§1 案例1

习题1：证明若$u \in C^4(\bar{I})$,则有

$$|R_i(u)| \leq Ch^2, i = 1, \dots, n-1$$

其中$C = \max\limits_{x \in \bar{I}} |\frac{d^4 u(x)}{dx^4}|$ 是与h无关的正常数.

证明: 首先对区间$\bar{I}$做均匀网格剖分，剖分步长记为h,选取内部节点x_{i-1}, x_i, x_{i+1}为模板点，利用待定系数法求解.设

$$u_i'' = \alpha_i u_i + \alpha_{i+1} u_{i+1} + \alpha_{i-1} u_{i-1} \tag{1}$$

$Taylor$展开，有

$$u_{i+1} = u_i + hu_i' + \frac{h^2}{2} u_i'' + \frac{h^3}{6} u_i''' + O(h^4) \tag{2}$$

$$u_{i-1} = u_i - hu_i' + \frac{h^2}{2} u_i'' - \frac{h^3}{6} u_i''' + O(h^4) \tag{3}$$

将(2),(3)式代入(1),合并同类项有

$$\begin{aligned} u_i'' &= (\alpha_i + \alpha_{i+1} + \alpha_{i-1}) u_i + h(\alpha_{i+1} - \alpha_{i-1}) u_i' \\ &+ \frac{h^2}{2} (\alpha_{i+1} + \alpha_{i-1}) u_i'' + \frac{h^3}{6} (\alpha_{i+1} - \alpha_{i-1}) u_i''' \\ &+ (\alpha_{i+1} + \alpha_{i-1}) O(h^4) \end{aligned}$$

比较公式两边系数，有

$$\begin{cases} \alpha_i + \alpha_{i+1} + \alpha_{i-1} = 0 \\ \alpha_{i+1} - \alpha_{i-1} = 0 \\ \frac{h^2}{2} (\alpha_{i+1} + \alpha_{i-1}) = 1 \\ \frac{h^3}{6} (\alpha_{i+1} - \alpha_{i-1}) = 0 \end{cases}$$

解此线性方程组，得到

$$\begin{cases} \alpha_i = -\frac{2}{h^2} \\ \alpha_{i-1} = \alpha_{i+1} = \frac{1}{h^2} \end{cases}$$

故有

$$\begin{aligned}|R_i(u)| &= u''(x_i) - \alpha_i u(x_i) - \alpha_{i-1}u(x_{i-1}) - \alpha_{i+1}u(x_{i+1}) \\ &= O(h^2) \\ &\leq Ch^2, i = 1, 2, \ldots, n-1\end{aligned}$$

其中$C = \max\limits_{x\in\bar{I}} |\frac{d^4u(x)}{dx^4}|$ 是与h无关的正常数.

习题2 :利用极值定理证明差分方程

$$\begin{cases} L_h u_i = f_i \\ u_0 = \alpha, u_n = \beta \end{cases}$$

的适定性.

证明:

$$\begin{cases} L_h u_i = f_i \\ u_0 = \alpha, u_n = \beta \end{cases}$$

适定，等价于

$$\begin{cases} L_h u_i = 0 \\ u_0 = 0, u_n = 0 \end{cases} \tag{4}$$

只有零解.

将(4)式拆分为

$$\begin{cases} L_h u_i \leq 0 \\ u_0 = 0, u_n = 0 \end{cases}$$

和

$$\begin{cases} L_h u_i \geq 0 \\ u_0 = 0, u_n = 0 \end{cases}$$

由极值原理得，正的极大值与负的极小值都不在内部节点上，所以内部节点函数值都为0. 又边界值也为0，因此(4)式只有零解.命题得证.

习题3 :证明若

$$L_h u_i = f_i \geq 0 \text{ (或} \leq 0) \quad i = 1, \cdots, n-1$$

且

$$u_0 \geq 0\ (\text{或} \leq 0);\ u_n \geq 0\ (\text{或} \leq 0)$$

则

$$u_i \geq 0\ (\text{或} \leq 0),\ i = 1, \cdots, n-1$$

证明：只证明$L_h u_i = f_i \geq 0$的情况，$L_h u_i = f_i \leq 0$的情况类似证明. 由极值定理得，$L_h u_i = f_i \geq 0$时，内部节点不可能取负的极小值，并且$u_0 \geq 0, u_n \geq 0$,故有$u_i \geq 0$.

