OPTIMIZATION FOR AI

GLOBAL AND MULTI-OBJECTIVE OPTIMIZATION

Luca Manzoni

TEAM OF THE COURSE

Code:

bpplxhk

The slides and material of the lectures will be uploaded on the Teams of the course

EXAM STRUCTURE

- **Project + oral exam**
  - Project should be sent about a week before the oral exam
  - For December and January there will be some fixed dates for the exam…
  - …but after that all exams can be “by appointment”, so take the time that you need

EXAM STRUCTURE

- Kinds of projects:
  - Implement an existing paper and reproduce the results
  - Apply evolutionary algorithms / swarm intelligence algorithms to an existing problem
  - Literature review on a specific topic
- A set of projects will be presented later in the course, but we can discuss personalized project

EXAM STRUCTURE

- Oral exam:
  - First part: presentation (15-20 minutes max) of your project
  - Second part: questions about all topics of the course
- Both parts are essential!

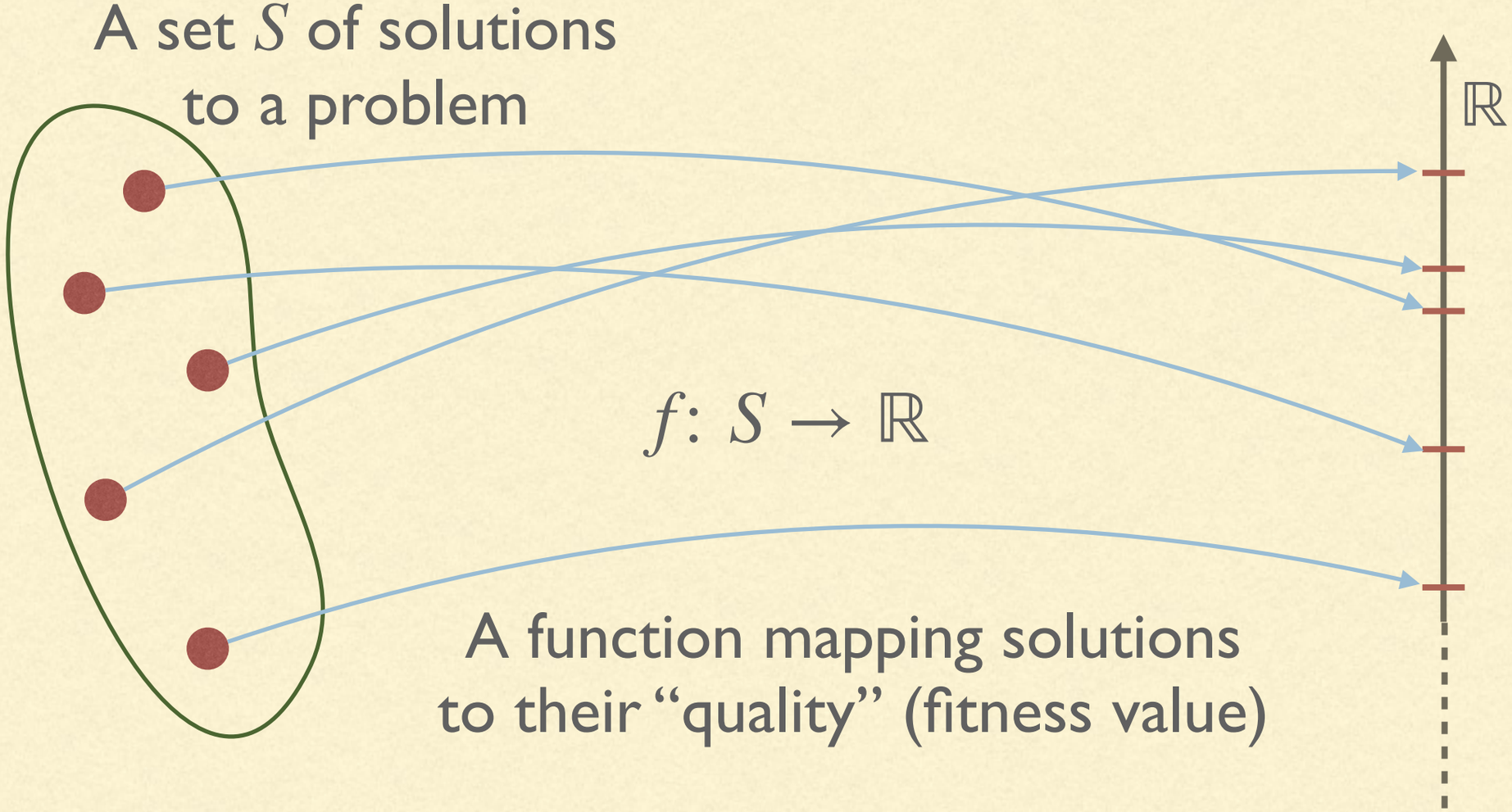
EXAM CHECKLIST

- **Project**
  - Select your project
  - Do your project
  - Fix a date for the oral exam
  - Send the project for evaluation one week before the oral exam
- **Oral Exam**
  - Present your project
  - Answer questions on all the topics of the course

OUTLINE

- What are population-based optimization methods
- Outline of the course
- Reference material
- Genetic Algorithms
- Evolution Strategies

WHAT ARE WE TALKING ABOUT?



We want to find $\text{argmax}_{x \in S} f(x)$ or $\text{argmin}_{x \in S} f(x)$

HOW CAN WE FIND THE OPTIMUM?

TYPICALLY GRADIEN DESCENT IS GOOD ENOUGH

- We might be unable to solve the problem analytically
- S might be too large to perform an exhaustive search
- We might have very few assumption on f, i.e., f is a "black box"
- It might be OK to return "good enough" solutions

A TRIVIAL PROBLEM: ONEMAX

- Let $S = \{0,1\}^n$, hence the size of the space is 2^n

NUMBER

- Let $f(x) = \#$ of ones in x

- We want to maximise f

- Clearly, the optimum is 1^n with fitness value n

APPROACH #1: RANDOM SEARCH

- Select a solution b from S (it is not important how)
- Repeat the following until some termination criteria is met
  - Let x be a solution selected uniformly at random from S
  - If $f(x) \geq f(b)$ substitute b with x
- Return b

APPROACH #1: RANDOM SEARCH

- Even if we avoid sampling the same solution more than once, we will still need to explore a significant fraction of the search space

- Without repetitions, it is like exhaustive search for some choice of the order of enumeration of the solutions

- Generally unfeasible

- In **some** search spaces this is better than other kinds of search…

- …but hopefully this is not something that happens for real problems

APPROACH #2: HILL CLIMBING

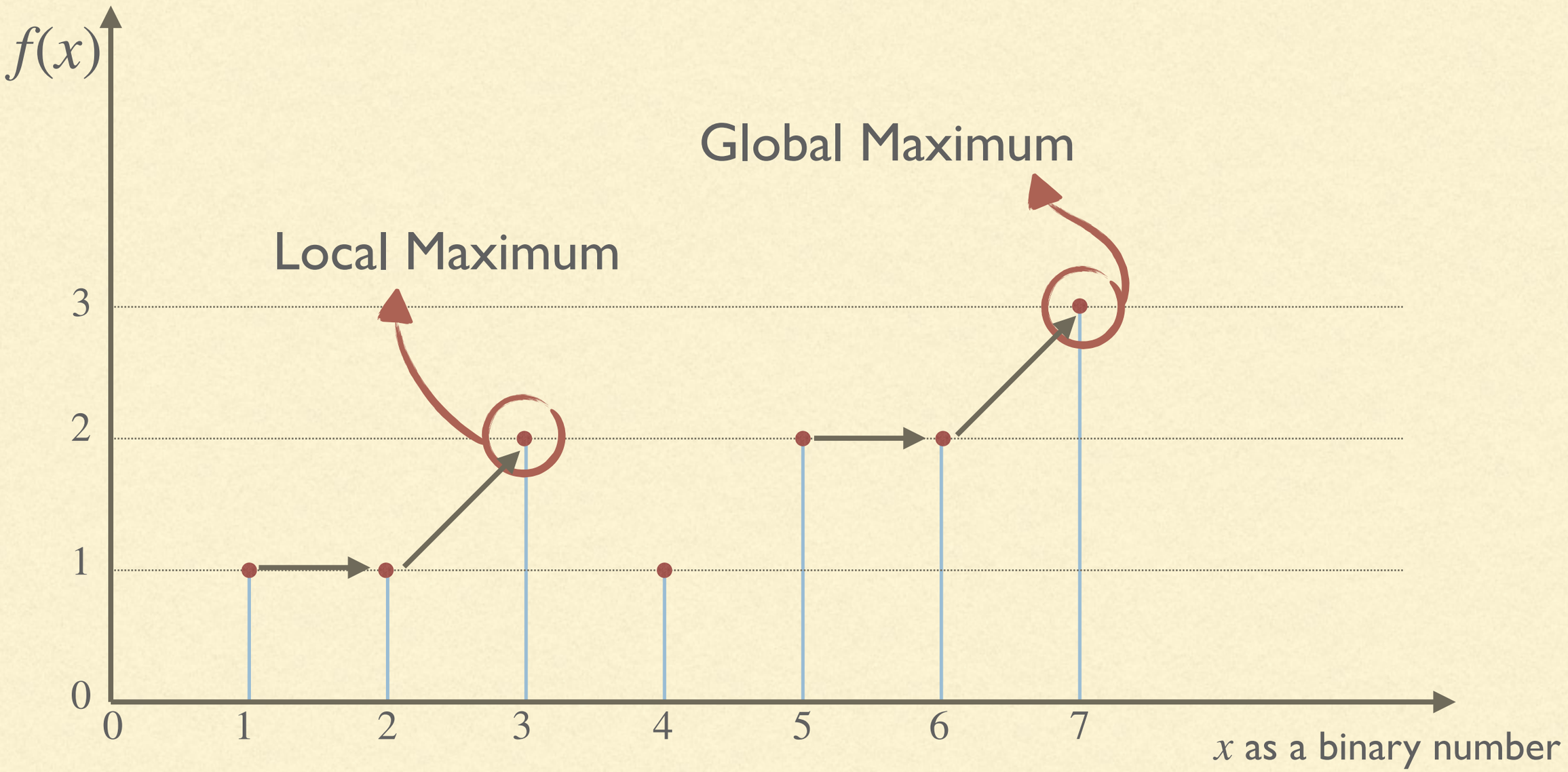
- Let b be an initial solution from S
- Repeat the following until some termination criteria is met
  - Let x be a **neighbor** of b
  - If $f(x) \geq f(b)$ then replace b with x
- Return b

