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Poincaré Sphere

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Polarized Light

> Assumption: The beam propagates along the z-direction, the amplitude is E_1 and E_2 in x,y-direction , respectively.

Under this assumption:

$$\begin{cases} E_x = E_1 \cos(kz - \omega t + \delta_1) \\ E_y = E_2 \cos(kz - \omega t + \delta_2) \end{cases}$$

We will use $\tau = kz - wt$ to simpfy the proof.

$$\begin{cases} E_x = E_1 \cos(\tau + \delta_1) \\ E_y = E_2 \cos(\tau + \delta_2) \end{cases}$$

then:

$$\frac{E_x}{E_1} = \cos\tau\cos\delta_1 - \sin\tau\sin\delta_1$$
$$\frac{E_y}{E_2} = \cos\tau\cos\delta_2 - \sin\tau\sin\delta_2$$

Polarized Light

do these calculation:

$$\sin\delta_2 \cdot \frac{E_x}{E_1} - \sin\delta_1 \frac{E_y}{E_2} = \cos\tau(\sin\delta_2\cos\delta_1 - \sin\delta_1\cos\delta_2) \tag{1}$$

$$\cos\delta_2 \cdot \frac{E_x}{E_1} - \cos\delta_1 \frac{E_y}{E_2} = \sin\tau(\sin\delta_2\cos\delta_1 - \sin\delta_1\cos\delta_2) \tag{2}$$

Combining eq(1) and eq(2), we can find:

$$\left(\frac{E_x}{E_1}\right)^2 - 2\cos(\delta_1-\delta_2)\cdot\frac{E_xE_y}{E_1E_2} + \left(\frac{E_y}{E_2}\right)^2 = \sin^2(\delta_1-\delta_2) \tag{3}$$

We can use $\delta = \delta_1 - \delta_2$ to simpfy eq(3) (Although we can also use $\delta = \delta_2 - \delta_1$ to describe without changing the form of equation, it will keep a well and same style of later derivation.)

$$\boxed{\left(\frac{E_x}{E_1}\right)^2 - 2\cos(\delta)\cdot\frac{E_xE_y}{E_1E_2} + \left(\frac{E_y}{E_2}\right)^2 = \sin^2(\delta)} \tag{4}$$

Polarized Light

We can use elliptical discimaninate equation to check:

$$\begin{vmatrix} \dfrac{1}{E_1^2} & -\dfrac{\cos\delta}{E_1 E_2} \\ -\dfrac{\cos\delta}{E_1 E_2} & \dfrac{1}{E_2^2} \end{vmatrix} = \frac{\sin^2\delta}{E_1^2 E_2^2} \geqslant 0$$

Coordinates transformation of the elliptic

We can use a better coordinate system which is the major axis and minor axis to discribe the elliptical light.

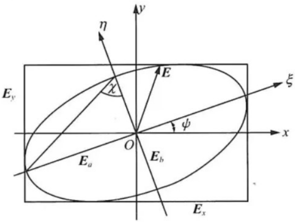


图 1: cooridnates transformation - the derivation for a better method to discribe the polarization

Suppose:

$$\begin{cases} E_\xi = E_a \cos(\tau + \varphi) \\ E_\eta = E_b \sin(\tau + \varphi) \end{cases}$$

Coordinates transformation of the elliptic

Using the rotation matrix to describe:

$$\begin{bmatrix} E_\xi \\ E_\eta \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Expanding the equation:

$$\begin{cases} E_a\left(\cos\tau\cos\varphi - \sin\tau\sin\varphi\right) = E_1\left(\cos\tau\cos\delta_1 - \sin\tau\sin\delta_1\right)\cos\psi \\ +E_2\left(\cos\tau\cos\delta_2 - \sin\tau\sin\delta_2\right)\sin\psi \\ E_b\left(\sin\tau\cos\varphi + \cos\tau\sin\varphi\right) = -E_1\left(\cos\tau\cos\delta_1 - \sin\tau\sin\delta_1\right)\sin\psi \\ +E_2\left(\cos\tau\cos\delta_2 - \sin\tau\sin\delta_2\right)\cos\psi \end{cases}$$

