

Then $H^m(T^n)$ is a Hilbert space.

If f and $D^\alpha f$ are in $L^2(T^n)$, one may compute the Fourier coefficients of $D^\alpha f$ to be

$$\begin{aligned}(D^\alpha f)\hat{}(k) &= (2\pi)^{-n} \int_{T^n} D^\alpha f(x) e^{-ik\cdot x}\, dx \\ &= (2\pi)^{-n}(-1)^{|\alpha|} \int_{T^n} f(x) D^\alpha (e^{-ik\cdot x})\, dx \\ &= k^\alpha \hat{f}(k)\end{aligned}$$

by the definition of weak derivative, since $e^{ik\cdot x} \in C^\infty(T^n)$. This motivates the following characterization of $H^m(T^n)$ in terms of Fourier coefficients.

LEMMA 2.6. *Let $f \in L^2(T^n)$. Then $f \in H^m(T^n)$ if and only if $\langle k\rangle^m \hat{f} \in \ell^2(\mathbf{Z}^n)$, where $\langle k\rangle = (1 + k_1^2 + \ldots + k_n)^{1/2}$.*

PROOF. One has

$$\begin{aligned} f \in H^m(T^n) &\Leftrightarrow D^\alpha f \in L^2(T^n) \quad \text{for } |\alpha| \leq m \\ &\Leftrightarrow k^\alpha \hat{f}(k) \in \ell^2(\mathbf{Z}^n) \quad \text{for } |\alpha| \leq m \\ &\Leftrightarrow (k_1^2, \ldots, k_n^2)^\alpha |\hat{f}(k)|^2 \in \ell^1(\mathbf{Z}^n) \quad \text{for } |\alpha| \leq m. \end{aligned}$$

If the last condition is satisfied, then

$$\langle k\rangle^{2m} |\hat{f}(k)|^2 = \sum_{|\beta| \leq m} c_\beta (k_1^2, \ldots, k_n^2)^\beta |\hat{f}(k)|^2 \in \ell^1(\mathbf{Z}^n),$$

consequently $\langle k\rangle^m \hat{f}(k) \in \ell^2(\mathbf{Z}^n)$. Conversely, if $\langle k\rangle^m \hat{f}(k) \in \ell^2(\mathbf{Z}^n)$, then $k^\alpha \hat{f}(k) \in \ell^2(\mathbf{Z}^n)$ for $|\alpha| \leq m$ because $|k_j| \leq \langle k\rangle$. $\square$

It is now easy to prove a version of the Sobolev embedding theorem.

THEOREM 2.7. *If $m > n/2$ then $H^m(T^n) \subseteq C(T^n)$.*

PROOF. Let $f \in H^m(T^n)$, so that $\langle k\rangle^m \hat{f} \in \ell^2(\mathbf{Z}^n)$ and

$$f(x) = \sum_{k\in\mathbf{Z}^n} \hat{f}(k)e^{ik\cdot x}.$$

Let $M_k = |\hat{f}(k)e^{ik\cdot x}| = \langle k\rangle^{-m}(\langle k\rangle^m \hat{f}(k))$. We have

$$\sum_{k\in\mathbf{Z}^n} M_k \leq \|\langle k\rangle^{-m}\|_{\ell^2(\mathbf{Z}^n)}\|\langle k\rangle^m \hat{f}(k)\|_{\ell^2(\mathbf{Z}^n)} < \infty,$$

by Lemma 2.6 and since $m > n/2$. Since the terms in the Fourier series of f are continuous functions, this Fourier series converges absolutely and uniformly into a continuous function in T^n by the Weierstrass M-test. $\square$

The final result in this section will be elliptic regularity in the periodic case. Consider a second order differential operator $P(D)$ acting on 2π-periodic functions in $\mathbf{R}^n$,

$$P(D) = \sum_{|\alpha|\leq 2} a_\alpha D^\alpha,$$

where a_α are complex constants. The principal part of $P(D)$ is

$$P_2(D) = \sum_{|\alpha|=2} a_\alpha D^\alpha.$$

We say that $P(D)$ is *elliptic* if $P_2(D)$ has real coefficients, and

$$P_2(k) > 0$$

whenever $k \in \mathbf{Z}^n \setminus \{0\}$. The following proof also indicates how Fourier series are used in the solution of partial differential equations.

THEOREM 2.8. *Let $P(D)$ be an elliptic second order differential operator with constant coefficients, and assume that $u \in L^2(T^n)$ solves the equation*

$$P(D)u = f \quad \text{in } T^n,$$

for some $f \in L^2(T^n)$. Then $u \in H^2(T^n)$.

PROOF. Taking the Fourier coefficients on both sides of $P(D)u = f$ gives

$$P(k)\hat{u}(k) = \hat{f}(k), \quad k \in \mathbf{Z}^n. \tag{2.2}$$

We have

$$P_2(k) = |k|^2 P_2(k/|k|) \geq c|k|^2$$

for some $c > 0$, by ellipticity. Then for $k \neq 0$,

$$
\begin{aligned}
|P(k)| &= |P_2(k) + \sum_{j=1}^{n} a_j k_j + a_0| \\
&\geq |P_2(k)| - (\sum_{j=1}^{n} |a_j|)|k| - |a_0| \\
&\geq c|k|^2 - C|k|.
\end{aligned}
$$

If $C' > 0$ is sufficiently large, it follows that

$$|P(k)| \geq \frac{1}{2} c|k|^2, \qquad \text{for } |k| \geq C'.$$

From (2.2) we obtain

$$|\hat{u}(k)| = \left| \frac{\hat{f}(k)}{P(k)} \right| \leq \frac{2}{c|k|^2} |\hat{f}(k)|, \qquad |k| \geq C'.$$

Since $\hat{f}(k) \in \ell^2(\mathbf{Z}^n)$ this shows that $\langle k \rangle^2 \hat{u}(k) \in \ell^2(\mathbf{Z}^n)$, which implies $u \in H^2(T^n)$ as required. $\square$

CHAPTER 3

Uniqueness

In this chapter, we will discuss the proof of the following uniqueness result of Sylvester and Uhlmann.

THEOREM 3.1. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_1 and γ_2 be two positive functions in $C^2(\overline{\Omega})$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in Ω.*

In fact, this theorem will be reduced to a uniqueness result for the Schrödinger equation (also due to Sylvester and Uhlmann).

THEOREM 3.2. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let q_1 and q_2 be two functions in $L^\infty(\Omega)$ such that the Dirichlet problems for $-\Delta + q_1$ and $-\Delta + q_2$ in Ω are well posed. If $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$ in Ω.*

The reduction of Theorem 3.1 to Theorem 3.2 is presented in Section 3.1 along with the relevant definitions. The proof of the uniqueness results relies on complex geometrical optics (CGO) solutions, which are constructed in Section 3.2 for the Schrödinger equation by Fourier analysis and perturbation arguments. Section 3.3 includes an integral identity relating boundary measurements to interior information about the coefficients, and this identity is used to prove Theorem 3.2 by taking an asymptotic limit with suitable CGO solutions.

3.1. Reduction to Schrödinger equation

The first step in the proof is the reduction of the conductivity equation to a Schrödinger equation,

$$(-\Delta + q)u = 0 \quad \text{in } \Omega,$$

where q is a function in $L^\infty(\Omega)$. This equation turns out to be easier to handle, since the principal part is the Laplacian $-\Delta$.

LEMMA 3.3. *If $\gamma \in C^2(\overline{\Omega})$ and $u \in H^1(\Omega)$ then*

$$-\nabla \cdot \gamma \nabla(\gamma^{-1/2} u) = \gamma^{1/2}(-\Delta + q) u \tag{3.1}$$

in the weak sense, where

$$q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}.$$

PROOF. This is a direct computation: if $u \in C^2(\overline{\Omega})$ then

$$\begin{aligned} -\partial_j(\gamma \partial_j(\frac{1}{\sqrt{\gamma}} u)) &= -\partial_j(\sqrt{\gamma}\partial_j u) + \partial_j(\partial_j(\sqrt{\gamma}) u) \\ &= -\sqrt{\gamma}\partial_j^2 u - \partial_j(\sqrt{\gamma})\partial_j u + \partial_j(\sqrt{\gamma})\partial_j u + \partial_j^2(\sqrt{\gamma}) u. \end{aligned}$$

The claim follows by taking the sum over j. The case where $u \in H^1(\Omega)$ can be proved by approximation[1]. $\square$

If $q \in L^\infty(\Omega)$, we consider the Dirichlet problem for the Schrödinger equation,

$$\left\{ \begin{aligned} (-\Delta + q) u &= 0 && \text{in } \Omega, \\ u &= f && \text{on } \partial\Omega, \end{aligned} \right. \tag{3.2}$$

where $f \in H^{1/2}(\partial\Omega)$. We say that $u \in H^1(\Omega)$ is a weak solution of (3.2) if $u|_{\partial\Omega} = f$, and if for all $\varphi \in H_0^1(\Omega)$ one has

$$\int_\Omega (\nabla u \cdot \nabla \varphi + q u \varphi) \, dx = 0.$$

---

[1]Let $(u_k) \subseteq C^2(\overline{\Omega})$ be a sequence with $u_k \to u$ in $H^1(\Omega)$. We have

$$-\nabla \cdot \gamma \nabla(\gamma^{-1/2} u_k) = \gamma^{1/2}(-\Delta + q) u_k.$$

If $a \in C^1(\overline{\Omega})$ then $u \mapsto au$ is a continuous map on $H^1(\Omega)$, since

$$\|au\|_{H^1(\Omega)} = \|au\|_{L^2(\Omega)} + \|(\nabla a)u + a\nabla u\|_{L^2(\Omega)} \leq 2(\|a\|_{L^\infty(\Omega)} + \|\nabla a\|_{L^\infty(\Omega)})\|u\|_{H^1(\Omega)}.$$

Using that γ is in $C^2(\overline{\Omega})$ and that $\nabla : H^m(\Omega) \to H^{m-1}(\Omega)$ is a continuous map, we have

$$\begin{array}{rcl} & u_k \to u & \text{in } H^1(\Omega) \\ \Longrightarrow & \gamma^{-1/2} u_k \to \gamma^{-1/2} u & \text{in } H^1(\Omega) \\ \Longrightarrow & \nabla(\gamma^{-1/2} u_k) \to \nabla(\gamma^{-1/2} u) & \text{in } L^2(\Omega) \\ \Longrightarrow & \gamma\nabla(\gamma^{-1/2} u_k) \to \gamma\nabla(\gamma^{-1/2} u) & \text{in } L^2(\Omega) \\ \Longrightarrow & -\nabla \cdot \gamma\nabla(\gamma^{-1/2} u_k) \to -\nabla \cdot \gamma\nabla(\gamma^{-1/2} u) & \text{in } H^{-1}(\Omega). \end{array}$$

Similarly, $\gamma^{1/2}(-\Delta + q) u_k \to \gamma^{1/2}(-\Delta + q) u$ in $H^{-1}(\Omega)$, which shows that (3.1) holds in the sense of $H^{-1}(\Omega)$.

The problem (3.2) is said to be *well-posed* if the following three conditions hold:

1. (existence) there is a weak solution u in $H^1(\Omega)$ for any boundary value f in $H^{1/2}(\partial\Omega)$,
2. (uniqueness) the solution u is unique,
3. (stability) the solution operator $f \mapsto u$ is continuous from $H^{1/2}(\partial\Omega) \to H^1(\Omega)$, that is,
$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)}.$$

If the problem is well-posed, we can define a DN map formally by

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),\ f \mapsto \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

This is analogous to the conductivity equation, and also here the precise definition of the DN map is given by the weak formulation

$$(\Lambda_q f, g)_{\partial\Omega} = \int_\Omega (\nabla u \cdot \nabla v + quv)\, dx, \quad f, g \in H^{1/2}(\partial\Omega),$$

where u solves (3.2) and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. The next proof shows that this is a valid definition and Λ_q is a bounded linear map.

LEMMA 3.4. *If $q \in L^\infty(\Omega)$ is such that the problem* (3.2) *is well-posed, then Λ_q is a bounded linear map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, and satisfies*

$$(\Lambda_q f, g)_{\partial\Omega} = (f, \Lambda_q g)_{\partial\Omega}, \qquad f, g \in H^{1/2}(\partial\Omega).$$

PROOF. Fix $f \in H^{1/2}(\partial\Omega)$, and define a map $T : H^{1/2}(\partial\Omega) \to \mathbf{C}$ by

$$T(g) = \int_\Omega (\nabla u \cdot \nabla v + quv)\, dx,$$

where u solves (3.2) and v is any function in $H^1(\Omega)$ with $v|_{\partial\Omega} = g$. Since u is a solution, we have

$$\int_\Omega (\nabla u \cdot \nabla \varphi + qu\varphi)\, dx$$

for any $\varphi \in H_0^1(\Omega)$. Therefore we may replace v by $v + \varphi$ in the definition of $T(g)$. If $v, \tilde{v} \in H^1(\Omega)$ and $v|_{\partial\Omega} = \tilde{v}_{\partial\Omega} = g$, then $v - \tilde{v} \in H_0^1(\Omega)$, so indeed the definition does not depend on the particular choice of v (as long as $v \in H^1(\Omega)$ and $v|_{\partial\Omega} = g$).