A GRADIENT DESCENT WITHOUT COMPUTING THE EIGHENVALUE

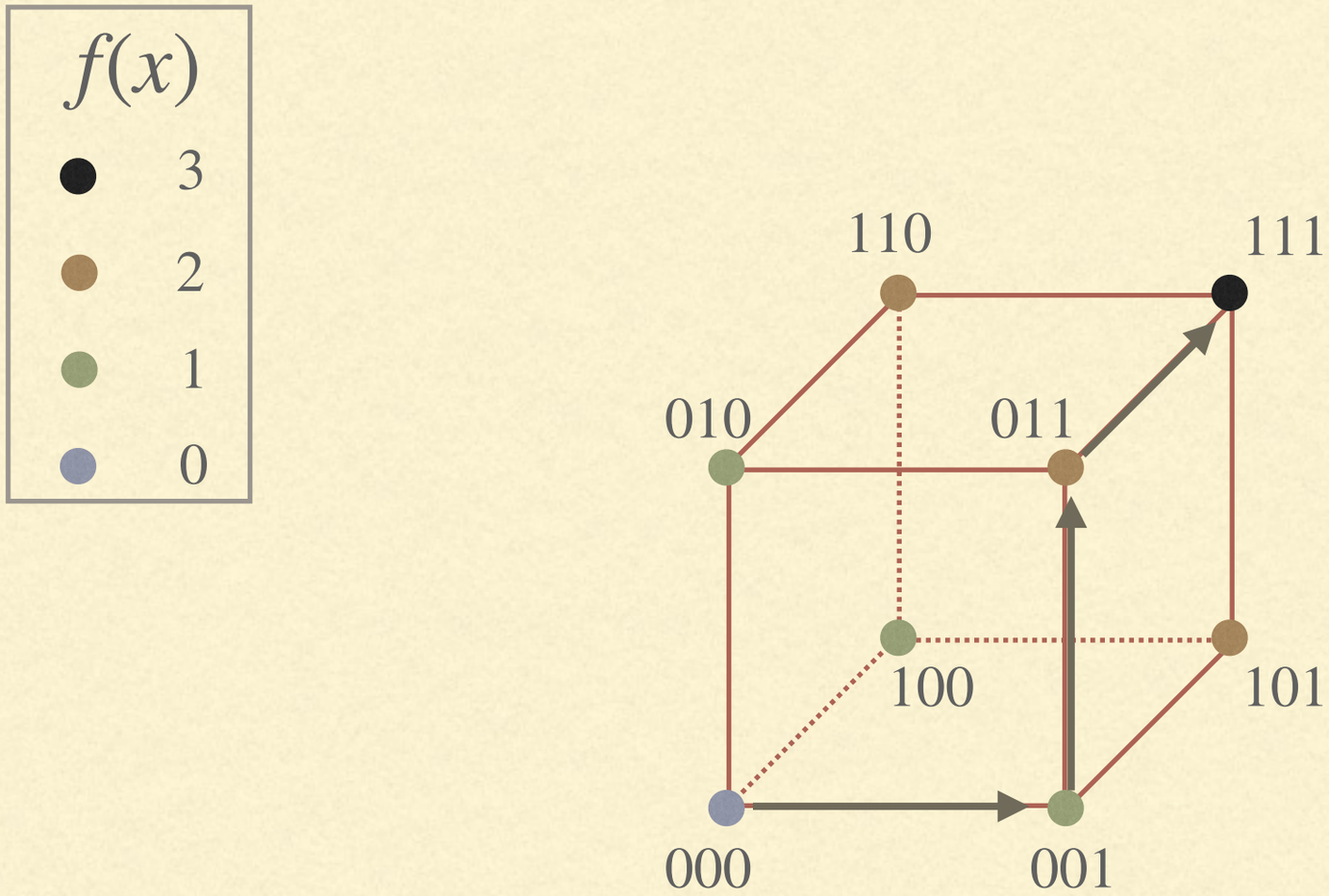
APPROACH #2: HILL CLIMBING

- We are defining a neighborhood structure on S
- The ability to find a global optimum depends on this structure:
  - OneMax with neighborhood of x given by $x + 1$ and $x - 1$
  - OneMax with neighborhood of x given by all binary strings at Hamming distance one.

LOCAL AND GLOBAL MAXIMA



LOCAL AND GLOBAL MAXIMA



Notice that there are no local optima

APPROACH #3: SIMULATED ANNEALING

THE CONCEPT OF SIMULATED ANNEALING IS "BEATING THE IRON WHILE IS HOT"

- Let b be an initial solution from S and T the "temperature"
- Repeat the following until some termination criteria is met
  - Let x be a **neighbor** of b
  - If $f(x) \geq f(b)$ then replace b with x
  - Otherwise replace b with x with probability $e^{\frac{f(x)-f(b)}{T}}$
  - Update T (by decreasing it) according to some schedule
- Return b

IT IS AN ANALOGY OF PYSHICAL SYSTEM. THE HIGHER THE TEMPERATURE (MORE ENERGY) THE HIGHER THE PROBABILITY.

APPROACH #3: SIMULATED ANNEALING

- The idea is to allow the selection of less fit solutions with some probability that depends from the time and the difference in fitness
- This allows to reduce the risk of getting stuck in a local optimum
- The choice of the schedule is important!

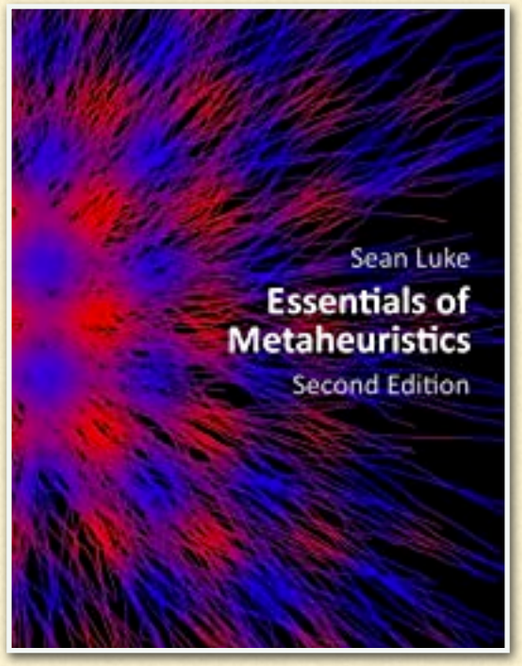
MULTIPLE RESTARTS

- For Hill Climbing and Simulated Annealing we can reduce the risk of getting stuck on a local optimum by repeating the process several times
- Each repetition is independent…
- …can we do better?
- The idea is to use a multiset of solutions instead of working with one solution at a time
- This time the different solutions "interact" in some way

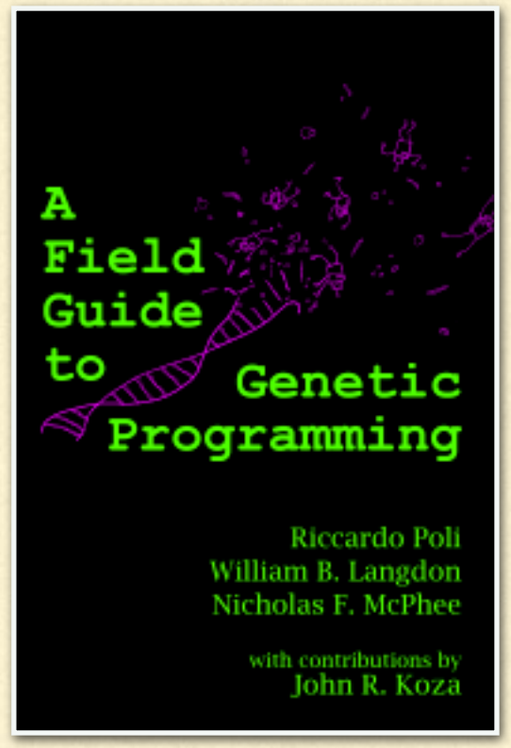
OUTLINE OF THE COURSE

- Genetic Algorithms
- Evolution Strategies
- Genetic Programming
- Particle Swarm Optimization and Ant-Colony Optimization
- Differential Evolution
- Neuroevolution => BASED ON NEURAL NETWORKS
- EDA and CMA-ES
- Parallel implementations
- Multi-objective optimization
- Coevolution
- Policy Optimization
- Theory of Evolutionary Computation

REFERENCE MATERIAL

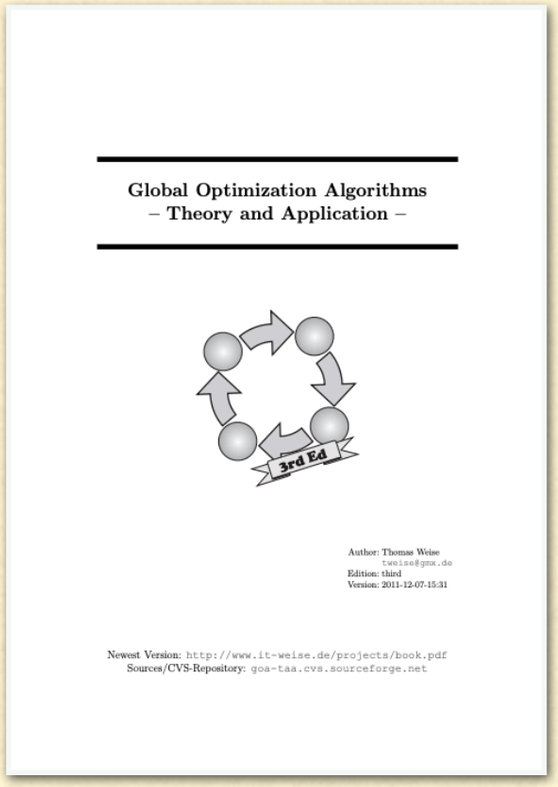


S. Luke
Essentials of Metaheuristics, 2nd Edition
https://cs.gmu.edu/~sean/book/metaheuristics/



R. Poli, W. R. Langdon, N. F. McPhee
A Field Guide to Genetic Programming
http://www.gp-field-guide.org.uk

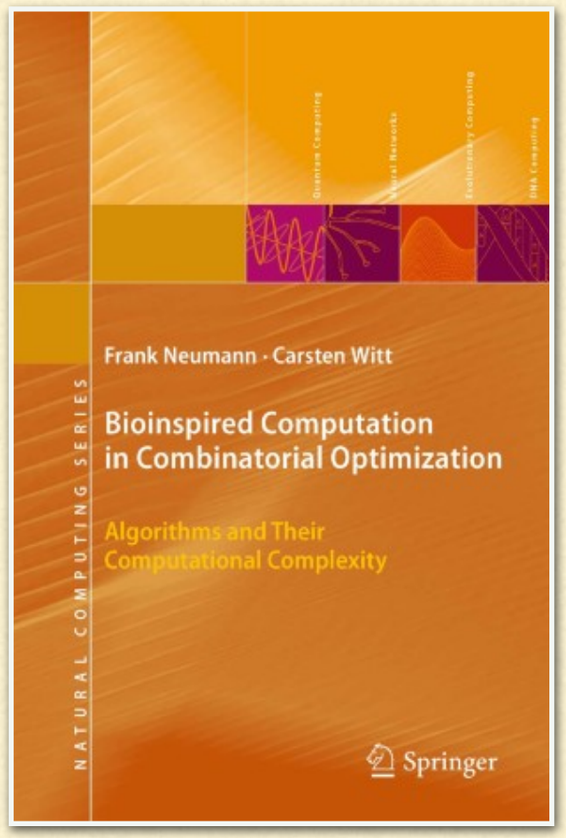
REFERENCE MATERIAL



T. Weise
Global Optimization Algorithms – Theory and Application, 3rd Edition
(Search for it online)

F. Neumann and C. Witt
Bioinspired Computation in Combinatorial Optimization
Algorithms and Their Computational Complexity

http://www.bioinspiredcomputation.com/self-archived-bookNeumannWitt.pdf



AN ITERATIVE WAY OF IMPROVING A POPULATION OF SAMPLE

GENETIC ALGORITHMS

GENETIC ALGORITHMS

- THE FIRT THOUGHT WAS BY VON NHOIMAN

- Invented by John Holland in the Seventies:
  John Holland, *Adaptation in Natural and Artificial Systems*, University of Michigan Press, 1975

- Inspired by the Darwinian theory of evolution

SURVIVE THROW
AN ENVIROMENT
HOW? EVOLVING AND
ADAPTING

- A multiset of solutions (called *population*, each solution is an *individual*) is iteratively evolved, with each iteration consisting of
  - Selection
  - Crossover and mutation

GENETIC ALGORITHMS

- Solutions are usually represented as binary strings of fixed length
- This is not a requirement, as we will see in one of the next lectures
- We must now distinguish between the **genotype** and the **phenotype** of a solution, since some operators works on the genotype and some on the phenotype

PHENOTYPE AND GENOTYPE

IS THE DNA

- The **genotype** is how a solution is represented. (OR ENCODED)
  E.g., a string of bits.