Since τ ranges from $-\pi$ to π, $\cos\tau$ and $\sin\tau$ can be treated as independent variables.

$$E_a\cos\varphi = E_1\cos\delta_1\cos\psi + E_2\cos\delta_2\sin\psi \tag{5}$$

$$-E_a\sin\varphi = -E_1\sin\delta_1\cos\psi - E_2\sin\delta_2\sin\psi \tag{6}$$

$$E_b\cos\varphi = E_1\sin\delta_1\sin\psi - E_2\sin\delta_2\cos\psi \tag{7}$$

$$E_b\sin\varphi = -E_1\cos\delta_1\sin\psi + E_2\cos\delta_2\cos\psi \tag{8}$$

Coordinates transformation of the elliptic

$(5)^2 + (6)^2 + (7)^2 + (8)^2$:

$$S_0 \equiv E_a^2 + E_b^2 = E_1^2 + E_2^2 \tag{9}$$

$(5) * (7) - (6) * (8)$:

$$E_a E_b = E_1 E_2 \sin(\delta_1 - \delta_2) = E_1 E_2 \sin\delta \tag{10}$$

$(5)/(6) + (7)/(8)$:

$$2E_1 E_2 \cos\delta \cos 2\psi = \left(E_1^2 - E_2^2\right) \sin 2\psi \tag{11}$$

Coordinates transformation of the elliptic

Now, we define:

$$\frac{E_1}{E_2} = \tan\alpha \tag{12}$$

From eq(11), we can get:

$$\tan 2\psi = \frac{2E_1E_2\cos\delta}{E_1^2 - E_2^2} = \frac{2\tan\alpha}{\tan^2\alpha - 1}\cos\delta = -\tan 2\alpha\cos\delta \tag{13}$$

Coordinates transformation of the elliptic

Another defination:

$$\frac{E_a}{E_b} = \tan\chi \tag{14}$$

(10)/(9):

$$\begin{aligned}
\frac{E_a E_b}{E_a^2 + E_b^2} &= \frac{E_1 E_2 \sin\delta}{E_1^2 + E_2^2} \\
\frac{2\tan\chi}{\tan^2\chi + 1} &= \frac{2\tan\alpha}{1 + \tan^2\alpha}\sin\delta \\
2\sin\chi\cos\chi &= 2\sin\alpha\cos\alpha\sin\delta \\
\sin 2\chi &= \sin 2\alpha \sin\delta
\end{aligned}$$

$$\sin 2\chi = \sin 2\alpha \sin\delta \tag{15}$$

Coordinates transformation of the elliptic

(i) if $\delta = \delta_1 - \delta_2 = m\pi, m = 0, \pm 1, \pm 2, \cdots$

$$\sin 2\chi = 0$$
$$\frac{2\tan\chi}{1+\tan^2\chi} = 0$$
$$\tan\chi = 0$$

Since $\tan\chi = 0$, we can get $E_a = 0$. This situation is called linear polarization.

Coordinates transformation of the elliptic

(ii) if $\delta = \delta_1 - \delta_2 = \frac{\pi}{2}(2n+1), n = 0, \pm 1, \pm 2, \cdots$
The euqation of polarized light will be:

$$\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1$$

if we add another condition, $E_1 = E_2 = E$, the equation will be:

$$E_x^2 + E_y^2 = E^2$$

It is called circular polarization. The orthogonal linear component of this situation is:

$$\begin{cases} E_x = E\cos(\tau) \\ E_y = \pm E\cos(\tau - \frac{\pi}{2}) \end{cases}$$

Coordinates transformation of the elliptic

An interesting trick: We can decompose a elliptical polarization to the combination of two circular polarization.

$$\begin{cases} E_\xi = E_a \cos(\tau + \varphi) = (E_r + E_l)\cos(\tau + \varphi) \\ E_\eta = E_b \sin(\tau + \varphi) = (E_r - E_l)\cos(\tau + \varphi - \frac{\pi}{2}) \end{cases}$$