Now, for $g \in H^{1/2}(\partial\Omega)$, use the one-sided inverse to the trace operator to obtain $v_g \in H^1(\Omega)$ with $\|v_g\|_{H^1(\Omega)} \leq C\|g\|_{H^{1/2}(\partial\Omega)}$. Then by Cauchy-Schwarz

$$\begin{aligned}|T(g)| &\leq \int_\Omega |\nabla u \cdot \nabla v_g + quv_g|\, dx \leq C'\|u\|_{H^1(\Omega)}\|v_g\|_{H^1(\Omega)} \\ &\leq C''\|f\|_{H^{1/2}(\partial\Omega)}\|g\|_{H^{1/2}(\partial\Omega)}.\end{aligned}$$

Thus $T : H^{1/2}(\partial\Omega) \to \mathbf{C}$ is a continuous map, and there is an element $\Lambda_q f \in H^{-1/2}(\partial\Omega)$ satisfying $(\Lambda_q f, g) = T(g)$. Consequently, the map $\Lambda_q : f \mapsto \Lambda_q f$ is a bounded linear map $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

To show the last identity, let $f, g \in H^{1/2}(\Omega)$, and let u, v be solutions of (3.2) with boundary values $u|_{\partial\Omega} = f$, $v|_{\partial\Omega} = g$. Then

$$(\Lambda_q f, g)_{\partial\Omega} = \int_\Omega (\nabla u \cdot \nabla v + quv)\, dx = \int_\Omega (\nabla v \cdot \nabla u + qvu)\, dx = (\Lambda_q g, f)$$

by the definition of Λ_q. □

The problem (3.2) is not always well-posed, consider for instance the case $q = -\lambda$ where $\lambda > 0$ is an eigenvalue of the Laplacian in Ω (then there exists a nonzero $u \in H^1(\Omega)$ with $-\Delta u = \lambda u$ in Ω and $u|_{\partial\Omega} = 0$). However, for potentials q coming from conductivities, the Dirichlet problem is always well-posed, and there is a relation between the DN maps Λ_γ and Λ_q.

LEMMA 3.5. *If $\gamma \in C^2(\overline{\Omega})$ and $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$, then the Dirichlet problem* (3.2) *is well-posed and*

$$\Lambda_q f = \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2} f) + \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu} f\Big|_{\partial\Omega}, \quad f \in H^{1/2}(\Omega).$$

PROOF. Fix $f \in H^{1/2}(\partial\Omega)$. We need to show that there is u in $H^1(\Omega)$ which solves (3.2). Motivated by Lemma 3.3, we take v be the solution of $\nabla \cdot \gamma\nabla v = 0$ in Ω with $v|_{\partial\Omega} = \gamma^{-1/2} f$. Then $u = \gamma^{1/2} v$ is a function in $H^1(\Omega)$ which solves $(-\Delta + q)u = 0$ with the right boundary values. The same argument with $f = 0$ shows that the solution is unique. Further, since $\gamma \in C^2(\overline{\Omega})$, we have the estimate

$$\|u\|_{H^1(\Omega)} \leq C\|v\|_{H^1(\Omega)} \leq C'\|\gamma^{-1/2} f\|_{H^{1/2}(\partial\Omega)} \leq C''\|f\|_{H^{1/2}(\partial\Omega)}.$$

If u solves (3.2), then $v = \gamma^{-1/2} u$ solves the conductivity equation and

$$\Lambda_\gamma(\gamma^{-1/2} f) = \gamma\frac{\partial v}{\partial\nu} = \gamma^{1/2}\frac{\partial u}{\partial\nu} - \frac{1}{2}\gamma^{-1/2}\frac{\partial\gamma}{\partial\nu} f.$$

Since $\Lambda_q f = \partial u/\partial \nu$, we obtain the relation between Λ_γ and Λ_q. $\square$

We may now show that uniqueness in the inverse conductivity problem can be deduced from the corresponding result for the Schrödinger equation.

Proof that Theorem 3.2 implies Theorem 3.1. Let γ_1, γ_2 be positive functions in $C^2(\overline{\Omega})$, and assume that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. If $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$, then $q_j \in L^\infty(\Omega)$ and by Theorem 3.5 the Dirichlet problems for $-\Delta + q_j$ are well posed. Also, the boundary reconstruction result ([**5**], Theorem 6.6.1) implies that $\gamma_1 = \gamma_2$ and $\partial\gamma_1/\partial\nu = \partial\gamma_2/\partial\nu$ on $\partial\Omega$. Thus, for any f in $H^{1/2}(\partial\Omega)$ one has

$$\begin{aligned}
\Lambda_{q_1} f &= \gamma_1^{-1/2}\Lambda_{\gamma_1}(\gamma_1^{-1/2} f) + \frac{1}{2}\gamma_1^{-1}\frac{\partial\gamma_1}{\partial\nu} f\Big|_{\partial\Omega} \\
&= \gamma_2^{-1/2}\Lambda_{\gamma_2}(\gamma_2^{-1/2} f) + \frac{1}{2}\gamma_2^{-1}\frac{\partial\gamma_2}{\partial\nu} f\Big|_{\partial\Omega} \\
&= \Lambda_{q_2} f.
\end{aligned}$$

Theorem 3.2 gives that $q_1 = q_2$ in Ω. Therefore,

$$\frac{\Delta\sqrt{\gamma_1}}{\sqrt{\gamma_1}} = \frac{\Delta\sqrt{\gamma_2}}{\sqrt{\gamma_2}} \quad \text{in } \Omega. \tag{3.3}$$

We would like to conclude from (3.3) that $\gamma_1 = \gamma_2$. Let $q = \gamma_1^{-1/2}\Delta\gamma_1 = \gamma_2^{-1/2}\Delta\gamma_2$, and consider the equation

$$(-\Delta + q)u = 0 \qquad \text{in } \Omega.$$

Both $\sqrt{\gamma_1}$ and $\sqrt{\gamma_2}$ solve this equation, and $\sqrt{\gamma_1}|_{\partial\Omega} = \sqrt{\gamma_2}|_{\partial\Omega}$ by boundary determination. Since the Schrödinger equation with potential q coming from a conductivity is well-posed, we obtain $\gamma_1 = \gamma_2$ by uniqueness of solutions. $\square$

Remark 3.6. We record for later use another argument showing that (3.3) implies $\gamma_1 = \gamma_2$. The equation (3.3) looks like a nonlinear PDE involving γ_1 and γ_2. To simplify the equation, we note that

$$\Delta(\log\sqrt{\gamma}) = \sum_{j=1}^n \partial_j\Big(\frac{1}{\sqrt{\gamma}}\partial_j\sqrt{\gamma}\Big) = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} - |\nabla(\log\sqrt{\gamma})|^2.$$

The equation becomes

$$\Delta(\log\sqrt{\gamma_1} - \log\sqrt{\gamma_2}) + |\nabla(\log\sqrt{\gamma_1})|^2 - |\nabla(\log\sqrt{\gamma_2})|^2 = 0.$$

Rewriting slightly, we get

$$\Delta(\log \frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}}) + \nabla a \cdot \nabla(\log \frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}}) = 0,$$

where $a = \log \sqrt{\gamma_1 \gamma_2}$. This is a linear equation for the C^2 function $v = \log \frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}}$. Further, noting the identity

$$\nabla \cdot (e^a \nabla v) = e^a(\Delta v + \nabla a \cdot \nabla v),$$

and using the fact that $\gamma_1 = \gamma_2$ on $\partial\Omega$, we see that v solves the Dirichlet problem

$$\begin{cases} \nabla \cdot ((\gamma_1\gamma_2)^{1/2} \nabla v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem is well-posed, and we get $v \equiv 0$ and $\gamma_1 \equiv \gamma_2$.

3.2. Complex geometrical optics

In this section, we will construct *complex geometrical optics* (CGO) solutions to the Schrödinger equation

$$(-\Delta + q)u = 0 \quad \text{in } \Omega.$$

The potential q is assumed to be in $L^\infty(\Omega)$.

Motivation. First let $q = 0$. We try as a solution to the equation $-\Delta u = 0$ the complex exponential at frequency $\zeta \in \mathbf{C}^n$,

$$u(x) = e^{i\zeta \cdot x}.$$

This satisfies

$$Du(x) = \zeta e^{i\zeta\cdot x}, \quad D^2 u(x) = (\zeta \cdot \zeta) e^{i\zeta \cdot x}.$$

Thus, if $\zeta \in \mathbf{C}^n$ satisfies $\zeta \cdot \zeta = 0$, then $u(x) = e^{i\zeta\cdot x}$ solves $-\Delta u = 0$. Writing $\zeta = \mathrm{Re}\,\zeta + i\,\mathrm{Im}\,\zeta$, we see that

$$\zeta \cdot \zeta = 0 \iff |\mathrm{Re}\,\zeta| = |\mathrm{Im}\,\zeta|,\ \mathrm{Re}\,\zeta \cdot \mathrm{Im}\,\zeta = 0.$$

Now suppose q is nonzero. The function $u = e^{i\zeta\cdot x}$ is not an exact solution of $(-\Delta + q)u = 0$ anymore, but we can find solutions which resemble complex exponentials. These are the CGO solutions, which have the form

$$u(x) = e^{i\zeta\cdot x}(1 + r(x)). \tag{3.4}$$

Here r is a *correction term* which is needed to convert the approximate solution $e^{i\zeta\cdot x}$ to an exact solution.

In fact, we are interested in solutions in the *asymptotic limit* as $|\zeta| \to \infty$. This follows the principle that it is usually not possible to obtain explicit formulas for solutions to general equations, but in suitable asymptotic limits explicit expressions for solutions may exist.

We note that (3.4) is a solution of $(-\Delta + q)u = 0$ iff

$$e^{-i\zeta\cdot x}(-\Delta + q)e^{i\zeta\cdot x}(1 + r) = 0. \tag{3.5}$$

It will be convenient to conjugate the exponentials $e^{i\zeta\cdot x}$ into the Laplacian. By this we mean that

$$e^{-i\zeta\cdot x}D_j(e^{i\zeta\cdot x}v) = (D_j + \zeta_j)v,$$
$$e^{-i\zeta\cdot x}D^2(e^{i\zeta\cdot x}v) = (D + \zeta)^2 v = (D^2 + 2\zeta \cdot D)v.$$

We can rewrite (3.5) as

$$(D^2 + 2\zeta \cdot D + q)(1 + r) = 0.$$

This implies the following equation for r:

$$(D^2 + 2\zeta \cdot D + q)r = -q. \tag{3.6}$$

The solvability of (3.6) is the most important step in the construction of CGO solutions. We proceed in several steps.

3.2.1. Basic estimate. We first consider the free case in which there is no potential on the left hand side of (3.6).

THEOREM 3.7. *There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbf{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$, and for any $f \in L^2(\Omega)$, the equation*

$$(D^2 + 2\zeta \cdot D)r = f \quad \text{in } \Omega \tag{3.7}$$

has a solution $r \in H^1(\Omega)$ satisfying

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|}\|f\|_{L^2(\Omega)},$$
$$\|\nabla r\|_{L^2(\Omega)} \leq C_0\|f\|_{L^2(\Omega)}.$$

The idea of the proof is that (3.7) is a linear equation with constant coefficients, so one can try to solve it by the Fourier transform. Since $(D_j u)\hat{}(\xi) = \xi_j \hat{u}(\xi)$, the Fourier transformed equation is

$$(\xi^2 + 2\zeta \cdot \xi)\hat{r}(\xi) = \hat{f}(\xi).$$

We would like to divide by $\xi^2 + 2\zeta \cdot \xi$ and use the inverse Fourier transform to get a solution r. However, the symbol $\xi^2 + 2\zeta \cdot \xi$ vanishes for some $\xi \in \mathbf{R}^n$, and the division cannot be done directly.

It turns out that we can divide by the symbol if we use Fourier series in a large cube instead of the Fourier transform, and moreover take the Fourier coefficients in a shifted lattice instead of the usual integer coordinate lattice.

Proof of Theorem 3.7. Write $\zeta = s(\omega_1 + i\omega_2)$ where $s = |\zeta|/\sqrt{2}$ and ω_1 and ω_2 are orthogonal unit vectors in $\mathbf{R}^n$. By rotating coordinates in a suitable way, we can assume that $\omega_1 = e_1$ and $\omega_2 = e_2$ (the first and second coordinate vectors). Thus we need to solve the equation

$$(D^2 + 2s(D_1 + iD_2))r = f.$$

We assume for simplicity that Ω is contained in the cube $Q = [-\pi, \pi]^n$. Extend f by zero outside Ω into Q, which gives a function in $L^2(Q)$ also denoted by f. We need to solve

$$(D^2 + 2s(D_1 + iD_2))r = f \quad \text{in } Q. \tag{3.8}$$

Let $w_k(x) = e^{i(k+\frac{1}{2}e_2)\cdot x}$ for $k \in \mathbf{Z}^n$. That is, we consider Fourier series in the lattice $\mathbf{Z}^n + \frac{1}{2}e_2$. Writing

$$(u, v) = (2\pi)^{-n} \int_Q u\bar{v}\, dx, \quad u, v \in L^2(Q),$$

we see that $(w_k, w_l) = 0$ if $k \neq l$ and $(w_k, w_k) = 1$, so $\{w_k\}$ is an orthonormal set in $L^2(Q)$. It is also complete: if $v \in L^2(Q)$ and $(v, w_k) = 0$ for all $k \in \mathbf{Z}^n$ then $(ve^{-\frac{1}{2}ix_2}, e^{ik\cdot x}) = 0$ for all $k \in \mathbf{Z}^n$, which implies $v = 0$.