习题4 :(关于边界值的稳定性)试证明差分方程

$$\begin{cases} L_h u_i = 0, & i = 1, \cdots, n-1 \\ u_0 = \alpha, & u_n = \beta \end{cases}$$

的解$\{u_i\}$ 满足估计式

$$||u_h||_C = \max_{1 \leq i \leq n-1} |u_i| \leq \max\{|\alpha|, |\beta|\}$$

证明：取

$$U_i = \max\{|\alpha|, |\beta|\}, i = 0, 1, \cdots, n-1, n$$

故

$$|L_h u_i| \leq |L_h U_i|, i = 1, 2, \cdots, n-1$$

且

$$|u_0| \leq U_0, |u_n| \leq U_n$$

由比较定理得，

$$|u_i| \leq U_i, i = 0, 1, \cdots, n-1, n$$

故

$$||u_h||_C = \max_{1 \leq i \leq n-1} |u_i| \leq \max\{|\alpha|, |\beta|\}$$

§2 案例2

习题1 :导出下列常系数线性抛物型方程初边值问题

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + f(x), 0 < x < l,\ 0 < t \le T \\ u(x,0) = \phi(x), 0 < x < l \\ u(0,t) = u(l,t) = 0, 0 \le t \le T \end{cases} \tag{5}$$

的向后差分格式

$$\frac{u_j^{k+1} - u_j^k}{\tau} = a\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + f_j \tag{6}$$

六点对称格式

$$\frac{u_j^{k+1} - u_j^k}{\tau} = \frac{a}{2}[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2}] + f_j \tag{7}$$

Richardson 格式

$$\frac{u_j^{k+1} - u_j^{k-1}}{2\tau} = a\frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2} + f_j \tag{8}$$

的截断误差.

解: 首先, 计算向后差分格式的截断误差

把(6) 中数值解换为真解, 并左端减右端, 有

$$R_j^k(u) = \frac{u(x_j,t_{k+1}) - u(x_j,t_k)}{\tau} - a\frac{u(x_{j+1},t_{k+1}) - 2u(x_j,t_{k+1}) + u(x_{j-1},t_{k+1})}{h^2} - f(x_j) \tag{9}$$

利用Taylor 展式, 有

$$\frac{u(x_j,t_{k+1}) - u(x_j,t_k)}{\tau} = \frac{\partial u}{\partial t}(x_j,t_k) + O(\tau) \tag{10}$$

$$\frac{u(x_{j+1},t_{k+1}) - 2u(x_j,t_{k+1}) + u(x_{j-1},t_{k+1})}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_j,t_{k+1}) + O(h^2) \tag{11}$$

将(49),(11)式代入(48),并利用微分方程(5),有

$$R_j^k(u)=\left(\frac{\partial u}{\partial t}-a\frac{\partial^2 u}{\partial x^2}-f\right)\Big|_{(x_j,t_k)}+O(\tau+h^2)=O(\tau+h^2) \tag{12}$$

其次, 计算六点对称格式的截断误差

把(7) 中数值解换为真解, 并左端减右端, 有

$$\begin{aligned}R_j^k(u) &= \frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}-\frac{a}{2}\Big(\frac{u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})}{h^2}\\ &\quad+\frac{u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)}{h^2}\Big)+f(x_j)\end{aligned} \tag{13}$$

利用Taylor 展式, 有

$$\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}=\frac{\partial u}{\partial t}(x_j,t_k)+\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_j,t_k)+O(\tau^2) \tag{14}$$

$$\begin{aligned}&\frac{1}{2}\Big(\frac{u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})}{h^2}\\ &+\frac{u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)}{h^2}\Big)\\ &=\frac{1}{2}\Big(\frac{\partial^2 u}{\partial x^2}(x_j,t_{k+1})+\frac{\partial^2 u}{\partial x^2}(x_j,t_k)\Big)+O(h^2)\\ &=\frac{\partial^2 u}{\partial x^2}(x_j,t_k)+\frac{\tau}{2}\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_k)+O(\tau^2)+O(h^2)\end{aligned} \tag{15}$$

把(14), (15) 代入(13), 并利用微分方程(5), 有

$$\begin{aligned}R_j^k(u) &= \left(\frac{\partial u}{\partial t}-a\frac{\partial^2 u}{\partial x^2}-f\right)\Big|_{(x_j,t_k)}\\ &\quad+\frac{\tau}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}-a\frac{\partial^2 u}{\partial x^2}-f\right)\Big|_{(x_j,t_k)}+O(\tau^2+h^2)\\ &= O(\tau^2+h^2)\end{aligned}$$

最后, 计算Richardson 格式的截断误差

把(47) 中数值解换为真解, 并左端减右端, 有

$$\begin{aligned}R_j^k(u) &= \frac{u(x_j,t_{k+1})-u(x_j,t_{k-1})}{2\tau}-a\frac{u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)}{h^2}\\ &\quad-f(x_j)\end{aligned} \tag{16}$$

