ELEMENTARY MATHEMATICS III

ICT MATHEMATICS TEAM
UNIVERSITY OF JOS, NIGERIA

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Module 1: Functions

1 1.1 Objectives:

In this module:

- We introduce with examples, the concept of a function and study some classes of functions.
- We discuss methods of finding the domain, range, zeros and singularities of some real-valued functions.
- We look at injective (one to one) and surjective (onto) functions and end the module by exploring the inverse of a function and the composition of functions.
- We stimulate students through classroom and other interactive activities to understand the concept of a function and then discover the differences and similarities between the several types of functions introduced.

2 1.2 Learning Outcomes:

At the end of this module students should be able to

- Define a function; identify different types of functions and Identify relations which are not functions;
- Identify different classes of functions, find the domain and range that makes a rule f (say), a function;
- Investigate the injectiveness, surjectiveness and the inverse of different functions;
- Establish the composition of two or more functions.

3 1.3 Learning Activities:

Students should:

- Explore notes and exercises, individually and in groups;
- Explore and use related materials on the Intranet, especially the e-granary and MIT open course ware;
- Explore and use related materials on the OLI Calculus course on the Internet(http://www.cmu.edu/OLI/courses/);
- Solve relevant questions from past MTH103 Examinations.

4 1.4 Introduction

A function is the means by which the real world can be described in mathematical terms. For instance, the area of a circle C is a function of its radius r. It is an input-output relation in which the function f, is the output with the variable x as the input. Thus a function f , with an input x, can be expressed as $f(x)$. For a particular input value $x = x_1$ the output value is $f(x_1)$. A function can also be referred to as a mapping or a transformation.

We shall consider the definition of a function, some classes of functions and their properties.

[1.4.1] (**Function)**

Let X and Y be two non-empty sets. A function f :$X \to Y$ is a rule which assigns or associates to each element x of the set X, a unique (one and only one) element y of the set Y,and is written as $y = f(x)$.

Alternatively, a function can be defined as a relation R from a set X to another set Y such that: (i) $\forall\, x \in X, \exists\, y \in Y$ such that x is related to y written as xRy (ii) if xRy_1 and xRy_2 then $y_1 = y_2$ (in other words the relation is called a function from X to Y if to each $x \in X$ there corresponds exactly one $y \in Y$).

[1.4.2] (**Domain, Codomain and Range of a Function)** The set X of all x-values allowed by the function $f : X \to Y$, is the domain of the function f. It is denoted by $Dom(f)$ or $D(f)$. x which may take any value in $Dom(f)$, is the independent variable. y is called the dependent variable since its value depends on the value of the independent variable x. For a particular value $x = x_1$ the value $y_1 = f(x_1)$ is called the image of x_1 under f while x_1 is called the preimage of y_1 under f. The set Y in the definition of the function $f : X \to Y$ is called the codomain of the function f. The set of all admissible y-values as x goes through all values in its domain, is called the range of the function. The range of a function f will be denoted by $Ran(f) = \{y \in Y : f(x) = y,\ x \in D(f)\}$ or simply $R(f)$.

Example 1 (1.4.1) $X = X_1 = \{1, 2, 3\}; Y = Y_1 = \{-\frac{7}{3}, -1, \frac{1}{2}, \sqrt{2}, 2, 4, 5, 6\}$ *and* $f(x) = 2x$. *These two sets and the given rule define a function because each element in the domain* X_1 *has a unique image in the codomain* Y_1.*The range of the function is* $R(f) = \{2, 4, 6\}$.

Example 2 (1.4.2) $X = X_2 = \{0, 4, \}; Y = Y_2 = \{-\frac{7}{3}, -2, -1, 0, \frac{1}{2}, \sqrt{2}, 2, 4, 5, 6\}$ *and* $g(x) = \pm\sqrt{x}$. *These two sets* X_2, Y_2 *and the given rule* $g(x) = \pm\sqrt{x}$ *do not* *define a function because the element 4 in the domain X_2 has two different images 2 and -2 in the codomain Y_2. This is despite the fact that 0 in the domain X_2 has a unique image 0 in the codomain Y_2.*

Example 3 (1.4.3) *Let $h : X_3 \to E$ with $X_3 = \{1, 2, 3, 4, 5\}$ and $E = \{2, 4, 6, ...\}$ and h is defined by $h(x) = 2x$.*

$$\begin{pmatrix} X_3 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \longrightarrow \begin{pmatrix} E \\ 2 \\ 4 \\ 6 \\ 8 \\ 10 \\ . \\ . \\ . \end{pmatrix}$$

Here, the rule is multiplication of the elements of X_3 by 2 giving in each case, a unique even number. In this example the domain of h is the set $X_3 = \{1, 2, 3, 4, 5\}$,the codomain is the set $E = \{2, 4, 6, ...\}$, while the range is the set $\{2, 4, 6, 8, 10\} = R(h)$.

Remark 4 (1.4.1) *Generally the range is a subset of the codomain as the examples above illustrate.*

We shall consider functions $f : \mathbb{R} \to \mathbb{R}$. A function f is represented algebraically if, for each x in the domain, $f(x)$ is given by some algebraic expression involving x. Since many algebraic operations give unique results, there is usually no problem in determining whether or not, an algebraic expression involving x defines a function. For instance, since the square of a number is a unique number, the expression $f(x) = x^2$ is a rule that defines a function from $\mathbb{R}$ to $\mathbb{R}$.

Also, since the sum of any two numbers is a unique number, the expression $g(x) = x + 5$ is a rule that defines a function from $\mathbb{R}$ to $\mathbb{R}$. Can you determine whether or not, the expression $s(x) = \sqrt{x}$ is a rule that defines a function from $\mathbb{R}$ to $\mathbb{R}$?

Remark 5 (1.4.2) *When a function f is defined by a formula $y = f(x)$ and the domain of the function is not explicitly given, then we consider its domain to consist of all values of x for which the formula can be evaluated as a real number. In particular, division by zero is not allowed, and square roots (or even roots) of negative numbers are not allowed.*

Remark 6 (1.4.3) *A function can be expressed in terms of ordered pairs which in turn can be represented as co-ordinates of points in a plane. Given $S = \{a, b, c\}$ and $T = \{1, 2\}$, then $f = \{(a, 1), (b, 1), (c, 2)\}$ is a rule and defines a* *function that assigns elements of T to elements of S as shown. In the third example given above, h can be written as*

$$h = \{(1,2),(2,4),(3,6),(4,8),(5,10)\}.$$

[1.4.4] (of rules that are not functions)

a Consider $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \pm\sqrt{x}$ or $f = \{(x,y) : y^2 = x\}$. This is not a function because an element of the domain can be mapped to two elements of the codomain, e.g. $f(9) = \pm\sqrt{9} = \pm 3$. Thus 9 is assigned to -3 and $+3$ at the same time, as such, f cannot be a function. The rule will be a function if we are allowed to take only the values in $(-\infty, 0)$ or $(0, \infty)$. i.e. restricting the codomain to one of $\mathbb{R}^-$ or $\mathbb{R}^+$ at a time.

b Consider $f : \mathbb{R} \longrightarrow [0, 2\pi]$ defined by $f(x) = sin^{-1}x$. Observe that $\frac{1}{2}$ $\in \mathbb{R}$ is mapped to $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ in $[0, 2\pi]$. As such, f is not a rule defining a function from $\mathbb{R}$ to $[0, 2\pi]$. The rule will define a function if the codomain is $[0, \frac{\pi}{2}]$.

c Let $f : S \to T$; $S = \{a, b, c\}$ and $T = \{1, 2\}$ then $f = \{(a,1),(a,2),(b,1),(c,2)\}$ does not define a function because a is mapped by f to both 1 and 2. i.e. to more than one element in $\{1, 2\}$.

5 1.5 Names of Some Classes of Functions

1. **Linear Functions**

   A linear function is of the form $y = f(x) = mx + c$ where m and c are constants. The graph of such a function is a straight line with gradient m and $y-$intercept c. For example $-6x + 3y = 5$ reduces to $y = 2x + \frac{5}{3}$

2. **Constant Functions:**

   A constant function is of the form $f(x) = c$, where c is the constant taken for every value of x in the domain of f. A constant function is a special case of a linear function with $m = 0$.

3. **Quadratic Functions:**

   A quadratic function is of the form $f(x) = ax^2 + bx + c$ where a, b, c are constants and $a \neq 0$.

4. **Power functions:**

   A power function is of the form $f(x) = x^r$ where r is any real number. $f(x) = x^3; g(x) = x^{\frac{3}{4}}$ and $h(x) = x^{-2}$ are examples of power functions.

5. **Polynomial Functions:**

   A polynomial function of degree n is a function of the form,

   $$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n \text{ with } a_n \neq 0$$

   where $a_0, a_1, \cdots, a_n$ are constants called the coefficients of P_n and n is a non-negative integer. For example $P_4(x) = x^4 + 3x^3 + 2x + 3$ is a polynomial of degree 4. Note that constant, linear and quadratic functions are polynomials of degrees $0, 1$ and 2 respectively.

6. **Rational Functions:**

   A rational function is of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with $Q(x) \neq 0$. For example the function f given by $f(x) = \frac{x^3+2x+1}{x^4+x+1}$ is a rational function.

7. **Trigonometric Functions:**

   Functions that involve one or more of the trigonometric ratios (sine, cosine, tangent, cosecant, secant, or cotangent) are called trigonometric functions. For example, $f(x) = \sin x$ and $g(x) = \cos^2 x + \cot x$ are trigonometric functions.

8. **Inverse Trigonometric Functions:**

   Functions that involve one or more of the inverses of the trigonometric ratios (arcsine, arccosine, arctangent, arccosecant, arcsecant, or arccotangent) are called inverse trigonometric functions. For example, $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x + \tan^{-1} x$ are inverse trigonometric functions.

9. **Exponential Functions:**

   The exponential functions are functions of the form $f(x) = e^x$ or $f(x) = a^x$,where a is any fixed positive number and x is any real number. The former type can be expressed as a series as,

   $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots =_{n=0}^{\infty} (\frac{x^n}{n!})$.

   Here $n! = n(n-1)(n-2)\cdots(2)(1)$ and $f(x) = e^x$ is always real and positive. Note that the trigonometric functions can be defined in terms of the exponential function as $\sin x = \frac{e^{ix}-e^{-ix}}{2}$, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\tan x = \frac{e^{ix}-e{-ix}}{e^{ix}+e^{-ix}}$

10. **Hyperbolic Functions:**

    These functions are defined in terms of the exponential function as $\sinh x = \frac{e^x-e^{-x}}{2}$, $\cosh x = \frac{e^x+e^{-x}}{2}$, $\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$, $cschx = \frac{2}{e^x-e^{-x}}$, $sechx = \frac{2}{e^x+e^{-x}}$ and $cothx = \frac{e^x+e^{-x}}{e^x-e^{-x}}$. Note that the trigonometric and hyperbolic functions are related by $\sinh x = -i\sin(ix)$, $\cosh x = \cos(ix)$, $\tanh x = -\tan(ix)$, where $i = \sqrt{-1}$ defines the imaginary unit.

11. **Logarithmic Functions:**

    These are functions that involve logarithm, for example, $f(x) = \log_{10} x$ or $h(x) = \log_e x = \ln |x|$

12. **Modulus or Absolute Value Functions:**

    This is a function of the form;

    $f(x) = |x| = \{^{x,\ x \geq 0}_{-x, x < 0}$ or $|x| = \sqrt{x^2}$.Note for instance that, $|-3| = 3, |0| = 0$,and $|2| = 2$.

13. **The Characteristic Function***:*

    This function is defined by $\chi_S(x) = \begin{cases} 0, & \text{if } x \notin S \\ 1, & \text{if } x \in S \end{cases}$

14. **The Floor Function** $\lfloor x \rfloor$:

    This function is defined by $f(x) = \lfloor x \rfloor$, for every real number x, and its value is the greatest integer which is less than or equal to x. For example $\lfloor 2.34 \rfloor = 2$.

15. **The Ceiling Function** $\lceil x \rceil$:

    This function is defined by $f(x) = \lceil x \rceil$, for every real number x, and its value is the smallest integer that is greater than or equal to x. For example $\lceil -4.781 \rceil = -4$.

16. **The Square Root Function:**

    This function is defined by $f(x) = \sqrt{x}$, where the symbol $\surd$ denotes the positive square root of x.

17. **Euler's Form of Trigonometric and Hyperbolic Functions:**

    The exponential function e^{ix} can be expressed as $e^{ix} = \cos x + i \sin x$. This is known as Euler's form (pronounced as Oiler). Also, $e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i \sin x$. From these two expressions, we can write the trigonometric functions in terms of the exponential function as $\sin x = \frac{e^{ix} - e^{-ix}}{2}$, $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\tan x = \frac{e^{ix} - e - ^{ix}}{e^{ix} + e^{-ix}}$ etc.

18. **Periodic Functions:**

A function f is said to be periodic if $\exists\ T > 0$ such that $f(x + T) = f(x)$ for every x in the domain of f , T is called the period of f. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is periodic with period $T = 2\pi$ because $\sin(x + 2\pi) = \sin x$, for every x in the domain of f.

19 **Odd Functions:** A function f is said to be odd if $f(-x) = -f(x)\ \forall x \in D(f)$.For example, $f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = -f(x), \forall x \in D(f)$.Similarly, $\sin x$ is odd because $\sin(-x) = -\sin x\ \forall$ real x.

20 **Even Functions:**

A function f is said to be even if $f(-x) = f(x)\ \forall x \in D(f)$. For example, $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)\ \forall x \in D(f)$.Also, $\cos x$ is even since $\cos(-x) = \cos x\ \forall x \in \mathbb{R}$.

Remark 7 (1.5.1) *Some functions are neither odd nor even, for example,(i)* $f(x) = x^2 + x + 1$ *(ii)* $g(x) = \frac{1}{x-1}$ *(iii)* $h(x) = \sin x + \cos x$.

The evenness or oddness of a function f is known as its parity.

Exercise (1.5.1): Determine which of the following functions is periodic, even, odd or neither even nor odd?

$\tan x$, $\sec x$, $\csc x$, $\cot x$, $\sinh x$, $\cosh x$, $\tanh x$, e^{ix}, $|x|$, $\frac{1}{2x}$, $\ln x$, $\log_{10} x^2$, e^{-x^2}, xe^{x^2}, $\sqrt{x}$.

6 1.6 Determination of Domain and Range of Functions.

Example (1.6.1) : Given the rules below with $\mathbb{R}$ as codomain, determine in each case, the domain so that a function is well defined.

1. $f(x) = x^2$.
2. $f(x) = 2x^2 - x + 1$.
3. $f(x) = \frac{1}{x}$.
4. $f(x) = \sqrt{4x^2 - 1}$
5. $f(x) = \frac{1}{+\sqrt{4-x^2}}$.
6. $f(x) = e^{2x+4}$.
7. $f(x) = \log(x^2 - 4)$.

Solutions

1. For any real number x, x^2 is real. Therefore, f maps every real number into a real number and so the domain of f is the set of all real numbers. i.e. $Dom(f) = \mathbb{R}$, where $\mathbb{R} = (-\infty, \infty)$.
2. It is not difficult to see that $2x^2 - x + 1$ is always real $\forall x \in \mathbb{R}$, since it is a polynomial. $D(f) = (-\infty, \infty) = \mathbb{R}$. In fact, all polynomial functions are real for real values of x. i.e. $D(p_n(x)) = \mathbb{R}$, for all polynomials $P_n(x)$.
3. Observe that the function $\frac{1}{x}$ is real iff the denominator is real and not zero. i.e.

$$D(f) = \{x \in \mathbb{R} : x \neq 0\} = \mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$$

4. $\sqrt{4x^2-1}$ is real iff $4x^2-1 \geq 0$. i.e..$(2x-1)(2x+1) \geq 0 \Longrightarrow$ either $2x-1 \geq 0$ and $2x+1 \geq 0$; or $2x-1 \leq 0$ and $2x+1 \leq 0$.The first pair of inequalities gives $x \geq \frac{1}{2}$ and $x \geq -\frac{1}{2}$; both combine to give $x \geq \frac{1}{2}$. The second pair of inequalities gives $x \leq \frac{1}{2}$ and $x \leq -\frac{1}{2}$; both combine to give $x \leq -\frac{1}{2}$.

   Thus, $Dom(f) = \{x \in \mathbb{R} : -\infty < x \leq -\frac{1}{2}\} \cup \{x \in \mathbb{R} : \frac{1}{2} \leq x < \infty\} = \mathbb{R} - (-\frac{1}{2}, \frac{1}{2})$.

5. Observe that $\frac{1}{\sqrt{4-x^2}}$ is real iff the denominator $\sqrt{4-x^2}$ is real and nonzero, i.e..$(2-x)(2+x) > 0 \Longrightarrow$ either $2-x > 0$ and $2+x > 0$; or $2-x < 0$ and $2+x < 0$.The first pair of inequalities gives $x < 2$ and $x > -2$; both combine to give $-2 < x < 2$. The second pair of inequalities gives $x > 2$ and $x < -2$; there are no values of x satisfying these last two inequalities at the same time.

   Hence, $Dom(f) = \{x \in \mathbb{R} : -2 < x < 2, \} = (-2, 2)$.

6. Observe that $\forall x \in \mathbb{R}, 2x+4$ is real and the exponential e^{2x+4} is also real. $\therefore\ Dom(f) = \mathbb{R}$.

7. Note that $\log_e(x^2-4)$ is real iff $x^2-4 > 0 \Longrightarrow (x-2)(x+2) > 0$

   $\Longrightarrow x-2 > 0$ and $x+2 > 0$ or $x-2 < 0$ and $x+2 < 0$

$\Rightarrow x > 2$ and $x > -2$ or $x < 2$ and $x < -2$.
This implies that $x^2-4 > 0$ when either $x < -2$ or $x > 2$. $\therefore Dom(f) = \{x \in \mathbb{R} : x < -2$ or $x > 2\ \} = (-\infty, -2) \cup (2, \infty)$
$= \mathbb{R} - [-2, 2]$.

Example 8 (1.6.2) *Determine the range of each of the following functions.*

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.
2. $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x^2 - x + 1$.
3. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$.
4. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{4x^2-1}$
5. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{+\sqrt{4-x^2}}$.
6. $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = e^{2x+4}$.
7. $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \log(x^2-4)$.

1 Observe that f is mapped to non-negative real numbers since the square of any real number (whether negative, zero or positive) is non-negative. Therefore, the range is the set of non-negative real numbers i.e.. $Ran(f) = \{y \in \mathbb{R} : 0 \leq y < \infty\} = [0, \infty)$.

Alternatively, to find the range of f we can let $f(x)$ to be y and obtain the possible values of y for which x is real, since f is a mapping of real numbers.

Let $f(x) = y \Rightarrow x^2 = y \Rightarrow x = \pm\sqrt{y}$. Observe that for x to be real (as required) y has to be non-negative real numbers. i.e. $y \geq 0 \therefore R(f) = \{y \in \mathbb{R} : 0 \leq y < \infty\} = [0, \infty)$.

2 Let $f(x) = y \Rightarrow 2x^2 - x + 1 = y \Rightarrow 2x^2 - x + 1 - y = 0$.Using the quadratic formula for the roots

$$\Rightarrow x = \frac{1 \pm \sqrt{1 - 8(1-y)}}{4}$$

Observe that x is real iff $1 - 8(1-y) \geq 0 \Rightarrow y \geq \frac{7}{8} \therefore Ran(f) = \{y \in \mathbb{R} : \frac{7}{8} \leq y < \infty\} = [\frac{7}{8}, \infty)$. In determining the range, it is sometimes possible to sketch the curve of the function and take all the possible values of y from the graph as the range of the function.

3 Let $f(x) = y$

$$\Rightarrow \frac{1}{x} = y$$

$$\Rightarrow x = \frac{1}{y}$$

Again x is real iff y is real and not zero. Therefore,

$$Ran(f) = \{y \in \mathbb{R} : y \neq 0\} = \mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty).$$

Here the domain of f is the same as its range.

4 Let $y = f(x)$, then $y = \sqrt{4x^2 - 1}$.It should be observed that y is non-negative iff $4x^2 - 1 \geq 0$. $\Longrightarrow$ for all values of x in the $Dom(f), y$ takes non-negative values. Thus,

$$Ran(f) = \{y \in \mathbb{R} : y \geq 0\} = \mathbb{R}^+$$

5 Let $f(x) = y$

$$\Rightarrow \frac{1}{\sqrt{4 - x^2}} = y$$

$$\Rightarrow y\sqrt{4 - x^2} = 1$$

$$\Rightarrow y^2(4 - x^2) = 1 \Rightarrow 4y^2 - y^2x^2 = 1$$

$$\Rightarrow -y^2x^2 = 1 - 4y^2 \Rightarrow y^2x^2 = 4y^2 - 1$$
$$\Rightarrow x^2 = \frac{4y^2-1}{y^2} \Rightarrow x = \frac{\sqrt{4y^2-1}}{y}, y \neq 0$$

Observe that x is real if $4y^2 - 1 \geq 0 \Longrightarrow (2y+1)(2y-1) \geq 0$.

Therefore, either $2y+1 \leq 0$ and $2y-1 \leq 0$ or $2y+1 \geq 0$ and $2y-1 \geq 0$. This implies that either $y \leq -\frac{1}{2}$ and $y \leq \frac{1}{2}$ or $y \geq -\frac{1}{2}$ and $y \geq \frac{1}{2}$.Either $y \leq -\frac{1}{2}$ or $y \geq \frac{1}{2}$ satisfy the inequality $4y^2 - 1 \geq 0$.Thus,

$$y \in (-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty)$$

Observe that f cannot be negative,

$\therefore Ran(f) = \{y \in \mathbb{R} : \frac{1}{2} \leq y < \infty, \} = [\frac{1}{2}, \infty)$.

6 Let $f(x) = y$ i.e. $e^{2x+4} = y$ and taking the natural logarithm of both sides,

$\Rightarrow 2x + 4 = \ln y$ i.e. $x = \frac{1}{2}(\ln y - 4) = \frac{1}{2}\ln y - 2$.
Observe that x is real if $y > 0 \therefore Ran(f) = \{y \in \mathbb{R} : y > 0\} = (0, \infty)$.

7 Let $f(x) = y \Rightarrow \ln(x^2 - 4) = y$ and taking exponential of both sides

$\Rightarrow x^2 - 4 = e^y \Rightarrow x^2 = e^y + 4$ i.e. $x = \pm\sqrt{e^y + 4}$,
x is real for all $e^y + 4 > 0$. Observe that this is true $\forall\ y \in \mathbb{R}$,
since $e^y > 0 > -4 \therefore Ran(f) = \mathbb{R} = (-\infty, \infty)$.

Exercise 1.6.1

Find the domain and range of each of the following functions.

(i) $g : \mathbb{R} \to \mathbb{R}$ s.t. $g(x) = \frac{1}{(36-x^2)^{\frac{1}{2}}}$
(ii) $g : \mathbb{R} \to \mathbb{R}$ s.t. $g(x) = \frac{1}{\sqrt{x^2-36}}$
(iii) $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \frac{7x-2}{2x+3}$
(iv) $p : \mathbb{R} \to \mathbb{R}$ s.t. $p(x) = \tan(x^2 - 1)$
(v) $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \tan^{-1} 2x$
(vi) $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \sin^{-1} x$
(vii)$f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \cos x$
(viii) $f : \mathbb{N} \to \mathbb{R}$ s.t $f(x) = 2x$,Here $\mathbb{N}$ denotes the set of natural numbers.
(ix) $f : \mathbb{Z} \to \mathbb{R}$ s.t. $f(x) = \frac{1}{3x}$, Here $\mathbb{Z}$ denotes the set of integers.
(x) $f : [-1, 1] \to \mathbb{R}$ s.t. $f(x) = x^2$
(xi) $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \tanh x$.

7 1.7 Monotonic Functions

Definition 9 (1.7.1) *A function f is said to be increasing on an interval $I = [a, b]$ if for x_1, x_2 in I, $f(x_1) \leq f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.*

Definition 10 (1.7.2) *A function f is said to be strictly increasing on an interval $I = [a, b]$ if for x_1, x_2 in I, $f(x_1) < f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.*

Definition 11 (1.7.3) *A function f is said to be decreasing on an interval $I = [a, b]$ if for x_1, x_2 in I, $f(x_1) \geq f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.*

Remark 12 Definition 13 (1.7.4) *A function f is said to be strictly decreasing on an interval $I = [a, b]$ if for x_1, x_2 in I, $f(x_1) > f(x_2)$ whenever $a \leq x_1 < x_2 \leq b$.*

Definition 14 (1.7.5) *A function f which is either increasing (or strictly increasing) or decreasing (strictly decreasing) is called a (monotone) monotonic function.*

Example 15 (1.7.1) *The function $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = 2x$ is monotonically increasing on $[0, 2]$,* *since*
$f(x + h) = 2(x + h) = 2x + 2h > 2x = f(x)$ $\forall x \in [0, 2], h > 0$.

For instance, $f(\frac{1}{2}) = 2(\frac{1}{2}) = 1 < 2 = 2(1) = f(1)$, as $\frac{1}{2} < 1$ where $\frac{1}{2}, 1 \in [0, 2]$

Definition 16 (1.7.6) *The function $g : [1, 3] \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$ is monotonically decreasing on $[1, 3]$, since $g(x + k) = \frac{1}{x+k} < \frac{1}{x} = g(x)$ $\forall x \in [1, 3], k > 0$.For example $g(2) = \frac{1}{2} < \frac{1}{1} = 1 = g(1)$ as $1 < 2, 1, 2 \in [1, 3]$.*

Exercise1.7.1
Investigate the monotonicity or otherwise of the following functions: $3x^2 - 6x - 1$, $\frac{1}{x^2+1}$, $2 \sin \pi x$, $\sec x$, $\tan \pi x$, $\cosh x$, e^x, xe^x, $|x|$, $\ln(x + 1)$, $\log_{10}(x^2 + 3x + 2)$, $\sqrt{x}$, on the interval [0,1].

8 1.8 Zeros and Singularities of Rational Functions.

The values of x for which a function f takes the value zero are called the zeros of the function f. For example, the zeros of $f(x) = x^2 + 3x + 2$ are -2 and -1.

Given a rational function $f(x) = \frac{P(x)}{Q(x)}$,the values of x for which the denominator $Q(x) = 0$ are called the singularities of f. For example, the singularities of the function,

$f(x) = \frac{x^2-25}{x^2-5x+6}$ are 2 and 3.

Remark 17 (1.8.1) *For a rational function $f(x) = \frac{P(x)}{Q(x)}$, the zeros of f are the values of x for which the numerator $P(x) = 0$ while its singularities are the values of x for which the denominator $Q(x) = 0$. Thus for the rational function $f(x) = \frac{x^2+7x+12}{x^2-5x+6}$, the zeros are -3 and -4 while the singularities are 2 and 3.*

9 1.9 Injective, Surjective and Bijective Functions.

A function $f : Dom(f) \to \mathbb{R}$ is injective (one to one) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for $x_1$and $x_2 \in Dom(f)$.

i.e. for one element $x \in Dom(f)$,there is one and only one element y in the codomain such that $y = f(x)$.

That is two preimages in the domain of f do not have one image in the codomain of f.

A function $f : Dom(f) \to \mathbb{R}$ is surjective (onto) if every element in the codomain of f has a preimage in the domain of f.

That is the codomain is exhausted or the range and codomain of f are the same.

A function $f : Dom(f) \to \mathbb{R}$ is bijective if f is both injective and surjective.

Example 18 (1.9.1) *Which of the following functions is injective, surjective or bijective?*

(a) $f : \mathbb{R} \to \mathbb{R}; f(x) = 2x + 1$.
(b) $f : \mathbb{R} \to \mathbb{R}; f(x) = e^x$.
(c) $f : \mathbb{R} \to \mathbb{R}^+; f(x) = e^x$.
(d) $f : \mathbb{R} \to \mathbb{R}^+; f(x) = \frac{x^2+2}{x^2+1}$.
(e) $f : \mathbb{R} \to \mathbb{R}; f(x) = \frac{2x}{\sqrt{(x^2+1}}$.

Solution 19

(a) Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$.Since $f(x) = 2x + 1$,then $f(a) = f(b)$ implies that $2a + 1 = 2b + 1$and $a = b, \therefore a = b$ whenever $f(a) = f(b)$. Thus f is injective.

To show that f is surjective, let $2x + 1 = y \in \mathbb{R}$. Then $x = \frac{y-1}{2}$ which is in $\mathbb{R}$ $\therefore$every y in $\mathbb{R}$ has a preimage x in $\mathbb{R}$.

Thus f is surjective. Since f is both injective and surjective, it is bijective.

Solution 20

(b) $f : \mathbb{R} \to \mathbb{R}; f(x) = e^x$. Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$.

Then $e^a = e^b$.Taking the $\log_e$ of both sides gives $\log_e e^a = \log_e e^b \implies$ $a\log_e e = b \log_e e$ and $a = b$.Thus $a = b$ whenever $f(a) = f(b)$ and so f is injective.

Let $e^x = y \in \mathbb{R}$. Then $x \log_e e = \log_e y$ (since $\log_e e = 1$) or $x = \log_e y$.

But y is any real number and $\log_e y$ is not defined if y is zero or negative. $\therefore$For y zero or negative there is no real x such that $e^x = y$.

Thus f is not surjective because its range is $\mathbb{R}^+$although its codomain is $\mathbb{R}$;the range is not equal to the codomain. The conclusion is that f is injective but not bijective.

Solution 21

(c) $f : \mathbb{R} \to \mathbb{R}^+,\ f(x) = e^x$

Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$ then $e^a = e^b$.Taking $\log_e$ of both sides$\Longrightarrow$

$\log_e e^a = \log_e e^b \Longrightarrow a\log_e e = b\log_e e \implies a = b$. $\therefore a = b$ whenever $f(a) = f(b)$.