Hilbert space theory gives that f can be written as the series $f = \sum_{k\in\mathbf{Z}^n} f_k w_k$, where $f_k = (f, w_k)$ and $\|f\|^2_{L^2(Q)} = \sum_{k\in\mathbf{Z}^n} |f_k|^2$. Seeking also r in the form $r = \sum_{k\in\mathbf{Z}^n} r_k w_k$, and using that

$$Dw_k = (k + \frac{1}{2}e_2)w_k,$$

the equation (3.8) results in

$$p_k r_k = f_k, \quad k \in \mathbf{Z}^n,$$

where

$$p_k := (k + \frac{1}{2}e_2)^2 + 2s(k_1 + i(k_2 + \frac{1}{2})).$$

Note that $\operatorname{Im} p_k = 2s(k_2 + \frac{1}{2})$ is never zero, which was the reason for considering the shifted lattice. We define

$$r_k := \frac{1}{p_k} f_k$$

and

$$r := \sum_{k \in \mathbf{Z}^n} r_k w_k.$$

The last series converges in $L^2(Q)$ to a function $r \in L^2(Q)$ since

$$|r_k| \leq \frac{1}{|p_k|}|f_k| \leq \frac{1}{|2s(k_2 + \frac{1}{2})|}|f_k| \leq \frac{1}{s}|f_k|,$$

and then

$$\|r\|_{L^2(Q)} = \Big(\sum_k |r_k|^2\Big)^{1/2} \leq \frac{1}{s}\Big(\sum_k |f_k|^2\Big)^{1/2} = \frac{1}{s}\|f\|_{L^2(Q)}.$$

This shows the desired estimate in $L^2(Q)$.

It remains to show that $Dr \in L^2(Q)$ with correct bounds. We have

$$Dr = \sum_{k \in \mathbf{Z}^n} (k + \frac{1}{2}e_2) r_k w_k.$$

The derivative is justified since this is a convergent series in $L^2(Q)$: we claim

$$|(k + \frac{1}{2}e_2) r_k| \leq 4|f_k|, \quad k \in \mathbf{Z}^n, \tag{3.9}$$

which implies that $\|Dr\|_{L^2(Q)} \leq 4\|f\|_{L^2(Q)}$. To show (3.9) we consider two cases: if $|k + \frac{1}{2}e_2| \leq 4s$ we have

$$|(k + \frac{1}{2}e_2) r_k| \leq \frac{4s}{2s|k_2 + 1/2|}|f_k| \leq 4|f_k|,$$

and if $|k + \frac{1}{2}e_2| \geq 4s$ then

$$||k + \frac{1}{2}e_2|^2 + 2sk_1| \geq |k + \frac{1}{2}e_2|^2 - 2s|k + \frac{1}{2}e_2| \geq \frac{1}{2}|k + \frac{1}{2}e_2|^2$$

which implies

$$|(k + \frac{1}{2}e_2) r_k| \leq \frac{|k + \frac{1}{2}e_2|}{\frac{1}{2}|k + \frac{1}{2}e_2|^2}|f_k| \leq \frac{1}{2s}|f_k|.$$

The statement is proved. □

3.2.2. Basic estimate with potential. Now we consider the solution of (3.6) in the presence of a nonzero potential. It will be convenient to give a name to the solution operator in the free case.

NOTATION. Let $\zeta \in \mathbf{C}^n$ satisfy $\zeta \cdot \zeta = 0$ and $|\zeta|$ sufficiently large. We denote by G_ζ the solution operator

$$G_\zeta : L^2(\Omega) \to H^1(\Omega), \ f \mapsto r,$$

where r is the solution to $(D^2 + 2\zeta \cdot D)r = f$ provided by Theorem 3.7.

THEOREM 3.8. *Let $q \in L^\infty(\Omega)$. There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbf{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq \max(C_0 \|q\|_{L^\infty(\Omega)}, 1)$, and for any $f \in L^2(\Omega)$, the equation*

$$(D^2 + 2\zeta \cdot D + q)r = f \quad \text{in } \Omega \tag{3.10}$$

has a solution $r \in H^1(\Omega)$ satisfying

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|} \|f\|_{L^2(\Omega)},$$
$$\|\nabla r\|_{L^2(\Omega)} \leq C_0 \|f\|_{L^2(\Omega)}.$$

PROOF. If one has $q = 0$, a solution would be given by $r = G_\zeta f$. Here q may be nonzero, so we try a solution of the form

$$r := G_\zeta \tilde{f}, \tag{3.11}$$

where $\tilde{f} \in L^2(\Omega)$ is a function to be determined. Inserting (3.11) in the equation (3.10), and using that $(D^2 + 2\zeta \cdot D)G_\zeta = I$, we see that $\tilde{f}$ should satisfy

$$(I + qG_\zeta)\tilde{f} = f \quad \text{in } \Omega. \tag{3.12}$$

We have the norm estimate

$$\|qG_\zeta\|_{L^2(\Omega) \to L^2(\Omega)} \leq \frac{C_0 \|q\|_{L^\infty(\Omega)}}{|\zeta|}.$$

If $|\zeta| \geq \max(2C_0 \|q\|_{L^\infty(\Omega)}, 1)$ then

$$\|qG_\zeta\|_{L^2(\Omega) \to L^2(\Omega)} \leq \frac{1}{2}.$$

It follows that $I + qG_\zeta$ is an invertible operator on $L^2(\Omega)$, and the equation (3.12) has a solution

$$\tilde{f} := (I + qG_\zeta)^{-1} f.$$

The definition (3.11) for r implies

$$(D^2 + 2\zeta \cdot D + q)r = \tilde{f} + qG_\zeta \tilde{f} = (I + qG_\zeta)\tilde{f} = f,$$

and r indeed solves the equation (3.10). Since $\|(I+qG_\zeta)^{-1}\|_{L^2(\Omega)\to L^2(\Omega)} \leq 2$, we have $\|\tilde{f}\|_{L^2(\Omega)} \leq 2\|f\|_{L^2(\Omega)}$. The norm estimates in Theorem 3.7 imply the desired estimates for r, if we replace C_0 by $2C_0$. $\square$

3.2.3. Construction of CGO solutions. It is now easy to give the main result on the existence of CGO solutions. Note that the constant function $a \equiv 1$ always satisfies $\zeta \cdot \nabla a = 0$, so as a special case one obtains the solutions $u = e^{i\zeta\cdot x}(1 + r)$ in (3.4).

THEOREM 3.9. *Let $q \in L^\infty(\Omega)$. There is a constant C_0 depending only on Ω and n, such that for any $\zeta \in \mathbf{C}^n$ satisfying $\zeta \cdot \zeta = 0$ and $|\zeta| \geq \max(C_0\|q\|_{L^\infty(\Omega)}, 1)$, and for any function $a \in H^2(\Omega)$ satisfying*

$$\zeta \cdot \nabla a = 0 \quad \text{in } \Omega,$$

the equation $(-\Delta + q)u = 0$ in Ω has a solution

$$u(x) = e^{i\zeta\cdot x}(a + r), \tag{3.13}$$

where $r \in H^1(\Omega)$ satisfies

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|}\|(-\Delta + q)a\|_{L^2(\Omega)},$$
$$\|\nabla r\|_{L^2(\Omega)} \leq C_0\|(-\Delta + q)a\|_{L^2(\Omega)}.$$

PROOF. The function (3.13) is a solution of $(-\Delta + q)u = 0$ iff

$$e^{-i\zeta\cdot x}(-\Delta + q)e^{i\zeta\cdot x}(a + r) = 0. \tag{3.14}$$

As in the beginning of this section, we conjugate the exponentials into the derivatives and rewrite (3.5) as

$$(D^2 + 2\zeta \cdot D + q)(a + r) = 0.$$

Since $\zeta \cdot Da = 0$, this implies the following equation for r:

$$(D^2 + 2\zeta \cdot D + q)r = -(D^2 + q)a.$$

Theorem 3.8 guarantees the existence of a solution r satisfying the norm estimates above. Then (3.13) is the required solution to $(-\Delta+q)u = 0$ in Ω. $\square$

3.3. Uniqueness proof

In this section we prove the Sylvester-Uhlmann uniqueness results. As shown in Section 3.1, uniqueness in the inverse conductivity problem (Theorem 3.1) follows from the uniqueness result for the Schrödinger equation, which we now recall.

THEOREM 3.2. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let q_1 and q_2 be two functions in $L^\infty(\Omega)$ such that the Dirichlet problems for $-\Delta + q_1$ and $-\Delta + q_2$ in Ω are well posed. If $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$ in Ω.*

The starting point is an integral identity which relates the difference of the boundary measurements $\Lambda_{q_1} - \Lambda_{q_2}$ to the difference of the potentials.

LEMMA 3.8. *Let q_1 and q_2 be two functions in $L^\infty(\Omega)$ such that the Dirichlet problems for $-\Delta + q_1$ and $-\Delta + q_2$ in Ω are well-posed. Then for any $f_1, f_2 \in H^{1/2}(\partial\Omega)$ one has*

$$\langle (\Lambda_{q_1} - \Lambda_{q_2}) f_1, f_2 \rangle = \int_\Omega (q_1 - q_2) u_1 u_2 \, dx,$$

where $u_j \in H^1(\Omega)$ is the solution of $(-\Delta + q_j) u_j = 0$ in Ω with boundary values $u_j|_{\partial\Omega} = f_j$, $j = 1, 2$.

PROOF. By the weak definition of the DN map, we have

$$\langle \Lambda_{q_1} f_1, f_2 \rangle = \int_\Omega (\nabla u_1 \cdot \nabla u_2 + q_1 u_1 u_2) \, dx$$

since u_1 is a solution with boundary values f_1, and u_2 has boundary values f_2. Also, since the DN map is self-adjoint,

$$\langle \Lambda_{q_2} f_1, f_2 \rangle = \langle f_1, \Lambda_{q_2} f_2 \rangle = \langle \Lambda_{q_2} f_2, f_1 \rangle = \int_\Omega (\nabla u_2 \cdot \nabla u_1 + q_2 u_2 u_1) \, dx.$$

The claim follows. $\square$

PROOF OF THEOREM 3.2. Since $\Lambda_{q_1} = \Lambda_{q_2}$, we know from Lemma 3.8 that

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = 0 \tag{3.15}$$

for *any* H^1 solutions u_j to the equations $(-\Delta + q_j) u_j = 0$, $j = 1, 2$. Thus, to prove that $q_1 = q_2$, it is enough to establish that products $u_1 u_2$ of such solutions are dense in $L^1(\Omega)$.

Fix $\xi \in \mathbf{R}^n$. We would like to choose the solutions in such a way that $u_1 u_2$ is close to $e^{ix\cdot\xi}$, since the functions $e^{ix\cdot\xi}$ form a dense set. We begin by taking unit vectors ω_1 and ω_2 in $\mathbf{R}^n$ such that $\{\omega_1, \omega_2, \xi\}$ is an orthogonal set (here we need that $n \geq 3$). Let

$$\zeta = s(\omega_1 + i\omega_2),$$

so that $\zeta \cdot \zeta = 0$. By Theorem 3.9, if s is sufficiently large there exist H^1 solutions u_1 and u_2 which satisfy $(-\Delta + q_j)u_j = 0$, and which are of the form

$$\begin{aligned} u_1 &= e^{i\zeta\cdot x}(e^{ix\cdot\xi} + r_1), \\ u_2 &= e^{-i\zeta\cdot x}(1 + r_2), \end{aligned}$$

where $\|r_j\|_{L^2(\Omega)} \leq C/s$ for $j = 1, 2$. For the first solution we chose $a = e^{ix\cdot\xi}$ which satisfies $\zeta \cdot \nabla a = (\zeta \cdot \xi)e^{ix\cdot\xi} = 0$ by orthogonality, and for the second solution we chose a to be constant.

Inserting these solutions in (3.15), we obtain

$$\int_\Omega (q_1 - q_2)(e^{ix\cdot\xi} + r_1)(1 + r_2)\,dx = 0. \tag{3.16}$$

In this identity, only r_1 and r_2 depend on s. Since the L^2 norms of r_1 and r_2 are bounded by C/s, it is possible to take the limit as $s \to \infty$ in (3.15), and then the terms involving r_1 and r_2 will vanish. Taking this limit in (3.16), we get

$$\int_\Omega (q_1 - q_2)e^{ix\cdot\xi}\,dx = 0.$$

This holds for every $\xi \in \mathbf{R}^n$. If $\tilde{q}$ is the function in $L^1(\mathbf{R}^n)$ which is equal to $q_1 - q_2$ in Ω and vanishes outside Ω, the last identity implies that the Fourier transform of $\tilde{q}$ vanishes for every frequency $\xi \in \mathbf{R}^n$. Consequently $\tilde{q} = 0$, and $q_1 = q_2$ in Ω. $\square$

CHAPTER 4

Stability

In the preceding chapter, we proved that if γ_1, γ_2 are two conductivities in $C^2(\overline{\Omega})$ such that Λ_{γ_1} is equal to Λ_{γ_2}, then $\gamma_1 \equiv \gamma_2$. Here we address the stability question: if γ_1, γ_2 are two conductivities such that Λ_{γ_1} is close to Λ_{γ_2}, does this imply that γ_1 is close to γ_2?