利用Taylor 展式, 有

$$\frac{u(x_j,t_{k+1})-u(x_j,t_{k-1})}{2\tau}=\frac{\partial u}{\partial t}(x_j,t_k)+O(\tau^2) \tag{17}$$

$$\frac{u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)}{h^2}=\frac{\partial^2 u}{\partial x^2}(x_j,t_k)+O(h^2) \tag{18}$$

把(17), (18) 代入(16), 并利用微分方程(5), 有

$$\begin{aligned} R_j^k(u) &= \left(\frac{\partial u}{\partial t}-a\frac{\partial^2 u}{\partial x^2}-f\right)\bigg|_{(x_j,t_k)}+O(\tau^2+h^2) \\ &= O(\tau^2+h^2) \end{aligned}$$

习题2 :将向前差分格式和向后差分格式作加权平均, 得到如下格式:

$$\frac{u_j^{k+1}-u_j^k}{\tau}=\frac{a}{h^2}\left[\theta\left(u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1}\right)+(1-\theta)\left(u_{j+1}^k-2u_j^k+u_{j-1}^k\right)\right] \tag{19}$$

其中$0\leqslant\theta\leqslant 1$. 试计算截断误差, 并证明当$\theta=\frac{1}{2}-\frac{1}{12r}$ 时, 截断误差的阶最高$(O(\tau^2+h^4))$.

证明: 分别记$R_j^k(u)$、$R_i^{u,b}$ 和$R_i^{u,f}$ 为加权平均、向后和向前差分格式的截断误差，易知

$$R_j^k(u)=\theta R_i^{u,b}+(1-\theta)R_i^{u,f} \tag{20}$$

容易证明:

1) 向后差分格式的截断误差(在(x_j,t_{k+1}) 处作Taylor 展开)

$$\begin{aligned} R_i^{u,b} &= \frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}-\frac{a}{h^2}\left[u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})\right] \\ &= \frac{\partial u(x_j,t_{k+1})}{\partial t}-\frac{\tau}{2}\frac{\partial^2 u(x_j,t_{k+1})}{\partial t^2}-a\left(\frac{\partial^2 u(x_j,t_{k+1})}{\partial x^2}+\frac{h^2}{12}\frac{\partial^4 u(x_j,t_{k+1})}{\partial x^4}\right) \\ &\quad +O(\tau^2+h^4) \\ &= -\left(\frac{\tau}{2}+\frac{h^2}{12a}\right)\frac{\partial^2 u(x_j,t_{k+1})}{\partial t^2}+O(\tau^2+h^4) \end{aligned} \tag{21}$$

2) 向前差分格式的截断误差(在(x_j,t_k) 处作Taylor 展开)

$$
\begin{aligned}
R_i^{u,f} &= \frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau} - \frac{a}{h^2}\left[u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)\right] \\
&= \frac{\partial u(x_j,t_k)}{\partial t} + \frac{\tau}{2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2} - a\left(\frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u(x_j,t_k)}{\partial x^4}\right) \\
&\quad + O(\tau^2+h^4) \\
&= (\frac{\tau}{2} - \frac{h^2}{12a})\frac{\partial^2 u(x_j,t_k)}{\partial t^2} + O(\tau^2+h^4)
\end{aligned} \tag{22}
$$

注：在式(21) 和(22)的推导过程中，利用了原微分方程$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$.

将(21) 和(22)式代入(20)，可得

$$
R_j^k(u) = -\theta(\frac{\tau}{2}+\frac{h^2}{12a})\frac{\partial^2 u(x_j,t_{k+1})}{\partial t^2} + (1-\theta)(\frac{\tau}{2}-\frac{h^2}{12a})\frac{\partial^2 u(x_j,t_k)}{\partial t^2} + O(\tau^2+h^4)
$$

对上式在$(x_j,t_{k+1/2})$ 处作Taylor 展开，有

$$
\begin{aligned}
R_j^k(u) &= -\theta(\frac{\tau}{2}+\frac{h^2}{12a})(\frac{\partial^2 u(x_j,t_{k+1/2})}{\partial t^2} + \frac{\tau}{2}\frac{\partial^3 u(x_j,t_{k+1/2})}{\partial t^3}) \\
&\quad + (1-\theta)(\frac{\tau}{2}-\frac{h^2}{12a})(\frac{\partial^2 u(x_j,t_{k+1/2})}{\partial t^2} - \frac{\tau}{2}\frac{\partial^3 u_j^{k+\frac{1}{2}}}{\partial t^3}) + O(\tau^2+h^4) \\
&= \left(\frac{\tau}{2} - \tau\theta - \frac{h^2}{12a}\right)\frac{\partial^2 u(x_j,t_{k+1/2})}{\partial t^2} + \frac{h^2\tau}{24a}(1-2\theta)\frac{\partial^3 u(x_j,t_{k+1/2})}{\partial t^3} \\
&\quad + O(\tau^2+h^4)
\end{aligned} \tag{23}
$$

