M525: HW4

20/20

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1. Here we create a Gibbs sampler for the general normal model.

$$\tau \sim \text{Gamma}(\phi, \psi)$$
$$\mu \sim \text{Normal}\left(M, \frac{1}{T}\right)$$
$$w_n|\mu, \tau \sim \text{Normal}\left(\mu, \frac{1}{\tau}\right) \qquad n = 1, \cdots, N$$

where ϕ, ψ, M, and T are known values.

(i) The Gibbs sampling scheme is as follows: first, we choose an value for $\tau^{(1)}$; we choose $\tau^{(1)} = 1$. Next, we use the conditional, $\mu^{(1)} \sim p(\mu|\tau^{(1)}, w_{1:N})$, which we derived in class. Next we use this $\mu^{(1)}$ to generate a new τ based on the conditional $\tau^{(2)} \sim p(\tau|\mu^{(1)}, w_{1:N})$, also derived in class. We then repeat this process J times. We then create a histogram using these values of μ and τ to approximate the posterior $p(\mu, \tau|w_{1:N})$.

(ii) We now implement this scheme using $\phi = 2$, $\psi = 0.5$, $M = 0$, $T = 0.5$, $J = 25000$, and $w_{1:N}$ as given in `gen_normal.mat`. The generated posterior can be seen in Figure 1.

(iii) See Figure 1.

(iv) We now use these generated samples along with the process of Monte–Carlo integration to approximate the expectations of μ and τ.

$$\mathbb{E}\mu = \int_{-\infty}^{\infty} \mu p(\mu|w_{1:N})\, d\mu \approx 1.8605.$$
$$\mathbb{E}\tau = \int_{0}^{\infty} \tau p(\tau|w_{1:N})\, d\tau \approx 1.1738.$$

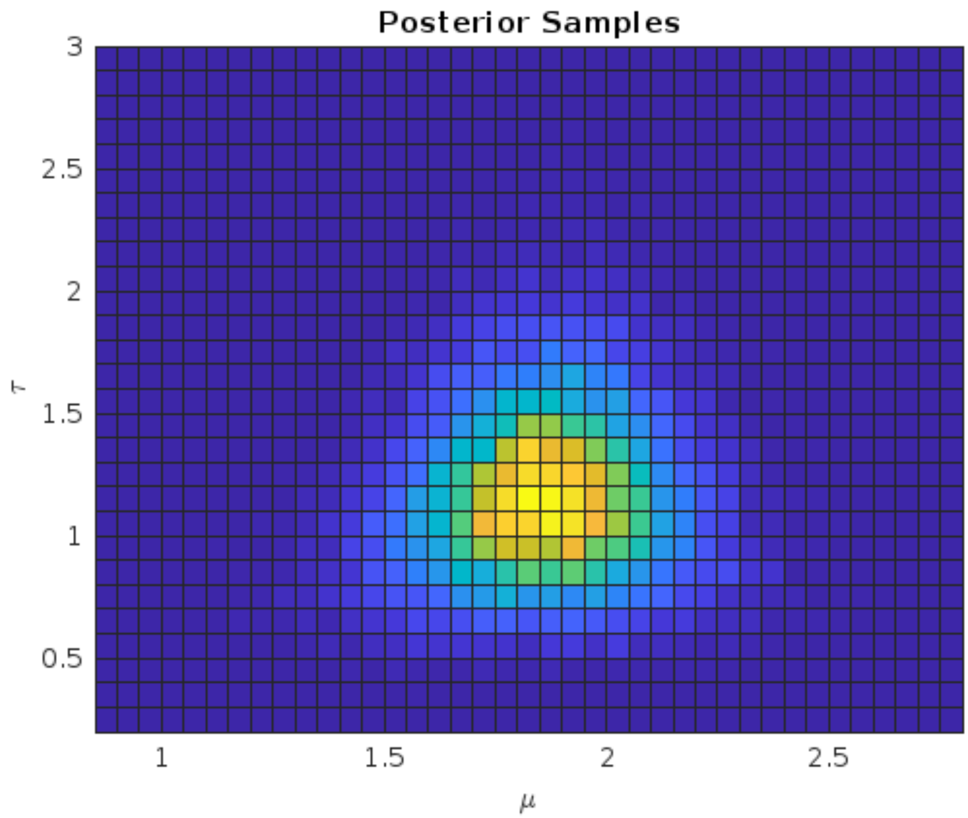

Figure 1: A histogram of 25000 points from the posterior $p(\mu, \tau | w_{1:N})$
. The points are generated using the Gibbs sampling scheme described in (i) and (ii).

