Green's Functions and the Solution to Maxwell's Equations

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1 Maxwell's Equations

We are going to look at solutions of the Maxwell's equations for various charge and potential distributions following an approach roughly adapted from Andrew Zangwill's textbook on electrodynamics. We start with Maxwell's equations for the electromagnetic fields:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

We can represent these fields in terms of our potentials using the formulas:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

We can choose to work in the Lorenz gauge where $\mathbf{A}$ and ϕ are related by:

$$\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \phi_L}{\partial t} = 0$$

Where the L subscript indicates that we are in the Lorenz gauge. We can use this as well as our earlier expressions to write the general equations governing our potentials as:

$$\nabla^2\phi_L - \frac{1}{c^2}\frac{\partial^2 \phi_L}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2\mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{j}$$

These equations (plus the Lorentz Force law) give us the full theory of E&M. Let's try to understand them.

2 Potentially Interesting

Our potentials (which govern all the physics of our system) obey the so called "inhomogenous wave equation", it is called a wave equation because the form of the differential operator on the left is the same as for the wave equation but it is inhomogeneous because we have extra terms on the right hand side of our equation. These extra terms are known as source terms because in some sense they are what allow the creation and destruction of waves. Why is this the case? This is easiest to see when we remember what happens when they are absent, when we don't have these terms our equation is just a regular wave equation and we get a wave propagating through our space and interacting with various boundary conditions. However, in the regular wave equation we don't get the creation or destruction of waves (unless we introduce them via boundary conditions). This tells us that some very important things are missing from our theory. Those very important things are our source terms, they control how waves are created from charges and how they interact with charges (which we previously modelled with boundary conditions).

I want to bring us back to electrostatics for a second. In electrostatics how did we get the solution to the general problem of a charge distribution in space? We summed up the contributions to the potential of all the infinitesimal charges in space to get this formula for our potential:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

In this case we summed over our source terms (static charge distribution) to get our final result. Can we do this same for our wave equation? How would we even model a single source term? Well the traditional model for modelling a single point source is fairly simple, we just let our charge distribution be represented by a dirac delta, mathematically we write:

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r}-\mathbf{r}')$$

What this equation is saying is that we have a single point charge at $\mathbf{r}'$. Ok, we've seen this before, but isn't the whole point of this to account for time dependence? How do we do that? Well we could just let our charge also exist as a delta function in time as well which would give us:

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

This should look a bit weird, I just made it so our charge only exists at t' and nowhere else. But this actually could let us get a model for our charges changing in time, we can just integrate over a bunch of different times letting our position vary accordingly to get our evolution. So the ρ I have written isn't particularly physical, but if we combine a particle at $\mathbf{r}'$ at t' and then move it to $\mathbf{r}' + \delta\mathbf{r}'$ at time $t' + \delta t'$ by summing many similar ρ we can get actual movement of a real physical charge. Ok, but that raises the question, why write it like this in the first place? The answer to that one is that it gives us the most general basis element of a source, something that exists only at one point in space and time, we can then combine many of these to get a real distribution. I should note that this type of source function can be physical, while it might be unrealistic for a charge distribution we can draw an analogy with water waves. If we poke a single point in the water at a single point in time we get a wave from a double delta source function like we have above as the poke happens at a single point in space and time (thanks to Robert G. Littlejohn for the analogy).

When we thought of writing out this charge distribution it was with the purpose of treating it as a source term in our equations. If we go and do that we get the equation governing ϕ as:

$$\nabla^2\phi_L - \frac{1}{c^2}\frac{\partial^2\phi_L}{\partial t^2} = -\frac{q}{\epsilon_0}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

If we can solve this type of equation we might be able to combine solutions for our entire charge distribution to get a general formula for ϕ. Let's try to solve this equation.