注意网格比$r=\frac{a\tau}{h^2}$ 为常数，因此有

$$
\tau = O(h^2) \tag{24}
$$

由(23) 和(24) 可得

$$
R_j^k(u) = \left(\frac{\tau}{2} - \tau\theta - \frac{h^2}{12a}\right)\frac{\partial^2 u(x_j,t_{k+1/2})}{\partial t^2} + O(\tau^2+h^4) \tag{25}
$$

利用(25)，并注意$r=\frac{a\tau}{h^2}$ 可知：当$\theta = \frac{1}{2}-\frac{1}{12r}$ 时，有

$$
\frac{\tau}{2} - \tau\theta - \frac{h^2}{12a} = 0 \ \Leftrightarrow\ R_j^k(u) = O(\tau^2+h^4)
$$

§3 案例3

习题1：分析下列两种差分格式的稳定性.

(1)

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^n-u_j^n}{h}=0 \tag{26}$$

(2)

$$\frac{u_j^{n+1}-u_j^n}{\tau}+a\frac{u_{j+1}^n-u_{j-1}^n}{2h}=0 \tag{27}$$

解: (1) 记$r=a\frac{\tau}{h}$, 则(26) 可以等价地写为

$$u_j^{n+1}=(1+r)u_j^n-ru_{j+1}^n \tag{28}$$

令

$$u_j^n=v_ne^{i\alpha x_j},\quad \alpha=2p\pi \tag{29}$$

将(29) 代入(28), 得:

$$v_{n+1}e^{i\alpha x_j}=(1+r)v_ne^{i\alpha x_j}-rv_ne^{i\alpha(x_j+h)}$$

两边约去因子$e^{i\alpha x_j}$, 可得

$$v_{n+1}=(1+r)v_n-rv_ne^{i\alpha h}=(1+r-re^{i\alpha h})v_n \tag{30}$$

由(30) 知, 差分格式(26) 的增长因子为

$$G(ph,\tau)=1+r-re^{i\alpha h}=(1+r-r\cos\alpha h)-i\cdot r\sin\alpha h$$

差分格式(26) 稳定的充要条件是增长因子满足Von Neumann 条件:

$$|G(ph,\tau)|\leqslant 1+M\tau$$

$\Leftrightarrow$

$$|(1+r-r\cos\alpha h)-i\cdot r\sin\alpha h|\leqslant 1$$

$\Leftrightarrow$

$$(1+r-r\cos\alpha h)^2+r^2\sin^2\alpha h\leqslant 1$$

$\Leftrightarrow$

$$r\cdot(r+1)\cdot(1-\cos\tau h)\leqslant 0$$

$\Leftrightarrow$

$$r\cdot(r+1)\leqslant 0$$

$\Leftrightarrow$

$$r^2\leqslant -r$$

$\Leftrightarrow$

$$(a\frac{\tau}{h})^2\leqslant -a\frac{\tau}{h}$$

$\Leftrightarrow$

$$a\leqslant 0\quad \text{且}\quad \left|a\frac{\tau}{h}\right|\leqslant 1 \tag{31}$$

即(31) 是差分格式(26) 稳定的充要条件.

(2) 记$r=a\frac{\tau}{h}$, 则(27) 可以等价地写为

$$u_j^{n+1}=u_j^n-\frac{r}{2}(u_{j+1}^n-u_{j-1}^n) \tag{32}$$

令

$$u_j^n=v_ne^{i\alpha x_j},\quad \alpha=2p\pi \tag{33}$$

将(33) 代入(32), 得:

$$v_{n+1}e^{i\alpha x_j}=v_ne^{i\alpha x_j}-\frac{r}{2}\big(v_ne^{i\alpha(x_j+h)}-v_ne^{i\alpha(x_j-h)}\big)$$

两边约去因子$e^{i\alpha x_j}$, 可得

$$\begin{aligned} v_{n+1} &= v_n-\frac{r}{2}\big(v_ne^{i\alpha h}-v_ne^{-i\alpha h}\big)\\ &= \left[1-\frac{r}{2}(e^{i\alpha h}-e^{-i\alpha h})\right]v_n\\ &= (1-i\cdot r\sin\alpha h)\cdot v_n \end{aligned} \tag{34}$$

由(34) 知, 差分格式(27) 的增长因子为

$$G(ph,\tau)=1-i\cdot r\sin\alpha h$$

差分格式(27) 稳定的充要条件是增长因子满足Von Neumann 条件:

$$|G(ph,\tau)|\leqslant 1+M\tau$$

$\Leftrightarrow$

$$|1-i\cdot r\sin\alpha h|\leqslant 1$$

$\Leftrightarrow$

$$1+r^2\sin^2\alpha h\leqslant 1$$

$\Leftrightarrow$

$$r^2\sin^2\alpha h\leqslant 0 \tag{35}$$

显然, (35) 对任意的$r\neq 0$ 均不成立, 因此, 差分格式(27) 对任意的$r\neq 0$ 均不稳定.

