M525: Midterm

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1. Consider the following Bayesian model:

$$\theta \sim \text{Gamma}(A, B)$$
$$w_n|\theta \sim \text{LN}\left(h, \theta\right) n = 1, 2, \ldots, N$$

where A, B, and h are known values.

(i) We first derive the posterior, $p(\theta|w_{1:N})$, of this model.

$$\begin{aligned} p(\theta|w_{1:N}) &\propto p(w_{1:N}|\theta)p(\theta) \\ &= \prod_{n=1}^{N} \left(p(w_n|\theta)\right) p(\theta) \\ &= \theta^{A-1} \exp\left\{-\frac{\theta}{B}\right\} \left(\sqrt{\frac{\theta}{2\pi}}\right)^N \exp\left\{\frac{\theta}{2} \sum_{n=1}^{N} \log\left(\frac{w_n}{h}\right)^2\right\}. \end{aligned}$$

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Letting $L = \sum_{n=1}^{N} \log\left(\frac{w_n}{h}\right)^2$, we find

$$\begin{aligned} p(\theta|w_{1:N}) &\propto \theta^{\left(A+\frac{N}{2}\right)-1} \exp\left\{-\theta\left(\frac{1}{B} - \frac{L}{2}\right)\right\} \\ \implies p(\theta|w_{1:N}) &= \text{Gamma}\left(A + \frac{N}{2}, \left(\frac{1}{B} - \frac{1}{2}\sum_{n=1}^{N} \log\left(\frac{w_n}{h}\right)^2\right)^{-1}\right). \end{aligned}$$

(ii) We now apply our known values, $A = 2.4$, $B = 5$, and $h = 2$, as well as our given $w_{1:N}$ and plot our prior against our posterior. See Figure 1

(iii) We now perform the process of maximum a posteriori estimation to find the "best" value of θ given our $w_{1:N}$. To accomplish this, we simply find the critical points of our posterior, then find the maximizer. We will find the maximizer of a general gamma distribution first, then apply the values from our found posterior.

$$\begin{aligned} \frac{d}{d\theta}\text{Gamma}(\theta; \phi, \psi) &\propto \frac{d}{d\theta}\theta^{\phi-1} \exp\left\{-\frac{\theta}{\psi}\right\} \overset{set}{=} 0 \\ \implies \theta^{\star} &= \psi(\phi - 1). \end{aligned}$$

Applying our paramaters we found in our posterior and our given data, we find $\theta^{\star} = 1.0682$. Note this aligns very well with our plotted posterior distribution.

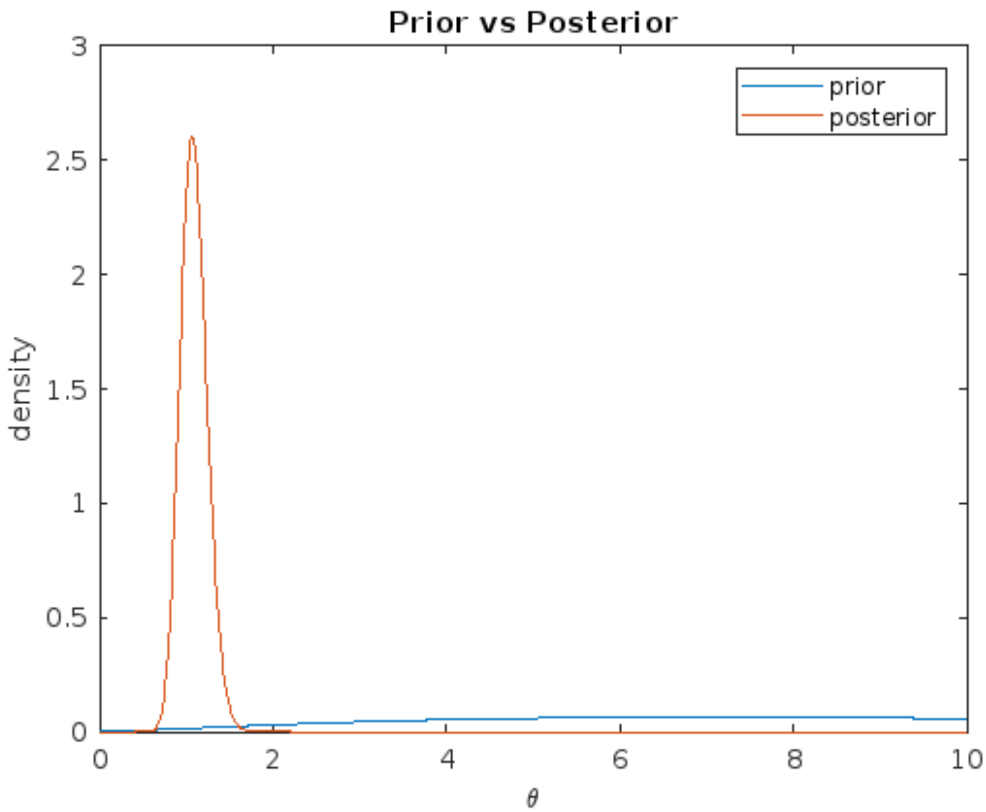

Figure 1: A plot of our against our posterior. Notice the prior is very diffuse. The posterior is much more concentrated and shifted far to the right of our prior.