3 Green's Functions and Ham

All of that earlier discussion exists to motivate the definition I am about to present. The Green's function for the wave equation is the function that satisfies:

$$\nabla^2 G(\mathbf{r}, t|\mathbf{r}', t') - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\mathbf{r}, t, \mathbf{r}', t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

From our earlier discussion we understand that this gives us the function that results from a single point source function in space and time. Now we have to start solving this equation. I should note that a lot of the steps in this derivation might make you think "How does anyone think to do that?". The answer is they spent a very long time on this problem and tried enough things that eventually led to an answer. That problem solving has been condensed to a tight sequence of steps for us to try and follow. With this in mind the first step of our solution is to guess the solution takes the form $G(\mathbf{r}-\mathbf{r}', t-t')$, the reason for making this guess is that the source function is only a function of the difference between our variables and not some independent relationship. With this assumption in hand we have:

$$\nabla^2 G(\mathbf{r}-\mathbf{r}', t-t') - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

We recognize that our equation on both sides now depends only on the difference between our variables. We can always shift our origin to let $\vec{r'} = 0$ and $t' = 0$, remembering that we can always bring this back by shifting back to our original origin we write our equation with this new origin as:

$$\nabla^2 G(\mathbf{r}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\mathbf{r}, t) = -\delta(\mathbf{r})\delta(t)$$

Our delta is at the origin of a spherically symmetric space. This symmetry tells us that $G(\mathbf{r}, t) = G(r, t)$ where we are now working in spherical coordinates. If we then plug in for the Laplacian in spherical and ignore angular derivatives (which will vanish when acting on G) we get:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rG) - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -\delta(\mathbf{r})\delta(t)$$

We can write this as a differential operator acting on our function. By an operator we just mean some object (call it $\hat{D}$) that sends functions to other functions. So in this case we have:

$$\hat{D} = \left\{\frac{\partial}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}$$

This object when acted onto a function will give you another function, i.e. it maps functions to functions. Operators are defined by how the act on functions. We can write our equation in terms of $\hat{D}$ as:

$$\frac{1}{r}\hat{D}(rG) = -\delta(\mathbf{r})\delta(t)$$

We can also note that if operators are defined by how they act on functions we can actually factor an operator in much the same way we might factor a polynomial. Consider:

$$\hat{D} = \left\{\frac{\partial}{\partial r} - \frac{1}{c}\frac{\partial}{\partial t}\right\}\left\{\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}\right\} = \hat{D}_-\hat{D}_+$$

If you act this on any function f it will give the same result as acting $\hat{D}$ as we defined earlier, so we know that this representation of our $\hat{D}$ operator is equivalent to $\hat{D}$ itself. We also understand that because the order we take partial derivatives doesn't matter we have:

$$\hat{D}_-\hat{D}_+ = \hat{D}_+\hat{D}_-$$

For our situation we write:

$$\frac{1}{r}\left\{\frac{\partial}{\partial r} - \frac{1}{c}\frac{\partial}{\partial t}\right\}\left\{\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}\right\}(rG) = \delta(\mathbf{r})\delta(t)$$

Now I'm going to define $g = rG$ and write:

$$\frac{1}{r}\left\{\frac{\partial}{\partial r} - \frac{1}{c}\frac{\partial}{\partial t}\right\}\left\{\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}\right\}(g) = \delta(\mathbf{r})\delta(t)$$

Now we're going to look at all points with $r > 0$, in that case our dirac delta will be 0 and we get:

$$\frac{1}{r}\left\{\frac{\partial}{\partial r} - \frac{1}{c}\frac{\partial}{\partial t}\right\}\left\{\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}\right\} g = 0,\ r > 0$$

Now we note that this differential equation is second order so it has a general solution composed of a combination of two independent solutions. Now, the reason we factored this operator and did all of this is that we can find our two independent solutions by the fact that they satisfy:

$$\hat{D}_- g = 0 \text{ and } \hat{D}_+ g = 0$$

If either of these conditions are satisfied we see that our equation above will be true because we can act $\hat{D}_-$ and $\hat{D}_+$ in whatever order we want (as we explained earlier) and so we can just act whichever one gives 0 first to solve:

$$\frac{1}{r}\hat{D}_-\hat{D}_+ g = 0$$

We can write the two general independent solutions for our $\hat{D}_-$ and $\hat{D}_+$ equations as:

$$g_\pm = g(t \pm r/c)$$

Using this we see that we have two independent Green's functions of the form:

$$G_\pm = \frac{1}{r} g_\pm(t \pm r/c)$$

And we note that in this form $g_\pm(s)$ can be any arbitrary function of s. The condition we have found is that $s = t \pm r/c$. Now we just act with the original wave equation operator on our solution.