习题2：试求下列混合问题的解(右边界条件)

$$\begin{cases} \frac{\partial u}{\partial t}-\frac{\partial u}{\partial x}=0,\ -\infty<x<0,\ t>0 \\ u(x,0)=|x+1|\ \ u(0,t)=1 \end{cases}$$

§4 案例5

习题1： 试对问题

$$\begin{cases} -\frac{d}{dx}(x\frac{du}{dx})+u=6, \quad 1<x<2 \\ u(1)=8, \quad u'(2)+2u(2)=3 \end{cases}$$

建立相应虚功原理。

解: 记$I=(1,2)$, 考察问题(A):

$$Lu:=-\frac{d}{dx}(x\frac{du}{dx})+u=6,\quad x\in I \tag{36}$$

$$u(1)=8,\quad u'(2)+2u(2)=3 \tag{37}$$

Step1:选取试探函数空间U 和检验函数空间V:

$$U=\{u|u\in H^1(I),u(1)=8\}$$

$$V=\{v|v\in H^1(I),v(1)=0\}$$

Step2:作L^2内积，并利用分部积分，将二阶导数项化为一阶，再根据边界条件化简

Step3:给出相应的虚功方程:

求$u\in U$，使得

$$a(u,v)=f(v),\forall v\in V$$

其中，

$$\begin{cases} a(u,v)=\int_1^2(x\frac{du}{dx}\frac{dv}{dx}+uv)dx+4u(2)v(2) \\ f(v)=6\int_1^2 vdx+6v(2) \end{cases} \tag{38}$$

Step4:证明解的等价性

必要性显然

（充分性）若u 是变分问题的解，即u 满足

$$a(u,v)=f(v),\forall v\in V$$

即

$$\begin{aligned} a(u,v)-f(v)= & \int_1^2(x\frac{du}{dx}\frac{dv}{dx}+uv)dx+4u(2)v(2) \\ & -6\int_1^2 vdx-6v(2)=0 \end{aligned} \tag{39}$$

由分部积分知:

$$\begin{aligned} \int_1^2 x\frac{du}{dx}\frac{dv}{dx}dx &= (x\frac{du}{dx})v|_1^2-\int_1^2\frac{d}{dx}(x\frac{du}{dx})vdx \\ &= 2u'(2)v(2)-\int_1^2\frac{d}{dx}(x\frac{du}{dx})vdx \end{aligned} \tag{40}$$

将(40) 式代入(39)

$$\int_1^2[-\frac{d}{dx}(x\frac{du}{dx})+u-6]vdx+2v(2)[u'(2)+2u(2)-3]=0 \tag{41}$$

特别取$v\in C_0^\infty(I)$，则$v(2)=0$，代入(41)，得

$$\int_1^2[-\frac{d}{dx}(x\frac{du}{dx})+u-6]vdx=0,\forall v\in C_0^\infty(I)$$

由变分法基本引理，有

$$-\frac{d}{dx}(x\frac{du}{dx})+u=6 \tag{42}$$

将(42) 代入(41) 有

$$2v(2)[u'(2)+2u(2)-3]=0,\forall v\in V$$

特别取$v(x)=x-1\in V$ 则有$u'(2)+2u(2)-3=0\Leftrightarrow u'(2)+2u(2)=3.$

下面给出极小位能原理

设$u \in U \cap C^2(I)$，则u 是问题(A) 的解的充分必要条件是，u 是如下变分问题(C) 的解：

求$u \in U$，使得

$$J(u) = \min_{v \in U} J(v)$$

其中，$J(v) = \frac{1}{2}a(v,v) - f(v)$，而

$$\begin{cases} a(u,v) = \int_1^2 (x\frac{du}{dx}\frac{dv}{dx} + uv)dx + 4u(2)v(2) \\ f(v) = 6\int_1^2 vdx + 6v(2) \end{cases}$$

证明：只需证明问题(C) 的解与问题(B) 的解的等价性，其证明过程与书中的两点边值模型完全一样。

设$u \in U \cap C^2(I)$，则u 是问题(C) 的解的充分必要条件是

$$J(u+tv) \geq J(u), \forall v \in V, t \in R$$

$\Leftrightarrow$

$$\frac{1}{2}a(u+tv, u+tv) - f(u+tv) \geq \frac{1}{2}a(u,u) - f(u), \forall v \in V, t \in R \tag{43}$$