We can calculate these parameters easilly:

$$\begin{cases} E_r = \frac{1}{2}(E_a + E_b) \\ E_l = \frac{1}{2}(E_a - E_b) \end{cases}$$

$$S_0 \equiv E_a^2 + E_b^2 = 2\left(E_r^2 + E_l^2\right)$$

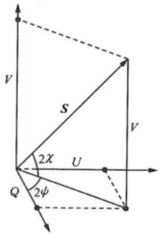
Poincaré Sphere

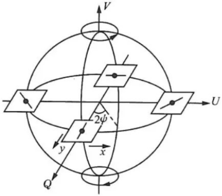
The cooridnates of Poincare sphere are:

$$\begin{cases} S_0 = I = E_a^2 + E_b^2 \\ S_1 = Q = S_0 \cos 2\chi \cos 2\psi \\ S_2 = U = S_0 \cos 2\chi \sin 2\psi \\ S_3 = V = S_0 \sin 2\chi \end{cases}$$

Attention: In fact, energy flux has a parameter: $\frac{1}{2}\epsilon$. But, it's just a easy method to describe the intensity of light. There just exits three independent variables:

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$
$$I^2 = Q^2 + U^2 + V^2$$





Poincaré Sphere

We can simpfy the Stokes parameters:

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{2E_1E_2}{E_1^2+E_2^2}$$

$$\sin 2\chi = \sin 2\alpha \cdot \sin\delta = \frac{2E_1E_2}{E_1^2+E_2^2}\sin\delta = \frac{2E_1E_2}{I}\cdot\sin\delta$$

$$|\cos 2\chi| = \sqrt{1-\sin^2 2\chi} = \frac{1}{I}\cdot\sqrt{I^2-(2E_1E_2)^2\sin^2\delta}$$

$$\tan 2\psi = -\tan 2\alpha\cdot\cos\delta = \frac{2E_1E_2}{E_1^2-E_2^2}\cdot\cos\delta$$

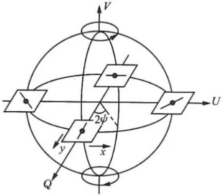
$$|\cos 2\psi| = \frac{1}{\sqrt{1+\tan^2 2\psi}} = \frac{|E_1^2-E_2^2|}{\sqrt{I^2-(2E_1E_2)^2\sin^2\delta}}$$

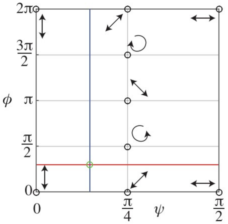
$$|\sin 2\psi| = \frac{|2E_1E_2\cos\delta|}{\sqrt{I^2-(2E_1E_2)^2\sin^2\delta}}$$

Poincaré Sphere

So, we can change the Stokes parameters to:

$$\begin{cases} S_0 = I = E_a^2 + E_b^2 = E_1^2 + E_2^2 \\ S_1 = Q = E_1^2 - E_2^2 \\ S_2 = U = 2E_1E_2\cos\delta \\ S_3 = V = 2E_1E_2\sin\delta \end{cases}$$





Poincaré Sphere

(i) counterclockwise circular polarized light(left-handed polarized light):

$$
\begin{aligned}
E_1 &= E_2 = E, \delta = \frac{\pi}{2} \\
S_0 &= I = S \\
S_1 &= Q = 0 \\
S_2 &= U = 0 \\
S_3 &= V = S
\end{aligned}
$$

(ii) clockwise circular polarized light(right-handed polarized light) :

$$
\begin{aligned}
E_1 &= E_2, \delta = -\frac{\pi}{2} \\
S_0 &= I = S \\
S_1 &= Q = 0 \\
S_2 &= U = 0 \\
S_3 &= V = -S
\end{aligned}
$$

Poincaré Sphere

(iii) linear polarized light:

$$
\begin{aligned}
E_1 =& E, E_2 = 0, \delta = 0 \\
S_0 &= I = S \\
S_1 &= Q = S \\
S_2 &= U = 0 \\
S_3 &= V = 0
\end{aligned}
$$

(iv) elliptical polarized light:

$$
\begin{aligned}
\delta &\neq 0 \\
S_0 &= I = S \\
S_1 &= Q = S \cos 2\chi \cos 2\psi \\
S_2 &= U = S \cos 2\chi \sin 2\psi \\
S_3 &= V = S \sin 2\chi
\end{aligned}
$$