$$\left\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}\frac{g_\pm(t \pm r/c)}{r}$$

We recognize the time derivative will just act on g independently of r while the ∇^2 will act on both of our terms, to see what happens there we just apply the product rule (to learn more look up vector calculus identities on Wikipedia) to get:

$$\nabla^2\left(\frac{g_\pm(t \pm r/c)}{r}\right) = \frac{1}{r}\nabla^2 g_\pm + \nabla g_\pm \cdot \nabla\left(\frac{1}{r}\right) + g_\pm \nabla^2\left(\frac{1}{r}\right)$$

We can expand this out and use $\nabla^2(r^{-1}) = -4\pi\delta(\mathbf{r})$ to get:

$$\nabla^2\left(\frac{g_\pm(t \pm r/c)}{r}\right) = -4\pi g_\pm \delta(\mathbf{r}) + \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^2\frac{\partial g_\pm}{\partial r}\right) - \frac{1}{r^2}\frac{\partial g_\pm}{\partial r}$$

$$\nabla^2\left(\frac{g_\pm(t \pm r/c)}{r}\right) = -4\pi g_\pm \delta(\mathbf{r}) + \frac{1}{r}\frac{\partial^2 g_\pm}{\partial r^2}$$

Plugging this into our wave equation we get:

$$\left\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}\frac{g_\pm(t \pm r/c)}{r} = -4\pi\delta(\mathbf{r}) + \frac{1}{r}\left\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\} g_\pm$$

But we know the term on the right is 0 from how we defined it so we get:

$$\left\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}\frac{g_\pm(t \pm r/c)}{r} = -4\pi\delta(\mathbf{r})g_\pm$$

$$\left\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}G_\pm = -4\pi\delta(\mathbf{r})g_\pm$$

But we also have:

$$\left\{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}G_\pm = -\delta(\mathbf{r})\delta(t)$$

We need these to equations to be equivalent everywhere for all time. We know that this is already satisfied for $r > 0$ (that's what we showed earlier!) so all we need to do is make it match at $r = 0$, in that case we need:

$$-4\pi\delta(\mathbf{r})g_\pm(t) = -\delta(\mathbf{r})\delta(t)$$

This equation reveals to us that we should have:

$$g_\pm(t) = \frac{\delta(t)}{4\pi}$$

When we look away from $r = 0$ we recognize that $t \to t \pm r/c$ which means that our general g has the functional form:

$$g_\pm(t \pm r/c) = \frac{\delta(t \pm r/c)}{4\pi}$$

This means that we get:

$$G_\pm = \frac{1}{4\pi r}\delta(t \pm r/c)$$

Now we go back ages ago and shift back to our original origin so $\mathbf{r} \to \mathbf{r} - \mathbf{r}'$ and $t \to t - t'$. This gives us our Green's functions for the wave equation in 3D:

$$G_\pm(\mathbf{r} - \mathbf{r}', t - t') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}\delta(t - t' \pm |\mathbf{r} - \mathbf{r}'|/c)$$

Alright, that was a long and confusing derivation, so let's back up to get the full picture. We started wanting to solve the inhomogenous wave equation and came up with the idea to try looking for a point source and then find the solution for that. We would then try and find some way to combine these solutions to get our final answer. We just finished finding the solution for a point source so the next step is using this mathematically to solve the inhomogenous wave equation. To do this we'll start by writing out the most general version of the equation that we wish to solve:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -f(\mathbf{r}, t)$$

Where ψ is any arbitrary scalar function satisfying this equation. Now we also have the equation for our Green's function:

$$\nabla^2 G(\mathbf{r} - \mathbf{r}', t - t') - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\mathbf{r} - \mathbf{r}', t - t') = -\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