Thus f is one to one To show that f is a surjection, let $y \in \mathbb{R}^+$such that $e^x = y$.Then y is a positive real number and $x\log_e e = \log_e y \implies x = \log_e y$.

Since y is a positive real number, then $\log_e y \in \mathbb{R}$ $\therefore$there exist a preimage x namely $\log_e y$ for which $y = e^x$.

Thus f is (onto) surjective. Since f is injective and surjective, it is therefore bijective.

Solution 22

(d) $f : \mathbb{R} \to \mathbb{R},\ f(x) = \frac{x^2+2}{x^2+1}$.Let $a, b, \in \mathbb{R}$ be such that $f(a) = f(b)$. Then $\frac{a^2+2}{a^2+1} = \frac{b^2+2}{b^2+1}$

$\Longrightarrow (a^2+1)(b^2+2) = (a^2+2)(b^2+1)$ or $a^2b^2+2a^2+b^2+2 = a^2b^2+2b^2+a^2+2$ or

$2a^2 + b^2 = 2b^2 + a^2$ or $a^2 = b^2$ or $a = \pm\sqrt{b^2} = \pm b$.Thus, $a = \pm b$, where $f(a) = f(b)$.

It is therefore possible to have $f(a) = f(b)$ when $a \neq b$. Take for instance, 2 and -2.

Although $f(2) = \frac{2^2+2}{2^2+1} = f(-2) = \frac{(-2)^2+2}{(-2)^2+1} = \frac{6}{5}$ but $2 \neq -2$.Thus, one image, $\frac{6}{5}$ had two preimages namely 2 and -2.Therefore, f is not injective .

Let $y \in \mathbb{R}$ such that $\frac{x^2+2}{x^2+1} = y \implies x^2 + 2 = y(x^2 + 1) = yx^2 + 2$

$x^2(1 - y) = y - 2,\ x = \sqrt{(\frac{y-2}{1-y})}$.

Observe that $\sqrt{(\frac{y-2}{1-y})}$ is not defined for all values of y. For instance if $y = 0$, there is no real x such that $x = \sqrt{(\frac{y-2}{1-y})}$.

In fact for all $y \geq 2$,there is no value of x for which $x = \sqrt{(\frac{y-2}{1-y})}$.Thus f is not a surjection since not every y in $\mathbb{R}$ has no preimage in $\mathbb{R}$. Hence f is not bijective.

Solution 23

(e) $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{2x}{\sqrt{(2x+1)}}$, Suppose $f(a) = f(b)$, then

$\frac{2a}{\sqrt{(a^2+1)}} = \frac{2b}{\sqrt{(b^2+1)}}$ or $2a\sqrt{(b^2 + 1)} = 2b\sqrt{(a^2 + 1)}$ and on squaring both sides, we get $4a^2(b^2 + 1) = 4b^2(a^2 + 1)$ which implies that

$a^2b^2 + a^2 = a^2b^2 + b^2$or $a^2 = b^2$or $a = \pm b$.

By proceeding as in example (d), students can verify that f is neither one to one nor surjective. Therefore f is not bijective.

Example 24 (1.9.2) *Let the absolute value relation be as defined below. Find the domain and range for this relation to define a function. Is the absolute value function bijective?*

$$f(x) = |x| = +\sqrt{x^2} = \{\begin{smallmatrix} x, & \text{if } x\geq 0 \\ -x, & \text{if } x<0 \end{smallmatrix}$$

Solution 25 *The absolute value is defined for all real values of x and as such the domain is $\mathbb{R}$. For any real $x, |x| = +\sqrt{x^2} \geq 0$. Thus the range is the set of all non-negative real numbers, i.e.Ran$(f) = [0, \infty)$.Therefore, the relation defines a function with $\mathbb{R}$ as its domain and $[0, \infty)$ as its range. Let $|a| = |b|$ with $a, b \in \mathbb{R}$. This does not imply that $a = b$ .For instance $|-2| = |2| = 2$ but $-2 \neq 2$. The modulus function is therefore not injective. However for any nonnegative real number y in the range $[0, \infty)$, there is always a real number x whose modulus is $|x|$ .Thus the modulus function $|.|$ is surjective, since the range is $[0, \infty)$. The modulus function is therefore not bijective.*

Example 26 (1.9.3) *What do you understand by $|x| \leq a$ where a is a non negative real number?*

Solution 27 *If $x \geq 0$,then $x = |x| \leq a$ and $x \leq a$. If $x < 0$,then $x = -|x|$; given $|x| \leq a, -|x| \geq -a$ and $x \geq -a$.*

$$\therefore |x| \leq a \Rightarrow -a \leq x \leq a$$

Example 28 (1.9.4) *If $A = \{a, b, c\}$, find (i) $\chi_A(a)$, (ii) $\chi_A(b)$, (iii)$\chi_A(k)$.*

Solution 29 *(i) $\chi_A(a) = 1$ since $a \in A$.(ii) $\chi_A(b) = 1$ since $b \in A$. (iii) $\chi_A(k) = 0$ since $k \notin A$*

Example 30 (1.9.5) *Write the Characteristic function of the interval $[-1, 1)$.*

Solution 31 $\chi_{[-1,1)}(x) = \{\begin{smallmatrix} 1, & \text{if } -1\leq x<1 \\ 0, & \text{if } x\geq 1 \text{ or } x<-1 \end{smallmatrix}$

Example 32 (1.9.6) *Show that* $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$

Solution 33 *Since* $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}, e^{-x} = \sum\limits_{n=0}^{\infty} \frac{(-x)^n}{n!}$

$$\begin{aligned} \therefore \ e^x - e^{-x} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{x^n - (-x)^n}{n!} \end{aligned}$$

If n is even, then $x^n - (-x)^n = x^n - x^n = 0$ but if n is odd, then $x^n - (-x)^n = 2x^n$.Considering odd values of n, we

$\therefore \frac{e^x - e^{-x}}{2} = \frac{1}{2} \sum\limits_{n=0}^{\infty} 2 \frac{x^{2n+1}}{(2n+1)!}$

$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$
$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$

Exercise 1.9.1

(i) $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = \log_e x$,(Ans:injective,surjective,bijective)

(ii) $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = \log_e x + 2$,(Ans:injective,surjective,bijective)

(iii) $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{2x^2-1}{x^2+1}$,(Ans:not injective,not surjective,not bijective)

(iv) $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x$,(Ans:not injective,not surjective, not bijective)

(v) $f : \mathbb{R} \to [-1, 1], f(x) = \sin x$,(Ans:not injective,surjective,not bijective)

(vi) Sketch the graph of the modulus function between $x = -3$ and $x = 3$.What are the images of -2 and 2? $(Ans.2)$

(vii) If Peter is a Maths student but Jane is not, what is
$\chi_{(\text{math students})}$(Peter)+$\chi_{(\text{non math students})}$(Jane)? $(Ans.2)$

(viii) Show that $\frac{e^x+e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$

(ix) If $\cosh x = \frac{e^x+e^{-x}}{2}$ and $\sinh x = \frac{e^x-e^{-x}}{2}$, find $\cosh 0 + \sinh 0$.
What is $\cosh x + \sinh x$? $(Ans.1; e^x)$.

10 1.10 Composite Functions

Let $f : X \to Y$ and $g : Y \to Z$.The function $h : X \to Z$ defined by $h(x) = (gof)(x) = g(f(x))$ is called the composition of g with f.

Example 34 (1.10.1) *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 2x + 3$ and $f(x) = x^2 + x$. Find (i) $g(f(5))$ (ii) $g(f(x))$*

Solution 35 $g(f(5)) = g(30)) = 2 \times 30 + 3 = 63$.
$g(f(x)) = g(x^2 + x) = 2(x^2 + x) + 3 = 2x^2 + 2x + 3$

Example 36 (1.10.2) *Verify that $f(g(x) \neq g(f(x))$ in general by evaluating the two functions at a point $x = b$ of your choice.*

Solution 37 *From the example above $g(f(5)) = g(30)) = 2 \times 30 + 3 = 63$.*
Similarly we evaluate $f(g(x))$ for $x = 5$; $f(g(5)) = f(13) = (13)^2+13 = 182$
Thus with $b = 5$ the result is established.

Example 38 (1.10.3) *If $k(x) = 2x + 3$ and $t(x) = x + 1$, prove that $t(k(x)) = 2x + 4$ and $k(t(x)) = 2x + 5$.For what values of x do we have $k(t(x)) = t(k(x))$?*

Solution 39 $t(k(x)) = t(2x + 3) = (2x + 3) + 1 = 2x + 4$;

$k(t(x)) = k(x + 1) = 2(x + 1) + 3 = 2x + 5$.

If $k(t(x)) = t(k(x))$, then $2x + 5 = 2x + 4$;but there is no value of x such that $2x + 5 = 2x + 4$.Therefore $k(t(x)) \neq t(k(x))$ for any value of x.

Example 40 (1.10.4) *Let $g:\mathbb{R}\to\mathbb{R}$ such that $g(x)=2x+3$ Find $g(x^2)$ and $g(g(x))$*

Solution 41 $g(x^2)=2(x^2)+3=2x^2+3$

and $g(g(x))=g(2x+3)=2(2x+3)+3 \ =4x+9.$
Exercise 1.10.1

(a) Evaluate the following: (i) $\sin(\cos\frac{\pi}{2})$ (ii)$\cos^{-1}(\sin\pi)$
(Ans :(i) 0;(ii) $\pi/2$).
(b) If $f(x)=x^2$ and $g(x)=\sqrt{x^2+1}$, Find (i) $f(g(x))$;(ii)$g(f(x))$.
(Ans. (i) x^2+1; (ii) $\sqrt{x^4+1}$
(c) Evaluate the following: (i) $\log_e(e^{\pi})$ (ii) $\log(\sin\frac{\pi}{2})$ (iii)$e^{\cos 2\pi}$
Ans.(i) π, (ii) 0; (iii) $e^1=e$.

11 1.11 Inverse Functions

Let f be a bijective function with domain X and range Y,then a function g with domain Y and range $X$is called the inverse function

of f if $g(f(x))=x$ for every x in X and $f(g(y))=y$ for every y in Y. The inverse function g of f if it exists, is usually denoted by f^{-1}.

Remark 42 (1.11.1) *The symbol $^{-1}$ as used in the notation f^{-1} should not be mistaken for an exponent;*

that is, $f^{-1}(x)$ does not mean $\frac{1}{[f(x)]}$. The reciprocal $\frac{1}{[f(x)]}$ is denoted by $[f(x)]^{-1}$.

Remark 43 (1.11.2) *It is absolutely essential that a function f be bijective for its inverse to be defined. This is one reason why we study bijective functions before inverses of functions are introduced.*

11.1 1.11.1 Guidelines for finding the inverse of some functions.

(a) Verify that f is bijective (i.e. one to one and onto). if it is not one can only find an inverse of a restriction of the function that is bijective..

(b) when it is bijective, solve the equation $y=f(x)$ for x in terms of y, obtaining an equation of the form $x=f^{-1}(y)$.

(c)Verify the two conditions $f^{-1}(f(x))=x$ and $f(f^{-1}(x))=x$ for every x in the domain of f and f^{-1} respectively.

Example 44 (1.11.1.1) *Find the inverse of $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=$* $3x+11$.

Solution 45 *First we know that the function f is linear (a straight line)with slope* 3*, y-intercept* 11*. It is bijective throughout* $\mathbb{R}$.

Thus, the inverse function f^{-1} exists. Moreover, since the domain and range of the bijective function f is $\mathbb{R}$, the same is true for f^{-1}.

Now solving $y = 3x + 11$ for x, we obtain $x = \frac{y-11}{3}$.We let $f^{-1}(y) = \frac{y-11}{3}$ and since the symbol used for the variable is immaterial, we can write $f^{-1}(x) = \frac{x-11}{3}$.Finally, we verify that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.Thus $f^{-1}(f(x)) = f^{-1}(3x + 11) = \frac{3x+11-11}{3} = \frac{3x}{3} = x$ and $f(f^{-1}(x)) = f(\frac{x-11}{3}) = 3(\frac{x-11}{3}) + 11 = x - 11 + 11 = x$.This proves that the inverse function of f is given by $f^{-1}(x) = \frac{x-11}{3}$.

Example 46 Example 47 (1.11.1.2) *Find the inverse of $g : \mathbb{R} - \{7\} \to \mathbb{R} - \{7\}$ given by $g(x) = \frac{3x+5}{x-7}$.*

Solution 48 *The function $g : \mathbb{R} - \{7\} \to \mathbb{R} - \{7\}$ is bijective.Set $y = \frac{3x+5}{x-7}$ and solving for x we obtain, $x = \frac{7y+5}{y-3}, y \neq 3$.*

Let $g^{-1}(y) = \frac{7y+5}{y-3}, y \neq 3$ so that $g^{-1}(x) = \frac{7x+5}{x-3}, x \neq 3$.Thus, $g(g^{-1}(x)) = \frac{3(\frac{7x+5}{x-3})+5}{\frac{7x+5}{x-3}-7} = \frac{26x}{26} = x = g^{-1}(g(x))$.

Therefore, $g^{-1}(x) = \frac{7x+5}{x-3}$ is the inverse of $f(x) = \frac{3x+5}{x-7}$

Example 49 (1.11.1.3) *Prove that $g(x) = -1 + \sqrt{x-1}$ is the inverse of a restriction of the function $f(x) = x^2 + 2x + 2$.*

Solution 50 *Since a quadratic equation always has two real roots, it should be observed that the function $f : \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = x^2 + 2x + 2$ is not bijective. For example, $f(x) = 5$ has two solutions $x = 1$ and $x = -3$. Thus $f(1) = 1^2 + 2(1) + 2 = 5 = (-3)^2 + 2(-3) + 2$. We note that $f : \mathbb{R} \to \{y : y > 1\}$ and $f(-1) = 1$.Thus we can restrict the function f and consider $f_1 : R_1 \to \{y : y > 1\}$ where $R_1 \subset (-1, \infty)$ with the same rule of definition i.e. $f_1(x) = x^2 + 2x + 2$. $f_1$is bijective and therefore has an inverse g. $g : \{y : y > 1\} \to R_1$. Thus, by taking the composition of f_1 with g or substituting the value of $g(x)$ in x^2+2x+2 we have $f(g(x)) = [-1+\sqrt{(x-1)}]^2+2[-1+\sqrt{x-1}]+2 = (1 - 2\sqrt{x-1} + x - 1) - 2 + 2\sqrt{x-1} + 2 = x$.*

Hence the function g is the inverse of f_1 .

Exercise 1.11.1.1 Given the rule of definition of $f(x)$ below, find a domain and range such that $f(x)$ defines a bijective function. Find the inverse $g(x)$ of an appropriate restriction of the bijective function identified.

(a) $f(x) = \frac{2x+3}{7x-2}$.
(b) $f(x) = \frac{1}{(1-x)^2}$.
(c) $f(x) = \sqrt{x^2 - 1}$.
(d) $f(x) = \sqrt{x^2 + 1}$.

Module 2: Limits and Continuity of Functions

12 2.1 Objectives:

In this module we give an elementary introduction to the concepts of limits and continuity of a function. We discuss methods of finding limits and properties of limits, including indeterminate forms. We then define and explore continuous functions. We state some useful results without proof and show how to apply them. The exploration of the learning activities will provide an opportunity for in-depth study of limits and continuity.

13 2.2 Learning outcomes:

At the end of this module students should be able to

- Find the limit of a function at a point;
- Identify indeterminate forms and use L'hospital's rule;
- Define and establish the continuity of simple functions

14 2.3 Learning Activities

- Solve relevant questions from past MTH 103 examinations;
- Explore and use related materials on the Intranet, especially the e-granary and MIT open courseware;
- Explore and use related materials on the Internet especially units 5 and 6 of Calculus of the OLI courses;

15 2.4 Limits and continuity of a function:

15.1 2.4.1 Limit of a functionn:

[2.4.1.1] Limit from the left: Let $f(x)$ be defined in an interval (a, b); we say $f(x)$ tends to a limit l as x tends to b from the left if: given $\epsilon > 0$, we can find $\delta > 0$ such that $|f(x) - l| < \epsilon$ provided that $b - \delta < x < b$.We write this as $f(x) \to l$ as $x \to b^-$; or

$\lim_{x \to b^-} f(x) = l$.

[2.4.1.2] Limit from the right: Let $f(x)$ be defined in an interval (a, b); we say $f(x)$ tends to a limit l as x tends to a from the right if: given $\epsilon > 0$, we can find $\delta > 0$ such that $|f(x) - l| < \epsilon$ provided that $a < x < a + \delta$.We write this as $f(x) \to l$ as $x \to a^+$ or $\lim_{x \to a^+} f(x) = l$.

Definition 51 (2.4.1.3) *The limit of a function:*

Let $f(x)$ be defined in an interval (a, c) except possibly for some point $b \in (a, c)$; we say $f(x)$ tends to a limit l as x tends to b if: given $\epsilon > 0$, we can find $\delta > 0$ such that $|f(x) - l| < \epsilon$ provided that $0 < |x - b| < \delta$.We write this as $f(x) \to l$ as $x \to b$; or

$\lim_{x \to b} f(x) = l$.

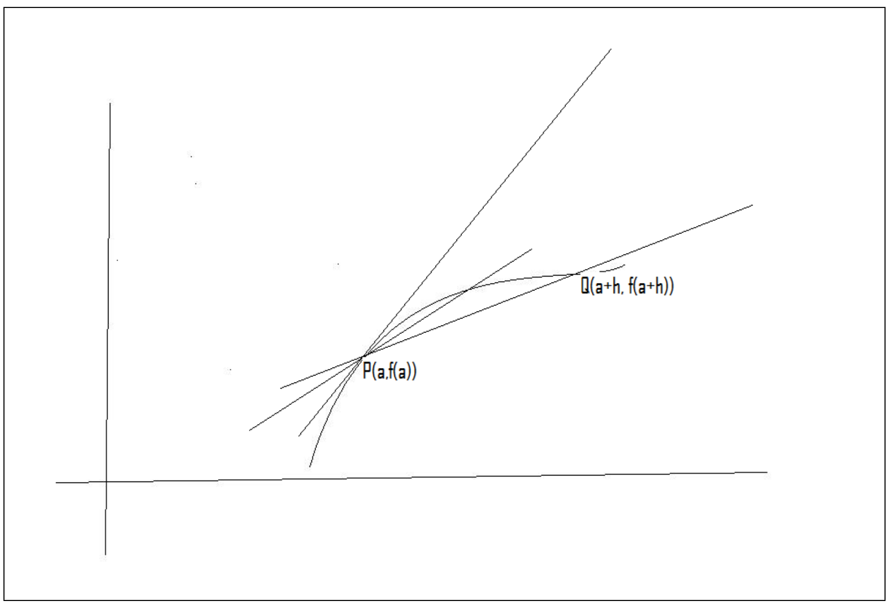
It follows that when $f(x)$ is defined in an interval (a, c) except possibly for some point $b \in (a, c)$; $f(x)$ tends to a limit l as x tends to b if and only if $f(x)$ tends to the limit l as x tends to b^+ and $f(x)$ tends to the limit l as x tends to b^-.

Let us explore the use of these definitions in the examples below.

[2.4.1.1] Examine the limit from the left and right of the function $f(x)$ given by $f(x) = x, x \leq 0$ and $f(x) = 1 + x, x > 0$. Does $f(x)$ tends to a limit l as x tends to 0? Sketch the graph of the function.

Solution 52 $\lim_{x \to 0^-} f(x) = 0$; $\lim_{x \to 0^+} f(x) = 1; f(0) = 0$. $\lim_{x \to 0} f(x)$ *does not exist, since the left and the right limits are different.*

$$y = \begin{cases} x & \text{if} \quad x \leq 0 \\ 1 + x & \text{if} \quad x > 0 \end{cases}$$

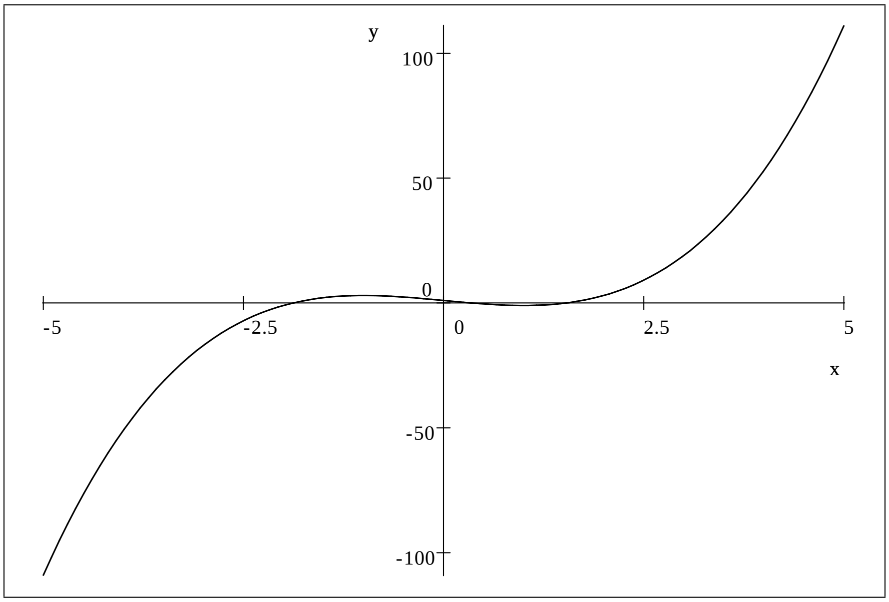


[2.4.1.2] Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$, $x < 0$; $f(x) = 1$, $x = 0$; $f(x) = x^2, 0 < x < 1$; $f(x) = 2x - 1, 1 \leq x$; Decide whether $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ exist. Find any limit which exists. Sketch the graph of the function.

Solution 53 $\lim_{x \to 0^-} f(x) = 0$; $\lim_{x \to 0^+} f(x) = 0; f(0) = 1$. $\lim_{x \to 0} f(x) = 0$.

$\lim_{x \to 1^-} f(x) = 1$; $\lim_{x \to 1^+} f(x) = 1; f(1) = 1$. $\lim_{x \to 1} f(x) = 1$. For all other values of x_0, $x_0 \neq 0$ or 1, $\lim_{x \to x_0} f(x)$ exists.

$$y = \begin{cases} \sin x & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ x^2 & \text{if} \quad 0 < x < 1 \\ 2x - 1 & \text{if} \quad 1 \le x \end{cases}$$



[2.4.1.3] Use the $\varepsilon-$ δ argument to show that $\lim\limits_{x\to 2} g(x) = 7$ if $g(x) = 2x + 3$

Solution 54 *Given* $\varepsilon > 0$, *we need to find a* $\delta(\epsilon) > 0$ *such that* $|g(x) - 7| < \varepsilon$ *whenever* $|x - 2| < \delta$.

Now, $|g(x) - 7| = |2x + 3 - 7| \ = |2x - 4| = |2(x - 2)| = 2\,|x - 2|$.
To get $|g(x) - 7| = 2\,|x - 2| < \epsilon$, We must choose $\delta = \frac{\varepsilon}{2}$.
[2.4.1.4] Use the $\varepsilon-$ δ argument to show that $\lim\limits_{x\to -2} \frac{x^2-4}{x+2} = -4$

Solution 55 *Given* $\varepsilon > 0$, *we need to find a* $\delta(\epsilon) > 0$ *such that* $\left|\frac{x^2-4}{x+2} - (-4)\right| <$ ε *whenever* $|x - (-2)| = |x + 2| < \delta$.

Now,

$$
\begin{aligned}
\left|\frac{x^2-4}{x+2}-(-4)\right| &= \left|\frac{x^2-4}{x+2}+4\right| \\
&= \left|\frac{x^2-4+4x+8}{x+2}\right| \\
&= \left|\frac{x^2+4x+4}{x+2}\right| \\
&= \left|\frac{(x+2)\,(x+2)}{x+2}\right| \\
&= |x+2|_{x\neq -2} \\
&= |x+2| < \varepsilon
\end{aligned}
$$

$\left|\frac{x^2-4}{x+2}-(-4)\right| < \varepsilon$ when δ is chosen equal to ε.

Definition 56 (2.4.1.4) *Let $f:\mathbb{R}\to\mathbb{R}$ we say $f(x)\to\infty$ as $x\to a$ if for $|x-a|<\delta, f(x)>A$ for any value of A however large. Let $f:\mathbb{R}\to\mathbb{R}$ we say $f(x)\to-\infty$ as $x\to a$ if for $|x-a|<\delta, f(x)<-A$ for any value of A however large. Similarly we can define $f(x)\to-\infty$ as $x\to\infty$ and $f(x)\to\infty$ as $x\to\infty$.*

15.2 2.4.2 Properties of Limit

1. If a function f has a limit l at a point a, then this limit is unique.

2. The limit of a constant function c equals the constant c.

3. If $f(x)\to l_1$ and $g(x)\to l_2$ as $x\to a$, then

   (a) $\lim\limits_{x\to a}[f(x)+g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = l_1+l_2$ (the limit of a sum is the sum of the limits) and

   (b) $\lim\limits_{x\to a}[f(x)-g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = l_1-l_2$ (the limit of a difference is the difference of the limits).

4. $\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a} f(x)\lim\limits_{x\to a} g(x) = l_1 l_2$ (the limit of a product is the product of the limits.

5. In the special case when $g(x)=k$, $lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$ where k is a constant.

6. $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{l_1}{l_2}$, provided $l_2\neq 0$, (the limit of a quotient is the quotient of the limits).

15.3 2.4.3 Continuity

[2.4.3.1] A function $f(x)$ is continuous at a point $x = a$ in its domain if $\lim\limits_{x\to a} f(x)$ exists and has value $f(a)$. If $f(x)$ is continuous at every point in its domain, then $f(x)$ is a continuous function.

Equivalently, a function $f(x)$ is continuous at a point a in its domain if , for each $\varepsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\mid f(x) - f(a) \mid< \varepsilon$ provided that $\mid x - a \mid< \delta$.

A function $f(x)$is discontinuous at a point a in its domain if it is not continuous there.

15.4 2.4.4 Limits of Polynomial and Rational Functions

[2.4.4.1] If $p(x)$ is any polynomial given by

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

then $\lim\limits_{x\to a} p(x) = p(a)$ and $p(x)$ is continuous at $x = a$.

[2.4.4.2] If $f(x)$ is any rational function with $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(a) \neq 0$, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ and $f(x)$is continuous at $x = a$.

15.5 2.4.5 Indeterminate Forms

If as $x \to x_0$, $\frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$, we say $\frac{f(x)}{g(x)}$ has an indeterminate form. This is also the case if as $x \to x_0$, $\frac{f(x)}{g(x)}$ takes the form $\frac{\infty}{\infty}$. The limit of some functions with indeterminate form can be evaluated using L'hospital's rule. The rule states that if the functions $f(x)$ and $g(x)$ have derivatives on an open interval (x_1, x_2) containing x_0, and $g'(x) \neq 0$ for $x =\ x_0$ and if $\frac{f(x)}{g(x)}$ has an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x \to x_0$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f\ '(x)}{g'(x)}$$

provided that

$$\lim_{x\to x_0} \frac{f\ '(x)}{g'(x)}$$

exists or

$$\lim_{x\to x_0} \frac{f\ '(x)}{g'(x)} = \infty$$

[2.4.5.1] Let $f : \mathbb{R}^+ - \{4\} \to \mathbb{R}$ be defined by $f(x) = \frac{x-4}{\sqrt{x}-2}$.Find the limit of the function $f(x)$ as x approaches 4 and sketch the graph of f .

Solution 57 *We rationalise the denominator to get:*

$\lim\limits_{x\to 4} f(x) = \lim\limits_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim\limits_{x\to 4} (\frac{x-4}{\sqrt{x}-2} . \frac{\sqrt{x}+2}{\sqrt{x}+2}) = \lim\limits_{x\to 4} (\frac{(\sqrt{x}+2)(x-4)}{x-4})$
$= \lim\limits_{x\to 4} \sqrt{x} + 2 = 4.$

This argument is valid since $x \neq 4$ and $\sqrt{x} + 2$ is continuous.

[2.4.5.2] Let $f : \mathbb{R} - \{2\} \to \mathbb{R}$ be defined by $f(x) = \frac{x^2-4}{x+2}$. Find the limit of $f(x)$ as $x \to -2$.

Solution 58 *We evaluate this limit by factorising the numerator. We then cancel the common factor since $x \neq -2$. Since $x - 2$ is a continuous function we get*

$$\lim_{x\to -2} \frac{x^2-4}{x+2} = \lim_{x\to -2} (x-2) = -2 - 2 = -4,$$

[2.4.5.3] Evaluate the following limits if they exist:

i) $\lim\limits_{x\to 0} \frac{e^x - e^{-x}}{2x}$ ii) $\lim\limits_{x\to 2} \frac{-2x}{-x(x^2+5)^{\frac{-1}{2}}}$

Solution 59 *i)* $\lim\limits_{x\to 0} \frac{e^x + e^{-x}}{2x} = \lim\limits_{x\to 0} \frac{e^x - e^{-x}}{2} = \frac{0}{2} = 0$

ii) $\lim\limits_{x\to 2} \frac{-2x}{-x(x^2+5)^{-\frac{1}{2}}} = \lim\limits_{x\to 2} 2.\sqrt{(x^2+5)} = 6.$

Problem [2.4.5.1]: Evaluate the limits below if they exist

i) $\lim\limits_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x}$ ii) $\lim\limits_{x\to 1} \frac{\ln x}{x^3-1}$ iii) $\lim\limits_{x\to \infty} \frac{e^x - 1}{3x-1}$

[2.4.5.4] Evaluate the following limits if they exists:

i) $\lim\limits_{x\to\infty} \frac{e^x}{\ln x}$ ii) $\lim\limits_{x\to\infty} \frac{e^{3x}-1}{x-\sin x}$ iii) $\lim\limits_{x\to 0} \frac{\cot 2x}{\cot 3x}$

Solution 60 *i)* $\lim\limits_{x\to\infty} \frac{e^x}{1/x} = \lim\limits_{x\to\infty} xe^x = \infty$

ii) $\lim\limits_{x\to\infty} \frac{3e^{3x}}{1-\cos x} = \lim\limits_{x\to\infty} \frac{9e^{3x}}{\sin x} = \lim\limits_{x\to\infty} \frac{27e^{3x}}{-\cos x} = \infty.$

Hence, the limit does not exist.

iii) $\lim\limits_{x\to 0} \frac{\cot 2x}{\cot 3x} = \lim\limits_{x\to 0} \frac{\cos 2x}{\sin 2x} . \frac{\sin 3x}{\cos 3x} = \lim\limits_{x\to 0} \frac{-2\sin 2x . \sin 3x + \cos 2x . 3\cos 3x}{-2\cos 2x . \cos 3x - 3\sin 2x . 3\sin 3x} = \frac{3}{-2}$

Remark 61 (2.4.5.1) *Other Indeterminate Forms*

1.If $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$ has the form $0.\infty$ or $\infty.0$, we say its form is also indeterminate. This form can be changed to the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by writing

$f(x)g(x) = \frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$ before L'Hospital's rule is applied.

