Blackbody Spectrum

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1 Catastrophe

A blackbody is an object that absorbs 100% of the light sent into it, hence the name of "black-body". Blackbody radiation is the radiation emitted by such an object in thermal equilibrium with its environment. For an ideal blackbody we treat it as if it emits every possible frequency of light it can. A crude model one can make is by looking at a cubic box of side length L with conducting walls. In the vacuum inside of the box the electric field follows the wave equation:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

We can apply separation of variables to this equation and impose the boundary condition that the electric field must vanish at the boundary to get that our allowed wavenumbers are:

$$k_x = \frac{n_x \pi}{L},\ k_y = \frac{n_y \pi}{L},\ k_z = \frac{n_z \pi}{L}$$

Where n_x, n_y, n_z are any positive integer. From our dispersion relation $\omega_n = c|k|_n$ where $|k| = \sqrt{k_x^2 + k_y^2 + k_z^2}$. Using this we see that our angular frequency for our electromagnetic waves is quantized into:

$$\omega_n = \frac{c\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

We have a problem. This means a blackbody emits an infinite amount of frequencies, this creates a problem because each photon it emits at a given frequency carries quadratic energy in its electric and magnetic fields. Because it emits an infinite amount of frequencies the blackbody also has infinitely many quadratic degrees of freedom (i.e. each frequency of photon it can emit gives a degree of freedom). This creates a problem, by the equipartition theorem we expect:

$$\langle E \rangle = \frac{f}{2} kT$$

Where f is the number of degrees of freedom of our system. But we just realized f is ∞ in our system so we expect an infinite amount of energy from our black body which is clearly a problem. Let's keep doing some analysis. We can describe our waves by their $\vec{k}$ values, we call this describing the wave in k-space. We recognize that ω is proportional to our radius in this k-space. The total number of states occuped up to a given ω is given by:

$$N = \frac{V_\omega}{V_s}$$

Where V_ω is our volume in k-space given by:

$$V_\omega = \frac{1}{8}\frac{4}{3}\pi\left(\frac{\omega}{c}\right)^3$$

And V_s is the volume per state in our space. From our earlier expressions for k we see:

$$V_s = \left(\frac{\pi}{L}\right)^3$$

From this we can plug in to get:

$$N = \frac{V_b}{6\pi^2c^3}\omega^3$$

Where $V_b = L^3$ is the volume of our box in real space. This tells us:

$$dN = \frac{V}{2\pi^2c^3}\omega^2 d\omega$$

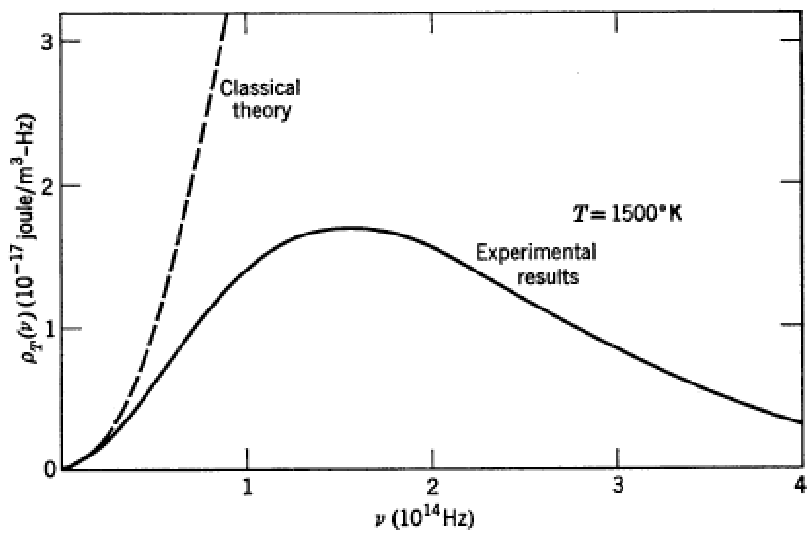
To get energy from this we have to find the energy per state. The energy per degree of freedom is $1/2kT$ but for EM waves you have to multiply by two because you have two different (independent!) polarizations and you have to multiply by two again to account for the fact that we are emitting electric and magnetic waves. All of these degrees of freedom carry quadratic energies so the energy/mode is $2kT$ according to equipartition. From this we get:

$$dU = \langle E\rangle dN = \frac{kTV}{\pi^2c^3}\omega^2 d\omega = u_\omega V d\omega$$