A GENOTYPE IS A SEQUENCE OF FLOATING POINT NUMBER OR MATRIX OR GRAPH ENCODED VIA MATRIX. IS HOW A SOLUTION IS RAPPRESENTED

  - Crossover and mutation works on the genotype of a solution
    THEY NEED TO COMBINE OR CHANGE, SO THEY HAVE TO WORK ON THE SOLUTION

- The **phenotype** is the actual solution. E.g., the number of ones, the number represented by a string of bits, etc.

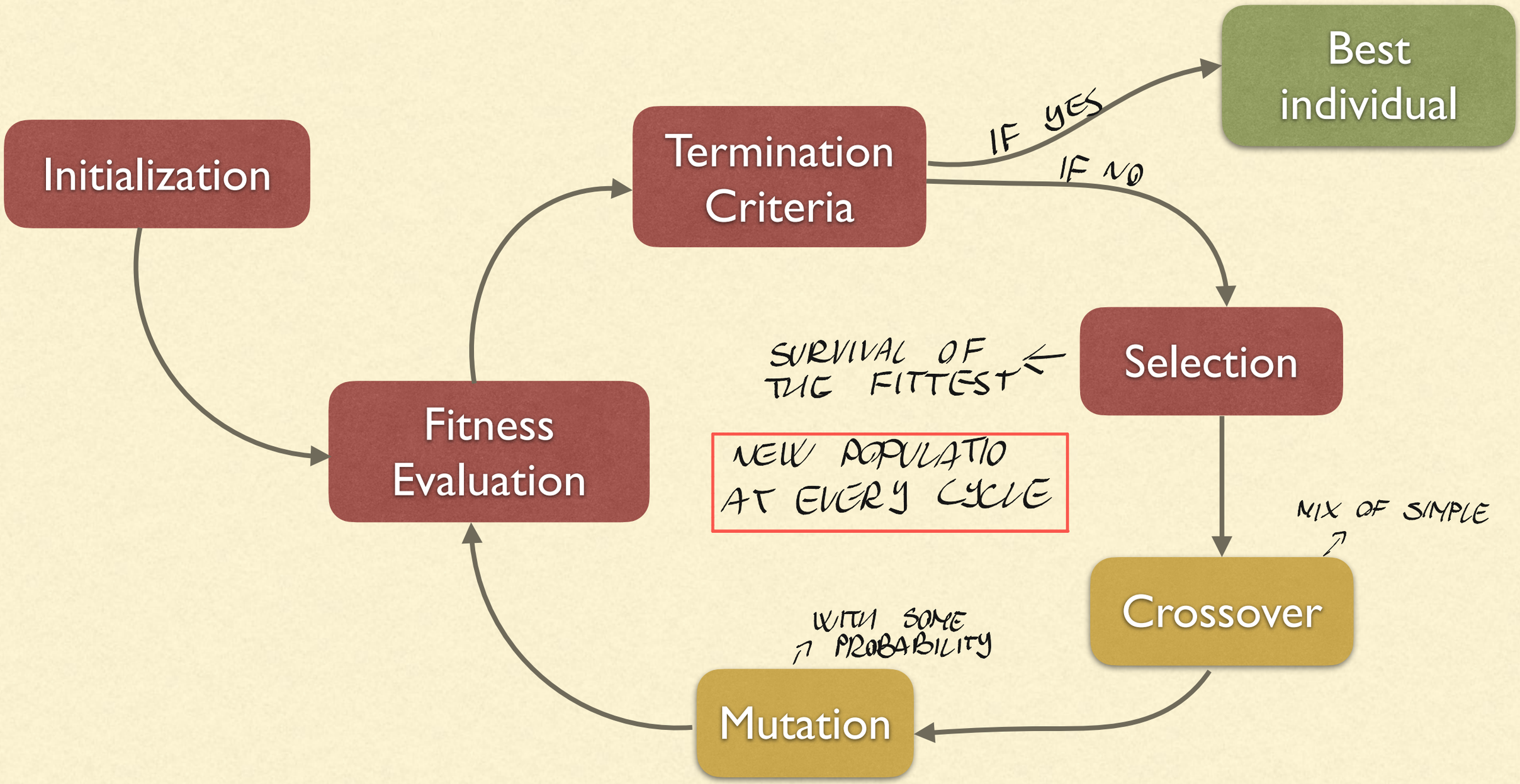
THE PHENOTYPE DON'T CHANGE AND YOU DON'T CONSIDER THE SOLUTION

YOU CHANGE THE RAPPRESENTATION, YOU CHANGE THE PHENOTYPE
THE GENOTYPE IS NOT BOTHERED OF THE CHANGE

  - The fitness computes how good the phenotype of a solution is, without considering the genotype

  - Selection operates only on the phenotype ignoring the genotype

EVOLUTION CYCLE



INITIALIZATION

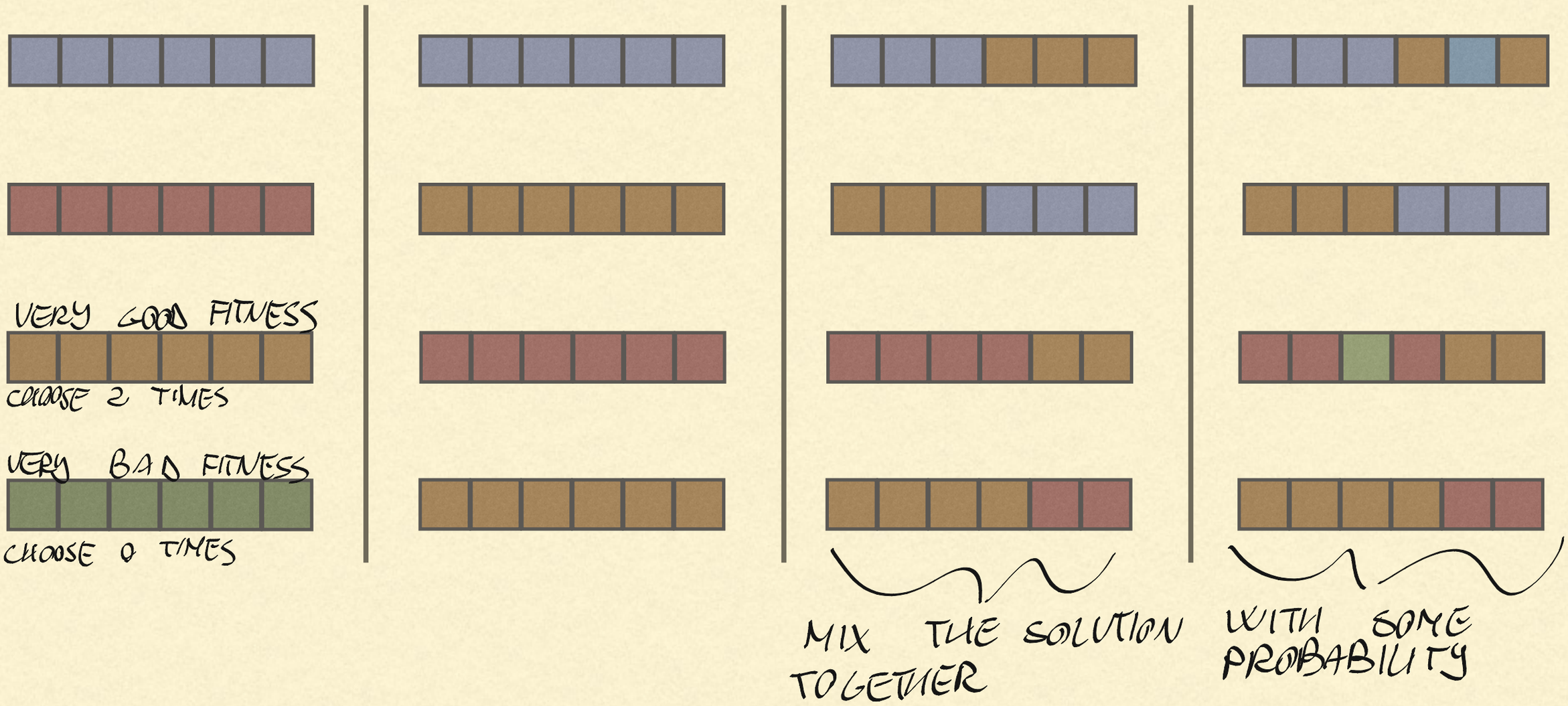
IT'S LESS IMPORTANT FOR GA CAUSE WE USUALLY USE A NORMALL DISTRIBUTION UNLESS WE KNOW SOMETHING ABOUT THE DOMAIN OF THE SPACE.

ONE OF THE POSSIBILITY IS CALLED CEADING:

IF YOU HAVE GOOD SOLUTIONS YOU CAN TRY TO GENERATE PART OF THE POPULATION AND ADD THE GOOD SOLUTIONS AS BOOST THE MISSION OF SEARCH.
THE PROBLEM IS THAT YOU RISK TO FIND SMALL VARIANTS OF THE SOLUTION THAT YOU ALREADY HAVE.
THEY ARE GOOD AN SO TENS TO DOMINATE THE OTHERS.