To combine a bunch of delta function point sources our first inclination is to integrate the delta functions. But we recognize that if we just do that we get nothing about the solution to our equations. So what do we do? We need to incorporate ψ somehow, well we want to find $\psi(\mathbf{r}, t)$ so one thing we realize is that if we bring in a $\psi(\mathbf{r}', t')$ and integrate over the all of d^3r' and dt' then the right hand side of the Green's function equation will give us $\psi(\mathbf{r}, t)$. To see this multiply both sides of the equation by $\psi(\mathbf{r}', t')d^3r'dt'$ and replace derivatives with primed counterparts (which we can do because delta function is symmetric) to get:

$$\left(\psi\nabla'^2 G - \frac{\psi}{c^2}\frac{\partial^2 G}{\partial t'^2}\right) d^3r'dt' = -\psi(\mathbf{r}', t')\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')d^3r'dt'$$

If we then integrate over some volume V from time t_1 to t_2 (with the time we are interested in somewhere in this range) and use the properties of the delta function we get:

$$\psi(\mathbf{r}, t) = -\int\limits_V \int\limits_{t_1}^{t_2} \left(\psi\nabla'^2 G - \frac{\psi}{c^2}\frac{\partial^2 G}{\partial t'^2}\right) d^3r'dt'$$

Now this equation as it currently stands is not useful it all, it relates ψ to itself via an integral which doesn't tell you much. But through some wizardry we may still yet recover an actual solution. To start we integrate our time derivative term by parts twice to get:

$$\int\limits_{t_1}^{t_2} \psi\frac{\partial^2 G}{\partial t'^2}dt' = \int\limits_{t_1}^{t_2} G\frac{\partial^2 \psi}{\partial t'^2}dt' + \left[\psi\frac{\partial G}{\partial t'} - G\frac{\partial \psi}{\partial t'}\right]\Bigg|_{t_1}^{t_2}$$

We also have:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + f(\mathbf{r}, t) = 0$$

We recognize that because it is zero we can add the integral of this times G (working in primed variables) to our original expression to get:

$$\begin{aligned}\psi(\mathbf{r}, t) = \int\limits_V \int\limits_{t_1}^{t_2} \left(\frac{1}{c^2}G\frac{\partial^2\psi}{\partial t'^2} + G\nabla'^2\psi - \psi\nabla'^2 G - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}G + Gf(\mathbf{r}', t')\right) d^3r'dt' \\ + \int\limits_V \frac{1}{c^2}\left[\psi\frac{\partial G}{\partial t'} - G\frac{\partial \psi}{\partial t'}\right]\Bigg|_{t_1}^{t_2} d^3r'\end{aligned} \tag{1}$$

After the cancellation of the second time derivative of *psi* we get three terms:

$$\begin{aligned}\psi(\mathbf{r}, t) = &\int\limits_V \int\limits_{t_1}^{t_2} Gf(\mathbf{r}, t)d^3r'dt' \\ &+ \int\limits_V \int\limits_{t_1}^{t_2} \left(G\nabla'^2\psi - \psi\nabla'^2 G\right) d^3r'dt' \\ &+ \int\limits_V \frac{1}{c^2}\left[\psi\frac{\partial G}{\partial t'} - G\frac{\partial \psi}{\partial t'}\right]\Bigg|_{t_1}^{t_2} d^3r'\end{aligned} \tag{2}$$

We can simplify this further, first we apply Green's second identity (mathematical theorem, see Griffith's or wikipedia) to write:

$$\int\limits_V \int\limits_{t_1}^{t_2} \left(G\nabla'^2\psi - \psi\nabla'^2 G\right) d^3r'dt' = \int\limits_S \int\limits_{t_1}^{t_2} \left(G\nabla'\psi - \psi\nabla' G\right) \cdot d\mathbf{S}'dt'$$