In this case u_ω is the spectral energy density of our blackbody radiation, it tells you the energy density per unit frequency of blackbody radiation and is given by the formula:

$$u_\omega = \frac{kT\omega^2}{\pi^2c^3}$$

We expect something to be wrong with this prediction because of the absurd results we got from equipartition theory earlier. It turns out this theory only matches experiment at low frequencies. This is illustrated in the graph below:



Source: https://aboutradiation.blogspot.com/2020/01/black-body-radiation-uv-catastrophe.html

2 Planck

We clearly have a problem but how do we fix it? Quite a few years passed and theorists tried many ideas but the one that worked (and turned out to be incredibly important) is an idea from Planck. He asked what would happen if you could only excite energy to discrete levels separated by an energy E_0. This means your energy levels are described by the formula:

$$E_n = E_0 n$$

Now we analyze this photon gas under this assumption using Boltzmann statistics. We start by writing our partition function:

$$Z = \sum_{n=0}^{\infty} e^{-\beta E_0 n} = \sum_{n=0}^{\infty} e^{-\beta E_0}$$

This is just a geometric series that we assume will converge by assuming $E_0 > 0$. This gives us the partition function:

$$Z = \frac{1}{1 - e^{-\beta E_0}}$$

Now we write the probability of a given energy from the Boltzmann distribution:

$$P(E_n) = \frac{1}{Z} e^{-\beta E_n}$$

Using this one can prove:

$$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}$$

By using the fact:

$$\langle E \rangle = \sum_{allE} P(E)E$$

Plugging in the partition function we just derived this gives us:

$$\langle E \rangle = \frac{E_0}{e^{\beta E_0} - 1}$$

This differs from the equipartition result, but we expect them to match somewhere. For low energies we can taylor expand our exponential to get:

$$\langle E \rangle \approx kT$$

But we want this to be $2kT$ in the calssical limit so we multiply our expected value of energy formula by 2 to get:

$$\langle E \rangle = \frac{2E_0}{e^{\beta E_0} - 1}$$

In this theory we assume that this E_0 depends on frequency somehow so this is the energy for a specific mode ω that corresponds to E_0 somehow. If we take this expected value of energy and plug it into our earlier equation for dU instead of the equipartition results we get:

$$dU = \frac{2E_0}{e^{\beta E_0} - 1}\frac{V\omega^2}{2\pi^2 c^3}d\omega$$

This tells us:

$$u_\omega = \frac{1}{\pi^2 c^3}\frac{E_0\omega^2}{e^{\beta E_0} - 1}$$

To find our u_ω we have to do a brief intermission. One can do a lot of work and show that you expect a spectral energy density of the form:

$$u_\omega(\omega, T) = \omega^3 \chi\left(\frac{\omega}{T}\right)$$

Where χ is some arbitrary function. In our theory E_0 is not dependent on temperature so for our u_ω to take this form we expect $E_0 \propto \omega$. We write this as a formula:

$$E = \hbar\omega$$

In this case $\hbar$ is a proportionality constant called Planck's constant. This gives us a final form of the spectral energy density of:

$$u_\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/kT} - 1}$$

This gives us our spectral energy density for radiation. We can integrate this to recover a famous result.

$$u = \int_0^\infty u_\omega d\omega$$

Letting $x = \hbar\omega/kT$ we can write this integral as:

$$u = \frac{k^4}{\pi^2 c^3 \hbar^3} T^4 \int_0^\infty \frac{x^3}{e^x - 1} dx$$

We can write this more succinctly as:

$$u = aT^4$$

This is the classic result for the energy density of blackbody radiation.