Lecture Notes, Introduction to Inverse Problems

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Chapter 1

Inverse problems and Fourier transforms

In this chapter, we consider two inverse problems that can be solved by simply taking inverse Fourier transforms. The applications are magnetic resonance imaging (MRI) and inverse scattering.

We then present classical results on the Fourier transform and introduce the Hilbert scale of functional spaces $H^s(\mathbb{R}^n)$. This scale is useful to understand a characteristic of many inverse problems, namely that unlike the inversion of a Fourier transform, they are *ill-posed*.

1.1 Magnetic Resonance Imaging (MRI)

MRI exploits the precession of the spin of protons (among others) in a magnetic field of strength $H(\mathbf{x})$. The axis of the precession is that of the magnetic field and the frequency of the precession is $\omega(\mathbf{x}) = \gamma H(\mathbf{x})$, where $\gamma = e/(2m)$ is called the gyromagnetic ratio, e is the electric charge of the proton and m its mass.

In a nutshell, MRI works as follows. We first impose a strong static magnetic field $\mathbf{H}_0 = H_0\mathbf{e}_z$ along the z axis. All protons end up with their spin parallel to $\mathbf{H}_0$ and slightly more so in the direction $\mathbf{H}_0$ than in $-\mathbf{H}_0$. This difference is responsible for a macroscopic *magnetization* $\mathbf{M}$ pointing in the same direction as $\mathbf{H}_0$.

In a second step, we send a spatially varying radio frequency magnetic wave pulse at the Larmor frequency $\omega_0 = \gamma|\mathbf{H}_0|$. In clinical MRI, the frequency is typically between 15 and 80 MHz (for hydrogen imaging). Notice that this corresponds to wavelengths between 20 and 120 m! (since $\omega = ck = 2\pi c/\lambda$ and $c \approx 3\,10^8$). So the wavelength is not what governs spatial resolution in MRI. For instance the pulse (assumed homogeneous in $\mathbf{x}$ to start with) could be of the form $\mathbf{H}_1(t) = 2H_1\cos(\omega_0 t)\mathbf{e}_x$ and turned on for a duration t_p. Because the field oscillates at the Larmor frequency, the spins of the protons are affected. The resulting effect on the macroscopic magnetization is that it precesses around the axis $\mathbf{e}_z$ at frequency ω_0 through an angle given by

$$\theta = \gamma H_1 t_p.$$

Generally, t_p is chosen such that $\theta = \pi/2$ or $\theta = \pi$. The corresponding pulses are called 90^0 and 180^0 pulses, respectively. Thus, after a 90^0 pulse, the magnetization oscillates

in the xy plane and after a 180^0 pulse, the magnetization is pointing in the direction $-\mathbf{H}_0$.

Finally once the radio frequency is shut off (but not the static field $\mathbf{H}_0$), protons tend to realign with the static field $\mathbf{H}_0$. By doing so, they emit a radio frequency wave at the Larmor frequency ω_0 that can be *measured*. This wave is called the free induction decay (FID) signal. The FID signal after a 90^0 pulse will have the form

$$S(t) = \rho \cos(\omega_0 t) e^{-t/T_2}. \tag{1.1}$$

Here ρ is the density of the magnetic moments and T_2 is the spin-spin relaxation time. (There is also a spin-lattice relaxation time $T_1 \ll T_2$, which cannot be imaged with 90^0 pulses and which we ignore.) The main reason for doing all this is that the density ρ and the relaxation time T_2 depend on the tissue sample. We restrict ourselves to the reconstruction of ρ here, knowing that similar experiments can be devised to image T_2 (and T_1) as well.

Now human tissues are not homogeneous, which makes imaging a lot more useful. The first trick that allows for very good spatial resolution is that only tissue samples under a static magnetic field $\mathbf{H}$ such that $|\mathbf{H}| = \gamma\omega_0$ will be affected by the radio frequency pulse $\mathbf{H}_1$. Let us thus impose the static field $\mathbf{H}(z) = \mathbf{H}_0 + G_z z \mathbf{e}_z$. Only protons in the slice where z is close to 0 will be affected by the pulse $\mathbf{H}_1$ (since we have assumed that $|\mathbf{H}_0| = \gamma\omega_0$). This allows us to image the average tissue density $\int_{\mathbb{R}^2} \rho(x, y, 0) dxdy$ in the plane $z = 0$ and by changing $\mathbf{H}_0$ or ω_0 in any z-plane we want. MRI is thus a tomographic technique (tomos meaning slice in Greek).

This does not quite allow us to obtain $\rho(x, y, 0)$ yet. The transversal discrimination is obtained by imposing a static field linearly varying in the x and y directions. Remember that after the 90^0 pulse, the magnetization $\mathbf{M}(x, y, 0)$ rotates with frequency ω_0 in the xy plane (i.e., is orthogonal to $\mathbf{e}_z$), and is actually independent of x and y. Let us now impose a static field $\mathbf{H}(y) = \mathbf{H}_0 + G_y y \mathbf{e}_z$ for a duration T. Since the frequency of precession is related to the magnetic field, the magnetization will rotation at position y with frequency $\omega(y) = \omega_0 + \gamma G_y y$. Therefore, compared to the magnetization at $z = 0$, the magnetization at z will accumulate a phase during the time T the field $G_y y \mathbf{e}_z$ is turned on given by $T(\omega(y) - \omega_0) = T\gamma G_y y$. Once the field $G_y y \mathbf{e}_z$ is turned off, the magnetization will again rotate everywhere with frequency ω_0. However the phase will depend on the position y. This part of the process is call *phase encoding*. A measurement of the FID would then give us a radio frequency signal of the form

$$S(t; T) = e^{i\omega_0 t} \int_{\mathbb{R}^2} e^{i\gamma G_y T y} \rho(x, y, 0) dxdy. \tag{1.2}$$

By varying the time T or the gradient G_y, we see that we can obtain the frequency content in y of the density $\rho(x, y, 0)$. We are still missing the frequency content in the x variable. However nothing prevents us from adding a $x-$dependent static field during the FID measurements. Let us assume that after time T (where we reset time to be $t = 0$), we impose a static field of the form $\mathbf{H}(x) = \mathbf{H}_0 + G_x x \mathbf{e}_z$. The magnetization will now precess around the z axis with $x-$dependent frequency $\omega(x) = \omega_0 + \gamma G_x x$. This implies that the measured signal will be of the form

$$S(t; T) = \int_{\mathbb{R}^2} e^{i\gamma G_y T y} e^{i(\omega_0 + \gamma G_x x)t} \rho(x, y, 0) dxdy. \tag{1.3}$$

We have thus access to the *measured data*

$$d(k_x, k_y) = e^{-i\omega_0 k_x/(\gamma G_x)} S(\frac{k_x}{\gamma G_x}; \frac{k_y}{\gamma G_y}) = \int_{\mathbb{R}^2} e^{ik_y y} e^{ik_x x} \rho(x, y, 0) dx dy. \tag{1.4}$$

By varying T (or G_y) and t and G_x, we can obtain the above information for essentially all values of k_x and k_y. The reconstruction of $\rho(x, y, 0)$ is thus equivalent to inverting a *Fourier transform* and is given by

$$\rho(x, y, 0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i(k_x x + k_y y)} d(k_x, k_y) dk_x dk_y. \tag{1.5}$$

Several approximations have been made to obtain this reconstruction formula. Within this framework, we see however that density reconstructions are rather simple: all we have to do is to invert a Fourier transform.

1.2 One dimensional inverse scattering problem

Let us consider the one dimensional wave equation

$$\frac{1}{v^2(x)} \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = \delta(t)\delta(x - x_s), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{1.6}$$

with delta source term at time $t = 0$ and position $x = x_0$. We assume causality so that $U(x, t; x_s) = 0$ for $t < 0$ and assume that U is bounded. We measure $U(x_s, t; x_s)$ and want to infer some properties about $v(x)$. Our set-up is such that the geophone is at the same position as the source term.

We want to analyze the problem in the frequency domain. Let us define $u(x, \omega; x_s)$ the *causal Fourier transform* of $U(x, t; x_s)$ in the time variable

$$u(x, \omega; x_s) = \int_0^\infty U(x, t; x_s) e^{i\omega t} dt. \tag{1.7}$$

This transform can be inverted as follows:

$$U(x, t; x_s) = \frac{1}{2\pi} \int_{-\infty}^\infty u(x, \omega; x_s) e^{-i\omega t} d\omega. \tag{1.8}$$

The equation for $u(x, \omega; x_s)$ is the well-known Helmholtz equation

$$\frac{d^2 u}{dx^2} + \frac{\omega^2}{v^2(x)} u = -\delta(x - x_s), \qquad \omega \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{1.9}$$

augmented with the following *radiation conditions*

$$\frac{du}{dx} \mp \frac{i\omega}{v(x)} u \ \rightarrow \ 0, \quad \text{as} \quad x \ \rightarrow \ \pm\infty. \tag{1.10}$$

Since $U(x_s, t; x_s)$ is measured, $u(x_s, \omega; x_s)$ is known by Fourier transform.

Let us make a few assumptions. We assume that $v(x)$ is known on $(-\infty, x_s)$ (in Earth profile reconstructions, one is interested in positive depths only) and that we have a good constant approximation c of $v(x)$ on (x_s, ∞). We recast the latter assumption as

$$\frac{1}{v^2(x)} = \frac{1}{c^2}(1 + \alpha(x)), \tag{1.11}$$

where $\alpha(x)$ is small compared to $v(x)$. In effect we *linearize* the problem of the reconstruction of $v(x)$ from the scattering measurements $u(x_s, \omega; x_s)$. Moreover our linearization is about a spatially independent velocity profile c.

The advantage is that the resulting problem is easy to invert and admits an explicit solution in the asymptotic regime of smallness of $\alpha(x)$. Let us define by u_i (i for incident) the solution of the unperturbed problem

$$\frac{d^2u_i}{dx^2} + \frac{\omega^2}{c^2}u_i = -\delta(x - x_s), \qquad \frac{du_i}{dx} \mp \frac{i\omega}{c}u_i \ \to\ 0, \quad \text{as} \quad x \ \to \ \pm\infty. \tag{1.12}$$

The solution to the above problem is nothing but the *Green function* of the Helmholtz equation with constant coefficients. It is given explicitly by

$$u_i(x, \omega; x_s) = g(x - x_s, \omega) = -\frac{ce^{i\omega|x-x_s|/c}}{2i\omega}. \tag{1.13}$$

This can be verified by inspection. Notice that the radiation conditions are also satisfied.

Let us now decompose the Helmholtz solution as the superposition of the incident field and the *scattered field*:

$$u(x, \omega; x_s) = u_i(x, \omega; x_s) + u_s(x, \omega; x_s).$$

From the equations for u and u_i, we easily verify that u_s satisfies the following equation

$$\frac{d^2u_s}{dx^2} + \frac{\omega^2}{c^2}u_s = -\frac{\omega^2}{c^2}\alpha(x)(u_i + u_s), \tag{1.14}$$

with appropriate radiation conditions. By the principle of superposition, this implies that

$$u_s(x, \omega; x_s) = \omega^2 \int_{x_s}^{\infty} \frac{\alpha(y)}{c^2}(u_s + u_i)(y, \omega; x_s)g(x - y, \omega)dy. \tag{1.15}$$

We have not used so far the assumption that $\alpha(x)$ is small. This approximation is called the Born approximation and allows us to obtain from the above equation that u_s is also small and of order α. This implies that αu_s is of order α^2, hence much smaller than the other contributions in (1.15). So neglecting u_s on the right hand side of (1.15) and replacing u_i and g by their expression in (12.4), we deduce that a good approximation of u_s is

$$u_s(x_s, \frac{ck}{2}; x_s) = -\int_{\mathbb{R}} \frac{\alpha(x)}{4}e^{ikx}dx, \qquad k \in \mathbb{R}. \tag{1.16}$$

This implies that the scattering data $u_s(x_s, \omega; x_s)$ uniquely determine the fluctuation α and that the reconstruction is *stable*: all we have to do is to take the inverse Fourier transform of u_s to obtain $\alpha(x)$. Namely, we have

$$\alpha(x) = -\frac{2}{\pi}\int_{\mathbb{R}} e^{-ikx}u_s(x_s, \frac{ck}{2}; x_s)dk. \tag{1.17}$$

Several assumptions have been made to arrive at this result. However as was the case with the MRI problem, we obtain in fine a very simple reconstruction procedure: all we have to do is to compute an inverse Fourier transform.

1.3 Fourier transforms and well-posedness

We recall in this section some important facts about the Fourier transform and some functional (Hilbert) spaces that we will use throughout this course.

Let $f(\mathbf{x})$ be a complex-valued function in $L^2(\mathbb{R}^n)$ for some $n \in \mathbb{N}^*$, which means a (measurable) function on $\mathbb{R}^n$ that is square integrable in the sense that

$$\|f\|^2 = \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 d\mathbf{x} < \infty. \tag{1.18}$$

Here $\|f\|$ is the $L^2(\mathbb{R}^n)$-norm of f and $d\mathbf{x}$ the Lebesgue (volume) measure on $\mathbb{R}^n$. We define the Fourier transform of f as

$$\hat{f}(\mathbf{k}) = [\mathcal{F}_{\mathbf{x}\to\mathbf{k}} f](\mathbf{k}) = \int_{\mathbb{R}^n} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) d\mathbf{x}. \tag{1.19}$$

We know that $\hat{f}(\mathbf{k}) \in L^2(\mathbb{R}^n)$ and the Fourier transform admits an inverse on $L^2(\mathbb{R}^n)$ given by

$$f(\mathbf{x}) = [\mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}} \hat{f}](\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{f}(\mathbf{k}) d\mathbf{k}. \tag{1.20}$$

More precisely we have the Parseval relation

$$(\hat{f}, \hat{g}) = (2\pi)^n (f, g), \tag{1.21}$$

where the Hermitian product is given by

$$(f, g) = \int_{\mathbb{R}^n} f(\mathbf{x}) \overline{g}(\mathbf{x}) d\mathbf{x}. \tag{1.22}$$

Here $\overline{g}$ is the complex conjugate to g. This implies that

$$\|\hat{f}\| = (2\pi)^{n/2} \|f\|. \tag{1.23}$$

So up to the factor $(2\pi)^{n/2}$, the Fourier transform and its inverse are isometries.

Important properties of the Fourier transform are how they interact with differentiation and convolutions. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multi-index of non-negative components $\alpha_j \geq 0$, $1 \leq j \leq n$ and let $|\alpha| = \sum_{i=1}^n \alpha_j$ be the length of the multi-index. We then define the differentiation D^α of degree $|\alpha|$ as

$$D^\alpha = \prod_{i=1}^n \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}. \tag{1.24}$$

We then deduce from the definition (1.19) that

$$\mathcal{F}_{\mathbf{x}\to\mathbf{k}}[D^\alpha f](\mathbf{k}) \;=\; \Big(\prod_{j=1}^n (ik_j)^{\alpha_j} \Big) \, [\mathcal{F}_{\mathbf{x}\to\mathbf{k}} f](\mathbf{k}). \tag{1.25}$$

Let us now define the convolution as

$$f * g(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})d\mathbf{y}. \tag{1.26}$$

We then verify that

$$\begin{array}{rclcrcl} \mathcal{F}_{\mathbf{x}\to\mathbf{k}}(f*g) & = & \mathcal{F}_{\mathbf{x}\to\mathbf{k}} f \mathcal{F}_{\mathbf{x}\to\mathbf{k}} g, & \text{i.e.} & \widehat{f*g} & = & \hat{f}\hat{g}, \\ \mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}}(\hat{f}*\hat{g}) & = & (2\pi)^n f\, g & \text{i.e.} & \hat{f}*\hat{g} & = & (2\pi)^d \widehat{fg}. \end{array} \tag{1.27}$$

So the Fourier transform *diagonalizes* differential operators (replaces them by multiplication in the Fourier domain). However Fourier transforms replace products by non-local convolutions.

For the two inverse problems we have seen in earlier sections, this means that the reconstruction is a *well-posed* problem. Let X and Y be Banach spaces (nice functional spaces with a norm) and let A be a linear operator from X to Y. For every $y \in Y$, we would like to solve the linear problem

$$\text{Find } x \text{ such that} \qquad Ax = y. \tag{1.28}$$

What we mean by a *well posed* problem is a problem such that A is invertible ($A^{-1}y$ is defined for all $y \in Y$) and of bounded inverse (i.e., $\|A^{-1}y\|_X \leq C\|y\|_Y$ for a constant C that depends on A but not on $y \in Y$).

The Fourier transform is a well-posed operator from $X = L^2(\mathbb{R}^n)$ to $Y = L^2(\mathbb{R}^n)$ since the inverse Fourier transform is also defined from $Y = L^2(\mathbb{R}^n)$ to $X = L^2(\mathbb{R}^n)$ and is bounded as shown in (1.23). The resulting important effect in practice is that the reconstructions encountered in the two preceding sections are *stable*. Stability is meant with respect to some noise in the data. Let us assume that we measure

$$\hat{d}(\mathbf{k}) = \hat{f}(\mathbf{k}) + \hat{N}(\mathbf{k}),$$

where we know that $\delta = \|\hat{N}\|$ is relatively small. Then the error in the reconstruction will also be of order δ in the $X = L^2(\mathbb{R}^n)$ norm. Indeed let $d(\mathbf{x})$ be the reconstructed function from the data $d(\mathbf{k})$ and $f(\mathbf{x})$ be the real function we are after. Then we have

$$\|d - f\| = \|\mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}}\hat{d} - \mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}}\hat{f}\| = \|\mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}}(\hat{d} - \hat{f})\| = (2\pi)^{-n/2}\delta. \tag{1.29}$$

In other words, measurements errors can still be seen in the reconstruction. The resulting image is not perfect. However the error due to the noise has not been amplified too drastically. Another way of saying the same thing with the notation of (1.28) is to say that the error between two solutions x_1 and x_2 corresponding to two sets of data y_1 and y_2 satisfies that

$$\|x_1 - x_2\|_X \leq C\|\hat{y}_1 - \hat{y}_2\|_Y. \tag{1.30}$$

Let us stress that the choice of the spaces X and Y and the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ matters. The definition and the boundedness operator A^{-1} depends upon these choices.

1.4 Hilbert scale and ill-posedness

The problems we considered in earlier sections are essentially the only examples of well-posed inversion problems we will encounter in this course. All the other ones will be therefore *ill-posed*. Being ill-posed **does not** mean that a problem cannot be solved. In the case of the linear problem (1.28), it means that either A is not invertible on the whole space Y (i.e., the range of A defined by $\text{Range}(A) = A(X)$ is a proper subset of Y; that is to say, is not equal to Y), or that A^{-1} is not bounded.

Among ill-posed problems, some are called *mildly ill-posed*. We will see many examples of such problems. The other ones are called *severely ill-posed*. We will also encounter some examples of such problems.

The main reason an inverse problem is ill-posed is because the forward operator is smoothing. The forward (or direct) operator is the operator that maps what we wish to reconstruct to the noise-free measured data. The operator A in (1.28) is an example of a forward operator. What we mean by smoothing is that Ax is "more regular" than x, in the sense that details (small scale structures) are attenuated by the forward mapping. This does not mean that the details cannot be reconstructed. In many cases they can. It means however that the reconstruction has to undo this smoothing, i.e. has to deregularize. This works as long as no noise is present in the data. However, as soon as the data are noisy (i.e., always in practice), the deregularization process has the effect of amplifying the noise in a way that cannot be controlled.

The answer to this problem requires to impose some regularity assumptions on the function we wish to reconstruct. These assumptions are not too drastic for *mildly ill-posed*. They are very drastic for *severely ill-posed* problems.

In order to quantify the degree of ill-posedness, we will use the following scale of function spaces. Let $s \geq 0$ be a non-negative real-number. We define the scale of Hilbert spaces $H^s(\mathbb{R}^n)$ as the space of measurable functions $f(\mathbf{x})$ such that

$$\|f\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\mathbf{k}|^2)^s |\mathcal{F}_{\mathbf{x}\to\mathbf{k}} f|^2(\mathbf{k}) d\mathbf{k} < \infty. \tag{1.31}$$

We verify that $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ since the Fourier transform is an isometry. We also verify that

$$\{f \in H^1(\mathbb{R})\} \quad \Longleftrightarrow \quad \Big\{ f \in L^2(\mathbb{R}) \text{ and } \frac{\partial f}{\partial x_i} \in L^2(\mathbb{R}), 1 \leq i \leq n \Big\}. \tag{1.32}$$

This results from (1.25). More generally the space $H^m(\mathbb{R}^n)$ for $m \in \mathbb{N}$ is the space of functions such that all partial derivatives of f of order up to m are square integrable. The advantage of the definition (1.31) is that it holds for real values of s. So $H^{1/2}(\mathbb{R}^n)$ is roughly the space of functions such that "half-derivatives" are square integrable. Notice also that s characterizes the degree of smoothness of a function $f(\mathbf{x})$. The larger s, the smoother the function $f \in H^s(\mathbb{R}^n)$, and the faster the decay of its Fourier transform $\hat{f}(\mathbf{k})$ as can be seen from the definition (1.31).

It is also useful to define the Hilbert scale for functions supported on subdomains of $\mathbb{R}^n$. Let Ω be a sufficiently smooth subdomain of $\mathbb{R}^n$. We define two scales. The first scale is $H^s_0(\Omega)$, defined as the closure of $C^\infty_0(\Omega)$, the space of functions of class C^∞ with support in Ω (so these functions and all their derivatives vanish at the boundary of Ω),

for the norm in $H^s(\mathbb{R}^n)$. Thus, $f \in H_0^s(\Omega)$ can be described as the limit of functions $f_n \in C_0^\infty(\mathbb{R})$ uniformly bounded in $H^s(\mathbb{R}^n)$. We also define $H^s(\Omega)$ as the space of functions f on Ω that can be extended to functions f^* in $H^s(\mathbb{R}^n)$ (i.e., $f = f^*\chi_\Omega$, where χ_Ω is the characteristic function of Ω) and $\|f\|_{H^s(\Omega)}$ is the lower bound of $\|f\|_{H^s(\mathbb{R}^n)}$ over all possible extensions. The are several (sometimes not exactly equivalent) ways to define the scale of Hilbert spaces $H^s(\Omega)$. We refer the reader to [2] for additional details.

Finally, it is also convenient to define H^s for negative values of s. We define $H^{-s}(\mathbb{R}^n)$ for $s \geq 0$ as the subspace of $\mathcal{S}'(\mathbb{R}^n)$, the space of *tempered distributions*, such that (1.31) holds. For bounded domains we define $H^{-s}(\Omega)$ as the dual to $H_0^s(\Omega)$ equipped with the norm

$$\|f\|_{H^{-s}(\Omega)} = \sup_{g \in H_0^s(\Omega)} \frac{\int_\Omega f g d\mathbf{x}}{\|g\|_{H_0^s(\Omega)}}. \tag{1.33}$$

We can verify that the two definitions agree when $\Omega = \mathbb{R}^n$, in which case $H_0^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

Let us demonstrate on a simple example how the Hilbert scale can be used to understand the ill-posedness of inverse problems. Let $f(\mathbf{x})$ be a function in $L^2(\mathbb{R})$ and let u be the solution in $L^2(\mathbb{R})$ of the following ODE

$$-u'' + u = f, \qquad x \in \mathbb{R}. \tag{1.34}$$

There is a unique solution in $L^2(\mathbb{R})$ to the above equation given by

$$u(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|y-x|} f(y) dy = (g * f)(x), \qquad g(x) = \frac{1}{2}e^{-|x|},$$

as can be verified by inspection. Notice that the above formula is nothing but (12.4) with $\omega/c = i$. Everything simplifies in the Fourier domain as

$$(1 + k^2)\hat{u}(k) = \hat{f}(k).$$

This implies that u is not only in $L^2(\mathbb{R})$ but also in $H^2(\mathbb{R})$ as is easily verified. Let us define the operator A as follows

$$A : \begin{array}{ccc} L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ f & \mapsto & Af = u, \end{array} \tag{1.35}$$

where u is the solution to (1.34). As such the operator A is not invertible on $L^2(\mathbb{R})$. Indeed the inverse of the operator A is formally defined by $A^{-1}u = -u'' + u$. However for $u \in L^2(\mathbb{R})$, $-u''$ is not a function in $L^2(\mathbb{R})$ but a distribution in $H^{-2}(\mathbb{R})$. The inverse problem consisting of reconstructing $f(\mathbf{x}) \in L^2(\mathbb{R})$ from $u(\mathbf{x}) \in L^2(\mathbb{R})$ is therefore *ill-posed*. The reason is that the operator A is regularizing.

However let us define the "same" operator

$$\tilde{A} : \begin{array}{ccc} L^2(\mathbb{R}) & \to & H^2(\mathbb{R}) \\ f & \mapsto & \tilde{A}f = u. \end{array} \tag{1.36}$$

Now $\tilde{A}$ is invertible from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ and its inverse is indeed given by $\tilde{A}^{-1}u = -u'' + u$. So $\tilde{A}$ is *well-posed* (from $L^2(\mathbb{R})$ to $H^2(\mathbb{R})$) as can be easily verified.

So why do we bother defining ill-posed problems? The main reason pertains to **noise**. If we assume that our noise (the error between measurement u_1 and measurement u_2) is small in the H^2-norm, so that $\|u_1 - u_2\|_{H^2(\mathbb{R})} \leq \delta$, then there is no problem. The reconstruction will also be accurate in the sense that $\|f_1 - f_2\|_{L^2(\mathbb{R})} \leq C\delta$, where $f_j = \tilde{A}^{-1}u_j$, $j = 1, 2$. However in general noise is not small in the strong norm $H^2(\mathbb{R})$, but rather in the weaker norm $L^2(\mathbb{R})$. The situation is then more complicated as noise can be arbitrarily amplified by the unbounded operator "A^{-1}". The inverse operator $\tilde{A}^{-1}$ cannot be applied directly to the data and rather needs to be *regularized*. This will be the object of a future lecture.

Let us conclude this section by a definition of *mildly* ill-posed problems. The problem (1.28) with $X = Y = L^2(\mathbb{R}^n)$ is said to be *mildly ill-posed* provided that there exists a positive constant C and $\alpha > 0$ such that

$$\|Af\|_{H^\alpha(\mathbb{R}^n)} \geq C\|f\|_{L^2(\mathbb{R}^n)}. \tag{1.37}$$

We define $\|Af\|_{H^\alpha(\mathbb{R}^n)} = +\infty$ if f does not belong to $H^\alpha(\mathbb{R}^n)$. We say that A is *mildly ill-posed of order* α if α is the smallest real number such that (1.37) holds for some $C = C(\alpha)$. Notice that we can choose any $\alpha \geq 2$ for $\tilde{A}$ so the operator that maps f to u solution of (1.34) is a mildly ill-posed problem of order 2. The operator $\tilde{A}$ in (1.36) essentially integrates twice the function f. Any operator that corresponds to a finite number of integrations is therefore essentially mildly ill-posed.

We call the inversion a *severely* ill-posed problems when no such α exists. We will see examples of such problems in later Lectures. A typical example is the following operator

$$Bf(x) = \mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}}[e^{-k^2}\mathcal{F}_{\mathbf{x}\to\mathbf{k}}f](x). \tag{1.38}$$

Physically this corresponds to solving the heat equation forward in time: a very smoothing operation. We easily verify that the operator B maps $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Hoverer it destroys high frequencies exponentially fast. This is faster than any integration would do (m integrations essentially multiply high frequencies by $|k|^{-m}$) so no $\alpha > 0$ in (1.37) exists for B. Notice that it does not mean that B is never invertible. Indeed for sufficiently smooth functions $g(x)$ (for instance such that $\hat{g}(k)$ has compact support), we can easily define the inverse operator

$$B^{-1}g(x) = \mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}}[e^{k^2}\mathcal{F}_{\mathbf{x}\to\mathbf{k}}f](x).$$

Physically, this corresponds to solving the heat equation backwards in time. It is clear that on a space of sufficiently smooth functions, we have $BB^{-1} = B^{-1}B = Id$. Yet, if a little bit of noise is added to the data, it will be amplified by e^{k^2} in the Fourier domain. This has devastating effects on the reconstruction.

The following lectures will focus on inverse problems in medical imaging and earth science that will have a structure close to the operator $\tilde{A}$ (for the nice ones) or close to the operator B (for the nasty ones). We will come back later to techniques to solve numerically ill-posed problems.

Chapter 2

The Radon transform

In this chapter we consider the simplest example of integral geometry: the integration of a two-dimensional function along all possible lines in the plane (the Radon transform) and its inversion. This is the mathematical framework for one of the most successful medical imaging techniques: computed (or computerized) tomography (CT).

2.1 Transmission Tomography

In transmission tomography, objects to be imaged are probed with non-radiating sources such as X-rays in medical imaging. X-rays are composed of sufficiently high energy photons so that they propagate in the object along straight lines unless they interact with the underlying medium and get absorbed. Let $\mathbf{x}$ denote spatial position and $\boldsymbol{\theta}$ orientation. We denote by $u(\mathbf{x}, \boldsymbol{\theta})$ the density of X-rays with position $\mathbf{x}$ and orientation $\boldsymbol{\theta}$, and by $a(\mathbf{x})$ the linear attenuation coefficient. Velocity is normalized to 1 so that locally the density $u(\mathbf{x}, \boldsymbol{\theta})$ satisfies the following equation

$$\boldsymbol{\theta} \cdot \nabla_{\mathbf{x}} u(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) = 0, \qquad \mathbf{x} \in \Omega, \boldsymbol{\theta} \in S^1. \tag{2.1}$$

Here Ω is the physical domain (assumed to be convex) of the object and S^1 is the unit circle. We identify any point $\boldsymbol{\theta} \in S^1$ with the angle $\theta \in [0, 2\pi)$ such that $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$. The advection operator is given by

$$\boldsymbol{\theta} \cdot \nabla_{\mathbf{x}} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

and models free convection of X-rays and $a(\mathbf{x})u(\mathbf{x}, \boldsymbol{\theta})$ models the number of absorbed photons per unit distance at $\mathbf{x}$.

The probing source is emitted at the boundary of the domain and takes the form

$$u(\mathbf{x}, \boldsymbol{\theta}) = \delta(\mathbf{x} - \mathbf{x}_0)\delta(\boldsymbol{\theta} - \boldsymbol{\theta}_0), \tag{2.2}$$

where $\mathbf{x}_0 \in \partial\Omega$ and $\boldsymbol{\theta}_0$ is *entering* the domain, i.e., $\boldsymbol{\theta}_0 \cdot \mathbf{n}(\mathbf{x}_0) < 0$ where $\mathbf{n}$ is the outward normal to Ω.

The solution to (2.1)-(2.2), which is a first-order ODE in the appropriate variables, is then given by

$$\begin{aligned} u(\mathbf{x}_0 + s\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) &= \exp\Big(- \int_0^s a(\mathbf{x}_0 + t\boldsymbol{\theta}_0) dt \Big), \qquad s > 0,\ \mathbf{x}_0 + s\boldsymbol{\theta}_0 \in \Omega \\ u(\mathbf{x}, \boldsymbol{\theta}) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{elsewhere.} \end{aligned} \tag{2.3}$$

Intensity measurements are made at the boundary of Ω, i.e., for this source term at the unique point $\mathbf{x}_1 \in \partial\Omega$ such that $\mathbf{x}_1 = \mathbf{x}_0 + \tau\boldsymbol{\theta}_0$ with positive travel time $\tau = \tau(\mathbf{x}_0, \boldsymbol{\theta}_0)$. By taking the logarithm of the measurements, we have thus access to

$$\int_0^{\tau(\mathbf{x}_0;\boldsymbol{\theta}_0)} a(\mathbf{x}_0 + t\boldsymbol{\theta}_0)dt.$$

This is the integral of a over the line segment $(\mathbf{x}_0, \mathbf{x}_1)$. By varying the point $\mathbf{x}_0$ and the direction of the incidence $\boldsymbol{\theta}_0$, we can have access to all possible integrals of $a(\mathbf{x})$ over segments (and since a can be extended by 0 outside Ω without changing the measurements over all lines) crossing the domain Ω.

The question in transmission tomography is thus how one can recover a function $a(\mathbf{x})$ from its integration over all possible lines in the plane $\mathbb{R}^2$. This will be the object of subsequent sections. Of course in practice one needs to consider integration over a finite number of lines. How these lines are chosen is crucial to obtain a rapid and practical inversion algorithm. We do not consider the problems of discretization in this course and refer the reader to the following excellent works in the literature [14, 15].

2.2 Radon transforms

We have seen that the problem of transmission tomography consisted of reconstructing a function from its integration along lines. We have considered the two-dimensional problem so far. Since X-rays do not scatter, the three dimensional problem can be treated by using the two-dimensional theory: it suffices to image the object slice by slice using the two dimensional reconstruction, as we did in MRI (Transmission Tomography is indeed a tomographic method).

We need to represent (parameterize) lines in the plane in a more convenient way than by describing them as the line joining $\mathbf{x}_0$ and $\mathbf{x}_1$. This is done by defining an origin $\mathbf{0} = (0, 0)$, a direction $(\cos\theta, \sin\theta) = \boldsymbol{\theta} \in S^1$, and a scalar s indicating the (signed) distance of the line to $\mathbf{0}$. The line is defined by the set of points $\mathbf{x}$ such that $\mathbf{x} \cdot \boldsymbol{\theta}^\perp = s$, where $\boldsymbol{\theta}^\perp$ is the rotation of $\boldsymbol{\theta}$ by $\pi/2$, i.e., the vector given by $\boldsymbol{\theta}^\perp = (-\sin\theta, \cos\theta)$. More precisely, for a smooth function $f(\mathbf{x})$ on $\mathbb{R}^2$, we define the Radon transform $Rf(s, \theta)$ for $(s, \theta) \in Z = \mathbb{R} \times (0, 2\pi)$ as

$$Rf(s, \theta) = \int_{\mathbb{R}} f(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})dt = \int_{\mathbb{R}^2} f(\mathbf{x})\delta(\mathbf{x} \cdot \boldsymbol{\theta}^\perp - s)d\mathbf{x}. \tag{2.4}$$

Notice that the cylinder Z is a double covering of the space of lines in the real plane $\mathbb{R}^2$. Indeed one easily verifies that

$$Rf(s, \theta) = Rf(-s, \theta + \pi), \qquad \text{as} \quad \{\mathbf{x} \cdot \boldsymbol{\theta}^\perp = s\} = \{\mathbf{x} \cdot (-\boldsymbol{\theta}^\perp) = (-s)\}.$$

Let us derive some important properties of the Radon transform. We first define the operator

$$R_\theta f(s) = Rf(s, \theta). \tag{2.5}$$

This notation will often be useful in the sequel. The first important result on the Radon transform is the *Fourier slice theorem*:

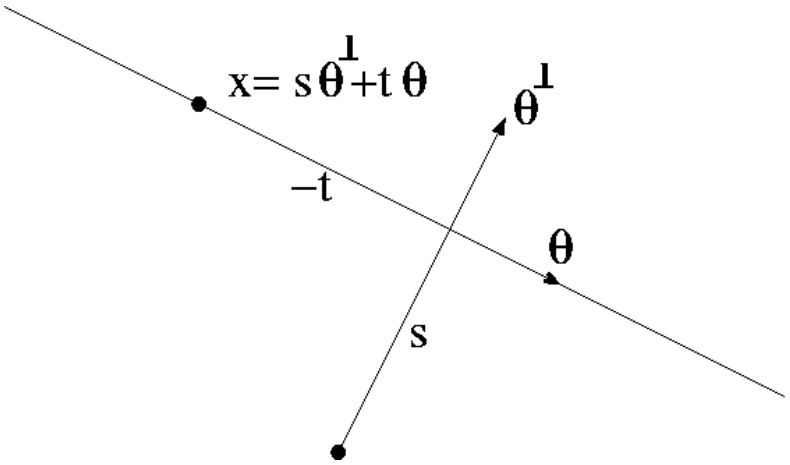

Figure 2.1: Geometry of the Radon transform.