Where S is the surface that bounds our volume. Using this our equation becomes:

$$\begin{aligned}\psi(\mathbf{r},t) &= \int\limits_V \int\limits_{t_1}^{t_2} G f(\mathbf{r},t) d^3r' dt' \\ &+ \int\limits_{t_1}^{t_2} \int\limits_S \left(G\nabla'\psi - \psi\nabla' G\right) \cdot d\mathbf{S}' dt' \\ &+ \int\limits_V \frac{1}{c^2} \left[\psi \frac{\partial G}{\partial t'} - G \frac{\partial \psi}{\partial t'}\right]\Bigg|_{t_1}^{t_2} d^3r'\end{aligned} \tag{3}$$

This is good because it eliminates relating ψ to an integral over the volume (in that term) and instead only requires knowledge of the boundary conditions. Boundary conditions we can specify in our problem statement. This gives us two terms that we can solve for, the first is just an integral over our space and the second is just spatial boundary conditions. The third term is related to our boundary conditions in time. It's about now that I should mention something important. We can add any function that satisfies the homogeneous wave equation for any boundary condition to our Green's function and still have it preserve the formula that got us here. This gives us a freedom of sorts with which we can control our boundary conditions on the Green's function. What this means is that for the spatial integral I can actually pick if I want G or $\nabla' G$ to vanish on the boundary leaving it so that I only need knowledge of one of ψ or $\nabla\psi$ on the boundary. This is an important part of the use of Green's functions but for this case it won't turn out to be important because we will set our boundary to ∞ and let the boundary term vanish promptly.

I want to pause here to look at the formula we have just obtained, we have related ψ to three integrals using the Green's function. The first just contains our inhomogenous function and the Green's function, I think it's astounding how simple this result is. I can take any function I want and make it my source function and all I need to do to figure out what this does to my wave function is an integral of my source function against my Green's function. Next we have an integral over the boundary conditions of our space, one of which we can specify and the other of which we can ignore via changing our Green's function. Finally, we have a term that works with boundary conditions in time, we'll either need to have knowledge of the initial or final state (and the other can be made to vanish by adjusting our Green's function) to get our answer. We have just solved the wave equation in complete generality. Give me any initial conditions, give me any boundary conditions, and give me any source terms and I just have to do a few integrals to tell you exactly what your solution will be at some point in space and time. This is the power of the Green's function approach.

Anyways, back to the problem at hand. We will look at situations where our boundary conditions vanish. Now if we drop the spatial term from our formula we get:

$$\psi(\mathbf{r},t) = \int\limits_V \int\limits_{t_1}^{t_2} G f(\mathbf{r}',t') d^3r' dt' + \int\limits_V \frac{1}{c^2} \left[\psi \frac{\partial G}{\partial t'} - G \frac{\partial \psi}{\partial t'}\right]\Bigg|_{t_1}^{t_2} d^3r' \tag{4}$$

We can plug in $G_\pm$ to the first term to get:

$$\int\limits_V \int\limits_{t_1}^{t_2} \frac{f(\mathbf{r}',t')\delta(t - t' \pm |\mathbf{r}-\mathbf{r}'|/c)}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3r' dt'$$

We can perform the time integral to get:

$$\int_V \frac{f(\mathbf{r}', t \pm |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3r$$

This is one simplification we can make with our Green's function. To simplify this further we need to look at the temporal behavior of our Green's function from earlier. Specifically we need to study what our $\delta(t - t' \pm |\mathbf{r} - \mathbf{r}'|/c)$ will do at the boundaries. For our G_+ function we will get:

$$\delta(t - t_2 + |\mathbf{r} - \mathbf{r}'|/c),\ \delta(t - t_1 + |\mathbf{r} - \mathbf{r}'|/c)$$

Now we are looking at $t_1 \leq t \leq t_2$ so $t - t_1 + |\mathbf{r} - \mathbf{r}'|/c > 0$ as $|\mathbf{r} - \mathbf{r}'|/c \geq 0$ this means that G_+ at the inner boundary is 0. So our G_+ solution gives us a solution after specifying $\psi(t_2)$ as that is the only boundary where it does not vanish. We can do a similar bit of logic on G_- to say that G_- is non-zero at the t_1 boundary but vanishes at the t_2 boundary. This means we have two distinct solutions! We can either specify $\psi(\mathbf{r}, t_1)$ or $\psi(\mathbf{r}, t_2)$ and either one will give a legitimate solution to our equations. We define two solutions, the first is called the retarded solution and it takes the form:

$$\psi_{ret}(\mathbf{r}, t) = \psi_{in}(\mathbf{r}, t) + \frac{1}{4\pi} \int_V \frac{f(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3r$$

