Superballs

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1 The Base Case

In physics we tend to start with balls, in kinematics you throw balls, in dynamics you push balls, when rotation is added you start to spin balls, when you get to momentum you start bouncing balls, and then you get to electromagnetism and the balls develop magic properties. Many students are satisfied with this progression but for those who want to see the pinnacle of ball physics there is more. There are the Superballs.

The setup is simple, we start with two perfectly elastic balls one of mass m_1 and the other with mass m_2 where $m_1 << m_2$, we then put ball 1 on top of ball 2. The question is even simpler, if you drop these balls from a height h how high does m_1 rise?

Our First Solution

When solving this problem we start by considering what happens when the m_2 hits the ground, because m_2 is a perfectly elastic ball if it has speed v when it hits the ground it will bounce back with exactly this speed v (note that m_1 will also have exactly same the velocity, why?). We now enter the frame of m_2, in this frame we have just bounced off the ground and we see the much smaller m_1 approaching us at a speed of $2v$. We have a David vs Goliath type of battle, but this time Goliath wins (probably because David spends all his time doing physics). What this means is that when they collide in the m_2 frame because $m_2 >> m_1$ it is like m_1 hits a wall which means it reflects perfectly with speed $2v$ upwards, if we now leave the frame of m_2 we see this as $3v$ because previously we were moving up with m_2 at speed v. Now all we need to do is calculate v as a function of height. We know that:

$$\frac{1}{2}gt_{fall}^2 = h$$

Using the fact that $t_{fall} = v/g$ we get that:

$$\frac{g}{2}\frac{v^2}{g^2} = h$$

$$v = \sqrt{2gh}$$

We can now use this to find the height that m_1 will reach knowing that the time it will take to reach this height is $t_{up} = 3v/g$, plugging in for our h_{max} we get:

$$h_{max} = 9\frac{v^2}{g} - \frac{g}{2}\frac{9v^2}{g^2}$$

$$h_{max} = 9\frac{v^2}{g}(1 - \frac{1}{2})$$

$$h_{max} = 9h$$

This is an amazing result! Using just two balls you can reach heights never before seen. We have completed our journey and have found one of the classic results in physics!

2 There is More!

The unserious individuals reading this may have thought that we have just reached the end of the problem but we have only just started. The true Superball problem contains stops not at 1, or 2, or 3, or 100, the true Superball problem will have you analyzing as many (or as few) balls as you want, now we begin on our journey to find the true solution. Our new setup is very similar to the old one, we just have n balls stacked on top each other with every ball being significantly lighter than the ball below it. We once again drop them from a height h and see what happens.

Solution

We have actually done most of the heavy lifting in the last problem. First we can say is that when the balls all begin colliding they are all moving at speed v, now we can make the first big step in solving this problem. The major realization we need for this problem is that if you consider the nth ball colliding with the $n+1$th ball then you actually have exactly the same situation as the last problem. First we enter the frame of m_n and we will see m_{n+1} approaching at a speed $v_n + v$, it will then bounce off of m_n and keep the same speed but reverse directions which means that in the ground frame we know see that:

$$v_{n+1} = 2v_n + v$$

From this recursive formula we can figure out $v(n)$ (we can do this by inspecting the first few cases), this gives us a final formula of:

$$v(n) = v(2^n - 1)$$

Using this (plus our formulas from the first problem) we get that the final height we bounce to is:

$$h_{max} = (2^n - 1)^2 h$$

We have now found the true man's Superball answer, for a fun exercise to test out our results try figuring out how many balls you need to reach the escape velocity of the earth if you drop them from 1 meter.