Theorem 2.2.1 *Let $f(\mathbf{x})$ be a smooth function. Then for all $\boldsymbol{\theta} \in S^1$, we have*

$$[\mathcal{F}_{s\to\sigma} R_\theta f](\sigma) = \widehat{R_\theta f}(\sigma) = \hat{f}(\sigma\boldsymbol{\theta}^\perp), \qquad \sigma \in \mathbb{R}. \tag{2.6}$$

Proof. We have that

$$\widehat{R_\theta f}(\sigma) = \int_{\mathbb{R}} e^{-is\sigma} \int_{\mathbb{R}^2} f(\mathbf{x})\delta(\mathbf{x}\cdot\boldsymbol{\theta}^\perp - s)d\mathbf{x}ds = \int_{\mathbb{R}^2} e^{-i\mathbf{x}\cdot\boldsymbol{\theta}^\perp\sigma} f(\mathbf{x})d\mathbf{x}.$$

This concludes the proof. □

This result should not be surprising. For a given value of $\boldsymbol{\theta}$, the Radon transform gives us the integration of f over all lines parallel to $\boldsymbol{\theta}$. So obviously the oscillations in the direction $\boldsymbol{\theta}$ are lost, but not the oscillations in the orthogonal direction $\boldsymbol{\theta}^\perp$. Yet the oscillations of f in the direction $\boldsymbol{\theta}^\perp$ are precisely of the form $\hat{f}(\alpha\boldsymbol{\theta}^\perp)$ for $\alpha \in \mathbb{R}$. It is therefore not surprising that the latter can be retrieved from the Radon transform $R_\theta f$.

Notice that this result also gives us a *reconstruction* procedure. Indeed, all we have to do is to take the Fourier transform of $R_\theta f$ in the variable s to get the Fourier transform $\hat{f}(\sigma\boldsymbol{\theta}^\perp)$. It remains then to obtain the latter Fourier transform for all directions $\boldsymbol{\theta}^\perp$ to end up with the full $\hat{f}(\mathbf{k})$ for all $\mathbf{k} \in \mathbb{R}^2$. Then the object is simply reconstructed by using the fact that $f(\mathbf{x}) = (\mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}}\hat{f})(\mathbf{x})$. We will consider other (equivalent) reconstruction methods and explicit formulas later on.

Before doing so, we derive additional properties satisfied by Rf. From the Fourier slice theorem, we deduce that

$$R_\theta[\frac{\partial f}{\partial x_i}](s) = \theta_i^\perp \frac{d}{ds}(R_\theta f)(s). \tag{2.7}$$

Exercise 2.2.1 Verify (2.7).

This is the equivalent for Radon transforms to (1.25) for the Fourier transform.

Let us now look at regularizing properties of the Radon transform. To do so we need to introduce a function $\chi(\mathbf{x})$ of class $C_0^\infty(\mathbb{R}^2)$ (i.e., χ is infinitely many times differentiable) and with compact support (i.e. there is a radius R such that $\chi(\mathbf{x}) = 0$

for $|\mathbf{x}| > R$. When we are interested in an object defined on a domain Ω, we can choose $\chi(\mathbf{x}) = 1$ for $\mathbf{x} \in \Omega$.

As we did for $\mathbb{R}^n$ in the previous chapter, let us now define the Hilbert scale for the cylinder Z as follows. We say that $g(s,\theta)$ belongs to $H^s(Z)$ provided that

$$\|g\|^2_{H^s(Z)} = \int_0^{2\pi} \int_{\mathbb{R}} (1+\sigma^2)^s |\mathcal{F}_{s\to\sigma} g(\sigma)|^2 d\sigma d\theta < \infty. \tag{2.8}$$

This is a constraint stipulating that the Fourier transform in the s variable decays sufficiently fast at infinity. No constraint is imposed on the directional variable other than having a square-integrable function. We have then the following result:

Theorem 2.2.2 *Let $f(\mathbf{x})$ be a distribution in $H^s(\mathbb{R}^2)$ for some $s \in \mathbb{R}$. Then we have the following bounds*

$$\begin{array}{rcl} \sqrt{2}\|f\|_{H^s(\mathbb{R}^2)} & \leq & \|Rf\|_{H^{s+1/2}(Z)} \\ \|R(\chi f)\|_{H^{s+1/2}(Z)} & \leq & C_\chi \|\chi f\|_{H^s(\mathbb{R}^2)}. \end{array} \tag{2.9}$$

The constant C_χ depends on the function $\chi(\mathbf{x})$, and in particular on the size of its support.

Proof. From the Fourier slice theorem $\widehat{R_\theta(w)} = \widehat{w}(\sigma\boldsymbol{\theta})$, we deduce that

$$\begin{array}{rcl} \displaystyle\int_Z |\widehat{R(w)}(\sigma,\theta)|^2 (1+\sigma^2)^{s+1/2} d\sigma d\theta & = & \displaystyle\int_Z |\widehat{w}(\sigma\boldsymbol{\theta})|^2 (1+\sigma^2)^{s+1/2} d\sigma d\theta \\ & = & \displaystyle 2\int_{\mathbb{R}^2} |\widehat{w}(\mathbf{k})|^2 \frac{(1+|\mathbf{k}|^2)^{s+1/2}}{|\mathbf{k}|} d\mathbf{k}, \end{array}$$

using the change of variables from polar to Cartesian coordinates so that $d\mathbf{k} = \sigma d\sigma d\theta$ and recalling that $\hat{f}(\sigma\boldsymbol{\theta}) = \hat{f}((-\sigma)(-\boldsymbol{\theta}))$. The first inequality in (2.9) then follows from the fact that $|\mathbf{k}|^{-1} \geq (1+|\mathbf{k}|^2)^{-1/2}$ using $w(\mathbf{x}) = f(\mathbf{x})$. The second inequality is slightly more difficult because of the presence of $|\mathbf{k}|^{-1}$. We now choose $w(\mathbf{x}) = f(\mathbf{x})\chi(\mathbf{x})$. Let us split the integral into $I_1 + I_2$, where I_1 accounts for the integration over $|\mathbf{k}| > 1$ and I_2 for the integration over $|\mathbf{k}| < 1$. Since $(1+|\mathbf{k}|^2)^{1/2} \leq \sqrt{2}|\mathbf{k}|$ for $|\mathbf{k}| > 1$, we have that

$$I_1 \leq 2\sqrt{2} \int_{\mathbb{R}^2} |\widehat{\chi f}(\mathbf{k})|^2 (1+|\mathbf{k}|^2)^s d\mathbf{k} \leq 2\sqrt{2}\|\chi f\|^2_{H^s}.$$

It remains to deal with I_2. We deduce from (1.26) that

$$I_2 \leq C\|\widehat{\chi f}\|^2_{L^\infty(\mathbb{R}^2)}$$

$\psi = 1$ on the support of χ so that $\psi\chi f = \chi f$ and let us define $\psi_{\mathbf{k}}(\mathbf{x}) = e^{-i\mathbf{x}\cdot\mathbf{k}}\psi(\mathbf{x})$. Upon using the definition (1.19), the Parseval relation (1.21) and the Cauchy Schwarz inequality $(f,g) \leq \|f\|\|g\|$, we deduce that

$$\begin{array}{l} |\widehat{\chi f}|(\mathbf{k}) = |\widehat{\psi\chi f}|(\mathbf{k}) = \Big| \displaystyle\int_{\mathbb{R}^2} \psi_{\mathbf{k}}(\mathbf{x})(\chi f)(\mathbf{x}) d\mathbf{x} \Big| = (2\pi)^{-2} \Big| \int_{\mathbb{R}^2} \widehat{\psi_{\mathbf{k}}}(\boldsymbol{\xi}) \widehat{\chi f}(\boldsymbol{\xi}) d\boldsymbol{\xi} \Big| \\ = (2\pi)^{-2} \Big| \displaystyle\int_{\mathbb{R}^2} \frac{\widehat{\psi_{\mathbf{k}}}(\boldsymbol{\xi})}{(1+|\boldsymbol{\xi}|^2)^{s/2}} (1+|\boldsymbol{\xi}|^2)^{s/2} \widehat{\chi f}(\boldsymbol{\xi}) d\boldsymbol{\xi} \Big| \leq \|\psi_{\mathbf{k}}\|_{H^{-s}(\mathbb{R}^2)} \|\chi f\|_{H^s(\mathbb{R}^2)}. \end{array}$$

Since $\psi(\mathbf{x})$ is smooth, then so is $\psi_{\mathbf{k}}$ uniformly in $|\mathbf{k}| < 1$ so that $\psi_{\mathbf{k}}$ belongs to $H^{-s}(\mathbb{R}^2)$ for all $s \in \mathbb{R}$ uniformly $|\mathbf{k}| < 1$. This implies that

$$I_2 \leq C\|\widehat{\chi f}\|^2_{L^\infty(\mathbb{R}^2)} \leq C_\chi \|\chi f\|^2_{H^s(\mathbb{R}^2)},$$

where the function depends on ψ, which depends on the support of χ. This concludes the proof. □

The theorem should be interpreted as follows. Assume that the function (or more generally a distribution) $f(\mathbf{x})$ has compact support. Then we can find a function χ which is of class C^∞, with compact support, and which is equal to 1 on the support of f. In that case, we can use the above theorem with $\chi f = f$. The constant C_χ depends then implicitly on the size of the support of $f(\mathbf{x})$. The above inequalities show that R is a *smoothing* operator. This is not really surprising as an integration over lines is involved in the definition of the Radon transform. However, the result tells us that the Radon transform is smoothing by exactly one half of a derivative. The second inequality in (2.9) tells us that the factor $1/2$ is optimal, in the sense that the Radon transform does not regularize more than one half of a derivative. Moreover this corresponds to (1.37) with $\alpha = 1/2$, which shows that the inversion of the Radon transform in two dimensions is a mildly ill-posed problem of order $\alpha = 1/2$: when we reconstruct f from Rf, a differentiation of order one half is involved.

Let us now consider such *explicit* reconstruction formulas. In order to do so, we need to introduce two new operators, the adjoint operator R^* and the Hilbert transform H. The adjoint operator R^* to R (with respect to the usual inner products $(\cdot,\cdot)_{\mathbb{R}^2}$ and $(\cdot,\cdot)_z$ on $L^2(\mathbb{R})$ and $L^2(Z)$, respectively) is given for every smooth function $g(s,\theta)$ on Z by

$$(R^*g)(\mathbf{x}) = \int_0^{2\pi} g(\mathbf{x}\cdot\boldsymbol{\theta}^\perp, \theta)d\theta, \qquad \mathbf{x} \in \mathbb{R}^2. \tag{2.10}$$

That R^* is indeed the adjoint operator to R is verified as follows

$$\begin{aligned}
(R^*g, f)_{\mathbb{R}^2} &= \int_{\mathbb{R}^2} f(\mathbf{x}) \int_0^{2\pi} g(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)d\theta d\mathbf{x} \\
&= \int_{\mathbb{R}^2} f(\mathbf{x}) \int_0^{2\pi}\int_{\mathbb{R}} \delta(s - \mathbf{x}\cdot\boldsymbol{\theta}^\perp) g(s,\theta) ds d\theta d\mathbf{x} \\
&= \int_0^{2\pi}\int_{\mathbb{R}} g(s,\theta)\int_{\mathbb{R}^2} f(\mathbf{x})\delta(s-\mathbf{x}\cdot\boldsymbol{\theta}^\perp)d\mathbf{x}dsd\theta \\
&= \int_0^{2\pi}\int_{\mathbb{R}} g(s,\theta)(Rf)(s,\theta)dsd\theta \ = (g, Rf)_Z.
\end{aligned}$$

We also introduce the Hilbert transform defined for smooth functions $f(t)$ on $\mathbb{R}$ by

$$Hf(t) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(s)}{t-s}ds. \tag{2.11}$$

Here p.v. means that the integral is understood in the principal value sense, which in this context is equivalent to

$$Hf(t) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus(t-\varepsilon,t+\varepsilon)} \frac{f(s)}{t-s}ds.$$

Both operators turn out to be *local* in the Fourier domain in the sense that they are multiplications in the Fourier domain. More precisely we have the following lemma.

Lemma 2.2.3 *We have in the Fourier domain the following relations:*

$$\begin{aligned}(\mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}}R^*g)(\boldsymbol{\xi}) &= \frac{2\pi}{|\boldsymbol{\xi}|}\Big((\mathcal{F}_{s\to\sigma}g)(-|\boldsymbol{\xi}|,\hat{\boldsymbol{\xi}}^{\perp}) + (\mathcal{F}_{s\to\sigma}g)(|\boldsymbol{\xi}|,-\hat{\boldsymbol{\xi}}^{\perp})\Big)\\ (\mathcal{F}_{t\to\sigma}Hf)(\sigma) &= -i\,\mathrm{sign}(\sigma)\mathcal{F}_{\mathrm{t}\to\sigma}\mathrm{f}(\sigma).\end{aligned} \tag{2.12}$$

We have used the notation $\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}/|\boldsymbol{\xi}|$. For $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$ with $\boldsymbol{\theta} \in S^1$ and $\theta \in (0, 2\pi)$, we also identify the functions $g(\theta) = g(\boldsymbol{\theta})$. Assuming that $g(s,\theta) = g(-s,\theta+\pi)$, which is the case in the image of the Radon transform (i.e., when there exists f such that $g = Rf$), and which implies that $\hat{g}(\sigma,\boldsymbol{\theta}) = \hat{g}(-\sigma,-\boldsymbol{\theta})$ we have using shorter notation the equivalent statement:

$$\begin{aligned}\widehat{R^*g}(\boldsymbol{\xi}) &= \frac{4\pi}{|\boldsymbol{\xi}|}\,\hat{g}(|\boldsymbol{\xi}|,-\hat{\boldsymbol{\xi}}^{\perp})\\ \widehat{Hf}(\sigma) &= -i\,\mathrm{sign}(\sigma)\hat{\mathrm{f}}(\sigma).\end{aligned} \tag{2.13}$$

Proof. Let us begin with R^*g. We compute

$$\begin{aligned}&\widehat{R^*g}(\boldsymbol{\xi}) = \int e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}g(\mathbf{x}\cdot\boldsymbol{\theta}^{\perp},\theta)d\theta d\mathbf{x} = \int e^{-is\boldsymbol{\xi}\cdot\boldsymbol{\theta}^{\perp}}g(s,\theta)dsd\theta e^{-it\boldsymbol{\xi}\cdot\boldsymbol{\theta}}dt\\ &= \int 2\pi\delta(|\boldsymbol{\xi}|\hat{\boldsymbol{\xi}}\cdot\boldsymbol{\theta})\hat{g}(\boldsymbol{\xi}\cdot\boldsymbol{\theta}^{\perp},\theta)d\theta = \int \frac{2\pi}{|\boldsymbol{\xi}|}\delta(\hat{\boldsymbol{\xi}}\cdot\boldsymbol{\theta})\hat{g}(\boldsymbol{\xi}\cdot\boldsymbol{\theta}^{\perp},\theta)d\theta\\ &= \frac{2\pi}{|\boldsymbol{\xi}|}\Big(\hat{g}(-|\boldsymbol{\xi}|,\hat{\boldsymbol{\xi}}^{\perp}) + \hat{g}(|\boldsymbol{\xi}|,-\hat{\boldsymbol{\xi}}^{\perp})\Big).\end{aligned}$$

In the proof we have used that $\delta(\alpha x) = \alpha^{-1}\delta(x)$ and the fact that there are two directions, namely $\hat{\boldsymbol{\xi}}$ and $-\hat{\boldsymbol{\xi}}$ on the unit circle, which are orthogonal to $\hat{\boldsymbol{\xi}}^{\perp}$. When g is in the form $g = Rf$, we have $\hat{g}(-|\boldsymbol{\xi}|,\hat{\boldsymbol{\xi}}^{\perp}) = \hat{g}(|\boldsymbol{\xi}|,-\hat{\boldsymbol{\xi}}^{\perp})$, which explains the shorter notation (2.13).

The computation of the second operator goes as follows. We verify that

$$Hf(t) = \frac{1}{\pi}\Big(\frac{1}{x} * f(x)\Big)(t).$$

So in the Fourier domain we have

$$\widehat{Hf}(\sigma) = \frac{1}{\pi}\widehat{\frac{1}{x}}(\sigma)\hat{f}(\sigma) = -i\mathrm{sign}(\sigma)\hat{f}(\sigma).$$

The latter is a result of the following calculation

$$\frac{1}{2}\mathrm{sign}(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{ix\xi}}{i\xi}d\xi.$$

This concludes the proof of the lemma. □

The above calculations also show that $H^2 = H\circ H = -Id$, where Id is the identity operator, as can easily be seen in from its expression in the Fourier domain. We are now ready to introduce some reconstruction formulas.

Theorem 2.2.4 *Let $f(\mathbf{x})$ be a smooth function and let $g(s,\theta) = Rf(s,\theta)$ be its Radon transform. Then, f can explicitly be reconstructed from its Radon transform as follows:*

$$f(\mathbf{x}) = \frac{1}{4\pi} R^* \Big(\frac{\partial}{\partial s} H g(s,\theta) \Big)(\mathbf{x}). \tag{2.14}$$

In the above formula, the Hilbert transform H acts on the s variable.

Proof. The simplest way to verify the inversion formula is to do it in the Fourier domain. Let us denote by

$$w(s,\theta) = \frac{\partial}{\partial s} H g(s,\theta).$$

Since $g(-s,\theta+\pi) = g(s,\theta)$, we verify that the same property holds for w in the sense that $w(-s,\theta+\pi) = w(s,\theta)$. Therefore (2.14) is equivalent to the statement:

$$\hat{f}(\boldsymbol{\xi}) = \frac{1}{|\boldsymbol{\xi}|} \hat{w}(|\boldsymbol{\xi}|, -\hat{\boldsymbol{\xi}}^{\perp}), \tag{2.15}$$

according to (2.13). Notice that $\hat{w}$ is the Fourier transform of w in the *first variable* only.

Since in the Fourier domain, the derivation with respect to s is given by multiplication by $i\sigma$ and the Hilbert transform H is given by multiplication by $-i\text{sign}(\sigma)$, we obtain that

$$\mathcal{F}^{-1}_{\sigma \to s} \frac{\partial}{\partial s} H \mathcal{F}_{s \to \sigma} = |\sigma|.$$

In other words, we have

$$\hat{w}(\sigma,\theta) = |\sigma| \hat{g}(\sigma,\theta).$$

Thus (2.15) is equivalent to

$$\hat{f}(\boldsymbol{\xi}) = \hat{g}(|\boldsymbol{\xi}|, -\hat{\boldsymbol{\xi}}^{\perp}).$$

This, however, is nothing but the Fourier slice theorem stated in Theorem 2.2.1 since $(-\hat{\boldsymbol{\xi}}^{\perp})^{\perp} = \hat{\boldsymbol{\xi}}$ and $\boldsymbol{\xi} = |\boldsymbol{\xi}|\hat{\boldsymbol{\xi}}$. This concludes the proof of the reconstruction. □
The theorem can equivalently be stated as

$$Id = \frac{1}{4\pi} R^* \frac{\partial}{\partial s} H R = \frac{1}{4\pi} R^* H \frac{\partial}{\partial s} R. \tag{2.16}$$

The latter equality comes from the fact that H and ∂_s commute as can easily be observed in the Fourier domain (where they are both multiplications). Here, Id is the identity operator, which maps a function $f(\mathbf{x})$ to itself $Id(f) = f$.

Here is some additional useful notation in the manipulation of the Radon transform. Recall that $R_\theta f(s)$ is defined as in (2.5) by

$$R_\theta f(s) = Rf(s,\theta).$$

The adjoint R_θ (with respect to the inner products in $L^2_s(\mathbb{R})$ and $L^2_{\mathbf{x}}(\mathbb{R}^2)$) is given by

$$(R^*_\theta g)(\mathbf{x}) = g(\mathbf{x} \cdot \boldsymbol{\theta}^{\perp}). \tag{2.17}$$

Indeed (since $\boldsymbol{\theta}$ is *frozen* here) we have

$$\int_{\mathbb{R}} (R_\theta f)(s) g(s) ds = \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(\mathbf{x}) \delta(s - \mathbf{x} \cdot \boldsymbol{\theta}^\perp) g(s) d\mathbf{x} ds = \int_{\mathbb{R}^2} g(\mathbf{x} \cdot \boldsymbol{\theta}^\perp) f(\mathbf{x}) d\mathbf{x},$$

showing that $(R_\theta f, g)_{L^2(\mathbb{R})} = (f, R_\theta^* g)_{L^2(\mathbb{R}^2)}$. We can then recast the inversion formula as

$$Id = \frac{1}{4\pi} \int_0^{2\pi} \boldsymbol{\theta}^\perp \cdot \nabla R_\theta^* H R_\theta d\theta. \tag{2.18}$$

The only thing to prove here compared to previous formulas is that R_θ^* and the derivation commute, i.e., for any function $g(s)$ for $s \in \mathbb{R}$, we have

$$\boldsymbol{\theta}^\perp \cdot \nabla (R_\theta^* g)(\mathbf{x}) = (R_\theta^* \frac{\partial}{\partial s} g)(\mathbf{x}).$$

This results from the fact that both terms are equal to $g'(\mathbf{x} \cdot \boldsymbol{\theta}^\perp)$.

One remark on the smoothing properties of the Radon transform. We have seen that the Radon transform is a smoothing operator in the sense that the Radon transform is half of a derivative smoother than the original function. The adjoint operator R^* enjoys exactly the same property: it regularizes by half of a derivative. It is not surprising that these two half derivatives are exactly canceled by the appearance of a full derivation in the reconstruction formula. Notice that the Hilbert transform (which corresponds to multiplication by the smooth function $i\text{sign}(\sigma)$ in the Fourier domain) is a bounded operator with bounded inverse in $L^2(\mathbb{R})$ (since $H^{-1} = -H$).

Exercise 2.2.2 (easy). Show that

$$f(\mathbf{x}) = \frac{1}{4\pi} \int_0^{2\pi} (Hg')(\mathbf{x} \cdot \boldsymbol{\theta}^\perp, \theta) d\theta.$$

Here g' means first derivative of g with respect to the s variable only.

Exercise 2.2.3 (easy). Show that

$$f(\mathbf{x}) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{\mathbb{R}} \frac{\frac{d}{ds} g(s, \theta)}{\mathbf{x} \cdot \boldsymbol{\theta}^\perp - s} ds d\theta.$$

This is Radon's original inversion formula [17].

Exercise 2.2.4 (moderately difficult). Starting from the definition

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\mathbf{k} \cdot \mathbf{x}} \hat{f}(\mathbf{k}) d\mathbf{k},$$

and writing it in polar coordinates (with change of measure $d\mathbf{k} = |\mathbf{k}| d|\mathbf{k}| d\hat{\mathbf{k}}$), deduce the above reconstruction formulas by using the Fourier slice theorem.

2.3 Three dimensional Radon transform

Let us briefly mention the case of the Radon transform in three dimensions (the generalization to higher dimensions being also possible). The Radon transform in three dimensions consists of integrating a function $f(\mathbf{x})$ over all possible planes. A plane $\mathcal{P}(s,\boldsymbol{\theta})$ in $\mathbb{R}^3$ is characterized by its direction $\boldsymbol{\theta} \in S^2$, where S^2 is the unit sphere, and by its signed distance to the origin s. Notice again the double covering in the sense that $\mathcal{P}(s,\boldsymbol{\theta}) = \mathcal{P}(-s,-\boldsymbol{\theta})$. The Radon transform is then defined as

$$Rf(s,\boldsymbol{\theta}) = \int_{\mathbb{R}^3} f(\mathbf{x})\delta(\mathbf{x}\cdot\boldsymbol{\theta} - s)d\mathbf{x} = \int_{\mathcal{P}(s,\theta)} f d\sigma. \tag{2.19}$$

Notice the change of notation compared to the two-dimensional case. The Fourier slice theorem still holds

$$\widehat{Rf}(\sigma,\boldsymbol{\theta}) = \hat{f}(\sigma\boldsymbol{\theta}), \tag{2.20}$$

as can be easily verified. We check that $Rf(s,\boldsymbol{\theta}) = Rf(-s,-\boldsymbol{\theta})$. The reconstruction formula is then given by

$$f(\mathbf{x}) = \frac{-1}{8\pi^2}\int_{S^2} g''(\mathbf{x}\cdot\boldsymbol{\theta},\boldsymbol{\theta})d\boldsymbol{\theta}. \tag{2.21}$$

Here $d\boldsymbol{\theta}$ is the usual (Lebesgue) surface measure on the unit sphere.

The result can be obtained as follows. We denote by $S^2/2$ half of the unit sphere (for instance the vectors $\boldsymbol{\theta}$ such that $\boldsymbol{\theta}\cdot\mathbf{e}_z > 0$).

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{(2\pi)^3}\int_{\frac{S^2}{2}}\int_{\mathbb{R}} \hat{f}(r\boldsymbol{\theta})e^{ir\boldsymbol{\theta}\cdot\mathbf{x}}|r|^2 dr d\boldsymbol{\theta} = \frac{1}{(2\pi)^3}\int_{\frac{S^2}{2}}\int_{\mathbb{R}} \widehat{Rf}(r,\boldsymbol{\theta})e^{ir\boldsymbol{\theta}\cdot\mathbf{x}}|r|^2 dr d\boldsymbol{\theta} \\ &= \frac{1}{(2\pi)^2}\int_{\frac{S^2}{2}} (-g)''(\boldsymbol{\theta}\cdot\mathbf{x},\boldsymbol{\theta})d\boldsymbol{\theta} = \frac{1}{2}\frac{-1}{(2\pi)^2}\int_{S^2} g''(\boldsymbol{\theta}\cdot\mathbf{x},\boldsymbol{\theta})d\boldsymbol{\theta}. \end{aligned}$$

Here we have used the fact that the inverse Fourier transform of $r^2\hat{f}$ is $-f''$.

Exercise 2.3.1 Generalize Theorem 2.2.2 and prove the following result:

Theorem 2.3.1 *There exists a constant C_χ independent of $f(\mathbf{x})$ such that*

$$\begin{aligned} \sqrt{2}\|f\|_{H^s(\mathbb{R}^3)} &\leq \|Rf\|_{H^{s+1}(Z)} \\ \|R(\chi f)\|_{H^{s+1}(Z)} &\leq C_\chi\|\chi f\|_{H^s(\mathbb{R}^3)}, \end{aligned} \tag{2.22}$$

where $Z = \mathbb{R}\times S^2$ and $H^s(Z)$ is defined in the spirit of (2.8).

The above result shows that the Radon transform is more smoothing in three dimensions than it is in two dimensions. In three dimensions, the Radon transform smoothes by a full derivative rather than a half derivative.

Notice however that the inversion of the three dimensional Radon transform (2.21) is *local*, whereas this is not the case in two dimensions. What is meant by local is the following: the reconstruction of $f(\mathbf{x})$ depends on $g(s,\boldsymbol{\theta})$ only for the planes $\mathcal{P}(s,\boldsymbol{\theta})$ that pass through $\mathbf{x}$ (and an infinitely small neighborhood so that the second derivative can

be calculated). Indeed, we verify that $\mathbf{x} \in \mathcal{P}(\mathbf{x}\cdot\boldsymbol{\theta}, \boldsymbol{\theta})$ and that all the planes passing by $\mathbf{x}$ are of the form $\mathcal{P}(\mathbf{x}\cdot\boldsymbol{\theta}, \boldsymbol{\theta})$. The two dimensional transform involves the Hilbert transform, which unlike differentiations, is a *non-local* operation. Thus the reconstruction of f at a point $\mathbf{x}$ requires knowledge of *all* line integrals $g(s, \theta)$, and not only for those lines passing through $\mathbf{x}$.

Exercise 2.3.2 Calculate R^*, the adjoint operator to R (with respect to the usual L^2 inner products). Generalize the formula (2.16) to the three dimensional case.

Chapter 3

Inverse kinematic problem

This chapter concerns the reconstruction of velocity fields from travel time measurements. The problem can also be recast as the reconstruction of a Riemannian metric from the integration of arc length along its geodesics. We start with a problem with spherical symmetry in which explicit reconstruction formulas are available.

3.1 Spherical symmetry

We want to reconstruct the velocity field $c(r)$ inside the Earth assuming spherical symmetry from travel time measurements. To simplify we assume that the Earth radius is normalized to 1. We assume that $c(1)$ is known. We want to reconstruct $c(r)$ from the time it takes to travel along all possible geodesics. Because the geodesics depend on $c(r)$, the travel time is a *nonlinear* functional of the velocity profile $c(r)$. We thus have to solve a nonlinear inverse problem.

Let us consider the classical mechanics approximation to wave propagation through the Earth. Unlike what we did in Chapter 1, we treat waves here in their geometrical optics limit. This means that we assume that the wavelength is sufficiently large compared to other length scales in the system that the waves can be approximated by particles. The particles satisfy then dynamics given by classical dynamics. They are represented by position $\mathbf{x}$ and wavenumber $\mathbf{k}$ and their dynamics are governed by Hamilton's equations:

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \dot{\mathbf{x}} = \nabla_{\mathbf{k}} H(\mathbf{x}(t), \mathbf{k}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \frac{d\mathbf{k}}{dt} &= \dot{\mathbf{k}} = -\nabla_{\mathbf{x}} H(\mathbf{x}(t), \mathbf{k}(t)), \quad \mathbf{k}(0) = \mathbf{k}_0. \end{aligned} \tag{3.1}$$

In wave propagation the Hamiltonian is given by

$$H(\mathbf{x}, \mathbf{k}) = c(\mathbf{x})|\mathbf{k}| = c(|\mathbf{x}|)|\mathbf{k}|. \tag{3.2}$$

The latter relation holds because of the assumption of spherical symmetry. Let us denote $\mathbf{x} = r\hat{\mathbf{x}}$ and $\mathbf{k} = |\mathbf{k}|\hat{\mathbf{k}}$. The Hamiltonian dynamics take the form

$$\dot{\mathbf{x}} = c(r)\hat{\mathbf{k}}, \qquad \dot{\mathbf{k}} = -c'(r)|\mathbf{k}|\hat{\mathbf{x}}. \tag{3.3}$$

We are interested in calculating the travel times between points at the boundary of the domain $r = |\mathbf{x}| < 1$. This implies integrating dt along particle trajectories. Since we want to reconstruct $c(r)$, we perform a change of variables from dt to dr. This will allow us to obtain integrals of the velocity $c(r)$ along curves. The objective will then be to obtain a reconstruction formula for $c(r)$.

In order to perform the change of variables from dt to dr, we need to know where the particles are. Indeed the change of variables should only involve position r and no longer time t. This implies to solve the problem $t \mapsto r(t)$. As usual it is useful to find invariants of the dynamical system. The first invariant is the Hamiltonian itself:

$$\frac{dH(\mathbf{x}(t), \mathbf{k}(t))}{dt} = 0,$$

as can be deduced from (3.1). The second invariant is *angular momentum* and is obtained as follows. Let us first introduce the basis $(\hat{\mathbf{x}}, \hat{\mathbf{x}}^\perp)$ for two dimensional vectors (this is the usual basis $(\mathbf{e}_r, \mathbf{e}_\theta)$ in polar coordinates). We decompose $\mathbf{k} = k_r\hat{\mathbf{x}} + k_\theta\hat{\mathbf{x}}^\perp$ and $\hat{\mathbf{k}} = \hat{k}_r\hat{\mathbf{x}} + \hat{k}_\theta\hat{\mathbf{x}}^\perp$. We verify that

$$\dot{r} = c(r)\hat{k}_r \qquad \text{since} \qquad \dot{\mathbf{x}} = \dot{r}\hat{\mathbf{x}} + r\dot{\hat{\mathbf{x}}} = c(r)\hat{\mathbf{k}}. \tag{3.4}$$

We also verify that

$$\frac{d(rk_\theta)}{dt} = \frac{d\mathbf{x}^\perp \cdot \mathbf{k}}{dt} = \dot{\mathbf{x}}^\perp \cdot \mathbf{k} + \mathbf{x} \cdot \dot{\mathbf{k}}^\perp = c(r)\hat{\mathbf{k}}^\perp \cdot \mathbf{k} - c'(r)|\mathbf{k}|\mathbf{x} \cdot \hat{\mathbf{x}}^\perp = 0. \tag{3.5}$$

This is conservation of angular momentum and implies that

$$r(t)k_\theta(t) = k_\theta(0),$$

since $r(0) = 1$.

By symmetry, we observe that the travel time is decomposed into two identical components: the time it takes to go down the Earth until $k_r = 0$, and the time it takes to go back up. On the way up to the surface, k_r is non-negative. Let us denote $p = \hat{k}_\theta(0)$ with $0 < p < 1$. The lowest point is reached when $\hat{k}_\theta = 1$. This means at a point r_p such that

$$\frac{r_p}{c(r_p)} = \frac{p}{c(1)}.$$

To make sure that such a point is uniquely defined, we impose that the function $rc^{-1}(r)$ be *increasing* on $(0, 1)$ since it cannot be decreasing. This is equivalent to the constraint:

$$c'(r) < \frac{c(r)}{r}, \qquad 0 < r < 1. \tag{3.6}$$

This assumption ensures the uniqueness of a point r_p such that $pc(r_p) = c(1)r_p$.

Since the Hamiltonian $c(r)|\mathbf{k}|$ is conserved, we deduce that

$$\dot{r} = c(r)\sqrt{1 - \hat{k}_\theta^2} = c(r)\sqrt{1 - \Big(\frac{\hat{k}_\theta(0)c(r)}{rc(1)}\Big)^2},$$

so that

$$dt = \frac{dr}{c(r)\sqrt{1 - \Big(\frac{\hat{k}_\theta(0)c(r)}{rc(1)}\Big)^2}}. \tag{3.7}$$

Notice that the right-hand side depends only on r and no longer on functions such as $\hat{k}_r$ that depend on time. The travel time as a function of $p = \hat{k}_\theta(0)$ is now given by twice the time it takes to go back to the surface:

$$T(p) = 2\int_{t(r_p)}^{1} dt = 2\int_{r_p}^{1} \frac{dr}{c(r)\sqrt{1 - \Big(\frac{\hat{k}_\theta(0)c(r)}{rc(1)}\Big)^2}}. \tag{3.8}$$

Our measurements are $T(p)$ for $0 < p < 1$ and our objective is to reconstruct $c(r)$ on $(0, 1)$. We need a theory to invert this integral transform. Let us define

$$u = \frac{c^2(1)r^2}{c^2(r)} \quad \text{so that} \quad du = \frac{2rc^2(1)}{c^2(r)}\Big(1 - \frac{rc'(r)}{c(r)}\Big)dr.$$

Upon using this change of variables we deduce that

$$T(p) = 2\int_{p^2}^{1} \Big(\frac{dr}{du}\frac{u}{r}\Big)(u)\frac{du}{\sqrt{u - p^2}}. \tag{3.9}$$

It turns out that the function in parenthesis in the above expression can be reconstructed from $T(p)$. This is an *Abel integral*. Before inverting the integral, we need to ensure that the change of variables $r \mapsto u(r)$ is a diffeomorphism (a continuous function with continuous inverse). This implies that du/dr is positive, which in turn is equivalent to (3.6). The constraint (3.6) is therefore useful both to obtain the existence of a minimal point r_p and to ensure that the above change of variables is admissible. The constraint essentially ensures that no rays are trapped in the dynamics so that energy entering the system will eventually exit it. We can certainly consider velocity profiles such that the energy is attracted at the origin. In such situation the velocity profile cannot be reconstructed.