// "IS LIKE PUTING AN ANIMAL IN AN EVIROMENT WITHOUT A PREDATOR THAT HUNT IT"

A GA GENERATION



GA PARAMETERS

- Population size N ~100 → ~1000 THE POPULATION HAVE TO BE VARIAGATED IF TOO SIMILAR THEY ARE GONNA DO THE SAME THINGS
- Type of selection
- Type of crossover
- Crossover probability p_{cross}, usually near one
- Type of mutation
- Mutation probability p_{mut}, usually small
- Elitism or others additional modifications

SELECTION METHODS

IT DEPENDS ALL ON THE POPULATION

- There are multiple ways to decide which individuals are selected:
  - Roulette wheel selection
  - Ranked selection
  - Tournament selection

ROULETTE WHEEL SELECTION

The probability of an individual to be selected is proportional to its fitness

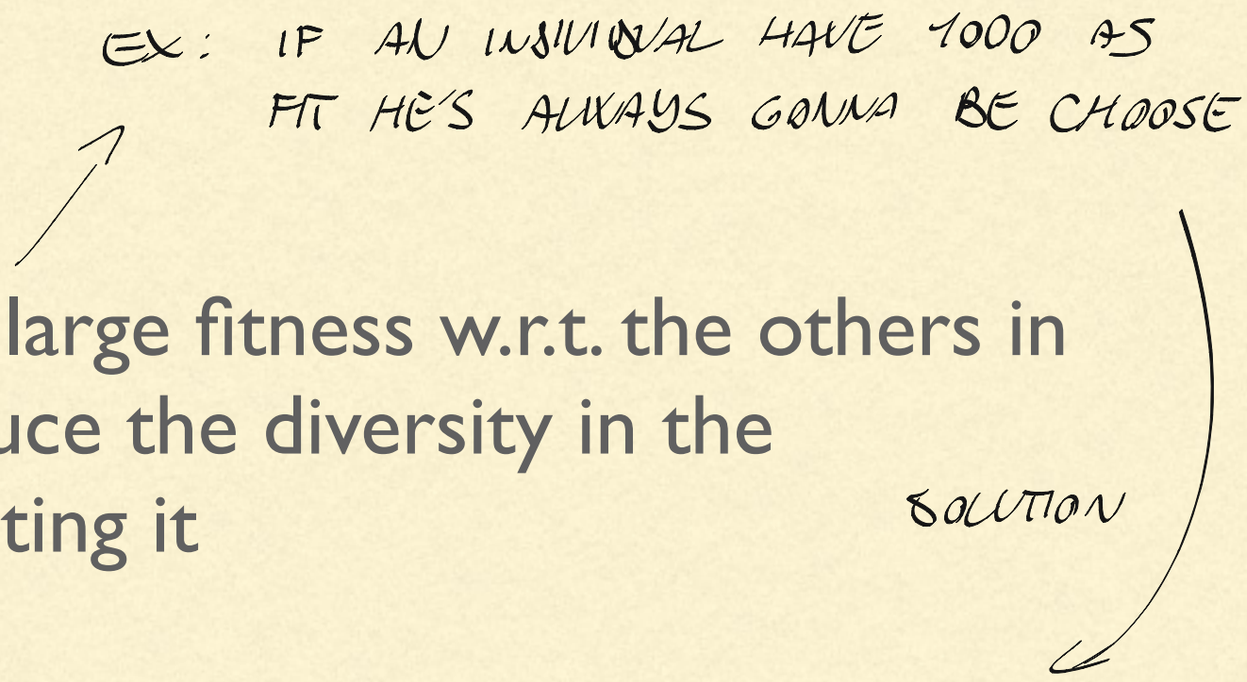
FUNCTIO TO MAXMISE

$$p_{x,P} = \frac{f(x)}{\sum_{y \in P} f(y)}$$

Probability of x getting selected given that the current population is P.

ROULETTE WHEEL SELECTION

- **Pro**: easy to implement



- **Cons**: if an individual has a very large fitness w.r.t. the others in the population then we may reduce the diversity in the population by continuously selecting it

- We may want something that depends on the *ranking* of the fitnesses, not on their raw value

RANKED SELECTION

- We compute the fitness of all individuals in the population
- The individuals are ranked w.r.t. their fitness values
- Each rank has a fixed probability of being selected
  - E.g., the individual with the i^{th} best fitness will be selected with probability $\frac{N-i+1}{r_{\text{sum}}}$ where r_{sum} is a normalisation factor

WE LIKE NORMALIZED VALUES

- WE STILL NEED A FUNCTION THAT CONVERT RANK TO PROBABILITY
  WE NEED TO BUILD A FAMILY OF FUNCTIONS

TOURNAMENT SELECTION

A BATTLE ROYAL OF SOLUTIONS WHERE THE FITTEST SURVIVE

- Tournament selection is the most used kind of selection in GA
- Let $t \in \mathbb{N}^+$ be the *tournament size*
- Extract t individuals from the current population (with reinsertion). THANKS TO REINSERTION WORT INDIVIDUAL ARE ABLE TO SURVIVE.
- Select the one with the best fitness

TOURNAMENT SELECTION

- Has the advantage of providing a "ranked" selection without specifying the function to assign the probabilities

- The "selection pressure" (how difficult it is for a non-fit individual to be selected) can easily be tuned via the tournament size

  MORE COMPETITION

  - Large tournament size → high selection pressure

  LESS COMPETITION

  - Low tournament size → low selection pressure

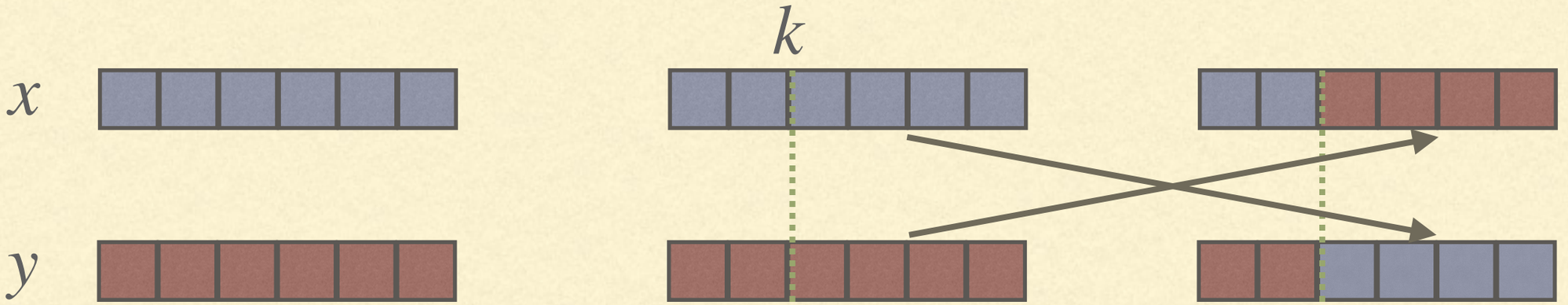
THE TOURNAMENT SIZE IS AN HYPERPARAMETER THAT HAVE TO BE TUNED. IT DEPENDS A LOT ON THE SIZE OF THE POPULATION. SOMETIMES IT CHANGE DURING THE EXECUTION OF THE CYCLE

CROSSOVERS

MIX SOLUTIONS: · GOOD ONE , VERY GOOD ONE > POTENTIALLY THE OPTIMUM ONE

- Some common crossovers:
  - One-point crossover
  - m-points crossover
  - Uniform crossover

ONE-POINT CROSSOVER



- Given the two parent $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$
- A crossover point $k \in \{1, \ldots, n\}$ is selected uniformly at random

WE ARE NOT INTRESTED IN THE FITEST

- The two offsprings are $x_1 \cdots x_k y_{k+1} \cdots y_n$ and $y_1 \cdots y_k x_{k+1} \cdots x_n$

m-POINTS CROSSOVER

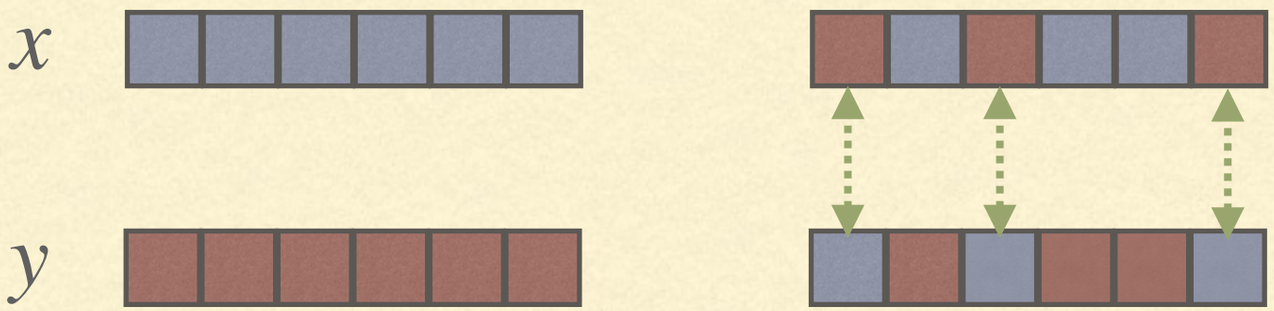
- A direct generalization of one-point crossover where more than one point is selected

- $k_1 \ldots k_m \in \{1, \ldots, n\}$ crossover points are selected (with $k_i \leq k_{i+1}$ for all $i \in \{1, \ldots, m-1\}$)

- The elements of the two parents are swapped between each pair of consecutive crossover points

UNIFORM CROSSOVER

MOST INTERSTING: · MOST RANDOMIC
· BEST PERFORMING



On average half of the bits will be exchanged

- Given the two parent $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$
- For each $i \in \{1, \ldots, n\}$ with probability $1/2$ the i^{th} element in the first (resp., second) offspring will be x_i (resp., y_i) and otherwise y_i (resp., x_i)

BIT-FLIP MUTATION

x 

- Given an individual $x = x_1 x_2 \cdots x_n$
- For each $i \in \{1, \ldots, n\}$ flip the bit x_i with probability p_{mut}
- Usually $p_{\text{mut}} = \frac{1}{n}$ to mutate (on average) one bit per individual

CROSSOVER AND MUTATION

MIXING BIT

FLIPPING BIT

⚠ THE BIT IS JUST AN EXMPLE. IT COULD BE ANYTHING ELSE

- Crossover is **not** "global" mutation
- That is, given two Boolean vectors x and y it is not possible to obtain every possible vector from them. Why?
- If $x_i = y_i = 1$ then the result of crossover will never have $x_i = 0$
- This will be important for the definition of the topological properties of crossover and mutation

EXPLOITATIVE VARIANTS

- **Elitism**. A certain fraction of the individuals with the best fitness are preserved in the next generation unless better solutions are found
  - Keep only the best individual found so far
  - Keep the top k individuals
  - Keep the top $p\,\%$ of individual
- Elitism changes the selection pressure: new individuals have better "competitors" in the next selection phase

STEADY STATE GA

EX: WE REPLACE THE 10% OF THE POPULATION

- Instead of creating a completely new population at each iteration we only replace some of the individuals in the existing population

↳ BY OTHER INDIVIDUALS. WE ARE NOT REMOVING

- Which individuals to replace?
  - The worst ones
  - A random selection

EACH INDIVIDUA HAVE A CHANCE OF REMAINING IN THE POPULATION

HYBRID GA

LOCAL SEARCH + GA

WE MIX THE CONCEPT OF SEARCHING THE BESTS INDIVIDUAL WITH EXPLORING THE ENVIRONMENT FOR THE BEST SOLUTION.