More precisely, we are looking for an estimate of the form

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*), \tag{4.1}$$

where $\|\cdot\|_* = \|\cdot\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}$ is the natural operator norm for the DN maps, and $\omega : [0,\infty) \to [0,\infty)$ is a *modulus of continuity*, that is, a continuous nondecreasing function satisfying $\omega(t) \to 0$ as $t \to 0^+$.

We begin with an example due to Alessandrini.

Example. Let $\mathbb{D}$ be the unit disc in $\mathbf{R}^2$, and let $\gamma_1, \gamma_2 \in L^\infty(\mathbb{D})$ be two conductivities such that $\gamma_1 \equiv 1$ in $\mathbb{D}$, and

$$\gamma_2(x) = \begin{cases} 1 + A, & |x| < r_0, \\ 1, & r_0 < |x| < 1, \end{cases}$$

where A is a positive constant and $r_0 \in (0,1)$. If $f \in H^{1/2}(\partial\mathbb{D})$, then f may be written as Fourier series

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta}.$$

It can be shown (exercise) that

$$\Lambda_{\gamma_1} f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} |k| \hat{f}(k) e^{ik\theta},$$

$$\Lambda_{\gamma_2} f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} |k| \frac{2 + A(1 + r_0^{2|k|})}{2 + A(1 - r_0^{2|k|})} \hat{f}(k) e^{ik\theta}.$$

Then

$$\|(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})f\|^2_{H^{-1/2}} = \sum_{k=-\infty}^{\infty}(1+k^2)^{-1/2}k^2\left|1-\left(\frac{2+A(1+r_0^{2|k|})}{2+A(1-r_0^{2|k|})}\right)^2\right|^2|\hat{f}(k)|^2$$

$$= \sum_{k=-\infty}^{\infty}\frac{k^2}{1+k^2}\left(\frac{2Ar_0^{2|k|}}{2+A(1-r_0^{2|k|})}\right)^2(1+k^2)^{1/2}|\hat{f}(k)|^2.$$

The sum is actually over $k \neq 0$, and then the expression in parentheses is $\leq Ar_0^2$ since it attains its maximum value when $|k| = 1$. It follows that

$$\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_* = \sup_{\|f\|_{H^{1/2}}=1}\|(\Lambda_{\gamma_1}-\Lambda_{\gamma_2})f\|_{H^{-1/2}} \leq Ar_0^2.$$

Now $\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_* \to 0$ as $r_0 \to 0$, but

$$\|\gamma_1-\gamma_2\|_{L^\infty(\mathbb{D})} = A.$$

Thus, an estimate of the form (4.1) can not be valid if one only assumes that $\gamma_1, \gamma_2 \in L^\infty$.

It turns out that under certain *a priori* assumptions on γ_1 and γ_2, it is possible to prove a stability estimate with logarithmic modulus of continuity.

THEOREM 4.1. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_j, $j = 1, 2$, be two positive functions in $H^{s+2}(\Omega)$ with $s > n/2$, satisfying*

$$\frac{1}{M} \leq \gamma_j \leq M, \tag{4.2}$$

$$\|\gamma_j\|_{H^{s+2}(\Omega)} \leq M. \tag{4.3}$$

There are constants $C = C(\Omega, n, M, s) > 0$ and $\sigma = \sigma(n, s) \in (0, 1)$ such that

$$\|\gamma_1-\gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1}-\Lambda_{\gamma_2}\|_*)$$

where ω is a modulus of continuity satisfying

$$\omega(t) \leq C|\log t|^{-\sigma}, \quad 0 < t < 1/e.$$

Note that $\gamma_j \in H^{s+2}(\Omega)$ with $s > n/2$ implies that $\gamma_j \in C^2(\overline{\Omega})$ by Sobolev embedding. The logarithmic modulus of continuity is rather weak, in the sense that even small changes in γ can result in large changes in Λ_γ. However, it has been proved that the logarithmic modulus is optimal, and for instance Hölder type stability can not hold for conductivities of general form.

4.1. Schrödinger equation

As in the uniqueness proof, we will use the inverse problem for the Schrödinger equation to study the stability question. In the following, $\Omega \subseteq \mathbf{R}^n$ is a bounded open set with C^∞ boundary, and $n \geq 3$.

THEOREM 4.2. *Let $q_j \in L^\infty(\Omega)$ be two potentials such that the Dirichlet problems for $-\Delta + q_j$ are well-posed. Further, assume that*

$$\|q_j\|_{L^\infty(\Omega)} \leq M.$$

There is a constant $C = C(\Omega, n, M)$ such that

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq \omega(\|\Lambda_{q_1} - \Lambda_{q_2}\|_*)$$

where ω is a modulus of continuity satisfying

$$\omega(t) \leq C|\log t|^{-\frac{2}{n+2}}, \quad 0 < t < 1/e.$$

PROOF. Let $\xi \in \mathbf{R}^n$. We start from the identity in Lemma 3.8, which states that

$$\int_\Omega (q_1 - q_2)u_1 u_2 \, dx = ((\Lambda_{q_1} - \Lambda_{q_2})(u_1|_{\partial\Omega}), u_2|_{\partial\Omega})_{\partial\Omega}, \tag{4.4}$$

for any $u_j \in H^1(\Omega)$ which solve $(-\Delta + q_j)u_j = 0$ in Ω. As in the proof of Theorem 3.2, let ω_1 and ω_2 be unit vectors such that $\{\omega_1, \omega_2, \xi\}$ is an orthogonal set. The choice of complex vectors is slightly different (we make this choice to obtain better constants), we take

$$\zeta_1 = \frac{s}{\sqrt{2}}\Big(\sqrt{1 - \frac{|\xi|^2}{2s^2}}\omega_1 + \frac{1}{\sqrt{2}s}\xi + i\omega_2\Big),$$
$$\zeta_2 = -\frac{s}{\sqrt{2}}\Big(\sqrt{1 - \frac{|\xi|^2}{2s^2}}\omega_1 - \frac{1}{\sqrt{2}s}\xi + i\omega_2\Big).$$

These satisfy $\zeta_j \cdot \zeta_j = 0$ and $|\zeta_1| = |\zeta_2| = s$. Theorem 3.9 ensures the existence of solutions u_j to $(-\Delta + q_j)u_j = 0$, provided that $s \geq \max(C_0 M, 1)$, of the form

$$u_1 = e^{i\zeta_1 \cdot x}(1 + r_1),$$
$$u_2 = e^{i\zeta_2 \cdot x}(1 + r_2),$$

with $\|r_j\|_{L^2(\Omega)} \leq \frac{C_0\|q_j\|_{L^\infty}}{s}$, and $C_0 = C_0(\Omega, n)$.

Inserting u_1 and u_2 in (4.4) and using that $e^{i(\zeta_1+\zeta_2)\cdot x} = e^{ix\cdot\xi}$, we obtain

$$\begin{aligned}\Big|\int_\Omega (q_1-q_2)e^{ix\cdot\xi}\,dx\Big| &\leq \|\Lambda_{q_1}-\Lambda_{q_2}\|_*\|u_1\|_{H^{1/2}(\partial\Omega)}\|u_2\|_{H^{1/2}(\partial\Omega)} \\ &\quad + \Big|\int_\Omega (q_1-q_2)e^{ix\cdot\xi}(r_1+r_2+r_1r_2)\,dx\Big| \\ &\leq \|\Lambda_{q_1}-\Lambda_{q_2}\|_*\|u_1\|_{H^1}\|u_2\|_{H^1} + C(\|r_1\|_{L^2}+\|r_2\|_{L^2}+\|r_1\|_{L^2}\|r_2\|_{L^2})\end{aligned}$$

with $C = C(\Omega, n, M)$. If $\Omega \subseteq B(0,R)$ then

$$\begin{aligned}\|u_j\|_{H^1} &\leq \|e^{i\zeta_j\cdot x}(1+r_j)\|_{L^2} + \sum_{k=1}^n \|\partial_k(e^{i\zeta_j\cdot x})(1+r_j) + e^{i\zeta_j\cdot x}\partial_k r_j\|_{L^2} \\ &\leq Cse^{Rs}.\end{aligned}$$

We assume that s is so large that $s \leq e^{Rs}$, and have

$$|(\tilde{q}_1-\tilde{q}_2)\hat{\ }(\xi)| \leq C\Big(e^{4Rs}\|\Lambda_{q_1}-\Lambda_{q_2}\|_* + \frac{1}{s}\Big), \tag{4.5}$$

where $\tilde{q}_j$ is the extension of q_j to $\mathbf{R}^n$ by zero (thus $\tilde{q}_j \in L^1(\mathbf{R}^n)$).

So far, we have proved that there are constants C and C', depending on Ω, n, and M, such that (4.5) holds whenever $s \geq C'$. It is possible to obtain a bound for $q_1 - q_2$ in H^{-1} by using (4.5) and the L^∞ bounds for q_j. If $\rho > 0$ is a constant which will be determined later, we have

$$\begin{aligned}\|q_1-q_2\|^2_{H^{-1}(\Omega)} \leq \|q_1-q_2\|^2_{H^{-1}(\mathbf{R}^n)} &= \Big(\int_{|\xi|\leq\rho} + \int_{|\xi|>\rho}\Big)\frac{|(q_1-q_2)\hat{\ }(\xi)|^2}{1+|\xi|^2}\,d\xi \\ &\leq C\rho^n\Big(e^{8Rs}\|\Lambda_{q_1}-\Lambda_{q_2}\|_*^2 + \frac{1}{s^2}\Big) + (1+\rho^2)^{-1}\|q_1-q_2\|^2_{L^2(\mathbf{R}^n)} \\ &\leq C\rho^n e^{8Rs}\|\Lambda_{q_1}-\Lambda_{q_2}\|_*^2 + \frac{C\rho^n}{s^2} + \frac{C}{\rho^2}.\end{aligned}$$

To make the last two terms of equal size, we choose

$$\rho = s^{\frac{2}{n+2}}.$$

Then

$$\|q_1-q_2\|^2_{H^{-1}(\Omega)} \leq Ce^{16Rs}\|\Lambda_{q_1}-\Lambda_{q_2}\|_*^2 + Cs^{-\frac{4}{n+2}}$$

for $s \geq C'(\Omega, n, M)$. We make the final choice

$$s = \frac{1}{16R}|\log\|\Lambda_{q_1}-\Lambda_{q_2}\|_*|$$

where we assume that

$$0 < \|\Lambda_{q_1} - \Lambda_{q_2}\|_* < c'(\Omega, n, M)$$

with c' chosen so that $s \geq C'$. With this assumption, it follows that

$$\|q_1 - q_2\|^2_{H^{-1}(\Omega)} \leq C(\|\Lambda_{q_1} - \Lambda_{q_2}\|_* + |\log\|\Lambda_{q_1} - \Lambda_{q_2}\|_*|^{-\frac{4}{n+2}}).$$

The claim is an easy consequence. □

4.2. More facts on Sobolev spaces

To reduce the stability result for the conductivity equation to Theorem 4.2, we will need more properties of Sobolev spaces. There are three settings to consider: Sobolev spaces in $\mathbf{R}^n$, Sobolev spaces in bounded C^∞ domains $\Omega \subseteq \mathbf{R}^n$, and Sobolev spaces on $(n-1)$-dimensional boundaries $\partial\Omega$. Further, we want to consider the case where s may not be an integer.

The philosophy is that $H^s(\mathbf{R}^n)$ may be defined via the Fourier transform, $H^s(\Omega)$ is the restriction of $H^s(\mathbf{R}^n)$ to Ω, and $H^s(\partial\Omega)$ can be defined by locally flattening the boundary and reducing matters to $H^s(\mathbf{R}^{n-1})$. We now give some specifics, see [**7**] for more details.

Definition. If $s \geq 0$, let

$$H^s(\mathbf{R}^n) = \{u \in L^2(\mathbf{R}^n)\,;\, \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbf{R}^n)\},$$

with norm

$$\|u\|_{H^s(\mathbf{R}^n)} = \|\langle\xi\rangle^s \hat{u}\|_{L^2(\mathbf{R}^n)}.$$

Here $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbf{R}^n$.

The space $H^s(\mathbf{R}^n)$ is in fact a Hilbert space, with inner product $(u, v)_{H^s} = \int \langle\xi\rangle^{2s} \hat{u}(\xi)\overline{\hat{v}(\xi)}$. Recall from [**5**] that if $k \geq 0$ is an integer, there is the equivalent norm

$$\|u\|_{W^{k,2}(\mathbf{R}^n)} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^2(\mathbf{R}^n)} \sim \|u\|_{H^k(\mathbf{R}^n)}, \tag{4.6}$$

where $A \sim B$ means that $c^{-1}B \leq A \leq cB$ for some constant $c > 0$ (independent of u).

The following properties of Sobolev spaces are the main point in this section. Recall that $C^k(\mathbf{R}^n)$ is the space of k times continuously differentiable functions on $\mathbf{R}^n$, such that all partial derivatives up to order k are bounded. The norm is $\|u\|_{C^k(\mathbf{R}^n)} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\mathbf{R}^n)}$.

THEOREM 4.3.