Let us denote by $f = \frac{dr}{du}\frac{u}{r}$. We will show in the following section that $f(u)$ can be reconstructed from $T(p)$ and is given by

$$f(u) = -\frac{2}{\pi}\frac{d}{du}\int_u^1 \frac{T(\sqrt{p})}{\sqrt{p - u}}dp. \tag{3.10}$$

Now we reconstruct $r(u)$ from the relations

$$\frac{f(u)}{u}du = \frac{dr}{r}, \qquad u(1) = 1, \quad \text{so that} \quad r(u) = \exp\Big(\int_u^1 \frac{f(v)dv}{v}\Big).$$

Upon inverting this diffeomorphism, we obtain $u(r)$ and $g(r) = f(u(r))$. Since

$$g(r) = \frac{1}{2}\frac{1}{1 - rc'/c},$$

we now know rc'/c, hence $(\log c)'$. It suffices to integrate $\log c$ from 1 to 0 to obtain $c(r)$ everywhere. This concludes the proof of the reconstruction.

3.2 Abel integral and Abel transform

For a smooth function $f(x)$ (continuous will do) defined on the interval $(0,1)$, we define the Abel transform as

$$g(x) = \int_x^1 \frac{f(y)}{(y-x)^{1/2}} dy. \tag{3.11}$$

This transform can be inverted as follows:

Lemma 3.2.1 *The Abel transform (3.11) admits the following inversion*

$$f(y) = -\frac{1}{\pi}\frac{d}{dy}\int_y^1 \frac{g(x)}{(x-y)^{1/2}} dx. \tag{3.12}$$

Proof. Let us calculate

$$\int_z^1 \frac{g(x)}{(x-z)^{1/2}} dx = \int_z^1 \int_x^1 \frac{f(y)}{(x-z)^{1/2}(y-x)^{1/2}} dxdy = \int_z^1 dy f(y) k(z,y) dy.$$

The kernel $k(z,y)$ is given by

$$k(z,y) = \int_z^y \frac{dx}{(x-z)^{1/2}(y-x)^{1/2}} = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \pi.$$

The latter equality comes from differentiating arccos. Thus we have

$$\int_z^1 \frac{g(x)}{(x-z)^{1/2}} dx = \pi \int_z^1 f(y) dy.$$

Upon differentiating both sides with respect to z, we obtain the desired result. □

We can also ask ourselves how well-posed the inversion of the Abel transform is. Since the transforms are defined on bounded intervals, using the Hilbert scale introduced in Chapter 1 would require a few modifications. Instead we will count the number of differentiations. The reconstruction formula shows that the Abel transform is applied once more to g before the result is differentiated. We can thus conclude that the Abel transform regularizes as one half of an integration (since it takes one differentiation to compensate for two Abel transforms). We therefore deduce that the Abel transform is a smoothing operator of order $\alpha = 1/2$ using the terminology introduced in Chapter 1. Inverting the Abel transform is a *mildly ill-posed* problem.

3.3 Kinematic Inverse Source Problem

We have seen that arbitrary spherically symmetric velocity profiles satisfying the condition (3.6) could be reconstructed from travel time measurements. Moreover an explicit reconstruction formula based on the (ill-posed) Abel inverse transform is available.

We now generalize the result to more general velocity profiles that do not satisfy the assumption of spherical symmetry. We start with a somewhat simpler (and linear) problem. Assuming that we have an approximation of the velocity profile and want to obtain a better solution based on travel time measurements. We can linearize the

problem about the known velocity. Once the linearization is carried out we end up with averaging the velocity fluctuations over the integral curves of the known approximation. This is a problem therefore very similar to the Radon transform except that the integration is performed over curves instead than lines.

More generally we give ourselves a set of curves in *two space dimensions* and assume that the integrals of a function $f(\mathbf{x})$ over all curves are known. The question is whether the function $f(\mathbf{x})$ can uniquely be reconstructed. The answer will be yes, except that no explicit formula can be obtained in general. Therefore our result will be a *uniqueness* result.

Let Ω be a simply connected domain (i.e., $\mathbb{R}^2\backslash\Omega$ is a connected domain) with smooth boundary $\Sigma = \partial\Omega$. We denote $\overline{\Omega} = \Omega \cup \Sigma$. We parameterize Σ by arc length

$$\mathbf{x} = \boldsymbol{\sigma}(t), \qquad 0 \leq t \leq T.$$

Here T is the total length of the boundary Σ. Obviously, $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}(T)$.

We give ourselves a *regular family of curves* Γ, i.e., satisfying the following hypotheses:

1. Two points of $\overline{\Omega}$ are joined by a unique curve in Γ.

2. The endpoints of $\gamma \in \Gamma$ belong to Σ, the inner points to Ω, and the length of γ is uniformly bounded.

3. For every point $\mathbf{x}_0 \in \Omega$ and direction $\boldsymbol{\theta} \in S^1$, there is a unique curve passing through $\mathbf{x}_0$ with tangent vector given by $\boldsymbol{\theta}$. The curve between $\mathbf{x}_0$ and Σ is parameterized by

$$\mathbf{x}(s) = \boldsymbol{\gamma}(\mathbf{x}_0, \boldsymbol{\theta}, s), \qquad 0 \leq s \leq S(\mathbf{x}_0, \boldsymbol{\theta}), \tag{3.13}$$

   where s indicates arc length on γ and S is the distance along γ from $\mathbf{x}_0$ to Σ in direction $\boldsymbol{\theta}$.

4. The function $\boldsymbol{\gamma}(\mathbf{x}_0, \boldsymbol{\theta}, s)$ is of class C^3 of its domain of definition and

$$\frac{1}{s}\boldsymbol{\theta}^{\perp}\nabla_{\boldsymbol{\theta}}\gamma \geq C > 0.$$

These assumptions mean that the curves behave similarly to the set of straight lines in the plane, which satisfy these hypotheses.

Exercise 3.3.1 Show that the lines in the plane satisfy the above hypotheses.

Assuming that we now have the integral of a function $f(\mathbf{x})$ over all curves in Γ joining two points $\boldsymbol{\sigma}(t_1)$ and $\boldsymbol{\sigma}(t_2)$. Let $\gamma(t_1, t_2)$ be this curve and $\mathbf{x}(s; t_1, t_2)$ the points on the curve. The integral is thus given by

$$g(t_1, t_2) = \int_{\gamma(t_1,t_2)} f(\mathbf{x}(s; t_1, t_2))ds, \tag{3.14}$$

where $ds = \sqrt{dx^2 + dy^2}$ is the usual Lebesgue measure. Our objective is to show that $g(t_1, t_2)$ characterizes $f(\mathbf{x})$. In other words the reconstruction of $f(\mathbf{x})$ is uniquely determined by $g(t_1, t_2)$:

Theorem 3.3.1 *Under the above hypotheses for the family of curves Γ, a function $f \in C^2(\overline{\Omega})$ is uniquely determined by its integrals $g(t_1, t_2)$ given by (3.14) along the curves of Γ. Moreover we have the stability estimate*

$$\|f\|_{L^2(\Omega)} \leq C \|\frac{\partial g(t_1, t_2)}{\partial t_1}\|_{L^2((0,T)\times(0,T))}. \tag{3.15}$$

Proof. We first introduce the function

$$u(\mathbf{x}, t) = \int_{\tilde{\gamma}(\mathbf{x},t)} f ds, \tag{3.16}$$

for $\mathbf{x} \in \overline{\Omega}$ and $0 \leq t \leq T$, where $\tilde{\gamma}(\mathbf{x}, t)$ is the unique segment of curve in Γ joining $\mathbf{x} \in \Omega$ and $\boldsymbol{\sigma}(t) \in \Sigma$. We denote by $\boldsymbol{\theta}(\mathbf{x}, t)$ the tangent vector to $\tilde{\gamma}(\mathbf{x}, t)$ at $\mathbf{x}$. We verify that

$$\boldsymbol{\theta} \cdot \nabla_{\mathbf{x}} u(\mathbf{x}, t) = f(\mathbf{x}). \tag{3.17}$$

This is obtained by differentiating (3.16) with respect to arc length. Notice that $u(\boldsymbol{\sigma}(t_2), t_1) = g(t_1, t_2)$. We now differentiate (3.17) with respect to t and obtain

$$Lu \equiv \frac{\partial}{\partial t}\boldsymbol{\theta} \cdot \nabla_{\mathbf{x}} u = 0. \tag{3.18}$$

As usual we identify $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$. This implies

$$\frac{\partial}{\partial t}\boldsymbol{\theta} = \dot{\theta}\boldsymbol{\theta}^{\perp}, \qquad \frac{\partial}{\partial t}\boldsymbol{\theta}^{\perp} = -\dot{\theta}\boldsymbol{\theta}.$$

Here $\dot{\theta}$ means partial derivative of θ with respect to t. We calculate

$$\boldsymbol{\theta}^{\perp} \cdot \nabla u \frac{\partial}{\partial t}\boldsymbol{\theta} \cdot \nabla u = \dot{\theta}(\boldsymbol{\theta}^{\perp} \cdot \nabla u)^2 + \boldsymbol{\theta}^{\perp} \cdot \nabla u \boldsymbol{\theta} \cdot \nabla(u_t)$$

Here u_t stands for partial derivative of u with respect to t. The same equality with $\boldsymbol{\theta}$ replaced by $\boldsymbol{\theta}^{\perp}$ yields

$$-\boldsymbol{\theta} \cdot \nabla u \frac{\partial}{\partial t}\boldsymbol{\theta}^{\perp} \cdot \nabla u = \dot{\theta}(\boldsymbol{\theta} \cdot \nabla u)^2 - \boldsymbol{\theta} \cdot \nabla u \boldsymbol{\theta}^{\perp} \cdot \nabla(u_t)$$

Upon adding these two equalities, we obtain

$$\begin{aligned} &2\boldsymbol{\theta}^{\perp} \cdot \nabla u \frac{\partial}{\partial t}\boldsymbol{\theta} \cdot \nabla u - \frac{\partial}{\partial t}\Big(\boldsymbol{\theta}^{\perp} \cdot \nabla u \boldsymbol{\theta} \cdot \nabla u\Big) \\ &= \dot{\theta}|\nabla u|^2 + \boldsymbol{\theta} \cdot \nabla(\boldsymbol{\theta}^{\perp} \cdot \nabla u u_t) - \boldsymbol{\theta}^{\perp} \cdot \nabla(\boldsymbol{\theta} \cdot \nabla u u_t). \end{aligned}$$

Why is this relation useful? The answer is the following: one term vanishes thanks to (3.18), one term is positive as soon as u is not the trivial function, and all the other contributions are in divergence form, i.e. are written as derivatives of certain quantities, and can thus be estimated in terms of functions defined at the boundary of the domain (in space $\mathbf{x}$ and arc length t) by applying the Gauss-Ostrogradsky theorem.

More precisely we obtain that

$$\int_0^T \int_\Omega \dot{\theta} |\nabla u|^2 d\mathbf{x} dt = \int_0^T \int_\Sigma [\boldsymbol{\theta}^\perp \cdot \mathbf{n}(\boldsymbol{\theta} \cdot \nabla u) - \boldsymbol{\theta} \cdot \mathbf{n}(\boldsymbol{\theta}^\perp \cdot \nabla u)](\sigma) u_t(\sigma) d\sigma dt.$$

Here $d\sigma$ is the surface (Lebesgue) measure on Σ and $\mathbf{n}$ the outward unit normal vector. Since Σ is a closed curve the term in ∂_t vanishes. The above formula substantially simplifies. Let us assume that Σ has "counterclockwise" orientation. The tangent vector along the contour is $\dot{\boldsymbol{\sigma}}(t)$ so that $\mathbf{n} = -\dot{\boldsymbol{\sigma}}(t)^\perp$, or equivalently $\dot{\boldsymbol{\sigma}}(t) = \mathbf{n}^\perp$. Let us decompose $\boldsymbol{\theta} = \theta_n \mathbf{n} + \theta_\perp \mathbf{n}^\perp$ so that $\boldsymbol{\theta}^\perp = -\theta_\perp \mathbf{n} + \theta_n \mathbf{n}^\perp$. We then verify that

$$\boldsymbol{\theta}^\perp \cdot \mathbf{n}(\boldsymbol{\theta} \cdot \nabla u) - \boldsymbol{\theta} \cdot \mathbf{n}(\boldsymbol{\theta}^\perp \cdot \nabla u) = -\mathbf{n}^\perp \cdot \nabla u = -\dot{\boldsymbol{\sigma}}(t) \cdot \nabla u.$$

However $\dot{\boldsymbol{\sigma}}(t) \cdot \nabla u d\sigma = u_t(\boldsymbol{\sigma}, s) ds$ on Σ so that

$$\int_0^T \int_\Omega \dot{\theta} |\nabla u|^2 d\mathbf{x} dt = -\int_0^T \int_0^T u_t(s) u_t(t) ds dt.$$

This is equivalent to the relation

$$\int_0^T \int_\Omega \dot{\theta} |\nabla u|^2 d\mathbf{x} dt = -\int_0^T \int_0^T \frac{\partial g(t_1, t_2)}{\partial t_1} \frac{\partial g(t_1, t_2)}{\partial t_2} dt_1 dt_2. \tag{3.19}$$

This implies uniqueness of the reconstruction. Indeed, the problem is linear, so uniqueness amounts to showing that $f(\mathbf{x}) \equiv 0$ when $g(t_1, t_2) \equiv 0$. However the above relation implies that $\nabla u \equiv 0$ because $\dot{\theta} > 0$ by assumption on the family of curves. Upon using the transport equation (3.17), we observe that $f(\mathbf{x}) \equiv 0$ as well.

The same reasoning gives us the stability estimate. Indeed we deduce from (3.17) that $|f(\mathbf{x})| \leq |\nabla u(\mathbf{x}, t)|$ so that

$$\int_0^T \int_\Omega \dot{\theta} |f(\mathbf{x})|^2 d\mathbf{x} dt = 2\pi \int_\Omega |f(\mathbf{x})|^2 d\mathbf{x} \leq \int_0^T \int_\Omega \dot{\theta} |\nabla u|^2 d\mathbf{x} dt.$$

Upon using (3.19) and the Cauchy-Schwarz inequality $(f, g) \leq \|f\| \|g\|$, we deduce the result with $C = (2\pi)^{-1/2}$. □

3.4 Kinematic velocity Inverse Problem

Let us return to the construction of the Earth velocity from boundary measurements. Here we consider a "two-dimensional" Earth. Local travel time and distance are related by

$$d\tau^2 = \frac{1}{c^2(\mathbf{x})}(dx^2 + dy^2) = n^2(\mathbf{x})(dx^2 + dy^2).$$

which defines a Riemannian metric with tensor proportional to the 2×2 identity matrix. We are interested in reconstructing $n(\mathbf{x})$ from the knowledge of

$$G(t_1, t_2) = \int_{\gamma(t_1, t_2)} d\tau = \int_{\gamma(t_1, t_2)} n(\mathbf{x}) \sqrt{dx^2 + dy^2}, \tag{3.20}$$

for every possible boundary points t_1 and t_2, where $\gamma(t_1, t_2)$ is an extremal for the above functional, i.e., a geodesic of the Riemannian metric $d\tau^2$. Notice that since the extremals (the geodesics) of the Riemannian metric depend on $n(\mathbf{x})$, the above problem is *non-linear*.

Let Γ_k for $k = 1, 2$ correspond to measurements for two slownesses n_k, $k = 1, 2$. We then have the following result

Theorem 3.4.1 *Let n_k be smooth positive functions on Ω such that the family of extremals are sufficiently regular. Then n_k can uniquely be reconstructed from $G_k(t_1, t_2)$ and we have the stability estimate*

$$\|n_1 - n_2\|_{L^2(\Omega)} \leq C\|\frac{\partial}{\partial t_1}(G_1 - G_2)\|_{L^2((0,T)\times(0,T))}. \tag{3.21}$$

Proof. The proof is similar to that of Theorem 3.3.1. The regular family of curves Γ is defined as the geodesics of the Riemannian metric $d\tau^2$. Indeed let us define

$$\tau(\mathbf{x}, t) = \int_{\tilde{\gamma}(\mathbf{x},t)} n ds, \tag{3.22}$$

so that as before $\tau(\boldsymbol{\sigma}(t_1), t_2) = G(t_1, t_2)$. We deduce as before that

$$\boldsymbol{\theta} \cdot \nabla_{\mathbf{x}} \tau = n(\mathbf{x}). \tag{3.23}$$

Because τ is an integration along an extremal curve, we deduce that

$$\boldsymbol{\theta}^\perp \cdot \nabla\tau = 0, \quad \text{so that} \quad \nabla_{\mathbf{x}}\tau = n(\mathbf{x})\boldsymbol{\theta} \quad \text{and} \quad |\nabla_{\mathbf{x}}\tau|^2(\mathbf{x}, t) = n^2(\mathbf{x}).$$

Upon differentiating the latter equality we obtain

$$\frac{\partial}{\partial t}|\nabla_{\mathbf{x}}\tau|^2 = 0.$$

Let us now define $u = \tau_1 - \tau_2$ so that $\nabla u = n_1\boldsymbol{\theta}_1 - n_2\boldsymbol{\theta}_2$. We deduce from the above expression that

$$\frac{\partial}{\partial t}(\nabla u \cdot (\boldsymbol{\theta}_1 + \frac{n_2}{n_1}\boldsymbol{\theta}_2)) = 0.$$

We multiply the above expression by $2\boldsymbol{\theta}_1^\perp \cdot \nabla u$ and express the product in divergence form. We obtain as in the preceding section that

$$\begin{aligned} &2\boldsymbol{\theta}_1^\perp \cdot \nabla u \frac{\partial}{\partial t}\boldsymbol{\theta}_1 \cdot \nabla u - \frac{\partial}{\partial t}\Big(\boldsymbol{\theta}_1^\perp \cdot \nabla u \boldsymbol{\theta}_1 \cdot \nabla u\Big) \\ &= \dot{\theta}_1 |\nabla u|^2 + \boldsymbol{\theta}_1 \cdot \nabla(\boldsymbol{\theta}_1^\perp \cdot \nabla u u_t) - \boldsymbol{\theta}_1^\perp \cdot \nabla(\boldsymbol{\theta}_1 \cdot \nabla u u_t). \end{aligned}$$

We now show that the second contribution can also be put in divergence form. More precisely, we obtain, since n_1 and n_2 are independent of t, that

$$\begin{aligned} &2\boldsymbol{\theta}_1^\perp \cdot \nabla u \frac{\partial}{\partial t}(\frac{n_2}{n_1}\boldsymbol{\theta}_2 \cdot \nabla u) = 2\boldsymbol{\theta}_1^\perp \cdot (n_1\boldsymbol{\theta}_1 - n_2\boldsymbol{\theta}_2)\frac{\partial}{\partial t}(n_2\boldsymbol{\theta}_1 \cdot \boldsymbol{\theta}_2 - \frac{n_2^2}{n_1}) \\ &= -2n_2^2\boldsymbol{\theta}_1^\perp \cdot \boldsymbol{\theta}_2(\boldsymbol{\theta}_1 \cdot \boldsymbol{\theta}_2) = -2n_2^2(\boldsymbol{\theta}_1^\perp \cdot \boldsymbol{\theta}_2)^2\frac{\partial(\theta_1 - \theta_2)}{\partial t} = \\ &-2n_2^2 \sin^2(\theta_1 - \theta_2)\frac{\partial(\theta_1 - \theta_2)}{\partial t} = \frac{\partial}{\partial t}\Big(n_2^2\big[\frac{\sin(2(\theta_1 - \theta_2))}{2} - (\theta_1 - \theta_2)\big]\Big). \end{aligned}$$

The integration of the above term over $\Omega \times (0,T)$ yields thus a vanishing contribution. Following the same derivation as in the preceding section, we deduce that

$$\int_0^T \int_\Omega \frac{\partial \theta_1}{\partial t} |\nabla u|^2 d\mathbf{x}dt = \int_0^T \int_0^T \frac{\partial G(t_1,t_2)}{\partial t_1} \frac{\partial G(t_1,t_2)}{\partial t_2} dt_1 dt_2, \tag{3.24}$$

where we have defined $G = G_1 - G_2$. To conclude the proof, notice that

$$\begin{aligned} &\nabla u \cdot \nabla u = |n_1 \boldsymbol{\theta}_1 - n_2 \boldsymbol{\theta}_2|^2 = n_1^2 + n_2^2 - 2n_1 n_2 \boldsymbol{\theta}_1 \cdot \boldsymbol{\theta}_2 \\ &\geq n_1^2 + n_2^2 - 2n_1 n_2 = (n_1 - n_2)^2, \end{aligned}$$

since both n_1 and n_2 are non-negative. With (3.24), this implies that $n_1 = n_2$ when $G_1 = G_2$ and using again the Cauchy-Schwarz inequality yields the stability estimate (3.21). □

Chapter 4

Attenuated Radon Transform

In the two previous chapters, the integration over lines (for the Radon transform) or over more general curves was not weighted. We could more generally ask whether integrals of the form

$$\int_{\mathbb{R}} f(s\boldsymbol{\theta}^{\perp} + t\boldsymbol{\theta})\alpha(s\boldsymbol{\theta}^{\perp} + t\boldsymbol{\theta}, \boldsymbol{\theta})d\boldsymbol{\theta},$$

over all possible lines parameterized by $s \in \mathbb{R}$ and $\boldsymbol{\theta} \in S^1$ and assuming that the weight $\alpha(\mathbf{x}, \boldsymbol{\theta})$ is known, uniquely determine $f(\mathbf{x})$. This is a much more delicate question for which only partial answers are known. We concentrate here on one example of weighted Radon transform, namely the attenuated Radon transform, for which an inversion formula was obtained only very recently.

4.1 Single Photon Emission Computed Tomography

An important application for the attenuated Radon transform is SPECT, single photon emission computed tomography. The principle is the following: radioactive particles are injected in a domain. These particles emit then some radiation. The radiation propagates through the tissues and gets partially absorbed. The amount of radiation reaching the boundary of the domain can be measured. The imaging technique consists then of reconstructing the location of the radioactive particles from the boundary measurements.

We model the density of radiated photons by $u(\mathbf{x}, \theta)$ and the source of radiated photons by $f(\mathbf{x})$. The absorption of photons (by the human tissues in the medical imaging application) is modeled by $a(\mathbf{x})$. We assume that $a(\mathbf{x})$ is *known* here. The absorption can be obtained, for instance, by transmission tomography as we saw in Chapter 2. The density $u(\mathbf{x}, \theta)$ satisfies then the following transport equation

$$\boldsymbol{\theta} \cdot \nabla u(\mathbf{x}, \theta) + a(\mathbf{x})u(\mathbf{x}, \theta) = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2, \boldsymbol{\theta} \in S^1. \tag{4.1}$$

We assume that $f(\mathbf{x})$ is compactly supported and impose that no radiation comes from infinity:

$$\lim_{s \to \infty} u(\mathbf{x} - s\boldsymbol{\theta}, \theta) = 0. \tag{4.2}$$

The transport equation (4.1) with conditions (4.2) admits a unique solution that can be obtained by the method of characteristics. Let us define the following *symmetrized*

beam transform

$$D_\theta a(\mathbf{x}) = \frac{1}{2}\int_0^\infty [a(\mathbf{x}-t\boldsymbol{\theta}) - a(\mathbf{x}+t\boldsymbol{\theta})]dt = \frac{1}{2}\int_{\mathbb{R}} \mathrm{sign}(t) a(\mathbf{x}-t\boldsymbol{\theta})dt. \tag{4.3}$$

We verify that $\boldsymbol{\theta}\cdot\nabla D_\theta a(\mathbf{x}) = a(\mathbf{x})$ so that $e^{D_\theta a(\mathbf{x})}$ is an integrating factor for (4.1) in the sense that

$$\boldsymbol{\theta}\cdot\nabla(e^{D_\theta a(\mathbf{x})}u(\mathbf{x},\theta)) = (e^{D_\theta a} f)(\mathbf{x},\theta).$$

Therefore the solution $u(\mathbf{x},\theta)$ is given by

$$e^{D_\theta a(\mathbf{x})}u(\mathbf{x},\theta) = \int_0^\infty (e^{D_\theta a} f)(\mathbf{x}-t\boldsymbol{\theta},\theta)dt. \tag{4.4}$$

We recall that $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$ and that $\boldsymbol{\theta}^\perp = (-\sin\theta, \cos\theta)$ and decompose $\mathbf{x} = s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}$. We deduce from (4.4) that

$$\lim_{t\to+\infty} e^{D_\theta a(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})}u(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta},\theta) = \int_{\mathbb{R}} (e^{D_\theta a} f)(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta},\theta)dt.$$

In the above expression the left hand side is know from the measurements. Indeed $u(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta},\theta)$ is the radiation outside of the domain to image and is thus measured and $e^{D_\theta a(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})}$ involves the attenuation coefficient $a(\mathbf{x})$ which we have assumed is known. The objective is thus to reconstruct $f(\mathbf{x})$ from the right hand side of the above relation, which we recast as

$$(R_a f)(s,\theta) = (R_{a,\theta} f)(s) = (R_\theta(e^{D_\theta a} f))(s), \tag{4.5}$$

where R_θ is the Radon transform defined for a function of $f(\mathbf{x},\theta)$ as

$$R_\theta f(s) = \int_{\mathbb{R}} f(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta},\theta)dt = \int_{\mathbb{R}^2} f(\mathbf{x},\theta)\delta(\mathbf{x}\cdot\boldsymbol{\theta}^\perp - s)d\mathbf{x}.$$

When $a \equiv 0$, we recover that the measurements involve the Radon transform of $f(\mathbf{x})$ as defined in (2.4). Thus in the absence of absorption, SPECT can be handled by inverting the Radon transform as we saw in Chapter 2. When absorption is constant, an inversion formula has been known for quite some time [20]. The inversion formula for non-constant absorption is more recent and was obtained independently by two different techniques [5, 16]. We do not consider here the method of $A-$analytic functions developed in [5]. We will present the method developed in [16] based on the extension of the transport equation in the complex domain and on the solution of a Riemann Hilbert problem.

4.2 Riemann Hilbert problem

Riemann Hilbert problems find many applications in complex analysis. We consider here the simplest of Riemann Hilbert problems and refer the reader to [1] for much more general cases and applications.

Let T be a smooth closed curve in the complex plane, which in our application will be the unit circle, i.e., the complex numbers λ such that $|\lambda| = 1$. The reason why we

choose the notation λ to represent complex numbers will appear more clearly in the next section. We denote by D^+ the open bounded domain inside the curve T, i.e., in our application the unit disk $\{\lambda \in \mathbb{C}, |\lambda| < 1\}$, and by D^- the open unbounded domain outside of the curve T, i.e., in our application $\{\lambda \in \mathbb{C}, |\lambda| > 1\}$. The orientation of the curve T is chosen so that D^+ is on the "left" of the curve T.

For a smooth function $\phi(\lambda)$ defined on $D^+ \cup D^-$, we denote by $\phi^+(t)$ and $\phi^-(t)$ the traces of ϕ on T from D^+ and D^-, respectively. So in the case where T is the unit circle, we have

$$\phi^+(t) = \lim_{0<\varepsilon\to 0} \phi((1-\varepsilon)t), \qquad \phi^-(t) = \lim_{0<\varepsilon\to 0} \phi((1+\varepsilon)t).$$

We define $\varphi(t)$ on T as the jump of ϕ, i.e.,

$$\varphi(t) = \phi^+(t) - \phi^-(t). \tag{4.6}$$

Let $\varphi(t)$ be a smooth function defined on T. The Riemann Hilbert problem is stated as follows. Find a function $\phi(\lambda)$ on $D^+ \cup D^-$ such that

1. $\phi(\lambda)$ is analytic on D^+ and analytic on D^-
2. $\lambda\phi(\lambda)$ is bounded as $|\lambda| \to \infty$ on D^-
3. the jump of ϕ is given by $\varphi(t) = \phi^+(t) - \phi^-(t)$.

The solution to the above Riemann Hilbert problem is *unique* and is given by the Cauchy formula

$$\phi(\lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(t)}{t-\lambda}dt, \qquad \lambda \in \mathbb{C}\backslash T = D^+ \cup D^-. \tag{4.7}$$

This is the form of the Riemann Hilbert problem we will use in the sequel. We refer the reader to [1] for the theory.

4.3 Inversion of the Attenuated Radon Transform

We now want to apply the theory of Riemann Hilbert problems to invert the attenuated Radon transform (AtRT). The first step is to extend the transport equation to the complex domain as follows. We parameterize the unit circle in the complex plans as

$$\lambda = e^{i\theta}, \qquad \theta \in (0, 2\pi). \tag{4.8}$$

The new parameter takes values on the unit circle T for $\theta \in (0, 2\pi)$. It can also be seen more generally as an arbitrary complex number $\lambda \in \mathbb{C}$. With the notation $\mathbf{x} = (x, y)$, the transport equation (4.1) can be recast as

$$\Big(\frac{\lambda + \lambda^{-1}}{2}\frac{\partial}{\partial x} + \frac{\lambda - \lambda^{-1}}{2i}\frac{\partial}{\partial y} + a(\mathbf{x})\Big)u(\mathbf{x}, \lambda) = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2, \quad \lambda \in T. \tag{4.9}$$

We can simplify the above equation by identifying $\mathbf{x}$ with $z = x + iy$ and by defining

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \tag{4.10}$$

The transport equation (4.9) is then equivalent to

$$\Big(\lambda\frac{\partial}{\partial z}+\lambda^{-1}\frac{\partial}{\partial \bar{z}}+a(z)\Big)u(z,\lambda)=f(z),\qquad z\in\mathbb{C},\quad \lambda\in T. \tag{4.11}$$

The same boundary conditions (4.2) that no information comes from infinity need to be added in the new variables as well.

The above equation can also be generalized to $\lambda\in\mathbb{C}$ instead of T. It is in this framework that the Riemann Hilbert problem theory is used to invert the attenuated Radon transform. This will be done in three steps

(i) We show that $u(z,\lambda)$ is *analytic* in $D^+\cup D^-=\mathbb{C}\backslash T$ and that $\lambda u(z,\lambda)$ is bounded as $\lambda\to\infty$.

(ii) We verify that $\varphi(\mathbf{x},\theta)=u^+(\mathbf{x},\theta)-u^-(\mathbf{x},\theta)$, the jump of u at $\lambda=e^{i\theta}$ can be written as a function of the measured data $R_af(s,\theta)$.

(iii) We solve the Riemann Hilbert problem using (4.7) and evaluate (4.11) at $\lambda=0$ to obtain a reconstruction formula for $f(z)=f(\mathbf{x})$.

4.4 Step (i): The $\overline{\partial}$ problem, an elliptic equation

Let us now analyze (4.11). In the absence of absorption the fundamental solution of (4.11) solves the following equation

$$\Big(\lambda\frac{\partial}{\partial z}+\lambda^{-1}\frac{\partial}{\partial \bar{z}}\Big)G(z,\lambda)=\delta(z),\qquad |G(z,\lambda)|\to 0 \text{ as } |z|\to\infty, \tag{4.12}$$

for $\lambda\in\mathbb{C}\backslash(T\cup\{0\})$.

Lemma 4.4.1 *The unique solution to (4.12) is given by*

$$G(z,\lambda)=\frac{\operatorname{sign}(|\lambda|-1)}{\pi(\lambda\bar{z}-\lambda^{-1}z)},\qquad \lambda\notin(T\cup\{0\}). \tag{4.13}$$

Proof. The formula can be verified by inspection. A more constructive derivation is the following. Let us define the change of variables

$$\zeta=\lambda^{-1}z-\lambda\bar{z},\qquad\qquad \bar{\zeta}=\overline{\lambda^{-1}}\bar{z}-\bar{\lambda}z. \tag{4.14}$$

Let us assume that $|\lambda|>1$. The Jacobian of the transformation is $|\lambda|^2-|\lambda|^{-2}$. We verify that

$$\lambda\frac{\partial}{\partial z}+\lambda^{-1}\frac{\partial}{\partial \bar{z}}=(|\lambda|^{-2}-|\lambda|^2)\frac{\partial}{\partial\bar{\zeta}}.$$

The change of variables (4.14) has been precisely tailored so that the above holds. Denoting $\tilde{G}(\zeta)=G(z)$, we thus obtain

$$\frac{\partial}{\partial\bar{\zeta}}\tilde{G}(\zeta)=\frac{1}{|\lambda|^{-2}-|\lambda|^2}\delta(z(\zeta))=-\delta(\zeta).$$

So $-\tilde{G}(\zeta)$ is the fundamental solution of the $\overline{\partial}$ operator $\dfrac{\partial}{\partial\bar{\zeta}}$. We verify that

$$\frac{\partial}{\partial\bar{\zeta}}\frac{1}{\zeta} = \pi\delta(\zeta). \tag{4.15}$$

Indeed let $\psi(z)$ be a smooth test function in $C_0^\infty(\mathbb{R}^2)$ and let $d\mu(\zeta)$ be the Lebesgue measure $dxdy$ in $\mathbb{C}\sim\mathbb{R}^2$. Then

$$\begin{aligned}\int_{\mathbb{C}}\psi(\zeta)\frac{\partial}{\partial\bar{\zeta}}\frac{1}{\zeta}d\mu(\zeta) &= -\int_{\mathbb{C}}\frac{\partial\psi}{\partial\bar{\zeta}}\frac{1}{\zeta}d\mu(\zeta) = -\lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus\{|\zeta|<\varepsilon\}}\frac{\partial\psi}{\partial\bar{\zeta}}\frac{1}{\zeta}d\mu(\zeta)\\ &= -\lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus\{|\zeta|<\varepsilon\}}\frac{\partial\zeta^{-1}\psi}{\partial\bar{\zeta}}d\mu(\zeta) = \frac{1}{2i}\int_{|\zeta|=\varepsilon}\frac{\psi}{\zeta}d\mu(\zeta),\end{aligned}$$

by the Green formula with complex variables:

$$\int_{\partial\Omega}udz = \int_{\partial\Omega}(udx+iudy) = \int_{\Omega}(i\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y})dxdy = 2i\int_{\Omega}\frac{\partial u}{\partial\bar{z}}d\mu(z).$$

Sending ε to 0, we find in the limit

$$\int_{\mathbb{C}}\psi(\zeta)\frac{\partial}{\partial\bar{\zeta}}\frac{1}{\zeta}d\mu(\zeta) = \frac{1}{2i}2\pi i\psi(0) = \pi\int_{\mathbb{R}^2}\psi(\zeta)\delta(\zeta)d\mu(\zeta).$$

This implies that $\tilde{G}(\zeta) = (-\pi\zeta)^{-1}$, hence $G(z) = (-\pi\zeta)^{-1} = (\pi(\lambda\bar{z}-\lambda^{-1}z))^{-1}$ This is (4.13) for $|\lambda|>1$. Now for $|\lambda|<1$, we verify that the Jacobian of the transformation $z\to\zeta(z)$ now becomes $|\lambda|^{-2}-|\lambda|^2$ so that

$$\frac{\partial}{\partial\bar{\zeta}}\tilde{G}(\zeta) = \frac{1}{|\lambda|^{-2}-|\lambda|^2}\delta(z(\zeta)) = \delta(\zeta).$$

This yields (4.13) for $|\lambda|<1$. □

The above proof shows that $(\pi z)^{-1}$ is the fundamental solution of the $\bar{\delta}$ operator. This implies that the solution to the following $\bar{\delta}$ problem

$$\frac{\partial}{\partial\bar{z}}f(z) = g(z), \qquad z\in\mathbb{C}, \tag{4.16}$$

such that $f(z)$ vanishes at infinity is given by convolution by the fundamental solution, i.e.,

$$f(z) = \frac{1}{\pi}\int_{\mathbb{C}}\frac{g(\zeta)}{z-\zeta}d\mu(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{C}}\frac{g(\zeta)}{z-\zeta}d\zeta\wedge d\bar{\zeta}. \tag{4.17}$$

Here we have used that $dz\wedge d\bar{z} = (dx+idy)\wedge(dx-idy) = 2idx\wedge dy = 2id\mu(z)$, whence the change of 2-form in the above integrations.