... -> CROSS OVER -> MUTATION -> LOCAL SEARCH -> NEW GEN ->...

- After each generation all individuals are improved with a local search algorithm (e.g., hill climbing)

IT'S LIKE THE POPULATION "TRAIN" A LITTLE BIT

- Called "Memetic Algorithms", better names are "Lamarckian algorithms" or "Baldwin Effect Algorithms"

- Parameters:
  - WICH LOCAL SEARCH TO USE
  - How frequently the local search is performed
  - How many iterations of local search are performed each time

IT APPLY WELL IN NEURALNETWORK: I DON'T WANT TO OPTIMIZE WITH GRADIENT DESCENT, SO WE COMBINE THE TWO

REPRESENTATION

- **Beyond Binary**. It is possible to generalize GA to use symbols from a finite set/alphabet Σ instead of using only $\{0,1\}$
  - THE SELECTION DON'T CARE ABOUT THE TYPE. THE CYCLE IS THE SAME.
- The only difference is that mutation will have to select (usually uniformly at random) between $|\Sigma| - 1$ symbols instead of simply flipping a bit

$\Sigma = \{A, B, C, D, \ldots, Z\}$

SAMPLE = A F G H 1 0 Z $\overset{\text{MUTATION}}{\Rightarrow}$ B F G 1 1 P Z

- If there is an ordering between the elements of Σ (e.g., $\Sigma = \{0,1,2,3\}$) then mutation can be changed to be, for example "add or subtract one".

SPECIAL REPRESENTATIONS FOR GENETIC ALGORITHMS

REAL-VALUED GA

- Until now we have seen binary valued (or integer valued) GA
- We can represent each floating point numbers as 32/64 binary genes…

ONE SMALL PROBLEM:

- …but this means that different bits have different impact on the encoded number

MAYBE YOU ARE CHANGING A LESS IMPORTANT BIT LIKE 7037.84137 AND FLIP THE LAST BIT 7 TO 8, OR I MAKE AN ERROR AND MAKE A NaN

- If each gene is a floating point value then mutation and crossover should be adapted

WHAT WE NEED? CHANGE THE FITNESS VALUE

SELECTION (PHENOTYPE) DON'T CARE, BUT MUTATION DOES (GENOTYPES)

CROSSOVER

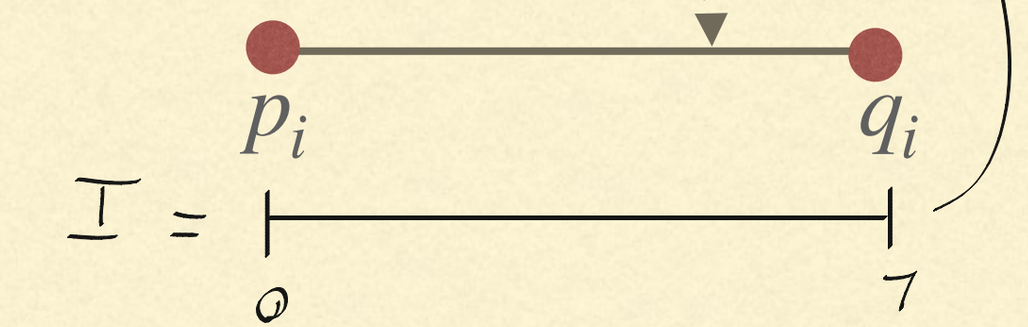
$p_1 = 3$
$q_1 = 5$
$\alpha_1 = 0.5$

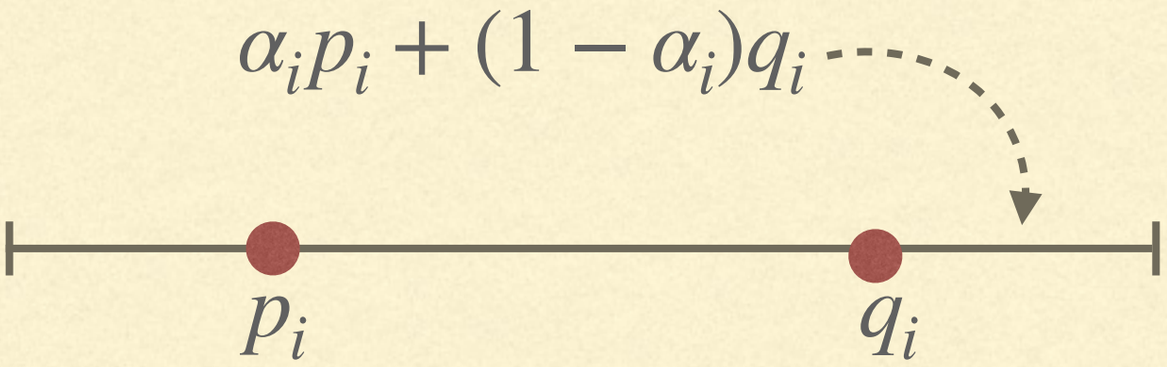
$> = 0.5(3) + (1-0.5)5$

$= 1.5 + 2.5 = 4$

$\alpha_i p_i + (1 - \alpha_i) q_i$

IT MOVES THE WEIGHT TO LEFT OR RIGHT





Intermediate recombination

$$\alpha_i \leftarrow \text{random}(0,1)$$

JUST WEIGHT THE VALUES WITH SOME PROBABILITY

$p_1 = 3$
$q_1 = 5$
$\alpha_1 = 2$

$> = 2(3) + (1-2)5$

$= 6 + 5 = 11$

Line recombination

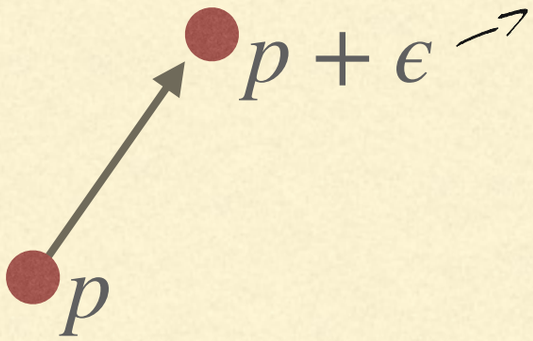
$$\alpha_i \leftarrow \text{random}(-k, 1+k)$$

I ALLOWE THE NUMBER TO MOVE OUTSIDE THE TWO INDIVIDUALS

MUTATION

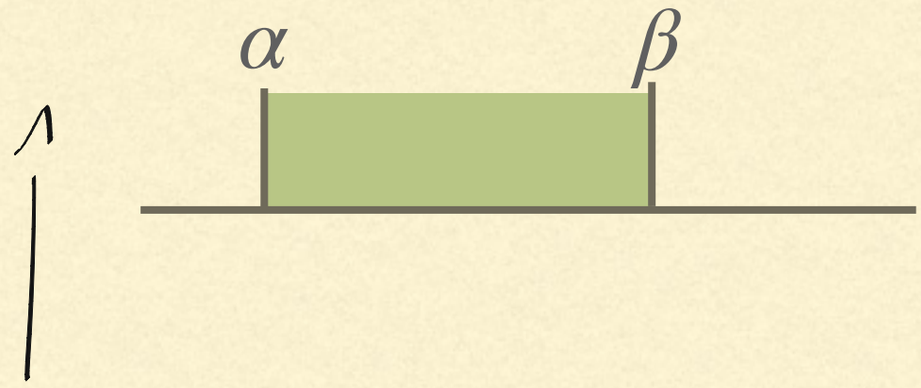
INSTEAD OF BIT FLIP YOU ADD A (SMALL) PERTURBATION

Add a small value
to each coordinate
of the individual

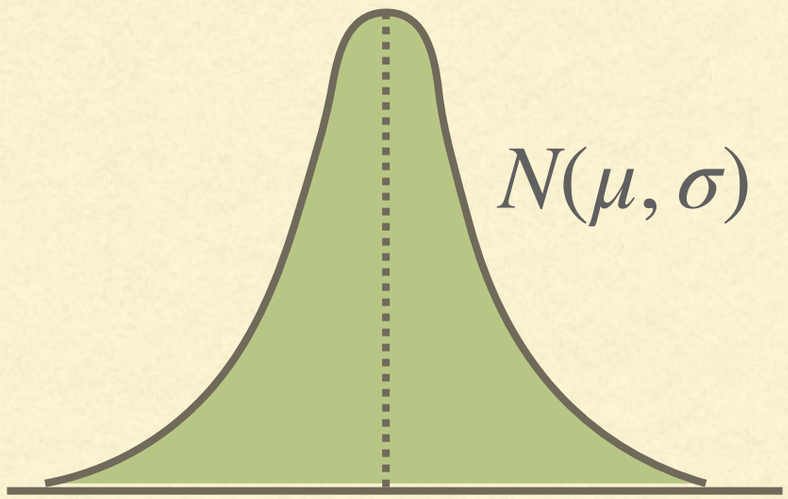


GIVEN BY OR →

Uniform between two values



→ Gaussian



POTENTIALLY YOU CAN MIX CONTINUE AND DESCRET.
IT DOESN'T AFFECT SELECTION,
BUT MUTATION NEED A PERTURBATION ADAPTED TO IF IS DISCRETE AN CONTINUOS.

CYCLE AND PMX CROSSOVERS

- Sometimes we need additional constraints in the representation of an individual by GA

- One usual constraint is that each individual must be a permutation of the numbers from 1 to n

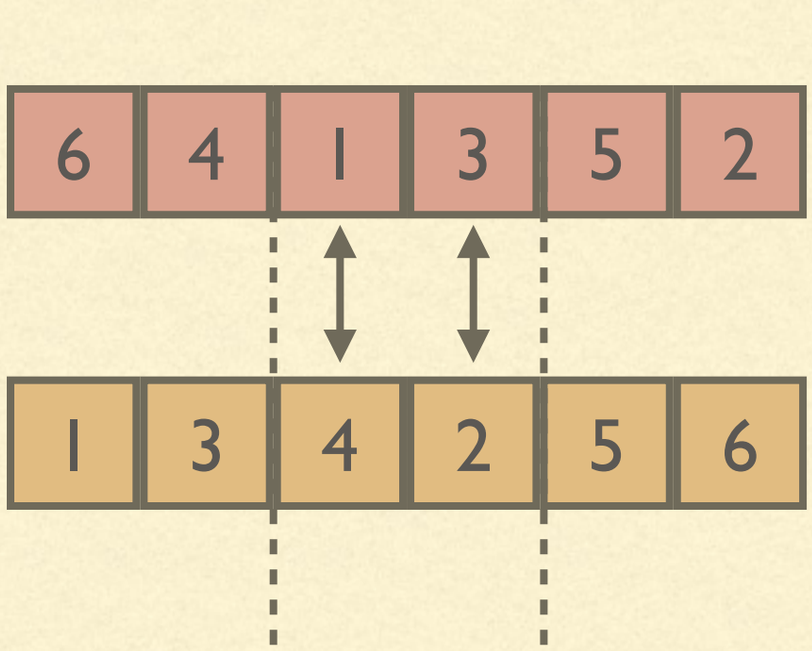
- Mutation can be performed by swapping two positions

