20/20

M526: HW1

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1. Here we create a Bayesian model employing a Gaussian process to estimate the depth profile along the channel's cross–section.

$$f(\cdot) \sim \text{GP}\left(h(\cdot), C(\cdot,\cdot)\right).$$
$$y_n^{\#}|f(\cdot) \sim \text{Normal}\left(f(x_n^{\#}), v_n^{\#}\right) \qquad n = 1, \cdots, N.$$

We choose

$$h(x) = -10$$
$$C(x, x') = \lambda^2 \exp\left\{-\frac{(x-x')^2}{2l}\right\}$$

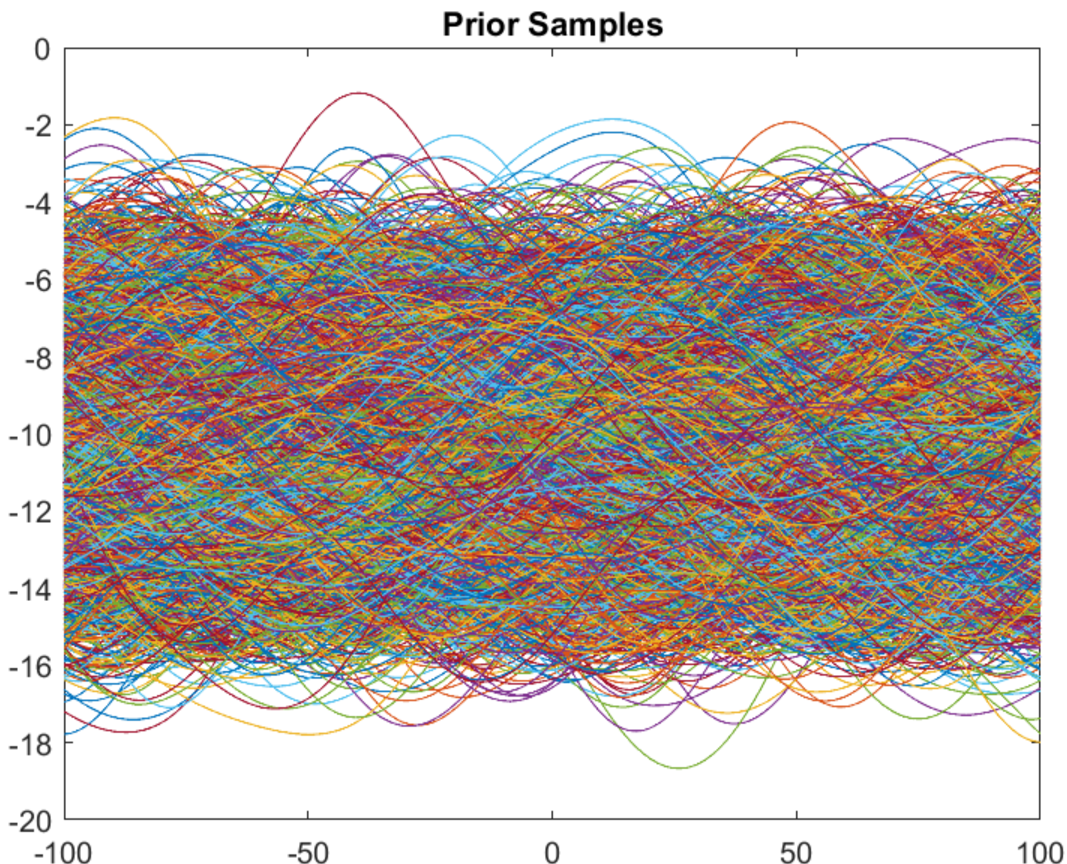
We choose $h(x) = -10$ since, visually, the stream seems to be on average -10ft deep, and we want a weakly informative prior. We pick $\lambda = 2$ to limit the model's ability to have large values, and pick $l = 25$ to ensure smooth functions with low frequency.

2. First, we generate 10000 samples from our prior distribution with the parameters defined above. See Figure 1.

3. Next, we generate 10000 samples from our posterior distribution, derived in class, and using the stream bed data provided to us. See Figure 2.

4. We derive the probability that the the stream is less than 10ft deep at the points $x_A = -20$, $x_B = -10$, $x_C = 0$, $x_D = 10$, and $x_E = 20$. Consider the point x_I. Since we have the joint posterior distribution, we simply marginalize out all of the test points besides x_I and are left with the parameters of a one dimensional normal distribution with mean derived in class, and variance as seen as the diagonal entry of our posterior covariance matrix corresponding to the point x_I. We now simply calculate the cdf of this function evaluated at the point -10 and take the complimentary probability. This give an integral of the following form:

$$p(x_I > -10) = 1 - \int_{-\infty}^{-10} \text{Normal}(f^*(x_I); h(x_I) + G^{*\#}(y^{\#} - h^{\#}), C^{**} - G^{*\#}C^{\#*})\, df^*(x_I).$$



labels?

here it whould be useful to superimpose the datapoints as well

Figure 1: A graph of 10000 samples from our previously defined Gaussian Process prior. Note that the functions tend to be centered about $y = -10$ since $h(x) = -10$. Also the majority of the functions seem to be bounded by -4 and -16, this is set largely by our choice of λ.