Notice that the Green function $G(z,\lambda)$ tends to 0 as $z\to\infty$ for $\lambda\notin T$. This is clearly not true when $\lambda\in T$, where $G(z,\lambda)=\delta(l_\theta(z))$, where $l_\theta(z)$ is the segment $\{t\boldsymbol{\theta}, t>0\}$. The reason is that for $\lambda\notin(T\cup\{0\})$,

$$\lambda\frac{\partial}{\partial z}+\lambda^{-1}\frac{\partial}{\partial\bar{z}} \qquad \text{and} \qquad \frac{\partial}{\partial z},$$

are elliptic operators, in the sense that in the Fourier domain, their symbol given by $\lambda k_z + \lambda^{-1}\overline{k_z}$ and $\overline{k_z}$, respectively, are positive provided that k_z is not 0. Indeed we verify that

$$\lambda k_z + \lambda^{-1}\overline{k_z} = 0 \quad \text{implies} \quad |\lambda|^2 = 1,$$

when $k_z \neq 0$ since $|k_z| = |\overline{k_z}| \neq 0$.

Let us now define $h(z,\lambda)$ as the solution to

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}}\Big)h(z,\lambda) = a(z), \qquad |h(z,\lambda)| \to 0 \text{ as } |z| \to \infty, \tag{4.18}$$

for $\lambda \notin (T \cup \{0\})$. The solution is given by

$$h(z,\lambda) = \int_{\mathbb{R}^2} G(z-\zeta,\lambda)a(\zeta)d\mu(\zeta). \tag{4.19}$$

We now verify that

$$\Big(\lambda\frac{\partial}{\partial z} + \lambda^{-1}\frac{\partial}{\partial \bar{z}}\Big)(e^{h(z,\lambda)}u(z,\lambda)) = e^{h(z,\lambda)}f(z),$$

so that for $\lambda \notin (T \cup \{0\})$, the solution of (4.11) is given by

$$u(z,\lambda) = e^{-h(z,\lambda)}\int_{\mathbb{R}^2} G(z-\zeta,\lambda)e^{h(\zeta,\lambda)}f(\zeta)d\mu(\zeta). \tag{4.20}$$

We verify that $G(z,\lambda)$, $h(z,\lambda)$ and $u(z,\lambda)$ are defined by continuity at $z = 0$ since $G(z,\lambda) = 0$ by continuity. We now verify that $G(z,\lambda)$ is analytic in D^+ (including at $z = 0$) and in D^-. Assuming that $a(z)$ and $f(z)$ are smooth functions, this is also therefore the case for $h(z,\lambda)$ and $u(z,\lambda)$. Moreover we easily deduce from (4.20) that $\lambda u(z,\lambda)$ is bounded on D^-. The solution $u(z,\lambda)$ of the transport equation extended to the complex plane is therefore a good candidate to apply the Riemann Hilbert theorem.

4.5 Step (ii): jump conditions

We now want to find the limit of $u(z,\lambda)$ as λ approaches T from above (in D^-) and below (in D^+). Let us write $\lambda = re^{i\theta}$ and let us send $r-1$ to ∓ 0 on $D^\pm$. The Green function behaves according to the following result

Lemma 4.5.1 *As $r - 1 \to \mp 0$, the Green function $G(\mathbf{x},\lambda)$ tends to*

$$G_\pm(\mathbf{x},\theta) = \frac{\pm 1}{2\pi i(\boldsymbol{\theta}^\perp \cdot \mathbf{x} \mp i0\,\mathrm{sign}(\boldsymbol{\theta}\cdot\mathbf{x}))}. \tag{4.21}$$

Proof. Let us assume that $|\lambda| > 1$, i.e., $r = 1+\varepsilon$ with $\varepsilon > 0$. We then find

$$\begin{aligned} G(z,re^{i\theta}) &= \frac{1}{\pi}\frac{1}{re^{i\theta}\bar{z} - \frac{e^{-i\theta}}{r}z} = \frac{1}{\pi}\frac{1}{(1+\varepsilon)e^{i\theta}\bar{z} - e^{-i\theta}(1-\varepsilon)z + o(\varepsilon)} \\ &= \frac{1}{2\pi}\frac{1}{-i\mathcal{I}(e^{-i\theta}z) + \varepsilon\mathcal{R}(e^{-i\theta}z) + o(\varepsilon)} = \frac{1}{2i\pi}\frac{-1}{\boldsymbol{\theta}^\perp\cdot\mathbf{x} + i\varepsilon(\boldsymbol{\theta}\cdot\mathbf{x}) + o(\varepsilon)}. \end{aligned}$$

Passing to the limit $\varepsilon \to 0$, we obtain

$$G_-(\mathbf{x},\theta) = \frac{1}{2i\pi}\frac{-1}{\boldsymbol{\theta}^\perp\cdot\mathbf{x} + i0\mathrm{sign}(\boldsymbol{\theta}\cdot\mathbf{x})}.$$

Here by convention ± 0 is the limit of $\pm\varepsilon$ as $0 < \varepsilon \to 0$. The limit on D^+ is treated similarly. □

We have chosen to define $G_\pm(\mathbf{x},\theta)$ as functions of $\theta \in (0, 2\pi)$ instead of functions of $e^{i\theta}$. We have also identified $\mathbf{x} = (x, y)$ with $z = x + iy$. The above lemma gives us a convergence in the sense of distributions. We can equivalently say that for all smooth function $\psi(\mathbf{x})$, we have

$$\int_{\mathbb{R}^2} G_\pm(\mathbf{x}-\mathbf{y},\theta)\psi(\mathbf{y})d\mathbf{y} = \pm\frac{1}{2i}(HR_\theta\psi)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp) + (D_\theta\psi)(\mathbf{x}). \tag{4.22}$$

We recall that the Hilbert transform H is defined in (2.11) and the Radon transform in (2.4)-(2.5).

Proof. The derivation is based on the following result. For any $f(x) \in C_0^\infty(\mathbb{R})$, we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{f(x)}{ix+\varepsilon}dx = -i\mathrm{p.v.}\int_{\mathbb{R}}\frac{f(x)}{x}dx + \mathrm{sign}(\varepsilon)\pi f(0). \tag{4.23}$$

Exercise 4.5.1 Prove the above limit called Plemelj's formula.

Let us denote $\mathbf{x} = \sigma\boldsymbol{\theta}^\perp + \tau\boldsymbol{\theta}$ and $\mathbf{y} = s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^2} G_+(\mathbf{y})\psi(\mathbf{x}-\mathbf{y})d\mathbf{y} = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\psi((\sigma-s)\boldsymbol{\theta}^\perp + (\tau-t)\boldsymbol{\theta})}{is + 0\mathrm{sign}(t)}dsdt\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{p.v.}\int_{\mathbb{R}}\frac{\psi((\sigma-s)\boldsymbol{\theta}^\perp + (\tau-t)\boldsymbol{\theta})}{is}dsdt + \frac{1}{2}\int_{\mathbb{R}}\mathrm{sign}(t)\psi(\sigma\boldsymbol{\theta}^\perp + (\tau-t)\boldsymbol{\theta})dt\\
&= \frac{1}{2\pi}\mathrm{p.v.}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\psi((\sigma-s)\boldsymbol{\theta}^\perp + (\tau-t)\boldsymbol{\theta})}{is}dtds + \frac{1}{2}\int_{\mathbb{R}}\mathrm{sign}(t)\psi(\mathbf{x}-t\boldsymbol{\theta})dt\\
&= \frac{1}{2i}(HR_\theta\psi)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp) + (D_\theta\psi)(\mathbf{x}).
\end{aligned}$$

A similar derivation yields the limit on D^-. □

We deduce that the function $h(z,\lambda)$ defined in (4.18) admits the limits

$$h_\pm(\mathbf{x},\theta) = \pm\frac{1}{2i}(HR_\theta a)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp) + (D_\theta a)(\mathbf{x}). \tag{4.24}$$

Notice that R_θ and D_θ involve integrations in the direction $\boldsymbol{\theta}$ only so that

$$R_\theta[u(\mathbf{x})v(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)](s) = v(s)R_\theta[u](s), \quad D_\theta[u(\mathbf{x})v(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)](\mathbf{x}) = v(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)D_\theta[u](\mathbf{x}).$$

Using this result and (4.22), we deduce that the limits of the solution $u(z,\lambda)$ to (4.9) are given by

$$\begin{aligned}
u^\pm(\mathbf{x},\theta) = {}& e^{-D_\theta a}e^{\frac{\mp 1}{2i}(HR_\theta a)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)}\frac{\mp 1}{2i}H\Big(e^{\frac{\pm 1}{2i}(HR_\theta a)(s)}R_\theta(e^{D_\theta a}f)\Big)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp)\\
&+e^{-D_\theta a}D_\theta(e^{D_\theta a}f)(\mathbf{x}).
\end{aligned} \tag{4.25}$$

We recall that $R_\theta(e^{D_\theta a} f) = R_{a,\theta} f(s)$ are our measurements. So whereas u^+ and u^- do not depend only on the measurements (they depend on $D_\theta(e^{D_\theta a} f)(\mathbf{x})$ which is not measured), the difference $u^+ - u^-$ depends *only* on the measurements. This is the property that allows us to invert the attenuated Radon transform. More precisely, let us define

$$\varphi(\mathbf{x}, \theta) = (u^+ - u^-)(\mathbf{x}, \theta). \tag{4.26}$$

Using (4.25) we deduce that

$$i\varphi(\mathbf{x}, \theta) = R^*_{-a,\theta} H_a R_{a,\theta} f(\mathbf{x}), \tag{4.27}$$

where we have defined the following operators

$$\begin{aligned} &R^*_{a,\theta} g(\mathbf{x}) = e^{D_\theta a(\mathbf{x})} g(\mathbf{x}\cdot\boldsymbol{\theta}^\perp), \qquad H_a = C_c H C_c + C_s H C_s \\ &C_c g(s,\theta) = g(s,\theta)\cos(\frac{H R_\theta a(s)}{2}), \quad C_s g(s,\theta) = g(s,\theta)\sin(\frac{H R_\theta a(s)}{2}). \end{aligned} \tag{4.28}$$

Here $R^*_{a,\theta}$ is the formal adjoint operator to $R_{a,\theta}$.

The above derivation shows that $i\varphi(\mathbf{x}, \theta)$ is real-valued and of the form $e^{-D_\theta a(\mathbf{x})} M(\mathbf{x}\cdot\boldsymbol{\theta}^\perp, \theta)$ for some function M. We deduce therefore that

$$\boldsymbol{\theta}\cdot\nabla\varphi(\mathbf{x}, \theta) + a\varphi(\mathbf{x}, \theta) = 0. \tag{4.29}$$

4.6 Step (iii): reconstruction formulas

We have seen that $u(z, \lambda)$ is analytic in λ on $D^+ \cup D^-$ and is of order $O(z^{-1})$ at infinity. Moreover the jump of $u(z, \lambda)$ across T is given by $\varphi(\mathbf{x}, \theta)$ for $0 \le \theta < 2\pi$. We thus deduce from the Cauchy formula (4.7) that

$$u(\mathbf{x}, \lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(\mathbf{x}, t)}{t - \lambda} dt, \qquad \lambda \in D^+ \cup D^-. \tag{4.30}$$

where we identify $\varphi(\mathbf{x}, t)$ with $\varphi(\mathbf{x}, \theta)$ for $t = e^{i\theta}$. We now deduce from (4.11) in the vicinity of $\lambda = 0$ that

$$f(\mathbf{x}) = \lim_{\lambda\to 0} \lambda^{-1} \frac{\partial}{\partial\bar{z}} u(\mathbf{x}, \lambda). \tag{4.31}$$

Indeed we verify that $u(z, \lambda) = O(\lambda)$ on D^+ so that $a(\mathbf{x})u(\mathbf{x}, \lambda) \to 0$ as $\lambda \to 0$. Since $u(\mathbf{x}, \lambda)$ is known thanks to (4.30) in terms of the boundary measurements $R_{a,\theta} f(s)$, this is our reconstruction formula. Let us be more specific. We verify that

$$u(\mathbf{x}, \lambda) = \frac{1}{2\pi i}\int_T \frac{\varphi(\mathbf{x}, t)}{t} dt + \lambda \frac{1}{2\pi i}\int_T \frac{\varphi(\mathbf{x}, t)}{t^2} dt + O(\lambda^2). \tag{4.32}$$

We thus deduce from (4.31) and the fact that $u(\mathbf{x}, \lambda) = O(\lambda)$ on D^+ that

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\int_T \frac{\varphi(\mathbf{x}, t)}{t} dt \\ f(\mathbf{x}) &= \frac{1}{2\pi i}\int_T \frac{\partial\varphi}{\partial\bar{z}}(\mathbf{x}, t)\frac{1}{t^2} dt. \end{aligned} \tag{4.33}$$

The second equality is the reconstruction formula we were looking for since φ defined in (4.27) in terms of the measurements $R_{a,\theta}f(s)$. The first equality is a *compatibility* conditions that $i\varphi$ must satisfy in order for the data to be the attenuated Radon transform of a function $f(\mathbf{x})$. This compatibility condition is similar to the condition $g(s,\theta) = g(-s,\theta+\pi)$ satisfied by the Radon transform in the absence of absorption. These compatibility conditions are much more difficult to visualize when absorption does not vanish because the integral along the line $\{s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}; t \in \mathbb{R}\}$ differs depending on the direction of integration.

Let us recast the reconstruction formula so that it only involves real-valued quantities.

Exercise 4.6.1 Using $t = e^{i\theta}$ and $dt = ie^{i\theta}$, deduce that

$$\begin{aligned} \frac{1}{2\pi i}\int_T \frac{\varphi(\mathbf{x},t)}{t}dt &= \frac{1}{2\pi}\int_0^{2\pi}\varphi(\mathbf{x},\theta)d\theta \\ \frac{1}{2\pi i}\int_T \frac{\partial\varphi}{\partial\bar{z}}(\mathbf{x},t)\frac{1}{t^2}dt &= \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla(i\varphi)(\mathbf{x},\theta)d\theta + \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}\cdot\nabla\varphi(\mathbf{x},\theta)d\theta. \end{aligned}$$

Use (4.29) and (4.33) to show that

$$f(\mathbf{x}) = \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla(i\varphi)(\mathbf{x},\theta)d\theta. \tag{4.34}$$

Let us denote by $g_a(s,\theta) = R_a f(s,\theta)$ the SPECT measurements. From the above results we recast (4.34) as

$$f(\mathbf{x}) = [\mathcal{N}g](\mathbf{x}) \equiv \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla(R^*_{-a,\theta}H_a g)(\mathbf{x},\theta)d\theta. \tag{4.35}$$

Exercise 4.6.2 Show that (4.35) simplifies to (2.14) when $a \equiv 0$.

Exercise 4.6.3 (Reconstruction with constant absorption.) We assume that $f(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq 1$ and that $a(\mathbf{x}) = \mu$ for $|\mathbf{x}| < 1$. This corresponds thus to the case of a constant absorption coefficient on the support of the source term $f(\mathbf{x})$.
(i) Show that

$$e^{D_\theta a(\mathbf{x})} = e^{\mu\mathbf{x}\cdot\boldsymbol{\theta}}, \qquad |\mathbf{x}| < 1.$$

Deduce that

$$\boldsymbol{\theta}^\perp\cdot\nabla(e^{D_\theta a(\mathbf{x})}g(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)) = e^{\mu\mathbf{x}\cdot\boldsymbol{\theta}}\frac{\partial g}{\partial s}(\mathbf{x}\cdot\boldsymbol{\theta}^\perp).$$

(ii) Verify that the operator H_μ defined by $H_\mu = H_a$ for a constant is diagonal in the Fourier domain and that

$$\widehat{H_\mu u}(\sigma) = -i\mathrm{sign}_\mu(\sigma)\hat{u}(\sigma), \qquad \mathrm{sign}_\mu(\sigma) = \begin{cases} \mathrm{sign}(\sigma) & |\sigma| \geq \mu, \\ 0 & |\sigma| < \mu. \end{cases}$$

(iii) Show that

$$\begin{aligned} g_\mu(s,\theta) &= R_\theta(e^{\mu\mathbf{x}\cdot\boldsymbol{\theta}}f)(s), \\ f(\mathbf{x}) &= \frac{1}{4\pi}\int_0^{2\pi} e^{-\mu\mathbf{x}\cdot\boldsymbol{\theta}}(H_\mu\frac{\partial}{\partial s}g_\mu)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)d\theta. \end{aligned} \tag{4.36}$$

Verify that in the limit $\mu \to 0$, we recover the inversion for the Radon transform.

4.7 Source problem with scattering

We now consider the same problem as above except that we now account for possible scattering. To simplify the presentation we shall assume that photon propagation is two dimensional. This approximation, which is valid in the absence of scattering because photons propagate along straight lines, is not physical in three space dimensions in the presence of scattering.

One interesting application of source transport problems with scattering is optical fluorescence. Fluorescent particles are injected into a body and emit (near infra red) light under certain conditions. The emitted light is measured at the boundary of the domain. The objective is then to reconstruct the density of fluorescent particles. The latter gives some indication about the tissues to which the fluorescent particles are attached.

The two dimensional source transport equation with scattering takes the following form

$$\boldsymbol{\theta} \cdot \nabla u(\mathbf{x}, \theta) + a(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) = \int_{S^1} \sigma_s(\mathbf{x}, \boldsymbol{\theta} \cdot \boldsymbol{\theta}') u(\mathbf{x}, \theta') d\theta' + f(\mathbf{x}). \tag{4.37}$$

Here $\sigma_s(\mathbf{x}, \boldsymbol{\theta} \cdot \boldsymbol{\theta}') \equiv \sigma_s(\mathbf{x}, \theta - \theta')$ is the scattering coefficient. It is assumed to be known. As before we identify $\boldsymbol{\theta} \in S^1$ with $\theta \in [0, 2\pi)$ such that $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$.

The boundary conditions are still given by

$$\lim_{t \to +\infty} u(\mathbf{x} - t\boldsymbol{\theta}, \theta) = 0, \qquad \mathbf{x} \in \Omega, \quad \boldsymbol{\theta} \in S^1.$$

The data we consider are given by

$$g(s, \theta) = \lim_{t \to \infty} [e^{D_\theta a} u](s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}, \theta). \tag{4.38}$$

What is physically measured is $\lim_{t\to\infty} u(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}, \theta)$. However since a is known (as well as σ), we work with the modified data in (4.38). Let us introduce the scattering operator

$$Ku(\mathbf{x}, \theta) = \int_{S^1} \sigma_s(\mathbf{x}, \boldsymbol{\theta} \cdot \boldsymbol{\theta}') u(\mathbf{x}, \theta') d\theta', \tag{4.39}$$

and the classical beam transform

$$Sg(\mathbf{x}, \theta) = \int_{-\infty}^{0} g(\mathbf{x} + s\boldsymbol{\theta}) ds. \tag{4.40}$$

We verify that $w(\mathbf{x}, \theta) = e^{D_\theta a}(\mathbf{x}) u(\mathbf{x}, \theta)$ solves the following integral equation

$$w(\mathbf{x}, \theta) = [Se^{D_\theta a} K e^{-D_\theta a} w](\mathbf{x}, \theta) + [Se^{D_\theta a} f](\mathbf{x}, \theta). \tag{4.41}$$

Exercise 4.7.1 Check (4.41).

Let us assume that the scattering operator is *subcritical*, which for us means that in a certain norm (that we do not describe here),

$$|||Se^{D_\theta a} K e^{-D_\theta a}||| < 1. \tag{4.42}$$

This allows us to invert (4.41) as

$$w(\mathbf{x},\theta) = Tf(\mathbf{x},\theta) = Se^{D_\theta a} f + Se^{D_\theta a} Ke^{-D_\theta a} Tf, \tag{4.43}$$

where we have defined the operator

$$T = [I - Se^{D_\theta a} Ke^{-D_\theta a}]^{-1} Se^{D_\theta a}. \tag{4.44}$$

Replacing $\mathbf{x} = s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}$ and passing to the limit $t \to \infty$ in (4.43), we obtain that

$$g(s,\theta) = R_a f(s,\theta) + Re^{D_\theta a} Ke^{-D_\theta a} Tf(s,\theta). \tag{4.45}$$

Applying the Novikov inversion operator $\mathcal{N}$ defined in (4.35) on both sides, we obtain that

$$f(\mathbf{x}) + \mathcal{N} Re^{D_\theta a} Ke^{-D_\theta a} Tf(\mathbf{x}) = \mathcal{N} g(\mathbf{x}). \tag{4.46}$$

Provided that the operator $\mathcal{N} Re^{D_\theta a} Ke^{-D_\theta a} T$ is of norm less than 1 (in an appropriate norm not defined here), we observe that the above equation can be inverted and that the solution is given in terms of the following Neumann expansion

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} [-\mathcal{N} Re^{D_\theta a} Ke^{-D_\theta a} T]^k \mathcal{N} g(\mathbf{x}). \tag{4.47}$$

This gives us a reconstruction formula and an algorithm to reconstruct the source term $f(\mathbf{x})$ from boundary measurements in the presence of sufficiently small scattering. How small scattering should be for the theory to apply is governed by the constraint that $\mathcal{N} Re^{D_\theta a} Ke^{-D_\theta a} T$ is of norm less than 1 in an appropriate norm.

Exercise 4.7.2 We consider in this exercise an explicit algorithm to reconstruct $f(\mathbf{x})$ from the data $g(s,\theta)$ iteratively. Let us define

$$f^0(\mathbf{x}) = \mathcal{N} g(\mathbf{x}). \tag{4.48}$$

For $k \geq 0$, we define u^k as the solution to

$$\boldsymbol{\theta} \cdot \nabla u^k(\mathbf{x},\theta) + a(\mathbf{x}) u^k(\mathbf{x},\theta) = Ku^k(\mathbf{x},\theta) + f^k(\mathbf{x}), \tag{4.49}$$

and update $f(\mathbf{x})$ as

$$f^{k+1}(\mathbf{x}) = \mathcal{N} g(\mathbf{x}) - \mathcal{N} R_a e^{D_\theta a} Ku^k(\mathbf{x}). \tag{4.50}$$

Assuming that scattering is sufficiently small as indicated above, show that $f^k(\mathbf{x})$ converges to $f(\mathbf{x})$ as $k \to \infty$.

In the above theory, we have assumed that the total scattering operator K was sufficiently small. In fact, we need only assume that the *anisotropic* part of the scattering operator is small. This is shown as follows.

Exercise 4.7.3 Let us split the scattering operator as

$$Ku(\mathbf{x},\theta) = K_0 u(\mathbf{x}) + K_1 u(\mathbf{x},\theta), \quad K_0 u(\mathbf{x}) = \sigma_0(\mathbf{x}) \frac{1}{2\pi} \int_0^{2\pi} u(\mathbf{x},\theta') d\theta'. \tag{4.51}$$

Here K_1 is chosen to be anisotropic in the sense that $\int_0^{2\pi} K_1 u(\mathbf{x}, \theta) d\theta = 0$. In many applications, the norm of the anisotropic part of scattering K_1 is substantially smaller than the full scattering K. It is smaller in the fluorescence application.

Let $u(\mathbf{x}, \theta)$ be the solution to (4.37) and $g(s, \theta)$ the associated measured data at the boundary of the domain. Find $h(\mathbf{x})$ so that the transport equation (4.37) is recast as

$$\boldsymbol{\theta} \cdot \nabla u(\mathbf{x}, \theta) + a(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) = K_1 u(\mathbf{x}, \theta) + h(\mathbf{x}).$$

Assuming that K_1 is sufficiently small, find $h(\mathbf{x})$ from the boundary measurements $g(s, \theta)$.

Find $f(\mathbf{x})$ from the boundary measurements (you need to solve yet another transport equation to do so.

Show that $f(\mathbf{x})$ can be uniquely reconstructed with no assumption on the scattering operator K (other than the subcriticality condition (4.42) to ensure that the transport equation (4.37) admits a unique solution) when scattering is isotropic, i.e., the component $K_1 \equiv 0$.

Chapter 5

Diffraction tomography

In this chapter we consider the probing of an object by waves rather than particles. The waves are typically acoustic waves and micro waves. Such waves are characterized by a wavelength small enough so that the features of the object one is interested in can indeed be imaged, but large enough that the high frequency geometrical optics limit (whereby waves are replaced by particles) does not hold. As the frequency increases, the geometrical optics assumption becomes more and more accurate. We will see that the reconstruction in that limit gets closer to the particle reconstruction, i.e., involves inverting a Radon transform.

5.1 Scattering problem

Let us consider the wave equation in dimension $n = 2, 3$ given by

$$\frac{1}{v^2(\mathbf{x})}\frac{\partial^2 U}{\partial t^2} - \Delta U = 0, \qquad \mathbf{x} \in \mathbb{R}^n, \quad t > 0, \tag{5.1}$$

with appropriate initial conditions. The velocity $v(\mathbf{x})$ is unknown. Let us assume that $v(\mathbf{x}) = c$ outside of the object we want to image, where c is a constant. We assume that $U = 0$ for $t < 0$ and pass to the frequency domain by introducing

$$u(\mathbf{x}, \omega) = \int_0^\infty e^{i\omega t} U(\mathbf{x}, t) dt, \qquad U(\mathbf{x}, t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t} u(\mathbf{x}, \omega) d\omega. \tag{5.2}$$

The equation for u is then the following *Helmholtz* equation

$$\begin{aligned} &(\Delta + \frac{\omega^2}{v^2(\mathbf{x})})u(\mathbf{x}, \omega) = 0, \qquad \mathbf{x} \in \mathbb{R}^n, \quad \omega \in \mathbb{R}, \\ &\hat{\mathbf{x}} \cdot \nabla u(\mathbf{x}, \omega) - i\frac{\omega}{c} u(\mathbf{x}, \omega) = o(|\mathbf{x}|^{-(n-1)/2}). \end{aligned} \tag{5.3}$$

As usual $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ and the second equation is the *radiation* condition, ensuring that no energy comes from infinity (only waves radiating *out* are allowed at infinity). The notation $o(x)$ means a quantity such that $o(x)/x \to 0$ as $0 < x \to 0$. So the decay should be faster than $|\mathbf{x}|^{-1/2}$ in two dimensions and faster than $|\mathbf{x}|^{-1}$ in three dimensions.

Let us now introduce the following expression for the velocity and the frequency:

$$\frac{1}{v^2(\mathbf{x})} = \frac{1}{c^2}(1+\alpha(\mathbf{x})), \qquad k = \frac{\omega}{c}. \tag{5.4}$$

We recast the Helmholtz equation as

$$\begin{aligned} &(\Delta + k^2)u(\mathbf{x},\omega) = -\alpha(\mathbf{x})k^2 u(\mathbf{x},\omega), \\ &\hat{\mathbf{x}}\cdot\nabla u(\mathbf{x},\omega) - iku(\mathbf{x},\omega) = o(|\mathbf{x}|^{-(n-1)/2}). \end{aligned} \tag{5.5}$$

Let $\boldsymbol{\theta} \in S^n$ be a unit vector. We verify that

$$(\Delta + k^2)u_i(\mathbf{x},\omega;\boldsymbol{\theta}) = 0, \qquad \text{where} \quad u_i(\mathbf{x},\omega;\boldsymbol{\theta}) = e^{ik\boldsymbol{\theta}\cdot\mathbf{x}}. \tag{5.6}$$

Thus plane waves with the right wavenumber $k = |k\boldsymbol{\theta}|$ are solutions of the homogeneous Helmholtz equation. Notice however that they *do not* satisfy the radiation conditions (they do radiate out in the direction $\boldsymbol{\theta}$ but certainly not in the direction $-\boldsymbol{\theta}$).

The forward problem we are interested in is the following: we have a plane wave coming from infinity and want to find a solution $u_s(\mathbf{x},\omega)$ modeling the response of the system. Therefore we ask that u_s does not radiate at infinity (i.e., satisfies the radiation condition) and that the whole field $u = u_i + u_s$ satisfies the Helmholtz equation as required by physics. We thus end up with solving the following *scattering* problem

$$\begin{aligned} &(\Delta + k^2)u_s(\mathbf{x},\omega) = -\alpha(\mathbf{x})k^2(u_s+u_i)(\mathbf{x},\omega), \\ &\hat{\mathbf{x}}\cdot\nabla u_s(\mathbf{x},\omega) - iku_s(\mathbf{x},\omega) = o(|\mathbf{x}|^{-(n-1)/2}). \end{aligned} \tag{5.7}$$

In the above equation we have used (5.6). Under some assumptions on $\alpha(\mathbf{x})$, the above equation admits a unique solution [8]. The *inverse scattering* problem consists then of reconstructing $\alpha(\mathbf{x})$ from measurements of u_s at infinity in all possible directions $\mathbf{x}$ for all possible plane waves $\boldsymbol{\theta} \in S^n$. We will not be concerned with this general problem, which is fairly well-understood for $n = 3$ and much less so for $n = 2$. We refer the reader to [8] for details on this difficult theory. What we will do instead is to concentrate on the linearization of the problem about the constant velocity profile $v(\mathbf{x}) = c$. Namely, we will assume that α is small (in some appropriate sense (norm) that we do not describe here). As a consequence, u_s is also small as can be seen from (5.7). We can therefore neglect the term αu_s, which is second-order, as a first approximation. This approximation is called the Born approximation. It also has the advantage of linearizing the problem of reconstructing $\alpha(\mathbf{x})$ from scattering measurements. We are thus now concerned with

$$\begin{aligned} &(\Delta + k^2)u_s(\mathbf{x},\omega) = -\alpha(\mathbf{x})k^2 u_i(\mathbf{x},\omega), \\ &\hat{\mathbf{x}}\cdot\nabla u_s(\mathbf{x},\omega) - iku_s(\mathbf{x},\omega) = o(|\mathbf{x}|^{-(n-1)/2}). \end{aligned} \tag{5.8}$$

This equation can be solved explicitly as

$$u_s(\mathbf{x},\omega) = k^2 \int_{\mathbb{R}^n} \alpha(\mathbf{y})u_i(\mathbf{y},\omega)g_n(\mathbf{x}-\mathbf{y})d\mathbf{y}, \tag{5.9}$$

where g_n is the Green function solving the following equation

$$\begin{aligned}&(\Delta + k^2)g_n(\mathbf{x}) = \delta(\mathbf{x})\\ &\hat{\mathbf{x}} \cdot \nabla u_s(\mathbf{x}, \omega) - iku_s(\mathbf{x}, \omega) = o(|\mathbf{x}|^{-(n-1)/2}),\end{aligned} \tag{5.10}$$

and is given for $n = 2, 3$ by

$$g_2(\mathbf{x}) = \frac{i}{4}H_0(k|\mathbf{x}|), \qquad g_3(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}. \tag{5.11}$$

Here, H_0 is the 0th order Hankel function of the first kind, given by

$$H_0(k|\mathbf{x}|) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{\sqrt{k^2 - p^2}} e^{i(px + \sqrt{k^2-p^2}y)}dp, \tag{5.12}$$

where we have decomposed $\mathbf{x} = (x, y)$ in Cartesian coordinates.

5.2 Far field data and reconstruction

The measurements we consider are the far field scattering data. They correspond to the waves propagating outwards at infinity. This simplification amounts to saying that the other component of the radiating field composed of the *evanescent* waves, does not make it to infinity, and thus cannot be measured. Mathematically, this means that we consider the asymptotic limit of u_s as $\mathbf{x} \to \infty$. Let us consider the three dimensional case first. Since $\mathbf{x}$ goes to infinity, $|\mathbf{x} - \mathbf{y}|$ is equal to $|\mathbf{x}|$ plus a smaller order correction. So we have

$$u_s(\mathbf{x}, \omega) = \frac{k^2}{4\pi|\mathbf{x}|}\int_{\mathbb{R}^3} \alpha(\mathbf{y}) e^{ik\boldsymbol{\theta}\cdot\mathbf{y}} e^{ik|\mathbf{x}-\mathbf{y}|}d\mathbf{y} \ + \ \text{l.o.t.} \ .$$

Upon using the following approximation

$$|\mathbf{x} - \mathbf{y}| = |\mathbf{x}||\hat{\mathbf{x}} - \frac{\mathbf{y}}{|\mathbf{x}|}| = |\mathbf{x}|(1 + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} - 2\frac{\hat{\mathbf{x}}\cdot\mathbf{y}}{|\mathbf{x}|^2})^{1/2} = |\mathbf{x}| - \hat{\mathbf{x}}\cdot\mathbf{y} + \ \text{l.o.t.},$$

we obtain

$$u_s(\mathbf{x}, \omega) = \frac{k^2 e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}\int_{\mathbb{R}^3} \alpha(\mathbf{y}) e^{ik(\boldsymbol{\theta}-\hat{\mathbf{x}})\cdot\mathbf{y}}d\mathbf{y} \ + \ \text{l.o.t.} \ .$$

We thus see that

$$\begin{aligned}u_s(\mathbf{x}, \omega) &= \frac{k^2 e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}A(\hat{\mathbf{x}}) + o\Big(\frac{1}{|\mathbf{x}|}\Big),\\ A(\hat{\mathbf{x}}) &= \hat{\alpha}\big(k(\hat{\mathbf{x}} - \boldsymbol{\theta})\big) = \int_{\mathbb{R}^3} \alpha(\mathbf{y}) e^{ik(\boldsymbol{\theta}-\hat{\mathbf{x}})\cdot\mathbf{y}}d\mathbf{y}.\end{aligned} \tag{5.13}$$

Recall that $\omega = ck$. So for a plane wave at a given frequency ω, i.e., at a given wavenumber k, and direction $\boldsymbol{\theta}$, the far field measurement is $A(\hat{\mathbf{x}}) = A(\hat{\mathbf{x}}; k, \boldsymbol{\theta})$ in the direction $\hat{\mathbf{x}}$ (obtained by multiplying the measured signal by $4\pi|\mathbf{x}|e^{-ik|\mathbf{x}|}/k^2$).