- *(Sobolev embedding theorem) If $u \in H^{s+k}(\mathbf{R}^n)$ where $s > n/2$ and k is a nonnegative integer, then $u \in C^k(\mathbf{R}^n)$ and*
$$\|u\|_{C^k(\mathbf{R}^n)} \le C\|u\|_{H^{s+k}(\mathbf{R}^n)}.$$
- *(Multiplication by functions) If $u \in H^s(\mathbf{R}^n)$ and $s \ge 0$, and if $f \in C^k(\mathbf{R}^n)$ where k is an integer $\ge s$, then $fu \in H^s(\mathbf{R}^n)$ and*
$$\|fu\|_{H^s(\mathbf{R}^n)} \le \|f\|_{C^k(\mathbf{R}^n)}\|u\|_{H^s(\mathbf{R}^n)}.$$
- *(Logarithmic convexity of Sobolev norms) If $0 \le \alpha \le \beta$ and $0 \le t \le 1$, then*
$$\|u\|_{H^\gamma(\mathbf{R}^n)} \le \|u\|_{H^\alpha(\mathbf{R}^n)}^{1-t}\|u\|_{H^\beta(\mathbf{R}^n)}^t, \quad u \in H^\beta(\mathbf{R}^n),$$
where $\gamma = (1-t)\alpha + t\beta$.

PROOF. The first statement was proved in [**5**]. The second fact follows, if s is an integer, by using the equivalent norm (4.6) and the Leibniz rule. If s is not an integer, the most convenient way to prove the result is by interpolation: if $s = l - \varepsilon$ where l is an integer and $0 < \varepsilon < 1$, the claim is true if s is replaced by $l-1$ or l, and the claim follows for s by interpolation between these two cases.

We now prove the third statement. The claim is trivial if $t = 0$ or $t = 1$, so assume that $0 < t < 1$. Then

$$\begin{aligned}
\|u\|_{H^\gamma(\mathbf{R}^n)}^2 &= \int \langle\xi\rangle^{2((1-t)\alpha+t\beta)}|\hat{u}(\xi)|^2\,d\xi \\
&= \int (\langle\xi\rangle^{2\alpha}|\hat{u}(\xi)|^2)^{1-t}(\langle\xi\rangle^{2\beta}|\hat{u}(\xi)|^2)^t\,d\xi \\
&\le \Big(\int \langle\xi\rangle^{2\alpha}|\hat{u}(\xi)|^2\,d\xi\Big)^{1-t}\Big(\int \langle\xi\rangle^{2\beta}|\hat{u}(\xi)|^2\,d\xi\Big)^t = \|u\|_{H^\alpha(\mathbf{R}^n)}^{2(1-t)}\|u\|_{H^\beta(\mathbf{R}^n)}^{2t}
\end{aligned}$$

by using Hölder's inequality with $p = \frac{1}{1-t}$ and $p' = \frac{1}{t}$. $\square$

We would like to use similar results for Sobolev spaces in bounded domains. In the following, let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with C^∞ boundary. In [**5**] one has the Sobolev spaces in Ω, denoted here by $W^{k,2}(\Omega)$, consisting of the functions in $L^2(\Omega)$ whose all weak partial derivatives up to order k are in $L^2(\Omega)$. The norm is

$$\|u\|_{W^{k,2}(\Omega)} := \sum_{|\alpha|\le k}\|\partial^\alpha u\|_{L^2(\Omega)}.$$

We wish to relate these to Sobolev spaces in $\mathbf{R}^n$. The main tool for doing this is the extension operator.

THEOREM 4.4. *(Extension operator) If k is a nonnegative integer, there is a bounded linear operator $E : W^{k,2}(\Omega) \to H^k(\mathbf{R}^n)$ satisfying $Eu|_\Omega = u$.*

PROOF. See the exercises. $\square$

DEFINITION. If $s \geq 0$, let $H^s(\Omega)$ be the set of those $u \in L^2(\Omega)$ such that $u = v|_\Omega$ for some $v \in H^s(\mathbf{R}^n)$. The norm is the quotient norm

$$\|u\|_{H^s(\Omega)} = \inf_{v \in H^s(\mathbf{R}^n), v|_\Omega = u} \|v\|_{H^s(\mathbf{R}^n)}.$$

The space $H^s(\Omega)$ is a Hilbert space. The definition is justified by the fact that

$$H^k(\Omega) = W^{k,2}(\Omega),$$

with equivalent norms, if $k \geq 0$ is an integer. To see this, note that

$$\|u\|_{H^k(\Omega)} \leq \|Eu\|_{H^k(\mathbf{R}^n)} \leq C\|u\|_{W^{k,2}(\Omega)}.$$

Further, if $u \in W^{k,2}(\Omega)$ and if $v \in H^k(\mathbf{R}^n)$ with $v|_\Omega = u$, then $\|u\|_{W^{k,2}(\Omega)} \leq \|v\|_{W^{k,2}(\mathbf{R}^n)}$. Therefore

$$\|u\|_{W^{k,2}(\Omega)} \leq C\|u\|_{H^k(\Omega)}.$$

On domains, the following results correspond to the ones above.

THEOREM 4.5.

- *(Sobolev embedding theorem) If $u \in H^{s+k}(\Omega)$ where $s > n/2$ and k is a nonnegative integer, then $u \in C^k(\overline{\Omega})$ and*

$$\|u\|_{C^k(\overline{\Omega})} \leq C\|u\|_{H^{s+k}(\Omega)}.$$

- *(Multiplication by functions) If $u \in H^s(\Omega)$ and $s \geq 0$, and if $f \in C^k(\overline{\Omega})$ where k is an integer $\geq s$, then $fu \in H^s(\Omega)$ and*

$$\|fu\|_{H^s(\Omega)} \leq \|f\|_{C^k(\overline{\Omega})}\|u\|_{H^s(\Omega)}.$$

- *(Logarithmic convexity of Sobolev norms) If $0 \leq \alpha \leq \beta$ and $0 \leq t \leq 1$, then*

$$\|u\|_{H^{(1-t)\alpha + t\beta}(\Omega)} \leq \|u\|_{H^\alpha(\Omega)}^{1-t}\|u\|_{H^\beta(\Omega)}^t, \quad u \in H^\beta(\Omega).$$

Proof. The first item follows from the corresponding result on $\mathbf{R}^n$ by using the extension operator: if $u \in H^{s+k}(\Omega)$, there is $v \in H^{s+k}(\mathbf{R}^n)$ with $v|_\Omega = u$ and $\|v\|_{H^{s+k}(\mathbf{R}^n)} \leq C\|u\|_{H^{s+k}(\Omega)}$. Sobolev embedding in $\mathbf{R}^n$ implies $v \in C^k(\mathbf{R}^n)$ with $\|v\|_{C^k(\mathbf{R}^n)} \leq C\|v\|_{H^{s+k}(\mathbf{R}^n)}$. Then

$$\|u\|_{C^k(\overline{\Omega})} \leq \|v\|_{C^k(\mathbf{R}^n)} \leq C\|v\|_{H^{s+k}(\mathbf{R}^n)} \leq C\|u\|_{H^{s+k}(\Omega)}.$$

The second item is again a direct computation for integer s and in general can be proved by interpolation, and this is also true for the third item. $\square$

Next consider the spaces $H^s(\partial\Omega)$, where $\partial\Omega$ is the boundary of a bounded C^∞ domain Ω. As described in [**5**], one can define $H^s(\partial\Omega)$ via the spaces $H^s(\mathbf{R}^{n-1})$, by using a partition of unity and diffeomorphisms which locally flatten the boundary. The above properties carry over to the spaces $H^s(\partial\Omega)$. Note that in the Sobolev embedding theorem, the condition is $s > \frac{n-1}{2}$ since $\partial\Omega$ is an $(n-1)$-dimensional manifold.

Theorem 4.6.

- *(Sobolev embedding theorem) If $u \in H^{s+k}(\partial\Omega)$ where $s > \frac{n-1}{2}$ and k is a nonnegative integer, then $u \in C^k(\partial\Omega)$ and*

$$\|u\|_{C^k(\partial\Omega)} \leq C\|u\|_{H^{s+k}(\partial\Omega)}.$$

- *(Multiplication by functions) If $u \in H^s(\partial\Omega)$ and $s \geq 0$, and if $f \in C^k(\partial\Omega)$ where k is an integer $\geq s$, then $fu \in H^s(\partial\Omega)$ and*

$$\|fu\|_{H^s(\partial\Omega)} \leq \|f\|_{C^k(\partial\Omega)}\|u\|_{H^s(\partial\Omega)}.$$

- *(Logarithmic convexity of Sobolev norms) If $0 \leq \alpha \leq \beta$ and $0 \leq t \leq 1$, then*

$$\|u\|_{H^{(1-t)\alpha+t\beta}(\partial\Omega)} \leq \|u\|_{H^\alpha(\partial\Omega)}^{1-t}\|u\|_{H^\beta(\partial\Omega)}^t, \quad u \in H^\beta(\partial\Omega).$$

Finally, a remark about negative index Sobolev spaces $H^s(\mathbf{R}^n)$, $H^s(\Omega)$, $H^s(\partial\Omega)$, where $s < 0$. These spaces contain elements which are no longer L^2 functions, and in fact are not functions at all. The most convenient definition uses the space of tempered distributions $\mathscr{S}'(\mathbf{R}^n)$, which contains for instance all L^p functions and polynomially bounded measures, and which is closed under the Fourier transform. Then, if $s \in \mathbf{R}$, one defines

$$H^s(\mathbf{R}^n) = \{u \in \mathscr{S}'(\mathbf{R}^n)\,;\, \langle\xi\rangle^s\hat{u}(\xi) \in L^2(\mathbf{R}^n)\},$$
$$\|u\|_{H^s(\mathbf{R}^n)} = \|\langle\xi\rangle^s\hat{u}(\xi)\|_{L^2(\mathbf{R}^n)}.$$

Further, $H^s(\Omega)$ can be defined via restriction as above, and $H^s(\partial\Omega)$ by reducing to $H^s(\mathbf{R}^{n-1})$. We will use the fact that multiplication by C^k functions is bounded on H^s for $|s| \leq k$, which can be proved by duality arguments.

4.3. Conductivity equation

We proceed to prove the main result on stability for the conductivity equation, which we recall here.

THEOREM 4.1. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_j, $j = 1, 2$, be two positive functions in $H^{s+2}(\Omega)$ with $s > n/2$, satisfying*

$$\tfrac{1}{M} \leq \gamma_j \leq M, \tag{4.7}$$

$$\|\gamma_j\|_{H^{s+2}(\Omega)} \leq M. \tag{4.8}$$

There are constants $C = C(\Omega, n, M, s) > 0$ and $\sigma = \sigma(n, s) \in (0, 1)$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*)$$

where ω is a modulus of continuity satisfying

$$\omega(t) \leq C|\log t|^{-\sigma}, \quad 0 < t < 1/e.$$

The proof will involve a stability result at the boundary. This is an easier problem, and in fact one has stability with a Lipschitz modulus of continuity.

THEOREM 4.7. *(Boundary stability) Under the assumptions in Theorem 4.1, one has*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*$$

where $C = C(\Omega, n, M, s)$.

PROOF. Follows by using the same method as in the proof of the boundary determination result in [**5**], see the exercises. $\square$

We need to relate the difference of DN maps for the conductivity equation to a corresponding quantity in the Schrödinger case.

LEMMA 4.8. *Under the assumptions in Theorem 4.1, if*

$$q_j = \frac{\Delta\sqrt{\gamma_j}}{\sqrt{\gamma_j}},$$

we have

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_* \leq C(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{2}{2s+3}}),$$

with $C = C(\Omega, n, M, s)$.

PROOF. We use the identity in Lemma 3.5,

$$\Lambda_{q_j} f = \gamma_j^{-1/2} \Lambda_{\gamma_j}(\gamma_j^{-1/2} f) + \frac{1}{2}\gamma_j^{-1}\frac{\partial \gamma_j}{\partial \nu} f\Big|_{\partial\Omega}.$$

If $f \in H^{1/2}(\partial\Omega)$, we obtain

$$\begin{aligned}(\Lambda_{q_1} - \Lambda_{q_2})f = (\gamma_1^{-1/2} - \gamma_2^{-1/2})\Lambda_{\gamma_1}(\gamma_1^{-1/2} f) \\ + \gamma_2^{-1/2}(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})(\gamma_1^{-1/2} f) + \gamma_2^{-1/2}\Lambda_{\gamma_2}((\gamma_1^{-1/2} - \gamma_2^{-1/2})f) \\ + \frac{1}{2}(\gamma_1^{-1} - \gamma_2^{-1})\frac{\partial \gamma_1}{\partial \nu} f + \frac{1}{2}\gamma_2^{-1}\Big(\frac{\partial \gamma_1}{\partial \nu} - \frac{\partial \gamma_2}{\partial \nu}\Big) f.\end{aligned}$$

We estimate the $H^{-1/2}$ norm of this expression by the triangle inequality. For the first three terms we use the estimate

$$\|au\|_{H^{-1/2}(\partial\Omega)} \leq \|a\|_{C^1(\partial\Omega)} \|u\|_{H^{-1/2}(\partial\Omega)},$$

and for the last two terms we use that

$$\|au\|_{H^{-1/2}(\partial\Omega)} \leq \|a\|_{L^\infty(\partial\Omega)} \|u\|_{H^{1/2}(\partial\Omega)}.$$

The a priori estimates for γ_j imply

$$\begin{gathered}\|\gamma_1^p - \gamma_2^p\|_{C^1(\partial\Omega)} \leq C\|\gamma_1 - \gamma_2\|_{C^1(\partial\Omega)}, \\ \|\gamma_j^p\|_{C^1(\partial\Omega)} \leq C, \quad \|\Lambda_{\gamma_j}\|_* \leq C,\end{gathered}$$

where $C = C(\Omega, n, M)$. Consequently

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_* \leq C(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_* + \|\gamma_1 - \gamma_2\|_{C^1(\partial\Omega)}).$$

We would like to use Lemma 4.7 to estimate the last term. By Sobolev embedding, logarithmic convexity of Sobolev norms, the trace theorem, and the a priori estimates for γ_j, we have

$$\begin{aligned}\|\gamma_1 - \gamma_2\|_{C^1(\partial\Omega)} &\leq C\|\gamma_1 - \gamma_2\|_{H^{s+\frac{1}{2}}(\partial\Omega)} \\ &\leq C\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)}^{\frac{2}{2s+3}} \|\gamma_1 - \gamma_2\|_{H^{s+\frac{3}{2}}(\partial\Omega)}^{\frac{2s+1}{2s+3}} \\ &\leq C\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)}^{\frac{2}{2s+3}} \|\gamma_1 - \gamma_2\|_{H^{s+2}(\Omega)}^{\frac{2s+1}{2s+3}} \\ &\leq C\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)}^{\frac{2}{2s+3}}.\end{aligned}$$

Now $\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)} \leq C\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)}$, and the claim follows by Lemma 4.7. $\square$

We can now give the proof of the main stability result. It will be convenient to use the approach in Remark 3.6 to reduce matters to the Schrödinger equation.

