Kepler's Laws Derivation

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1 Kepler's First Law

We start with the equation for an ellipse in polar coordinates

$$r = \frac{r_o}{1 - \epsilon cos(\theta)}$$

The task is now to show that the two-body problem reduces to this formula. We start with two masses m_1 and m_2 which have a force of gravity between them

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

Where G is the gravity constant, r is the distance between the two masses and $\hat{r}$ is the unit vector that points on the line between the two masses. Let the two masses have position vectors $\vec{r_1}$ and $\vec{r_2}$ respectively. Then the center of the mass of the system is then

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$$

We can then write their positions in terms of $\vec{r} = \vec{r_1} - \vec{r_2}$ which is the relative position vector between the two masses and let $M = m_1 + m_2$. This lets us say

$$\vec{r_1} = \vec{R} + \frac{m_2}{M}\vec{r} \text{ and } \vec{r_2} = \vec{R} - \frac{m_1}{M}\vec{r}$$

We also will use $\mu = \frac{m_1m_2}{m_1+m_2}$ as the reduced mass. If we make our origin the center of mass of the system then we can say $\vec{R} = \vec{0}$ and we can use Konig's 2nd Theorem to find the kinetic energy as:

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 \text{ which reduces to } T = \frac{1}{2}\mu\dot{\vec{r}}^2$$

while for potential energy we get

$$U(r)$$

Now we turn our attention to the angular momentum of the system (which we consider to be about our origin).

$$\vec{l} = m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$$

Using our identities for $\vec{r}_1$ and $\vec{r}_2$ we can rewrite this expression as

$$\vec{l} = \frac{m_1 m_2}{M}(\frac{m_1}{M} + \frac{m_2}{M})(\vec{r} \times \dot{\vec{r}})$$

We note that we have both a perpendicular and parallel part of $\dot{\vec{r}}$ and that in the cross product we will only end up multiplying by the perpendicular part of the cross product giving us a final answer of

$$\vec{l} = \mu r^2 \dot{\theta} \hat{k}$$

This quantity is important because it allows us to express the non-constant $\dot{\theta}$ in terms of the constant l. We now write the Lagrangian for the system

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$$

This will give us the Euler-Lagrange equations of motion of the form

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} - \frac{\partial U(r)}{\partial r}$$

We should note that from this you can define the effective potential (through bringing out the partial derivative to the entire right side) which is a useful quantity for working with orbits as it is equal at the two extremum of an orbit and it's graph can be used to determine stability and forces in orbits. It is defined as

$$U_{eff} = \frac{l^2}{2\mu r^2} + U(r)$$

But if we go back to our initial equation we should use the gravitational potential to get a full expression

$$U = -G\frac{m_1 m_2}{r} = -\frac{\alpha}{r}$$

We now have the equation

$$\mu \ddot{r} = \frac{l^2}{\mu r^3} - \frac{\alpha}{r^2}$$

This can be solved by making the substitution

$$u = \frac{1}{r} \text{ or equivalently } r = \frac{1}{u}$$

We then rewrite the time derivative of r in terms of the derivatives of θ to get

$$\dot{r} = r' \frac{l}{\mu r^2}$$

We then use u to get the rewrite our equations

$$r' = -\frac{1}{u^2}u' \text{ giving us } \dot{r} = -\frac{u'l}{\mu}$$

Using these we get that

$$\ddot{r} = -\frac{l^2}{\mu^2}u''u^2$$

Finally, this gives us the equation

$$-\frac{l^2}{\mu^2}u''u^2 = \frac{l^2}{\mu}u^3 - \alpha u^2 \text{ which becomes } u'' + \mu u = \frac{\alpha\mu^2}{l^2}$$

This is a solvable differential equation which yields the solution

$$u = Acos(\theta - \theta_o) + \frac{\mu\alpha}{l^2}.$$

Which can be turned into

$$r = \frac{1}{Acos(\theta - \theta_o) + \frac{\mu\alpha}{l^2}}.$$

Which can be rewritten by using ϵ as an arbitrary constant and $r_o = \frac{l^2}{\mu\alpha}$

$$r = \frac{r_o}{1 + \epsilon cos(\theta - \theta_o)}$$

2 Kepler's Second Law

We now turn our attention to the much nicer 2nd law. The proof for this is mostly geometric but the equations are fairly easy to get from visualization. First we consider a tiny section of an orbit that subtends an angle $d\theta$, we can then create an approximate triangle (which becomes exact with the use of infinitesimals), this triangle has area $dA = \frac{r^2 d\theta}{2}$, we then divide by dt to get $\frac{dA}{dt} = \frac{r^2\dot{\theta}}{2}$ this then gives us the equation

$$\frac{dA}{dt} = \frac{l}{2\mu}$$

The change in area with respect to time is constant. Or as it is commonly said the area swept by a planets orbit during the same amount of time is the same.

3 Kepler's Third Law

Finally we get to Kepler's third law, the proof is a but more involved, but it is based on the fact that

$$\frac{dA}{dt} = \frac{A}{T}$$

Where A is the area of the ellipse and T is the period of the orbit, using this we can get

$$l^2 = 4\mu^2 \frac{\pi^2 a^2 b^2}{T^2} = r_o \alpha \mu$$

We then use $b = \frac{r_o}{\sqrt{1-\epsilon^2}}$ and $a = \frac{r_o}{1-\epsilon^2}$ to obtain a final identity

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(m_1 + m_2)}$$

Which in the limit $m_1 >> m_2$ becomes

$$\frac{T^2}{a^3} \approx \frac{4\pi^2}{GM}$$