[2.4.5.5] i) $\lim\limits_{x\to 0} x \cot x$ ii) $\lim\limits_{x\to\infty} (x^2 - 1)e^{-x^2}$

iii) $\lim\limits_{x\to \frac{\pi}{2}} (x - \frac{\pi}{2}) \tan x$

Solution 62 *i)* $\lim\limits_{x\to 0} \frac{x}{\tan x} = \lim\limits_{x\to 0} \frac{1}{\sec x} = 1$

ii) $\lim\limits_{x\to\infty}\frac{(x^2-1)}{e^{x^2}} = \lim\limits_{x\to\infty}\frac{2x}{2xe^{x^2}} = \lim\limits_{x\to\infty}\frac{1}{e^{x^2}} = 0$

iii) $\lim\limits_{x\to\frac{\pi}{2}}(x-\frac{\pi}{2})\tan x = \lim\limits_{x\to\frac{\pi}{2}}\frac{(x-\frac{\pi}{2})}{\cot x} = \lim\limits_{x\to\frac{\pi}{2}}\frac{1}{-\cos ec^2 x} = \lim\limits_{x\to\frac{\pi}{2}}\frac{1}{-1/\sin^2 x} = -\lim\limits_{x\to\frac{\pi}{2}}\sin^2 x = -1$

[2.4.5.6] Determine whether the functions f and g as defined below are continuous or not at the points indicated.

(i) $f(x) = \{\begin{smallmatrix}\frac{x^2-x-6}{x-3} & \text{if } x\neq 3\\ 5, \text{ if } & x=3\end{smallmatrix}$ at $x = 3$.

(ii) $g(x) = \{\begin{smallmatrix}\frac{x^2-9}{x+3} & \text{if } x\neq -3\\ 10 & \text{if } x=-3\end{smallmatrix}$ at $x = -3$

Solution 63 *(i) Since* $x \neq 3, \lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{(x-3)(x+2)}{x-3}$

$= \lim_{x\to 3}(x+2) = 5 = f(3)$.

Consequently f is continuous at $x = 3$.

(ii) Since $x \neq -3, \lim_{x\to -3} g(x) = \lim_{x\to -3}\frac{(x-3)(x+3)}{x+3}$

$= \lim_{x\to -3}(x-3) = -6$.

But $g(-3) = 10$. Since $\lim\limits_{x\to -3} g(x) \neq g(-3)$, this function is not continuous at $x = -3$.

Suppose f and g are continuous functions,then their sum, $f + g$ (difference, $f - g$), product,fg and quotients $\frac{f}{g}$ are also continuous , except where the denominator is zero.

This result helped us prove a polynomial is continuous at every point at which it is defined.

15.5.1 2.4.6 Limits that arise frequently

We state without proof some limits that arise very frequently in Mathematics, Science, Technology and Engineering.

(i) $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

(ii) $\lim\limits_{\theta\to 0}\frac{\cos\theta-1}{\theta} = 0$.

(iii) $\lim\limits_{n\to\infty}\frac{\ln n}{n} = 0, \forall n \in \mathbb{N}$.

(iv) $\lim\limits_{n\to\infty}\sqrt[n]{n} = 1$.

(v) $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1, (\text{for } x > 0, \forall n \in \mathbb{N})$.

(vi) $\lim\limits_{n\to\infty} x^n = 0, (\text{for } |x| < 1, \forall n \in \mathbb{N})$.

(vii) $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x, (\text{for any } x, \forall n \in \mathbb{N})$.

(viii) $\lim\limits_{n\to\infty}\frac{x^n}{n!} = 0$, (for any $x, \forall n \in \mathbb{N}$).

(ix) $\lim\limits_{n\to\infty}\frac{1}{n} = 0$

Example 64 (2.4.6.1) *(1) If* $|x| < 1$, *then*

$\lim\limits_{n\to\infty} x^{n+4} = \lim\limits_{n\to\infty} x^4 . \lim\limits_{n\to\infty} x^n = x^4.0 = 0$.

(2) $\lim\limits_{n\to\infty}\sqrt[n]{2n} = \lim\limits_{n\to\infty}\sqrt[n]{2}\lim\limits_{n\to\infty}\sqrt[n]{n} = 1.1 = 1$

(3) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e.e = e^2$

(4) $\lim\limits_{n\to\infty}\frac{x^{n+1}}{(n+1)!} = \lim\limits_{n\to\infty}\frac{x}{(n+1)}\lim\limits_{n\to\infty}\frac{x^n}{n!} = 0.0 = 0$

MODULE THREE- The Derivative

16 3.1 Objectives

In this module, we expose students to the concept of differentiation emphasizing its dependence on limits. We discuss rules for differentiation of different combinations of functions. We explore the application of derivatives to illustrate the use of calculus in the solution of real life problems. Worked examples are provided to aid understanding of the material introduced. Group assignments, take home tests and laboratory sessions will complement power point presentations in tutorial type exchanges. All these activities expose the students to the concepts and techniques and lead them to find out when the different techniques are most appropriate for use.

17 3.2 Learning Outcomes

At the end of this chapter, students should be able to:

22c5 Appreciate the dependence of the derivative on limits

22c5 Apply the product, quotient and chain rules

22c5 Differentiate implicit, exponential, logarithm and hyperbolic functions

22c5 Use the first and second derivative tests to investigate the maxima and minima of functions.

22c5 Investigate the Rolle's and mean value theorems.

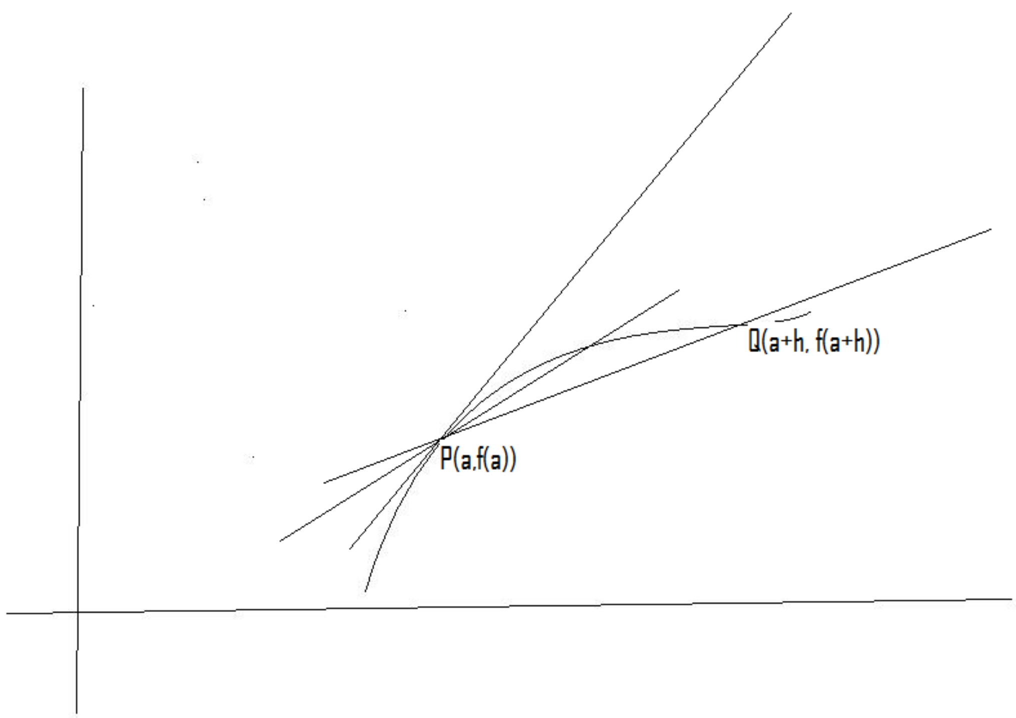
18 3.3 Learning Activities

Students should:

- Explore notes and exercises, individually and in groups;
- Explore and use related materials on the Intranet, especially the e-granary and MIT open course ware;
- Explore and use relevant materials from the MIT open course ware 18.01 Calculus;
- Solve relevant questions from past MTH103 Examinations.

19 3.4 Introduction

The slope of a curve at any point a is the slope of the tangent to the curve at a. Let $P(a, f(a))$ be any point on the graph of a continuous function f and $Q(a+h, f(a+h))$ be another point where h is the difference between the x coordinates of Q and P.


Q(a+h, f(a+h))
P(a,f(a))

The line PQ above is the secant line whose slope $M_{PQ} = \frac{f(a+h)-f(a)}{h}$ while the line L is the tangent line at P.

The tangent line at P is the limiting value of M_{PQ} as Q approaches P hence the slope M of the tangent of line L is given by: $M = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = f^{/}(a)$. The symbol $f'(a)$ read as f prime of a was introduced by Lagrange (1736-1813). The idea is that as Q changes along the curve PQ becomes shorter secant lines, and the lines through PQ, in the limiting situation becomes the tangent line L.

Definition 65 *(3.4.1): Let $f(x)$ be a function that is defined on an interval containing a. The slope M of the tangent line to the graph of f at the point $P(a, f(a))$ is $M = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = f^{/}(a)$.*

provided the limit exists. Thus the derivative is the value of the slope of the curve at point P.The concept above, of the derivative of a function at a point, was invented by Isaac Newton (1642-1727). This limit M, if it exists, is a function. This is the case since as a changes , so does $f(a)$ and each point in the domain of f has a unique image which we call $f^{/}(a)$. The new function is refered to as the derived function $f^{/}(x)$.When we find $f^{/}(x)$ using this definition and the limiting process, we say we are finding $f^{/}(x)$ from first principles.

Definition 66 *(3.4.2):(Rate of change of a function). Given y as a function of x (as x changes y also changes), the rate of change of the function y is the change in y divided by the corresponding change in x. That is, if x and $x+\Delta x$ are the values of x, with $\Delta x \neq 0$; and $y = f(x)$ and $y+\Delta y = f(x+\Delta x)$ are the corresponding values of y, then the rate of change of the function as x changes from x to $x+\Delta x$ is given by $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}$. The derivative of a function is therefore given by $\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$*

we write this last limit as $\frac{dy}{dx}$ and we call it a differential operator. Note that $\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$ is a limit and not a quotient. The differential operator $\frac{dy}{dx}$ was first used by Liebnitz to denote the derivative of a function. This notion of derivatives as the rate of change is applicable in a variety of disciplines including Biological, Social, Physical,and Economic theory.

In particular if a body moves with its displacement given by $s(t)$, the velocity $v(t)$ of the body at time t is given by $v(t) = \frac{ds}{dt} = \lim_{\Delta t\to 0} \frac{s(t+\Delta t)-s(t)}{\Delta t}$. The derivative of the velocity $\frac{dv}{dt}$is the acceleration a, of the body. $a(t) = \frac{dv}{dt}$.

20 3.5 Differentiability of a function at a point:

Definition 67 *(3.5.1): Let $f(x)$ be a function that is defined on an open interval containing a .The derivative of $f(x)$ at $x = a$ written as $f'(a)$ is given by $f'(a) = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$.*

If $f'(a)$ exists, we say that $f(x)$ is differentiable at a .If $f(x)$ is differentiable at a then $f'(a)$ is the slope of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$.

(3.5.1): We explore the derivative of $f(x) = x^n$ for specific and general values of n. Case $n = 1, f(x) = x,\ f(x+h) = x+h$, and $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{h}{h} = 1$. $f(x) = x^2,\ f(x+h) = (x+h)^2$, and $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{(x+h)^2-x^2}{h} = .\lim\limits_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x$. $f(x) = x^3, f(x+h) = (x+h)^3, and \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{(x+h)^3-x^3}{h} = .\lim\limits_{h\to 0} \frac{3x^2h+3xh^2}{h} = \lim_{h\to 0}(3x^2 + 3xh) = 3x^2$. $f(x) = x^n, f(x+h) = (x+h)^n, and \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{(x+h)n-x^n}{h} = .\lim\limits_{h\to 0}(nx^{n-1} + C_2^n x^{n-2}h + C_3^n x^{n-3}h^2 + ... + h^{n-1}) = nx^{n-1}$. Since $\frac{(x+h)n-x^n}{h} = \quad nx^{n-1} + C_2^n x^{n-2}h + C_3^n x^{n-3}h^2 + ... + h^{n-1}$

(3.5.2): We can show that when a function is differentiable at a point it is continuous there. This follows since $\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}[\frac{f(x)-f(a)]}{x-a}.(x-a)] \quad = \lim_{x\to a}[\frac{f(x)-f(a)]}{x-a}]. \lim_{x\to a}[(x-a)] \ = f^{/}(a).0 = 0$.

(3.5.3): State the converse of the result above. Show that is not true by examining the function $f(x) = |x|$.

(3.5.4): We can show that $sinx$ is differentiable with derivative $cosx$. Also $cosx$ is differentiable with derivative $-\sin x$.

21 3.6 Methods of Differentiation:

We shall in this section introduce some rules that simplify the task of finding derivatives of different classes of functions and display their use.

21.0.2 3.6.1. Differentiation of sums and powers:

The first set of rules that we introduce in this subsection enable us to differentiate any polynomial.

We can show by using first principles that:

(i). If $f(x) = c$ then $f^{/}(x) = 0$.

(ii).If $f(x) = cg(x)$ and $g(x)$ is differentiable with derivative $g^{/}(x)$ then $f^{/}(x) = [cg(x)]^{/} = cg^{/}(x)$.

(iii).If $f(x)$ and $g(x)$ are two differentiable functions with derivatives $f^{/}(x)$ and $g^{/}(x)$ respectively then $[f(x) \pm g(x)]^{/} = f^{/}(x) \pm g(x)^{/}$.

How do we use the rules above to show that every polynomial is differentiable?

Example 68 (3.6.1.1) *: Let us find the derivative of the functions below using the first principles or the methods introduced above:*

(i). $4x^{100}$;

(ii).$x^4 + 2x$.

(iii).$(x^4 + 2).(x^4 - 2)$.

$(iv).(x^2+2)^3$.

21.0.3 3.6.2.The product rule

This rule enables us to differentiate the product of two functions when we know the derivatives of each one of them. It can be written in several ways and we present two below.

[3.6.2.1] : The product rule can be written as: Given two differentiable functions $f(x)$ and $g(x)$, then $[f(x).g(x)]^{/} = [f(x)g'(x) + g(x).f'(x)]$ or Given $y = u(x).v(x)$ with u(x) and v(x) differentiable, the derivative of $y^{/}$of y is then $y^{/} = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$. A sketch of the proof of the result follows.

$[f(x).g(x)]^{/} = \lim_{h\to 0}[\frac{f(x+h).g(x+h)-f(x).g(x)]}{h} = \lim_{h\to 0}\left[\frac{f(x+h)\{g(x+h)-f(x+h)g(x)\}+f(x+h).g(x)-f(x).g(x)\}}{h}\right] =$ $\lim\limits_{h\to 0}\left(f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\right)$

$= \lim\limits_{h\to 0}f(x+h)\lim\limits_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right) + g(x)\lim\limits_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

Since f is differentiable and therefore continuous at x, we have $\lim\limits_{h\to 0}f(x+h) =$ $f(x)$ and the limit becomes $f(x)g'(x) + g(x)f'(x)$.

Example 69 (3.6.2.1) : *We find $k'(x)$ if $k(x) = (x^2-2)(3x^3+5x-2)$ using the product rule.*

Solution 70 ***:W****e let* $f(x) = x^2 - 2$ *and* $g(x) = 3x^3 + 5x - 2$

$f'(x) = 2x$ and $g'(x) = 9x^2 + 5$.

Thus $k'(x) = (3x^3+5x-2)(2x) + (x^2-2)(9x^2+5)$.This is the result. We simplify for convenience only and we get:

$= 6x^4 + 10x^2 - 4x + 9x^4 + 5x^2 - 18x^2 - 10 = 15x^4 - 3x^2 - 4x - 10$. We note that we can express the product as a single polynomial and differentiate term by term.

21.0.4 3.6.3.The quotient rule

This rule facilitates the differentiation of the quotient of two differentiable functions. We present it in two equivalent statements below.

[3.6.3.1] : The quotient rule can be written as: Given two differentiable functions $f(x)$ and $g(x)$, then $[\frac{f(x)}{g(x)}]^{/} = [\frac{g(x).f'(x)-f(x)g'(x)}{[g(x)]^2}]$ or Given $y = \frac{u(x)}{v(x)}$ with u(x) and v(x) differentiable, the derivative of $y^{/}$of y is then $y^{/} = \frac{v\frac{du}{dx}-u(x)\frac{dv}{dx}}{v^2}$. A sketch of the proof of the result follows.

If $y'(x)$ exists, $y'(x) = \lim\limits_{h\to 0}\frac{y(x+h)-y(x)}{h} = \lim\limits_{h\to 0}\left\{\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\right\}$

$= \lim\limits_{h\to 0}\left(\frac{f(x+h)g(x)-g(x+h)f(x)}{hg(x)g(x+h)}\right)$

We add and subtract $g(x)f(x)$ to the numerator to get $\lim\limits_{h\to 0}\left(\frac{g(x)f(x+h)-g(x)f(x)+g(x)f(x)-f(x)g(x+h)}{hg(x+h)g(x)}\right) =$ $\lim\limits_{h\to 0}\left(\frac{\frac{g(x)[f(x+h)-f(x)]}{h}-\frac{f(x)[g(x+h)-g(x)]}{h}}{g(x+h)g(x)}\right)$

Taking the limits gives $y'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$; $g(x) \neq 0$

Example 71 (3.6.3.1) *::Using the quotient rule we find $y'(x)$ if we are given $y(x) = \frac{4x^3-2x^2}{5x-3}$*

Solution 72 *: We let $f(x) = 4x^3 - 2x^2 \Rightarrow f'(x) = 12x^2 - 4x$*

and $g(x) = 5x - 3 \Rightarrow g'(x) = 5$

Then $y'(x) = \frac{(5x-3)(12x^2-4x)-(4x^3-2x^2)(5)}{[5x-3]^2} = \frac{40x^3-46x^2+12x}{(5x-3)^2}$

Corollary 73 (3.6.3.1) *: Let g be a function with $g(x) \neq 0$. Then $\left[\frac{1}{g(x)}\right]^/ = \frac{-[g(x)]^/}{[g(x)]^2}$*

We establish the above by letting $f(x) = 1$ in the quotient rule. Then $\left[\frac{1}{g(x)}\right]^/ = \frac{g(x).0-1g^/(x)}{[g(x)]^2} = \frac{g^/(x)}{[g(x)]^2}$

Example 74 (3.6.3.2) *: We differentiate the functions below and solve the other problems using methods of differentiation..*

(1) $y(x) = 14 - x + 3x^2 - 6x^4$
(2) $y(r) = r^3(4r^3 - 8r^2 + 2r)$
(3) $y(x) = (8x + 5x)(12x^2 - 6)$
(4) $h(x) = \frac{1}{1-x} + \frac{1-x}{x^2}$
(5) $g(z) = z^2(3z^2 - 4z + 1)(6z^2 - 8)$
(6) $f(t) = \frac{(\frac{3}{5}t-1)}{(\frac{2}{t^2}+7)}$

(7) Find an equation of the tangent line to the graph of $y = 3x^2 + 4x - 6$ that is parallel to the line $y = 5x - 2y - 1 = 0$

(8) A ball rolls down an inclined plane such that the distance in (cm) it rolls in t seconds is given by $s(t) = 3t^3 + 2t^2 + 4$ where $0 \leq t \leq 3$

(i) find its velocity at $t = 3$

(ii) find the time when the velocity is $35cm/\sec$.

(9) An object was thrown upward such that its distance $s(t)$ a above the ground during the first 15seconds is given by $s(t) = 8 - 6t + t^2$,where $s(t)$ is in metres and t in seconds. Find. (i) the velocity of the object at $t = 2$ and $t = 4$.

(ii) the velocity when the object is $50m$ above the ground.

21.1 3.6.4 The chain rule

The rules of differentiation discussed so far are limited as they can only be used for sums, differences,products and quotients of differentiable functions. They cannot be used for differentiating composite functions such as $(x^2 + 2x)^3$. The chain rule gives us a rule for differentiating composite functions. Let the function $y = f(x)$ be defined as $f(x) = h(g(x))$ with $h(x)$ and $g(x)$ differentiable. The chain rule states that $f'(x) = h'(g(x))g'(x)$.A sketch of the proof of the rule follows.

Let $f(x) = h(g(x))$, and let $u = g(x)$ and $y = h(u)$, then $y = h(u) = h(g(x)) = f(x)$ and $\frac{dy}{dx} = f'(x) = h'(g(x))g'(x) = h'(u)g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}$

Thus the chain rule becomes $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example 75 *(3.6.4.1): Using the chain rule differentiate the functions below and discuss whether their derivatives can be found in any other way.:*

(i) $y = \cos(x^2 + 3x)$, *(ii)* $y = (x^3 - 4x^2 + 6x)^{12}$, *(iii)* $y = \left(\frac{3x-4}{x-6}\right)^5$, *(iv)* $y = 6x^2(5x^2 - 7)^3$

Solution 76 *:*

(i). Let $u = x^2 + 3x$, then $y = \cos u$ and $\frac{du}{dx} = 2x + 3$, $\frac{dy}{du} = -\sin u$

thus $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -(2x + 3)\sin u = -(2x + 3)\sin(x^2 + 3x)$

(ii). Let $u = x^3 - 4x^2 + 6x$, then $y = u^{12}$ and $\frac{du}{dx} = 3x^2 - 8x + 6$, $\frac{dy}{du} = 12u^{11}$

thus $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (3x^2 - 8x + 6)12u^{11} = 12(3x^2 - 8x + 6)(x^3 - 4x^2 + 6x)^{11}$

(iii). Let $u = \frac{3x-4}{x-6}$, then $y = u^5$, and applying the quotient rule on u we have $\frac{du}{dx} = \frac{(x-6)3-(3x-4)}{(x-6)^2}$, $\frac{dy}{du} = 5u^4$

thus $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{(x-6)3-(3x-4)}{(x-6)^2} 5u^4 \Rightarrow \frac{dy}{dx} = 5\left(\frac{(x-6)3-(3x-4)}{(x-6)^2}\right)\left(\frac{3x-4}{x-6}\right)^4$

(iv). Let $u = 6x^2$ and $v = (5x^2 - 7)^3$.

Applying the product rule, we have that $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ with $\frac{du}{dx} = 12x$,

To differentiate v we apply the chain rule, thus let $k = 5x^2 - 7$, then $v = k^3$.$\frac{dk}{dx} = 10x$ and $\frac{dv}{dk} = 3k^2$

$\frac{dv}{dx} = \frac{dk}{dx} \times \frac{dv}{dk} = (10x\ )3k^2 = 30x(5x^2 - 7)^2$.

Substituting t gives $\frac{dy}{dx} = (5x^2 - 7)^3(12x) + 6x^2\ (30x(5x^2 - 7)) = 12x(5x^2 - 7)^3 + 180x^3(5x^2 - 7)$

Example 77 *(3.6.4.2): Let us find the derivatives of the functions below.*

(i) $y = \frac{2}{5x^3-6x+2}$

(ii) $y = \left(\frac{6x^3-7}{x+5}\right)^8$

(iii) $y = 6x^3\cos(6x - 4)$

(iv) $y = \sqrt{\frac{7x5}{6x^2-5}}$

(v) $y = (8x^4 - 5x^3 + 3x^2)(3x^8 + 2x^3 - 5)^3$

(vi) $y = \left(\frac{\sin 4x}{1-\cos x}\right)^3$

21.1.1 3.6.5 Leibnitz Rule

We have seen that if $f(x)$ is differentiable, we can find its derivative, a new function $f^{/}(x)$.If $f^{/}(x)$ is itself differentiable, we can form its derivative, called the second derivative of $f(x)$, and denoted by $f^{//}(x)$. As long as we have differentiability, we can continue in this way to get $f^{///}(x)$ and $f^{4}(x)$. We do not use dashes for more than the third derivative..

Example 78 (3.6.5.1) *: Find the following derivatives.*
(i).$f^{//}(x)$ for $f(x) = 7x^8 - 6x^5$.
(ii). $f^4(x)$ for $f(x) = ax^4$.
(iii).$f^3(x)$ for $f(x) = (7 + 2x)^2$.
(iv).$f^2(x)$ for $f(x) = \frac{ax+b}{cx+d}$.
(v).$f^{10}(x)$ for $f(x) = 7x^8 - 6x^5$.

[3.6.5.1] : Given $y = f(x) = U(x).V(x)$, with U and V, n-times differentiable functions, Leibnitz Rule gives us a formula for the nth derivative of the product $U.V$.

$y^n(x) = f^n(x) = [U(x).V(x)]^n = U^n(x).V(x) +^n C_1U^{n-1}(x)V^{/}(x) +^n C_2U^{n-2}(x)V^{//}(x) +^n C_3U^{n-3}(x)V^3(x) + ... + U(x)V^n(x)$.

where ${}^nC_r = \frac{n!}{(n-r)!r!}$.

Example 79 (3.6.5.2) *:*
(i) Find the fourth derivative of the function $y = x^5(2x^4 + 4x^2 - 6x)$ using the Leibnitz rule
(ii) Find the third derivative of $y = x^2 \cos x$.

Solution 80 *:*

(i) Let $U = x^5$ and $V = 2x^4 + 4x^2 - 6x$
$\frac{dU}{dx} = 5x^4$, $\frac{d^2U}{dx^2} = 20x^3$, $\frac{d^3U}{dx^3} = 60x^2$, $\frac{d^4U}{dx^4} = 120x$ and
$\frac{dV}{dx} = 8x^3 + 8x - 6$, $\frac{d^2V}{dx^2} = 24x^2 + 8$, $\frac{d^3V}{dx^3} = 48x$, $\frac{d^4V}{dx^4} = 48$.
Substituting the results into the formula gives
$\frac{d^4y}{dx^4} = 120x(2x^4 + 4x^2 - 6x) + 4(60x^2)(8x^3 + 8x - 6) + 6(20x^3)(24x^2 + 8) + 4(5x^4)(48x) + 48(x^5)$
$= 120x(2x^4+4x^2-6x)+240x^2(8x^3+8x-6)+120x^3(24x^2+8)+20x^4(48x)+48x^5$ $= 240x^5+480x^3-720x^2+1920x^5+1920x^3-1440x^2+2880x^5+960x^3 + 960x^5 + 48x^5$
$= 6048x^5 + 3360x^3 - 2160x^2$
(ii) Let $U = x^2$ and $V = \cos x$
$\frac{d^3y}{dx^3} = \frac{d^3U}{dx^3}V +^3 C_1 \frac{d^2U}{dx^2}\frac{dV}{dx} +^3 C_2 \frac{dU}{dx}\frac{d^2V}{dx^2} + U\frac{d^3V}{dx^3}$
$\frac{dU}{dx} = 2x$, $\frac{d^2U}{dx^2} = 2$, $\frac{d^3U}{dx^3} = 0$, and $\frac{dV}{dx} = \sin x$, $\frac{d^2V}{dx^2} = -\cos x$, $\frac{d^3V}{dx^3} = -\sin x$, substituting into the formula gives
$\frac{d^3y}{dx^3} = 6\ \sin x - 6x\cos x - \sin x = 5\sin x - 6x\cos x$

Example 81 (3.6.5.3) *: Find the nth derivatives of the following functions , as indicated, using the Leibnitz rule.*

(i). Fourth derivative of $[x^5(x^2+5x)]$
(ii).Third derivative of $[x^4(9x^2-x)]$
(iii).Second derivative of $(\sin x\cos x)$
(iv) Fourth derivative of $[\cos 3x(x^2+5x)]$
(v). Fifth derivative of $(x^5-4x)(x^2+5x)$

21.2 3.6.6. Implicit Differentiation

A function can be defined in two ways; either explicitly or implicitly. When a function is expressed solely in terms of the independent variable, such a function is called an explicit function. An example of an explicit function is $f(x) = x^2+3x+3$. When a function is not expressed solely in terms of the independent variable, such a function is said to be implicit. In other words, an implicit function is one which does not present an explicit formula for its values, but rather defines it by giving conditions that it satisfies. Thus its values, must be inferred as consequences of the definition. An example of an implicit function is $3x^2+y^3+xy = 3xy^2$.In this section we shall consider implicit functions and how to differentiate them. We take the derivative of implicit functions just as we take that of the explicit functions. This is done by taking the derivative of each side of the equation, by applying the rules of differentiation and remembering to treat the independent variable as a function of the dependent variable, and solving for the derivative. Another alternative method is to first solve for the dependent variable and then differentiate. We illustrate this with examples.

Example 82 (3.6.6.1) *:*
(i) Suppose that $8xy-16y^2 = x^2$, *find* $\frac{dy}{dx}$.
(ii) Differentiate $4x^2y^2+3y^3+4x^2 = 16xy^2$.
(iii) Differentiate $x\sqrt{y+1} = xy^{\frac{3}{2}}+1$
(iv) Differentiate $\cos xy = y^2+x^2$
(v) Obtain the derivative of the function $3x^2+xy^3 = 4x^3y$ *at the point* $(-1,0)$.