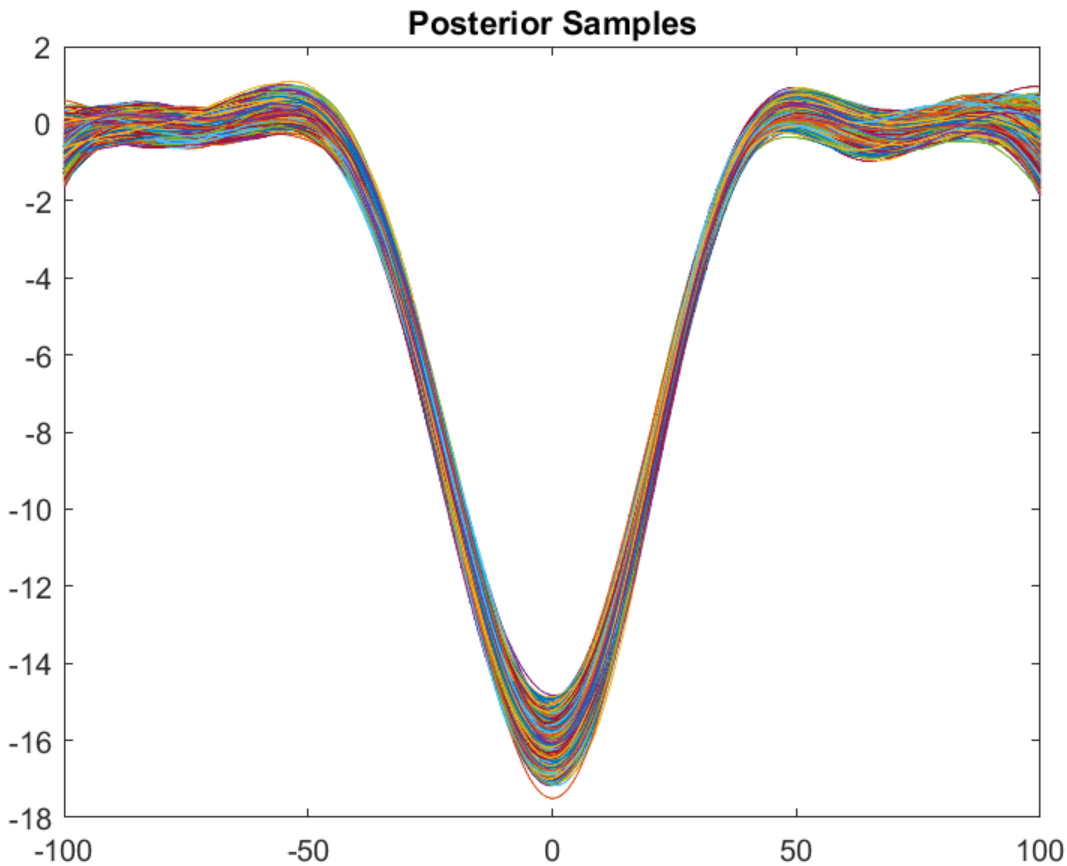


Figure 2: Note the smooth nature of the samples. The variance seems to increase near the edges of our data since that area is more sparse in data. The variance also seems to increase near the bottom of the stream due to the data bring more noisy in that region.

We find that $p(f^*(x_A) > -10) = 0.8648$, $p(f^*(x_B) > -10) = 0$, $p(f^*(x_C) > -10) = 0$, $p(f^*(x_D) > -10) = 0$, and $p(f^*(x_E) > -10) = 0.9999$. Using Monte Carlo integration with the 10000 samples generated in Figure 2 to estimate this, we find $p(f^*(x_A) > -10) \approx 0.8594$, $p(f^*(x_B) > -10) \approx 0$, $p(f^*(x_C) > -10) \approx 0$, $p(f^*(x_D) > -10) \approx 0$, and $p(f^*(x_E) > -10) \approx 0.9999$.

5. Finally, we use the law of total probability that the stream is less than 10ft at each x_I. $p(f^*(x_A) > -10 \& f^*(x_B) > -10 \& f^*(x_C) > -10 \& f^*(x_D) > -10 \& f^*(x_E) > -10) = p(f^*(x_A) > -10)p(f^*(x_B) > -10)p(f^*(x_C) > -10)p(f^*(x_D) > -10)p(f^*(x_E) > -10) = 0$. Using Monte-Carlo integration with the 10000 samples generated in Figure 2, we again find this probability to be 0.