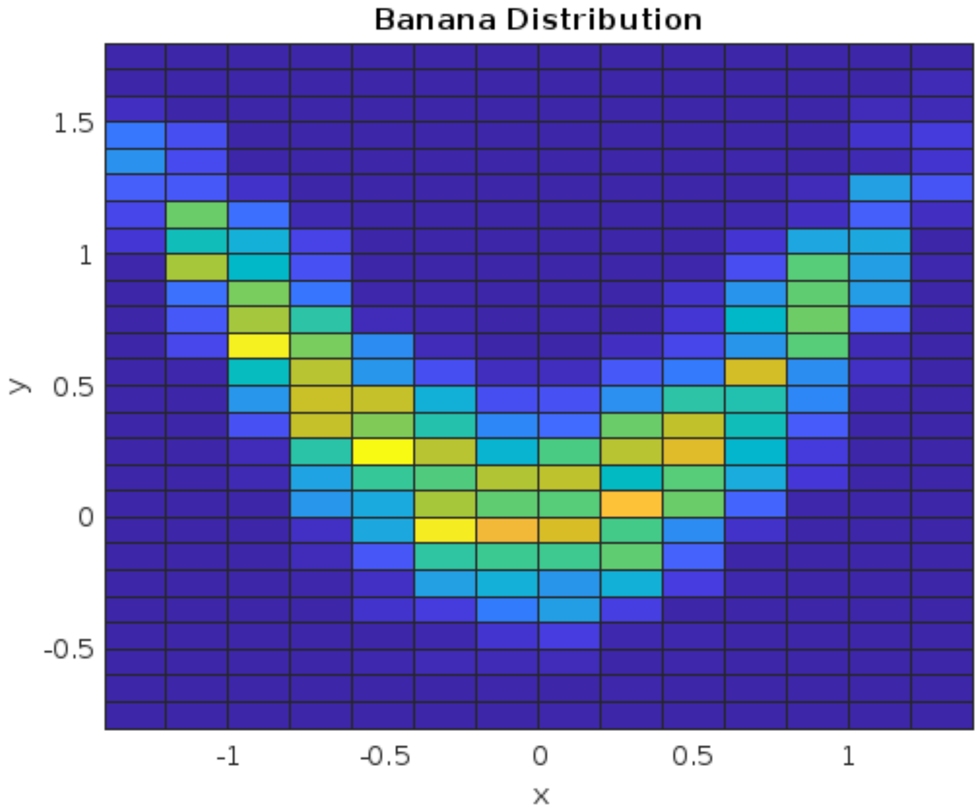

Figure 2: A histogram of $p(x, y)$ generated by the Metropolis–Hastings algorithm described in (i) and (ii). Note the clear "banana" shape.

2. Now we use the Metropolis–Hastings algorithm to sample points from the banana distribution as described up to normalization by

$$p(x, y) \propto \exp\left(-10(x^2 - y)^2 - \left(y - \frac{1}{4}\right)^4\right).$$

   (i) We will generate points $(x^{(j)}, y^{(j)})$ using the proposal distribution

$$q_{x^{\text{old}}, y^{\text{old}}}(x^{\text{prop}}, y^{\text{prop}}) = \text{Normal}(x^{\text{prop}}; x^{\text{old}}, s^2)\text{Normal}(y^{\text{prop}}; y^{\text{old}}, s^2).$$

   First, we choose an initial point $(x^{(1)}, y^{(1)})$. Then we generate $(x^{(prop)}, y^{(prop)}) \sim q(x^{(1)}, y^{(1)})$. Next, we calculate $q(x^{(1)}, y^{(1)})$ and $\alpha = \dfrac{p(x^{\text{prop}}, y^{\text{prop}})}{p(x^{(1)}, y^{(1)})}$. If $q(x^{(1)}, y^{(1)}) < \min\{\alpha, 1\}$, we let $(x^{(2)}, y^{(2)}) = (x^{(prop)}, y^{(prop)})$. Else, we let $(x^{(2)}, y^{(2)}) = (x^{(1)}, y^{(1)})$. We then repeat this process J times.

   (ii) Now we use $(x^{(1)}, y^{(1)}) = (1, 0)$, $s = 0.5$, and $J = 10^4$ to implement the algorithm described in (i) to approximate $p(x, y)$. See Figure 2.

   (iii) See Figure 2.

(iv) Now we use these samples and the Monte–Carlo integral to estimate

$$I = \int_0^\infty \int_0^\infty \sqrt{x^2 + y^2} p(x, y) \, dx \, dy \approx 0.3086.$$