$\Leftrightarrow$

$$\begin{aligned} &\tfrac{1}{2}[a(u,u) + ta(u,v) + ta(v,u) + t^2a(v,v)] - 6\int_1^2 (u+tv)dx \\ &-6[u(2) + tv(2)] \geq \tfrac{1}{2}a(u,u) - 6\int_1^2 udx - 6u(2), \forall v \in V, t \in R \end{aligned}$$

$\Leftrightarrow$

$$ta(u,v) + \frac{t^2}{2}a(v,v) - 6\int_1^2 tvdx - 6tv(2) \geq 0, \forall v \in V, t \in R$$

$\Leftrightarrow$

$$ta(u,v) + \frac{t^2}{2}a(v,v) - t(6\int_1^2 vdx - 6v(2)) \geq 0, \forall v \in V, t \in R$$

$\Leftrightarrow$

$$t[a(u,v) - f(v)] + \frac{t^2}{2}a(v,v) \geq 0, \forall v \in V, t \in R \tag{44}$$

$\Leftrightarrow$

$$a(u,v) = (f,v), \forall v \in V \tag{45}$$

(44) 和(45) 的等价性参考课件中的方法。

习题2 :$f\in C(\bar{I}), I=[a,b]$, f' 仅在$x_c=\frac{a+b}{2}$ 处有间断（即在其余点均连续），且该点左右极限存在，试证明

1）对$\forall\varphi\in C_0^\infty(I)$，以下积分恒等式成立。

$$\int_a^b f'(x)\varphi(x)dx=-\int_a^b f(x)\varphi'(x)dx, \forall\varphi\in C_0^\infty(I)$$

2）$f'\in L^2(I)$.

证明：1）对$\forall\phi\in C_0^\infty(I)$，有

$$\begin{aligned}
&\int_a^b f'(x)\phi(x)dx=\int_a^{x_c^-} f'(x)\phi(x)dx+\int_{x_c^+}^b f'(x)\phi(x)dx\\
=\ &\left[f(x)\phi(x)\Big|_a^{x_c^-}-\int_a^{x_c^-} f(x)\phi'(x)dx\right]\\
&+\left[f(x)\phi(x)\Big|_{x_c^+}^b-\int_{x_c^+}^b f(x)\phi'(x)dx\right]\\
=\ &\left[f(x_c^-)\phi(x_c^-)-\int_a^{x_c^-} f(x)\phi'(x)dx\right]\\
&+\left[-f(x_c^+)\phi(x_c^+)-\int_{x_c^+}^b f(x)\phi'(x)dx\right]\\
=\ &-\left[\int_a^{x_c^-} f(x)\phi'(x)dx+\int_{x_c^+}^b f(x)\phi'(x)dx\right]\qquad (f(x),\phi(x)\text{在}x_c\text{ 点处连续})\\
=\ &-\int_a^b f(x)\phi'(x)dx
\end{aligned}$$

2）由已知条件可知f' 在$[a,x_c^-)$ 和$(x_c^+,b]$ 上是连续的，因此$(f')^2$ 在$[a,x_c^-)$ 和$(x_c^+,b]$ 上也是连续的(复合函数的连续性)，所以$(f')^2$ 在$\bar{I}$ 上至多在$x_c=\frac{a+b}{2}$ 处有间断，且该点左右极限存在(极限的四则运算)。

又由连续性的定义可知，函数在各个点的值均有定义（即均为有限值），所以$(f')^2$ 在$\bar{I}$ 上为有界函数。由数学分析的知识“闭区间上只有有限个不连续点的有界函数必定可积”可知$(f')^2$ 是可积的，即$f'\in L^2(I)$。

习题3 :用线性元求下列两点边值问题的数值解：

$$\begin{cases} Lu=-u''+\frac{\pi^2}{4}u=\frac{\pi^2}{2}\sin\frac{\pi x}{2}, 0<x<1\\ u(0)=0, u'(1)=0\end{cases}\tag{46}$$

要求:

(1) 区间等距剖分成2 段或3 段;

(2) 计算总刚度矩阵和总荷载向量所涉及的定积分用两种方法:

1. 精确求解;

2. 用中矩形公式或Gauss 型求积公式近似计算。

解1): 按以下步骤求出线性有限元解函数$u_h(x)$.