In two space dimensions ($n = 2$), the final result is similar in the sense that u_s is proportional to $|\mathbf{x}|^{-1/2}$ at infinity with a coefficient of proportionality $A(\hat{\mathbf{x}})$ taking the same expression as given in (5.13).

We observe that each measurement gives us new information about the Fourier transform of the velocity fluctuation $\alpha(\mathbf{x})$. We can distinguish two types of measurements. The first ones correspond to directions of measurements $\mathbf{x}$ such that $\mathbf{x} \cdot \boldsymbol{\theta} > 0$. These measurements are called *transmission* measurements since they correspond to the radiated wave that have passed through the object to image. The second ones correspond to the directions such that $\mathbf{x} \cdot \boldsymbol{\theta} < 0$ and are called *reflection* measurements. In many settings, we have access to only one type of measurements.

Let us consider transmission measurements first. This means that we have access to $\hat{\alpha}(k(\hat{\mathbf{x}} - \boldsymbol{\theta}))$ for $\hat{\mathbf{x}} \cdot \boldsymbol{\theta} > 0$. In particular we obtain for $\hat{\mathbf{x}} = \hat{\boldsymbol{\theta}}$ the value $\hat{\alpha}(\mathbf{0})$, which is the average of the fluctuation $\alpha(\mathbf{x})$ over the whole domain. More generally as $\hat{\mathbf{x}}$ varies in S^1 such that $\hat{\mathbf{x}} \cdot \boldsymbol{\theta} > 0$, we obtain $\hat{\alpha}(\mathbf{k})$ over a half-circle passing through $\mathbf{0}$, of radius k and symmetric about the axis $\boldsymbol{\theta}$. As $\boldsymbol{\theta}$ varies on the unit circle, we observe that $\hat{\alpha}(k(\hat{\mathbf{x}} - \boldsymbol{\theta}))$ fills in the disk of radius $\sqrt{2}k$. At a fixed value of k, this is therefore all we can get: $\hat{\alpha}(\mathbf{k})$ for $\mathbf{k}$ such that $|\mathbf{k}| \leq \sqrt{2}k$.

The picture in three dimensions is very similar: for a given $\boldsymbol{\theta} \in S^2$, we have access to $\hat{\alpha}(\mathbf{k})$ for $\mathbf{k}$ on a half-sphere of radius $\sqrt{2}k$ passing through $\mathbf{0}$ and invariant by rotation about $\boldsymbol{\theta}$. As $\boldsymbol{\theta}$ varies over the sphere S^2, we thus get $\hat{\alpha}(\mathbf{k})$ for all $\mathbf{k}$ such that $|\mathbf{k}| \leq \sqrt{2}k$, as in the two-dimensional case.

The diffraction inverse problem is therefore *not unique*. All we can reconstruct from the measured data is a low-pass filter of the object $\alpha(\mathbf{x})$. The high frequencies are not measured. The high frequencies of α are encoded in the radiation field u_s. However they are the evanescent part of the waves. They decay therefore much more rapidly than $|\mathbf{x}|^{-1}$ (when $n = 3$), actually exponentially, and thus cannot be measured accurately in practice.

Let us now consider reconstruction formulas. Since frequencies above $\sqrt{2}k$ cannot be reconstructed, we assume that

$$\alpha(\mathbf{x}) = (\mathcal{F}^{-1}_{\mathbf{k}\to\mathbf{x}} \chi_{\sqrt{2}k}(\mathbf{k}) \mathcal{F}_{\mathbf{x}\to\mathbf{k}} \alpha)(\mathbf{x}), \tag{5.14}$$

where $\chi_{\sqrt{2}k}(\mathbf{k}) = 1$ when $|\mathbf{k}| < \sqrt{2}k$ and 0 otherwise,i.e. α does not have high wavenumbers. Then the reconstruction is obviously unique according to what we just saw. Let us consider the two-dimensional case. We want to reconstruct $\alpha(\mathbf{x})$ from $\hat{\alpha}(k(\hat{\mathbf{x}} - \boldsymbol{\theta}))$, where $\hat{\mathbf{x}}$ and $\boldsymbol{\theta}$ run over the unit circle S^1. Notice that we want to reconstruct a two-dimensional function and we have two dimensions of data (one in $\hat{\mathbf{x}}$ and one in $\boldsymbol{\theta}$). The inverse Fourier transform tells us that

$$\alpha(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i\mathbf{x}\cdot\mathbf{k}} \hat{\alpha}(\mathbf{k}) d\mathbf{k} = \frac{k^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{\sqrt{2}} e^{ik\rho\mathbf{x}\cdot\boldsymbol{\theta}} \hat{\alpha}(k\rho\boldsymbol{\theta}) \rho d\rho d\theta.$$

Observe that as $\boldsymbol{\theta}$ covers the unit circle, all points of the disk $|\mathbf{k}| < \sqrt{2}k$ are covered twice as $\hat{\mathbf{x}}$ varies, once for a point such that $\hat{\mathbf{k}} \cdot \boldsymbol{\theta}^\perp > 0$ and once for a point such that $\hat{\mathbf{k}} \cdot \boldsymbol{\theta}^\perp < 0$. Therefore the information corresponding to $\hat{\mathbf{k}} \cdot \boldsymbol{\theta}^\perp > 0$ is sufficient. This information is parameterized as follows: for a given $\boldsymbol{\theta}$ we write $\hat{\mathbf{x}}$ as

$$\hat{\mathbf{x}}(\phi, \boldsymbol{\theta}) = \sin\phi \boldsymbol{\theta} + \cos\phi \boldsymbol{\theta}^\perp, \qquad 0 \leq \phi \leq \frac{\pi}{2}. \tag{5.15}$$

We thus obtain that

$$\hat{\alpha}(k(\hat{\mathbf{x}}-\boldsymbol{\theta}))=\hat{\alpha}\big(k\rho(\phi)(\frac{\sin\phi-1}{\rho(\phi)}\boldsymbol{\theta}+\frac{\cos\phi}{\rho(\phi)}\boldsymbol{\theta}^{\perp})\big)=\hat{\alpha}\big(k\rho(\phi)\mathcal{R}(\phi)\boldsymbol{\theta}\big),$$

with $\rho(\phi)=\sqrt{2}\sqrt{1-\cos\phi}$ and $\mathcal{R}(\phi)$ an explicitly defined rotation depending on ϕ. Here is the rest of the reconstruction:

$$\begin{aligned}\alpha(\mathbf{x})&=\frac{k^2}{(2\pi)^2}\int_0^{2\pi}\int_0^{\pi/2}e^{ik\rho(\phi)\mathbf{x}\cdot\boldsymbol{\theta}}\hat{\alpha}(k\rho(\phi)\boldsymbol{\theta})\rho(\phi)\frac{d\rho(\phi)}{d\phi}d\phi d\theta\\&=\frac{k^2}{(2\pi)^2}\int_0^{2\pi}\int_0^{\pi/2}e^{ik\rho(\phi)\mathbf{x}\cdot\mathcal{R}(\phi)\boldsymbol{\theta}}\hat{\alpha}(k\rho(\phi)\mathcal{R}(\phi)\boldsymbol{\theta})\frac{1}{2}\frac{d\rho^2(\phi)}{d\phi}d\phi d\theta,\end{aligned}$$

so that finally

$$\alpha(\mathbf{x})=\frac{2k^2}{(2\pi)^2}\int_0^{2\pi}\int_0^{\pi/2}e^{ik\rho(\phi)\mathbf{x}\cdot\mathcal{R}(\phi)\boldsymbol{\theta}}\hat{\alpha}(k|\hat{\mathbf{x}}(\phi,\boldsymbol{\theta})-\boldsymbol{\theta}|)\sin\phi d\phi d\theta. \tag{5.16}$$

This is the reconstruction formula we were after.

Let us conclude the section with a few words about reflection tomography. In that case, we only measure data in directions $\hat{x}$ such that $\hat{x}\cdot\boldsymbol{\theta}<0$. Following the same techniques as above, we see that we can reconstruct wavenumbers of $\alpha(\mathbf{x})$ in the corona of wavenumbers $\mathbf{k}$ such that $\sqrt{2}k<|\mathbf{k}|<2k$. The reconstruction from reflection data is therefore by no means unique. We do not get low-frequency components of α and do not get very-high frequencies either. Assuming that the wavenumber content of $\alpha(\mathbf{x})$ is in the above corona, then the reconstruction is unique. A reconstruction formula similar to what we just obtained can also be derived. Notice that when both the transmission and reflection data can be measured, we can reconstruct all wavenumbers $\mathbf{k}$ of $\alpha(\mathbf{x})$ such that $|\mathbf{k}|<2k$.

All these result are in sharp contrast to the one-dimensional example we saw earlier. There, a given wavenumber k allows us to reconstruct one wavenumber of $\alpha(x)$. All wavenumbers are thus required (i.e. measurements for all frequencies ω) to reconstruct $\alpha(x)$. Here $\alpha(\mathbf{x})$ is also uniquely determined by reconstruction for all values of k (since each value of k allows us to reconstruct all wavenumbers $|\mathbf{k}|<2k$). However because of the multidimensional nature of the measurements (the variable $\hat{x}$ is discrete in one-dimension instead of living on the unit sphere S^n), measurements for all values of k is very redundant: once we have obtained measurements at a given value of k, all measurements at values less than k are redundant.

5.3 Comparison to X-ray tomography

Let us consider the case of transmission data in two space dimensions. We have seen that wavenumbers of $\alpha(\mathbf{x})$ up to $\sqrt{2}k$ could be reconstructed. However as k tends to ∞, this essentially means that all wavenumbers of $\alpha(\mathbf{x})$ can be reconstructed.

Indeed in that limit we observe that the half circle of radius k becomes the full line orthogonal to $\boldsymbol{\theta}$. That is, as $k\to\infty$, the measurements tend to

$$\hat{\alpha}(\sigma\boldsymbol{\theta}^{\perp})=R\alpha(\sigma,\theta+\pi/2).$$

Exercise 5.3.1 Show that the reconstruction formula (5.16) indeed converges to the inverse Radon transform as $k \to \infty$.

In the limit of infinite frequency, we therefore obtain that the transmission measurements tend to the Radon transform of α. We know that the knowledge of $R_\alpha(\sigma, \theta + \pi/2)$ for all values of σ and θ is sufficient to uniquely reconstruct the fluctuation $\alpha(\mathbf{x})$.

So how should we consider the inverse diffraction problem? How ill-posed is it? As we already mentioned, the first problem with diffraction tomography is that for a fixed frequency ω, the function $\alpha(\mathbf{x})$ cannot uniquely be reconstructed. Only the wavenumbers below $\sqrt{2}k$ (below $2k$) in the case of transmission (transmission and reflection) measurements can be reconstructed. However in the class of functions $\alpha(\mathbf{x}) \in L^2(\mathbb{R}^2)$ such that (5.14) holds, we have uniqueness of the reconstruction. In this class we can perform a similar analysis to what we obtained in Theorem 2.2.2.

Let us consider the measurements $d(\phi, \boldsymbol{\theta}) = \hat{\alpha}(k(\hat{\mathbf{x}} - \boldsymbol{\theta}))$ for $\boldsymbol{\theta} \in S^1$ and $0 \leq \phi \leq \pi/2$ using (5.15). We verify that $1 \leq \rho'(\phi) \leq \sqrt{2}$ for $0 \leq \phi \leq \pi/2$.

Let us assume that the error we make is of the same order for every angle ϕ and every angle $\boldsymbol{\theta}$. An estimate of the total error will thus involve

$$\begin{aligned}
&\int_{S^1} \int_0^{\pi/2} |d(\phi, \boldsymbol{\theta})|^2 d\phi d\boldsymbol{\theta} = \int_{S^1} \int_0^{\pi/2} |\hat{\alpha}(k\rho(\phi)\mathcal{R}(\phi)\boldsymbol{\theta})|^2 d\phi d\boldsymbol{\theta} \\
&= \int_{S^1} \int_0^{\sqrt{2}} |\hat{\alpha}(k\rho\boldsymbol{\theta})|^2 (\rho')^{-1} d\rho d\boldsymbol{\theta} \sim \int_{S^1} \int_0^{\sqrt{2}} |\hat{\alpha}(k\rho\boldsymbol{\theta})|^2 d\rho d\boldsymbol{\theta} \\
&\sim \frac{1}{k} \int_{S^1} \int_0^{\sqrt{2}k} |\hat{\alpha}(u\boldsymbol{\theta})|^2 du d\boldsymbol{\theta} \sim \frac{1}{k} \int_{S^1} \int_0^{\sqrt{2}k} \frac{|\hat{\alpha}(u\boldsymbol{\theta})|^2 u \; du \; d\boldsymbol{\theta}}{u} \\
&\geq \frac{1}{k} \|\alpha\|^2_{H^{-1/2}(\mathbb{R}^2)}.
\end{aligned}$$

In some sense, the above formula also shows that the data $d(\phi, \boldsymbol{\theta})$ are more regular than the function $\alpha(\mathbf{x})$ by half of a derivative. This is consistent with the Radon transform in the limit as $k \to \infty$. To be more consistent with the Radon transform, notice in the limit $k \to \infty$ that $k \cos\phi \sim \sigma$ so that $k \sin\phi d\phi \sim d\sigma$ as the half circle converges to the real line. Since $\sin\phi \sim 1$ for most of the wavenumbers σ as $k \to \infty$, this implies that $k d\phi \sim d\sigma$. Therefore a total error in the angular measurements in diffraction tomography consistent with the measurement errors for the Radon transform is given by

$$\int_{S^1} \int_0^{\pi/2} |d(\phi, \boldsymbol{\theta})|^2 k d\phi d\boldsymbol{\theta} \geq \|\alpha\|^2_{H^{-1/2}(\mathbb{R}^2)}.$$

We recover in this limit that the measurements in diffraction tomography smooth the function α by half of a derivative.

We see here again that the ill-posedness of a problem very much depends on the norm in which the error on the data is measured.

Chapter 6

Cauchy Problem

We now come to a very classical example of severely ill-posed problem: the Cauchy problem for elliptic equations. We consider an application in medical imaging: the monitoring of the electrical activity of the heart. The electric problem is modeled by a Laplace equation. We first describe in more detail the medical application and next analyze the Cauchy problem for the Laplace equation.

6.1 Electrocardiac potential

Let us consider the application of imaging the electrical activity of the heart. The problem consists of measuring the electric potential on the endocardial surface (the inside of the cardial wall). A possible method consists of sending a probe inside the heart and to measure the potential on the probe. The inverse problem consists then of reconstructing the potential on the endocardial surface from the measurements.

The problem is modeled as follows. Let Γ_0 be a closed smooth surface in $\mathbb{R}^3$ representing the endocardial surface and let Γ_1 be the closed smooth surface inside the volume enclosed by Γ_0 where the measurements are performed. We denote by Ω the domain with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$; see Fig.6.1. The electric potential solves the

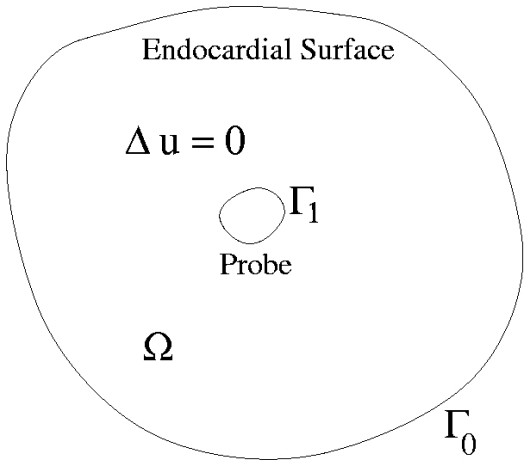

Figure 6.1: Geometry of endocardial measurements

following Laplace equation:

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ u &= u_1 && \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 && \text{on } \Gamma_1. \end{aligned} \tag{6.1}$$

The function u_1 models the measurements at the surface of the probe, which is assumed to be insulated so that $\mathbf{n} \cdot \nabla u = 0$ on Γ_1. The objective is then to find $u = u_0$ on Γ_0. As we will see, this is a severely ill-posed problem.

6.2 Half Space Problem

In this section we first consider the case of data given on the real line $y = 0$. In section 6.2.1 we aim at reconstructing the potential on a parallel line $y = \rho$. We show in section 6.2.2 that the same method can be used to analytically continue an analytic function described on the real line.

6.2.1 Electrocardiac application

Let us consider the following two dimensional problem in the upper half plane:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, && x \in \mathbb{R},\ y > 0 \\ u(x,0) &= u_0(x), && x \in \mathbb{R} \\ \frac{\partial u}{\partial y}(x,0) &= g_0(x), && x \in \mathbb{R}. \end{aligned} \tag{6.2}$$

Let us denote by

$$\hat{u}(k_x, y) = (\mathcal{F}_{x \to k_x} u)(k_x, y). \tag{6.3}$$

Upon Fourier transforming (6.2) in the x variable, we obtain

$$\begin{aligned} -k_x^2 \hat{u} + \frac{\partial^2 \hat{u}}{\partial y^2} &= 0, && k_x \in \mathbb{R},\ y > 0 \\ \hat{u}(k_x, 0) &= \hat{u}_0(k_x), && k_x \in \mathbb{R}, \\ \frac{\partial \hat{u}}{\partial y}(k_x, 0) &= \hat{g}_0(k_x), && k_x \in \mathbb{R}. \end{aligned} \tag{6.4}$$

The solution of the above ODE is given by

$$\hat{u}(k_x, y) = \hat{u}_0(k_x) \cosh(|k_x| y) + \frac{\hat{g}_0(k_x)}{|k_x|} \sinh(|k_x| y). \tag{6.5}$$

Let us now assume that

$$|k_x| \hat{u}_0(k_x) + \hat{g}_0(k_x) = 0, \tag{6.6}$$

so that

$$\hat{u}(k_x, y) = \hat{u}_0(k_x) e^{-|k_x| y}. \tag{6.7}$$

Upon inverting the Fourier transform, we obtain that the solution $u(x,y)$ is given by

$$u(x,y) = (u_0 * \frac{1}{\pi}\frac{y}{x^2+y^2})(x) = \frac{1}{\pi}\int_{\mathbb{R}} u_0(x-z)\frac{y}{z^2+y^2}dz. \tag{6.8}$$

Exercise 6.2.1 (i) Prove (6.8). Hint: Use (1.27) and show that

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-y|k_x|}e^{ixk_x}dk_x = \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

(ii) Show that $\frac{1}{\pi}\frac{y}{x^2+y^2}$ is the fundamental solution of (6.2) with $u_0(x) = \delta(x)$. Calculate the corresponding value of $g_0(x)$.

Notice that the compatibility condition (6.6) can be recast in the physical domain as

$$g_0(x) = -Hu_0'(x), \tag{6.9}$$

where H is the Hilbert transform defined in (2.11). Provided the above compatibility condition is met, the problem (6.2) admits a unique solution and is well-posed, for instance in the sense that

$$\int_{\mathbb{R}} u^2(x,y)dx \leq \int_{\mathbb{R}} u_0^2(x)dx, \qquad \text{for all } y > 0.$$

This is an immediate consequence of (6.7) and the Parseval relation.

However in general, (6.9) is not satisfied. For instance in the electrocardial potential application, we have $g_0 = 0$. In the Fourier domain the solution of the Laplace equation (6.9) with $g_0 = 0$ is thus given by

$$\hat{u}(k_x,y) = \hat{u}_0(k_x)\cosh(|k_x|y). \tag{6.10}$$

This implies that high frequencies are exponentially amplified when $y > 0$. Let us assume that we are interested in the potential at $y = y_0$. The forward problem, in which $u(x,y_0)$ is known and $u(x,0) = A[u(x,y_0)]$ is measured, is a well-posed problem in $L^2(\mathbb{R})$. Indeed we easily verify from (6.10) that

$$\|Av\|_{L^2(\mathbb{R})} \leq \|v\|_{L^2(\mathbb{R})}.$$

The inverse problem, which to $u_0(x)$ maps $u(x,y_0)$ is however severely ill-posed. Indeed since $\cosh(|k_x|y_0)$ cannot be bounded by $C(1+|k_x|^2)^{\alpha/2}$ for any value of α, we see that (1.37) cannot hold for any α. The reason is that the forward operator A is more smoothing than an arbitrary number of anti-differentiations.

The inverse operator A^{-1} can be defined only for sufficiently smooth functions. Indeed let us consider the space of functions

$$X_y(\mathbb{R}) = \{u \in L^2(\mathbb{R}); \cosh(|k_x|y)\hat{u}(k_x) \in L^2(\mathbb{R})\}. \tag{6.11}$$

We verify that A is continuous in $\mathcal{L}(L^2(\mathbb{R}), X_{y_0}(\mathbb{R}))$ and that its inverse A^{-1} is in $\mathcal{L}(X_{y_0}(\mathbb{R}), L^2(\mathbb{R}))$ and is given by

$$A^{-1}u = \mathcal{F}^{-1}_{k_x\to x}\cosh(|k_x|y)\mathcal{F}_{x\to k_x}u. \tag{6.12}$$

Unless noise in the data belongs to X_{y_0}, the noise will be amplified by (6.12) in the reconstruction.

6.2.2 Analytic continuation

Let us now apply the same type of technique to the analytic continuation of an analytic function given on the real line.

Let $f(z) = g(z) + ih(z)$ be an analytic function with $g(z)$ and $h(z)$ real valued-functions. Let us assume that $g(z)$ and $h(z)$ are known on the real line $\Im(z) = 0$. The objective is to find them for arbitrary values of z. We identify $z = x + iy$ and assume that $g(x, 0)$ and $h(x, 0)$ are distributions in $\mathcal{S}'(\mathbb{R})$ so that their Fourier transform is defined. Since $f(z)$ is analytic, we have

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

or equivalently that

$$\frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} = 0, \qquad \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x} = 0. \tag{6.13}$$

These Cauchy-Riemann relations imply that g and h are harmonic, i.e., $\Delta g = \Delta h = 0$. They thus solve the following problems

$$\begin{array}{ll} \Delta g = 0, \quad y > 0, & \Delta h = 0, \quad y > 0, \\ \dfrac{\partial g}{\partial y}(x, 0) = -\dfrac{\partial h}{\partial x}(x, 0), & \dfrac{\partial h}{\partial y}(x, 0) = \dfrac{\partial g}{\partial x}(x, 0), \\ g(x, 0) \text{ known}, & h(x, 0) \text{ known}. \end{array} \tag{6.14}$$

Both problems are of the form (6.2). The solutions in the Fourier domain are given by

$$\begin{aligned} \hat{g}(k_x, y) &= \hat{g}(k_x, 0)\cosh(|k_x|y) - i\text{sign}(k_x)\hat{h}(k_x, 0)\sinh(|k_x|y) \\ \hat{h}(k_x, y) &= \hat{h}(k_x, 0)\cosh(|k_x|y) + i\text{sign}(k_x)\hat{g}(k_x, 0)\sinh(|k_x|y) \end{aligned} \tag{6.15}$$

We verify that the problem is well-posed provided that (6.9) is verified, which in this context means

$$\frac{\partial h}{\partial x} = H\frac{\partial g}{\partial x}, \qquad \frac{\partial g}{\partial x} = -H\frac{\partial h}{\partial x}. \tag{6.16}$$

Notice that $H^2 = -I$ so that both equalities above are equivalent. When the above conditions are met, then the analytic continuation is a stable process. When they are not met, we have seen that high frequencies exponentially increase as y increases, which renders the analytic continuation process a severely ill-posed problem.

6.3 General two dimensional case

We now consider the arbitrary two-dimensional geometries described in section 6.1. We use the Riemann mapping theorem to map such geometries conformally to an annulus (the region lying between two concentric circles) in the plane. We then solve the problem on the annulus. The Riemann mapping gives us a stable way to transform the original domain to an annulus and back. We will see that solving the problem on the annulus is severely ill-posed similarly to what we saw in the preceding section.

6.3.1 Laplace equation on an annulus

Let us begin with a presentation of the problem on the annulus. We assume that the inner circle has radius 1 and the outer circle radius $\rho > 0$. By simple dilation, this is equivalent to the more general case of two circles of arbitrary radius a and b. In polar coordinates the Laplacian takes the form

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{6.17}$$

The general solution to the above equation periodic in θ is decomposed in Fourier modes as

$$u(r,\theta) = a_0 + b_0 \ln r + \sum_{n\in\mathbb{N}^*} \Big(\frac{a_n}{2} r^n + \frac{b_n}{2} r^{-n}\Big) e^{in\theta}. \tag{6.18}$$

Since $\mathbf{n}\cdot\nabla u = 0$ on at $r = 1$, we deduce that b_0 and $b_n - a_n$ vanish. We then find that the solution to (6.1) on the annulus is given by

$$u(r,\theta) = \sum_{n\in\mathbb{N}} \Big(\frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi} u_1(\phi) d\phi\Big) \frac{r^n + r^{-n}}{2} e^{in\theta}. \tag{6.19}$$

The above solution holds for all $r > 1$.

Exercise 6.3.1 Find the relation that $u_1(\theta) = u(1,\theta)$ and $g_1(\theta) = \mathbf{n}\cdot\nabla u(1,\theta)$ must satisfy so that the problem

$$\begin{aligned} &\Delta u = 0, \qquad |r| > 1 \\ &u(1,\theta) = u_1(\theta), \quad \mathbf{n}\cdot\nabla u(1,\theta) = g_1(\theta), \qquad 0 \le \theta < 2\pi, \end{aligned}$$

is well-posed (in the sense that the energy of $\theta \to u(\rho,\theta)$ is bounded by the energy of $u_1(\theta)$). Compare to (6.6) and (6.9).

We verify that an error of size δ in the measurement of the coefficient a_n is amplified into an error of order

$$A_n(\rho) = \frac{\delta}{2} e^{n\ln\rho} \qquad \text{at} \quad r = \rho.$$

We verify that A_n cannot be bounded by any Cn^α for all $\alpha > 0$, which would correspond to differentiating the noise level α times. This implies that the reconstruction of $u(\rho,\theta)$ from $u_1(\theta)$ using (6.19) is a severely ill-posed problem.

6.3.2 Riemann mapping theorem

Let Ω be an open smooth two dimensional domain with smooth boundary having two smooth connected components. We denote by Γ_0 the outer component of $\partial\Omega$ and Γ_1 the inner component; see Fig. 6.1.

For $z \in \Omega \subset \mathbb{C}$ we construct a holomorphic function $\Psi(z)$ (i.e., a function such that $\frac{\partial\Psi}{\partial z} = 0$) mapping Ω to an annulus of the form $1 < r < \rho$. The function is constructed as follows. Let first v be the unique solution to the following Dirichlet problem

$$\Delta v = 0 \ \text{ on } \Omega, \quad v_{|\Gamma_0} = 1, \qquad v_{|\Gamma_1} = 0. \tag{6.20}$$

For some c to be fixed later, let $G = cv$. We verify that

$$I = \int_{\Gamma_1} -\frac{\partial G}{\partial y}dx + \frac{\partial G}{\partial x}dy = -c\int_{\Gamma_1} \frac{\partial v}{\partial \boldsymbol{\nu}}ds > 0$$

by the maximum principle (v is positive inside Ω and its normal derivative must be negative on Γ_1). We fix c such that $I = 2\pi$. In that case we can define a function $H(z)$, a conjugate harmonic of $G(z)$ on Ω, by

$$H(z) = \int_p^z -\frac{\partial G}{\partial y}dx + \frac{\partial G}{\partial x}dy, \tag{6.21}$$

where p is an arbitrary point in Ω. Since G is harmonic, we verify that the definition of H is independent of the path chosen between p and z. Moreover we verify that

$$\frac{\partial H}{\partial x} = -\frac{\partial G}{\partial y}, \qquad \frac{\partial H}{\partial y} = \frac{\partial G}{\partial x},$$

so that $G + iH$ is a holomorphic function on Ω. Then so is

$$\Psi(z) = e^{G(z)+iH(z)}. \tag{6.22}$$

We verify that $\Psi(z)$ maps Γ_0 to the circle $|z| = e^c$ and Γ_1 to the circle $|z| = 1$. Moreover Ψ is a diffeomorphism between Ω and the annulus $U_c = \{z \in \mathbb{C},\, 1 \leq |z| \leq e^c\}$. Finally we verify that $\Delta\Psi(z) = 0$ on Ω since Ψ is holomorphic. Therefore we have replaced the problem (6.1) on Ω by solving the Laplace equation on U_c with boundary conditions $u_1(\Psi(z))$ on the circle $r = 1$ and vanishing Neumann boundary conditions (we verify that Neumann boundary conditions are preserved by the map Ψ).

Chapter 7

Atmospheric gas concentration reconstructions

In this chapter we consider the reconstruction of gas concentration profiles in the atmosphere from radiation measurements carried out by satellites above the atmosphere. This is a severely ill-posed problem based on inverting the *Laplace transform*. We present the problem in section 7.1, solve the inverse problem in section 7.2, and concentrate on the inverse Laplace transform in section 7.3.

7.1 Radiation modeling

We consider a simplified model for the radiation of gases in the atmosphere. Let us denote by $\mathcal{Z} = (0, Z)$ the atmosphere, where Z is the "top" of the atmosphere. We consider one gas of interest, such as ozone, and assume the existence of a range of wavenumbers that only the gas of interest can absorb. This assumption is relatively accurate in some cases. We note $\mathcal{N} = [\nu_{\min}, \nu_{\max}]$ the range of wavenumbers measured by the Infrared spectrometer mounted on the satellite.

The density of infrared radiation is modeled by $L(z, \nu)$, where z is altitude and ν is wavenumber. The source term of radiation is given by $a(z, \nu)B(z, \nu)$, where $a(z, \nu)$ is the absorption profile of the gas and $B(z, \nu)$ is the Planck function of Black body radiation. In the regime of interest, we can assume that the Planck function has the following form

$$B(z, \nu) = \frac{2k\nu^2}{c^2} T(z), \tag{7.1}$$

where k is the Boltzmann constant, c the light-speed, and $T(z)$ is the temperature profile that we assume here is *known*. The radiation density satisfies the following transport equation

$$\begin{aligned} &\frac{\partial L(z, \nu)}{\partial z} + a(z, \nu)L(z, \nu) = a(z, \nu)B(z, \nu), && (z, \nu) \in \mathcal{Z} \times \mathcal{N}, \\ &L(0, \nu) = \varepsilon B(0, \nu), && \nu \in \mathcal{N}, \end{aligned} \tag{7.2}$$

where ε is a constant modeling the radiation at the Earth surface. We assume that the absorption profile of the gas takes the form

$$\alpha(z, \nu) = c(z)g(z)\mu(\nu), \tag{7.3}$$

where $c(z)$ is the concentration profile we want to reconstruct (this is the only unknown quantity in this chapter), and $g(z)$ and $\mu(\nu)$ are known functions. The range of μ is denoted by $\mathcal{M} = \mu(\mathcal{N})$. We assume that $\mathcal{M}$ is an interval in $\mathbb{R}^+$. The assumption on $g(z)$ is that it is a positive and continuous known function.

Based on these assumptions we can simplify the forward model as follows. It is first convenient to consider the following quantity

$$H(z,\nu) = \frac{c^2}{2k\nu^2}\big(L(z,\nu) - B(z,\nu)\big), \tag{7.4}$$

modeling the scaled departure from the black-body radiation equilibrium. It satisfies the equation

$$\begin{aligned} &\frac{\partial H(z,\nu)}{\partial z} + c(z)g(z)\mu(\nu)H(z,\nu) = -\frac{\partial T(z)}{\partial z}, && (z,\nu) \in \mathcal{Z}\times\mathcal{N}, \\ &H(0,\nu) = \gamma T(0), && \nu \in \mathcal{N}, \end{aligned} \tag{7.5}$$

with $\gamma = \varepsilon - 1$. We easily verify that $H(z,\nu_1) = H(z,\nu_2)$ when $\mu(\nu_1) = \mu(\nu_2)$. We thus uniquely define $D(z,\mu)$ such that $D(z,\mu(\nu)) = H(\nu)$ and verify that it solves the equation

$$\begin{aligned} &\frac{\partial D(z,\mu)}{\partial z} + c(z)g(z)\mu D(z,\mu) = -\frac{\partial T(z)}{\partial z} \equiv S(z), && (z,\mu) \in \mathcal{Z}\times\mathcal{M}, \\ &D(0,\mu) = \gamma T(0), && \mu \in \mathcal{M}. \end{aligned} \tag{7.6}$$

An important remark concerning the above equation is that the absorption coefficient separates in the z and μ variables. This allows us to use a change of variables so that the absorption coefficient becomes linear in μ.

Let us define the optical length

$$\alpha(z) = \int_z^Z c(\zeta)g(\zeta)d\zeta. \tag{7.7}$$

We can then invert the first-order ordinary differential equation (7.6) as

$$D(Z,\mu) = D(0)e^{-\mu\alpha(0)} + \int_0^Z S(z)e^{-\mu\alpha(z)}dz, \qquad \mu \in \mathcal{M}. \tag{7.8}$$

Here $D(Z,\mu)$ represents the measurements for $\mu \in \mathcal{M}$. We recall that the positive function $g(z)$ and the temperature profile $T(z)$ (hence $S(z)$) are *known* a priori here.

7.2 Reconstruction of gas concentrations

The inverse problem can thus be stated as follows:

(IP) Determine the positive function $c(z) \in C^0(\mathcal{Z})$ from the measurements $D_m(\mu) = D(Z,\mu)$ for $\mu \in \mathcal{M}$.

The main result we want to present is the following

Theorem 7.2.1 *Let us assume that $S(z)$ is a Lipschitz function on $\mathcal{Z}$, which vanishes at a finite (possibly zero) number of points in $\mathcal{Z}$. Then there is a unique strictly positive function $c(z) \in C^0(\mathcal{Z})$ solving* (IP).

Proof. The proof of the theorem is based on recasting the inverse problem as an inverse Laplace transform. Since $c(z)$ and $g(z)$ are both positive functions on $\mathcal{Z} = (0, Z)$, we deduce that $\alpha'(z) = -c(z)g(z) < 0$ on $\mathcal{Z}$. We can then change variables $z \mapsto \alpha(z)$ and define the continuously differentiable inverse map $\alpha \mapsto z(\alpha)$. Verifying that $\alpha(Z) = 0$, we then recast (7.8) as

$$D(Z, \mu) \;=\; D(0)e^{-\mu\alpha(0)} + \int_0^{\alpha(0)} S(z(\alpha))\Big|\frac{dz}{d\alpha}\Big|e^{-\mu\alpha}d\alpha, \quad \mu \in \mathcal{M}. \tag{7.9}$$

The data $D(Z, \mu)$ is thus the *Laplace transform* of the distribution

$$h(\alpha) \equiv S(z(\alpha))\Big|\frac{dz}{d\alpha}\Big| + D(0)\delta(\alpha - \alpha(0)) \tag{7.10}$$

The above distribution has compact support in $[0, \alpha(0)]$ so its Fourier transform $\hat{h}(\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\zeta\alpha}h(\alpha)d\alpha$ is analytic in ζ. It is therefore sufficient to know $D(Z, \mu)$ on a set with at least one accumulation point to uniquely define $\hat{h}(\zeta)$ for all $\zeta \in \mathbb{C}$ by analytic continuation.