Proof of Theorem 4.1. As in Remark 3.6, we introduce the function

$$v = \log \frac{\sqrt{\gamma_1}}{\sqrt{\gamma_2}} = \frac{1}{2}(\log \gamma_1 - \log \gamma_2).$$

This is a C^2 function in $\overline{\Omega}$, and satisfies

$$\left\{ \begin{array}{rll} \nabla \cdot (\gamma_1\gamma_2)^{1/2}\nabla v &= (\gamma_1\gamma_2)^{1/2}(q_1 - q_2) & \text{in } \Omega, \\ v &= \frac{1}{2}(\log \gamma_1 - \log \gamma_2) & \text{on } \partial\Omega, \end{array} \right.$$

where $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$. Therefore

$$\begin{aligned} &\frac{1}{2}\|\log \gamma_1 - \log \gamma_2\|_{H^1(\Omega)} = \|v\|_{H^1(\Omega)} \\ &\leq \|(\gamma_1\gamma_2)^{1/2}(q_1 - q_2)\|_{H^{-1}(\Omega)} + \frac{1}{2}\|\log \gamma_1 - \log \gamma_2\|_{H^{1/2}(\partial\Omega)} \\ &\leq C\|q_1 - q_2\|_{H^{-1}(\Omega)} + C\|\log \gamma_1 - \log \gamma_2\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

By Theorem 4.2 and Lemma 4.8, if $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*$ is small we have

$$\begin{aligned} \|q_1 - q_2\|_{H^{-1}(\Omega)} &\leq C|\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_*|^{-\frac{2}{n+2}} \\ &\leq C|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^{\frac{2}{2s+3}}|^{-\frac{2}{n+2}} \\ &\leq C|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*|^{-\frac{2}{n+2}}. \end{aligned}$$

We obtain

$$\begin{aligned} \|\log \gamma_1 - \log \gamma_2\|_{H^1(\Omega)} \leq C|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*|^{-\frac{2}{n+2}} \\ + C\|\log \gamma_1 - \log \gamma_2\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

We would like to change the norms of $\log \gamma_1 - \log \gamma_2$ on both sides to L^∞ norms. As before, this can be done by Sobolev embedding, logarithmic convexity of Sobolev norms, and the a priori bounds on γ_j:

$$\begin{aligned} \|\log \gamma_1 - \log \gamma_2\|_{L^\infty(\Omega)} &\leq C\|\log \gamma_1 - \log \gamma_2\|_{H^s(\Omega)} \\ &\leq C\|\log \gamma_1 - \log \gamma_2\|_{H^1(\Omega)}^{\frac{2}{s+1}}\|\log \gamma_1 - \log \gamma_2\|_{H^{s+2}(\Omega)}^{\frac{s-1}{s+1}} \\ &\leq C\|\log \gamma_1 - \log \gamma_2\|_{H^1(\Omega)}^{\frac{2}{s+1}}, \end{aligned}$$

and

$$
\begin{aligned}
\|\log\gamma_1 - \log\gamma_2\|_{H^{1/2}(\partial\Omega)} &\le C\|\log\gamma_1 - \log\gamma_2\|_{L^2(\partial\Omega)}^{\frac{2s}{2s+3}}\|\log\gamma_1 - \log\gamma_2\|_{H^{s+\frac{3}{2}}(\Omega)}^{\frac{1}{2s+3}} \\
&\le C\|\log\gamma_1 - \log\gamma_2\|_{L^\infty(\partial\Omega)}^{\frac{2s}{2s+3}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|\log\gamma_1 - \log\gamma_2\|_{L^\infty(\Omega)}^{\frac{s+1}{2}} \le C|\log\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*|^{-\frac{2}{n+2}} \\
+ C\|\log\gamma_1 - \log\gamma_2\|_{L^\infty(\partial\Omega)}^{\frac{2s}{2s+3}}.
\end{aligned}
$$

Finally, we obtain bounds in terms of $\gamma_1 - \gamma_2$ by using that

$$
\begin{aligned}
\log\gamma_1 - \log\gamma_2 &= \int_0^1 \frac{d}{dt}\log\left((1-t)\gamma_2 + t\gamma_1\right) dt \\
&= \Big(\int_0^1 \frac{1}{(1-t)\gamma_2 + t\gamma_1}\Big)(\gamma_1 - \gamma_2).
\end{aligned}
$$

The a priori bounds on γ_j imply that

$$
\begin{aligned}
\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} &\le C\|\log\gamma_1 - \log\gamma_2\|_{L^\infty(\Omega)}, \\
\|\log\gamma_1 - \log\gamma_2\|_{L^\infty(\partial\Omega)} &\le C\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)}.
\end{aligned}
$$

This shows that

$$
\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}^{\frac{s+1}{2}} \le C|\log\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*|^{-\frac{2}{n+2}} + C\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)}^{\frac{2s}{2s+3}}.
$$

The result follows by Theorem 4.7. $\square$

CHAPTER 5

Partial data

In Chapter 3, we showed that if the boundary measurements for two C^2 conductivities coincide on the whole boundary, then the conductivities are equal. Here we consider the case where measurements are made only on part of the boundary.

The first result in this direction was proved by Bukhgeim and Uhlmann. It involves a unit vector α in $\mathbf{R}^n$ and the subset of the boundary

$$\partial\Omega_{-,\varepsilon} = \{x \in \partial\Omega\,;\, \alpha \cdot \nu(x) < \varepsilon\}.$$

The theorem is as follows.

THEOREM 5.1. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set with smooth boundary, where $n \geq 3$, and let γ_1 and γ_2 be two positive functions in $C^2(\overline{\Omega})$. If $\alpha \in \mathbf{R}^n$ is a unit vector, if $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$, and if for some $\varepsilon > 0$ one has*

$$\Lambda_{\gamma_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{\gamma_2} f|_{\partial\Omega_{-,\varepsilon}} \quad \text{for all } f \in H^{1/2}(\partial\Omega),$$

then $\gamma_1 = \gamma_2$ in Ω.

The proof is based on complex geometrical optics solutions, but requires new elements since we need some control of the solutions on parts of the boundary. The main tool is a weighted norm estimate known as a *Carleman estimate*. This estimate also gives rise to a new construction of complex geometrical optics solutions, which does not involve Fourier analysis.

5.1. Carleman estimates

Again, we first consider the Schrödinger equation, $(-\Delta+q)u = 0$ in Ω, where $q \in L^\infty(\Omega)$ and $\Omega \subseteq \mathbf{R}^n$ is a bounded open set with smooth boundary.

MOTIVATION. Recall from Theorem 3.8 that in the construction of complex geometrical optics solutions, which depend on a large vector

$\zeta \in \mathbf{C}^n$ satisfying $\zeta \cdot \zeta = 0$, we needed to solve equations of the form

$$(D^2 + 2\zeta \cdot D + q)r = f \qquad \text{in } \Omega,$$

or written in another way,

$$e^{-i\zeta \cdot x}(-\Delta + q)e^{i\zeta \cdot x} r = f \qquad \text{in } \Omega.$$

In particular, Theorem 3.8 shows the existence of a solution and implies the estimate

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\zeta|}\|f\|_{L^2(\Omega)}.$$

We write

$$\zeta = \frac{1}{h}(\beta + i\alpha),$$

where α and β are orthogonal unit vectors in $\mathbf{R}^n$, and $h > 0$ is a *small parameter*. The estimate for r may be written as

$$\|r\|_{L^2(\Omega)} \leq C_0 h \|e^{\frac{1}{h}\alpha \cdot x}(-\Delta + q)e^{-\frac{1}{h}\alpha \cdot x} r\|_{L^2(\Omega)}.$$

It is possible to view this as a uniqueness result: if the right hand side is zero, then the solution r also vanishes. It turns out that such a uniqueness result can be proved directly without Fourier analysis, and this is sufficient to imply also existence of a solution.

REMARK 5.2. We will systematically use a small parameter h instead of a large parameter $|\zeta|$ (these are related by $h = \frac{\sqrt{2}}{|\zeta|}$). This is of course just a matter of convention, but has the benefit of being consistent with *semiclassical calculus* which is a well-developed theory for the analysis of certain asymptotic limits. We will also arrange so that our basic partial derivatives will be hD_j instead of $\frac{\partial}{\partial x_j}$. The usefulness of these choices will hopefully be evident below.

5.1.1. Carleman estimates for test functions. We begin with the simplest Carleman estimate, which is valid for test functions and does not involve boundary terms.

THEOREM 5.3. *(Carleman estimate) Let $q \in L^\infty(\Omega)$, let α be a unit vector in $\mathbf{R}^n$, and let $\varphi(x) = \alpha \cdot x$. There exist constants $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, we have*

$$\|u\|_{L^2(\Omega)} \leq Ch\|e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}u\|_{L^2(\Omega)}, \quad u \in C_c^\infty(\Omega).$$

We introduce some notation which will be used in the proof and also later. If $u, v \in L^2(\Omega)$ we write

$$(u|v) = \int_\Omega u\bar{v}\,dx,$$
$$\|u\| = (u|u)^{1/2} = \|u\|_{L^2(\Omega)}.$$

Consider the semiclassical Laplacian

$$P_0 = -h^2\Delta = (hD)^2,$$

and the corresponding Schrödinger operator

$$P = h^2(-\Delta + q) = P_0 + h^2 q.$$

The operators conjugated with exponential weights will be denoted by

$$P_{0,\varphi} = e^{\varphi/h} P_0 e^{-\varphi/h},$$
$$P_\varphi = e^{\varphi/h} P e^{-\varphi/h} = P_{0,\varphi} + h^2 q.$$

We will also need the concept of adjoints of differential operators. If

$$L = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$$

is a differential operator in Ω, with $a_\alpha \in W^{|\alpha|,\infty}(\Omega)$ (that is, all partial derivatives up to order $|\alpha|$ are in $L^\infty(\Omega)$), then L^* is the differential operator which satisfies

$$(Lu|v) = (u|L^*v), \qquad u, v \in C_c^\infty(\Omega).$$

For L of the above form, an integration by parts shows that

$$L^*v = \sum_{|\alpha|\leq m} D^\alpha(\overline{a_\alpha(x)}v).$$

Proof of Theorem 5.3. Using the notation above, the desired estimate can be written as

$$h\|u\| \leq C\|P_\varphi u\|, \qquad u \in C_c^\infty(\Omega).$$

First consider the case $q = 0$, that is, the estimate

$$h\|u\| \leq C\|P_{0,\varphi} u\|, \qquad u \in C_c^\infty(\Omega).$$

We need an explicit expression for $P_{0,\varphi}$. On the level of operators, one has

$$e^{\varphi/h} hD_j e^{-\varphi/h} = hD_j + i\partial_j\varphi.$$

Since $\varphi(x) = \alpha \cdot x$ where α is a unit vector, we obtain

$$\begin{aligned}
P_{0,\varphi} &= \sum_{j=1}^{n} (e^{\varphi/h} hD_j e^{-\varphi/h})(e^{\varphi/h} hD_j e^{-\varphi/h}) = \sum_{j=1}^{n} (hD_j + i\alpha_j)^2 \\
&= (hD)^2 - 1 + 2i\alpha \cdot hD.
\end{aligned}$$

The objective is to prove a positive lower bound for

$$\|P_{0,\varphi} u\|^2 = (P_{0,\varphi} u | P_{0,\varphi} u).$$

To this end, we decompose $P_{0,\varphi}$ in a way which is useful for determining which parts in the inner product are positive and which may be negative. Write

$$P_{0,\varphi} = A + iB$$

where $A^* = A$ and $B^* = B$. Here, A and iB are the self-adjoint and skew-adjoint parts of $P_{0,\varphi}$. Since

$$\begin{aligned}
P_{0,\varphi}^* &= (e^{\varphi/h} P_0 e^{-\varphi/h})^* = e^{-\varphi/h} P_0 e^{\varphi/h} = P_{0,-\varphi} \\
&= (hD)^2 - 1 - 2i\alpha \cdot hD,
\end{aligned}$$

we obtain A and B from the formulas (cf. the real and imaginary parts of a complex number)

$$\begin{aligned}
A &= \frac{P_{0,\varphi} + P_{0,\varphi}^*}{2} = (hD)^2 - 1, \\
B &= \frac{P_{0,\varphi} - P_{0,\varphi}^*}{2i} = 2\alpha \cdot hD.
\end{aligned}$$

Now we have

$$\begin{aligned}
\|P_{0,\varphi} u\|^2 &= (P_{0,\varphi} u | P_{0,\varphi} u) = ((A + iB)u | (A + iB)u) \\
&= (Au|Au) + (Bu|Bu) + i(Bu|Au) - i(Au|Bu) \\
&= \|Au\|^2 + \|Bu\|^2 + (i[A, B]u|u),
\end{aligned}$$

where $[A, B] = AB - BA$ is the commutator of A and B. This argument used integration by parts and the fact that $A^* = A$ and $B^* = B$. There are no boundary terms since $u \in C_c^\infty(\Omega)$.