Step 1. 写出原问题(5) 的基于虚功原理的变分形式

求$u \in H_E^1(I)$，使得

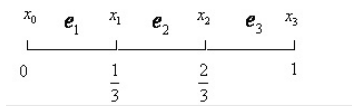
$$a(u,v) = f(v), \forall v \in H_E^1(I) \tag{47}$$

其中

$$\begin{aligned} H_E^1(I) &= \{u | u \in H^1(I), u(0) = 0\} \\ a(u,v) &= \int_0^1 (u'v' + \frac{\pi^2}{4} uv) dx \\ f(v) &= \frac{\pi^2}{2} \int_0^1 \sin\frac{\pi x}{2} v dx \end{aligned}$$

Step 2. 构造线性有限元空间

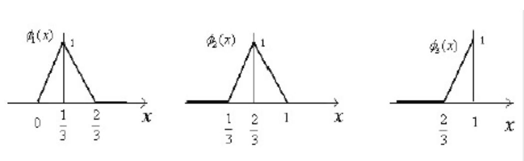
(2.1) 对区间I 作3 段等距剖分



(2.2) 定义线性Lagrange 有限元空间

$$V_E^h = \{u_h \in C(\bar{I}) : u_h|_{e_i} \in P_1(e_i), i = 1,2,3, u_h(0) = 0\}$$

(2.3) 写出V_E^h 的节点基函数



$$\phi_1(x) = \begin{cases} 3x, & x \in [0, \frac{1}{3}] \\ 2 - 3x, & x \in [\frac{1}{3}, \frac{2}{3}] \\ 0, & \text{在别处} \end{cases}$$

$$\phi_2(x) = \begin{cases} 3x - 1, & x \in [\frac{1}{3}, \frac{2}{3}] \\ 3 - 3x, & x \in [\frac{2}{3}, 1] \\ 0, & \text{在别处} \end{cases}$$

$$\phi_3(x) = \begin{cases} 3x - 2, & x \in [\frac{2}{3}, 1] \\ 0, & \text{在别处} \end{cases}$$

(2.4) 给出空间V_E^h 中元素的（整体）表示，对$\forall u_h \in V_E^h$，有

$$u_h(x) = \sum_{j=1}^{3} u_j \phi_j(x) \tag{48}$$

其中，$u_i = u_h(x_i), i = 1, 2, 3.$

Step 3. 写出线性有限元方程

线性元变分问题：求$u_h \in V_E^h$，使得

$$a(u_h, v_h) = f(v_h), \forall v_h \in V_E^h \tag{49}$$

利用(48) 并将v_h 取为$\phi_i(x), i = 1, 2, 3$，则变分问题(49) 等价于：求$u_1, u_2, u_3 \in R$，使得

$$a(\sum_{j=1}^{3} u_j \phi_j, \phi_i) = f(\phi_i), i = 1, 2, 3$$

$\Leftrightarrow$

$$\sum_{j=1}^{3} a(\phi_j, \phi_i) u_j = f(\phi_i), i = 1, 2, 3$$

$\Leftrightarrow$

$$\sum_{j=1}^{3} a(\phi_i, \phi_j) u_j = f(\phi_i), i = 1, 2, 3$$

因此，线性有限元方程为

$$AU = b$$

其中

$$A=\begin{bmatrix} a(\phi_1,\phi_1) & a(\phi_1,\phi_2) & a(\phi_1,\phi_3)\\ a(\phi_2,\phi_1) & a(\phi_2,\phi_2) & a(\phi_2,\phi_3)\\ a(\phi_3,\phi_1) & a(\phi_3,\phi_2) & a(\phi_3,\phi_3)\end{bmatrix}$$

$$U=\begin{bmatrix} u_1\\ u_2\\ u_3\end{bmatrix},\quad b=\begin{bmatrix} f(\phi_1)\\ f(\phi_2)\\ f(\phi_3)\end{bmatrix}$$

解2): 显然$a(\phi_1,\phi_3)=a(\phi_3,\phi_1)=0$, 下面计算其它元素.

(a) 精确求解. 先考虑$a(\phi_1,\phi_1)$ 和$f(\phi_1)$.

$$\begin{aligned} a(\phi_1,\phi_1) &= \int_0^1 [({\phi'}_1)^2+\frac{\pi^2}{4}\phi_1^2]dx\\ &= \int_0^{\frac{1}{3}} [({\phi'}_1)^2+\frac{\pi^2}{4}\phi_1^2]dx+\int_{\frac{1}{3}}^{\frac{2}{3}} [({\phi'}_1)^2+\frac{\pi^2}{4}\phi_1^2]dx\\ &= \int_0^{\frac{1}{3}} [3^2+\frac{\pi^2}{4}(3x)^2]dx+\int_{\frac{1}{3}}^{\frac{2}{3}} [(-3)^2+\frac{\pi^2}{4}(2-3x)^2]dx\\ &= 6+\frac{\pi^2}{18}\end{aligned}$$

$$\begin{aligned} f(\phi_1) &= \frac{\pi^2}{2}\int_0^1 \sin\frac{\pi x}{2}\phi_1 dx\\ &= \frac{\pi^2}{2}\int_0^{\frac{1}{3}} \sin\frac{\pi x}{2}(3x)dx+\frac{\pi^2}{2}\int_{\frac{1}{3}}^{\frac{2}{3}} \sin\frac{\pi x}{2}(2-3x)dx\\ &= 6-3\sqrt{3}\end{aligned}$$