Thus $h(\alpha)$ is uniquely defined. Since $\mathcal{M}$ is an interval in our model, we can thus reconstruct $\alpha(0)$, H_0, and $S(z(\alpha))$ on $(0, \alpha(0))$ from the measurements $D(Z, \mu)$.

It remains to invert the change of variables from z to α. We can recast (7.10) for $z \in \mathcal{Z}$ as

$$\frac{dz(\alpha)}{d\alpha} = -\frac{h(\alpha)}{S(z(\alpha))}, \tag{7.11}$$

which holds all but possibly a finite number of points in $\mathcal{Z}$ by assumption on $S(z)$ and can be extended by continuity to the whole interval since $z(\alpha)$ is a C^1 diffeomorphism. We thus uniquely recover the diffeomorphism $z(\alpha)$ from (7.11) since $h(\alpha)$ is continuous on $(0, \alpha(0))$ as can be seen in (7.10) and $S(z)$ is a Lipschitz function. This also uniquely defines its inverse $\alpha(z)$. Once α is reconstructed we uniquely reconstruct $c(z)$ by differentiating formula (7.7). This concludes our reconstruction and proves Theorem 7.2.1. □

The reconstruction is based on changes of variables, integrations and differentiations, which are at worst mildly ill-posed problems. It is also based on analytic continuation and the inversion of the Laplace transform. Notice that analytic continuation is not necessary if we assume that $\mathcal{M}$ is the whole half line $\mathbb{R}^+$. However in all cases, the reconstruction is based on inverting a Laplace transform. As we will see in the next section, this is a severely ill-posed problem. This allows us to conclude that one should not expect too much accuracy on the concentration profile reconstructions from satellite radiation measurements.

7.3 Inverse Laplace transform

The Laplace transform of a sufficiently smooth function $f(t)$ is given by

$$F(s) = \mathcal{L}_{t\to s}f(s) = \int_0^\infty f(t)e^{-st}dt, \qquad \Re(s) > 0. \tag{7.12}$$

For $\sigma > 0$ we verify that

$$F(\sigma + i\xi) = \int_0^\infty f(t)e^{-s\sigma}e^{-i\xi t} = e^{-s\sigma}\hat{f}(\xi), \qquad \text{for all} \quad \xi \in \mathbb{R},$$

where $\hat{f}(\xi)$ is the Fourier transform of $\chi_{\mathbb{R}^+}(t)f(t)$. We thus obtain the reconstruction formula for all $\sigma > 0$:

$$f(t) = \frac{e^{\sigma t}}{2\pi}\int_{\mathbb{R}} F(\sigma + i\xi)e^{i\xi t}dt = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(p)e^{pt}dp. \tag{7.13}$$

Notice that on the line in the complex plane $\{\sigma + it; t \in \mathbb{R}\}$, we have $dp = i|dp|$.

The problem with the above inversion is that the Laplace transform is required on a line of the form $\{\sigma + it; t \in \mathbb{R}\}$ whereas is it usually known for $F(p)$ with $p > 0$. It thus requires to first analytically continue the function $F(p)$ on the line and then apply an inverse Fourier transform. We have seen that analytic continuation is a severely ill-posed problem. It turns out that the inversion of the Laplace transform is also a severely ill-posed problem no matter how we perform the inversion. To convince ourselves of this, let us consider a non-trivial compactly supported smooth function $f(t)$ that vanishes on $(0, \gamma)$ for some $\gamma > 0$ and that we extend by 0 on $\mathbb{R}^-$. We then verify that

$$|\mathcal{L}_{t\to s}f(s)| \leq e^{-\gamma s}\|f\|_{L^1(\mathbb{R})}$$

so that $\mathcal{L}_{t\to s}f(s) \in L^2(\mathbb{R}^+)$. This implies the existence of a constant $C_{f,\alpha}$ for all $\alpha \geq 0$ such that

$$\|\mathcal{L}_{t\to s}f(s)\|_{L^2(\mathbb{R}^+)} = C_{f,\alpha}\|f\|_{H^\alpha(\mathbb{R})}.$$

We deduce from the definition of the Laplace transform that

$$\mathcal{L}_{t\to s}(f(t-\beta))(s) = e^{-\beta s}\mathcal{L}_{t\to s}f(s). \tag{7.14}$$

As a consequence we deduce that

$$\lim_{\tau\to+\infty}\|\mathcal{L}_{t\to s}(f(t-\tau))(s)\|_{L^2(\mathbb{R}^+)} = \lim_{\tau\to+\infty}\|e^{-s\tau}\mathcal{L}_{t\to s}f(s)\|_{L^2(\mathbb{R}^+)} = 0,$$

by the Lebesgue dominated convergence theorem. However we obviously have that

$$\|f(t)\|_{H^\alpha(\mathbb{R})} = \|f(t-\tau)\|_{H^\alpha(\mathbb{R})}, \qquad \text{for all} \quad \tau \geq 0.$$

We deduce that there *cannot* be a constant C_α independent of the function $f \in H^\alpha(\mathbb{R})$ such that

$$\|\mathcal{L}f\|_{L^2(\mathbb{R}^+)} \geq C_\alpha\|f\|_{H^\alpha(\mathbb{R})}.$$

Thus from our definition (1.37), we deduce that the inversion of the Laplace transform is a severely ill-posed problem.

The above reconstruction formula (7.13) requires analytic continuation. The analytic continuation we have to perform is however not totally arbitrary: we want to extend an analytic function defined on the positive real axis to its value on a line parallel to the imaginary axis. There are explicit transforms to do this (though as we have seen they necessarily involve severely ill-posed steps).

Let us define the Mellin transform as

$$\tilde{F}(r) = \mathcal{M}_{s\to r}F(s) = \int_0^\infty F(s)s^{r-1}ds. \tag{7.15}$$

Using (7.12) we can recast this as

$$\tilde{F}(r) = \Gamma(r)\int_0^\infty f(t)t^{-r}dt = \Gamma(r)\tilde{f}(1-r), \qquad \Gamma(r) = \int_0^\infty e^{-u}u^{r-1}du. \tag{7.16}$$

Here we recognize the Gamma function $\Gamma(r)$. This gives us a formula for the Mellin transform of f

$$\tilde{f}(r) = \frac{\tilde{F}(1-r)}{\Gamma(1-r)}. \tag{7.17}$$

Assuming now that $\Re(r) = \sigma \in (0,1)$, the above formula holds (the integrals are well-defined) and we have the following inversion formula for the Mellin transform

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\tilde{F}(1-r)}{\Gamma(1-r)}t^{-r}dr. \tag{7.18}$$

Exercise 7.3.1 Derive the above formula for the inverse Mellin transform.

This gives us explicit integrals to estimate in order to perform the inverse Laplace transform. Notice that the analytic continuation of an arbitrary smooth analytic function $F(s)$ on the domain $\Re(s) > 0$ from the knowledge for $s > 0$ can be performed similarly: we first construct the inverse Laplace transform $f(t)$ for $t \geq 0$ and then apply the Laplace transform (7.12) to calculate $F(s)$ for $\Re(s) > 0$. See [3] for some numerical simulations.

Chapter 8

Regularization of ill-posed problems

We have seen that many of the inverse problems we have considered so far are ill-posed, either mildly ill-posed or severely ill-posed. We present here some techniques to regularize them.

We will consider two relatively simple but practically quite general settings: that of compact operators defined on Hilbert spaces and of smoothing operator in the Hilbert scale $H^s(\mathbb{R}^n)$. We refer the reader to [10, 12, 14, 15, 22] for different and more general frameworks.

8.1 Ill-posed problems and compact operators

Let A be an injective and compact operator defined on an infinite dimensional Hilbert space H with range Range(A) in H:

$$A : H \rightarrow \text{Range}(A) \subset H. \tag{8.1}$$

We recall that compact operators map the unit ball in H to a subset of H whose closure (with respect to the usual norm in H) is compact, i.e., verifies that every bounded (with respect to the usual norm in H) family of points admits a converging (with respect to the usual norm in H) subsequence in the compact set.

Since A is injective (i.e., $Ax = 0$ implies $x = 0$), we can define the inverse operator A^{-1} with domain of definition Range(A) and Range H:

$$A^{-1} : D(A^{-1}) = \text{Range}(A) \rightarrow H. \tag{8.2}$$

The problem is that A^{-1} is *never* a continuous operator from Range(A) to H when both spaces are equipped with the usual norm in H:

Lemma 8.1.1 *For A as above, there exists a sequence x_n such that*

$$\|x_n\|_H = 1, \qquad \|Ax_n\|_H \rightarrow 0. \tag{8.3}$$

The same holds true with $\|x_n\|_H \rightarrow \infty$.

Proof. The proof holds in more complicated settings than Hilbert spaces. The Hilbert structure gives us a very simple proof and is based on the existence of an orthonormal

basis in H, i.e., vectors x_n such that $\|x_n\|_H = 1$ and $(x_n, x_m)_H = 0$ for $n \neq m$. Since these vectors belong to the unit ball, we deduce that $y_n = Ax_n$ is a converging sequence (up to taking subsequences), say to $y \in H$. Take now $\tilde{x}_n = 2^{-1/2}(x_n - x_{n+1})$. We verify that $\tilde{x}_n$ satisfies (8.3). Now define $z_n = \tilde{x}_n / \|A\tilde{x}_n\|_H^{1/2}$ when the latter denominator does not vanish and $z_n = n\tilde{x}_n$ otherwise. Then Az_n still converges to 0 while $\|z_n\|_H$ converges to ∞. □

This simple lemma shows that inverting a compact operator can never be a well-posed problem in the sense that A^{-1} is not continuous from $D(A^{-1})$ to H with the H norm. Indeed take the sequence $y_n = Ax_n/\|Ax_n\|$ in $D(A^{-1})$, where x_n is the sequence in (8.3). Then $\|y_n\|_H = 1$ while $\|A^{-1}y_n\|_H$ tends to ∞.

The implication for our inverse problem is the following. If δy_n is our measurement noise for n large, then $\delta A^{-1}y_n$ will be the error in our reconstruction, which may thus be arbitrarily larger than the norm of the true object we aim to reconstruct. More precisely, if $Ax = b$ is the real problem and $A\tilde{x} = \tilde{b}$ is the exact reconstruction from noisy data, then arbitrarily small errors $\|b - \tilde{b}\|$ in the measurements is still compatible with arbitrary large errors $\|x - \tilde{x}\|$ in the space of objects to reconstruct. This shows that the problem needs to be regularized before any inversion is carried out.

8.2 Regularity assumptions and error bound

You should be convinced by the calculations we have carried out in the preceding section that an ill-posed inverse problem cannot satisfactorily be solved if no other *assumptions* on the problem are added. A very reasonable and practically useful assumption is to impose, before we start the reconstruction process, that the object we want to reconstruct is itself *sufficiently smooth*. Then, we will be able to filter out high frequencies that may appear in the reconstruction because we know they are part of the noise and not of the object we want to reconstruct. We present two types of theories that put this idea into a more mathematical framework.

In the first framework, we first introduce the adjoint operator A^* to A, defined from H to Range(A^*) by the relation

$$(Ax, y)_H = (x, A^*y)_H, \qquad \text{for all } x, y \in H.$$

Since A is compact and injective, then so is A^*. We can also define the inverse operator $A^{-*} = (A^*)^{-1}$ from Range(A^*) to H.

We may now assume that x, the object we aim at reconstructing, is sufficiently smooth that is belongs to the range of A^*, i.e., there exists y such that $x = A^*y$. Since A and A^* are compact operators, hence smoothing operators, the above hypothesis means that we assume *a priori* that x is smoother than being merely an element in H. We then define the *stronger* norm

$$\|x\|_1 = \|A^{-*}x\|_H. \tag{8.4}$$

We may also assume that the object x is even smoother than being in the range of A^*. For instance let us assume that x belongs to the range of A^*A), i.e., there exists y such that $x = A^*Ay$. Note that since both A and A^* are smoothing operators (because

they are compact), the assumption on x is stronger than simply being in the range of A^*. We define the even stronger norm

$$\|x\|_2 = \|(A^*A)^{-1}x\|_H. \tag{8.5}$$

Then we have the following result.

Theorem 8.2.1 *Let $x \in H$ such that $\|x\|_1 \leq E$ and $\|Ax\|_H \leq \delta$. Then we have the bound*

$$\|x\|_H \leq \sqrt{E\delta}. \tag{8.6}$$

If we now assume that $\|x\|_2 \leq E$ instead, we obtain the better bound

$$\|x\|_H \leq E^{1/3}\delta^{2/3}. \tag{8.7}$$

Proof. Let $y = A^{-*}x$ so that $\|y\|_H \leq E$. We have then

$$\|x\|_H^2 = (x, A^*y) = (Ax, y) \leq \|Ax\|_H\|y\|_H \leq \delta E.$$

This proves (8.6). For the second bound let $z = (A^*A)^{-1}x$ so that $\|z\|_H \leq E$ and compute:

$$\|x\|_H^2 = (x, A^*Az) = (Ax, Az) \leq \delta\|Az\| = \delta(Az, Az)^{\frac{1}{2}} \leq \delta(z, x)^{\frac{1}{2}} \leq \delta E^{\frac{1}{2}}\|x\|_H^{\frac{1}{2}}.$$

This proves the second bound (8.7). □

The theorem should be interpreted as follows. Consider that Ax is the noise level in the measured data and that $\|x\|_1 < E$ or $\|x\|_2 < E$ is *a priori* smoothness information we have on the object we want to reconstruct. Then the worst error we can make on the reconstruction (provided we find an appropriate inversion method; see below) is given by the bounds (8.6) and (8.7). Notice that the latter bound is better (since $\delta^{2/3} \ll \delta^{1/2}$). This results from a more stringent assumption on the image x.

Let us now consider smoothing operators in the framework of the Hilbert scale $H^s(\mathbb{R})$ we have introduced in Chapter 1. Then we have the following result

Theorem 8.2.2 *Let us assume that the operator A is mildly ill-posed of order $\alpha > 0$ so that*

$$\|Af\|_{L^2(\mathbb{R})} \geq m\|f\|_{H^{-\alpha}(\mathbb{R})}. \tag{8.8}$$

Suppose now that the measurement error is small and that the function we want to reconstruct is regular in the sense that

$$\|Af\|_{L^2(\mathbb{R})} \leq \delta m, \qquad \text{and} \qquad \|f\|_{H^\beta(\mathbb{R})} \leq E, \tag{8.9}$$

for some $\delta > 0$, $\beta > 0$, and $E > 0$. Then we have

$$\|f\|_{L^2(\mathbb{R})} \leq \delta^{\frac{\beta}{\alpha+\beta}} E^{\frac{\alpha}{\alpha+\beta}}. \tag{8.10}$$

Proof. The proof is an interesting exercise in interpolation theory. Notice that the hypotheses are

$$\|f\|_{H^\beta(\mathbb{R})} \leq E, \qquad \text{and} \qquad \|f\|_{H^{-\alpha}(\mathbb{R})} \leq \delta,$$

and our objective is to find a bound for $\|f\|_{L^2(\mathbb{R})}$. Let us denote $\langle \boldsymbol{\xi} \rangle = (1+|\boldsymbol{\xi}|^2)^{1/2}$. We have

$$\begin{aligned}(2\pi)^n \|f\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^{2\theta} \langle \boldsymbol{\xi} \rangle^{2\gamma} |\hat{f}(\boldsymbol{\xi})|^{2(1-\theta)} \langle \boldsymbol{\xi} \rangle^{-2\gamma} d\boldsymbol{\xi} \\ &\leq \Big(\int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2 \langle \boldsymbol{\xi} \rangle^{2\gamma/\theta} d\boldsymbol{\xi} \Big)^{\theta} \Big(\int_{\mathbb{R}^n} |\hat{f}(\boldsymbol{\xi})|^2 \langle \boldsymbol{\xi} \rangle^{-2\gamma/(1-\theta)} d\boldsymbol{\xi} \Big)^{1-\theta},\end{aligned}$$

thanks to Hölder's inequality

$$\Big| \int_{\mathbb{R}} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} \Big| \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})},$$

which holds for all $p \geq 1$ and $q \geq 1$ such that $p^{-1} + q^{-1} = 1$, where we have defined for all $p \geq 1$,

$$\|f\|_{L^p(\mathbb{R})} = \Big(\int_{\mathbb{R}} |f(\mathbf{x})|^p d\mathbf{x} \Big)^{1/p}. \tag{8.11}$$

Choosing $\theta = \frac{\alpha}{\alpha+\beta}$ and $\gamma = \frac{\alpha\beta}{\alpha+\beta}$ gives (8.10).

Let us briefly recall the proof of the Hölder's inequality [19]. We first verify that

$$x^{1/p} \leq \frac{x}{p} + \frac{1}{q}, \qquad x > 0,$$

for $p^{-1} + q^{-1} = 1$ and $p \geq 1$, since $x^{1/p} - \frac{x}{p}$ attains its maximum at $x = 1$ where it is equal to q^{-1}. For $y > 0$ we use the above inequality for x/y and multiply by y to obtain

$$x^{1/p} y^{1/q} \leq \frac{x}{p} + \frac{y}{q}, \qquad x > 0,\, y > 0. \tag{8.12}$$

Choosing $x = |tf(\mathbf{x})|^p$ and $y = |t^{-1} g(\mathbf{x})|^q$, we deduce that

$$\int_{\mathbb{R}^n} |f(\mathbf{x}) g(\mathbf{x})| d\mathbf{x} \leq \frac{1}{p} \|tf\|^p_{L^p(\mathbb{R})} + \frac{1}{q} \|t^{-1} g\|^q_{L^q(\mathbb{R})} = \frac{t^p}{p} \|f\|^p_{L^p(\mathbb{R})} + \frac{t^q}{q} \|g\|^q_{L^q(\mathbb{R})},$$

for all $t > 0$. Maximizing over t gives the Hölder inequality. □

The last theorem applies to a less general class of operators than compact operators (although it applies to operators that are not necessarily compact) but gives us more accurate results. We should still consider δ as the noise level and E as an a priori bound we have on the object we want to reconstruct. Then depending on the a priori smoothness of the object, we obtain different accuracies. What is important is the relative regularity of the object compared to the smoothing effect of the operator A. When $\beta = \alpha$, this corresponds to assuming the same regularity as $\|x\|_1 \leq E$ in Theorem 8.2.1. We thus obtain an accuracy of order $\delta^{1/2}$ in the reconstruction. When $\beta = 2\alpha$, this corresponds to $\|x\|_2 \leq E$ since $f = (A^*A)^{-1} g$ for some $g \in L^2(\mathbb{R})$ means that f is twice as regular as A is smoothing. We thus recover the accuracy of order $\delta^{2/3}$ as in Theorem 8.2.1. Theorem 8.2.2 allows us to deal with arbitrary values of β. Notice that as $\beta \to \infty$, we recover that the problem is almost well-posed since the error in the reconstruction is asymptotically of the same order δ as the noise level.

8.3 Regularization methods

Now that we know how noise can optimally be controlled in the reconstruction based on the regularity of the object we want to reconstruct, we need to devise algorithms that indeed control noise amplification in the reconstruction.

Since A^{-1} is an unbounded operator with domain of definition Range(A), a proper subset of H, we first need to introduce approximations of the inverse operator. We denote by R_γ defined from H to H for $\gamma > 0$ a sequence of regularizations of A^{-1} such that

$$\lim_{\gamma \to 0} R_\gamma A x = x \qquad \text{for all } x \in H. \tag{8.13}$$

Under the hypotheses of Lemma 8.1.1, we can show that the sequence of operators R_γ is not uniformly bounded.

Exercise 8.3.1 Prove this.

A uniform bound would indeed imply that A^{-1} is bounded. Similarly, $R_\gamma A$ does not converge to identity uniformly. Roughly speaking, convergence is uniform for smooth functions x and becomes less and less uniform as x becomes more singular.

One of the main objectives of the regularization technique is to handle noise in an optimal fashion. Let us denote by y^δ our measurements and assume that $\|y^\delta - Ax\|_H \leq \delta$. We then define

$$x^{\gamma,\delta} = R_\gamma y^\delta. \tag{8.14}$$

We want to find sequences R_γ that deal with noise in an optimal fashion. For instance assuming that $\|x\|_1 \leq E$ and that $\|y^\delta - Ax\|_H \leq \delta$, we want to be able to show that

$$\|x - x^{\gamma,\delta}\| \leq C\sqrt{E\delta},$$

at least for some values of γ. We know from Theorem 8.2.2 that such a bound is optimal. We will consider three regularization techniques: singular value decomposition, Tikhonov regularization, and Landweber iterations.

The choice of a parameter γ is then obviously of crucial importance as the above bound will not hold independently of γ. More precisely, the reconstruction error can be decomposed as

$$\|x^{\gamma,\delta} - x\|_H \leq \delta \|R_\gamma\|_H + \|R_\gamma A x - x\|_H. \tag{8.15}$$

Exercise 8.3.2 Prove this. The operator norm $\|R_\gamma\|_H$ is defined as the supremum of $\|R_\gamma x\|_H$ under the constraint $\|x\|_H \leq 1$.

We thus observe that two competing effects enter (8.15). The first effect is the ill-posedness effect: as $\gamma \to 0$, the norm $\|R_\gamma\|_H$ tends to ∞ so γ should not be chosen too small. The second effect is the regularization effect: as γ increases, $R_\gamma A$ becomes a less accurate approximation of identity so γ should not be chosen too large. Only intermediate values of γ will provide an optimal reconstruction.

8.3.1 Singular Value Decomposition

For a compact and injective operator A defined on an infinite dimensional Hilbert space H, let us assume that we know it singular value decomposition defined as follows. Let A^* be the adjoint operator to A and $\lambda_j > 0$, $j \in \mathbb{N}$ the eigenvalues of the symmetric operator A^*A. Then the sequence $\mu_j = \sqrt{\lambda_j}$ for $j \in \mathbb{N}$ are called the *singular values* of A. Since $\mu_j \leq \|A\|_H$, we order the singular values such that

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots > 0.$$

Multiple eigenvalues are repeated as many times as their multiplicity (which is necessarily finite since the associated eigenspace for A^*A needs to be compact).

There exists then two orthonormal systems $(x_j)_{j\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ in H such that

$$Ax_j = \mu_j y_j \qquad \text{and} \qquad A^*y_j = \mu_j x_j, \qquad \text{for all} \quad j \in J. \tag{8.16}$$

We call (μ_j, x_j, y_j) the singular system for A. Notice that

$$Ax = \sum_{j=1}^{\infty} \mu_j (x, x_j) y_j, \qquad A^*y = \sum_{j=1}^{\infty} \mu_j (y, y_j) x_j.$$

Here (x, x_j) is the inner product in H, $(x, x_j)_H$. We have then the very useful characterization of the Range of the compact and injective operator A:

Lemma 8.3.1 (Picard) *The equation $Ax = y$ is solvable in H if and only if*

$$\sum_{j\in\mathbb{N}} \frac{1}{\mu_j^2} |(y, y_j)|^2 < \infty, \tag{8.17}$$

in which case the solution is given by

$$x = A^{-1}y = \sum_{j\in\mathbb{N}} \frac{1}{\mu_j} (y, y_j) x_j. \tag{8.18}$$

The ill-posedness of the inverse problem appears very clearly in the singular value decomposition. As $j \to \infty$, the singular values μ_j tend to 0. And they do so all the faster that the inverse problem is ill-posed. We can extend the definition of ill-posed problems in the sense that a compact operator generates a mildly ill-posed inverse problem of order $\alpha > 0$ when the singular values decay like $j^{-\alpha}$ and generates a severely ill-posed problem when the singular values decay faster than any j^{-m} for $m \in \mathbb{N}$.

So in order to regularize the problem, all we have to do is to replace too small singular values by larger values. Let us define $q(\gamma, \mu)$ for $\gamma > 0$ and $\mu \in [0, \|A\|]$ such that

$$|q(\gamma, \mu)| < 1, \quad |q(\gamma, \mu)| \leq c(\gamma)\mu, \quad \text{and} \quad q(\gamma, \mu) - 1 \to 0 \quad \text{as} \quad \gamma \to 0, \tag{8.19}$$

(not uniformly in μ obviously). Then we define the regularizing sequence

$$R_\gamma y = \sum_{j\in\mathbb{N}} \frac{q(\gamma, \mu_j)}{\mu_j} (y, y_j) x_j. \tag{8.20}$$

Compare to (8.18). As $\gamma \to 0$, R_γ converges to A^{-1} pointwise. We are interested in estimating (8.15) and showing that the error is optimal based on the assumed regularity of x. The total error is estimated by using

$$\|R_\gamma\|_H \leq c(\gamma), \qquad \|R_\gamma Ax - x\|_H = \sum_{j=1}^{\infty} \big(q(\gamma,\mu_j) - 1\big)^2 |(x, x_j)|^2. \tag{8.21}$$

Exercise 8.3.3 Prove these relations.

We can now prove the following results

Theorem 8.3.2 *(i) Let us assume that $x = A^*z$ with $\|z\|_H \leq E$ and that $\|y^\delta - Ax\| \leq \delta$, where y^δ is the measurements. Choose $q(\gamma,\mu)$ and γ such that*

$$|q(\gamma,\mu) - 1| \leq C_1 \frac{\sqrt{\gamma}}{\mu}, \qquad c(\gamma) \leq \frac{C_2}{\sqrt{\gamma}}, \qquad \gamma = \frac{C_3\delta}{E}. \tag{8.22}$$

Then we have that

$$\|x^{\gamma,\delta} - x\|_H \leq \Big(\frac{C_2}{\sqrt{C_3}} + C_1\sqrt{C_3}\Big)\sqrt{\delta E}. \tag{8.23}$$

(ii) Let us assume that $x = A^*Az$ with $\|z\|_H \leq E$ and that $\|y^\delta - Ax\| \leq \delta$, where y^δ is the measurements. Choose $q(\gamma,\mu)$ and γ such that

$$|q(\gamma,\mu) - 1| \leq C_4 \frac{\gamma}{\mu^2}, \qquad c(\gamma) \leq \frac{C_5}{\sqrt{\gamma}}, \qquad \gamma = C_6\Big(\frac{\delta}{E}\Big)^{2/3}. \tag{8.24}$$

Then we have that

$$\|x^{\gamma,\delta} - x\|_H \leq \Big(\frac{C_5}{\sqrt{C_6}} + C_4C_6\Big)\delta^{2/3}E^{1/3}. \tag{8.25}$$

Proof. Since $x = A^*z$, we verify that $(x, x_j) = \mu_j(y, y_j)$ so that

$$\|R_\gamma Ax - x\|_H^2 = \sum_{j=1}^{\infty} \big(q(\gamma,\mu_j) - 1\big)^2 |(z, y_j)|^2 \leq C_1^2\gamma\|z\|_H^2.$$

This implies that

$$\delta\|R_\gamma\|_H + \|R_\gamma Ax - x\|_H \leq \frac{C_2\delta}{\sqrt{\gamma}} + C_1\sqrt{\gamma}E.$$

Using (8.15) and the expression for γ yields (8.23).

Exercise 8.3.4 Using similar arguments, prove (8.25).

This concludes the proof. □

We have thus defined an optimal regularization scheme for the inversion of $Ax = y$. Indeed from the theory in Theorem 8.2.1 we know that up to some multiplicative constants, the above estimates are optimal.

It remains to find filters $q(\gamma,\mu)$ satisfying the above hypotheses. We propose two:

$$q(\gamma,\mu) = \frac{\mu^2}{\gamma + \mu^2}, \tag{8.26}$$

$$q(\gamma,\mu) = \begin{cases} 1, & \mu^2 \geq \gamma, \\ 0, & \mu^2 < \gamma. \end{cases} \tag{8.27}$$

Exercise 8.3.5 Show that the above choices verify the hypotheses of Theorem 8.3.2.

8.3.2 Tikhonov Regularization

One of the main drawbacks of the theory presented in the preceding section is that in most cases, the singular value decomposition of the operator is not analytically available (although it is for the Radon transform; see [14, 15]), and is quite expensive to compute numerically once the continuous problem has been discretized. It is therefore useful to consider regularization techniques that do not depend on the SVD. One of the most famous regularization techniques is the Tikhonov-Phillips regularization technique.

Solving $Ax = y$ corresponds to minimizing $\|Ax - y\|_H$. Instead one may want to minimize the regularized Tikhonov functional

$$J_\gamma(x) = \|Ax - y\|_H^2 + \gamma\|x\|_H^2, \qquad x \in H. \tag{8.28}$$

For $\alpha > 0$ and A a linear bounded operator on H, we can show that the above functional admits a unique minimizer x^γ solving the following normal equations

$$A^*Ax^\gamma + \gamma x^\gamma = A^*y. \tag{8.29}$$

We can thus define the regularizing sequence

$$R_\gamma = (\gamma + A^*A)^{-1}A^*. \tag{8.30}$$

The operator is bounded in H by $\|R_\gamma\|_H \leq C\gamma^{-1/2}$ for all $\gamma > 0$. Notice that for a compact operator A, we verify that the singular value decomposition of R_γ is

$$R_\gamma y = \sum_{j=1}^{\infty} \frac{\mu_j}{\gamma + \mu_j^2}(y, y_j)x_j. \tag{8.31}$$

This means that the Tikhonov regularization corresponds to the SVD regularization with filter given by (8.26) and implies that the Tikhonov regularization is optimal to inverse problem with a priori regularity $\|x\|_1 \leq E$ or $\|x\|_2 \leq E$. It is interesting to observe that the Tikhonov regularization is no longer optimal when the a priori regularity of x is better than $\|x\|_2 \leq E$ (see [12]).

Let us make this observation more explicit. Let us consider the operator A given in the Fourier domain by multiplication by $\langle\boldsymbol{\xi}\rangle^{-\alpha}$ for some $\alpha > 0$. We verify that $A^* = A$ and that R_γ is given in the Fourier domain by

$$R_\gamma = \mathcal{F}^{-1}_{\boldsymbol{\xi}\to\mathbf{x}} \frac{\langle\boldsymbol{\xi}\rangle^{-\alpha}}{\langle\boldsymbol{\xi}\rangle^{-2\alpha} + \gamma}\mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}}, \qquad \text{so that} \qquad \|R_\gamma\| \leq \frac{1}{2\sqrt{\gamma}}.$$

Indeed, we check that $x/(x^2 + \gamma) \leq 1/(2\sqrt{\gamma})$ and attains its maximum at $x = \sqrt{\gamma}$. We now verify that

$$I - R_\gamma A = \mathcal{F}^{-1}_{\boldsymbol{\xi}\to\mathbf{x}} \frac{\gamma}{\langle\boldsymbol{\xi}\rangle^{-2\alpha} + \gamma}\mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}},$$

so that for a function $f \in H^\beta(\mathbb{R}^n)$, we have

$$\|f - R_\gamma Af\| \leq \sup_{\langle\boldsymbol{\xi}\rangle\geq 1} \frac{\gamma\langle\boldsymbol{\xi}\rangle^{-\beta}}{\langle\boldsymbol{\xi}\rangle^{-2\alpha} + \gamma}\|f\|_{H^\beta(\mathbb{R}^n)}.$$

Moreover the inequality is sharp in the sense that there exists functions f such that the reverse inequality holds (up to a multiplicative constant independent of γ; Check this). For $\beta > 2\alpha$, the best estimate we can have for the above multiplier is that it is of order $O(\gamma)$ (choose for instance $\langle\boldsymbol{\xi}\rangle = 1$).

Exercise 8.3.6 Using (8.12) show that

$$\frac{\gamma\langle\boldsymbol{\xi}\rangle^{-2\alpha\theta}}{\langle\boldsymbol{\xi}\rangle^{-2\alpha}+\gamma} \leq \gamma^{\theta}, \qquad 0 \leq \theta \leq 1.$$

Show that the above inequality is sharp.

Let $Af = g$ be the problem we want to solve and g^{δ} the measurements so that $\|Af - g^{\delta}\|_{L^2(\mathbb{R}^n)} \leq \delta$. Let us assume that f belongs to $H^{\beta}(\mathbb{R}^n)$. We verify using (8.15) that the error of the regularized problem is given by

$$\|f - R_{\gamma} g^{\delta}\| \leq \frac{\delta}{2\sqrt{\gamma}} + \gamma^{\frac{\beta}{2\alpha}\wedge 1}\|f\|_{H^{\beta}(\mathbb{R}^n)}. \tag{8.32}$$

Here, $a \wedge b = \min(a, b)$. This implies that

$$\|f - R_{\gamma} g^{\delta}\| \leq C\delta^{\frac{\beta}{\alpha+\beta}\wedge\frac{2}{3}}\|f\|^{\frac{\alpha}{\alpha+\beta}\wedge\frac{1}{3}}_{H^{\beta}(\mathbb{R}^n)}, \tag{8.33}$$

for a universal constant C. We therefore obtain that the Tikhonov regularization is optimal according to Theorem 8.2.2 when $0 < \beta \leq 2\alpha$. However, for all $\beta > 2\alpha$, the error between the Tikhonov regularization and the exact solution will be of order $\delta^{2/3}$ instead of $\delta^{\frac{\beta}{\beta+\alpha}}$.

Exercise 8.3.7 More generally, consider an operator A with symbol $a(\boldsymbol{\xi})$, i.e.,

$$A = \mathcal{F}^{-1}_{\boldsymbol{\xi}\to\mathbf{x}} a(\boldsymbol{\xi}) \mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}},$$

such that $0 < a(\boldsymbol{\xi}) \in C^{\infty}(\mathbb{R}^n)$ and for some $\alpha > 0$ and $a_{\infty} \neq 0$,

$$\frac{a(\boldsymbol{\xi})}{\langle\boldsymbol{\xi}\rangle^{\alpha}} \to a_{\infty}, \qquad \text{as} \qquad |\boldsymbol{\xi}| \to \infty. \tag{8.34}$$

(i) Show that A^*, the adjoint of A for the $L^2(\mathbb{R}^n)$ inner product, satisfies the same hypothesis (8.34).
(ii) Show that R_{γ} and $S_{\gamma} = R_{\gamma}A - I$ are bounded operator with symbols given by

$$r_{\gamma}(\boldsymbol{\xi}) = (|a(\boldsymbol{\xi})|^2 + \gamma)^{-1}\bar{a}(\boldsymbol{\xi}), \qquad s_{\gamma}(\boldsymbol{\xi}) = \gamma(|a(\boldsymbol{\xi})|^2 + \gamma)^{-1},$$

respectively.
(iii) Assuming that $f \in H^{\beta}(\mathbb{R})$, show that (8.33) holds.

These results show that for the Radon transform, an a priori regularity of the function $f(\mathbf{x})$ in $H^1(\mathbb{R}^2)$ is sufficient to obtain an error of order $\delta^{2/3}$. When the function is smoother, a different technique from Tikhonov regularization is necessary to get a more accurate reconstruction.

