Euler's Equation

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1 Material Derivative

Let's start by considering a fluid with velocity field $\vec{v}$. Now consider a piece of this fluid with mass dm and position $\vec{r}$ at some time t. After a time dt the fluid is now at $\vec{r} + d\vec{r}$. Let us now consider the velocity of the fluid:

$$d\vec{v} = \vec{v}(\vec{r} + d\vec{r}, t + dt) - \vec{v}(\vec{r}, t)$$

This is actually just the total derivative of $d\vec{v}$ which is:

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + \frac{\partial \vec{v}}{\partial x_i} dx_i$$

Now what we want to know is what the acceleration is of the particle along the stream line, this is called the material derivative and is denoted:

$$\vec{a} = \frac{D\vec{v}}{Dt}$$

We get this value by dividing by dt which will give us:

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x_i} v_i$$

Now let's interpret our index notation to get a vector calculus expression. We have a velocity dotted with our differential operator acting on a velocity, this gives us:

$$\frac{\partial \vec{v}}{\partial x_i} v_i = (\vec{v} \cdot \nabla)\vec{v}$$

2 Our Forces

Now we want to analyze our forces so we can set up our Newton's second law. First we define a gravitational field potential (analogous to the electric potential) as:

$$\phi = \frac{U_G}{m}$$

This basically lets us model the potential due to a gravitational field without requiring a tests mass to be present in the formulas. From this we get that gravitational field is:

$$\vec{g} = -\vec{\nabla}\phi$$

The next thing we are going to do is derive the relationship between pressure and force. We first consider a 3d box and look at the pressure on the two sides whose normal vector is parallel to the x-axis. We know that the net force on the block is:

$$-P(x+dx)A + P(x)A = F_x$$

The reason we have the negative sign is because the pressure on a side produces a pressure inwards so at the point $x + dx$ when we increase the pressure we actually increase the force in the $-x$ direction as we have a pressure inwards. Now let's multiply our equation by dx/dx, this gives us:

$$-Adx\left(\frac{P(x+dx) - P(x)}{dx}\right) = F_x$$

$$-\frac{F_x}{V} = \frac{\partial P}{\partial x}$$

We can repeat this procedure for y and z to get:

$$\vec{\nabla}P = -\frac{d\vec{F}}{dV}$$

Now one can also consider frictional forces, in doing this you will derive the Navier-Stokes equations, this is done later in the document but right now we derive Euler's equation.

3 Euler's Equation

We start with Newton's second law:

$$\vec{F} = m\vec{a}$$

We know our force is the sum of force due to pressure and the force due to gravity, our mass is the tiny volume dV of our piece of fluid times its density ρ, and our acceleration comes from the expression earlier. Plugging all of this in we get:

$$-\vec{\nabla}PdV - \rho\vec{\nabla}\phi dV = \rho dV\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right)$$

Dividing by dV we get that:

$$-\vec{\nabla}P - \rho\vec{\nabla}\phi = \rho\frac{\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v}$$

This is Euler's equation. Now let's play with it. We can use the vector calculus identity that:

$$\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A$$

This can be used to represent our $(v \cdot \nabla)v$ term as:

$$\frac{1}{2}\nabla(v^2) - \vec{v} \times \Omega$$

Where $\Omega = \nabla \times \vec{v}$ is called the vorticity. Using this we can

$$-\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi = \frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\nabla(v^2) - \vec{v} \times \Omega$$

4 Solar Corona

If we consider a fluid in equilibrium then all of our $\vec{v}$ terms will disappear. This leaves you with the formula:

$$\nabla P = -\rho \nabla \phi$$

We will use this to analyze the solar corona. This is the gas at the edge of the sun that you can see during an eclipse. If we assume its spherical and has negligible mass compared to the rest of the sun then the force on it is due to the gravity of the sun. We can get the density of our corona from the ideal gas law (where m represent the mass of an atom of corona). Plugging this in (and working in spherical coordinates) we get:

$$\frac{dP}{dr} = -\frac{GM_S}{r^2}\frac{m}{k_B}\frac{p}{T}$$

Now from the theory of plasmas and an energy equation in fluids it turns out that the temperature as a function of r follows the formula for our situation:

$$T = T_o\left(\frac{r_o}{r}\right)^{2/7}$$

Where $T = T_o$ at $r = r_o$. If we substitute into our original expression we get:

$$\frac{dP}{p} = \frac{GM_S m}{k_B T r_o^{2/7}}\frac{dr}{r^{12/7}}$$

If you solve this equation you get:

$$P = P_o e^{\frac{7GM_S m}{5k_B T_o r_o}\left(\left(\frac{r_o}{r}\right)^{5/7} - 1\right)}$$

$$P = P_o \exp\left[\frac{7GM_S m}{5k_B T_o r_o}\left\{\left(\frac{r_o}{r}\right)^{5/7} - 1\right\}\right]$$

An interesting thing about this equation is that pressure does not go to 0 as r approaches infinity. This would imply that the Corona applies a pressure on empty space (which of course applies a negligible pressure back) that would cause it to expand. This fact was used to predict the existence of the solar winds as they account for this ”expansion” of the corona that happens.

Appendix: Incompressible Fluids

Consider Euler's equation:

$$-\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi = \frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\nabla(v^2) - \vec{v} \times \Omega$$

Now take the curl of this equation to get:

$$\frac{\partial \Omega}{\partial t} = \nabla \times (\vec{v} \times \Omega) + \frac{1}{\rho^2}\nabla\rho \times \nabla P$$

Note that all terms that are the gradient of something else will disappear because of the nature of curl. If we have an incompressible fluid, i.e. one where $\nabla \cdot v = 0$ then we can say first an foremost that our density is constant over space. If we apply these conditions then Euler's equation becomes:

$$\frac{\partial \Omega}{\partial t} = \nabla \times (\vec{v} \times \Omega)$$

Which is very nice to work with. Especially because it relates $\vec{v}$ and its derivatives only.