Derivation of upper-bound on the value of f

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Refer to the DoubleDice Token documentation for an explanation of the context of this derivation.

1 Definitions

Define:

$$\begin{aligned}
T &\mid T > 0 && = \text{initial (max) total supply } (= \texttt{initTotalSupply}) \\
C &\mid 0 < C \leqslant T && = \text{initial circulating supply } (= \texttt{initTotalSupply - totalYieldAmount}) \\
B &\mid B \geqslant 0 && = \text{total yield supply that is burned rather than distributed} \\
d_i &\mid d_i \geqslant 0 && = \text{the amount of yield distributed during the } i^{\text{th}} \text{ yield-distribution} \\
n &\mid n \geqslant 0 && = \text{number of rounds over which the unburned yield supply is distributed}
\end{aligned}$$

All together, these values satisfy the equation:

$$T = C \; + \; \sum_{i=1}^{n} d_i \; + \; B$$

Let the *circulating supply* be defined as the portion of the total supply that is eligible for receiving distributed yield.

Suppose that before the i^{th} yield-distribution round, we "remove" a proportion $\varepsilon_i < 1$ from the circulating supply, so that it does not receive yield as part of that round (e.g. the portion of the circulating supply held by a DEX). We then proceed to distribute an amount d_i of yield to the remaining proportion $\alpha_i = 1 - \varepsilon_i$ of the original circulating supply, thus incrementing the remaining circulating supply by d_i, and finally we "place" the excluded supply back into circulation.

If f_i is the value of f just after the i^{th} yield-distribution, we define a sequence f_i by recurrence as follows:

$$
\begin{aligned}
f_0 &= 1 \\
f_{i+1} &= f_i \cdot \frac{\text{circulating supply just after } i^{\text{th}} \text{ distribution}}{\text{circulating supply just before } i^{\text{th}} \text{ distribution}}
\end{aligned}
$$

The "circulating supply" appearing in the sequence f_i above *excludes* the proportion ε_i.

2 Statement of the problem

We want to find an upper bound for f_n, subject to the constraints:

$$C + \sum_{i=1}^{n} d_i + B = T, \quad B \geqslant 0, \quad d_i \geqslant 0 \quad \forall i$$

3 Solution

We have:

$$
\begin{aligned}
f_n &= 1 \cdot \frac{\alpha_1 C + d_1}{\alpha_1 C} \cdot \frac{\alpha_2(C + d_1) + d_2}{\alpha_2(C + d_1)} \cdot \frac{\alpha_3(C + d_1 + d_2) + d_3}{\alpha_3(C + d_1 + d_2)} \cdot \cdots \cdot \frac{\alpha_n(C + d_1 + d_2 + \cdots + d_{n-1}) + d_n}{\alpha_n(C + d_1 + d_2 + \cdots + d_{n-1})} \\
&= \left(1 + \frac{d_1}{\alpha_1 C}\right) \cdot \left(1 + \frac{d_2}{\alpha_2(C + d_1)}\right) \cdot \left(1 + \frac{d_3}{\alpha_3(C + d_1 + d_2)}\right) \cdot \cdots \cdot \left(1 + \frac{d_n}{\alpha_n(C + \sum_{i=1}^{n-1} d_i)}\right).
\end{aligned}
$$

It is clear from the sequence above that if no limit is placed on the value of α_i, the value of f can start growing very rapidly. To this end, we impose a constraint that the proportion of the circulating supply excluded from a distribution can never exceed $\varepsilon_{\max}$, or:

$$
\begin{aligned}
\varepsilon_i &\leqslant \varepsilon_{\max} \quad \forall i \\
1 - \alpha_i &\leqslant \varepsilon_{\max} \\
\alpha_i &\geqslant 1 - \varepsilon_{\max} \\
\frac{1}{\alpha_i} &\leqslant \frac{1}{1 - \varepsilon_{\max}} = \gamma
\end{aligned}
$$

Hence

$$f_n \leqslant \left(1 + \gamma \frac{d_1}{C}\right) \left(1 + \gamma \frac{d_2}{C + d_1}\right) \left(1 + \gamma \frac{d_3}{C + d_1 + d_2}\right) \cdots \left(1 + \gamma \frac{d_n}{C + \sum_{i=1}^{n-1} d_i}\right).$$

For the k^{th} term we have

$$\left(1+\gamma\frac{d_k}{C+\sum_{i=1}^{k-1} d_i}\right) \leqslant \left(1+\gamma\frac{d_k}{C}\right),$$

therefore

$$f_n \leqslant \left(1+\frac{\gamma}{C}d_1\right)\left(1+\frac{\gamma}{C}d_2\right)\left(1+\frac{\gamma}{C}d_3\right)\cdots\left(1+\frac{\gamma}{C}d_n\right).$$

But by the Arithmetic Mean–Geometric Mean Inequality

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \leqslant \frac{\sum_{i=1}^{n} a_i}{n}, \qquad \prod_{i=1}^{n} a_i \leqslant \left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^n,$$

hence

$$\begin{aligned} f_n &\leqslant \left(\frac{(1+\frac{\gamma}{C}d_1)+(1+\frac{\gamma}{C}d_2)+\cdots+(1+\frac{\gamma}{C}d_n)}{n}\right)^n \\ &\leqslant \left(1+\frac{\gamma}{nC}(d_1+d_2+\cdots+d_n)\right)^n \\ &\leqslant \left(1+\frac{\gamma(T-C-B)}{nC}\right)^n. \end{aligned}$$

Since this upper bound will be largest when no yield is burned, we substitute $B=0$ to obtain

$$f_n \leqslant \left(1 \;+\; \frac{1}{n}\cdot\frac{\frac{T}{C}-1}{1-\varepsilon_{\max}}\right)^n.$$

As long as $\sum_{i=1}^{n} d_i = T-C-B$, the value n may be any non-negative integer. If the yield supply $(T-C)$ is exhausted during the k^{th} distribution round, this would simply mean that $d_i = 0 \;\; \forall i > k$. Therefore it is valid to consider $\lim_{n\to\infty} f_i$.

Now if $A > 0$ the sequence $\left(1+\frac{A}{n}\right)^n$ is increasing and $\lim_{n\to\infty}\left(1+\frac{A}{n}\right)^n = \exp(A)$, hence $\left(1+\frac{A}{n}\right)^n < \exp(A)$. Applied to our case this gives

$$\boxed{f_\infty \;<\; \exp\left(\frac{\frac{T}{C}-1}{1-\varepsilon_{\max}}\right)}$$