Solution 83 *:*

(i) To solve this problem, differentiate across the entire equation.
$8\frac{d}{dx}(xy)-16\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2)$. Use a combination of chain rule and product rule to get,
$8[x\frac{dy}{dx}+y]-16(2y\frac{dy}{dx}) = 2x$
$8x\frac{dy}{dx}+8y-32y\frac{dy}{dx} = 2x$. Rearrange, and collect the like terms together.
$8x\frac{dy}{dx}-32y\frac{dy}{dx} = 2x-8y$
$(8x-32y)\frac{dy}{dx} = 2x-8y$
Therefore, $\frac{dy}{dx} = \frac{2x-8y}{8x-32y}$
$\frac{dy}{dx} = \frac{2(x-4y)}{8(x-4y)} = \frac{1}{4}$.
Alternative method. Rearrange $8xy-16y^2 = x^2$ to give ,
$0 = x^2-8xy+16y^2 = (x-4y)^2$ Taking the square root, ~~we have~~,

$0 = x - 4y \Rightarrow y = \frac{x}{4}$
Therefore, $\frac{dy}{dx} = \frac{1}{4}$.
(ii) Differentiate both sides of the equation, $4\frac{d}{dx}(x^2y^2)+3\frac{d}{dx}(y^3)+4\frac{d}{dx}(x^2) = 16\frac{d}{dx}(xy^2)$
Using product and chain rule, we have
$4[2xy^2 + x^2(2y\frac{dy}{dx})] + 3(3y^2)\frac{dy}{dx} + 4(2x) = 16[y^2 + 2xy\frac{dy}{dx}]$
$8xy^2 + 8x^2y\frac{dy}{dx} + 9y^2\frac{dy}{dx} + 8x = 16y^2 + 32xy\frac{dy}{dx}$
$8x^2y\frac{dy}{dx} + 9y^2\frac{dy}{dx} - 32xy\frac{dy}{dx} = 16y^2 - 8xy^2 - 8x$
$(8x^2y + 9y^2 - 32xy)\frac{dy}{dx} = 16y^2 - 8xy^2 - 8x$
Therefore, $\frac{dy}{dx} = \frac{16y^2-8xy^2-8x}{8x^2y+9y^2-32xy}$
(iii) Differentiate across the entire equation implicitly. $\frac{d}{dx}(x\sqrt{y+1}) = \frac{d}{dx}(xy^{\frac{3}{2}})+\frac{d}{dx}(1)$
By product rule we have that, $x\frac{d}{dx}(\sqrt{y+1}) + \sqrt{y+1}\frac{d}{dx}(x) = x\frac{d}{dx}(y^{\frac{3}{2}}) + y^{\frac{3}{2}}\frac{d}{dx}(x)$
$\frac{x}{2\sqrt{y+1}}\frac{dy}{dx} + \sqrt{y+1} = \frac{3x\sqrt{y}}{2}\frac{dy}{dx} + y^{\frac{3}{2}}$
This implies that $(\frac{x}{2\sqrt{y+1}} - \frac{3x\sqrt{y}}{2})\frac{dy}{dx} = y^{\frac{3}{2}} - \sqrt{y+1}$
Therefore, $\frac{dy}{dx} = \frac{y^{\frac{3}{2}} - \sqrt{y+1}}{\left(\frac{x}{2\sqrt{y+1}} - \frac{3x\sqrt{y}}{2}\right)} = \frac{y^{\frac{3}{2}} - \sqrt{y+1}}{\left(\frac{x}{2\sqrt{y+1}} - \frac{3x\sqrt{y(y+1)}}{2\sqrt{y+1}}\right)} = \frac{\left(y^{\frac{3}{2}} - \sqrt{y+1}\right)\sqrt{y+1}}{\left(x - 3x\sqrt{y(y+1)}\right)}$.
(iv) Differentiate across the entire equation implicitly,
$-(\sin(xy))\left(y + x\frac{dy}{dx}\right) = 2y\frac{dy}{dx} + 2x$
$-x\sin xy\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x + y\sin xy$
factorise the derivative, to get $(-x\sin xy - 2y)\frac{dy}{dx} = 2x + y\sin xy \Rightarrow \frac{dy}{dx} = \frac{2x+y\sin xy}{-x\sin xy-2y} = \frac{2x+y\sin xy}{-(x\sin xy+2y)}$.
(v) Differentiate across the entire equation, to have,
$3\frac{d}{dx}(x^2) + \frac{d}{dx}(xy^3) = 4\frac{d}{dx}(x^3y)$
$6x + 3xy\frac{dy}{dx} + y^3 = 4x^3\frac{dy}{dx} + 12x^2y$
$3xy\frac{dy}{dx} - 4x^3\frac{dy}{dx} = 12x^2y - 6x$
$(3xy - 4x^3)\frac{dy}{dx} = 12x^2y - 6x$
$\frac{dy}{dx} = \frac{12x^2y-6x}{3xy-4x^3}$.At $(-1,0)$, $\frac{dy}{dx}|_{(-1,0)} = \frac{12x^2y-6x}{3xy-4x^3}|_{(-1,0)}$
$= \frac{3}{2}$.

22 3.7 Derivatives of logarithmic functions

We recall the definitions of the exponential and logarithm functions and the relationship between them. Let $f : \Re \to \Re$ be a real-valued function, then f is said to be a logarithmic function if $f(x) = \log_a u(x)$, where $u(x) > 0$, is a function of x and a is the base. If $a = e$, then we have the natural logarithmic function and we sometimes denote it as $\ln u$. When $u(x) = x$, we have $f(x) = \ln x$. In this section, we explore the derivatives of the logarithmic functions.

Suppose that the function f is given by $f(x)=\ln x$. We find the derivative of the function using first principles. Given that $f(x)=\ln x$.

Then $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \quad =\lim\limits_{h\to 0}\frac{1}{h}\ln(\frac{x+h}{x})$ (from the laws of logarithm.) $=\lim\limits_{h\to 0}\frac{1}{h}\ln(1+\frac{h}{x})$.

Letting $z=\frac{1}{x}$, we have $\quad f'(x)=\lim\limits_{h\to 0}\frac{1}{h}\ln(1+zh)$.

Taking the exponential of both sides we have,

$e^{f'(x)}=\lim\limits_{h\to 0}e^{\ln(1+zh)^{\frac{1}{h}}} \quad =\lim\limits_{h\to 0}(1+zh)^{\frac{1}{h}}$.If we set $n=\frac{1}{h}$, then $\lim\limits_{n\to\infty}(1+\frac{z}{n})^n=e^z$, hence $e^{f'(x)}=e^z$. Since they have the same base, it means that $f'(x)=z$;

But then $z=\frac{1}{x},\Rightarrow f'(x)=\frac{1}{x}$. Therefore the derivative of the function $f(x)=\ln x$ is $\frac{1}{x}$.

An alternative method of finding the derivative of $f(x)=\ln x$ is to use the concept of implicit differentiation. We proceed as follows:

Let $y=\ln x$. Then taking the exponential of both sides we have, $e^y=e^{\ln x}\Rightarrow e^y=x$

Use implicit differentiation with respect to x to get $e^y\frac{dy}{dx}=1$. Making $\frac{dy}{dx}$ the subject of the formula, we have

$\frac{dy}{dx}=\frac{1}{e^y}$, $\quad$ but $e^y=x$. Therefore $\frac{dy}{dx}=\frac{1}{x}$.

In general, if u is differentiable function of x then $\frac{dy}{dx}=\frac{1}{u(x)}\frac{du}{dx}$.

Let $y=\ln u(x)$,then $e^y=e^{\ln u(x)}$. This means that $e^y=u(x)$. By implicit differentiation,$e^y\frac{dy}{dx}=\frac{du}{dx}$,hence $\frac{dy}{dx}=\frac{1}{e^y}\frac{du}{dx}$. But $e^y=u(x)\Rightarrow\frac{dy}{dx}=\frac{1}{u(x)}\frac{du}{dx}$.

Example 84 (3.7.1) *:*

(i) Find the derivative of the function $f(x)=\ln(x^2+3)$;
(ii) Differentiate $\ln[(x^2-1)(x^3+2)]$;
(iii) Find the derivative of the following function $f(x)=\ln\sqrt{x}$;
(iv) Differentiate $\log_3(2x^3-3x+1)$;
(v) Differentiate $\log_2 x$.

Solution 85 *:*

(i) Let $y=f(x)$ and $u(x)=x^2+3$. This means that $y=\ln u$ and $\frac{du}{dx}=2x$.
Since $\frac{dy}{dx}=\frac{1}{u(x)}\frac{du}{dx}$ it means that $\frac{dy}{dx}=\frac{1}{x^2+3}\frac{d}{dx}\left(x^2+3\right)=\frac{2x}{x^2+3}$
(ii) Let $y=\ln[(x^2-1)(x^3+2)]$ then $\quad y=\ln(x^2-1)+\ln(x^3+2)$
Differentiating the function with respect to x we have

$$\frac{dy}{dx}=\frac{1}{(x^2-1)}(2x)+\frac{1}{(x^3+2)}(3x) \quad =\frac{2x(x^3+2)+3x(x^2-1)}{(x^2-1)(x^3+2)} \quad =\frac{2x^4+3x^2-x}{(x^2-1)(x^3+2)}.$$

(iii) $f'(x)=\frac{1}{\sqrt{x}}\frac{d}{dx}(\sqrt{x}) \quad =\frac{1}{\sqrt{x}}.\frac{1}{2}x^{-\frac{1}{2}} \quad =\frac{1}{2x}$.

The change of base formula allows us to rewrite a logarithm of any base in terms of natural logarithm. For instance $\log_a u(x)=\frac{\log_e u(x)}{\log_e a}$. The derivative of $y=\log_a u(x)$ is $\frac{dy}{dx}=\frac{d}{dx}(\frac{\log_e u(x)}{\log_e a})$

$= \frac{1}{\log_e a}\frac{d}{dx}(\log_e u(x)) \qquad = \frac{1}{u(x)\log_e a}\frac{du}{dx}$. Note that $\ln u(x)$ is the same as $\log_e u(x)$.

(iv) Let $y = \log_3 u$, where $u(x) = 2x^3 - 3x + 1$. This means that $\frac{du}{dx} = 6x^2 - 3$, and so $\frac{dy}{dx} = \frac{1}{(2x^3-3x+1)\ln 3}(6x^2 - 3)$.

(v) Let $y = \log_2 x = \frac{\ln x}{\ln 2}$.

Therefore $\frac{dy}{dx} = \frac{d}{dx}(\frac{\ln x}{\ln 2}) = \frac{1}{\ln 2}\frac{d}{dx}(\ln x) = \frac{1}{x\ln 2}$

The function $\log x$ is a monotonic function which assumes all values between $-\infty$ and $+\infty$ as the independent variable x ranges over the continuum of positive numbers.

Remark 86 (3.7.1)

Logarithmic differentiation is a technique that is employed to take the derivatives of functions involving complicated combinations of products, powers and especially quotients. To carry out differentiation using this technique, we carry out the following steps.

(1) Take natural logarithms of both sides of the equation $y = f(x)$.

(2) Differentiate implicitly with respect to x.

(3) Solve for $\frac{dy}{dx}$, replacing y by $f(x)$ every where it occurs in the final result.

Example 87 (3.7.2) *: (i) Differentiate* $y = x^{\sin x}$*;(ii) Differentiate* $y = \frac{(x-2)^{\frac{1}{2}}}{(x^3+2)^{\frac{3}{4}}(x^6+3x-1)^2}$*;(iii) Differentiate* $y = \ln(\sec x + \tan x)$*; (iv) Differentiate* $w = \frac{\ln x}{1+x^2}$.

Solution 88 *: (i) We shall use logarithmic differentiation.*

$y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} \Rightarrow \ln y = \sin x \ln x \Rightarrow$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\sin x + \ln x \cos x$

$\frac{dy}{dx} = y(\frac{1}{x}\sin x + \ln x \cos x)$

$\frac{dy}{dx} = x^{\sin x}(\frac{1}{x}\sin x + \ln x \cos x)$.

(ii) Taking the logarithm to base e of both sides, we have

$\ln y = \ln\left(\frac{(x-2)^{\frac{1}{2}}}{(x^3+2)^{\frac{3}{4}}(x^6+3x-1)^2}\right) = \frac{1}{2}\ln(x-2) - \frac{3}{4}\ln\left(x^3+2\right) - 2\ln\left(x^6+3x-1\right)$

$\ln y = \frac{1}{2}\ln(x-2) - \frac{3}{4}\ln\left(x^3+2\right) - 2\ln\left(x^6+3x-1\right)$

Example 89 (3.7.3) . $\frac{y'}{y} = \frac{1}{2(x-2)} - \frac{9}{4}\frac{x^2}{x^3+2} - 2\frac{6x^5+3}{x^6+3x-1}$

$y' = y(\frac{1}{2(x-2)} - \frac{9}{4}\frac{x^2}{x^3+2} - 2\frac{6x^5+3}{x^6+3x-1})$; $y' = \frac{(x-2)^{\frac{1}{2}}}{(x^3+2)^{\frac{3}{4}}(x^6+3x-1)^2}(\frac{1}{2(x-2)} - \frac{9}{4}\frac{x^2}{x^3+2} - 2\frac{6x^5+3}{x^6+3x-1})$

$y' = -\frac{1}{4\sqrt{(x-2)}\left(\sqrt[4]{(x^3+2)}\right)^7(x^6+3x-1)^3}\left(55x^9 + 45x^4 - 109x^3 + 92x^6 + 36x - 92 - 114x^8 + 18x^2 - 192x^5\right)$

(iii) Taking the logarithm of both sides to base e, we have

$\ln y = \ln\ln(\sec x + \tan x)$.*differentiate implicitly to have*

$\frac{y'}{y} = \frac{\sec x\tan x + 1 + \tan^2 x}{(\sec x + \tan x)\ln(\sec x + \tan x)}$

$y' = y\frac{\sec x\tan x + 1 + \tan^2 x}{(\sec x + \tan x)\ln(\sec x + \tan x)}$

$y' = \ln(\sec x + \tan x)\frac{\sec x \tan x + 1 + \tan^2 x}{(\sec x + \tan x)\ln(\sec x + \tan x)}$
$y' = \frac{\sec x \tan x + \sec^2 x}{(\sec x + \tan x)} = \frac{\sec x(\tan x + \sec x)}{(\sec x + \tan x)} = \sec x$
(iv) Taking the logarithm to base e of both sides, we have
$\ln w = \ln\left(\frac{\ln x}{1+x^2}\right) = \ln(\ln x) - \ln\left(1+x^2\right)$
$\frac{w'}{w} = \frac{1}{x\ln x} - 2\frac{x}{1+x^2}$
$w' = w\left(\frac{1}{x\ln x} - 2\frac{x}{1+x^2}\right) = \frac{\ln x}{1+x^2}\left(\frac{1}{x\ln x} - 2\frac{x}{1+x^2}\right) = -\frac{-1-x^2+2x^2\ln x}{(1+x^2)^2 x}$

23 3.8 Differentiation of Exponential Functions.

Given $\dot{a} > 0$, the function $f : \Re \to \Re^+$, defined by $f(x) = a^x$, ,(where $x \in \Re$) is called an exponential function.

Using first principles, $f'(x) = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim\limits_{h\to 0}\frac{a^x(a^h-1)}{h}$
$= a^x \lim\limits_{h\to 0}\frac{(a^h-1)}{h}$

But $\lim\limits_{h\to 0}\frac{(a^h-1)}{h}$ is a constant depending on the value of the base a. By power series expansion,
$a^h = 1 + (\ln a)h + \left(\frac{1}{2}\ln^2 a\right)h^2 + \left(\frac{1}{6}\ln^3 a\right)h^3 + \left(\frac{1}{24}\ln^4 a\right)h^4 + ...$
$(a^h - 1) = (\ln a)h + \left(\frac{1}{2}\ln^2 a\right)h^2 + \left(\frac{1}{6}\ln^3 a\right)h^3 + \left(\frac{1}{24}\ln^4 a\right)h^4 + ...$
$\frac{1}{h}(a^h - 1) = (\ln a) + \left(\frac{1}{2}\ln^2 a\right)h + \left(\frac{1}{6}\ln^3 a\right)h^2 + \left(\frac{1}{24}\ln^4 a\right)h^3 + ...$
$\lim\limits_{h\to 0}\frac{1}{h}(a^h - 1) = \lim\limits_{h\to 0}(\ln a) + \lim\limits_{h\to 0}\left(\frac{1}{2}\ln^2 a\right)h + \lim\limits_{h\to 0}\left(\frac{1}{6}\ln^3 a\right)h^2 + ...$
Therefore, $\lim\limits_{h\to 0}\frac{1}{h}(a^h - 1) = (\ln a)$.

Now returning to our definition, we have that $f'(x) = a^x \ln a$. A special case of this occurs when $a = e = 2.71828...$ That is if $y = e^x$, then $\frac{dy}{dx} = e^x$.

In general, if u is a differentiable function of x, and $y = f(x)$ such that $y = a^{u(x)}$ then the following results hold:
(a) $\frac{dy}{dx} = a^{u(x)}\ln a\frac{du}{dx}$
(b) $\frac{dy}{dx} = e^{u(x)}\frac{du}{dx}$ (In this case $y = e^{u(x)}$).

Example 90 (3.8.1) *:*

Differentiate the following functions.
(i) (a) $y = a^{3x^2+4x+1}$ (b) $f(x) = 4^{x^3+2}$ (c) $f(x) = 5^{\sin(x^3+2)}$
(ii) Differentiate the following functions.
(a) e^{3x} (b) $e^{-4x}(x^3+1)$
(iii) A radioactive substance decays according to the formula $q(t) = q_0 e^{-ct}$, where q_0 is the initial amount of substance, c is a positive constant and $q(t)$ is the amount remaining after time t. Show that the rate at which the substance decays is proportional to $q(t)$.

Solution 91 *:*

(i) (a) Given that $y = a^{3x^2+4x+1}$. Here $u(x) = 3x^2+4x+1$. This means that $\frac{du}{dx} = 6x+4$. Therefore,
$\frac{dy}{dx} = a^{u(x)} \ln a \frac{du}{dx}$
$\frac{dy}{dx} = (a^{3x^2+4x+1}) \ln a(6x+4) = 2(a^{3x^2+4x+1})(3x+2) \ln a = 2 \ln a(a^{3x^2+4x+1})(3x+2)$

(b) Let $y = f(x) = 4^{x^3+2}$, then $\frac{dy}{dx} = \frac{d}{dx}(4^{x^3+2})$. This implies that
$\frac{dy}{dx} = (4^{x^3+2}) \ln 4(3x^2) = (3x^2)(4^{x^3+2}) \ln 4$.

(c) Let $y = 5^{\sin(x^3+2)}$. Then taking the natural logarithm of both sides, we have $\ln y = \sin(x^3+2) \ln 5$. Now differentiating both sides implicitly, we have $\frac{1}{y}\frac{dy}{dx} = \cos(x^3+2)\frac{d}{dx}(x^3+2) \ln 5$
$\frac{dy}{dx} = y[3x \ln 5 \cos(x^3+2)] = 5^{\sin(x^3+2)}\,[3x \ln 5 \cos(x^3+2)]$.

(ii) (a) Let $y = e^{3x}$, then $\frac{dy}{dx} = e^{3x}\frac{d}{dx}(3x)$. This means that $\frac{dy}{dx} = 3e^{3x}$.

(b) Let $y = e^{-4x}(x^3+1)$. This particular problem requires the usage of product rule and exponential differentiation.
$\frac{dy}{dx} = e^{-4x}\frac{d}{dx}(x^3+1) + (x^3+1)\frac{d}{dx}(e^{-4x})$.
$\frac{dy}{dx} = 3x^2e^{-4x} + (-4e^{-4x})(x^3+1) = 3x^2e^{-4x} - 4e^{-4x}(x^3+1) = (3x^2-4(x^3+1))e^{-4x} = -(4x^3-3x^2-4))e^{-4x}$.

(iii) To obtain the rate at which the substance is decaying, is equivalent to finding the first derivative of the function $q(t) = q_0 e^{-ct}$ with respect to the independent variable t.
$\frac{dq}{dt} = \frac{d}{dt}(q_0 e^{-ct}) \Rightarrow$
$\frac{dq}{dt} = q_0 e^{-ct}\frac{d}{dt}(-ct) \Rightarrow \frac{dq}{dt} = -cq_0 e^{-ct}$

Example 92 (3.8.2)

Find the derivative of the following functions:
1. (i) $y = \ln(x^6+3x^2+1)$ (ii) $y = \ln 2x\sqrt{x^2+4}$
(iii) $\ln\sqrt{\left(\frac{4x^4-3}{(5x^3+2)^{-3}}\right)}$
(iv) $y = \log_{10}(x^2-\sqrt{x})$ (v) $y = \ln(\frac{e^{x^6}+3x^2+1}{\cos 3x - \ln 4x^2})$
(vi) $\ln(\sec x + \tan x)$ (vii) $\ln(\cos x)$
2. Using logarithmic differentiation, differentiate with respect to x:
(i) $\frac{(x^2-3x-1)^4(\cos^2 x)}{(x^6-3x^2)^{\frac{3}{2}}}$ (ii) $\sqrt{3-4^x}$ (iii) $\frac{\ln x}{1+x^2}$ (iv) $\cos(x)^{\sin x}$
(v) $\ln\ln(x)$ (vi) $x^y = y^x$. (vii) $\frac{e^x x^5\sqrt{x^3+2}}{(x+2)^{\frac{5}{2}}(x^2+3)^3}$
3. Find the derivative of following functions with respect to the independent variable indicated.
(a) $e^{\frac{1}{x}} + \frac{1}{e^x}$ (b) $(e^{3x}-e^{-3x})^4$ (c) $e^{3x^{\sin x}}(\cos 3x)$ (d) $\sec^2(e^{-3x})$
(e) $\frac{e^{-(x^2-1)^2}}{x^2+1}$ (f) $e^{\cos x}(\ln\left(x^3-1\right)^{\frac{3}{2}})$.
4. If a drug is injected into the bloodstream, then its a concentration t minutes later is given by
$c(t) = \frac{\alpha}{a-b}(e^{-bt}-e^{-at})$
for positive constants a, b and α.

(a) At what time does the maximum concentration occur?
(b) What can be said about the concentration after a long period of time?

24 3.9 Derivatives of Hyperbolic Functions

Recall that the six hyperbolic functions are:

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}, \qquad \cosh x \equiv \frac{e^x + e^{-x}}{2},$$
$$\tanh x \equiv \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \coth x \equiv \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$
$$\mathrm{sech} x \equiv \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad \cos ech x \equiv \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

The following identities are true of the hyperbolic functions.

$$\cosh^2 x - \sinh^2 y = 1 \quad , \quad \mathrm{sech}^2 x + \tanh^2 x = 1 \text{ and}$$
$$\coth^2 y - \cos ech^2 y = 1.$$

The derivatives of the hyperbolic functions can be obtained easily when written in terms of exponentials. Since the derivative of e^x is still e^x and the derivative of e^{-x} is $-e^{-x}$, we can differentiate the above equations to have:

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}[\frac{e^x - e^{-x}}{2}] = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$
$$\frac{d}{dx}[\cosh x] = \frac{d}{dx}[\frac{e^x + e^{-x}}{2}] = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

Notice that the derivative of hyperbolic trigonometric $\sinh x$ is $\cosh x$ and the derivative of $\cosh x$ is $\sinh x$. This is unlike the trigonometric version, where the derivative of $\cos x$ is $-\sin x$.

The derivatives of the remaining four functions can be obtained in a similar manner. That is $\frac{d}{dx}[\tanh x] = \frac{d}{dx}[\frac{\sinh x}{\cosh x}] = \frac{\cosh x \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{[\cosh x]^2}$ (quotient rule).

$$= \frac{\cosh^2 x - \sinh^2 y}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \mathrm{sech}^2 x.$$

$$\frac{d}{dx}[\coth x] = \frac{d}{dx}[\frac{\cosh x}{\sinh x}]$$
$$= \frac{\sinh x \frac{d}{dx}(\cosh x) - \cosh x \frac{d}{dx}(\sinh x)}{[\sinh x]^2} \quad \text{(quotient rule).}$$
$$= \frac{\sinh^2 y - \cosh^2 x}{\sinh^2 x} = -\frac{(\cosh^2 x - \sinh^2 y)}{\sinh^2 x} = -\frac{1}{\sinh^2 x} = -\,\mathrm{cosech}^2 x.$$

$$\frac{d}{dx}[\mathrm{sech} x] = \frac{d}{dx}[\frac{1}{\cosh x}]$$
$$= \frac{\cosh x \frac{d}{dx}(1) - (1)\frac{d}{dx}(\cosh x)}{[\cosh x]^2}$$
$$= -\frac{\sinh x}{[\cosh x]^2}$$
$$= -\,\mathrm{sech} x \tanh x.$$

$$\frac{d}{dx}[\mathrm{cosech} x] = \frac{d}{dx}[\frac{1}{\sinh x}]$$
$$= \frac{\sinh x \frac{d}{dx}(1) - (1)\frac{d}{dx}(\sinh x)}{[\sinh x]^2}$$
$$= -\frac{\cosh x}{[\sinh x]^2}$$
$$= -\coth x \,\mathrm{cosech} x.$$

In general if u is a differentiable function of x, then

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx}$$
$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx}$$
$$\frac{d}{dx}[\tanh u] = \mathrm{sech}^2 u \frac{du}{dx}.$$
$$\frac{d}{dx}[\coth x] = -\,\mathrm{cosech}^2 u \frac{du}{dx}.$$

$\frac{d}{dx}[\text{sech}(u)] = \text{sech}(u)\tanh u \frac{du}{dx}$.
$\frac{d}{dx}[\text{cosech}(u)] = -\coth u \,\text{csch}(u)\frac{du}{dx}$.

Example 93 (3.9.1) *:*

Differentiate the following hyperbolic functions.
(i) $\sinh(4x^2)$ (ii) $\cosh(e^x + x)$ (iii) $\tanh(3x)$

Solution 94 *:*

(i) Let $y = \sinh(4x^2)$ and $u(x) = 4x^2$, then $\frac{dy}{dx} = \cosh(4x^2)\frac{d}{dx}(4x^2) = \cosh(4x^2)(8x) = 8x\cosh(4x^2)$.

(ii) Let $y = \cosh(e^x + x)$ and $u(x) = e^x + x$, then $\frac{dy}{dx} = \sinh(e^x + x)\frac{d}{dx}(e^x + x) = (e^x + 1)\sinh(e^x + x)$.

(iii) Let $y = \tanh(3x)$ and $u(x) = 3x$ then
$\frac{dy}{dx} = \text{sech}^2(3x)\frac{d}{dx}(3x)$
$= 3\,\text{sech}^2(3x)$.

25 3.10 Derivatives of the Inverse Trigonometric functions

The inverse trigonometric functions are:
(i) $y = \sin^{-1} x$ if $x = \sin y$, and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
(ii) $y = \cos^{-1} x$ if $x = \cos y$, and $0 < y < \pi$.
(iii) $y = \tan^{-1} x$ if $x = \tan y$, and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
(iv) $y = \cot^{-1} x$ if $x = \cot y$, and $0 < y < \pi$.
(v) $y = \sec^{-1} x$ if $x = \sec y$, and $0 \le y < \frac{\pi}{2}$ and $\frac{\pi}{2} < y \le \pi$.
(vi) $y = \csc^{-1} x$ if $x = \csc y$, and $-\frac{\pi}{2} \le y < 0$ and $0 < y \le \frac{\pi}{2}$.
To obtain the derivative of the six inverse trigonometric functions, we proceed as follows.

(i) Let $y = \sin^{-1} x$ where $x \in [-1, 1]$. Rearrange the expression to have
$x = \sin y$. Differentiate implicitly to have
$1 = \cos y\frac{dy}{dx}$. This means that $\frac{dy}{dx} = \frac{1}{\cos y}$. But $\cos y = \sqrt{1 - \sin^2 y}$
Thus $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$, (since $\sin^2 y = x^2$).

(ii) Let $y = \cos^{-1} x$ where $x \in [-1, 1]$. Rearrange the expression to have
$x = \cos y$. Differentiate implicitly to have
$1 = -\sin y\frac{dy}{dx}$. This means that $\frac{dy}{dx} = -\frac{1}{\sin y}$. But $\sin y = \sqrt{1 - \cos^2 y}$
Thus $\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$, (since $\cos^2 y = x^2$).

(iii) Let $y = \tan^{-1} x$ where $x \in (-\infty, \infty)$. Rearrange the expression to have
$x = \tan y$. Differentiate implicitly to have
$1 = \sec^2 y\frac{dy}{dx}$. This means that $\frac{dy}{dx} = \frac{1}{\sec^2 y}$. But $\sec^2 y = 1 + \tan^2 y$
Thus $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$, (since $\tan^2 y = x^2$).

(iv) Let $y = \cot^{-1} x$ where $x \in (-\infty, \infty)$. Rearrange the expression to have

$x = \cot y$. Differentiate implicitly to have
$1 = -\csc^2 y \frac{dy}{dx}$. This means that $\frac{dy}{dx} = -\frac{1}{\csc^2 y}$. But $\csc^2 y = 1 + \cot^2 y$
Thus $\frac{dy}{dx} = -\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}$, (since $\cot^2 y = x^2$).

(v) Let $y = \sec^{-1} x$ where $\mid x \mid\geqslant 1$. Rearrange the expression to have
$x = \sec y$. Differentiate implicitly to have
$1 = \sec y \tan y \frac{dy}{dx}$. This means that $\frac{dy}{dx} = \frac{1}{\sec y \tan y}$. But $\tan y = \frac{+}{-}\sqrt{\sec^2 y - 1}$
Thus $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{+}{-}\frac{1}{\left(\sqrt{\sec^2 y-1}\right)\sec y} = \frac{+}{-}\frac{1}{\left(x\sqrt{x^2-1}\right)}$, (since $\sec^2 y = x^2$).
$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$, $\mid x \mid > 1$.