The terms $\|Au\|^2$ and $\|Bu\|^2$ are nonnegative, so the only negative contributions could come from the commutator term. But in our case A and B are constant coefficient differential operators, and these operators always satisfy

$$[A, B] \equiv 0.$$

Therefore
$$\|P_{0,\varphi}u\|^2 = \|Au\|^2 + \|Bu\|^2.$$
By the Poincaré inequality (see [**5**]) [1],
$$\|Bu\| = 2h\|\alpha \cdot Du\| \geq ch\|u\|,$$
where c depends on Ω. This shows that for any $h > 0$, one has
$$h\|u\| \leq C\|P_{0,\varphi}u\|, \qquad u \in C_c^\infty(\Omega).$$

Finally, consider the case where q may be nonzero. The last estimate implies that for $u \in C_c^\infty(\Omega)$, one has
$$\begin{aligned} h\|u\| &\leq C\|P_{0,\varphi}u\| \leq C\|(P_{0,\varphi} + h^2 q)u\| + C\|h^2 qu\| \\ &\leq C\|P_\varphi u\| + Ch^2\|q\|_{L^\infty(\Omega)}\|u\|. \end{aligned}$$
Choose h_0 so that $C\|q\|_{L^\infty(\Omega)}h_0 = \frac{1}{2}$, that is,
$$h_0 = \frac{1}{2C\|q\|_{L^\infty(\Omega)}}.$$
Then, if $0 < h \leq h_0$,
$$h\|u\| \leq C\|P_\varphi u\| + \frac{1}{2}h\|u\|.$$
The last term may be absorbed in the left hand side, which completes the proof. $\square$

5.1.2. Complex geometrical optics solutions. Here, we show how the Carleman estimate gives a new method for constructing complex geometrical optics solutions. We first establish an existence result for an inhomogeneous equation, analogous to Theorem 3.8.

THEOREM 5.4. *Let $q \in L^\infty(\Omega)$, let α be a unit vector in $\mathbf{R}^n$, and let $\varphi(x) = \alpha \cdot x$. There exist constants $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, the equation*
$$e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}r = f \quad \text{in } \Omega$$
has a solution $r \in L^2(\Omega)$ for any $f \in L^2(\Omega)$, satisfying
$$\|r\|_{L^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}.$$

[1] In fact, if $\alpha \in \mathbf{R}^n$ is a unit vector, then the proof given in [**5**] implies the following Poincaré inequality in the unbounded strip $S = \{x \in \mathbf{R}^n\,;\, a < x \cdot \alpha < b\}$:
$$\|u\|_{L^2(S)} \leq \frac{b-a}{\sqrt{2}}\|\alpha \cdot Du\|_{L^2(S)}, \quad u \in C_c^\infty(S).$$

Remark 5.5. With some knowledge of unbounded operators on Hilbert space, the proof is immediate. Consider $P_\varphi^* : L^2(\Omega) \to L^2(\Omega)$ with domain $C_c^\infty(\Omega)$. It is a general fact that

$$\begin{cases} T \text{ injective} \\ \text{range of } T \text{ closed} \end{cases} \implies T^* \text{ surjective.}$$

Since the Carleman estimate is valid for P_φ^* one obtains injectivity and closed range for P_φ^*, and thus solvability for P_φ. Below we give a direct proof based on duality and the Hahn-Banach theorem, and also obtain the norm bound.

Proof of Theorem 5.4. Note that $P_\varphi^* = P_{0,-\varphi} + h^2\bar{q}$. If h_0 is as in Theorem 5.3, for $h \leq h_0$ we have

$$\|u\| \leq \frac{C}{h}\|P_\varphi^* u\|, \quad u \in C_c^\infty(\Omega).$$

Let $D = P_\varphi^* C_c^\infty(\Omega)$ be a subspace of $L^2(\Omega)$, and consider the linear functional

$$L : D \to \mathbf{C}, \quad L(P_\varphi^* v) = (v|f), \quad \text{for } v \in C_c^\infty(\Omega).$$

This is well defined since any element of D has a unique representation as $P_\varphi^* v$ with $v \in C_c^\infty(\Omega)$, by the Carleman estimate. Also, the Carleman estimate implies

$$|L(P_\varphi^* v)| \leq \|v\|\|f\| \leq \frac{C}{h}\|f\|\|P_\varphi^* v\|.$$

Thus L is a bounded linear functional on D.

The Hahn-Banach theorem ensures that there is a bounded linear functional $\hat{L} : L^2(\Omega) \to \mathbf{C}$ satisfying $\hat{L}|_D = L$ and $\|\hat{L}\| \leq Ch^{-1}\|f\|$. By the Riesz representation theorem, there is $\tilde{r} \in L^2(\Omega)$ such that

$$\hat{L}(w) = (w|\tilde{r}), \quad w \in L^2(\Omega),$$

and $\|\tilde{r}\| \leq Ch^{-1}\|f\|$. Then, for $v \in C_c^\infty(\Omega)$, by the definition of weak derivatives we have

$$(v|P_\varphi \tilde{r}) = (P_\varphi^* v|\tilde{r}) = \hat{L}(P_\varphi^* v) = L(P_\varphi^* v) = (v|f),$$

which shows that $P_\varphi \tilde{r} = f$ in the weak sense.

Finally, set $r = h^2\tilde{r}$. This satisfies $e^{\varphi/h}(-\Delta + q)e^{-\varphi/h} r = f$ in Ω, and $\|r\| \leq Ch\|f\|$. $\square$

We now give a construction of complex geometrical optics solutions to the equation $(-\Delta + q)u = 0$ in Ω, based on Theorem 5.4. This is slightly more general than the discussion in Chapter 3, and is analogous to the *WKB construction* used in finding geometrical optics solutions for the wave equation.

Our solutions are of the form

$$u = e^{-\frac{1}{h}(\varphi + i\psi)}(a + r). \tag{5.1}$$

Here $h > 0$ is small and $\varphi(x) = \alpha \cdot x$ as before, ψ is a real valued phase function, a is a complex amplitude, and r is a correction term which is small when h is small.

Writing $\rho = \varphi + i\psi$ for the complex phase, using the formula

$$e^{\rho/h} hD_j e^{-\rho/h} = hD_j + i\partial_j \rho$$

which is valid for operators, and inserting (5.1) in the equation, we have

$$\begin{aligned} & (-\Delta + q)u = 0 \\ \Leftrightarrow \quad & e^{\rho/h}((hD)^2 + h^2 q)e^{-\rho/h}(a + r) = 0 \\ \Leftrightarrow \quad & e^{\rho/h}((hD)^2 + h^2 q)e^{-\rho/h} r = -((hD + i\nabla\rho)^2 + h^2 q)a \end{aligned}$$

The last equation may be written as

$$e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}(e^{-i\psi/h} r) = f$$

where

$$f = -e^{-i\psi/h}(-h^{-2}(\nabla\rho)^2 + h^{-1}[2\nabla\rho \cdot \nabla + \Delta\rho] + (-\Delta + q))a.$$

Now, Theorem 5.4 ensures that one can find a correction term r satisfying $\|r\| \leq Ch$, thus showing the existence of complex geometrical optics solutions, provided that

$$\|f\| \leq C$$

with C independent of h. Looking at the expression for f, we see that it is enough to choose ψ and a in such a way that

$$\begin{aligned} (\nabla\rho)^2 &= 0, \\ 2\nabla\rho \cdot \nabla a + (\Delta\rho)a &= 0. \end{aligned}$$

Since $\varphi(x) = \alpha \cdot x$ with α a unit vector, expanding the square in $(\nabla\rho)^2 = 0$ gives the following equations for ψ:

$$|\nabla\psi|^2 = 1, \qquad \alpha \cdot \nabla\psi = 0.$$

This is an *eikonal equation* (a certain nonlinear first order PDE) for ψ. We obtain one solution by choosing $\psi(x) = \beta \cdot x$ where $\beta \in \mathbf{R}^n$ is a unit vector satisfying $\alpha \cdot \beta = 0$. It would be possible to use other solutions ψ, but this choice is close to the discussion in Chapter 3.

If $\psi(x) = \beta \cdot x$, then the second equation becomes

$$(\alpha + i\beta) \cdot \nabla a = 0.$$

This is a complex *transport equation* (a first order linear equation) for a, analogous to the equation for a in Theorem 3.9. One solution is given by $a \equiv 1$. Again, other choices would be possible.

This ends the construction of complex geometrical optics solutions based on Carleman estimates. There is one additional difference with the analogous result in Theorem 3.9: the correction term r given by this argument is only in $L^2(\Omega)$, not in $H^1(\Omega)$. The same is true for the solution u. One can in fact obtain r and u in $H^1(\Omega)$ (and even in $H^2(\Omega)$), but this requires a slightly stronger Carleman estimate and some additional work. Some details for this were given in the exercises and lectures.

5.1.3. Carleman estimates with boundary terms. We will continue by deriving a Carleman estimate for functions which vanish at the boundary but are not compactly supported. The estimate will include terms involving the normal derivative. We will use the notation

$$(u|v)_{\partial\Omega} = \int_{\partial\Omega} u\bar{v}\,dS,$$
$$\partial_\nu u = \nabla u \cdot \nu|_{\partial\Omega}$$

and

$$\partial\Omega_\pm = \partial\Omega_\pm(\alpha) = \{x \in \partial\Omega\,;\, \pm\alpha \cdot \nu(x) \geq 0\}.$$

THEOREM 5.6. *(Carleman estimate with boundary terms) Let $q \in L^\infty(\Omega)$, let α be a unit vector in $\mathbf{R}^n$, and let $\varphi(x) = \alpha \cdot x$. There exist constants $C > 0$ and $h_0 > 0$ such that whenever $0 < h \leq h_0$, we have*

$$\begin{aligned} &- h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} + \|u\|_{L^2(\Omega)}^2 \\ &\qquad \leq Ch^2\|e^{\varphi/h}(-\Delta + q)e^{-\varphi/h}u\|_{L^2(\Omega)}^2 + Ch((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+} \end{aligned}$$

for any $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$.

Note that the sign of $\alpha \cdot \nu$ on $\partial\Omega_\pm$ ensures that all terms in the Carleman estimate are nonnegative.

Proof. We first claim that

$$ch^2\|u\|^2 - 2h^3((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega} \leq \|P_{0,\varphi}u\|^2 \tag{5.2}$$

for $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$. It is easy to see that this implies the desired estimate in the case $q = 0$.

As in the proof of Theorem 5.3, we decompose

$$P_{0,\varphi} = A + iB$$

where $A = (hD)^2 - 1$ and $B = 2\alpha \cdot hD$, and $A^* = A$, $B^* = B$. Then

$$\begin{aligned}\|P_{0,\varphi}u\|^2 &= (P_{0,\varphi}u|P_{0,\varphi}u) = ((A+iB)u|(A+iB)u)\\ &= \|Au\|^2 + \|Bu\|^2 + i(Bu|Au) - i(Au|Bu).\end{aligned}$$

We wish to integrate by parts to obtain the commutator term involving $i[A,B]$, but this time boundary terms will arise. We have

$$\begin{aligned}i(Bu|(hD)^2u) &= \sum_{j=1}^n i(Bu|(hD_j)^2u)\\ &= \sum_{j=1}^n \Big[i(Bu|\frac{h}{i}\nu_j hD_j u)_{\partial\Omega} + i(hD_jBu|hD_ju)\Big]\\ &= -2h^3(\alpha\cdot\nabla u|\partial_\nu u)_{\partial\Omega} + \sum_{j=1}^n \Big[i(hD_jBu|\frac{h}{i}\nu_j u)_{\partial\Omega} + i((hD_j)^2Bu|u)\Big].\end{aligned}$$

But $u|_{\partial\Omega} = 0$, so the boundary term involving $\frac{h}{i}\nu_j u$ is zero. For the first boundary term we use the decomposition

$$\nabla u|_{\partial\Omega} = (\partial_\nu u)\nu + (\nabla u)_{\mathrm{tan}}$$

where $(\nabla u)_{\mathrm{tan}} := \nabla u - (\nabla u\cdot\nu)\nu|_{\partial\Omega}$ is the tangential part of ∇u, which vanishes since $u|_{\partial\Omega} = 0$. By these facts, we obtain

$$i(Bu|Au) = i(ABu|u) - 2h^3((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega}.$$

Similarly, using that $u|_{\partial\Omega} = 0$,

$$\begin{aligned}i(Au|Bu) &= i(Au|2\alpha\cdot\frac{h}{i}\nu u)_{\partial\Omega} + i(BAu|u)\\ &= i(BAu|u).\end{aligned}$$

We have proved that

$$\|P_{0,\varphi}u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u|u) - 2h^3((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega}.$$

Again, since A and B are constant coefficient operators, we have $[A,B] = AB - BA \equiv 0$. The Poincaré inequality gives $\|Bu\| \geq ch\|u\|$, which proves (5.2).