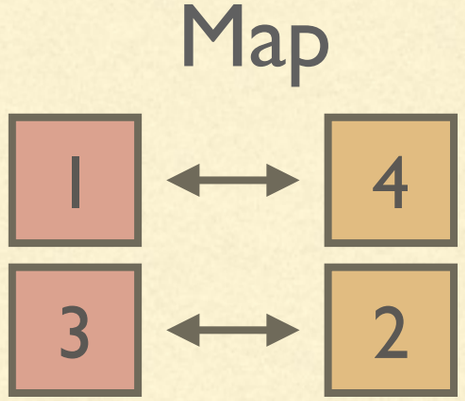
- Traditional crossover usually do not respect the constraints

HOW DO YOU MAINTAINE? REPEAT CROSSOVER UNTIL IT RESPECT THE CONSTRAINTS BUT IT'S INEFFICENT OR ASSIGN AN EXTREMLY BAD FITNESS THAT NOT FIT CONSTRAINTS BUT IT'S STILL INEFFICENT

PARTIALLY MAPPED CROSSOVER (PMX)

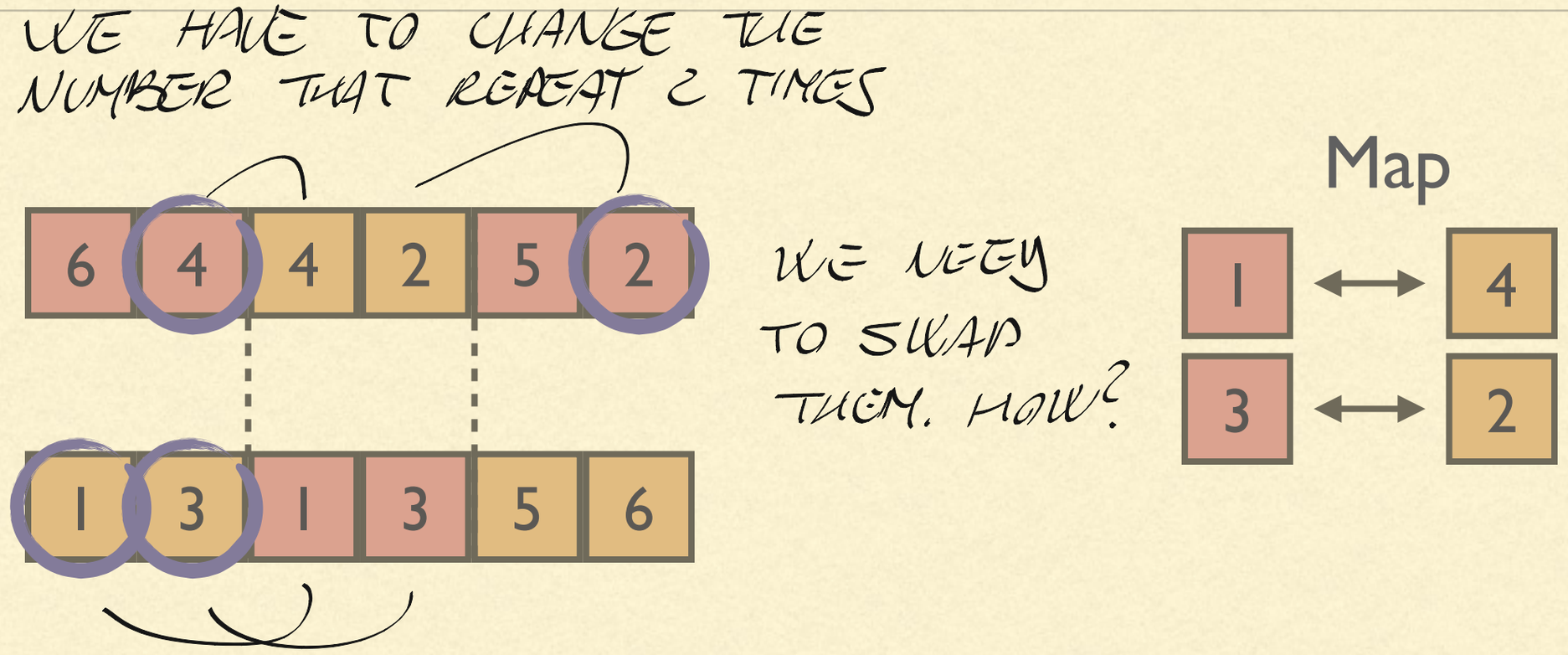
A 2 POINTS CROSSOVER
AND THEN A CORRECTION PROCEDURE





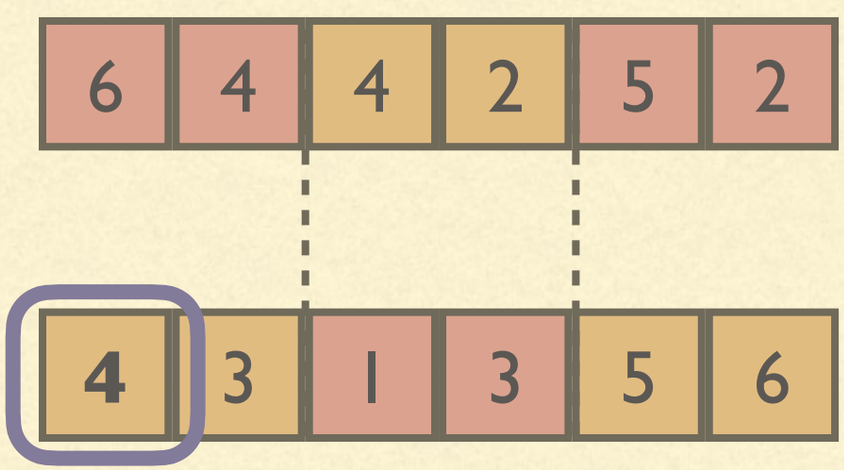
We select two crossover point and we build a map

PARTIALLY MAPPED CROSSOVER (PMX)

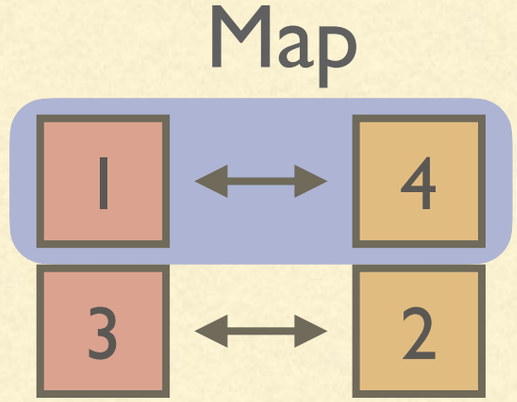


We perform the exchange but the offspring are not valid

PARTIALLY MAPPED CROSSOVER (PMX)

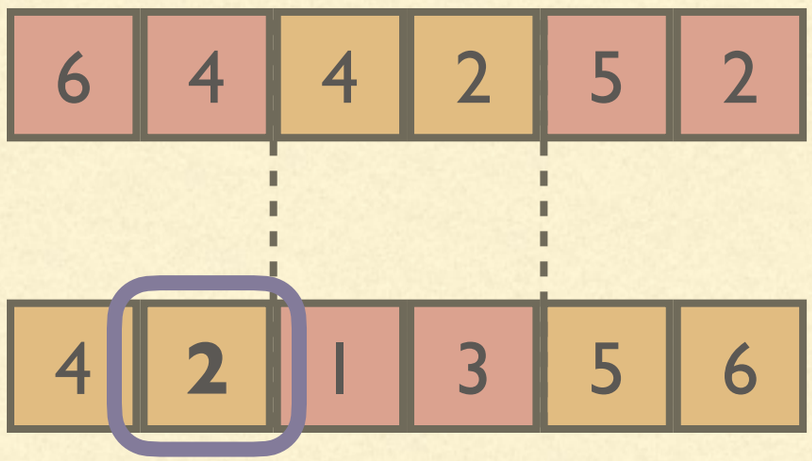


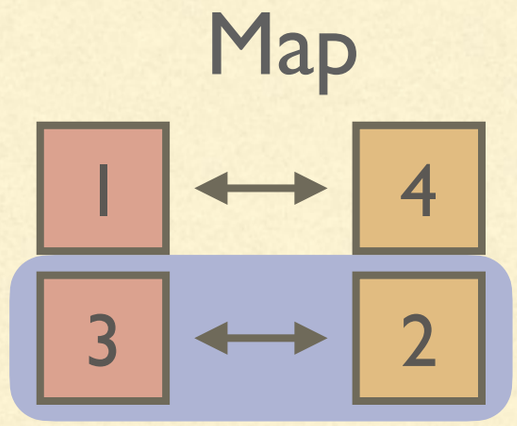
WE JUST USE THE VALUE



We iterate the map until we have resolved the conflicts

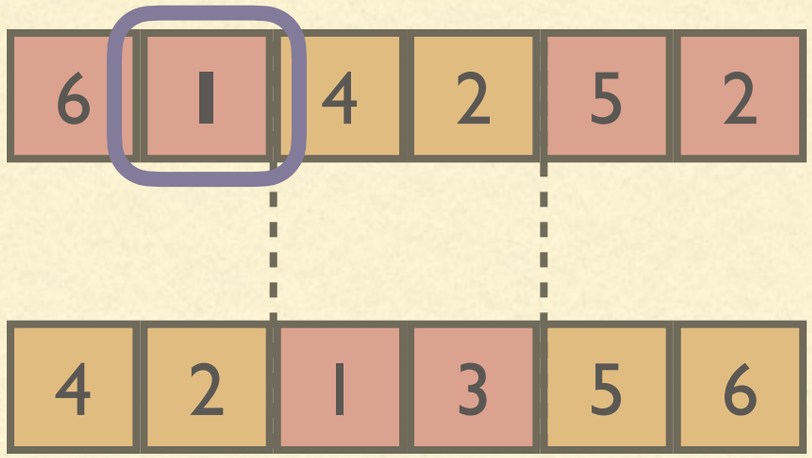
PARTIALLY MAPPED CROSSOVER (PMX)

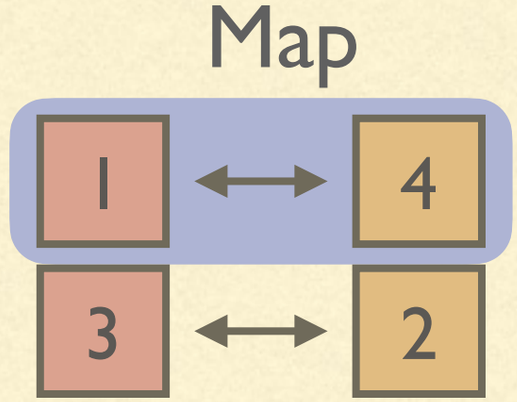




We iterate the map until we have resolved the conflicts

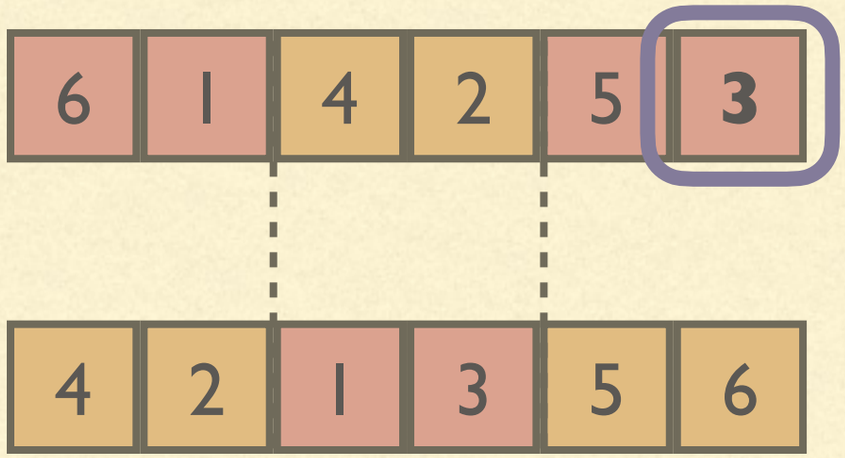
PARTIALLY MAPPED CROSSOVER (PMX)