(vi) Let $y = \csc^{-1} x$ where $\mid x \mid\geqslant 1$. Rearrange the expression to have
$x = \csc y$. Differentiate implicitly to have
$1 = -\csc y \cot y \frac{dy}{dx}$. This means that $\frac{dy}{dx} = -\frac{1}{\csc y \cot y}$. But $\cot y = \frac{+}{-}\sqrt{\csc^2 y - 1}$
Thus $\frac{dy}{dx} = -\frac{1}{\csc y \cot y} = \frac{+}{-}\frac{1}{\left(\sqrt{\csc^2 y-1}\right)\csc y} = -\frac{1}{\left(|x|\sqrt{x^2-1}\right)}$, (since $\csc^2 y = x^2$).

Generally if u is a differentiable function of x, then

$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, $\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$,
$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}$ $\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2}\frac{du}{dx}$,
$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$, $\frac{d}{dx}(\csc^{-1} u) = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

We shall attempt to prove the first two and leave the rest as exercises.

(i) Let $y = \sin^{-1} u$, where u is a function of x. Then
$u = \sin y$. By implicit differentiation, we have
$\frac{du}{dx} = \cos y \frac{dy}{dx}$. Making $\frac{dy}{dx}$, the subject we have
$\frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, (since $u^2 = \sin^2 y$).
This gives the formula
$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$.

(ii) Let $y = \cos^{-1} u$, where u is a function of x. Then
$u = \cos y$. By implicit differentiation, we have
$\frac{du}{dx} = -\sin y \frac{dy}{dx}$. Making $\frac{dy}{dx}$, the subject we have
$\frac{dy}{dx} = -\frac{1}{\sin y}\frac{du}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}\frac{du}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, (since $u^2 = \cos^2 y$).
This gives the formula
$\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$.

Example 95 (3.10.1) :

Differentiate the following inverse trigonometric functions.
(a) $y = x^3 \cos^{-1}(4x^3 - 1)$ (b) $y = e^x \sin^{-1}(x^2 - 2)$ (c) $y = \frac{1+\tan^{-1} x}{4-5\tan^{-1} x}$
(d) $y = \left(\sin^{-1}(x^3)\right)^4$.

Solution 96 :

(a) Let $y = x^3\cos^{-1}(4x^3-1)$. Apply the product rule. Then

$$\frac{dy}{dx} = x^3\frac{d}{dx}\left(\cos^{-1}(4x^3-1)\right) + \cos^{-1}(4x^3-1)\frac{d}{dx}(x^3)$$
$$= x^3\left(-\frac{1}{\sqrt{1-(4x^3-1)^2}}\right)\frac{d}{dx}\left(4x^3-1\right) + 3x^2\cos^{-1}(4x^3-1)$$
$$= x^3\left(-\frac{1}{\sqrt{1-(4x^3-1)^2}}\right)\left(12x^2\right) + 3x^2\cos^{-1}(4x^3-1)$$
$$= -\frac{12x^5}{\sqrt{1-(4x^3-1)^2}} + 3x^2\cos^{-1}(4x^3-1) = 3x^2\left(\cos^{-1}(4x^3-1) - \frac{4x^3}{\sqrt{1-(4x^3-1)^2}}\right). =$$
$$-3x^2\frac{-\left(\cos^{-1}\left(4x^3-1\right)\right)\sqrt{(-4x^4+2x)}+2x^2}{\sqrt{(-4x^4+2x)}}$$

(b) Let $y = e^x\sin^{-1}(x^2-2)$. Apply the product rule. Then

$$\frac{dy}{dx} = e^x\frac{d}{dx}\left(\sin^{-1}(x^2-2)\right) + \sin^{-1}(x^2-2)\frac{d}{dx}(e^x)$$
$$= e^x\left(\frac{1}{\sqrt{1-(x^2-2)^2}}\right)\frac{d}{dx}\left(x^2-2\right) + \sin^{-1}(x^2-2)e^x = \frac{2x\ e^x}{\sqrt{1-(x^2-2)^2}} +$$
$$\sin^{-1}(x^2-2)e^x. = e^x\frac{2x+\left(\sin^{-1}\left(x^2-2\right)\right)\sqrt{(-3-x^4+4x^2)}}{\sqrt{(-3-x^4+4x^2)}}.$$

(c) Let $y = \frac{1+\tan^{-1}x}{4-5\tan^{-1}x}$. Apply the quotient rule. Then

$$\frac{dy}{dx} = \frac{\left(4-5\tan^{-1}x\right)\frac{d}{dx}\left(1+\tan^{-1}x\right) - \frac{d}{dx}\left(4-5\tan^{-1}x\right)}{(4-5\tan^{-1}x)^2}$$
$$\frac{dy}{dx} = \frac{\left(4-5\tan^{-1}x\right)\left(\frac{1}{1+x^2}\right) - \left(1+\tan^{-1}x\right)\left((-5)\left(\frac{1}{1+x^2}\right)\right)}{(4-5\tan^{-1}x)^2} = \frac{9}{(1+x^2)(-4+5\tan^{-1}x)^2}.$$

(d) Let $y = \left(\sin^{-1}(x^3)\right)^4$. Apply the chain rule. Then

$$\frac{dy}{dx} = 4\left(\sin^{-1}(x^3)\right)^3\frac{d}{dx}\left(\sin^{-1}(x^3)\right)$$
$$= 4\left(\sin^{-1}(x^3)\right)^3\left(\frac{1}{\sqrt{1-(x^3)^2}}\frac{d}{dx}\left(x^3\right)\right)$$
$$= 4\left(\sin^{-1}(x^3)\right)^3\left(\frac{3x^2}{\sqrt{1-x^6}}\right) = \frac{12x^2\left(\sin^{-1}(x^3)\right)^3}{\sqrt{1-x^6}}.$$

Example 97 (3.10.2)

1. Differentiate the following inverse trigonometric functions with respect to x.

   (a) $\sin^{-1}(4x^3-3)$ (b) $\cos^{-1}(\sin x)$ (c) $\cos^{-1}(2x^3e^x)$ (d) $\tan^{-1}\left(\frac{2x}{1-x}\right)$

2. If $y = e^{\tan^{-1}x}$, show that $\left(1+x^2\right)\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} - y = 0$.

3 If $y = \left(1-x^2\right)^{\frac{1}{2}}\cos^{-1}x$, prove that $\left(1-x^2\right)\frac{dy}{dx} + xy + \left(1-x^2\right) = 0$.

4. Differentiate

   (a) $y = x^2\left(\tan^{-1}x\right)^3$ (b) $y = \tan^{-1}\left(e^{-x^2}\right)$ (c) $y = arccot\left(\frac{1}{x}\right) - \arctan x$

5. Find $\frac{dy}{dx}$ if $y = \exp\left(x\cos^{-1}(\ln x)\right)$.

26 3.11 Derivatives of the Inverse Hyperbolic Functions.

The inverse hyperbolic functions can be differentiated by the following technique.

1) Write the relationship in such a way that the independent variable becomes the dependent variable. For example if $y = \sinh^{-1} x$, then rewrite as $x = \sinh y$.

2) Differentiate the equation implicitly

3) Rearrange to make derivative (in this case $\frac{dy}{dx}$),the subject.

4) Using an appropriate identity, write the expression in terms of the independent variable.

Example 98 (3.11.1) *: Differentiate the following*

(a) $y = \cosh^{-1} x$ (b) $y = \sinh^{-1} x$ (c) $y = \tanh^{-1} x$

(a) Let $y = \cosh^{-1} x$. This means that $x = \cosh y$

Differentiate implicitly to have

$1 = \sinh y \frac{dy}{dx}$

Since $\cosh y = \frac{e^y + e^{-y}}{2}$ and $\frac{d}{dx}(\cosh y) = \frac{1}{2}\frac{d}{dx}(e^y + e^{-y}) = \frac{1}{2}(e^y - e^{-y}) = \sinh y$

Therefore $\frac{dy}{dx} = \frac{1}{\sinh y}$

But $\cosh^2 y - \sin^2 y = 1$

$\Rightarrow \sin^2 y = \cosh^2 y - 1$ and $\sin y = \sqrt{\cosh^2 y - 1}$

$$\sin y = \sqrt{x^2 - 1}, \quad \text{Therefore } \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}.$$

(b) Given that $y = \sinh^{-1} x$. Then $x = \sinh y$

Differentiating implicitly, we have $1 = \cosh y \frac{dy}{dx}$

This implies that $\frac{dy}{dx} = \frac{1}{\cosh y}$

But $\cosh y = \sqrt{1 + \sin^2 y} = \sqrt{1 + x^2}$, since $\sinh^2 y = x^2$.

(c) Let $y = \tanh^{-1} x$, $-1 < x < 1$.

Then $x = \tanh y$,and $1 = \sec h^2 y \frac{dy}{dx}$.

Thus $\frac{dy}{dx} = \frac{1}{\sec h^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1-x^2}$, $x^2 < 1$.

Generally, if u is a differentiable function of x the derivative of the inverse hyperbolic functions can be obtained as follows:

(i) Let $y = \sinh^{-1} u$, then $u = \sinh y$. Using implicit differentiation $\frac{du}{dx} = \cosh y \frac{dy}{dx}$.

Hence, $\frac{dy}{dx} = \frac{1}{\cosh y}\frac{du}{dx} = \frac{1}{\sqrt{1+\sinh^2 y}}\frac{du}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$.

This gives the formula

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}.$$

(ii) Similarly, let $y = \cosh^{-1} u$, then $u = \cosh y$. Now using implicit differentiation, we have

$\frac{du}{dx} = \sinh y \frac{dy}{dx}$.

Hence, $\frac{dy}{dx} = \frac{1}{\sinh y}\frac{du}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}}\frac{du}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$.

This gives the formula

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx} \quad , \quad u > 1.$$

(iii) Let $y = \tanh^{-1} u$ then $u = \tanh y$, and

$\frac{du}{dx} = \sec h^2 y \frac{dy}{dx}$. Rearrange to have

$\frac{dy}{dx}=\frac{1}{\sec h^2 y}\frac{du}{dx}=\frac{1}{1-\tanh^2 y}\frac{du}{dx}=\frac{1}{1-u^2}\frac{du}{dx}$.

This gives the formula

$$\frac{d}{dx}(\tanh^{-1} u)=\frac{1}{1-u^2}\frac{du}{dx} \quad , \quad \mid u \mid < 1.$$

(iv) Let $y=\coth^{-1} u$, then $u=\coth y$. This means that $\frac{du}{dx}=-\operatorname{csch}^2 y\frac{dy}{dx}=\left(1-\coth^2(y)\right)\frac{dy}{dx}$; Since $\coth^2 y-\operatorname{cosech}^2 y=1$, we have $\frac{dy}{dx}=-\frac{1}{1-u^2}\frac{du}{dx}$. This gives the formula

$$\frac{d}{dx}(\coth^{-1} u)=-\frac{1}{1-u^2}\frac{du}{dx} \quad , \mid u \mid > 1$$

v) Let $y=\operatorname{sech}^{-1}u$ then $u=\operatorname{sech}y$. Differentiate implicitly to have $\frac{du}{dx}=-\operatorname{sech}y\tanh y\frac{dy}{dx}$. Rearrange to have $\frac{dy}{dx}=-\frac{1}{\sec \mathrm{h} y\tanh y}\frac{du}{dx}=-\frac{1}{\sec \mathrm{h} y\sqrt{1-\sec \mathrm{h}^2 u}}\frac{du}{dx}=-\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$. This gives the formula

$$\frac{d}{dx}(\sec h^{-1}u)=-\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx} \quad , \quad 0<u<1.$$

Finally, using similar arguments, it can be proved that

(vi) $$\frac{d}{dx}(\csc h^{-1}u)=-\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx} \quad , \quad u \neq 1.$$

Example 99 (3.11.2) *:*

Find the derivative of the following inverse hyperbolic functions with respect to x.

(i) $\cosh^{-1}(2x^2+3)$ (ii) $\sinh^{-1}(x^3)$ (iii) $\sinh^{-1}(\sqrt{x^2-1})$

(iv) $\tanh^{-1}(\frac{2x}{3x+x^2})$. (v) $\operatorname{sech}^{-1}(x^3-1)$

Solution 100 *: (i) Let $y=\cosh^{-1}(2x^2+3)$, and $u=2x^2+3$. From*

$\frac{d}{dx}(\cosh^{-1} u)=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$ we have that

$\frac{d}{dx}(\cosh^{-1}(2x^2+3))=\frac{1}{\sqrt{(2x^2+3)^2-1}}\frac{d}{dx}(2x^2+3) = \frac{4x}{\sqrt{(2x^2+3)^2-1}}=2\frac{x}{\sqrt{(x^4+3x^2+2)}}$.

(ii) Let $y=\sinh^{-1}(x^3)$, and $u=x^3$. Now from the relation

$\frac{d}{dx}(\sinh^{-1} u)=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$, we have

$\frac{d}{dx}(\sinh^{-1}(x^3))=\frac{1}{\sqrt{1+(x^3)^2}}\frac{d}{dx}(x^3) = \frac{3x^2}{\sqrt{1+(x^3)^2}}=3\frac{x^2}{\sqrt{(1+x^6)}}$.

(iii) Let $y=\sinh^{-1}(\sqrt{x^2-1})$, and $u=\sqrt{x^2-1}$, then

$\frac{d}{dx}(\sinh^{-1}(\sqrt{x^2-1}))=\frac{1}{\sqrt{1+(\sqrt{x^2-1})^2}}\frac{d}{dx}(\sqrt{x^2-1}) = \frac{\frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x)}{\sqrt{1+(\sqrt{x^2-1})^2}}=\frac{1}{\sqrt{(x^2-1)}}$.

(iv) Let $y=\tanh^{-1}(\frac{2x}{3x+x^2})$, and $u=\frac{2x}{3x+x^2}$, then from formula

$\frac{d}{dx}(\tanh^{-1} u)=\frac{1}{1-u^2}\frac{du}{dx}$, we have

$\frac{d}{dx}(\tanh^{-1}(\frac{2x}{3x+x^2}))=\frac{1}{1-(\frac{2x}{3x+x^2})^2}\frac{d}{dx}(\frac{2x}{3x+x^2})$.

$=\frac{1}{1-(\frac{2x}{3x+x^2})^2}\left(\frac{(3x+x^2)\frac{d}{dx}(2x)-2x\frac{d}{dx}(3x+x^2)}{(3x+x^2)^2}\right) = \frac{1}{1-(\frac{2x}{3x+x^2})^2}\left(\frac{2(3x+x^2)-2x(3+2x)}{(3x+x^2)^2}\right) = -\frac{2}{5+6x+x^2}$.

(v) Let $y = \text{sech}^{-1}(x^3-1)$, and $u = x^3-1$. Then from the formula $\frac{d}{dx}(\text{sech}^{-1}u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$, we have

$$\frac{d}{dx}(\text{sech}^{-1}(x^3-1)) = -\frac{1}{(x^3-1)\sqrt{1-(x^3-1)^2}}\frac{d}{dx}(x^3-1) \ = -\frac{3x^2}{(x^3-1)\sqrt{1-(x^3-1)^2}}$$

$$= -3\frac{x}{(x^3-1)\sqrt{(-x^4+2x)}}.$$

Example 101 (3.11.3) *: We explore the problems below:*

(a) Differentiate the following hyperbolic functions with respect to x.
(i) $\cosh(3x)$ (ii) $\cosh(e^x)$ (iii) $e^x \sinh(x^2+3x)$
(iv) $\tanh(3x^3-1)$ (v) $\coth\left(x^3\sqrt{x^2-1}\right)$
(vi) $\text{cosech}(e^x\left(x^2-2\right)$ (vii) $\sec h\left(\frac{x^2}{(3x^2-1)^2}\right)$

(b) If $u = e^x \cosh x$ and $v = e^x \sinh x$, show that $\frac{du}{dx} = \frac{dv}{dx}$ and give a relation between u and v that is independent of x.

(c) Find the derivative of each of the following functions.
(i) $\sinh^{-1}(x^4-2)$ (ii) $\sinh^{-1}(e^{-3x})$ (iii) $\sinh^{-1}(\cosh(x^4-2))$
(iv) $x\cosh^{-1}x + \sqrt{1+x^2}$ (v) $x^2\ln\left(\sqrt{x^2-1}\right) - x\cosh^{-1}x$
(vii) $\text{sech}^{-1}\left(\sqrt{1-x^2}\right)$ (viii) $x^2\coth^{-1}\left(x^3-1\right)$ (ix) $e^{\sinh^{-1}x}$

27 3.12 Applications of Differentiation

27.1 3.12.1 Critical Point

The value of c for which the derivative $f^{/}(c) = 0$, is called the critical (or turning) point of $f(x)$. Geometrically this means that the tangent to $f(x)$ at $(c, f(c))$ is horizontal. The critical point of a function $f(x)$ can therefore be found by calculating and solving for x,the equation $f'(x) = 0$.

Example 102 (3.12.1.1) *: Find the critical points of the functions below:*

(i) $f(x) = x^3 - 3x^2 - 24x + 8.$
(ii) $y = 1 - \frac{1}{x^2}.$
Solution:
(i) $f\,'(x) = 3x^2 - 6x - 24, \quad f\,'(x) = 0 \Rightarrow\ x = 4,\ -2.$
(ii) $y' = \frac{2}{x^3},\ y \neq 0$ for any value of x. So f has no critical point.

Definition 103 (3.12.1.1) *:*

A function $f(x)$ is said to have a local maximum $f(p)$ at the point p if $f(x) \leq f(p)$ for all x near p. This means the inequality holds in some interval containing the point p.

Similarly, $f(x)$ has a local minimum $f(q)$ at the point q if $f(q) \leq f(x)$ for all x near q. A local maximum or minimum can be, but is not always a critical point.

A function $f(x)$ is said to have an absolute maximum at r if $f(x) \leq f(r)$ for all x in the domain of $f(x)$.Similarly, $f(x)$ has an absolute minimum at s if $f(x) \geq f(s)$ for all x in the domain of f.

Example 104 (3.12.1.2) *:*

(i) The function $y = x^2$ if $(-4 < x < 4)$ has no local maximum. There is no interval containing p such that $f(x) \leq f(p)$. The function however has a local minimum of 0 at $x = 0$. $f(0) \leq f(x)$ in every interval containing 0.

(ii) However, the function $y = x^2$ if $(-1 \leq x \leq 2)$ has local maximum of 1, taken at the point $x = -1$ and 4 taken at $x = 2$, and an absolute maximum of 4, taken at $x = 2$. It has an absolute minimum value of 0 taken at $x = 0$.

(iii)The function $y = x^2$if $(-1 \leq x \leq 1)$ has an absolute maximum at both -1 and 1.

If $f(x)$ has a local maximum or a local minimum at p, then p need not be a critical point. Conversely, if p is a critical point, $f(p)$ need not be a local maximum or local minimum .

If, on the other hand, $f(x)$ is differentiable at p and has a local maximum or minimum at p, then, p is a critical point of f.

27.2 3.12.2 First DerivativeTest:

Suppose $f'(a) = 0$ and x is near a, then

(i) If $f'(x) \leq 0$ for all $x \leq a$ and $f'(x) \geq 0$ for all $x \geq a$, f is a local minimum at a. The gradient of the curve goes from a negative value through zero to a positive value.

(ii) If $f'(x) \geq 0$ for all $x \leq a$ and $f'(x) \leq 0$ for all $x \geq a$, f is a local maximum at a. The gradient of the curve goes from a positive value through zero to a negative value.

If neither of these cases applies, then the critical point is neither a local maximum nor a local minimum point.

Example 105 (3.12.2.1) *:*

Sketch the graph of $y = x^2 - 2x - 1$ using the first derivative test.

Solution 106 *:*

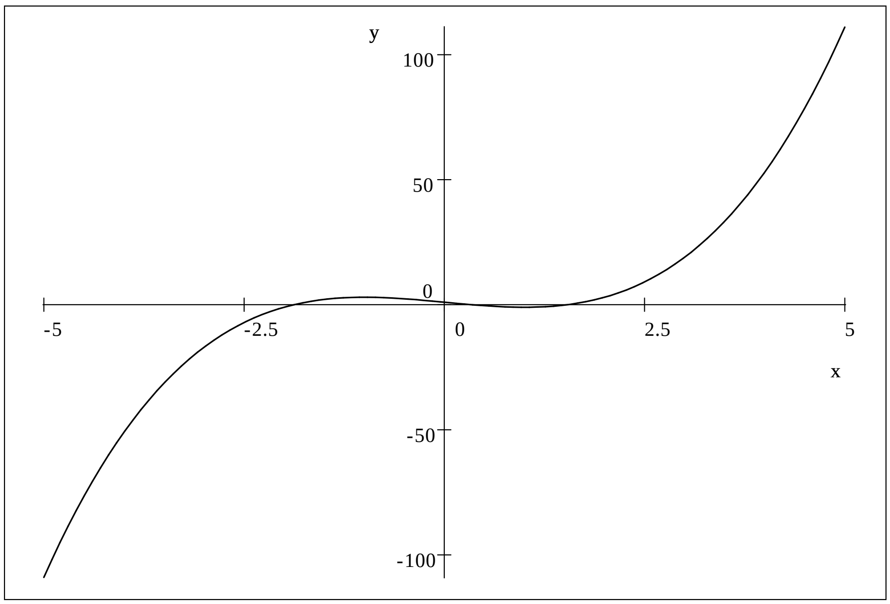
Since $\frac{dy}{dx} = 2x - 2$, we see that $x = 1$ is a critical point. Also $2x - 2 \leq 0$ if $x \leq 1$ and $2x - 2 \geq 0$ if $x \geq 1$.So the The gradient of the curve goes from a negative value through zero to a positive value.. The function has a minimum at $x = 1$..

Example 107 (3.12.2.2) *: Sketch the graph of $f(x) = x^3 - 3x + 1$.*

Solution 108 *:*

Note that $f\ '(x) = 3x^2 - 3 = 3(x^2 - 1)$.

Hence, $x = -1$ and $x = 1$ are critical points. For $x \in (-1, 1)$, we have $x^2 < 1$, so $f'(x) = 3(x^2 - 1) < 0$. $f'(x) = 3(x^2 - 1) > 0$ for $x \in (-\infty, -1)$ or $x \in (1, \infty)$. So f increases on these intervals. Hence $f(1) = -1$ is a local minimum and $f(-1) = 3$ is a local maximum. The graph appears as shown below.



27.3 3.12.3 Maximum and Minimum Problems

The problem often takes the form of a word problem to find the maximum and minimum quantity Q. It is helpful to draw and label its pertinent features. With the aid of the figure if possible, find an equation giving quantity Q in terms of other variables. Use relations among the variables to obtain Q in terms of a single variable $x : Q = Q(x)$. Find the derivative,$Q'(x)$. Set the derivative equal to zero, i.e. $Q'(x) = 0$ and solve to find critical points. The desired maximum or minimum usually occurs at a critical point. Select the proper critical point and apply the test. We have determined the value of x that either maximizes or minimizes Q. The desired value of Q is obtained by simply finding $Q = Q(x)$ for this value of x.

Example 109 (3.12.3.1) *: Show that the square has the greatest area amongst all rectangles of a given perimeter.*

Solution 110 *:*

Let P = the given perimeter, a constant, x = width, and y = the height of the rectangle. Then the area $A = xy$. Since $2x + 2y = P$, $y = \frac{P-2x}{2}$, so $A(x) = x\left(\frac{P-2x}{2}\right) = \frac{Px}{2} - x^2$.

(Note that $A(x)$ only makes physical sense for $0 \le x \le P$). We have,$A'(x) = \frac{p}{2} - 2x$, and set it equal to zero: $0 = \frac{p}{2} - 2x$. Solving, we find $x = \frac{P}{4}, y = \frac{P}{4}$ and the rectangle must be a square. It is clear from the problem that the maximum area occurs at $x = \frac{P}{4}$. To show this mathematically, we note that

If $A'(x) = \frac{1}{2}p - 2x \ge 0$ if $x \le \frac{p}{4}$, $A'(x) = \frac{1}{2}p - 2x \le 0$ if $x \ge \frac{p}{4}$

Hence, $A(x)$ increases on$[0, \frac{p}{4}]$ and decreases on, which proves that the maximum occurs at $x = \frac{p}{4}$.

Example 111 (3.12.3.2) *: A man,* 100 *m away from the base of a flag pole, starts walking towards the base at* 10 *m/sec just as a flag at the top of the pole is lowered at the rate of* 5 *m/sec. If the pole is* 7 *cm tall, find how the distance between the man and the flag is changing per unit of time at the end of* 2 *sec.*

Solution 112 *:*

Call x the distance the man is from the base, y the height of the flag, and z the distance between man and flag at any time t .

Then we are given

$\frac{dx}{dt} = -10cm/\sec$, $\quad \frac{dy}{dt} = -5cm/\sec$

(The minus sign is present because x and y are decreasing.) Moreover, $x = -10t + 100$ and $y = -5t + 70$. Now, always we have $z^2 = x^2 + y^2$ and differentiating this equation with respect to time gives

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$. We find that $x = 80m, y = 60m$ then $t = 2$; and then $z = \sqrt{x^2 + y^2} = \sqrt{80^2 + 60^2} = 100m$.

Hence, $100\frac{dz}{dt} = 80(-10) + 60(-5) = -1100. \Rightarrow \frac{dz}{dt} = -11m/\sec$. The minus sign says that the distance t is decreasing at $t = 2$.

Example 113 (3.12.3.3) *: We explore the problems below.*

1. Find the critical points of

i) $f(x) = x^3 - 3x + 7$. ii) $y = -2x^3 + 6x^2 - 12x + 4$.

2. Draw the graphs of the following functions:

i) $f(x) = \frac{1}{x-2}$ ii) $f(x) = \frac{x}{x^2-4}$

3. Show that $f(x) = x^3 + x + 1$ has no critical points and is always increasing.

4. A farmer has 100m of fence and wishes to enclose a rectangular plot of land. The land borders a river and no fence is required on that side. What should the dimensions of the rectangle be in order that it includes the largest possible area?

27.4 3.12.4 Second Derivative Test:

Suppose that a function $f(x)$ has a derivative on an open interval (x_1, x_2) containing p, and that $f'(p) = 0$. If $f''(p) < 0$, then $f(x)$ has a local maximum at the point $(p, f(p))$.

If $f''(p) > 0$, then $f(x)$ has a local minimum at the point $(p, f(p))$. The notion of concavity of the graph of f can be used to illustrate conditions above as follows.

If $f'(p) = 0$, the tangent line to the graph at $(p, f(p))$ is horizontal. In addition if $f''(p) < 0$, then the graph is concave down at p and hence there is an open interval (x_1, x_2) containing p such that the graph lies below the tangent line at p. It then follows that $f(p)$ is a relative (local) maximum. If $f'(p) = 0$, the tangent line to the graph at $(p.f(p))$ is also horizontal. If in addition , the graph is concave up at p and hence there is an open interval (x_1, x_2) containing p such that the graph lies above the tangent at p. Then $f(p)$ is a relative (local) minimum.

If $f\ '(p) = 0$, the second derivative test is not employed in that case, the first derivative test is employed. P is called a flex and $(p.f(p))$ point of inflexion.

Example 114 (3.12.4.1) *: Use concavity and second derivative test to examine the following curves for minima, maxima, flex point.*

(i). $x^5 - 2x^3 + x - 12$.
(ii). $x \ln x$.

Solution 115 *:*

(i) $\quad f(x) = x^5 - 2x^3 + x - 12$
$f'(p) = 5x^4 - 6x^2 + 1 = (5x^2 - 1)(x^2 - 1)$
$f'(p) = (5p^2 - 1)(p^2 - 1) = 0$
$p = \pm\frac{1}{\sqrt{5}}; \quad p = 1$ or $p = -1$.
. $\quad f\ ''(x) = 20x^3 - 12x. \qquad$ For $p = 1, \qquad f\ ''(1) = 8$ which is positive.

This implies that $(1, f(1))$ is a local minimum and the graph is concave up at $p = 1$.For $p = -1$,$f\ ''(-1) = -8$ which is negative.

Therefore, $f(-1)$ is a local maximum and the graph is concave down at $(-1, -12)$.

For $\quad p = -\frac{1}{\sqrt{5}} = -0.45$,, $f\ ''(-0.45) = 3.58$ which is positive.

Therefore, $f(-0.45)$ is a local minimum and the graph is concave upward at $(-0.45, -12.46)$.

For $\quad p = \frac{1}{\sqrt{5}} = 0.45 \qquad ,$

$f\ ''(0.45) = -3.58$ which is negative.

Therefore $f(0.45)$ is a local maximum and the graph is concave downward at $(0.45, -11.54)$.

For flex point, $f\ ''(p) = 0$ and this implies that $20p^2 - 12p = 0 \Rightarrow 4p[5p^2 - 3] = 0$.So either $p = 0$ or $p = \pm 3/5 = \pm 0.77$

Therefore, the points of inflection are

$(0, f(0))$, $\left(-\sqrt{\frac{3}{5}}, f(-\sqrt{\frac{3}{5}})\right)$ and $\left(\sqrt{\frac{3}{5}} \ , f(\sqrt{\frac{3}{5}})\right)$ or $(0, -12)$, $\left(-\sqrt{\frac{3}{5}} \ , -12.12\right)$ and $\left(\sqrt{\frac{3}{5}} \ , -12.5\right)$

ii) $\quad f(x) = x \ln x$

$f\ '(x) = \ln x + 1 \qquad f\ '(p) = \ln p + 1 = 0$
implies that: $\ln p = -1$ and $\quad p = e^{-1}.f\ ''(p) = \frac{1}{p} = \frac{1}{e^{-1}} = e$.Hence

$f\ ''(e^{-1}) = e$, is positive.

