Math 526 – Statistics II, Spring 2023

Final exam

Due: Monday, May 15, by 8:00PM

Problem: The provided dataset contains successive measurements $w_{1:N}$ of a highly corrupted signal with 5 dynamical states $\sigma_{1:5}$. The dataset is accurately represented by the hidden Markov model with hidden states $s_{1:N}$ and Poisson measurements

$$w|s \sim \text{Poisson}\left(\phi_s\right)$$

with state-specific parameters $\phi_{\sigma_1}, \phi_{\sigma_2}\phi_{\sigma_3}\phi_{\sigma_4}, \phi_{\sigma_5}$.

1. Implement the normalized forward filtering algorithm and evaluate the marginal likelihood under the following parameter choices

$$\begin{array}{lllll}
\phi_{\sigma_1} = 8, & \phi_{\sigma_2} = 15, & \phi_{\sigma_3} = 24, & \phi_{\sigma_4} = 31, & \phi_{\sigma_5} = 42 \\
\rho_{\sigma_1} = \frac{1}{5}, & \rho_{\sigma_2} = \frac{1}{5}, & \rho_{\sigma_3} = \frac{1}{5}, & \rho_{\sigma_4} = \frac{1}{5}, & \rho_{\sigma_5} = \frac{1}{5} \\
\pi_{\sigma_1\to\sigma_1} = \frac{25}{26}, & \pi_{\sigma_1\to\sigma_2} = \frac{1}{26}, & \pi_{\sigma_1\to\sigma_3} = 0, & \pi_{\sigma_1\to\sigma_4} = 0, & \pi_{\sigma_1\to\sigma_5} = 0 \\
\pi_{\sigma_2\to\sigma_1} = \frac{1}{27}, & \pi_{\sigma_2\to\sigma_2} = \frac{25}{27}, & \pi_{\sigma_2\to\sigma_3} = \frac{1}{27}, & \pi_{\sigma_2\to\sigma_4} = 0, & \pi_{\sigma_2\to\sigma_5} = 0 \\
\pi_{\sigma_3\to\sigma_1} = 0, & \pi_{\sigma_3\to\sigma_2} = \frac{1}{27}, & \pi_{\sigma_3\to\sigma_3} = \frac{25}{27}, & \pi_{\sigma_3\to\sigma_4} = \frac{1}{27}, & \pi_{\sigma_3\to\sigma_5} = 0 \\
\pi_{\sigma_4\to\sigma_1} = 0, & \pi_{\sigma_4\to\sigma_2} = 0, & \pi_{\sigma_4\to\sigma_3} = \frac{1}{27}, & \pi_{\sigma_4\to\sigma_4} = \frac{25}{27}, & \pi_{\sigma_4\to\sigma_5} = \frac{1}{27} \\
\pi_{\sigma_5\to\sigma_1} = 0, & \pi_{\sigma_5\to\sigma_2} = 0, & \pi_{\sigma_5\to\sigma_3} = 0, & \pi_{\sigma_5\to\sigma_4} = \frac{1}{26}, & \pi_{\sigma_5\to\sigma_5} = \frac{25}{26}
\end{array}$$

   Report your results.

2. Implement the Viterbi algorithm and compute the optimal hidden state sequence under the same parameter choices. Summarize graphically your results.

3. Set up a Bayesian model to estimate all model parameters. These include $\phi_{\sigma_{1:5}}, \tilde{\rho}_{\sigma_{1:5}}, \tilde{\pi}_{\sigma_{1:5}}$.

4. Describe a Markov chain Monte Carlo sampler to sample the model's posterior probability distribution. You do not need to implement your sampler.

Associated data: The dataset shown above is provided in `dyn_counts.mat`.