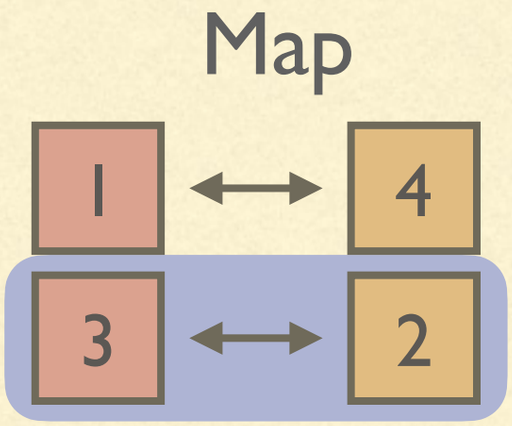




We iterate the map until we have resolved the conflicts

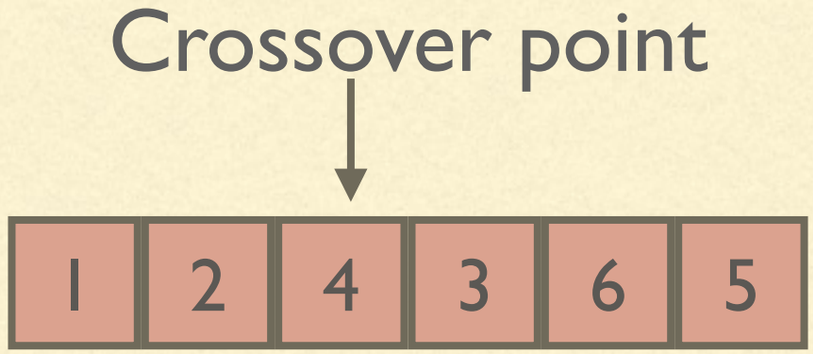
PARTIALLY MAPPED CROSSOVER (PMX)





We iterate the map until we have resolved the conflicts

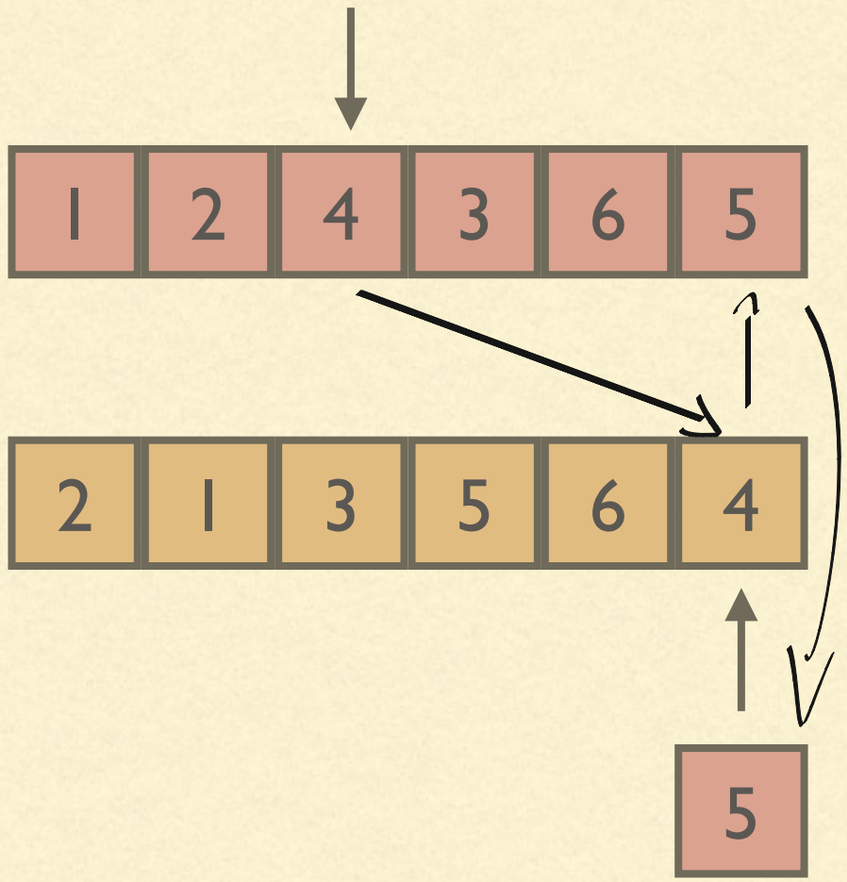
CYCLE CROSSOVER



SO, WE HAVE 2 TYPE OF PERMUTATION:
THE FIRST BETWEEN INDIVIDUALS
AND THE SECON BETWEEN NUMBERS INSIDE
THE SINGLE INDIVIDUAL

We select a single starting point
and we search che same value in the second parent

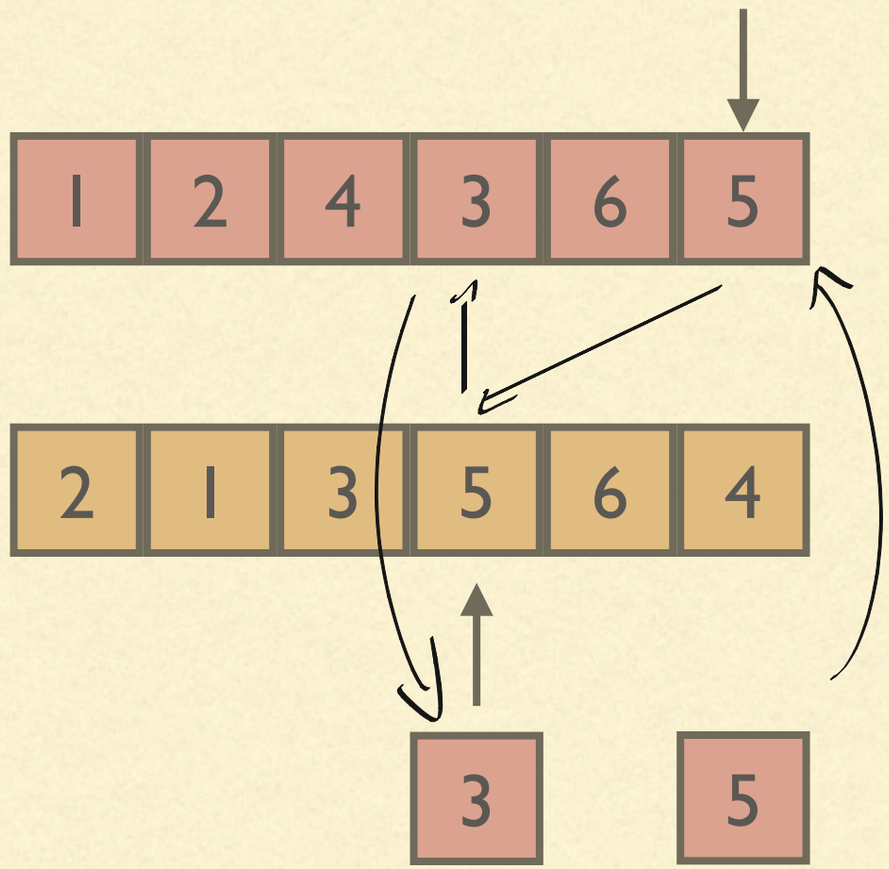
CYCLE CROSSOVER



- VALUE FIRST SAMPLE
- SEARCHE THAT VALUE ON THE SECOND SAMPLE
- EXCHANGE THE VALUE THAT HAVE THE SAME POSITION IN THE FIRST SAMPLE WITH THE POSITION OF VALUE WE FIND ABOVE
- USE THE VALUE WE EXTRACT AS INDEX
- REPEAT

Once found we copy the value from the first parent.
Repeat until we return to the beginning

CYCLE CROSSOVER



Once found we copy the value from the first parent.
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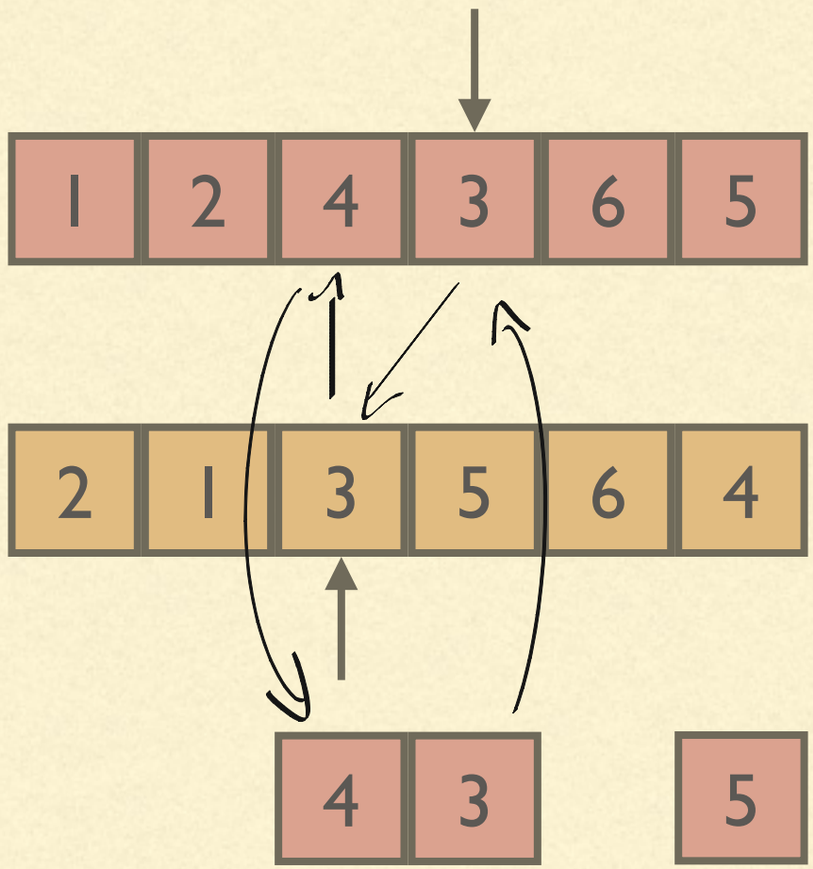
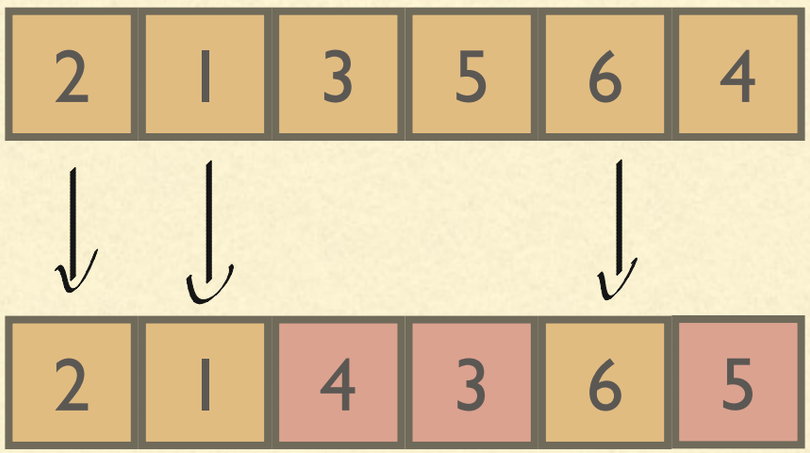
CYCLE CROSSOVER



Once found we copy the value from the first parent.
Repeat until we return to the beginning

CYCLE CROSSOVER



We copy the remaining elements from the second parent

REPRESENTING GRAPHS

- You might want to represent graphs. Possibly because the are ubiquitous in computer science.