Therefore, $f(e^{-1})$ is a local minimum, and the graph is concave up at $p = e^{-1}$.

For flex point,

$f\ ''(p) = 0$ implies that $\frac{1}{p} = 0 \Rightarrow 1 = 0$ which is a contradiction, and so $p = \infty$. Hence there is no flex point.

Example 116 *3.12.4.2]: Use the concavity and second derivative test to examine the following curves for minimum, maximum and flex point.*

i) $12 + 2x^2 - x^4$, ii) $x^5 - 5x^3$, iii) $1 - x^{1/3}$

Solution 117 *:*

i) $f(x) = 12 + 2x^2 - x^4$

$f\ '(x) = 4x - 4x^3 = 4x[1 - x^2]$
$f\ '(p) = 4p[1 - p^2] = 0 \quad \Rightarrow p = 0$ or $p = \pm 1$.
$f\ ''(p) = 4 - 12p^2$ .For $p = 0$, $\quad f\ ''(0) = 4$, is positive. Therefore, $f(0)$ is a local minimum and the graph is concave upward at $(0, 12)$.

For $p = 1$, $f\ ''(1) = -8$ is negative. Therefore, $f(1)$ is a relative (local) maximum, and the graph is concave down at $(1, 13)$.

For $p = -1$, $f\ ''(1) = -8$ is negative. Therefore, $f(-1)$ is a local maximum, and the graph is concave down at $(-1, 13)$.

For the flex point $f\ '(p) = 0$ and this implies that $\quad 4 - 12p^2 = 0, \quad p^2 = \frac{1}{3}$

$\Rightarrow p = \pm\frac{1}{\sqrt{3}} = \pm 0.58$

Therefore, the points of inflection are

$(-0.58, 12.56)$ and $(0.58, 12.56)$.

ii) $\quad f\ (x) = x^5 - 5x^3$

$f\ '\ (x) = 5x^4 - 15x^2 = 5x^2[x^2 - 3]$

$f\ '\ (p) = 5p^2[p^2 - 3] = 0$

$\Rightarrow p = 0$ and $p = \pm\sqrt{3} = \pm 1.73$

$f\ ''\ (p) = 20p^3 - 30p$.For $p = 0 \quad f\ ''\ (0) = 0$.Therefore the second derivative test is not applicable

For $p = 1.73$

$f\ ''\ (1.73) = 46.36$ is positive. Therefore, $f\ (1.73)$ is a local minimum and the graph is concave up at $(1.73,\ -10.37)$

For $p = -1.73$, $f\ ''\ (-1.73) = -51.65$ for positive.

Therefore, $f\ (-1.73)$ is a local maximum, and the graph is concave down at $(-1.73,\ -10.37)$.

For the flex point $f\ ''\ (p) = 0$

Therefore $20p^3 - 30p = 0. \Rightarrow 10p[2p^2 - 3] = 0 \Rightarrow p = 0$ or $p = \pm\sqrt{\frac{3}{2}} = \pm 1.22$.So the points of inflection are $(0,0), (-1.22, 6.38)$ and $(1.22, -6.38)$

Example 118 (3.12.4.3) *: We explore the problems below.*

(i) $x^4 - 6x^2 + 18x + 10$, (ii) $(x+5)^4(2x-3)^3$, (iii) $x^3 + 2x^2 + x + 1$.

Definition 119 (3.12.4.1) *:*

Suppose the function f is defined at p and for all values of x near p. Then the function is said to be continuous at p provided that $\lim\limits_{x\to p} f(x) = f(p)$.

Example 120 (3.12.4.4) *: Let us define $f(x) = \frac{\sin x}{x}$ if $x \neq 0$. This definition of $f(x)$ has no meaning if $x = 0$, since division by 0 is undefined. However, let us make the additional definition $f(0) = 1$. With this definition $f(x)$ is continuous at $x = 0$.*

For $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

We have based the concept of continuity directly upon the concept of a limit. A condition for continuity of a function may be given directly in terms of inequalities, just as we defined a limit in terms of inequalities.

Definition 121 (3.12.4.2) *: Thus, if f is defined throughout some interval containing p and all points near p, f is continuous at p if to each positive $\varepsilon > 0$ corresponds some positive $\delta > 0$ such that $\mid f(x) - f(b) \mid < \varepsilon$ whenever $\mid x - p \mid < \delta$. This form of the condition for continuity is equivalent to the original definition.*

Example 122 (3.12.4.5) *: Let f be a continuous function on a closed interval $[x_1, x_2]$, and differentiable on an open interval (x_1, x_2), and $f(x_1) = f(x_2)$. Then there exists at least one number p in (x_1, x_2) such that $f'(p) = 0$. This is called Rolle's theorem.*

(i) $f(x) = f(x_1)$, (ii) $f(x) < f(x_1)$ and (iii) $f(x) > f(x_1)$

Now if

(i) $f(x) = f(x_1)$, it implies that $f(x)$ is a constant function and hence $f'(x) = 0$ for every p and for every point p in (x_1, x_2), $f'(p) = 0$.

(ii) $f(x) < f(x_1)$ for some x in (x_1, x_2), this implies that the minimum value of $f(x)$ on $[x_1, x_2]$ is less than $f(x_1)$ and therefore must occur at some number p in (x_1, x_2). Since the function is continuous, the derivative exists throughout (x_1, x_2), which implies that $f'(p) = 0$.

(iii) If $f(x) > f(x_1)$, the argument is similar to (ii).

Example 123 (3.12.4.6) *:*

Show that the curve $3x^2-12x+11$ satisfies the hypothesis of Rolle's theorem on the interval $[0,4]$ and find all number(s) p in $(0,4)$ such that $f\,'(p)=0$.

Solution 124 *:*

Since $f(x)=3x^2-12x+11$ is a polynomial it is continuous and differentiable on every interval, therefore $f(x)$ is continuous on $[0,4]$ and differentiable on $(0,4)$ and $f(0)=f(4)=11$. Also,
$f\,'(x)=6x-12$
$f\,'(p)=6p-12=0$
$\Rightarrow p=2$
$f\,'(2)=0$ and $0<2<4$..
Hence, $f(x)$ satisfies the hypothesis of Rolle's theorem on $[0,4]$.

Example 125 *:[3.12.4.7] Verify that the function* x^4+4x^2+1 *satisfies the hypothesis of Rolle's theorem on the interval* $[-3,3]$ *and find all number(s)* p *in* $(-3,3)$ *such that* $f\,'(p)=0$.

Solution 126 *:*

$f(x)$ being a polynomial, it is continuous on $[-3,3]$, and differentiable on $(-3,3)$. And $f(-3)=f(3)=118$.

$f\,'(x)=4x^3+8x$
$\Rightarrow f\,'(p)=4p^3+8p=0$

$\Rightarrow 4p(p^2+2)=0$
$\Rightarrow p=0$ and $(p=\pm i\sqrt{2})$ which you can neglect. So $f\,'(0)=0$ and $-3<0<3$.Hence $f(x)$ satisfies the hypothesis of Rolle's theorem.

Example 127 (3.12.4.8)

Determine whether the following functions satisfy the hypothesis of Rolle's theorem on the interval $[x_1,x_2]$. If so, find all number(s) p in (x_1,x_2).
i) $f(x)=4x^2-20x+29$ $[1,4]$
ii) $f(x)=5-12x-2x^2$ $[-7,1]$
iii) $f(x)=x^3-x$ $[-1,1]$
[3.12.4.1] : Let f be a function which is continuous at each point of the closed interval $a\le x\le b$, and let it have a derivative at each point of the open interval $a<x<b$. Then there is a point $x=p$ in the open interval $(a<p<b)$ such that $f(b)-f(a)=(b-a)f\,'(p)$ ...(1)
The theorem has a geometrical interpretation. Represent the function graphically by the curve $y=f(x)$, and let A,B be points on the curve corresponding to $x=a, x=b$, respectively. The formula (1) states that there is some point on the curve, with the abscissa $x=p$, at which the tangent is parallel to the line AB. There may be more than one suitable value of p; the essential thing is that there is always at least one.

Remark 128 (3.12.4.1) *: It is worth noting that (1) remains true if we exchange a and b, for both sides merely change signs when this is done. Thus, suppose x_1, x_2 are the endpoints of an interval on which the conditions of the mean value theorem are satisfied, then we can write*

$f(x_2) - f(x_1) = (x_2 - x_1)f\,'(\xi)$ where $x = \xi$ is some point between x_1 and x_2. In writing this formula we do not need to know which of the numbers x_1, x_2 is larger.

Example 129 (3.12.4.9) *: Suppose $f(x) = x^3$, find a suitable value for p in formula (1) when $a = -1$ and $b = 2$.*

Solution 130 *: We have $f\,'(x) = 3x^2$, the mean value theorem takes the form*

$b^3 - a^3 = (b-a)3p^2$ in the present case. With $a = -1, b = 2$ we have$8 - (-1)^3 = (2-(-1)).3p^2$, or $p^2 = 1$. Solving, we find $p = \pm 1$. We want a value of p such that $a < p < b$, i.e.
$-1 < p < 2$. Hence, $p = 1$ is the suitable value in question.

Example 131 (3.12.4.10) *: If $f(x) = x^2$, show that the suitable value of p in the mean value theorem is $p = \frac{a+b}{2}$.*

Solution 132 *: We have $f\,'(x) = 2x$. Hence, the mean value theorem becomes*

$b^2 - a^2 = (b-a).2p$, or $p = \frac{a+b}{2}$. Where is this point located in relation to a and b?

Example 133 (3.12.4.11) *: If $f(x) = \sin x$ and $x_1 = 0, x_2 = \frac{5\pi}{6}$, find ξ such that*

$x_1 < \xi < x_2$ and formula (2) holds.

Solution 134 *: We have $f\,'(x) = \cos x$ and $\sin\frac{5\pi}{6} = \frac{1}{2}$. Hence formula (2) takes the form*

$\sin\frac{5\pi}{6} - \sin 0 = \frac{5\pi}{6}\cos\xi$
or $\frac{1}{2} = \frac{5\pi}{6}\cos\xi$
$\cos\xi = \frac{3}{5\pi} = 0.19099$.
since $0 < \xi < \frac{5\pi}{6}$, we find $\xi = \cos^{-1}(0.1909) = 1.37863$.

Example 135 (3.12.4.12) *: Show that $0 < \log 1.5 < 1$*

Solution 136 *: This may be done as follows: We take $f(x) = \log x, a = 1$,*

$b = 1.5$, then $f(x) = \frac{1}{x}$, and by the mean value theorem
$\log 1.5 - \log 1 = \frac{(1.5-1)}{p} = \frac{0.5}{p}$,where $1 < p < 1.5$. It follows that $\frac{0.5}{1.5} < \frac{0.5}{p} < \frac{0.5}{1}$.This gives the desired result.

Example 137 (3.12.4.13) *: Use the mean value theorem to show that*

$\frac{\log x}{x} < \frac{\log a}{x} + \frac{1}{a}$
if $0 < a < x$.

Solution 138 *: We use (1) with $f(x) = \log x$ and b replaced by x. Then* $\log x - \log a = \frac{x-a}{p}$,

$a < p < x$
Hence,

$\frac{\log x}{x} = \frac{\log a}{x} + \frac{x-a}{p}$,Now, $\frac{x-a}{x} . \frac{1}{p} < \frac{1}{p} < \frac{1}{a}$

and so the desired result.
Module 4: Integration and the Indefinite Integral

28 4.1 Objectives:

In this module we introduce with examples, the concept of integration as an anti-derivative. This is followed by worked examples and exercises to facilitate consolidation of appreciation of the concept. Various methods of evaluating integrals are treated with relevant worked examples for their easy understanding. Group assignments, take home tests and laboratory sessions will complement power point presentations in tutorial type exchanges. All these activities expose the students to the concepts and techniques and lead them to find out when the different techniques are most appropriate for use.

29 4.2 Learning Outcomes:

At the end of this module students should be able to decide when the different rules best apply and to:

- evaluate integrals using the antiderivative property;
- evaluate integrals using substitution, integration by parts, partial fractions and reduction methods.

30 4.3 Learning Activities:

Students should:

- explore notes and exercises, individually and in groups;
- explore and use related materials on the Intranet, especially the e-granary and MIT open course ware;
- explore and use related materials on the OLI Calculus course on the Internet(http://www.cmu.edu/oLI/courses/);
- solve relevant questions from past MTH103 Examinations.

31 4.4 Introduction

Integration can be considered from two perspectives:

(i) "to find a function whose derivative is given". In this sense, integration is referred to as finding the anti-derivative.

(ii) "to give the sum or total". In this sense integration measures the area bounded by curves. We introduce and study the indefinite integral from the anti derivative perspective. The definite integral is introduced in the next module as area under a curve.

31.1 4.4.1 The Indefinite Integral

Suppose we have the derivative $\frac{dy}{dx}$of a function $y = F(x)$ with

$\frac{dy}{dx} = f(x)$; $a < x < b$ and we are asked to find $y = F(x)$. We illustrate how to do so following the process below.

Let $\frac{dy}{dx} = f(x) = 2x$; from our knowledge of derivatives, we can find one answer, $y = F(x) = x^2$. This is so because if $y = x^2$, $\frac{dy}{dx} = 2x$. But this is not the only answer; $y = x^2 + 2, y = x^2 - \sqrt{3}, y = x^2 + 5\pi$ are all valid answers.

Hence $y = x^2 + C$ is an answer where C is an arbitrary constant.

Thus if we differentiate a function $F(x)$ with respect to x and obtain another function $f(x)$, that is $\frac{d}{dx}F(x) = f(x)$, we say that $F(x)$ is an integral of $f(x)$ with respect to x.

If $F(x)$ is an integral of $f(x)$ with respect to x, then

$F(x) + C$ is also such an integral where C is an arbitrary constant. This is true since

$$\begin{aligned}\frac{d}{dx}[F(x) + C] &= \frac{dF(x)}{dx} + \frac{dC}{dx} \\ &= f(x) + 0 = f(x).\end{aligned}$$

Thus if $\frac{d}{dx}[F(x) + C] = f(x)$, then $\int f(x)dx = F(x) + C$ where the symbol $\int$ is called an "integral sign" and $\int f(x)dx$ is called "the integral of $f(x)$ with respect to x.

We state some standard formulae for integrals using standard formulae for derivatives.

$\frac{d(x)}{dx} = 1$	$\int 1dx = x + c$
$\frac{d(x^n)}{dx} = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
$\frac{d(\ln x)}{dx} = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln x + c$
$\frac{d(\sin x)}{dx} = \cos x$	$\int \cos x dx = \sin x + c$
$\frac{d(\cos x)}{dx} = -\sin x$	$\int \sin x dx = -\cos x + c$
$\frac{d(\tan x)}{dx} = \sec^2 x$	$\int \sec^2 x dx = \tan x + c$
$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{(1-x^2)}}$	$\int \frac{1}{\sqrt{(1-x^2)}}dx = \sin^{-1} x + c$
$\frac{d(e^x)}{dx} = e^x$	$\int e^x dx = e^x + c$
$\frac{d(\cosh x)}{dx} = \sinh x$	$\int \sinh x dx = \cosh x + c.$
$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1} x + c$

We use the combination rules (sum and difference) for the derivatives to get similar rules for integrals and illustrate them below.

Example 139 *[4.4.1.1]Evaluate the following integrals:*

(i) $\int(5x - x^2 + 2)dx$
(ii) $\int x^{\frac{1}{2}}dx$
(iii) $\int(3x + 2e^x + \cos x)dx$
(iv) $\int(3e^x + 4\sinh x)dx$

Solution 140 *(i)*$\int(5x - x^2 + 2)dx = \int 5xdx - \int x^2dx + \int 2dx = 5\int xdx - \int x^2dx + 2\int dx = 5\frac{x^2}{2} - \frac{x^3}{3} + 2x + C.$

(ii) $\int x^{\frac{1}{2}}dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}x^{\frac{3}{2}} + C.$
(iii) $\int(3x + 2e^x + \cos x)dx = \int 3xdx + \int 2e^xdx + \int \cos xdx$
$= 3\int xdx + 2\int e^xdx + \int \cos dx = \frac{3x^2}{2} + 2e^x + \sin x + C.$
(iv) $\int(3e^x + 4\sinh x)dx = 3\int e^xdx + 4\int \sinh xdx$
$= 3e^x + 4\cosh x + C$

Example 141 *[4.4.1.2]: We evaluate the following integrals.*

(1) $\int (2x + 3)\,dx$
(2) $\int \left(x^2 - \sqrt{x}\right)dx$
(3) $\int (5e^x + \sin x)\,dx$
(4) $\int \left(3\cosh x + x^{\frac{1}{2}}\right)dx$
(5) $\int \left(x^4 - e^x + \cos x + \sinh x\right)dx$

32 4.5 Methods of Integration

The integrals evaluated so far have been quite direct and simple. We could easily figure out the function differentiated to obtain the integral. For more complicated functions we need to introduce rules and guidelines for easy operations. These are the various methods of integration for the evaluation of integrals.

32.1 4.5.1 The Method of Substitution or Integration by Change of Variable

The method of substitution is used to evaluate integrals of composite functions. It is derived from the results below, which are generalizations of the list of integrals given earlier. The method of substitution is the converse of differentiating a function of a function.
$\frac{d(ax+b)^n}{dx} = na(ax + b)^{n-1}$ $\int(ax + b)^ndx = \frac{(ax+b)^{n+1}}{(n+1)a} + c;\ n \neq -1.$

$\frac{d(\sin ax)}{dx} = a\cos ax$ $\quad\int \cos axdx = \frac{1}{a}\sin ax + c$
$\frac{d(\tan ax)}{dx} = a\sec^2 x$ $\quad\int \sec^2 axdx = \frac{1}{a}\tan ax + c$
$\frac{d(\sin^{-1}\frac{x}{a})}{dx} = \frac{1}{\sqrt{(a^2-x^2)}}$ $\quad\int \frac{1}{\sqrt{(a^2-x^2)}}dx = \sin^{-1}\frac{x}{a} + c$
$\frac{d(e^{ax})}{dx} = ae^{ax}$ $\quad\int e^{ax}dx = \frac{1}{a}e^{ax} + c$
$\frac{d(\cosh ax)}{dx} = a\sinh ax$ $\quad\int \sinh axdx = \frac{1}{a}\cosh ax + c.$

$\frac{d(\tan^{-1}\frac{x}{a})}{dx} = \frac{a}{a^2+x^2}$ $\quad\int \frac{1}{1+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
Let us consider $I = \int \varphi(x)dx$ where $x = x(t)$.
By definition of I, $\frac{dI}{dx} = \varphi(x)$ and $\frac{dI}{dt} = \frac{dI}{dx}.\frac{dx}{dt} = \varphi(x)\frac{dx}{dt}$.
Thus $\int \varphi(x)dx = \int \varphi(x)\frac{dx}{dt}dt$. This is the substitution method.
More explicitly it can be written as $\int \varphi(x[t])dx = \int \varphi(x[t])\frac{dx}{dt}dt$.
We now illustrate the use of these generalizations and the related method of substitution by solving some examples.

Example 142 (4.5.1.1) *:Evaluate the following integrals.*

(i) $\int \sqrt{(2x+1)dx}$
(ii) $\int (3x-1)^{234}dx$
(iii) $\int \frac{3rdr}{\sqrt{(1-r^2)}}$
(iv) $\int \sin^2 x\cos xdx$
(v) $\int e^{5x}dx$
(vi) $\int \tan 3xdx$

Solution 143 *(i)* $\int \sqrt{((2x)+1)dx} = \frac{1}{2}\frac{2(2x+1)^{\frac{3}{2}}}{3} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$ *directly.*

Alternatively, let $\varphi(x) = (2x+1)^{\frac{1}{2}}$ with $2x+1 = t$ and $2\frac{dx}{dt} = 1$.
$\int \varphi(x)dx = \int \varphi(x)\frac{dx}{dt}dt = \int t^{\frac{1}{2}}.\frac{1}{2}dt = \frac{1}{3}t^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$
(ii) $\int (3x-1)^{234}dx = \frac{1}{3}\frac{(3x-1)^{235}}{235} + C$ directly.
Alternatively, let $\varphi(x) = (3x-1)^{234}$ with $3x-1 = t$ and $3\frac{dx}{dt} = 1$.
$\int \varphi(x)dx = \int \varphi(x)\frac{dx}{dt}dt = \int t^{234}.\frac{1}{3}dt = \frac{1}{3}.\frac{1}{235}t^{235} + C = \frac{1}{3}.\frac{(3x-1)^{235}}{235} + C$
(iii) $\int \frac{3rdr}{\sqrt{(1-r^2)}} = -3\sqrt{(1-r^2)} + C.$
(iv) $\int \sin^2 x\cos xdx = \frac{\sin^3 x}{3} + C$ directly.
Alternatively let $\varphi(x) = \sin^2 x\cos x$ with $\sin x = t$ and $\cos x\frac{dx}{dt} = 1$.
$\int \varphi(x)dx = \int \varphi(x)\frac{dx}{dt}dt = \int \sin^2 x\cos x\frac{1}{\cos x}dt = \int \sin^2 xdt = \int t^2dt$
$= \frac{1}{3}t^3 + C = \frac{1}{3}\sin^3 x + C.$
(v)$\int e^{5x}dx = \frac{1}{5}e^{5x} + C.$
(vi) $\tan 3x = \frac{\sin 3x}{\cos 3x}$.
Let $\varphi(x) = \frac{\sin 3x}{\cos 3x}$ with $\sin 3x = t$ and $3\cos 3x\frac{dx}{dt} = 1$.
$\int \varphi(x)dx = \int \varphi(x)\frac{dx}{dt}dt = \int \frac{\sin 3x}{\cos 3x}\frac{1}{3\cos 3x}dt = \frac{1}{3}\int \frac{\sin 3x}{1-\sin^2 3x}dt = \frac{1}{3}\int \frac{t}{1-t^2}dt$
$= \frac{1}{3}\ln(1-t^2).\frac{-1}{2} + C = \frac{-1}{6}\ln\cos^2 3x + C = \frac{-1}{3}\ln|\cos 3x| + C.$

Example 144 (4.5.1.2) *:We explore the evaluation of the following integrals.*

(i) $\int x^2\sqrt{(x^3+5)dx}$
(ii) $\int e^{\cos x}\sin x dx$
(iii) $\int \frac{\cos x}{\sin x}dx$
(iv) $\int \sinh x \cosh x dx$
(v) $\int (1+\cos\theta)^3 \sin\theta d\theta$
(vi) $\int \frac{tdt}{\sqrt{(4t^2-1)}}$
(vii) $\int \frac{xdx}{4x^2+1}$
(viii) $\int \frac{2vdv}{\sqrt{(1-v^2)}}$

32.2 4.5.2 The Method of Integration by Parts

Integration by parts is a consequence of the product rule for derivatives and the anti derivative concept.

The product rule states that $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Using the antiderivative concept $uv = \int (u\frac{dv}{dx} + v\frac{du}{dx})dx$

and $uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$.

Hence $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ which is the formula for integration by parts. In using this method, care has to be taken in the choice of u and v as we illustrate in the following examples.

Example 145 (4.5.2.1) *Evaluate the following integrals:*

(i) $\int xe^x dx$
(ii) $\int \ln x dx$
(iii) $\int \tan^{-1} x dx$
(iv) $\int x^2 e^x dx$

Solution 146 *(i)*$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Choose $u = x$ and $\frac{dv}{dx} = e^x$;then $\frac{du}{dx} = 1$ and $v = e^x$.

Thus $\int xe^x dx = x.e^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$.

It is important to note that the alternative choice of u and $\frac{dv}{dx}$ does not lead to an easier integral and a solution.

(ii) Let $u = \ln x$ and $\frac{dv}{dx} = 1$. $\frac{du}{dx} = \frac{1}{x}$ and $v = x$.

Thus $\int \ln x dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C = x(\ln x - 1) + C$.

(iii)Let $u = \tan^{-1} x$ and $\frac{dv}{dx} = 1$; $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = x$;

Thus $\int \tan^{-1} x dx = x\tan^{-1} x - \int \frac{x}{1+x^2}dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

The next example illustrates the repeated use of the method of integration by parts.

(iv) Let $u = x^2$ and $\frac{dv}{dx} = e^x$; $\frac{du}{dx} = 2x$ and $v = e^x$.

Thus $\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$

We evaluated $\int xe^x dx$ in (i) above using the same method and can recall that result to conclude the solution as;

$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2[e^x(x-1)] + C$.

Example 147 (4.5.2.2) *We explore the evaluation of the following integrals:*

(i) $\int x \ln x dx$
(ii) $\int x \sin ax dx$
(iii) $\int x \cos(2x+1) dx$
(iv) $\int x^2 \cos ax dx$
(v) $\int \sin^{-1} ax dx$
(vi) $\int x^2 \tan^{-1} x dx$

32.3 4.5.3 Integration by the method of partial fractions

We recall that $\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)+3(x+1)}{(x+1)(x-3)} = \frac{5x-3}{x^2-2x-3}$

The reverse process of finding constants A and B such that

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

where A and B are constants is called resolution into partial fractions.

In the particular example above, we simplify both sides and explore the identity that results from the numerators.

Either we consider the identity for special values of x such as $x = 3$ and $x = -1$ or we equate coefficients of like powers to get $A = 2$ and $B = 3$

Remark 148 (4.5.3.1) *Generally, to write a rational function $\frac{f(x)}{g(x)}$ in terms of its partial fractions, the following guidelines should be observed.*

(1) The degree of the numerator $f(x)$ should be less than the degree of the denominator $g(x)$. If not, we first perform a long division and get a remainder $r(x)$ with degree less than the degree of $g(x)$.

(2) The factors of $g(x)$ should be known and the steps below followed.

(a) Let $x - r$ be a linear factor of $g(x)$. Suppose m is the highest power of $x-r$ that divides $g(x)$. Then assign the sum of m partial fractions to this factor thus:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + ... + \frac{A_m}{(x-r)^m}$$

This should be done for each distinct linear factor of $g(x)$.

(b) Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose n is the highest power of the quadratic factor $(x^2 + px + q)$ that divides $g(x)$. Then, to this factor, assign the sum of n partial fractions

$$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + ... + \frac{B_nx+C_n}{(x^2+px+q)^n}$$

This should be done for each distinct quadratic factor of $g(x)$.

(c) Put the original rational function $\frac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Simplify both sides and examine the numerators.

(d) Equate the coefficients of the corresponding powers of x or use particular values on the identities to get the unknown coefficients.

By resolving rational functions into partial fractions we have simpler functions that are much easier to integrate. We illustrate the use of the method of integrating by partial fractions in the examples below.

(i) Express $\frac{f(x)}{g(x)} = \frac{-2x+4}{(x^2+1)(x-1)^2}$ and find $\int \frac{4-2x}{(x^2+1)(x-1)^2} dx$

(ii) Evaluate $\int \frac{x+4}{x^3+3x^2-10x}dx$
(iii) Evaluate $\int \frac{dx}{1-x^2}$
(iv) Evaluate $\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}dx$

(i) The degree of $f(x)$ is less than that of $g(x)$ which is already written as a product of linear and quadratic factors. So let
$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$,then,
$-2x+4 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$
$= (A+C)x^2 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D).$

Equating coefficients, we have

$$A+C=0$$
$$-2A+B-C+D=0$$
$$A-2B+C=-2$$
$$B-C+D=4$$

Solving these equations simultaneously to have

$$A=2, B=1, C=-2, D=1.$$
$$\therefore \quad \frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} + \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Alternatively we could use the identity

$4-2x = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$ which is true for all values of x. When $x=1$ gives $2=2D$ and $D=1$. We have to equate coefficients to get A, B and C.

Thus $\int \frac{4-2x}{(x^2+1)(x-1)^2}dx = \int \frac{2x+1}{x^2+1}dx + \int \frac{2}{x-1}dx + \int \frac{1}{(x-1)^2}dx$
$= \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx + \int \frac{2}{x-1}dx + \int \frac{1}{(x-1)^2}dx$
$= \ln(x^2+1) + \tan^{-1}x + 2\ln(x-1) - \frac{1}{x-1} + C.$

(ii) The degree of $f(x) = x+4$ is less than the degree of $g(x) = x^3+3x^2-10x$ and factors of $g(x)$ are $x(x-2)(x+5)$.

Thus,

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+5}.$$

Solve simultaneously we obtain

$A = \frac{-2}{5}, B = \frac{3}{7}$ and $C = -\frac{1}{35}$. So $\frac{x+4}{x(x-2)(x+5)} = \frac{-2}{5x} + \frac{3}{7(x-2)} + \frac{1}{35(x+5)} = \int \frac{-2}{5x}dx + \int \frac{3}{7(x-2)}dx + \int \frac{1}{35(x+5)}dx$

$\therefore \int \frac{x+4}{x(x-2)(x+5)}dx = \frac{-2}{5}\ln|x| + \frac{3}{7}\ln|x-2| + \frac{1}{35}\ln|x+5| + C.$

(iii) $\frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)}.$

Let $\frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x}$

$$1 = A(1-x) + B(1+x)$$
$$1 = A - Ax + B + Bx = (B-A)x + (B+A).$$

Equating coefficients yields

$$B-A=0 \text{ and } B+A=1$$
$$\Longrightarrow B=A=\frac{1}{2}.$$

$\therefore \int \frac{1}{1-x^2}dx = \int \frac{1}{2(1+x)}dx + \int \frac{1}{2(1-x)}dx = \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + C = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$

(iv) The degree of $f(x)$ is greater than the degree of $g(x)$, so, we first do a long division to get

$\frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1} = x+1+\frac{-2x+4}{x^4-2x^3+2x^2-2x+1}$

We factor the denominator of the remainder term to get $(x^2+1)(x-1)^2$. Thus

$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$

$\Rightarrow \int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}dx = \int[x+1+\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}]dx$

$= \int xdx + \int 1dx + \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx - \int \frac{2}{x-1}dx + \int \frac{1}{(x-1)^2}dx$

$= \frac{x^2}{2} + x + \ln(x^2+1) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + C.$

Example 149 (4.5.3.1) *We explore the evaluation of the following integrals:*

(i) $\int \frac{d\theta}{\theta^2+\theta^2-2\theta}$

(ii) $\int \frac{xdx}{x^2+4x-5}$

(iii) $\int \frac{(x+1)dx}{x^2+4x-5}$

(iv) $\int \frac{\cos xdx}{\sin^2 x+\sin x-6}$

(v) $\int \frac{dx}{x(x^2+x+1)}$

(vi) $\int \frac{dx}{(x+1)(x^2+1)}$

32.4 4.5.4 Integration by the method of reduction

This method involves the reduction of higher powers of an integrand to lower powers in order to facilitate the integration. We explain the process by solving some examples.