8.3.3 Landweber iterations

The drawback of the Tikhonov regularization is that it requires to invert the regularization of the normal operator $\gamma + A^*A$. This inversion may be very costly in practice. The Landweber iteration method is an iterative technique in which no inversion is necessary. It is defined to solve the equation $Ax = y$ as follows

$$x_0 = 0, \qquad x_{n+1} = (I - rA^*A)x_n + rA^*y, \qquad n \geq 0, \tag{8.35}$$

for some $r > 0$. By induction, we verify that $x_n = R_n y$, where

$$R_n = r\sum_{k=0}^{n-1}(I - rA^*A)^k A^*, \qquad n \geq 1. \tag{8.36}$$

Consider a compact operator A with singular system (μ_j, x_j, y_j). We thus verify that

$$R_n y = \sum_{j=1}^{\infty} \frac{1}{\mu_j}\big(1 - (1 - r\mu_j^2)^n\big)(y, y_j)x_j. \tag{8.37}$$

Exercise 8.3.8 Check (8.37).

This implies that R_n is of the form R_γ in (8.20) with $\gamma = n^{-1}$ and

$$q(\gamma, \mu) = 1 - (1 - r\mu^2)^{1/\gamma}.$$

Exercise 8.3.9 *Show that the above filter verifies the hypotheses (8.19) and those of Theorem 8.3.2.*

This implies that the Landweber iteration method is an optimal inversion method by Theorem 8.3.2.

Exercise 8.3.10 Show that the hypotheses of Theorem 8.3.2 are met provided that the number of iterations n is chosen as

$$n = c\frac{E}{\delta}, \qquad n = c\Big(\frac{E}{\delta}\Big)^{2/3},$$

when $\|x\|_1 \leq E$ and $\|x\|_2 \leq E$, respectively.

The above result shows that the number of iterations should be chosen carefully: when n is too small, $R_\gamma A$ is a poor approximation of I, and when n is too large, $\|R_\gamma\|_H$ is too large. Unlike the Tikhonov regularization, we can show that the Landweber iteration method is also optimal for stronger regularity assumptions on x than those given in Theorem 8.3.2 (see [12] for instance).

Let us come back to the operator A with symbol $a(\boldsymbol{\xi}) = \langle\boldsymbol{\xi}\rangle^{-\alpha}$. We verify that R_n and $S_n = R_nA - I$ have respective symbols

$$r_n(\boldsymbol{\xi}) = \frac{1 - (1 - r\langle\boldsymbol{\xi}\rangle^{-2\alpha})^n}{\langle\boldsymbol{\xi}\rangle^{-\alpha}}, \qquad s_n(\boldsymbol{\xi}) = -(1 - r\langle\boldsymbol{\xi}\rangle^{-2\alpha})^n.$$

Exercise 8.3.11 (i) Show that $s_n(\mathbf{x})\langle\boldsymbol{\xi}\rangle^{-\beta}$ is bounded by $Cn^{-\beta/(2\alpha)}$ for $\langle\boldsymbol{\xi}\rangle$ of order $n^{1/(2\alpha)}$. Deduce that for $f \in H^\beta(\mathbb{R}^n)$, we have

$$\|S_n f\| \leq Cn^{-\frac{\beta}{2\alpha}}\|f\|_{H^\beta(\mathbb{R}^n)}.$$

(ii) Show that provided that n is chosen as

$$n = C\delta^{\frac{-2\alpha}{\alpha+\beta}}\|f\|^{\frac{2\alpha}{\alpha+\beta}}_{H^\beta(\mathbb{R}^n)},$$

we have the estimate

$$\delta\|R_n\| + \|S_n f\| \leq C\delta^{\frac{\beta}{\alpha+\beta}}\|f\|^{\frac{\alpha}{\alpha+\beta}}_{H^\beta(\mathbb{R}^n)}. \tag{8.38}$$

(iii) Deduce that the Landweber iteration method is an optimal regularization technique for all $\beta > 0$.
(iv) Generalize the above results for the operators described in Exercise 8.3.7.

We have thus the striking result that unlike the Tikhonov regularization method described in (8.33), the Landweber iteration regularization can be made optimal (by choosing the appropriate number of iterations n) for all choices on the regularity in $H^\beta(\mathbb{R}^n)$ of the object f.

Chapter 9

Transport equations

This chapter concerns the reconstruction of absorption and scattering coefficients from boundary measurements. This is a generalization of the reconstruction of the absorption coefficient treated in chapter 2 by inversion of the Radon transform.

In these notes we remain at a very formal level. The technical (but not difficult) proofs are not included.

9.1 Transport equation

We consider the following transport equation in three space dimensions

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla u(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) u(\mathbf{x}, \boldsymbol{\theta}) = \int_{S^2} k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta}) u(\mathbf{x}, \boldsymbol{\theta}') d\boldsymbol{\theta}', \qquad (\mathbf{x}, \boldsymbol{\theta}) \in \Omega \times S^2 \\ &u(\mathbf{x}, \boldsymbol{\theta}) = g(\mathbf{x}, \boldsymbol{\theta}), \qquad (\mathbf{x}, \boldsymbol{\theta}) \in \Gamma_-(\Omega). \end{aligned} \tag{9.1}$$

The domain $\Omega \subset \mathbb{R}^3$ is supposed convex with smooth boundary $\partial\Omega$. The domains $\Gamma_\pm(\Omega)$ are defined by

$$\Gamma_\pm(\Omega) = \{(\mathbf{x}, \boldsymbol{\theta}) \in \partial\Omega \times S^2, \text{ such that } \pm \boldsymbol{\theta} \cdot \mathbf{n}(\mathbf{x}) > 0\}, \tag{9.2}$$

where $\mathbf{n}(\mathbf{x})$ is the outward normal to Ω at $\mathbf{x} \in \partial\Omega$. The *Albedo* operator maps the incoming conditions to the outgoing radiation:

$$\mathcal{A} : g \mapsto u|_{\Gamma_+(\Omega)}, \tag{9.3}$$

where u is the solution to (9.1.

The absorption coefficient $a(\mathbf{x})$ and the scattering coefficient $k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta})$ satisfy the following hypotheses.

(H1) $0 \leq a(\mathbf{x}) \in L^\infty(\Omega)$.

(H2) $0 \leq k(\mathbf{x}, \boldsymbol{\theta}', \cdot) \in L^1(S^2)$ for almost all (with respect to the Lebesgue measure) $(\mathbf{x}, \boldsymbol{\theta}') \in \Omega \times S^2$ and $\int_{S^2} k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta}) d\boldsymbol{\theta} \in L^\infty(\Omega \times S^2)$.

(H3) $a(\mathbf{x}) - \int_{S^2} k(\mathbf{x}, \boldsymbol{\theta}', \boldsymbol{\theta}) d\boldsymbol{\theta} \geq \alpha > 0$ for almost all $(\mathbf{x}, \boldsymbol{\theta}) \in \Omega \times S^2$.

The latter hypothesis imposes that the problem be subcritical, i.e., that at least as many particles be absorbed as created.

For $\mathbf{x} \in \Omega$ and $\boldsymbol{\theta} \in S^2$, we introduce the travel times

$$\tau_\pm(\mathbf{x}, \boldsymbol{\theta}) = \min\{t \geq 0, \text{ such that } \mathbf{x} \pm t\boldsymbol{\theta} \in \partial\Omega\}. \tag{9.4}$$

We also define two measures on $\Gamma_\pm$: $d\xi = |\mathbf{n}(\mathbf{x})\cdot\boldsymbol{\theta}| d\mu(\mathbf{x}) d\boldsymbol{\theta}$, where $d\mu(\mathbf{x})$ is the Lebesgue measure on $\partial\Omega$, and $d\tilde{\xi} = \tau(\mathbf{x}, \boldsymbol{\theta}) d\xi$. Then we can show that the albedo operator $\mathcal{A}$ is bounded from $L^1(\Gamma_-, d\tilde{\xi})$ to $L^1(\Gamma_+, d\tilde{\xi})$:

$$\|\mathcal{A}\|_{\mathcal{L}(L^1(\Gamma_-, d\tilde{\xi}), L^1(\Gamma_+, d\tilde{\xi}))} \leq C < \infty. \tag{9.5}$$

This shows that the *forward* problem in (9.1) is well-posed [9].

9.2 Decomposition into singular components

We now show that the albedo operator can be split into three terms, each one being more singular than the next. The first term will allow us to reconstruct the absorption parameter $a(\mathbf{x})$ by inversion of a Radon transform. The second term will allow us to reconstruct the scattering term $k(\mathbf{x}, \boldsymbol{\theta}_0, \boldsymbol{\theta})$.

The decomposition of the albedo operator is performed as follows. Let $(\mathbf{x}_0, \boldsymbol{\theta}_0) \in \Gamma_-$. We construct a special solution to (9.1) with the boundary condition

$$g(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0) = \delta_{\mathbf{x}_0}(\mathbf{x}) \delta_{\boldsymbol{\theta}_0}(\boldsymbol{\theta}), \tag{9.6}$$

where the first delta function is in $\partial\Omega$ and the second delta function in S^2 so that for a smooth test function ϕ with compact support on Γ_-, we have

$$\int_{\Gamma_-} \phi(\mathbf{x}, \boldsymbol{\theta}) \delta_{\mathbf{x}_0}(\mathbf{x}) \delta_{\boldsymbol{\theta}_0}(\boldsymbol{\theta}) d\mu(\mathbf{x}) d\boldsymbol{\theta} = \phi(\mathbf{x}_0, \boldsymbol{\theta}_0).$$

Assuming that $k = 0$ in (9.1), we would obtain the solution

$$u_1(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0) = \exp\Big(- \int_0^{\tau_-(\mathbf{x}, \boldsymbol{\theta})} a(\mathbf{x} - s\boldsymbol{\theta}) ds \Big) \delta\big(\mathbf{x} - \tau_-(\mathbf{x}, \boldsymbol{\theta})\boldsymbol{\theta} - \mathbf{x}_0\big) \delta\big(\boldsymbol{\theta} - \boldsymbol{\theta}_0\big). \tag{9.7}$$

It turns out that u_1 is the most singular part of the solution $u(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0)$. Indeed let us define $v(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0) = u(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0) = -u_1(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0)$. Then v solves

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla v(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) v(\mathbf{x}, \boldsymbol{\theta}) = \int_{S^2} k(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\theta}') [v(\mathbf{x}, \boldsymbol{\theta}') + u_1(\mathbf{x}, \boldsymbol{\theta}')] d\boldsymbol{\theta}', \\ &v(\mathbf{x}, \boldsymbol{\theta}) = 0, \qquad (\mathbf{x}, \boldsymbol{\theta}) \in \Gamma_-(\Omega). \end{aligned} \tag{9.8}$$

This is an equation with a source term supported on the line defined by $\mathbf{x} - \tau_-(\mathbf{x}, \boldsymbol{\theta})\boldsymbol{\theta} = \mathbf{x}_0$. We verify that v is thus less singular than u_1. More precisely, we recast v as $v(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0) = u_2(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0) + w(\mathbf{x}, \boldsymbol{\theta}; \mathbf{x}_0, \boldsymbol{\theta}_0)$, where u_2 and w are the solutions to the following equations

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla u_2(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) u_2(\mathbf{x}, \boldsymbol{\theta}) = \int_{S^2} k(\mathbf{x}, \boldsymbol{\theta}_0, \boldsymbol{\theta}) u_1(\mathbf{x}, \boldsymbol{\theta}') d\boldsymbol{\theta}', \\ &\boldsymbol{\theta} \cdot \nabla w(\mathbf{x}, \boldsymbol{\theta}) + a(\mathbf{x}) w(\mathbf{x}, \boldsymbol{\theta}) = \int_{S^2} k(\mathbf{x}, \boldsymbol{\theta}_0, \boldsymbol{\theta}) [w(\mathbf{x}, \boldsymbol{\theta}') + u_2(\mathbf{x}, \boldsymbol{\theta}')] d\boldsymbol{\theta}', \\ &w(\mathbf{x}, \boldsymbol{\theta}) = u_2(\mathbf{x}, \boldsymbol{\theta}) = 0, \qquad (\mathbf{x}, \boldsymbol{\theta}) \in \Gamma_-(\Omega). \end{aligned} \tag{9.9}$$

It is shown in [7] that

$$u_2(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0) = \int_0^{\tau_-(\mathbf{x},\boldsymbol{\theta})} \exp\Big(-\int_0^s a(\mathbf{x}-p\boldsymbol{\theta})dp - \int_0^{\tau_-(\mathbf{x}-s\boldsymbol{\theta},\boldsymbol{\theta}_0)} a(\mathbf{x}-s\boldsymbol{\theta}-p\boldsymbol{\theta}_0)dp\Big) \\ k(\mathbf{x}-s\boldsymbol{\theta},\boldsymbol{\theta}_0,\boldsymbol{\theta})\delta\big(\mathbf{x}-s\boldsymbol{\theta}-\tau_-(\mathbf{x}-s\boldsymbol{\theta},\boldsymbol{\theta}_0)\boldsymbol{\theta}_0-\mathbf{x}_0\big)ds, \tag{9.10}$$

and that

$$\tau(\mathbf{x}_0,\boldsymbol{\theta}_0)^{-1}|\mathbf{n}(\mathbf{x}_0)\cdot\boldsymbol{\theta}_0|^{-1}w(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0) \in L^\infty(\Gamma_-;L^1(\Gamma_+,d\tilde{\xi})). \tag{9.11}$$

We thus obtain that w is a function, whereas u_1 and u_2 are more singular distributions because the space dimension $n \geq 3$ (verify that u_2 is also a function in two space dimensions). Moreover, u_1 is a more singular distribution than u_2 as can easily be verified. By analyzing the singularities of the albedo operator, we can thus have access to u_1 and u_2 from boundary measurements. We recognize in u_1 the exponential of the Radon transform of $a(\mathbf{x})$ along the line parameterized by $\mathbf{x}_0$ and $\boldsymbol{\theta}_0$. We thus recover $a(\mathbf{x})$ by inverse Radon transform. Note that the same reasoning allows us to reconstruct $a(\mathbf{x})$ also in two-space dimensions.

Once a is known, we verify that u_2 is precisely what is required to reconstruct $k(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\theta}')$. Indeed, the delta function in the definition of u_2 implies that

$$\mathbf{x}_0 + t\boldsymbol{\theta}_0 = \mathbf{x} - s\boldsymbol{\theta},$$

where we have introduced $t = \tau_-(\mathbf{x}-s\boldsymbol{\theta},\boldsymbol{\theta}_0)$, so that $\boldsymbol{\theta}$, $\mathbf{x}_0-\mathbf{x}$, and $\boldsymbol{\theta}_0$ must be coplanar, i.e.,

$$\boldsymbol{\theta}\cdot((\mathbf{x}_0-\mathbf{x})\wedge\boldsymbol{\theta}_0)) = 0. \tag{9.12}$$

In three space dimensions, this implies that the support of $u_2(\mathbf{x},\boldsymbol{\theta};\mathbf{x}_0,\boldsymbol{\theta}_0)$ is singular in $\Gamma_-\times\Gamma_+$. Note that such is not the case in two-space dimensions. The reconstruction of the scattering kernel in two space dimensions remains open in full generality.

Let us now assume that $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_0$ are not collinear, i.e., that $|\boldsymbol{\theta}\cdot\boldsymbol{\theta}_0| < 1$. We then verify that provided that the constraint (9.12) holds, we have

$$\begin{aligned} t &= \frac{(\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\theta}_0 - (\boldsymbol{\theta}\cdot\boldsymbol{\theta}_0)(\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\theta}}{1-(\boldsymbol{\theta}\cdot\boldsymbol{\theta}_0)^2} \\ s &= \frac{(\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\theta} - (\boldsymbol{\theta}\cdot\boldsymbol{\theta}_0)(\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\theta}_0}{1-(\boldsymbol{\theta}\cdot\boldsymbol{\theta}_0)^2}. \end{aligned} \tag{9.13}$$

Conversely, for a given point $\mathbf{y} = \mathbf{x} - s\boldsymbol{\theta}$ inside Ω and $\boldsymbol{\theta}_0 \in S^2$, we find a point $\mathbf{x}_0 = \mathbf{x} - s\boldsymbol{\theta} - t\boldsymbol{\theta}_0 \in \partial\Omega$, where t is given by the above formula. This uniquely defines t and we have $(\mathbf{x}_0,\boldsymbol{\theta}_0) \in \Gamma$. For this choice of t and $\mathbf{x}$, we verify that

$$\begin{aligned} &u_2(\mathbf{y}+s\boldsymbol{\theta},\boldsymbol{\theta};\mathbf{y}-t\boldsymbol{\theta}_0,\boldsymbol{\theta}_0) \\ &= \frac{1}{\sqrt{1-(\boldsymbol{\theta}\cdot\boldsymbol{\theta}_0)^2}} \exp\Big(-\int_0^s a(\mathbf{y}+p\boldsymbol{\theta})dp - \int_0^t a(\mathbf{y}-p\boldsymbol{\theta}_0)dp\Big)k(\mathbf{y},\boldsymbol{\theta}_0,\boldsymbol{\theta}). \end{aligned} \tag{9.14}$$

Since a is known, this gives us an explicit expression for $k(\mathbf{x},\boldsymbol{\theta},\boldsymbol{\theta}_0)$.

Exercise 9.2.1 Verify the above formula.

We thus obtain explicit reconstruction formulas for the absorption and scattering coefficients. The reconstruction of the absorption coefficient involves the inversion of a Radon transform. This is therefore a mildly ill-posed problem of order $\alpha = 1/2$. Once the absorption is known, the above formula (9.14) provides an explicit reconstruction for the scattering kernel k.

The main problem with the above theory is that it requires to separate the singular components of the albedo operator from the rest. Although this is possible theoretically, such a separation is usually difficult to implement in practice. One main reason is that physically realistic sources are not singular. A second reason is that the ballistic part necessary to reconstruct the absorption parameter and the single scattering term necessary to reconstruct the scattering kernel, may be relatively small in the regime of high scattering, where both a and k are relatively large. Indeed the exponential factors appearing in the reconstruction formulas become quite large, thus rendering the reconstructions quite unstable with respect to noise.

Chapter 10

Diffusion Equations

We now consider the problem of *electrical impedance tomography*, which is a non invasive technique that consists of reconstructing the conductivity coefficient of a diffusion equation from current and voltage boundary measurements. The notes below are taken from [21]. The technical proofs of some theorems are not reproduced.

10.1 Introduction

More precisely consider the Dirichlet problem

$$\begin{aligned} &L_\gamma u(\mathbf{x}) \equiv \nabla \cdot \gamma(\mathbf{x}) \nabla u(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega \\ &u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega, \end{aligned} \tag{10.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. The Dirichlet-to-Neumann or voltage-to-current map is given by

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}. \tag{10.2}$$

The inverse problem consists of reconstructing γ from the knowledge of Λ_γ.

We first assume that γ is a smooth function and perform the following change of variables:

$$\gamma^{-1/2} L_\gamma \gamma^{-1/2} = \Delta - q, \qquad q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}. \tag{10.3}$$

Here Δ is the usual Laplacian operator.

Exercise 10.1.1 Prove (10.3).

Therefore provided that γ is known at the boundary of the domain (we shall come back to this point later), it is sufficient to reconstruct $q(x)$ from the boundary measurements to reconstruct $\gamma(\mathbf{x})$ thanks to (10.3).

10.2 Exponential solutions

Let $\boldsymbol{\rho}$ be a complex-valued vector in $\mathbb{R}^n$ for $n \geq 2$ such that $\boldsymbol{\rho} \cdot \boldsymbol{\rho} = 0$. We then verify that $e^{\mathbf{x}\cdot\boldsymbol{\rho}}$ is harmonic, i.e.,

$$\Delta e^{\mathbf{x}\cdot\boldsymbol{\rho}} = \boldsymbol{\rho} \cdot \boldsymbol{\rho}\ e^{\mathbf{x}\cdot\boldsymbol{\rho}} = 0. \tag{10.4}$$

It turns out that there are solutions of $\Delta - q$ that are not very different from $e^{\mathbf{x}\cdot\boldsymbol{\rho}}$. Here is the technical result (without proof)

Theorem 10.2.1 *Let $q \in L^\infty(\mathbb{R}^n)$ such that $q(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq R > 0$. Let $-1 < \delta < 0$. Then there exists $\varepsilon(\delta)$ such that for every $\boldsymbol{\rho} \in \mathbb{C}^n$ satisfying the two constraints:*

$$\boldsymbol{\rho} \cdot \boldsymbol{\rho} = 0, \quad \text{and} \quad |\boldsymbol{\rho}| \geq \frac{1}{\varepsilon(\delta)} \|(1 + |\mathbf{x}|^2)^{1/2} q\|_{L^\infty(\mathbb{R}^n)},$$

there exists a unique solution to

$$(\Delta - q)u = 0,$$

of the form

$$u(\mathbf{x}) = e^{\mathbf{x}\cdot\boldsymbol{\rho}}\big(1 + \psi(\mathbf{x};\boldsymbol{\rho})\big), \tag{10.5}$$

where $\psi(\mathbf{x};\boldsymbol{\rho}) \in H^2_\delta(\mathbb{R}^n)$. Moreover for $0 \leq s \leq 1$, we have

$$\|\psi(\mathbf{x};\boldsymbol{\rho})\|_{H^s_\delta(\mathbb{R}^n)} \leq \frac{C}{|\boldsymbol{\rho}|^{1-s}}. \tag{10.6}$$

We use here the following Hilbert space

$$L^2_\delta(\mathbb{R}^n) = \Big\{ f; \|f\|^2_{L^2_\delta(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\mathbf{x}|^2)^\delta |f(\mathbf{x})|^2 d\mathbf{x} < \infty \Big\}, \tag{10.7}$$

and $H^s_\delta(\mathbb{R}^n)$ the associated Sobolev space of order s.

The above theorem shows that complex exponentials are almost in the kernel of $(\Delta - q)$, up to a factor of size $|\boldsymbol{\rho}|^{-1}$ in the $L^2_\delta(\mathbb{R}^n)$ sense.

10.3 The potential problem

Let $q \in L^\infty(\Omega)$, which we extend to 0 outside Ω. Let us assume that the dimension $n \geq 3$. We define the Cauchy data of the potential problem as the set

$$\mathcal{C}_q = \Big\{ \big(u\big|_{\partial\Omega}, \frac{\partial u}{\partial \boldsymbol{\nu}}\Big|_{\partial\Omega}\big) \Big\}, \tag{10.8}$$

where $u \in H^1(\Omega)$ is a solution of $(\Delta - q)u = 0$. Then we have the following uniqueness result

Theorem 10.3.1 *Let us assume that $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. Then $q_1 = q_2$.*

Proof. For $i = 1, 2$ we define $u_i \in H^1(\Omega)$ as a solution to

$$(\Delta - q_i)u_i = 0.$$

We thus deduce from the divergence theorem that

$$\int_\Omega (q_1 - q_2)u_1 u_2 d\mathbf{x} = \int_{\partial\Omega} \big(\frac{\partial u_1}{\partial \boldsymbol{\nu}} u_2 - \frac{\partial u_2}{\partial \boldsymbol{\nu}} u_1\big) d\sigma = 0, \tag{10.9}$$

since $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. We want to show that the product of two arbitrary solutions u_1 and u_2 of the above equations is dense in $L^2(\mathbb{R}^n)$. This would indeed imply then that $q_1 = q_2$ which is what we want to prove. The idea is that products of harmonic solutions are indeed dense in $L^2(\mathbb{R}^n)$ for $n \geq 3$. This is done as follows. We choose $\boldsymbol{\rho}_{1,2}$ as

$$\boldsymbol{\rho}_1 = \frac{\mathbf{m}}{2} + i\frac{\mathbf{k}+\mathbf{l}}{2}, \qquad \boldsymbol{\rho}_2 = -\frac{\mathbf{m}}{2} + i\frac{\mathbf{k}-\mathbf{l}}{2}, \tag{10.10}$$

where the three real-valued vectors $\mathbf{k}$, $\mathbf{l}$, and $\mathbf{m}$ are chosen in $\mathbb{R}^n$ such that

$$\mathbf{m}\cdot\mathbf{k} = \mathbf{m}\cdot\mathbf{l} = \mathbf{k}\cdot\mathbf{l} = 0, \qquad |\mathbf{m}|^2 = |\mathbf{k}|^2 + |\mathbf{l}|^2. \tag{10.11}$$

We verify that $\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_i = 0$ and that $|\boldsymbol{\rho}_i|^2 = \frac{1}{2}(|\mathbf{k}|^2 + |\mathbf{l}|^2)$. The solutions to $(\Delta - q)u_i$ are then chosen of the form

$$u_i(\mathbf{x}) = e^{\mathbf{x}\cdot\boldsymbol{\rho}_i}(1 + \psi_{q_i}(\mathbf{x})),$$

so that (10.9) can be recast as

$$\widehat{q_1 - q_2}(-\mathbf{k}) = -\int_\Omega e^{i\mathbf{x}\cdot\mathbf{k}}(q_1 - q_2)(\psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2})d\mathbf{x}. \tag{10.12}$$

We now let $|\mathbf{l}| \to \infty$ at fixed value of $\mathbf{k}$ and deduce in the limit that $\widehat{q_1 - q_2}(\mathbf{k}) = 0$ for all $\mathbf{k} \in \mathbb{R}^n$. This concludes the proof of the theorem. □

10.4 Inverse conductivity problem

Let us come back to the inversion of γ from Λ_γ. We verify from the change of variables (10.3) that

$$\mathcal{C}_{q_i} = \Big\{ \Big(f, \frac{1}{2}\gamma_i^{-1/2}\frac{\partial \gamma_i}{\partial \boldsymbol{\nu}}\Big|_{\partial\Omega} f + \gamma_i^{-1/2}\Big|_{\partial\Omega} \Lambda_{\gamma_i}\big(\gamma_i^{-1/2}\Big|_{\partial\Omega} f\big)\Big), \quad f \in H^{1/2}(\partial\Omega)\Big\}.$$

Therefore we can conclude that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies that $\gamma_1 = \gamma_2$ provided that we can show that γ_1 and γ_2 as well as their normal derivatives agree on $\partial\Omega$. We have the following result

Theorem 10.4.1 *Let us assume that $0 < \gamma_i \in C^m(\bar{\Omega})$ and that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then we can show that for all $|\alpha| < m$, we have*

$$\partial^\alpha \gamma_1\Big|_{\partial\Omega} = \partial^\alpha \gamma_2\Big|_{\partial\Omega}. \tag{10.13}$$

This results implies that when γ_i is real analytic (hence defined by its Taylor expansion at the boundary $\partial\Omega$), the knowledge of Λ_γ uniquely determines γ [13]. The results obtained in the preceding section show that even in the case of $\gamma \in C^2(\bar{\Omega})$, the knowledge of Λ_γ uniquely determines γ. This fundamental result was obtained in [18].

10.5 Stability result

We have seen a uniqueness result in the reconstruction of the diffusion coefficient in an elliptic equation. The stability of the reconstruction is however extremely poor. A result expected to be optimal obtained in [4] shows that

$$\|\gamma(\mathbf{x}) - \gamma'(\mathbf{x})\|_{L^\infty(\Omega)} \leq C \Big| \log \|\Lambda - \Lambda'\|_X \Big|^{-\delta}. \tag{10.14}$$

Here Λ and Λ' are the maps corresponding to the diffusion equations with coefficients γ and γ', respectively, $X = \mathcal{L}(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))$, and $\delta \in (0,1)$ is a constant that only depends on the spatial dimension (greater than 3). This formula implies that even very small errors of measurements may have quite large effects on the reconstructed diffusion coefficient.

The reconstruction of the conductivity γ at the boundary of the domain $\partial\Omega$ is actually much more stable. We can show the following result. For γ_1 and γ_2 of class $C^\infty(\bar{\Omega})$ such that $0 < E^{-1} \leq \gamma_i \leq E$, there exists C such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}. \tag{10.15}$$

Moreover for all $0 < \sigma < \frac{1}{n+1}$, there exists C_σ such that

$$\|\frac{\partial\gamma_1}{\partial\boldsymbol{\nu}} - \frac{\partial\gamma_2}{\partial\boldsymbol{\nu}}\|_{L^\infty(\partial\Omega)} \leq C_\sigma \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^\sigma_{\frac{1}{2},-\frac{1}{2}}. \tag{10.16}$$

The next two chapters are devoted to simplifications of the reconstruction problem: since we cannot expect to reconstruct more than a few coefficients from even almost-noisefree data, we should aim at reconstructing the coefficients we are the most interested in. This requires some sort of parameterization of those parameters, or equivalently to some a priori assumptions on the shape of the coefficients we wish to reconstruct.

Chapter 11

Reconstructing the domain of inclusions

The reconstruction of physical parameters in an elliptic equation from boundary measurements, such as the Neumann-to-Dirichlet map, is a severely ill-posed problem. One should therefore not expect to reconstruct much more than a few coefficients modeling the physical parameters, such as for instance the first Fourier modes in a Fourier series expansion.

In certain applications, knowing the first few coefficients in a Fourier series expansion is not what one is interested in. One may have additional *a priori* knowledge about the object to be reconstructed and may want to use this knowledge to look for more appropriate parameters. In this chapter, we assume that the physical parameters are given by a background, which is known, and an inclusion, from which we only know that it differs from the background. Moreover, we are not so much interested in the detailed structure of the inclusion as in its location. We thus wish to reconstruct an interface separating the background from the inclusion.

To reconstruct this interface, we use the method of *factorization*. The method provides a constructive technique to obtain the support of the inclusion from the Neumann-to-Dirichlet (NtD) boundary measurements. Notice that the NtD measurements allow us a priori to reconstruct much more than the support of the inclusion. However, because we restrict ourselves to this specific reconstruction, we can expect to obtain more accurate results on location of the inclusion than by directly reconstructing the physical parameters on the whole domain.

11.1 Forward Problem

We consider here the problem in impedance tomography. The theory generalizes to a certain extent to problems in impedance tomography.

Let $\gamma(\mathbf{x})$ be a conductivity tensor in an open bounded domain $\Omega \in \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. We define Σ, a smooth surface in Ω, as the boundary of the inclusion. We denote by D the simply connected bounded open domain such that $\Sigma = \partial D$. This means that D is the domain "inside" the surface Σ. We also define $D^c = \Omega \backslash \overline{D}$, of boundary $\partial D^c = \partial\Omega \cup \Sigma$. We assume that $\gamma(\mathbf{x})$ is a smooth known background $\gamma(\mathbf{x}) = \gamma_0(\mathbf{x})$ on D^c, and that γ and γ_0 are smooth but different on D. For γ_0 a smooth known tensor

on the full domain Ω, this means that γ jumps across Σ so that Σ is the surface of discontinuity of γ. More precisely, we assume that the $n \times n$ symmetric tensor $\gamma_0(\mathbf{x})$ is of class $C^2(\overline{\Omega})$ and positive definite such that $\xi_i \xi_j \gamma_{0ij}(\mathbf{x}) \geq \alpha_0 > 0$ uniformly in $\mathbf{x} \in \Omega$ and in $\{\xi_i\}_{i=1}^n = \boldsymbol{\xi} \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$. Similarly, the $n \times n$ symmetric tensor $\gamma(\mathbf{x})$ is of class $C^2(\overline{D}) \otimes C^2(\overline{D^c})$ (in the sense that $\gamma(\mathbf{x})|_D$ can be extended as a function of class $C^2(D)$ and same thing for $\gamma(\mathbf{x})|_{D^c}$) and positive definite such that $\xi_i \xi_j \gamma_{ij}(\mathbf{x}) \geq \alpha_0 > 0$ uniformly in $\mathbf{x} \in \Omega$ and in $\{\xi_i\}_{i=1}^n = \boldsymbol{\xi} \in S^{n-1}$.

The equation for the electric potential $u(\mathbf{x})$ is given by

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0, && \text{in } \Omega \\ \mathbf{n} \cdot \gamma \nabla u &= g && \text{on } \partial\Omega \\ \int_{\partial\Omega} u \, d\sigma &= 0. \end{aligned} \tag{11.1}$$

Here, $\mathbf{n}(\mathbf{x})$ is the outward unit normal to Ω at $\mathbf{x} \in \partial\Omega$. We also denote by $\mathbf{n}(\mathbf{x})$ the outward unit normal to D at $\mathbf{x} \in \Sigma$. Finally $g(\mathbf{x})$ is a mean-zero current, i.e., $\int_{\partial\Omega} g d\sigma = 0$, imposed at the boundary of the domain.

The above problem admits a unique solution $H_0^1(\Omega)$, the space of functions in $u \in H^1(\Omega)$ such that $\int_{\partial\Omega} u d\sigma = 0$. This results from the variational formulation of the above equation

$$b(u, \phi) \equiv \int_\Omega \gamma \nabla u \cdot \nabla \phi d\mathbf{x} = \int_{\partial\Omega} g \phi d\sigma(\mathbf{x}) \equiv l(\phi), \tag{11.2}$$

holding for any test function $\phi \in H_0^1(\Omega)$. Indeed from a Poincaré-like inequality, we deduce that $b(u, v)$ is a coercive and bounded bilinear form on $H_0^1(\Omega)$ and the existence result follows from the Lax-Milgram theorem. Classical trace estimates show that $u|_{\partial\Omega} \in H_0^{1/2}(\partial\Omega)$, the space of functions $v \in H^{1/2}(\partial\Omega)$ such that $\int_{\partial\Omega} v d\sigma = 0$.

We define the Neumann-to-Dirichlet operator Λ_Σ, depending on the location of the discontinuity Σ, as

$$\Lambda_\Sigma \; : \; H_0^{-1/2}(\partial\Omega) \; \longrightarrow \; H_0^{1/2}(\partial\Omega), \qquad g \; \longmapsto \; u_{|\partial\Omega}, \tag{11.3}$$

where $u(\mathbf{x})$ is the solution to (11.1) with boundary normal current $g(\mathbf{x})$. Similarly, we introduce the "background" Neumann-to-Dirichlet operator Λ_0 defined as above with γ replaced by the known background γ_0. To model that the inclusion has a different conductivity from the background, we assume that γ satisfies either one of the following hypotheses

$$\gamma(\mathbf{x}) - \gamma_0(\mathbf{x}) \geq \alpha_1 > 0 \quad \text{on } D, \qquad \gamma_0(\mathbf{x}) = \gamma(\mathbf{x}), \quad \text{on } D^c, \tag{11.4}$$

$$\gamma_0(\mathbf{x}) - \gamma(\mathbf{x}) \geq \alpha_1 > 0 \quad \text{on } D, \qquad \gamma_0(\mathbf{x}) = \gamma(\mathbf{x}), \quad \text{on } D^c, \tag{11.5}$$

for some constant positive definite tensor α_1. The tensor inequality $\gamma_1 \geq \gamma_2$ is meant in the sense that $\xi_i \xi_j (\gamma_{1,ij} - \gamma_{2,ij}) \geq 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^n$.