类似可以计算

$$a(\phi_1,\phi_2)=a(\phi_2,\phi_1)=\frac{\pi^2}{72}-3$$

$$a(\phi_2,\phi_2)=6+\frac{\pi^2}{18}$$

$$a(\phi_2,\phi_3)=a(\phi_3,\phi_2)=\frac{\pi^2}{72}-3$$

$$a(\phi_3,\phi_3)=3+\frac{\pi^2}{36}$$

(b) 中矩形公式: $\int_a^b g(x)dx \approx (b-a)g(\frac{a+b}{2})$.

以$a(\phi_1,\phi_1)$ 和$f(\phi_1)$ 的计算为例:

$$\begin{aligned}
a(\phi_1,\phi_1) &\approx \frac{1}{3}\times[3^2+\frac{\pi^2}{4}\times(3\times\frac{1}{6})^2] \\
&\quad +\frac{1}{3}\times[(-3)^2+\frac{\pi^2}{4}\times(2-3\times\frac{1}{2})^2] \\
&= \frac{1}{3}\times(9+\frac{\pi^2}{16})+\frac{1}{3}\times(9+\frac{\pi^2}{16}) \\
&= \frac{2}{3}\times(9+\frac{\pi^2}{16})\approx 6.411233517
\end{aligned}$$

$$\begin{aligned}
f_h(\phi_1) &\approx \frac{\pi^2}{2}\times\frac{1}{3}\times(\sin\frac{\pi\times\frac{1}{6}}{2})\times(3\times\frac{1}{6}) \\
&\quad +\frac{\pi^2}{2}\times\frac{1}{3}\times(\sin\frac{\pi\times\frac{1}{2}}{2})\times(2-3\times\frac{1}{2}) \\
&= \frac{\pi^2}{12}\times\sin\frac{\pi}{12}+\frac{\pi^2}{12}\times\sin\frac{\pi}{4}\approx 0.7944421
\end{aligned}$$

Step 4: 求解线性有限元方程

有限元方程在各节点处的近似解为:

$$u_h=(0.5057, 0.8759, 1.0114)'.$$

在各节点处真解为:

$$u=(0.5, 0.8660254, 1)'.$$

(下面说明误差函数$(u-u_h)(x)$ 在某些点逼近情况.

在$x=\frac{2}{3}$ 节点处:

$$\begin{aligned}
|(u-u_h)(x)| &= |0.8660254-0.8759| \\
&\approx 0.00987\le\frac{1}{2}\times10^{-1}.
\end{aligned}$$

习题4 :设u 是两点边值问题的二次连续可微解, 证明u_h 一致收敛到u, 收敛阶为$O(h)$(即给出$\|\cdot\|_\infty$ 下的线性有限元函数的误差估计).

知识点回顾：设$\{S_n(x)\}(x\in D)$ 是一函数序列，若对任意给定的$\varepsilon>0$，存在仅与ε 有关的正整数$N(\varepsilon)$，当$n>N(\varepsilon)$ 时

$$|S_n(x)-S(x)|<\varepsilon$$

对一切$x\in D$ 成立，则称$\{S_n(x)\}$ 在D 上一致收敛于$S(x)$.

解：由本章第二节的知识可知

$$\|u-u_h\|_1\le\beta Ch\|u''\|_{\infty,\bar{I}}=O(h\|u''\|_{\infty,\bar{I}})\tag{50}$$

利用$u(a)-u_h(a)=0$，对$\forall x\in[a,b]$，有

$$u(x)-u_h(x)=\int_a^x(u'-{u'}_h)dt$$

因此

$$\begin{aligned}|u(x)-u_h(x)| &\le \int_a^x|u'-{u'}_h|dt\\ &\le \int_a^b 1\cdot|u'-{u'}_h|dx\\ &\le (\int_a^b 1^2dx)^{\frac{1}{2}}\cdot(\int_a^b|u'-{u'}_h|^2dx)^{1/2}\quad (Schwarz\text{不等式})\\ &= (b-a)^{\frac{1}{2}}\cdot(\int_a^b|u'-{u'}_h|^2dx)^{1/2}\\ &\le (b-a)^{\frac{1}{2}}\cdot\|u-u_h\|_{1,\bar{I}}\end{aligned}$$

结合(50) 可知

$$\|u-u_h\|_\infty=\max_{x\in\bar{I}}|u(x)-u_h(x)|\le(b-a)^{\frac{1}{2}}\beta Ch\|u''\|_{\infty,\bar{I}}=O(h)$$