Example 150 (4.5.4.1) *(i) Obtain a reduction formula for* $I_n = \int \tan^n xdx$ *and hence evaluate* $\int \tan^4 xdx$.

(ii) Obtain a reduction formula for $J_n = \int \cos^n xdx$, then evaluate $\int \cos^4 xdx$.

(iii) Evaluate $\int \sin^n ax \cos axdx$.

(i) $I_n = \int \tan^n xdx = \int \tan^{n-2} x \tan^2 xdx, n \neq 1$

$= \int \tan^{n-2} x(\sec^2 x - 1)dx$

$= \int \tan^{n-2} x \sec^2 xdx - \int \tan^{n-2} xdx$

$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} xdx = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}.$

$\int \tan^4 xdx$ is evaluated as follows:

Substitute $n=4$; $I_4 = \frac{\tan 3x}{3} - I_2.$

For $n=2$; $I_2 = \frac{\tan x}{1} - I_0$. But $I_0 = \int \tan^0 xdx = \int 1dx = x + C_1.$

Thus $I_4 = \int \tan^4 xdx = \frac{1}{3}\tan^3 x - [\tan x - x + C_1]$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$,where $C = -C_1$.

(ii) $J_n = \int \cos^n xdx = \int \cos^{n-1} x \cos xdx.$

$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} xdx$,using integrating by parts.

$J_n = \int \cos^n xdx = \cos^{n-1} x \sin x + (n-1)\int(1-\cos^2)x \cos^{n-2} xdx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} xdx - (n-1)\int \cos^n xdx$

$= \cos^{n-1} x \sin x + (n-1)J_{n-2} - (n-1)J_n$

and $[1+(n-1)]J_n = nJ_n = \cos^{n-1} x \sin x + (n-1)J_{n-2}$

Hence $J_n = \int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} J_{n-2}$.
With $n = 4$, $J_4 = \int \cos^4 x dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} J_2$.
where $J_2 = \int \cos^2 x dx = \frac{\cos x \sin x}{2} + \frac{1}{2} \int dx$
and $J_4 = \int \cos^4 x dx = \frac{\cos^3 x \sin x}{3} + \frac{3}{4}(\frac{\cos x \sin x}{2}) + \frac{1}{2}x + C$.
(iii)$\int \sin^n ax \cos ax dx = \frac{\sin^{n+1} ax}{(n+1)a} + C$, $n \neq -1$.Directly.
For $n = -1$, $\int \cot ax dx = \int \frac{\cos ax}{\sin ax} dx = \frac{1}{a} \ln |\sin ax| + C$.
Similarly,
$\int \cos^n ax \sin ax dx = -\frac{\cos^{n+1} ax}{(n+1)a} + C$, $n \neq -1$ but $n = -1$ gives
$\int \tan ax dx = \int \frac{\sin ax}{\cos ax} dx = -\frac{1}{a} \ln |\cos ax| + C$.

Example 151 (4.5.4.2) *We explore the following problems:*

$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$ and use it to evaluate $\int \sin^4 x dx$.

(ii) Show that $\int x^m (\ln x)^n dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$; hence, evaluate $\int x^3 (\ln x)^2 dx$.
(iii) Show that $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$ and hence evaluate $\int x^3 e^x dx$.
(4) Show that $\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, n \neq 1$ and hence evaluate$\int \sec^3 x dx$.

Module 5:The Definite Integral and Its Applications

33 5.1 Objectives:

In this module, we shift our attention to the definite integral and its applications after having studied the indefinite integral in the previous one. Students are stimulated through on-line, classroom and laboratory activities, to discover that this introduction of the integral depends on the limit concept, which is the fundamental basis of calculus.

34 5.2 Learning Outcomes:

At the end of this module, students should be able to:

- Appreciate the limit property of integrals and evaluate definite integrals;
- Identify the definite integral with the area under a curve;
- Identify the definite integral with length of an arc of a curve;
- Identify the definite integral with volumes of a solid of revolution;
- Use the definite integral to investigate problems in physical sciences, engineering and economics;
- Use the definite integral to solve differential equations.

35 5.3 Learning Activities

Students should:

- explore notes and exercises, individually and in groups;
- explore and use related materials on the Intranet, especially the e-granary and MIT open course ware;
- explore and use related materials on the OLI Calculus course on the Internet(http://www.cmu.edu/oLI/courses/);
- solve relevant questions from past MTH103 Examinations.

36 5.4 Introduction:

The Definite Integral as area under a curve

Let us consider a function $y = f(x)$ which is positive and continuous in the range $a \leq x \leq b$. I. Let the

interval $I = [a, b]$ be divided or partition into n parts by arbitrarily choosing points $a = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = b$

on the $x-$ axis such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Suppose further that ξ_r represents a point on the $x-$ axis lying between the $(r-1)$th and rth points in the division of the interval (a, b) with

$$x_{r-1} < \xi_r < x_r$$

and

$$\delta_r = x_r - x_{r-1}.$$

Then an approximation to the area of the rth strip is given by

$$f(\xi_r)\delta_r$$

The whole area bounded by the curve, the $x-$axis and the lines $x = a, x = b$ is approximately represented by the sum S_n of n expressions and is denoted by S_n, where

$$S_n = \sum_{r=1}^{n} f(\xi_r)\delta_r.$$

If the limit of this sum as n tends to infinity exists independently of the way of choosing x_r and ξ_r and as δ_r tends to zero then this limit-sum is called the definite integral of $f(x)$ from $x = a$ to $x = b$ and is written as

$$\lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)\delta_r = \int_a^b f(x)dx.$$

The relationship between this integral and the indefinite integral may be found by considering the variation of the function $f(x)$ over a small interval $x = \xi$ to $x = \xi + \delta\xi$.

If $f_{\min}$ and $f_{\max}$ are respectively the least and greatest value of $f(x)$ in this interval, the elemental area δA (shown shaded) must satisfy the inequality

$$f_{\min}\delta\xi < \delta A < f_{\max}\delta\xi.$$

As $\delta\xi$ tends to zero, $f_{\min}$ and $f_{\max}$ both approach $f(\xi)$ and hence

$$\frac{dA}{d\xi} = f(\xi).$$

The area $A(\xi)$ under the curve from $x = a$ to $x = \xi$ is therefore an indefinite integral of $f(\xi)$. Suppose now that $F(\xi)$ is any indefinite integral of $f(\xi)$; then

$$F(\xi) = A(\xi) + C$$

where C is an arbitrary constant. Since $A(a) = 0$, we have (with $\xi = b$)

$$\int_a^b f(x)dx = F(b) - F(a).$$

This is the formula for computing the definite integral where a and b are called the lower and upper limits of the integral, respectively. The value of the definite integral is the same as the area under the curve $y = f(x)$ between the lines, $x = a$ and $x = b$.

Equivalently we can explore the definition of the integral as area under the curve by defining the lower and upper sums and hence the concept of integrability. This can be done in the special case when the function is $f(x) = x$, by exploring the diagrams below. The student is encouraged to independently do so before the lecture on this module is delivered.

Example 152 *Evaluate the integrals: (i)* $\int_1^2 3xdx$, *(ii)* $\int_0^{\pi/2} \cos xdx$.

Solution 153 *(i) We use our knowledge of indefinite integrals and do the evaluation using the limits.* $\int 3xdx = \frac{3}{2}x^2 + C = F(x) + C$.

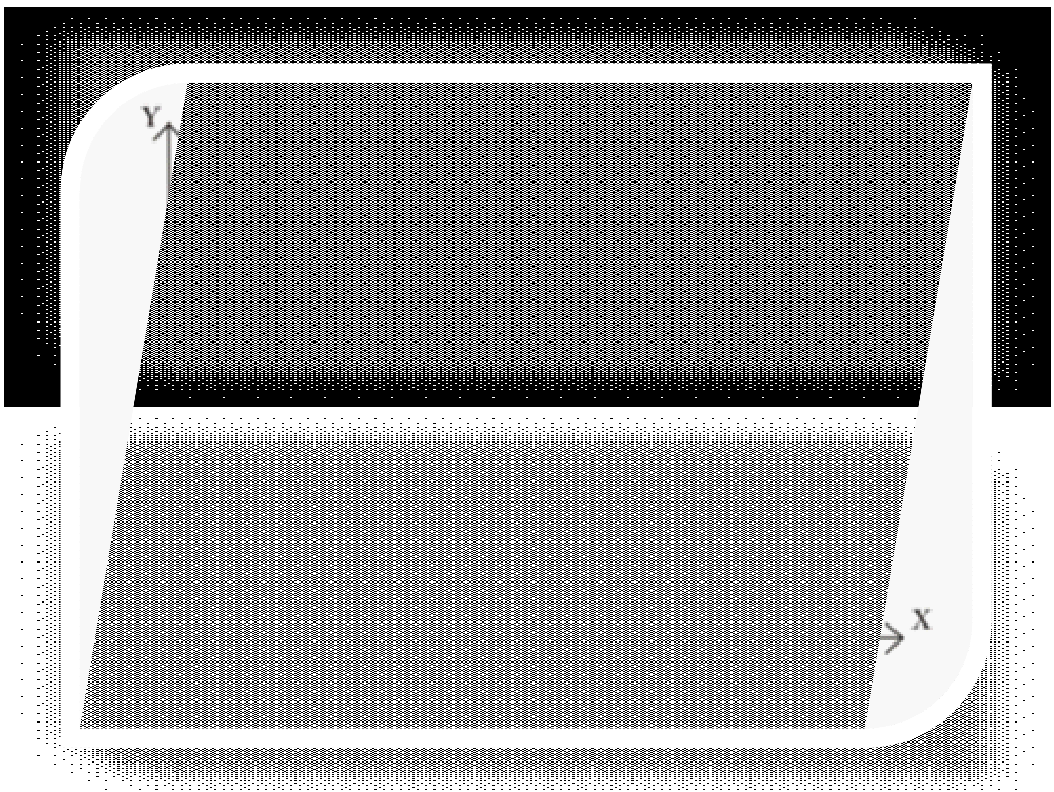
$\int_1^2 3xdx = F(2) + C - [F(1) + C] = 6 - \frac{3}{2} = \frac{9}{2}$.

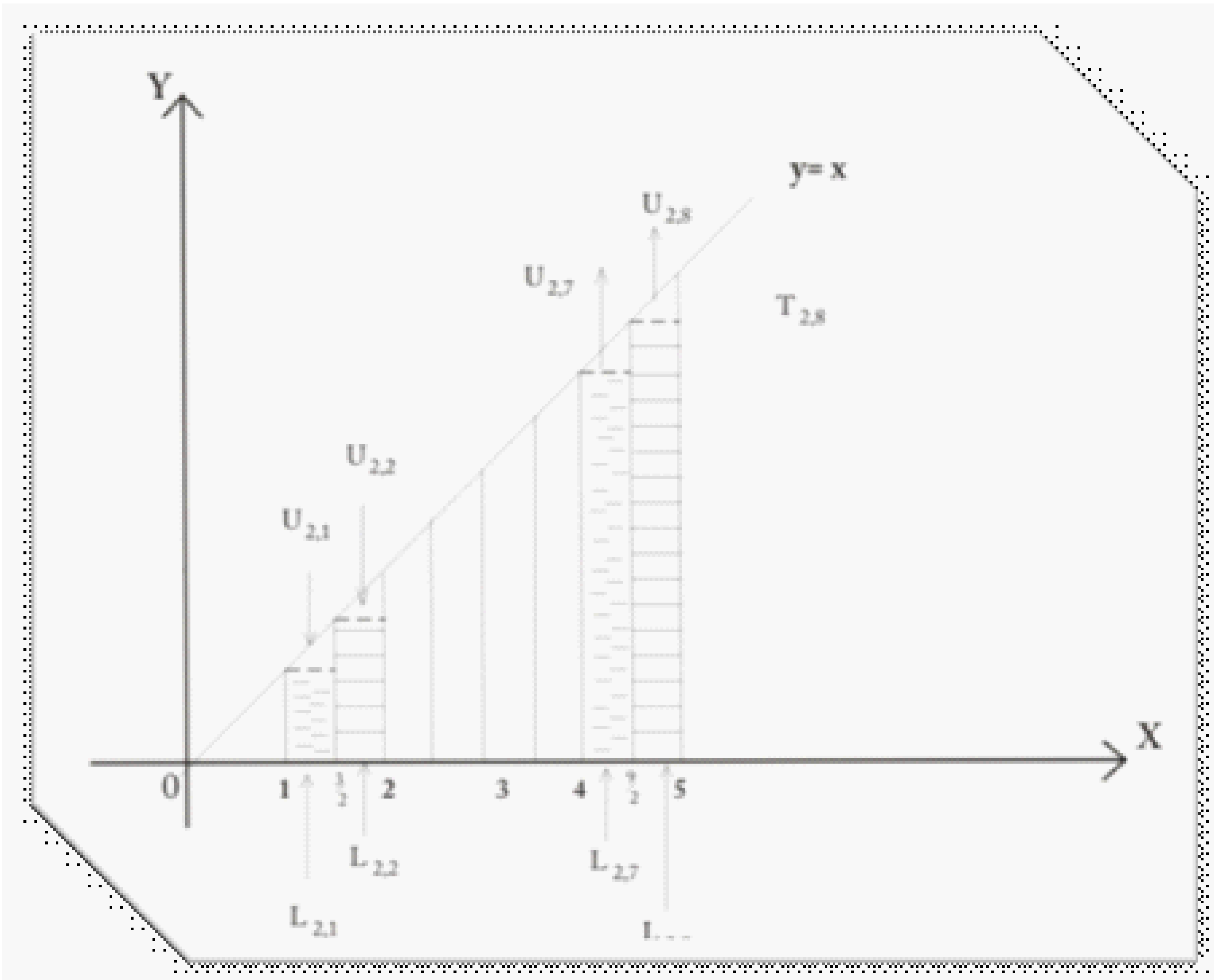
Solution 154 *(ii)* $\int_0^{\pi/2} \cos xdx = F(\frac{\pi}{2}) - F(0) = [\sin x]_0^{\pi/2} = \sin \pi/2 - \sin 0$
$= 1 - 0 = 1$.

36.1 5.4.1 Properties of the definite Integral

If $f(x)$ and $g(x)$ are integrable functions of x in $[a, b]$, then

(a) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ since $-\int_b^a f(x)dx = -[F(a) - F(b)] = F(b) - F(a) = \int_a^b f(x)dx$.


Y
X



Y
y= x
$U_{2,8}$
$U_{2,7}$
$T_{2,8}$
$U_{2,2}$
$U_{2,1}$
X
0
1
2
3
4
5
$L_{2,2}$
$L_{2,7}$
$L_{2,1}$

(b) For $a \leq c \leq b$, $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$. This follows since $\int_a^c f(x)dx + \int_c^b f(x)dx = [F(c) - F(a)] + [F(b) - F(c)]$
$= F(b) - F(a) = \int_a^b f(x)dx$

(c) $\int_a^b \{f(x) + g(x)\}dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ since $\int_a^b \{f(x) + g(x)\}dx = F(b) + G(b) - F(a) - G(a)$ which when rearranged becomes $\{F(b) - F(a)\} + \{G(b) - G(a)\} = \int_a^b f(x)dx + \int_a^b g(x)dx$

(d) If k is a constant, then $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

(e) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. We prove this by letting $x = a - y$ in the first integral and then using property (a) we have $\int_0^a f(x)dx = -\int_a^0 f(a-y)dy = \int_0^a f(a-y)dy$. Whether we write y or x in the last integral is quite irrelevant, since the value of the integral depends only on the form of the integrand and the values of the limits of integration.

(f) If x is a variable upper limit of integration, a is a constant, and $f(x)$ is a continuous function, then $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$. This follows from the definition of the definite integral, for $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = \frac{d}{dx}\{F(x) - F(a)\} = f(x)$.

Example 155 (5.4.1.1) *Evaluate the following (1)* $\int_1^2 3xdx$, *(2)* $\int_0^{\frac{\pi}{2}} \cos xdx$, *(3)* $\frac{d}{dx}\int_{\frac{\pi}{4}}^x \sin^2 tdt$, *(4)* $\int_1^4 (x^2 + x^3)dx$,

(5) Express the area under the curve, $y = 9 - x^2$ *between the lines* $x = 0$ *and* $x = 1$ *as a definite*

integral and then calculate it. The student is encouraged to sketch a graph of the curves for insight.

Solution 156 *(i)*$\int_1^2 3xdx = 3\int_1^2 xdx = \frac{3}{2}\left[x^2\right]_1^2 = \frac{3}{2}\left[2^2 - 1^2\right] = \frac{9}{2}$.

(2) $\int_0^{\frac{\pi}{2}} \cos xdx = [\sin x]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$.

(3) $\frac{d}{dx}\int_{\frac{\pi}{4}}^x \sin^2 tdt = \sin^2 x$. *This follows from property (f), that is* $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

(4)$\int_1^4 (x^2 + x^3)dx = \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_1^4 = \left[\frac{4^3}{3} + \frac{4^4}{4}\right] - \left[\frac{1^3}{3} + \frac{1^4}{4}\right] = \frac{64}{3} + 64 - \left(\frac{1}{3} + \frac{1}{4}\right)$
$= \frac{256}{3} - \frac{7}{12} = \frac{1024-7}{12} = \frac{339}{4} = 84\frac{3}{4}$

(5)The area bounded by the curve $y = f(x)$, *the* $x-$ *axis and the ordinates* $x = a, x = b$ *is given by* $\int_a^b ydx = \int_a^b f(x)dx$.*Thus the area bounded by the curve*

$y = 9 - x^2$ *between* $x = 0$ *and* $x = 1$ *is* $\int_0^1 (9 - x^2)dx$. $\int_0^1 (9 - x^2)dx = \left[9x - \frac{x^3}{3}\right]_0^1 = 9 - \frac{1}{3} = \frac{26}{3}$

$= 8\frac{2}{3}$ *square units.*

Solution 157 $y = 9 - x^2$

Example 158 (5.4.1.2) *(i) Calculate the area bounded by the curve* $y = x^2$ *and the* $x-$*axis and between the lines* $x = 1$*and* $x = 3$.

(ii) Calculate the area between the curve $y = 3x(x-4)$ *and the* $x-$*axis*

(iii) Find the area bounded by the curve $y = \cos x$*, the* $x-$*axis and the ordinates* $x = 0$ *and* $x = \pi$.

Solution 159 *(i) The student is encouraged to sketch the curves and display the areas under investigation.*

The required area is the definite integral $\int_1^3 x^2 dx$.

$\therefore\ A = \int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3}$*square units.*

$$\begin{aligned} Area &= \int_0^4 y dx \\ &= \int_0^4 3x\,(x-4)\,dx \\ &= \int_0^4 (3x^2 - 12x) dx \\ &= \left[x^3 - 6x^2\right]_0^4 \\ &= -34 \textit{ square units.} \end{aligned}$$

The negative sign should be discarded (i.e. Area = 34 square units). The negative sign only explains the fact that the curve lies below the x-axis for the range of x *under consideration.*

Area $AOB = \int_0^{\frac{\pi}{2}} \cos x dx$

$= [\sin x]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1$*square unit.*

Area $BCD = \int_{\frac{\pi}{2}}^{\pi} \cos x dx = [\sin x]_{\frac{\pi}{2}}^{\pi}$

$= \sin\pi - \sin\frac{\pi}{2} = -1$ *square unit. We discard the negative sign as it only explains that the area is below the x-axis. Area* $BCD = 1$. *The area bounded by the curve* $y = \cos x$*, the* $x-$*axis and the ordinates* $x = 0$ *and* $x = \pi$ *is equal to the Area* $AOB+$ *Area* $BCD = 1$*square unit*$+1$*square unit* $= 2$ *square units*

Note that Area required $= \int_0^{\frac{\pi}{2}} \cos x dx - \int_{\frac{\pi}{2}}^{\pi} \cos x dx$.

Note that $\int_0^{\pi} \cos x dx = [\sin x]_0^{\pi} = \sin\pi - \sin 0 = 0$.

The area required is not given by the definite integral, $\int_0^{\pi} \cos x dx$.

37 5.5 Area Between Two Curves

Given two curves $y = f(x)$ and $y = g(x)$ such that $f(x) > g(x)$, where $f(x)$ and $g(x)$ are non negative, then the area between them bounded by the ordinates $x = a$ and $x = b$, is

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

Example 160 (5.5.1) *Calculate the area between the curve $y = x^2$ and the line $y = x$.*

Solution 161 *Sketch the diagram for insight :*

The curves $y = x^2$ and $y = x$ intersect at the origin $(0, 0)$ and at the point $P(1, 1)$.

Therefore the area required is given by $\int_0^1 (x - x^2)dx = \int_0^1 xdx - \int_0^1 x^2dx$
$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ *square units.*

Example 162 (5.5.2) *: We explore the following exercises.*

(1) Calculate the area enclosed by the curve $y = x^2(1 - x)$,the axis of x and the ordinate $x = 0$ and $x = 1$

(2) Find the area enclosed by the axis of x and that part of the curve $y = 5x - 6 - x^2$ for which y is positive.

(3) Find the area enclosed between the curve $y = \sqrt{x^3}$ and the straight line $y = 2x$.

(4) Find the area enclosed by $y^2 = 4x$ and $x^2 = 4y$

(5) Show that the area contained by the curve $y = a + bx + cx^2 + dx^3$ the $x-$ axis and the ordinates at $x = \pm h$ is equal to $h(y_1 + y_2)$ where y_1, y_2 are the values when $x = \pm\frac{h}{3}$

38 5.6 Arc Length

Consider the function $y = f(x)$ whose curve passes through the point P and Q with the ordinates $x = a$ and $x = b$ respectively.

The length of the arc PQ is given by

$$L(PQ) = \int_a^b \left[1 + (f'(x))^2\right]^{\frac{1}{2}} dx.$$

Example 163 (5.6.1) *Find the length of the arc of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ between the ordinates $x = 3$ and $x = 8$.*

Solution 164 *Now $f(x) = \frac{2}{3}x^{\frac{3}{2}}$ $\therefore f'(x) = \frac{2}{3}\frac{3}{2}x^{\frac{1}{2}} = x^{\frac{1}{2}}$. Take $a = 3$ and $b = 8$. $f'(x) = x^{\frac{1}{2}}$. Thus, the*

length of the arc is $\int_3^8 \left[1 + \left(x^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} dx = \int_3^8 [1 + x]^{\frac{1}{2}} dx$

$= \left[\frac{2(1+x)^{\frac{3}{2}}}{3}\right]_3^8 = \frac{2(1+8)^{\frac{3}{2}}}{3} - \frac{2(1+3)^{\frac{3}{2}}}{3} = 16$ *units.*

39 5.7 Volumes of Solids of Revolution

(A) Volume formed by rotating a curve around the $x-$ axis.

If a region below a curve $y = f(x)$ of a continuous function f between $x = a$ and $x = b$ is rotated about the $x-$axis, it generates a solid figure called a solid of revolution. Because of the symmetrical way in which it is generated (by a rotation of a plane figure) it is easy to compute its volume. Such a region will be shown in class.

Let $[a, b]$ be divided (partition) into n subintervals of equal lengths. Consider the contribution to the solid of revolution given by revolving the strip about the x-axis. The strip sweeps out a slab , where the minimum radius is $m_i = f(t_{i-1})$ and the maximum radius is $M_i = f(t_i)$.The thickness of the slab is $\frac{b-a}{n} = \delta x$. The volume of a circular slab of radius r and thickness h is of course $\pi r^2 h$.Since the radius of the slab varies from m_i to M_i, its volume V_{slab} satisfies

$\pi m_i^2(\frac{b-a}{n}) \leq V_{slab} \leq \pi M_i^2(\frac{b-a}{n})$. Thus for the volume V of the whole solid of revolution, we have $(\frac{b-a}{n})\sum\limits_{i=1}^{n} \pi m_i^2 \leq V \leq (\frac{b-a}{n})\sum\limits_{i=1}^{n} \pi M_i^2$ for all n.The extremes of the inequality approach $\int\limits_a^b \pi f(x)^2 dx$ as $n \to \infty$, and the desired volume V is given by

$$V = \lim_{\delta x \to 0} \sum \pi y^2 \delta x = \int_a^b \pi y^2 dx = \pi \int_a^b y^2 dx$$

This is the formula for the volume of solid of revolution around the $x-$ axis.

(B) Volume of solid by rotating of curve around the $y-$ axis.

Using similar argument outlined above, the volume of revolution about the y-axis can also be derived and it is given by $V = \int_a^b \pi x^2 dy$

Example 165 (5.7.1) *(i) The area enclosed by the curve $x = 3(y^2 - 1)$ and the line $x = 0, x = 24$ is rotated through 4 right angles around the $x-$axis. Find the volume of the solid generated.*

(ii) The portion of the curve $y = x^2 + 1$ between the ordinates $x = 1$ and $x = 3$ is rotated

through $360°$ around the $y-$axis, calculate the volume of revolution.

Solution 166 *(i) Here $y^2 = 1 + \frac{1}{3}x$ and the required volume is*

$V = \pi \int_0^{24} \left(1 + \frac{1}{3}x\right) dx = \pi \left(24 + \frac{576}{6}\right) = 120\pi$ *cubic units.*

(ii) Now $y = x^2 + 1, a = y\,(x = 1) = 1^2 + 1 = 2$ $b = y(x = 3) = 3^2 + 1 = 10$

$\therefore V = \int_2^{10} \pi x^2 dx,$ *but* $y - 1 = x^2 \therefore x = \sqrt{y-1}.$

$V = \int_2^{10} \pi \left(\sqrt{y-1}\right)^2 dy = \pi \int_2^{10} (y-1)\, dy = \pi \left(\frac{y^2}{2} - y\right)_2^{10}$

$= \pi \left(\frac{10^2}{2} - 10 - (\frac{(2^2)}{2} - 2)\right) = 40\pi$ *cubic units.*

40 5.8 Mean or Average Value of a Function

Suppose that y is an integrable function of x over $[a, b]$ .Suppose that the range AB from $x = a$ and $x = b$ is divided (partition) into n equal subrange each of width δx. Let $y_1, y_{2,} \cdots, y_n$ be the values of y of the middle point of each subrange. The arithmetic mean of these n values of y is $\frac{1}{n}\left(y_1+y_2+\cdots+y_n\right)$ and since $\delta x=\frac{b-a}{n}$, this can be written $\frac{y_1+y_2+\cdots+y_n}{b-a}$

If this expression has a limiting value as $\delta x \to 0$, this limiting value is

$$\frac{1}{b-a}\int_a^b ydx$$

and this is called the mean value of y over the range of $[a,b]$.

Example 167 (5.8.1) *Find the mean value of* $\sin x$ *over the range* $x=0$ *and* $x=\frac{1}{2}\pi$.

Solution 168 *Here* $y=\sin x, a=0, b=\frac{1}{2}\pi$ *and from the formula, the mean value* $\bar{y}$ *of* y *is*

$\bar{y}=\frac{1}{\frac{1}{2}\pi-0}\int_0^{\frac{\pi}{2}}\sin xdx=\frac{2}{\pi}\left[-\cos x\right]_0^{\frac{\pi}{2}}=\frac{2}{\pi}=0.637.$

Example 169 (5.8.2) *: We explore the problems below drawing appropriate diagrams.*

(1) Evaluate the integrals (a) $\int_0^1 x\left(1-x\right)^5 dx$ (b) $\int_0^2\sqrt{1+2x^2}dx$

(2) Show that $\int_0^{\frac{\Pi}{2}} xsin^2xdx=\frac{1}{16}\left(\Pi^2+4\right)$

(3) By means of substitution $x^2=\frac{1}{u}$ show that $\int_1^2\frac{dx}{x^2\sqrt{5x^2-1}}=\frac{1}{2}\sqrt{19}-2$

(4) Calculate the area between the curve $y=4+2x-x^2$ and line $y=4$

(5) Sketch the curve $y=x\left(x-1\right)\left(x-2\right)$ and find the area enclosed by the curve and axis of x between $x=0$ and $x=2$

(6) Sketch the curve $y=x\left(3-x\right)$ and find the area contained between the curve $X-$ axis and ordinates $x=0, x=5$

(7) Find the mean value of the ordinate of the curve $y=4-x^2$ over the range $-2\leq x\leq 2$

(8) Find the mean value of x varied from 0 to π of (a) $sinx^2$ (b) $xsinx$

(9) The pressure P kilograms per square centimeter and the volume V cubic centimeter of quantity of gas are related by the laws $PV^{1.2}=1000$. Fond the mean pressure as the gas increases from $3cm^3$ to $8cm^3$.

(10) The portion of the curve $xy=8$ from $x=2$ to $x=4$ is rotated about the x-axis. find the volume generated.