11.2 Factorization method

The purpose of the factorization method is to show that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L, \tag{11.6}$$

where L and L^* are operators in duality that depend only on $\gamma|_D = (\gamma_0)|_D$ and $\pm F$ is an operator that generates a coercive form on $H_0^{1/2}(\Sigma)$ when (11.4) and (11.5) are satisfied, respectively. The operators are constructed as follows. Let v and w be the solutions of

$$\begin{array}{rcll} \nabla\cdot\gamma\nabla v &=& 0, & \text{in } D^c \\ \mathbf{n}\cdot\gamma\nabla v &=& \phi & \text{on } \partial\Omega \\ \mathbf{n}\cdot\gamma\nabla v &=& 0 & \text{on } \Sigma \\ \displaystyle\int_\Sigma v d\sigma &=& 0, & \end{array} \qquad \begin{array}{rcll} \nabla\cdot\gamma\nabla w &=& 0, & \text{in } D^c \\ \mathbf{n}\cdot\gamma\nabla w &=& 0 & \text{on } \partial\Omega \\ \mathbf{n}\cdot\gamma\nabla w &=& -\phi & \text{on } \Sigma \\ \displaystyle\int_\Sigma w d\sigma &=& 0. & \end{array} \tag{11.7}$$

These equations are well-posed in the sense that they admit solutions in $H^1(D^c)$ with traces in $H^{1/2}(\Sigma)$ and in $H^{1/2}(\partial\Omega)$ at the boundary of D^c. We then define the operator L, which maps $\phi \in H_0^{-1/2}(\partial\Omega)$ to $v|_\Sigma \in H_0^{1/2}(\Sigma)$, where v is the unique solution to the left equation in (11.7), and the operator L^*, which maps $\phi \in H_0^{-1/2}(\Sigma)$ to $w|_{\partial\Omega}$, where w is the unique solution to the right equation in (11.7). We verify that both operators are in duality in the sense that

$$(L\phi,\psi)_\Sigma \equiv \int_\Sigma \psi L\phi \; d\sigma = \int_{\partial\Omega} \phi L^*\psi \; d\sigma \equiv (\phi, L^*\psi)_{\partial\Omega}.$$

Let us now define two operators G_Σ and G_Σ^* as follows. For any quantity f defined on $D \cup D^c$, we denote by $f^+(\mathbf{x})$ for $\mathbf{x} \in \Sigma$ the limit of $f(\mathbf{y})$ as $\mathbf{y} \to \mathbf{x}$ and $\mathbf{y} \in D^c$, and by $f^-(\mathbf{x})$ the limit of $f(\mathbf{y})$ as $\mathbf{y} \to \mathbf{x}$ and $\mathbf{y} \in D$. Let v and w be the unique solutions to the following problems

$$\begin{array}{rcll} \nabla\cdot\gamma\nabla v &=& 0, & \text{in } \Omega\backslash\Sigma \\ [v] &=& 0, & \text{on } \Sigma \\ [\mathbf{n}\cdot\gamma\nabla v] &=& 0 & \text{on } \Sigma \\ \mathbf{n}\cdot\gamma\nabla v &=& g & \text{on } \partial\Omega \\ \displaystyle\int_\Sigma v \; d\sigma &=& 0 & \end{array} \qquad \begin{array}{rcll} \nabla\cdot\gamma\nabla w &=& 0, & \text{in } \Omega\backslash\Sigma \\ [w] &=& \phi, & \text{on } \Sigma \\ [\mathbf{n}\cdot\gamma\nabla w] &=& 0 & \text{on } \Sigma \\ \mathbf{n}\cdot\gamma\nabla w &=& 0 & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} w \; d\sigma &=& 0. & \end{array} \tag{11.8}$$

We define G_Σ as the operator mapping $g \in H_0^{-1/2}(\partial\Omega)$ to $G_\Sigma g = \mathbf{n}\cdot\gamma\nabla v^+_{|\Sigma} \in H_0^{-1/2}(\Sigma)$ and the G_Σ^* as the operator mapping $\phi \in H_0^{1/2}(\Sigma)$ to $G_\Sigma^*\phi = w_{|\partial\Omega} \in H_0^{1/2}(\partial\Omega)$, where v and w are the unique solutions to the above equations (11.8).

Except for the normalization $\int_\Sigma v \; d\sigma = 0$, the equation for v is the same as (11.1) and thus admits a unique solution in $H^1(\Omega)$, say. Moreover integrations by parts on D^c imply that

$$\int_\Sigma \mathbf{n}\cdot\gamma\nabla v^+ \; d\sigma = \int_{\partial\Omega} g \; d\sigma = 0.$$

This justifies the well-posedness of the operator G_Σ as it is described above. The operator G_Σ^* is more delicate. We first obtain that for any smooth test function ψ,

$$\begin{array}{rcl} \displaystyle\int_{D^c} \gamma\nabla w\cdot\nabla\psi \; d\mathbf{x} + \int_\Sigma \mathbf{n}\cdot\gamma\nabla w\psi^+ \; d\sigma &=& 0 \\ \displaystyle\int_D \gamma\nabla w\cdot\nabla\psi \; d\mathbf{x} - \int_\Sigma \mathbf{n}\cdot\gamma\nabla w\psi^- \; d\sigma &=& 0, \end{array}$$

so that

$$\int_\Omega \gamma\nabla w\cdot\nabla\psi\ d\mathbf{x} = \int_\Sigma (-\mathbf{n}\cdot\gamma\nabla w)\ [\psi]\ d\sigma. \tag{11.9}$$

It turns out that $\|\mathbf{n}\cdot\gamma\nabla w\|_{H_0^{-1/2}(\Sigma)}$ is bounded by the norm of $\gamma\nabla w$ in $H(\mathrm{div},\Omega)$ (see [11]). This and a Poincaré-type inequality shows that the above right-hand side with $\psi = w$ is bounded by $C\|\phi\|^2_{H_0^{1/2}(\Sigma)}$. Existence and uniqueness of the solution $w\in H^1(D)\otimes H^1(D^c)$ to (11.8) is then ensured by application of the Lax-Milgram theory. This also shows that the operator G_Σ^* as defined above is well-posed.

Integrations by parts in the equation for v in (11.8) by a test function φ yields

$$\begin{array}{rcl} \displaystyle\int_D \gamma\nabla u\cdot\nabla\varphi\ d\mathbf{x} - \int_\Sigma \mathbf{n}\cdot\gamma\nabla v\varphi^- d\sigma &=& 0 \\ \displaystyle\int_{D^c} \gamma\nabla v\cdot\nabla\varphi\ d\mathbf{x} + \int_\Sigma \mathbf{n}\cdot\gamma\nabla v\varphi^+ d\sigma &=& \displaystyle\int_{\partial\Omega} g\varphi\ d\sigma, \end{array} \tag{11.10}$$

from which we deduce that

$$\int_\Omega \gamma\nabla v\cdot\nabla\varphi = \int_{\partial\Omega} g\varphi - \int_\Sigma (G_\Sigma g)[\varphi]. \tag{11.11}$$

That G_Σ and G_Σ^* are in duality in the sense that

$$\int_\Sigma G_\Sigma g\ \phi\ d\sigma = \int_{\partial\Omega} g\ G_\Sigma^*\phi\ d\sigma, \tag{11.12}$$

follows from (11.11) with $\varphi = w$ and (11.9) with $\psi = v$ since $[v] = 0$.

We finally define F_Σ as the operator mapping $\phi\in H_0^{1/2}(\Sigma)$ to $F_\Sigma\phi = -\mathbf{n}\cdot\gamma\nabla w\in H_0^{-1/2}(\Sigma)$, where w is the solution to (11.8). Based on the above results, this is well-posed operator. Moreover, we deduce from (11.9) that

$$(F_\Sigma[w],[\psi])_\Sigma = \int_\Omega \gamma\nabla w\cdot\nabla\psi\ d\mathbf{x} = ([w],F_\Sigma[\psi])_\Sigma, \tag{11.13}$$

so that $F_\Sigma = F_\Sigma^*$. Upon choosing $[w] = [\psi]$, we find that F_Σ is coercive on $H_0^{1/2}(\Sigma)$. This implies among other things that F_Σ is injective.

We now notice that

$$G_\Sigma^* = L^* F_\Sigma.$$

This follows from the uniqueness of the solution to the elliptic problem on D^c with conditions defined on $\partial D^c = \Sigma\cup\partial\Omega$. By duality, this also implies that $G_\Sigma = F_\Sigma L$. The operators G_0 and F_0 are defined similarly except that γ is replaced by γ_0 in equations (11.8). Let us finally define the operator M, which maps $g\in H_0^{-1/2}(\partial\Omega)$ to $u_{|\partial\Omega}\in H_0^{1/2}(\partial\Omega)$, where u is the solution to

$$\begin{array}{rcll} \nabla\cdot\gamma\nabla u &=& 0, & \text{in } D^c \\ \mathbf{n}\cdot\gamma\nabla u &=& 0, & \text{on } \Sigma \\ \mathbf{n}\cdot\gamma\nabla u &=& g, & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} u\ d\sigma &=& 0. & \end{array} \tag{11.14}$$

Except for the normalization, the operator M is the same as the operator L (so that $L - M$ is proportional to identity) and is thus well-posed. We now verify from the linearity of the elliptic problems that

$$\Lambda_\Sigma = M - L^* G_\Sigma = M - L^* F_\Sigma L, \qquad \Lambda_0 = M - L^* G_0 = M - L^* F_0 L. \tag{11.15}$$

We thus deduce the main factorization result of this section, namely that

$$\Lambda_0 - \Lambda_\Sigma = L^* F L, \qquad F = F_\Sigma - F_0. \tag{11.16}$$

The above result would not be very useful if F did not have specific properties. We now show that F or $-F$ generates a coercive form on $H_0^{1/2}(\Sigma)$ and may be written as B^*B for some operator B^* *surjective*. Note that $F^* = F$ since both F_Σ and F_0 are self-adjoint.

We denote by w_Σ the solution w to (11.8) and by w_0 the solution to the same equation with γ replaced by γ_0. Upon multiplying the equation for w_Σ by w_0 and subtracting the equation for w_0 multiplied by w_Σ, we obtain since $\gamma = \gamma_0$ on D^c that

$$\begin{aligned} \int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma \, d\mathbf{x} &= \int_\Sigma (\mathbf{n} \cdot \gamma \nabla w_\Sigma w_0^- - \mathbf{n} \cdot \gamma \nabla w_0 w_\Sigma^-) d\sigma \\ 0 &= \int_\Sigma (\mathbf{n} \cdot \gamma \nabla w_\Sigma w_0^+ - \mathbf{n} \cdot \gamma \nabla w_0 w_\Sigma^+) d\sigma. \end{aligned}$$

Notice that both γ and ∇w_Σ jump across Σ but that $\mathbf{n} \cdot \gamma \nabla w_\Sigma$ does not. This yields that

$$\int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma \, d\mathbf{x} = \int_\Sigma (F_\Sigma - F_0) \phi \, \phi \, d\sigma = \int_\Sigma F \phi \, \phi \, d\sigma. \tag{11.17}$$

Let us now introduce $\delta w = w_0 - w_\Sigma$. Upon multiplying $\nabla \cdot \gamma_0 \nabla \delta w + \nabla \cdot (\gamma_0 - \gamma) \nabla w_\Sigma = 0$ by δw and integrating by parts on D^c and D we deduce that

$$\int_\Omega \gamma_0 \nabla \delta w \cdot \nabla \delta w \, d\mathbf{x} + \int_D (\gamma - \gamma_0) \nabla w_\Sigma \cdot \nabla w_\Sigma \, d\mathbf{x} = \int_D (\gamma - \gamma_0) \nabla w_0 \cdot \nabla w_\Sigma \, d\mathbf{x}.$$

By exchanging the roles of the indices Σ and 0 we also obtain

$$\int_\Omega \gamma \nabla \delta w \cdot \nabla \delta w \, d\mathbf{x} + \int_D (\gamma_0 - \gamma) \nabla w_0 \cdot \nabla w_0 \, d\mathbf{x} = \int_D (\gamma_0 - \gamma) \nabla w_0 \cdot \nabla w_\Sigma \, d\mathbf{x}.$$

Combining these results with (11.17) we deduce that

$$\begin{aligned} \int_\Sigma F \phi \, \phi \, d\sigma &= \int_\Omega \gamma_0 \nabla \delta w \cdot \nabla \delta w \, d\mathbf{x} + \int_D (\gamma - \gamma_0) \nabla w_\Sigma \cdot \nabla w_\Sigma \, d\mathbf{x} \\ \int_\Sigma -F \phi \, \phi \, d\sigma &= \int_\Omega \gamma \nabla \delta w \cdot \nabla \delta w \, d\mathbf{x} + \int_D (\gamma_0 - \gamma) \nabla w_0 \cdot \nabla w_0 \, d\mathbf{x}. \end{aligned} \tag{11.18}$$

Let us assume that (11.4) holds. Then F generates a coercive form on $H_0^{1/2}(\Sigma)$. Indeed, let us assume that the right-hand side of the first equality above is bounded. Then by a Poincaré-type inequality, we have $\delta w \in H^1(\Omega)$ and $w_\Sigma|_D \in H^1(D)$ thanks to (11.4). This implies that $(\mathbf{n} \cdot \gamma \nabla w_\Sigma)_{|\Sigma} \in H^{-1/2}(\Sigma)$ and thus based on (11.8) that $w_\Sigma|_{D^c} \in H^1(D^c)$.

This in turn implies that both w_Σ^+ and w_Σ^- belong to $H^{1/2}(\Sigma)$ so that their difference $\phi \in H_0^{1/2}(\Sigma)$. Thus, we have shown the existence of a positive constant C such that

$$\|\phi\|_{H_0^{1/2}(\Sigma)} \leq C(F\phi, \phi)_\Sigma^{1/2}. \tag{11.19}$$

Exchanging the indices Σ and 0 also yields the existence of a constant C under hypothesis (11.5) such that

$$\|\phi\|_{H_0^{1/2}(\Sigma)} \leq C(-F\phi, \phi)_\Sigma^{1/2}. \tag{11.20}$$

In what follows, we assume that (11.4) and (11.19) hold to fix notation. The final results are not modified when (11.5) and (11.20) hold instead.

The operator F is defined from $H_0^{1/2}(\Sigma)$ to $H_0^{-1/2}(\Sigma)$, which have not been identified. So writing $F = B^*B$ requires a little bit of work. Let $\mathcal{I}$ be the canonical isomorphism between $H_0^{-1/2}(\Sigma)$ and $H_0^{1/2}(\Sigma)$. Since it is positive definite we can decompose it as

$$\mathcal{I} = \mathcal{J}^*\mathcal{J}, \qquad \mathcal{J} :\ H_0^{-1/2}(\Sigma) \ \rightarrow\ L_0^2(\Sigma), \quad \mathcal{J}^* :\ L_0^2(\Sigma) \ \rightarrow\ H_0^{1/2}(\Sigma).$$

Both $\mathcal{J}$ and $\mathcal{J}^*$ are isometries as defined above. We can thus recast the coercivity of F as

$$(F\phi, \phi) = (F\mathcal{J}^*u, \mathcal{J}^*u) = (\mathcal{J}F\mathcal{J}^*u, u) \geq \alpha\|\phi\|^2_{H_0^{1/2}(\Sigma)} = \alpha\|u\|^2_{L_0^2(\Sigma)}.$$

So $\mathcal{J}F\mathcal{J}^*$ as a self-adjoint positive definite operator on $L^2(\Sigma)$ can be written as C^*C, where C and C^* are bounded operators from $L_0^2(\Sigma)$ to $L_0^2(\Sigma)$. Since

$$\|Cu\|^2_{L_0^2(\Sigma)} \geq \alpha\|u\|^2_{L_0^2(\Sigma)},$$

we deduce that C^* is surjective. We thus obtain that $F = B^*B$ where $B = C(\mathcal{J}^*)^{-1}$ maps $H_0^{1/2}(\Sigma)$ to $L_0^2(\Sigma)$ and its adjoint operator $B^* = \mathcal{J}^{-1}C^*$ maps $L_0^2(\Sigma)$ to $H_0^{1/2}(\Sigma)$. Since $\mathcal{J}$ is an isomorphism, we deduce that B^* is surjective.

From the above calculations we obtain that

$$\Lambda_0 - \Lambda_\Sigma = L^*FL = L^*B^*(L^*B^*)^* = A^*A, \qquad A = BL.$$

Since the Range of $(A^*A)^{1/2}$ for A acting on Hilbert spaces is equal to the Range of A^*, we deduce that

$$\mathcal{R}((\Lambda_0 - \Lambda_\Sigma)^{1/2}) = \mathcal{R}(L^*B^*) = \mathcal{R}(L^*) \tag{11.21}$$

since B^* is surjective. The latter is shown as follows. We always have that $\mathcal{R}(L^*B^*) \subset \mathcal{R}(L^*)$. Now for $y \in \mathcal{R}(L^*)$ there is x such that $y = L^*x$ and since B^* is surjective u such that $y = L^*B^*x$ so that $y \in \mathcal{R}(L^*B^*)$; whence $\mathcal{R}(L^*) \subset \mathcal{R}(L^*B^*)$.

When (11.5) and (11.20) hold instead of (11.4) and (11.19), we deduce that

$$\mathcal{R}((\Lambda_\Sigma - \Lambda_0)^{1/2}) = \mathcal{R}(L^*), \tag{11.22}$$

instead of (11.21). In both cases we have shown the following result.

Theorem 11.2.1 *Provided that* (11.4) *or* (11.5) *holds, the range of the operator* L^* *defined in* (11.7) *is determined by the Neumann-to-Dirichlet operator* Λ_Σ.

11.3 Reconstruction of Σ

The above theorem gives us a technique to reconstruct Σ: we need to find a family of functions that belong to the Range of L^* when some parameter covers D and do not belong to it when the parameter covers D^c. Notice that the operator L^* does not depend on the domain D and thus only depends on the tensor γ_0 and the surface Σ. Consequently, the reconstruction of Σ will be independent of γ on γ_0, except for the existence of a positive definite tensor α_0 such that (11.4) or (11.5) holds.

Let us now introduce the family of functions $N(\cdot;\mathbf{y})$ indexed by the parameter $\mathbf{y} \in \Omega$ solution of

$$\begin{aligned} \nabla \cdot \gamma_0 \nabla N(\cdot;\mathbf{y}) &= \delta(\cdot - \mathbf{y}), && \text{in } \Omega \\ \mathbf{n} \cdot \gamma_0 \nabla N(\cdot;\mathbf{y}) &= 0 && \text{on } \partial\Omega \\ \int_{\partial\Omega} N(\cdot;\mathbf{y})\, d\sigma &= 0. \end{aligned} \tag{11.23}$$

We define the family of functions $g_{\mathbf{y}}(\mathbf{x}) = N(\mathbf{x};\mathbf{y})_{|\partial\Omega}$ on $\partial\Omega$. We then have

Theorem 11.3.1 *The function $g_{\mathbf{y}}(\mathbf{x})$ belongs to $\mathcal{R}(L^*)$ when $\mathbf{y} \in D$ and does not belong to $\mathcal{R}(L^*)$ when $\mathbf{y} \in D^c$.*

This theorem provides us with a constructive method to image $\Sigma = \partial D$. For each $\mathbf{y} \in \Omega$, all we have to do is to solve (11.23) and verify whether the trace on $\partial\Omega$ belongs to the Range of $(\pm(\Lambda_0 - \Lambda_\Sigma))^{1/2}$, which can be evaluated boundary measurements. Only when the verification is positive can we deduce that $\mathbf{y} \in D$.

Proof. The proof of the theorem is as follows. When $\mathbf{y} \in D$, we have that $\mathbf{n} \cdot \gamma \nabla N(\mathbf{x};\mathbf{y})_{|\Sigma} \in H_0^{-1/2}(\Sigma)$ and $\nabla \cdot \gamma_0 \nabla N(\cdot;\mathbf{y}) = 0$ on D^c so that $g_{\mathbf{y}} \in \mathcal{R}(L^*)$. Let us now assume that $\mathbf{y} \in D^c$ and $g_{\mathbf{y}}(\mathbf{x}) \in \mathcal{R}(L^*)$. Then there exists $\phi \in H_0^{-1/2}(\Sigma)$ such that $g_{\mathbf{y}} = L^*\phi = w_{|\partial\Omega}$, where w is the solution to (11.7).Let $B(\mathbf{y};\varepsilon)$ be the ball of radius ε centered at $\mathbf{y}$ for ε sufficiently small. On $D^c \backslash \overline{B_\varepsilon}$, both w and $g_{\mathbf{y}}$ satisfy the same equation. By uniqueness of the solution to the Cauchy problem imposing Dirichlet data and vanishing Neumann data on $\partial\Omega$, we deduce that $w = g_{\mathbf{y}}$ on $D^c \backslash \overline{B_\varepsilon}$. On $\omega_\varepsilon = B_{\varepsilon_0} \backslash \overline{B_\varepsilon}$ for some fixed $\varepsilon_0 > 0$, we verify that the $H^1(\omega_\varepsilon)$ norm of w remains bounded independently of ε, which is not the case for the fundamental solution $g_{\mathbf{y}}$; whence the contradiction implying that $g_{\mathbf{y}}$ is not in the Range of L^* when $\mathbf{y} \in D^c$. □

Chapter 12

Reconstructing small inclusions

This chapter concerns the reconstruction of small inclusions. We have seen that the reconstruction of diffusion or absorption coefficients in an elliptic equation resulted in a severely ill-posed problem. The previous chapter dealt with the issue by reconstructing the support of an inclusion instead of its detailed structure. Because the support of the inclusion remains an infinite dimensional object, the stability of the reconstruction is still a severely ill-posed problem. Here we further simplify the problem by assuming that the inclusions have small support. This introduces a small parameter allowing us to perform asymptotic expansions. We can then characterize the influence of the inclusion on the boundary measurements by successive terms in the expansion. The interest of such a procedure is the following. Since high-order terms in the expansion become small very quickly, the procedure tells us which parameters can be reconstruction from a given noise level in the measurements and which parameters cannot possibly be estimated. Moreover, these parameters come in small numbers. Each term in the asymptotic expansion is characterized by a finite number of parameters. This implies that by truncating the expansion, we are in the end looking at the reconstruction of only a finite number of parameters. Unlike the previous reconstructions, this becomes a well-posed problem since ill-posedness inevitably comes from the infinite dimensionality of the object we want to reconstruct, at least as long as the mapping from the object to be reconstructed to the noise-free measurements is one-to one (injective), which was always the case in the problems treated so far.

We consider in this chapter a mathematically very simple problem, namely the reconstruction of inclusions characterized by a variation in the absorption coefficient. We also restrict ourselves to the reconstruction from the leading term in the aforementioned asymptotic expansion. The interesting problem of variations of the diffusion coefficient is mathematically more difficult, although the main conclusions are in the end very similar. The presentation follows that in [6].

12.1 First-order effects

Let us consider the problem of optical tomography modeled by a diffusion equation on a domain Ω with current density $g(\mathbf{x})$ prescribed at the boundary $\partial\Omega$. We assume here that the diffusion coefficient is known and to simplify, is set to $D \equiv 1$. Our main hypothesis on the absorption coefficient is that is is the superposition of a background

absorption, to simplify the constant σ_0, and a finite number of fluctuations of arbitrary size $\sigma_m - \sigma_0$, with σ_m constant to simplify, but of small volume. Smallness of the volume of the inclusions compared to the volume of the whole domain is characterized by the small parameter $\varepsilon \ll 1$. The diffusion equation with small absorption inclusions takes then the form

$$\begin{aligned} &-\Delta u_\varepsilon(\mathbf{x}) + \sigma_\varepsilon(\mathbf{x}) u_\varepsilon(\mathbf{x}) = 0, \qquad \Omega \\ &\frac{\partial u_\varepsilon}{\partial \boldsymbol{\nu}} = g, \qquad \partial\Omega, \end{aligned} \tag{12.1}$$

where absorption is given by

$$\sigma_\varepsilon(\mathbf{x}) = \sigma_0 + \sum_{m=1}^{M} \sigma_m \chi_{\mathbf{z}_m + \varepsilon B_m}(\mathbf{x}). \tag{12.2}$$

We have introduced here εB_m as the shape of the mth inclusion centered at $\mathbf{z}_m$, and $\chi_{\mathbf{z}_m+\varepsilon B_m}(\mathbf{x}) = 1$ if $\mathbf{x} - \mathbf{z}_m \in \varepsilon B_m$ and 0 otherwise. The inclusions are centered at $\mathbf{0}$ in the sense that

$$\int_{B_m} \mathbf{x} d\mathbf{x} = \mathbf{0} \qquad \text{for all } m, \tag{12.3}$$

and are assumed to be at a distance greater than $d > 0$, independent of ε, of each-other and of the boundary $\partial\Omega$. The parameter ε is a measure of the diameter of the inclusions. In the three-dimensional setting, which we assume from now on, this implies that the volume of the inclusions is of order ε^3.

We want to derive an asymptotic expansion for u_ε in powers of ε and see which information on the inclusions we deduce from the first terms in the expansion. Let us first define the Green function of the corresponding homogeneous problem

$$\begin{aligned} &-\Delta G(\mathbf{x};\mathbf{z}) + \sigma_0 G(\mathbf{x};\mathbf{z}) = \delta(\mathbf{x}-\mathbf{z}), \qquad \Omega \\ &\frac{\partial G}{\partial \boldsymbol{\nu}}(\mathbf{x};\mathbf{z}) = 0, \qquad \partial\Omega, \end{aligned} \tag{12.4}$$

the homogeneous-domain solution $U(\mathbf{x})$ of

$$\begin{aligned} &-\Delta U(\mathbf{x}) + \sigma_0 U(\mathbf{x}) = 0, \qquad \Omega \\ &\frac{\partial U}{\partial \boldsymbol{\nu}}(\mathbf{x}) = g(\mathbf{x}), \qquad \partial\Omega. \end{aligned} \tag{12.5}$$

As $\varepsilon \to 0$, the volume of the inclusions tends to 0 and u_ε converges to U. To show this, we multiply (12.4) by u_ε and integrate by parts to obtain

$$u_\varepsilon(\mathbf{z}) = \int_{\partial\Omega} g(\mathbf{x}) G(\mathbf{x};\mathbf{z}) d\sigma(\mathbf{x}) - \sum_{m=1}^{M} \int_{\mathbf{z}_m+\varepsilon B_m} \sigma_m G(\mathbf{x};\mathbf{z}) u_\varepsilon(\mathbf{x}) d\mathbf{x}.$$

Using the same procedure for $U(\mathbf{x})$, we obtain

$$u_\varepsilon(\mathbf{z}) = U(\mathbf{z}) - \sum_{m=1}^{M} \int_{\mathbf{z}_m+\varepsilon B_m} \sigma_m G(\mathbf{x};\mathbf{z}) u_\varepsilon(\mathbf{x}) d\mathbf{x}. \tag{12.6}$$

In three space dimensions, the Green function is given by

$$G(\mathbf{x};\mathbf{z}) = \frac{e^{-\sqrt{\sigma_0}|\mathbf{z}-\mathbf{x}|}}{4\pi|\mathbf{z}-\mathbf{x}|} + w(\mathbf{x};\mathbf{z}), \tag{12.7}$$

where $w(\mathbf{x};\mathbf{z})$ is a smooth function (because it solves (12.5) with smooth boundary conditions) provided that $\partial\Omega$ is smooth. For $\mathbf{z}$ at a distance greater than $d > 0$ away from the inclusions $\mathbf{x}_m + \varepsilon B_m$, we then deduce from the L^∞ bound on u_ε (because g and $\partial\Omega$ are assumed to be sufficiently regular) that

$$u_\varepsilon(\mathbf{z}) = U(\mathbf{z}) + O(\varepsilon^3).$$

In the vicinity of the inclusions, we deduce from the relation

$$\int_{\mathbf{z}_m+\varepsilon B_m} G(\mathbf{x};\mathbf{z})d\mathbf{x} = O(\varepsilon^2), \qquad \mathbf{z}-\mathbf{z}_m \in \varepsilon B_m,$$

that $u_\varepsilon(\mathbf{z}) - U(\mathbf{z})$ is of order ε^2 when $\mathbf{z}$ is sufficiently close to an inclusion. This also shows that the operator

$$K_\varepsilon u_\varepsilon(\mathbf{z}) = -\sum_{m=1}^{M} \int_{\mathbf{z}_m+\varepsilon B_m} \sigma_m G(\mathbf{x};\mathbf{z}) u_\varepsilon(\mathbf{x})d\mathbf{x} \tag{12.8}$$

is a bounded linear operator in $\mathcal{L}(L^\infty(\Omega))$ with a norm of order ε^2. This implies that for sufficiently small values of ε, we can write

$$u_\varepsilon(\mathbf{z}) = \sum_{k=0}^{\infty} K_\varepsilon^k U(\mathbf{z}). \tag{12.9}$$

The above series converges fast when ε is small. Notice however that the series does not converge as fast as ε^3, the volume of the inclusions, because of the singular behavior of the Green function $G(\mathbf{x};\mathbf{z})$ when $\mathbf{x}$ is close to $\mathbf{z}$.

Let us now use that

$$\begin{aligned} u_\varepsilon(\mathbf{z}) = U(\mathbf{z}) &- \sum_{m=1}^{M} \int_{\mathbf{z}_m+\varepsilon B_m} \sigma_m G(\mathbf{x};\mathbf{z}) U(\mathbf{x})d\mathbf{x} \\ &+ \sum_{m=1}^{M}\sum_{n=1}^{M} \int_{\mathbf{z}_m+\varepsilon B_m} \int_{\mathbf{z}_n+\varepsilon B_n} \sigma_m\sigma_n G(\mathbf{x};\mathbf{z}) G(\mathbf{y};\mathbf{x}) u_\varepsilon(\mathbf{y})d\mathbf{y}d\mathbf{x}. \end{aligned} \tag{12.10}$$

For the same reasons as above, the last term is of order ε^5, and expanding smooth solutions $U(\mathbf{x})$ and $G(\mathbf{x};\mathbf{z})$ inside inclusions of diameter ε, we obtain that

$$u_\varepsilon(\mathbf{x}) = U(\mathbf{x}) - \sum_{m=1}^{M} G(\mathbf{z};\mathbf{z}_m) C_m U(\mathbf{z}_m) + O(\varepsilon^5), \tag{12.11}$$

where C_m is given by

$$C_m = \varepsilon^3 |B_m| \sigma_m. \tag{12.12}$$

The reason why we obtain a correction term of order ε^5 in (12.11) comes from the fact that (12.3) holds so that the terms of order ε^4, proportional to $\mathbf{x}\cdot\boldsymbol{\nabla}U$ or $\mathbf{x}\cdot\boldsymbol{\nabla}G$, vanish.

12.2 Stability of the reconstruction

The above analysis tells us the following. Provided that our measurement errors are of order $O(\varepsilon^5)$, the only information that can possibly be retrieved on the inclusions is its location $\mathbf{z}_m$ and the product $C_m = \varepsilon^3 \sigma_m B_m$ of the absorption fluctuation with the volume of the inclusion. More refined information requires data with less noise. This information on the inclusion may look unimpressive. Yet assuming that the inclusions are sufficiently small so that the above asymptotic expansion makes sense, no other information can be obtained in a stable fashion from the data.

Notice that the problem to solve is now finite-dimensional. Indeed, each inclusion is represented by four real numbers, namely the three components of the position $\mathbf{z}_m$ and the product C_m. Assuming that only M inclusions are present, this leaves us with $4M$ parameters to reconstruct. The main advantage of reconstructing a finite number of parameters is that it is natural to expect stability of the reconstruction. We can even show stability of the reconstruction from boundary measurements corresponding to one current density $g(\mathbf{x})$ provided that the homogeneous solution $U(\mathbf{x})$ is uniformly positive inside the domain. Here is how it can be proved.

Let us assume that the boundary measurements have an accuracy of order $O(\varepsilon^5)$, consistent with

$$u_\varepsilon(\mathbf{z}) = U(\mathbf{z}) - \sum_{m=1}^{M} C_m (G(\mathbf{z}_m; \mathbf{z}) U(\mathbf{z}_m)) + O(\varepsilon^5). \tag{12.13}$$

We denote by u_ε and u'_ε the solution of two problems with absorption coefficients σ_ε and σ'_ε of the form (12.2). Using (12.13), we obtain that

$$u_\varepsilon(\mathbf{z}) - u'_\varepsilon(\mathbf{z}) = F(\mathbf{z}) + O(\varepsilon^5),$$

with

$$F(\mathbf{z}) = -\sum_{m=1}^{M} \Big(C_m (G(\mathbf{z}_m; \mathbf{z}) U(\mathbf{z}_m)) - C'_m (G(\mathbf{z}'_m; \mathbf{z}) U(\mathbf{z}'_m)) \Big). \tag{12.14}$$

Here we use $M = \max(M, M')$ with a small abuse of notation; we will see shortly that $M = M'$. The function $F(\mathbf{z})$ satisfies the homogeneous equation $-\Delta F + \sigma_0 F = 0$ on Ω except at the points $\mathbf{z}_m$ and $\mathbf{z}'_m$. Moreover, we have that $\frac{\partial F}{\partial \nu} = 0$ at $\partial\Omega$. If $F = 0$ on $\partial\Omega$, we deduce from the uniqueness of the Cauchy problem for the operator $-\Delta + \sigma_0$ that $F \equiv 0$ in Ω. As $\varepsilon \to 0$ and $u_\varepsilon - u'_\varepsilon \to 0$, we deduce that $F(\mathbf{z})$ becomes small not only at $\partial\Omega$ but also inside Ω (the continuation of F from $\partial\Omega$ to $\Omega \backslash \{\mathbf{z}_m \cup \mathbf{z}'_m\}$ is independent of ε). However, the functions $G(\mathbf{z}_m; \mathbf{z}) U(\mathbf{z}_m)$ form an independent family. Each term must therefore be compensated by a term from the sum over the *prime* coefficients. We thus obtain that $M = M'$ and that

$$\Big| C_m (G(\mathbf{z}_m; \mathbf{z}) U(\mathbf{z}_m)) - C'_m (G(\mathbf{z}'_m; \mathbf{z}) U(\mathbf{z}'_m)) \Big| \;\leq\; C \| u_\varepsilon - u'_\varepsilon \|_{L^\infty(\partial\Omega)} + O(\varepsilon^5).$$

The left-hand side can be recast as

$$(C_m - C'_m) G(\mathbf{z}_m; \mathbf{z}) U(\mathbf{z}_m) + C'_m (\mathbf{z}_m - \mathbf{z}'_m) \partial_{\mathbf{z}_m} (G(\bar{\mathbf{z}}_m; \mathbf{z}) U(\bar{\mathbf{z}}_m))$$

where $\bar{\mathbf{z}}_m = \theta \mathbf{z}_m + (1-\theta)\mathbf{z}'_m$ for some $\theta \in (0,1)$. Again these two functions are linearly independent so we deduce that

$$|C_m - C'_m| + |C'_m||\mathbf{z}_m - \mathbf{z}'_m| \leq C\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial\Omega)} + O(\varepsilon^5).$$

Using (12.11) and (12.12), we then obtain assuming that $\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial\Omega)} \approx \varepsilon^5$, that

$$\left|B_m\sigma_m - B'_m\sigma'_m\right| + |\mathbf{z}_m - \mathbf{z}'_m| \leq C\varepsilon^{-3}\|u_\varepsilon - u'_\varepsilon\|_{L^\infty(\partial\Omega)} \approx \varepsilon^2. \tag{12.15}$$

Assuming that the accuracy of the measured data is compatible with the expansion (12.11), i.e. that the u_ε is known on $\partial\Omega$ up to an error term of order ε^5, we can then reconstruct the location $\mathbf{z}_m$ of the heterogeneities up to an error of order ε^2. The product of the volume of the inclusion and the absorption fluctuation is also known with the same accuracy.

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