- You can represent graph in two ways:

  THE COMMONE ONE

  - *Direct encoding*. By actually representing vertices and edges

  - *Indirect encoding*. By representing some "device" that builds a graph

IT CAN BE USEFULL BECAUSE THE "DEVICE" CAN BE EXTREMLY SMALLER THEN THE GRAPH AND IT CAN BE EASYER TO RAPPRESENT REGULARITIES
EX: BUILT A GRAPH AND REPEAT IT 20 TIMES

ADJACENCY MATRIX

Side of the matrix = max number of nodes

$$\begin{bmatrix} 0.4 & \text{no edge} & 0.6 & -5.3 \\ \text{no edge} & 5.6 & 0.1 & 0.2 \\ 2.4 & 0.8 & 4.1 & 8.3 \\ -0.2 & \text{no edge} & 0.5 & \text{no edge} \end{bmatrix}$$

Special value to represent missing edges

LARGE GRAPHS

$V = \{a, b, c, d, e\}$ We evolve the sets of vertices and edges

$E = \{(a, b), (a, c), (d, a), (e, e)\}$

NON EXISTING EDGE

Possible mutations:

- Add an edge
- Add a node
- Remove an edge
- Remove a node and all its edges
- …

Each one can have a different probability

Crossover is difficult to define and you might decide not to use it

IF USE A LINEAR RAPPRESENTATION ROW BY ROW OR PERFORM A 2D CROSSOVER ON THE MATRIX

BUT HOW WE PERFORM AN INTERACTIONS BETWEEN SOLUTION?

SELECTION PHASE : GOOD SOLUTION PROPAGATE AND BAD FADE AWAY

PRODUCTION RULES

THIS ONE IS AN UNJIRECT RAPPRESENATION

- There are terminal symbols and non-terminal symbols
- Production rules map a non-terminal symbol into a sequence/ matrix of non-terminal and terminal symbols
- We continue the expansion until the configuration is composed only of terminal symbols
- By using adeguate production rules we can encode indirectly a graph (i.e., rules that build a graph)

PRODUCTION RULES

$$S \rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$A \rightarrow \begin{bmatrix} c & p \\ a & c \end{bmatrix} \; B \rightarrow \begin{bmatrix} a & a \\ a & e \end{bmatrix} \; C \rightarrow \begin{bmatrix} a & a \\ a & a \end{bmatrix} \; D \rightarrow \begin{bmatrix} a & a \\ a & b \end{bmatrix}$$

$$a \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \; b \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \; c \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; e \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \; p \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

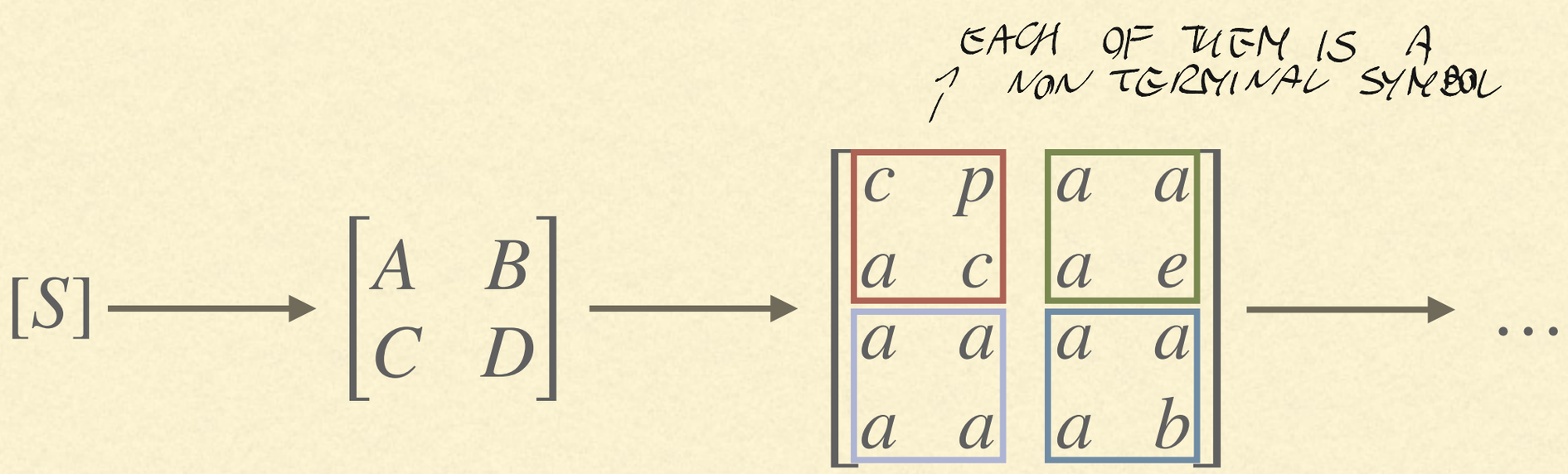
Ruleset used in an example in:

Hiroaki Kitano, Designing neural networks using a genetic algorithm with a graph generation system

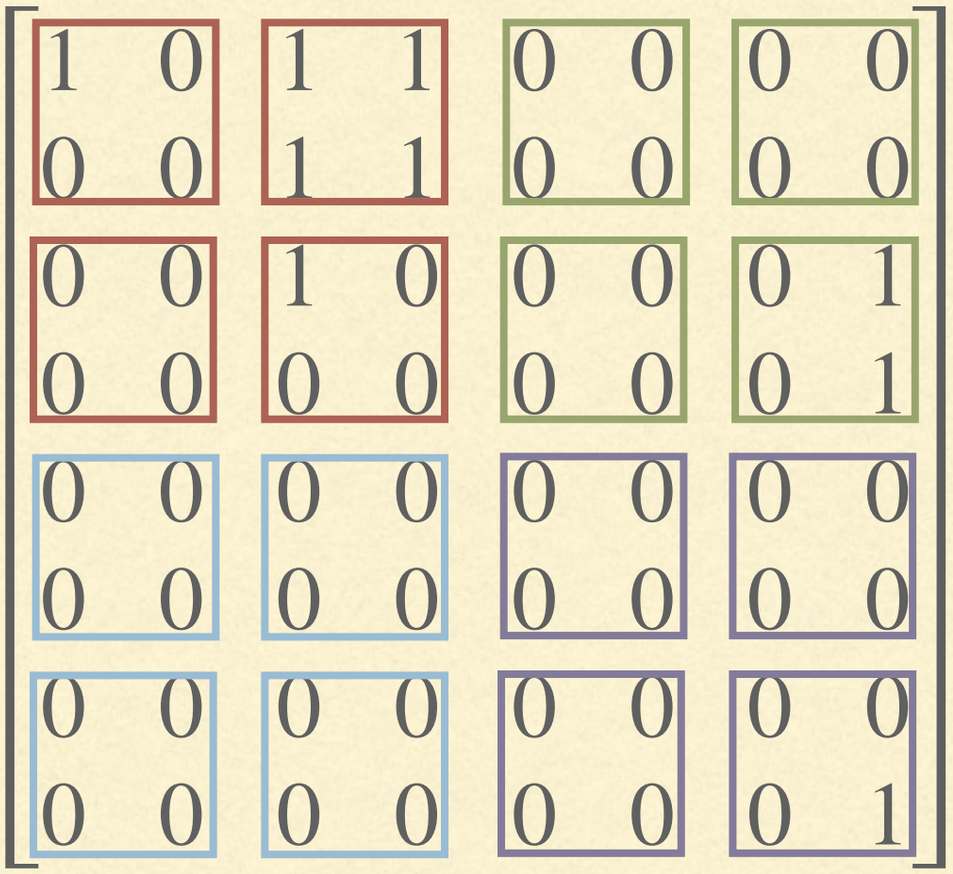
PRODUCTION RULES

Starting from an axiom [S] we can iterate the production rules

EACH OF THEM IS A NON TERMINAL SYMBOL



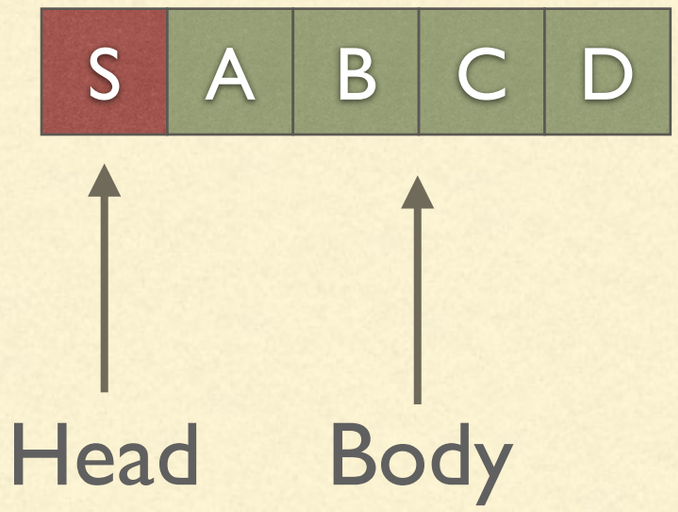
PRODUCTION RULES



A graph of 5 vertices
(8 encoded but 3 of them
are not connected to anything)

IF I CAN EVOLVE A SET OF PRODUCTION RULES, THEN I CAN GENERATE VERY LARGE GRAPH

PRODUCTION RULE: ENCODING



encodes $S \rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

IF WE FIX THE HEAD, WE JUST EVOLVE THE BODY AND CHANGE THE GRAPH WHILE MAINTAINING THE SCALABILITY

Since now we have a vector of fixed length,
to perform the evolution
we can apply traditional GA operators