(11) Find the volume generated if the area enclosed by $y=3x^2-x^3$ is rotated around the x axis

(13) The area bounded by the curve $y^2=20x$ and the line x=0 and the line x=10 is rotated about the axis of Y show that the volume of the solid obtained is 50Π and find the volume of the solid obtained by rotating the same area around the axis of x

41 5.9 Moment of Inertia

If $m_1, m_2, \cdots, m_n$ are masses of the system of particles situated at points $p_1, p_2, \cdots, p_n$ whose perpendicular distance from a straight line are $r_1, r_{2,}\cdots, r_n$, the sum of the product of each mass and the square of its distance from the line is called moment of inertia of the system with respect to the given line. This moment sometimes also called the second moment is conventionally denoted by I so that

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots . + m_n r_{n.}^2.$$

If we imagine the total mass $M = m_1 + m_2 + \cdots + m_n$ to be concentrated at a point at distance k from the given line such that this single mass has the same moment of inertia about the given line as the system of particles, then

$$Mk^2 = I = m_1 r_1^2 + m_2 r_2^2 + \cdots . + m_n r_n^2.$$

The distance k calculated from this equation is known as the radius of gyration of the system around the given line. The moment of inertia of a rigid body about an axis is required in dynamics to express the kinetic energy of the body and it is useful to set up two general theorems before discussing the moment of inertia of a specific rigid body. These theorems are as follows:

[5.9.1] : The parallel axis theorem

If the moment of inertia of a system of particles of the total mass M about an axis through the centre of mass is Mk^2, the moment of inertia of the system about a parallel axis distance a from the first is $M\left(k^2 + a^2\right)$.

Suppose the particles of masses $\mathrm{m}_1, m_2, ..., m_n$ and are at the distance $\mathrm{r}_1, r_2, ..., r_n$ from the axis through the centre of mass and distances $R_1, R_2, ..., R_n$ from the second (parallel) axis. Let the two axes meet a plane perpendicular to them in G and O respectively. If this plane passes the point P_1 occupied by first particle . Since OG =a, we shall have $R_1^2 = \mathrm{r}_1^2 + a^2 - 2ar_1 cos\theta$.

And similar relations between R and r hold for all the n particles of the system. The moment of inertia I of the system about the axis through O is given by $I = \sum mR^2 = \sum m\left(r^r + a^2 - 2arcos\theta\right)$ where $\sum$ denote summation over all the particles of the system and this can be written I= $\sum m\mathrm{r}^2 + a^2 \sum m - 2a \sum mrcos\theta$. Now $\sum m = M$, $\sum mr^2 = Mk^2$(the momentof inertia about the axis through G) and $\sum mrcos\theta = 0$ for this latter sum when divided by M gives the distance in the direction

G of the centre of mass from point G. Hence $I = Mk^2 + Ma^2$ and the theorem is proved.

(ii) The moment of inertia of the system of particles lying in the plane *about* two perpendicular axes in the plane meeting a point O are A and B respectively. The moment of inertia of the system about an axis through O perpendicular to the plane is $A + B$

From the diagram, we can see that, OX and OY are the given perpendicular axes in the plane and OZ and OY are the given perpendicular axes in the plane and OZ is the axis perpendicular to the plane. $\mathrm{P}_1, P_2, ..., P_n$ are the points (x_1, y_1), (x_2, y_2),....(x_n, y_n) occupied by particles of masses $m_1, m_2, ..., m_{n.}$ Since the moment of inertia about OY, OX are respectively A and B, A=$\mathrm{m}_1 x_1^2 +$

$m_2x_2^2 + ... + m_nx_n^2$ and by addition, A+B=m$_1\left(x_1^2+y_1^2\right)+m_2\left(x_2^2+y_2^2\right)+...+m_n\left(x_n^2+y_n^2\right)$. If OP$_1=r_1$, OP$_2=r_2,...,OP_n=r_n$, it is clear from the diagram that r$_1^2=x_1^2+y_1^2, r_2^2=x_2^2+y_2^2$, ...,r$_n^2=x_n^2+y_n^2$, hence A+B=m$_1r_1^2+m_2r_2^2+...+m_nr_n^2$ and this is the moment of inertia of the system around OZ

An extension from system of particles to continuous body can be obtained by replacing the particle by elements of the body and using integral in place of summation. We give below the methods of calculating the moment of inertia about specific axes of a thin rod, a circular disc and a sphere, with uniform density ρ in all cases.

An extension from system of particles to continuous body can be obtained by replacing the particle by elements of the body and using integral in place of summation. We give below the calculation for the moment of inertia about specific axes of a thin rod, a circular disc and a sphere, with uniform density ρ in all cases.

41.1 5.9.1 Thin Rod

Suppose (Fig.4.14) AB is a thin rod of length $2l$ lying along the axis OX with its midpoint at the origin O, the mass of an element PQ of length δx at the distance x from O is $\rho\delta x$ and its moment of inertia about the axis of y is $\rho\delta x \times x^2$.

Fig 4.14

Hence the moment of inertia I_0 of the whole rod about a line through its centre and perpendicular to its length is give by

$$I_0 = \int_{-l}^{l} \rho x^2 \delta x = p\left[\frac{1}{3}x^3\right]_{-l}^{l} = \frac{2}{3}\rho l^3.$$

Since the total mass M of the rod is $2\rho l$, then the moment of inertia can be written in the form $I_0 = \frac{1}{3}Ml^3$, showing that the radius of gyration k is given by $k^2 = \frac{1}{3}l^3$. If the moment of inertia I_A about an axis perpendicular to the length of the rod and passing through the end of A is required, the parallel axis theorem gives $I_A = I_0 + Ml^3 = \frac{4}{3}Ml^3$

41.2 5.9.2 Circular Disc

The moment of inertia of a circular disc about an axis through its centre O perpendicular to the plane of the disc can be found by considering the ring bounded by circles of radii r and $r+\delta r$ (Fig.4.15). The mass of the element is approximately $2\pi\rho r\delta r$ and its moment of inertia about the axis is $2\pi\rho r\delta r \times r^2$

Fig. 4.15

Hence if a is the radius of the disc, the total moment of inertia I_0 is

$$I_0 = \int_0^a 2\pi\rho r^3 dr = 2\pi\rho\left[\frac{1}{4}r^4\right]_0^a = \frac{1}{2}\pi\rho a^4.$$

Since the total mass M of the disc about a diameter is A, then by symmetry, the moment of inertia about a perpendicular diameter is also A. Therefore, by second of the general theorems, the moment of inertia about an axis perpendicular to the plane of the disc is $2A$ and we have $2A = \frac{1}{2}ma^2$ given $A = \frac{1}{4}ma^2$.

41.3 5.9.3 Sphere

Let the sphere be of radius r with its centre at the origin O. Consider an element in the form of a circular disc of radius y, thickness δx at distance x from the axis OY

The mass of this element is approximately $\pi\rho y^2\delta x$ and by the theorem (ii) above, the moment of inertia about OX of the whole sphere is therefore,

$$\begin{aligned}\int_{-r}^{r}\frac{1}{2}\pi\rho y^4dx &= \frac{1}{2}\pi\rho\int_{-r}^{r}\left(r^2-x^2\right)^2dx, \text{ since } x^2+y^2=r^2\\ &= \frac{1}{2}\pi\rho\left[r^4-\frac{2}{3}r^2x^3+\frac{1}{5}x^5\right]_{-r}^{r}\\ &= \frac{8}{15}\pi r^5.\end{aligned}$$

Since the mass M of the sphere is $\frac{4}{3}\pi\rho r^3$, then the moment of inertia of the sphere can be written as $\frac{2}{5}Mr^{2\cdot}$

Example 170 (5.9.3.1) *A uniform lamina is bounded by the curve $y = 8x^3$, the axis of x and an ordinate $x = 1$. Find the radius of gyration about a line perpendicular to its plane through the origin of coordinates.*

Solution 171 *An illustrative figure can show the lamina and an element strip of height y and width δx. If ρ is the surface density of the lamina, the mass of the strip is approximately $\rho y\delta x$ and its moment of inertia about the y axis is $\rho y\delta x \times x^2$. Hence the moment of inertia I_y of the whole lamina about the y axis is given by, since $y = 8x^3$ $I_y = \int_0^1 \rho x^2 y dx = 8\rho\int_0^1 x^5dx = 8\rho\left[\frac{1}{6}x^6\right]_0^1 = \frac{4}{3}\rho$.The moment of inertia of the strip about x axis is, using the result of thin rod, $\rho y dx \times \frac{1}{3}y^3$ and the moment of inertia about the x axis is $I_x = \int_0^1\frac{1}{3}\rho y^3 = \frac{512}{3}\rho\int_0^1 x^9dx = \frac{512}{3}\rho\left[\frac{1}{10}x^{10}\right]_0^1 = \frac{256}{15}\rho$. By the general theorem 2, the moment of inertia about an axis through O perpendicular to the plane of the lamina is $I_x + I_y = \left(\frac{256}{15}+\frac{4}{3}\right)\rho = \frac{92}{5}\rho$.*

Example 172 (5.9.3.2) *: We explore the problems below, sketching a diagram anytime you find it useful to do so.:*

(1)The smaller of the two areas bounded by the curve $y^2 = 4x$, the line $y = 2$ and the line $y = 4$ rotates through 4 right angles about the axis of x, prove that the volume so formed is 18π and that its radius of gyration about the axis is $2\sqrt{2}$

(2) Find by integration, the radius of gyration of a uniform semi circular-disc of radius a about its bounding diameter. Deduce its radius of gyration about

a parallel axis through the centroid G of the disc assuming G is a distant $\frac{4a}{3}\pi$ from the bounding diameter.

(3) Show that the moment of inertia about its axis of a uniform solid circular cone of mass M and base radius r is $\frac{3}{10}mr^2$.

(4)A uniform thin hemispherical shell has mass M and radius r. Show that its moment of inertia about the radius perpendicular to its base is$\frac{2}{3}Mr^2$

(5) The area enclosed by the curve $y^2 = 4ax$, the axis of x and the line $x = h$ is rotated through 4 right angles about the axis of x to form a solid. Find the radius of gyration about the x-axis of this solid

42 5.10 Application of Integration to Problems in Physical Sciences & Engineering.

42.1 5.10.1 Distance/Displacement and Velocity/Speed

If a body travels in a line with a constant velocity v, then the distance S, covered after a time t is given by the product $\mid v \mid t$, where $\mid v \mid$ represents the speed of the body. If a body is moving in a line with a velocity $v(t)$, starting from time t_0 to time t, then the distance from time t_0 to time t is given by

$$S = \int_{t_0}^{t} |v(t)| dt.$$

Example 173 (5.10.1.1) *(1) If the velocity at any time t seconds of a body travelling on a line is $v(t) = (2t + t^2)m/s$. Find how far the body travels from $t = 2s$ to $t = 4s$.*

(2) Suppose the velocity at time t of a body moving on a line is $v = \cos\frac{\pi t}{2}m/s$. Thus at time $t = 0$, the body is moving at $1m/s$ in the positive direction, while at time $t = 2$, the body is moving in the negative direction at the speed of $1m/s$.

(3) The velocity of a moving particle along the $x-$axis is given by $v(t) = (1 - 2t)(2 - t)$. Given that when $t = 0, x = 1$, find t when the velocity is a minimum, and also determine

(i) the displacement of the particle at this value of t, (ii) the total distance travelled and the displacement of the object when $t = 3$.

Solution 174 *(1) $S = \int_{t_0}^{t} \mid v(t) \mid dt = \int_2^4 (2t+t^2)dt = (4^2+\frac{4^3}{3})-(2^2+\frac{2^2}{3}) = \frac{92}{3}$. Thus the body travels $\frac{92}{3}m$ from $t = 2s$ to $t = 4s$.*

(2) Distance travelled from $t = 0$ to $t = 2$ is given by

$S = \int_0^2 \mid \cos\frac{\pi}{2}t \mid dt = \int_0^1 \cos\frac{\pi}{2}tdt + \int_1^2(-\cos\frac{\pi}{2}t)dt$

$= \frac{2}{\pi}\sin\frac{\pi}{2}(1) - \frac{2}{\pi}\sin\frac{\pi}{2}(0) - \left[\frac{2}{\pi}\sin\frac{\pi}{2}(2) - \frac{2}{\pi}\sin\frac{\pi}{2}(1)\right] = \frac{2}{\pi} + \frac{2}{\pi} = \frac{4}{\pi}$. *Thus, the total distance covered from $t = 0$ to $t = 2$ is $\frac{4}{\pi}m$.*

(3) Note that the velocity v, of an object at any point x is defined as the rate of change of the displacement x with respect to (w.r.t.) time t. Hence $v = \frac{dx}{dt}$, while the magnitude of $\frac{dx}{dt}$, i.e. $|v|$ is the

speed. v is minimum if $\frac{dv}{dt} = 0$. i.e., $\frac{dv}{dt} = -2(2-t)-(1-2t) = 4t-5 = 0$. $\therefore\ t = \frac{5}{4}$. But

$\frac{d^2v}{dt^2} = 4 > 0$. Hence the velocity is a minimum when $t = \frac{5}{4}$.

(i) Note that $v(t) = \frac{dx}{dt} = (1-2t)(2-t)$, $\therefore\ \int dx = \int(1-2t)(2-t)dt = \int(2t^2-5t+2)dt$;

$\therefore\ x = \frac{2t^3}{3} - \frac{5t^2}{2} + 2t + c$. $x = 1$, when $t = 0$, $\Rightarrow c = 1$. $\therefore\ x = \frac{2t^3}{3} - \frac{5t^2}{2} + 2t + 1$. This

is the expression for the displacement of the particle at $t = \frac{5}{4}$. The displacement at

$t = \frac{5}{4}$ is $x = \frac{2}{3}(\frac{5}{4})^3 - \frac{5}{2}(\frac{5}{4})^2 + 2(\frac{5}{4}) + 1 = \frac{43}{48}$.

(ii) Total distance travelled when $t = 3$ is

$S = \int_0^{\frac{1}{2}}(1-2t)(2-t)dt + \int_{\frac{1}{2}}^{2}(2t-1)(2-t)dt + \int_2^3(1-2t)(2-t)dt$

$= \int_0^{\frac{1}{2}}(2t^2-5t+2)dt + \int_{\frac{1}{2}}^{2}(-2t^2+5t-2)dt + \int_2^3(2t^2-5t+2)dt$

$= \left[\left(\frac{2}{3}(\frac{1}{2})^3 - \frac{5}{2}(\frac{1}{2})^2 + 2(\frac{1}{2})\right) - 0\right]$

$+ \left[\left(\frac{2}{3}(2)^3 - \frac{5}{2}(2)^2 + 2(2)\right) - \left(\frac{2}{3}(\frac{1}{2})^3 - \frac{5}{2}(\frac{1}{2})^2 + 2(\frac{1}{2})\right)\right]$

$+ \left[\left(\frac{2}{3}(3)^3 - \frac{5}{2}(3)^2 + 2(3)\right) - \left(\frac{2}{3}(2)^3 - \frac{5}{2}(2)^2 + 2(2)\right)\right]$

$= \frac{11}{24} + \frac{27}{24} + \frac{13}{6} = \frac{15}{4}$

The displacement at $t = 3$ is $x = \frac{2}{3}(3)^3 - \frac{5}{2}(3)^2 + 2(3) + 1 = \frac{5}{2}$.

42.2 5.10.2 Work

If the force acting on a body in the direction of its motion along a line is the function $F(s)$ of the position s of the body, then the work done by the force in moving the body from position $s = a$ to position $s = b$ is

$$W = \int_a^b F(s)ds.$$

Example 175 (5.10.2.1) *The force F required to stretch (or compress) a coil spring is proportional to the distance x it is stretched (or compressed) from its natural length. That is $F = kx$ for some constant k, the spring constant. Suppose that a spring is such that the force required to stretch it one metre from its natural length is $4N$. For a spring with constant $k = 4$, find the work done in stretching the spring a distance of $4m$ from its natural length.*

Solution 176 *Let x be the original or natural length of the spring, then*

$W = \int_0^4 4x dx = \left[2x^2\right]_0^4 = 2\left[(4)^2 - (0)^2\right] = 32$. *Thus $32J$ of work is done.*

43 5.11 Differential Equations

One area of application of integration is in the solution of differential equations. An equation involving differential coefficients such as $\frac{dy}{dx}, \frac{\partial f}{\partial x}, \frac{d^2y}{dx^2}, \frac{\partial f(x,y)}{\partial y}$ and so on, is called a **differential equation.** When the equation involves ordinary differential coefficients, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, *e.t.c.* it is referred to as an **ordinary**

differential equation (ODE), but if it involves partial derivatives such as $\frac{\partial f}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, *e.t.c.*the equation is called a **partial differential equation (PDE)**. The **order** of a differential equation is determined by the highest derivative appearing in the equation. The **degree** of a differential equation is the highest index (power) of the highest derivative in the equation. For example, $2\frac{dy}{dx} - y = x$, is a first order differential equation of degree 1, $\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{1}{y} = 0$, is of second order, and degree 1, $(\frac{d^2y}{dx^2})^3 - \frac{dy}{dx} + x = 0$, is of second order, and degree 3.

We shall solve first order equations using the method of separation of variables. A first order differential equation with variables separable is an equation which can be put into the form $\frac{dy}{dx} = f(x)g(y)$.

For example, consider the equation $\frac{dy}{dx} = xy$. The variables can be separated thus: $\frac{1}{y}\frac{dy}{dx} = x, \Rightarrow \frac{dy}{y} = xdx$. $\therefore$ $\int \frac{dy}{y} = \int xdx, \Rightarrow \ln y = \frac{x^2}{2} + c$ $\therefore$ $y = Ae^{\frac{x^2}{2}}$. This is called the **general solution** of the equation. If in addition, $y = 1$,when $x = 0$, that is $y(0) = 1$, called the **initial value,** we obtain $A = 1$. Hence $y = e^{\frac{x^2}{2}}$. This is called **particular solution** of the equation. The equation together with the initial values is said to be an **initial value problem (IVP).**

Example 177 (5.11.1) *Solve the following differential equations: (i)* $x\frac{dy}{dx} = 3y - 1$, *(ii)* $2(x+1)y\frac{dy}{dx} = x$,
(iii) $\frac{dx}{dt} = e^x \sin t$, *(iv)* $\frac{dy}{dx} + ty = y, y(1) = 3$, *(v)* $\frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8}$.

Solution 178 *(i)* $x\frac{dy}{dx} = 3y - 1$ $,\Rightarrow xdy = (3y-1)dx$, $\Rightarrow \frac{dy}{3y-1} = \frac{dx}{x}$. $\therefore$ $\int \frac{dy}{3y-1} = \int \frac{dx}{x}$, $\Rightarrow \frac{1}{3}\ln \mid 3y - 1 \mid= \ln \mid x \mid +k$. $\therefore$ $3y - 1 = e^{\ln |x|^3 + 3k} = |x|^3 e^{3k}$.
$\therefore$ $y = \frac{1}{3}x^3 e^{3k} + \frac{1}{3} = Ax^3 + \frac{1}{3}$, *where* $A = \frac{e^{3k}}{3}$.
(ii) $2(x+1)y\frac{dy}{dx} = x$, $\Rightarrow 2ydy = \frac{x}{1+x}dx = (1 - \frac{1}{1+x})dx$.
$\therefore$ $\int 2ydy = \int (1 - \frac{1}{1+x})dx = \int dx - \int \frac{dx}{1+x} = x - \ln \mid 1 + x \mid +k$
$\Rightarrow y^2 = x - \ln \mid 1 + x \mid +k = \ln \frac{e^x}{|1+x|} + k$, $\therefore$ $y^2 = \ln \frac{e^x}{|1+x|} + k$.
(iii) $\frac{dx}{dt} = e^x \sin t, \Rightarrow \frac{dx}{e^x} = \sin tdt$ $\therefore$ $\int e^{-x}dx = \int \sin tdt, \Rightarrow -e^{-x} = -\cos t + A$
$\therefore$ $e^{-x} = \cos t + c$, *where* $c = A$, *or* $x = -\ln \mid \cos t + c \mid$.
(iv) $\frac{dy}{dt} + ty = y$, $y(1) = 3, \Rightarrow \int \frac{dy}{y} = \int (1-t)dt$
$\therefore$ $\ln \mid y \mid= t - \frac{1}{2}t^2 + C, \Rightarrow y = Ae^{t(1-\frac{t}{2})}$. *But* $y(1) = 3, \Rightarrow 3 = Ae^{\frac{1}{2}}, \Rightarrow A = 3e^{-\frac{1}{2}}$.
$\therefore$ $y = 3e^{-\frac{1}{2}}e^{t(1-\frac{t}{2})} = 3e^{t(1-\frac{t}{2})-\frac{1}{2}}$.
(v) $\frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8}, \Rightarrow$ $\frac{dy}{dx} = \frac{(x-1)(y+3)}{(y-2)(x+4)}$.
$\therefore$ $(y-2)(x+4)dy =$ $(x-1)(y+3)dx, \Rightarrow \int \frac{y-2}{y+3}dy = \int \frac{x-1}{x+4}dx$.
$\Rightarrow \int \frac{y}{y+3}dy - \int \frac{2}{y+3}dy = \int \frac{x}{x+4}dx - \int \frac{dx}{x+4}$
$\therefore$ $\int (1 - \frac{3}{y+3})dy - 2\ln \mid y + 3 \mid= \int (1 - \frac{4}{x+4})dx - \ln \mid x + 4 \mid +C$
$\Rightarrow \int dy - \int \frac{3}{y+3}dy - 2\ln \mid y + 3 \mid= \int dx - \int \frac{4}{x+4}dx - \ln \mid x + 4 \mid +C$
$\therefore$ $y - 3\ln \mid y + 3 \mid -2\ln \mid y + 3 \mid= x - 4\ln \mid x + 4 \mid - \ln \mid x + 4 \mid +C$

$\Rightarrow y - 5\ln \mid y+3 \mid = x - 5\ln \mid x+4 \mid + C$
$\therefore\ 5(\ln \mid x+4 \mid - \ln \mid y+3 \mid) = x - y + C$
$\therefore\ 5\ln \mid \frac{x+4}{y+3} \mid = x - y + C, \Rightarrow \mid \frac{x+4}{y+3} \mid^5 = Ae^{x-y}.$

Example 179 (5.11.2) *: We explore the solution of the following equations.*

$(i)\ \frac{dy}{dx} = \frac{y}{2x}$ $(ii)\ \frac{dy}{dx} + y = x^2y$ $(iii)\ (1+x^2)\frac{dy}{dx} = 9 + y^2,\ y(0) = 0.$
$(iv)\ (x^2-1)\frac{dy}{dx} + 2y = 0,\ y(2) = 3.$ $(v)\ 2\sin\theta\frac{d\theta}{dr} = \cos\theta - \sin\theta.$
$(vi)\ \frac{dy}{dx} = \frac{xy-2y-x-2}{xy-3y+x-3}$ $(vii)\ (1+x^2)\frac{dy}{dx} = 1 + y^2,\ y(2) = 3$
$(viii)\ x(y^2-1)^{\frac{1}{2}}dx - y(x^2-1)^{\frac{1}{2}}dy = 0.$

44 5.12 Application of Integration to Laws of Growth and Decay

In nature, the rate of change of any given quantity of most physical phenomena is always proportional to the amount of the quantity at any instance. For instance, if the rate of change of a population of bacteria in a culture is proportional to the bacteria present in the culture at any time t, we have that $\frac{dp}{dt} \propto p, \Rightarrow \frac{dp}{dt} = kp$, where k is the proportionality constant. Solving the equation by separation of variables, we obtain $p = Ae^{kt}$. Given that the population at time $t = 0$ is p_0, we obtain $p_0 = Ae^{k\times 0} = A.\ \therefore\ p = p_0e^{kt}$.

Example 180 (5.12.1) *(1) The population of a colony doubles in 50 days. In how many days will the population triple?*

(2) The population of a country doubles in 50 years. If the present population is 20,000,000:

(a) when will its population reach 30,000,000?

(b) what will its population be in 10 years?

Solution 181 *(1) Let the population at any time t be p.*

$\therefore\ \frac{dp}{dt} = kp, \Rightarrow \int \frac{dp}{p} = \int ktdt\ \Rightarrow \ln p = kt + C, \Rightarrow p = Ae^{kt}$.*If the population at $t = 0$ is p_0, then $p_0 = Ae^0 = A, \Rightarrow p = p_0e^{kt}$. $p = 2p_0$ when $t = 50$,*

$\therefore\ 2p_0 = p_0e^{50k}, \Rightarrow \ln 2 = 50k, \Rightarrow k = \frac{\ln 2}{50}.$ $\therefore$ *when $p = 3p_0$, we have*
$3p_0 = p_0e^{\frac{\ln 2}{50}}, \Rightarrow t = \frac{50\ln 3}{\ln 2} \simeq 79$ *days.*

(2) (a) $2p_0 = p_0e^{50k}, \Rightarrow k = \frac{\ln 2}{50}.\ \therefore\ \ p = p_0e^{\frac{\ln 2}{50}},\ \Rightarrow p = 20,000,000e^{0.01386t}.$
$\therefore$ *when* $p = 30,000,000,$
$30,000,000 = 20,000,000e^{0.01386t},\ \Rightarrow 1.5 = e^{0.01386t}.$
$\therefore\ t = \frac{\ln 1.5}{0.01386} \simeq 29$ *. Thus, the population will be* $30,000,000$ *in* 29 *years.*
(b) For $t = 10,\ p = 20,000,000e^{0.01386\times 10} = 22,970,000.$

Note*: Disintegration of a substance is equivalent to decay, hence we obtain the*

equation $\frac{dQ}{dt} = -kQ$, *where Q represents the quantity of the substance present at time t,*

and the negative sign indicates decay. $\therefore\ \int \frac{dQ}{Q} = \int kdt,\ \Rightarrow \ln Q = -kt + C.$
$\therefore\ Q = Ae^{-kt}.$

Example 182 (5.12.2) *(3) Suppose the original quantity is* Q_0, *when* $t = 0$, *then* $Q_0 = Ae^0 = A$.
$\therefore\ Q = Q_0 e^{-kt}$.

Solution 183 10% *decays in* 100 *days implies* $\frac{1}{10}Q_0 = Q_0 e^{-100k}, \Rightarrow \ln 0.1 = -100k$.
$\therefore\ k = -\frac{\ln 0.1}{100} = \frac{\ln 10}{100}$, *hence* $Q = Q_0 e^{-\frac{\ln 10}{100}t}$. *Thus, for half-life of the substance,*
$\frac{1}{2}Q_0 = Q_0 e^{-\frac{\ln 10}{100}t}$. *Note that the **half-life** of a substance is the time it takes to disintegrate*
to half of its original size. $\therefore\ \ln 0.5 = -\frac{\ln 10}{100}t, \Rightarrow t = -\frac{100 \ln 0.5}{\ln 10} = 30.103$. *Hence, the half-life of the substance is approximately* 30.1 *days.*

Example 184 (5.12.3) *:We explore the solution of the problems below:*
(i) The bacteria count in a culture is 100,000. *In* $2\frac{1}{2}$*hours, the number has increased by* 10%.
(a) In how many hours will the count reach 200,000*?* *(b) What will the bacteria count be in* 10 *hours?*
(ii) Assume that the half-life of the radium in a piece of lead is 1500 *years. How much radium will remain in the lead after* 2500 *years?*
(iii) If 1.7% *of a substance decomposes in* 50 *years, what percentage of the substance will remain after* 100 *years? How many years will be required*
for 10% *to decompose?*
(iv) The bacteria count in a culture doubles in 3 *hours. At the end of* 15 *hours, the count is* 1,000,000. *How many bacteria were in the count initially?*
(v) The rate of natural increase of the population of a certain city is proportional to the population . If the population increases from 40,000 *to* 60,000 *in*
40 *years, when will the population be* 80,000*?*

45 5.13 Application of Integration to Problems in Economics

Integration can be applied in Economics in determining the total cost of production of a commodity when the marginal production cost is known. Recall that the total cost of producing number x items, is given by $C(x)$, therefore the marginal cost function is given by the rate of change of the total cost with respect to the items produced. That is, $\frac{dC}{dx}$ or $C^{/}(x)$. Thus the total cost function, $C(x) = \int C^{/}(x)dx$.

Example 185 (5.13.1) *(1) The cost of producing an article consists of £*1000 *fixed cost and £*2 *per unit variable*
cost. Find the total cost function and the cost of producing 2500 *units of the article.*

(2) Given that the marginal cost in dollars, of producing x units of a certain commodity

is $M(x) = 3x^2 - 4x + 5$, determine the total cost function and hence find (i) the cost of

producing 20 units, (ii) the cost of producing the 11th through the 15th units inclusive.

(3) The marginal revenue function for a certain commodity is

$M(x) = (500-30x)$, where x is the number of units produced. Determine the total revenue function $r(x)$, and hence, find the total revenue in naira, realized, when 20units of the commodity are produced and sold.

Solution 186 *(1) Let the total cost be $C(x)$. $\therefore$ $C(x) = \int 2dx = 2x + K$, where $K = 1000$. Hence,*

$C(x) = 2x+1000$. $\therefore$ the cost of producing 2500 units is $C(2500) = 2(2500) + 1000$

$= £6000$.

(2) Total cost function, $C(x) = \int(3x^2 - 4x + 5)dx = x^3 - 2x^2 + 5x + C$.

$\therefore$ $C(0) = 0 \Rightarrow C = 0$. Hence, $C(x) = x^3 - 2x^2 + 5x$.(i)Thus the cost of producing 20 units is $C(20) = 20^3 - 2(20)^2 + 5(20) = \$7300$. (ii) the cost of producing 11th through the

15th units is $\int_{10}^{15}(3x^2 - 4x + 5)dx = \left[x^3 - 2x^2 + 5x\right]_{10}^{15}$

$= \left[\left(15^3 - 2(15)^2 + 5(15)\right) - \left(10^3 - 2(10)^2 + 5(10)\right)\right] = \$2150$.

(3) $r(x) = \int (500 - 30x)\,dx = 500x - 15x^2 + C$. But $r(0) = 0 \Rightarrow C = 0$.

$\therefore$ $r(x) = 500x - 15x^2$. Hence, $r(20) = 500(20) - 15(20)^2 = N4000$.

The examples show the wide range of applications of calculus to real life problems. it is also the basic tool required for the study of more sophisticated mathematics.