Writing (5.2) in a different form, we have

$$\begin{aligned}&-2h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} + c\|u\|^2 \\ &\qquad\qquad \leq h^2\|e^{\varphi/h}(-\Delta)e^{-\varphi/h}u\|^2 + 2h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+}.\end{aligned}$$

Adding a potential, it follows that

$$\begin{aligned}&-2h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} + c\|u\|^2 \\ &\qquad \leq h^2\|e^{\varphi/h}(-\Delta+q)e^{-\varphi/h}u\|^2 + h^2\|q\|_{L^\infty(\Omega)}^2\|u\|^2 \\ &\qquad\qquad + 2h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+}.\end{aligned}$$

Choosing h small enough (depending on $\|q\|_{L^\infty(\Omega)}$), the term involving $\|u\|^2$ on the right can be absorbed to the left hand side. This concludes the proof. ☐

5.2. Uniqueness with partial data

Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary, where $n \geq 3$. If $\alpha \in \mathbf{R}^n$, recall the subsets of the boundary

$$\begin{aligned}\partial\Omega_\pm &= \{x \in \partial\Omega\,;\, \pm\alpha\cdot\nu(x) > 0\}, \\ \partial\Omega_{-,\varepsilon} &= \{x \in \partial\Omega\,;\, \alpha\cdot\nu(x) < \varepsilon\}.\end{aligned}$$

Also, let $\partial\Omega_{+,\varepsilon} = \{x \in \partial\Omega\,;\, \alpha\cdot\nu(x) > \varepsilon\}$. We first consider a partial data uniqueness result for the Schrödinger equation.

THEOREM 5.7. *Let q_1 and q_2 be two functions in $L^\infty(\Omega)$ such that the Dirichlet problems for $-\Delta + q_1$ and $-\Delta + q_2$ are well-posed. If α is a unit vector in $\mathbf{R}^n$, and if*

$$\Lambda_{q_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{q_2} f|_{\partial\Omega_{-,\varepsilon}} \quad \text{for all } f \in H^{1/2}(\partial\Omega),$$

then $q_1 = q_2$ in Ω.

Given this result, it is easy to prove the corresponding theorem for the conductivity equation.

PROOF THAT THEOREM 5.7 IMPLIES THEOREM 5.1. Define $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$. By Lemma 3.5 we have the relation

$$\Lambda_{q_j} f = \gamma_j^{-1/2}\Lambda_{\gamma_j}(\gamma_j^{-1/2} f) + \frac{1}{2}\gamma_j^{-1}\frac{\partial\gamma_j}{\partial\nu} f\Big|_{\partial\Omega}.$$

Since $\Lambda_{\gamma_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{\gamma_2} f|_{\partial\Omega_{-,\varepsilon}}$ for all f, boundary determination results (see [**5**]) imply that

$$\gamma_1|_{\partial\Omega_{-,\varepsilon}} = \gamma_2|_{\partial\Omega_{-,\varepsilon}}, \quad \frac{\partial\gamma_1}{\partial\nu}|_{\partial\Omega_{-,\varepsilon}} = \frac{\partial\gamma_2}{\partial\nu}|_{\partial\Omega_{-,\varepsilon}}.$$

Thus $\Lambda_{q_1} f|_{\partial\Omega_{-,\varepsilon}} = \Lambda_{q_2} f|_{\partial\Omega_{-,\varepsilon}}$ for all $f \in H^{1/2}(\partial\Omega)$. Theorem 5.7 then implies $q_1 = q_2$, or

$$\frac{\Delta\sqrt{\gamma_1}}{\sqrt{\gamma_1}} = \frac{\Delta\sqrt{\gamma_2}}{\sqrt{\gamma_2}} \quad \text{in } \Omega.$$

Now also $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$, so the arguments in Section 3.1 imply that $\gamma_1 = \gamma_2$ in Ω. $\square$

We proceed to the proof of Theorem 5.7. The main tool is the Carleman estimate in Theorem 5.6, which will be applied with the weight $-\varphi$ instead of φ. The estimate then has the form

$$\begin{aligned} &h((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_+} + \|u\|^2_{L^2(\Omega)} \\ &\qquad \leq Ch^2\|e^{-\varphi/h}(-\Delta+q)e^{\varphi/h}u\|^2_{L^2(\Omega)} - Ch((\alpha\cdot\nu)\partial_\nu u|\partial_\nu u)_{\partial\Omega_-} \end{aligned}$$

with $u \in C^\infty(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$. Choosing $v = e^{\varphi/h}u$ and noting that $v|_{\partial\Omega} = 0$, this may be written as

$$\begin{aligned} (5.3)\quad &h((\alpha\cdot\nu)e^{-\varphi/h}\partial_\nu v|e^{-\varphi/h}\partial_\nu v)_{\partial\Omega_+} + \|e^{-\varphi/h}v\|^2_{L^2(\Omega)} \\ &\leq Ch^2\|e^{-\varphi/h}(-\Delta+q)v\|^2_{L^2(\Omega)} - Ch((\alpha\cdot\nu)e^{-\varphi/h}\partial_\nu v|e^{-\varphi/h}\partial_\nu v)_{\partial\Omega_-}. \end{aligned}$$

This last estimate is valid for all $v \in H^2 \cap H^1_0(\Omega)$, which follows by an approximation argument (or can be proved directly).

Proof of Theorem 5.7. Recall from Lemma 3.8 that

$$\int_\Omega (q_1 - q_2)u_1u_2\,dx = \langle(\Lambda_{q_1} - \Lambda_{q_2})(u_1|_{\partial\Omega}), u_2|_{\partial\Omega}\rangle_{\partial\Omega}, \tag{5.4}$$

whenever $u_j \in H^1(\Omega)$ are solutions of $(-\Delta + q_j)u_j = 0$ in Ω. By the assumption on the DN maps, the boundary integral is really over $\partial\Omega_{+,\varepsilon}$. If further $u_1 \in H^2(\Omega)$, then

$$\Lambda_{q_1}(u_1|_{\partial\Omega}) = \partial_\nu u_1|_{\partial\Omega}$$

since $\nabla u_1 \in H^1(\Omega)$ and $\partial_\nu u_1|_{\partial\Omega} = (\mathrm{tr}\,\nabla u_1)\cdot\nu|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. Also,

$$\Lambda_{q_2}(u_1|_{\partial\Omega}) = \partial_\nu\tilde{u}_2|_{\partial\Omega},$$

where $\tilde{u}_2$ solves

$$\begin{cases} (-\Delta + q_2)\tilde{u}_2 = 0 & \text{in } \Omega, \\ \tilde{u}_2 = u_1 & \text{on } \partial\Omega. \end{cases}$$

We have $\tilde{u}_2 \in H^2(\Omega)$ since $u_1|_{\partial\Omega} \in H^{3/2}(\partial\Omega)$. Therefore, (5.4) implies

$$\int_\Omega (q_1 - q_2) u_1 u_2 \, dx = \int_{\partial\Omega_{+,\varepsilon}} \partial_\nu (u_1 - \tilde{u}_2) u_2 \, dS$$

for any $u_j \in H^2(\Omega)$ which solve $(-\Delta + q_j)u_j = 0$ in Ω.

Given the unit vector $\alpha \in \mathbf{R}^n$, let $\xi \in \mathbf{R}^n$ be a vector orthogonal to α, and let $\beta \in \mathbf{R}^n$ be a unit vector such that $\{\alpha, \beta, \xi\}$ is an orthogonal triplet. Write $\varphi(x) = \alpha \cdot x$ and $\psi(x) = \beta \cdot x$. Theorem 3.9 ensures that there exist CGO solutions to $(-\Delta + q_j)u_j = 0$ of the form

$$\begin{aligned} u_1 &= e^{\frac{1}{h}(\varphi + i\psi)} e^{ix\cdot\xi}(1 + r_1), \\ u_2 &= e^{-\frac{1}{h}(\varphi + i\psi)}(1 + r_2), \end{aligned}$$

where $\|r_j\| \leq Ch$, $\|\nabla r_j\| \leq C$, and $u_j \in H^2(\Omega)$ (the part that $r_j \in H^2(\Omega)$ was in the exercises). Then, writing $u := u_1 - \tilde{u}_2 \in H^2 \cap H_0^1(\Omega)$, we have

$$\int_\Omega e^{ix\dot{\xi}}(q_1 - q_2)(1 + r_1 + r_2 + r_1 r_2) \, dx = \int_{\partial\Omega_{+,\varepsilon}} (\partial_\nu u) u_2 \, dS. \tag{5.5}$$

By the estimates for r_j, the limit as $h \to 0$ of the left hand side is $\int_\Omega e^{ix\cdot\xi}(q_1 - q_2) \, dx$. We wish to show that the right hand side converges to zero as $h \to 0$.

By Cauchy-Schwarz, one has

$$\begin{aligned} \Big| \int_{\partial\Omega_{+,\varepsilon}} (\partial_\nu u) u_2 \, dS \Big|^2 &= \Big| \int_{\partial\Omega_{+,\varepsilon}} e^{-\varphi/h} (\partial_\nu u) e^{\varphi/h} u_2 \, dS \Big|^2 \\ &\leq \Big(\int_{\partial\Omega_{+,\varepsilon}} |e^{-\varphi/h} \partial_\nu u|^2 \, dS \Big) \Big(\int_{\partial\Omega_{+,\varepsilon}} |e^{\varphi/h} u_2|^2 \, dS \Big). \end{aligned} \tag{5.6}$$

To use the Carleman estimate, we note that $\varepsilon \leq \alpha \cdot \nu$ on $\partial\Omega_{+,\varepsilon}$, By (5.3) applied to u and with potential q_2, and using that $\partial_\nu u|_{\partial\Omega_{-,\varepsilon}} = 0$ by the assumption on DN maps, we obtain for small h that

$$\begin{aligned} \int_{\partial\Omega_{+,\varepsilon}} |e^{-\varphi/h} \partial_\nu u|^2 &\leq \frac{1}{\varepsilon} \int_{\partial\Omega_{+,\varepsilon}} (\alpha \cdot \nu) |e^{-\varphi/h} \partial_\nu u|^2 \, dS \\ &\leq \frac{1}{\varepsilon} Ch \|e^{-\varphi/h}(-\Delta + q_2)u\|_{L^2(\Omega)}^2. \end{aligned}$$

The reason for choosing the potential q_2 is that

$$(-\Delta + q_2)u = (-\Delta + q_2)u_1 = (q_2 - q_1)u_1.$$

Thus, the solution $\tilde{u}_2$ goes away, and we are left with an expression involving only u_1 for which we know exact asymptotics. We have

$$\int_{\partial\Omega_{+,\varepsilon}} |e^{-\varphi/h}\partial_\nu u|^2 \leq \frac{1}{\varepsilon}Ch\|(q_2 - q_1)e^{i\psi/h}e^{ix\cdot\xi}(1 + r_1)\|^2_{L^2(\Omega)} \leq Ch.$$

This takes care of the first term on the right hand side of (5.6). For the other term we compute

$$\begin{aligned}\int_{\partial\Omega_{+,\varepsilon}} |e^{\varphi/h}u_2|^2\,dS &= \int_{\partial\Omega_{+,\varepsilon}} |1 + r_2|^2\,dS \\ &\leq \frac{1}{2}\int_{\partial\Omega_{+,\varepsilon}} (1 + r_2^2)\,dS \leq C(1 + \|r_2\|^2_{L^2(\partial\Omega)}).\end{aligned}$$

By the trace theorem, $\|r_2\|_{L^2(\partial\Omega)} \leq C\|r_2\|_{H^1(\Omega)} \leq C$. Combining these estimates, we have for small h that

$$\Big|\int_{\partial\Omega_{+,\varepsilon}} (\partial_\nu u)u_2\,dS\Big| \leq C\sqrt{h}.$$

Taking the limit as $h \to 0$ in (5.5), we are left with

$$\int_\Omega e^{ix\cdot\xi}(q_1 - q_2)\,dx = 0. \tag{5.7}$$

This is true for all $\xi \in \mathbf{R}^n$ orthogonal to α. However, since the DN maps agree on $\partial\Omega_{-,\varepsilon}(\alpha)$ for a fixed constant $\varepsilon > 0$, they also agree on $\partial\Omega_{-,\varepsilon'}(\alpha')$ for α' sufficiently close to α on the unit sphere and for some smaller constant ε'. Thus, in particular, (5.7) holds for ξ in an open cone in $\mathbf{R}^n$. Writing q for the function which is equal to $q_1 - q_2$ in Ω and which is zero outside of Ω, this implies that the Fourier transform of q vanishes in an open set. But since q is compactly supported, the Fourier transform is analytic by the Paley-Wiener theorem, and this implies that $q \equiv 0$. We have proved that $q_1 \equiv q_2$. $\square$

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