Where ψ_{in} is the name we give to the boundary condition integral for G_-, it can be thought of as an initial condition term. On the other hand we can choose to work with G_+ and get the so called advanced solution:

$$\psi_{ret}(\mathbf{r}, t) = \psi_{out}(\mathbf{r}, t) + \frac{1}{4\pi} \int_V \frac{f(\mathbf{r}', t + |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3r$$

Where ψ_{out} is the value of the boundary condition integral for G_+. Comparing these solutions we see the differ in a boundary term, but they also differ fundamentally in the role of the source term. In the retarded case if I have an observer at $R = |\mathbf{r} - \mathbf{r}'|$ they will feel the effect of the source acting at an earlier or retarded time $t' = t - R/c$, this makes sense if we think about say, a water wave. If I start with some poke to the water to create a wave, if I am a distance R away from the wave I won't feel the effect of the poke at time $t = 0$ until a time of R/c as when $t = R/c$ is when my retarded time t' is 0. This reflects the finite propagation speed of solutions of the wave equation (and of physical signals), in this case ψ_{in} is an incoming wave solution of the homogeneous wave equation that describes the initial situation at $t = t_1$ before our source term contributes anything (so we let our source term turn on at $t = t_1$). On the other hand the advanced solution feels the influence of the source acting at the advanced time $t' = t + R/c$ so the behavior of the source in the future determines the behavior of our wave function currently.

We should note that there is no compelling mathematical reason to pick one of these solutions over the other. However, there are compelling physical reasons to pick G_-, the first is that in physics we usually start with a situation and want to see where it is going, we want to predict how a system evolves, this necessitates the use of the retarded solution. The retarded solution also follows our basic physical ideas of causality and the present following what happens in the past which is good. Next we can say that for physics we can let $\psi_{in} = 0$ and just specify initial conditions in our source function to represent some initial wave in our system. Finally, we usually let $f = 0$ until t_1 for analyzing the physical problem of potentials created by a finite charge distribution. In that case it is fair to ignore the spatial boundary terms at infinity as no signal from the source could have propagated that far. Using all of this we can write our final solution to the inhomogenous wave equation which we will use in E&M as:

$$\psi = \frac{1}{4\pi}\int\limits_V \frac{f(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3r$$

4 Just Solve Maxwell's Equations

Until now your journey through E&M has mostly been about looking at special cases and finding symmetries and using these to help you solve Maxwell's equations for your specific situation. Now we have developed the tools to just solve them for whatever problem you throw at me. If we go all the way back to the beginning we had our potentials following the equations:

$$\nabla^2\phi_L - \frac{1}{c^2}\frac{\partial^2\phi_L}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2\mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = -\mu_0\mathbf{j}$$

Now all we have to do is apply the formula we just derived to get the general solution to Maxwell's equations for a finite charge distribution:

$$\phi_L = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

$$\mathbf{A}_L = \frac{\epsilon_0}{4\pi}\int \frac{\mathbf{j}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

So we did it, we just solved E&M, that's it, give me charges, give me currents, and I'll tell you exactly what your electric and magnetic fields are. I also hope this derivation shed some light on the origin of the retarded time. It is not a mystical property me have to come up so our solutions work, it is a consequence of the fact that the fundamental equations governing our system are wave equations. And solutions to the wave equations all have this space and time connection via the speed of the wave. In our case the speed of our wave is the speed of light and it governs the communication between different elements of our system. It should be noted that while c is physically special due to the theory of special relativity plenty of physical situations involve retarded time in their models with their own wave speed v. Sound waves are an example of this, the speed of sound actually governs how signals propagate through our gas acoustically and they will obey similar equations with a source term whose time variable is replaced by the retarded time (Note: this only works for weak sound waves, stronger ones are nonlinear and do not obey these laws).