PS 0.

1. (a) $f(x) = \frac{1}{2} x^T A x + b^T x$

$\nabla f(x) = Ax + b$

(b) $\nabla f(x) = \begin{bmatrix} \frac{\partial g(h(x))}{\partial x_1} \\ \frac{\partial g(h(x))}{\partial x_2} \\ \vdots \\ \frac{\partial g(h(x))}{\partial x_n} \end{bmatrix}$

$\frac{\partial g(h(x))}{\partial x_i} = g'(h(x)) \cdot \frac{\partial h(x)}{\partial x_i}$

$\nabla f(x) = g'(h(x)) \nabla h(x)$

(c) $\nabla^2 f(x) = A$.

(d) $\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}$

$\frac{\partial f(x)}{\partial x_i} = \frac{\partial g(a^T x)}{\partial x_i} = g'(a^T x) \cdot \frac{\partial a^T x}{\partial x_i}$

$= g'(a^T x) \cdot a_i$

$\cancel{\partial f(x)}$ $\nabla f(x) = g'(a^T x)\, a$

$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial f(x)}{\partial x_1}\right) & \cdots & \frac{\partial}{\partial x_1}\left(\frac{\partial f(x)}{\partial x_n}\right) \\ \frac{\partial}{\partial x_2}\left(\frac{\partial f(x)}{\partial x_1}\right) & \cdots & \frac{\partial}{\partial x_2}\left(\frac{\partial f(x)}{\partial x_n}\right) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_n}\left(\frac{\partial f(x)}{\partial x_1}\right) & \cdots & \frac{\partial}{\partial x_n}\left(\frac{\partial f(x)}{\partial x_n}\right) \end{pmatrix}$

$\frac{\partial}{\partial x_j}\left(\frac{\partial f(x)}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(g'(a^T x) a_i\right)$

$= a_i \frac{\partial g'(a^T x)}{\partial x_j}$

$= a_i\, g''(a^T x) \frac{\partial a^T x}{\partial x_j}$

$= a_i a_j\, g''(a^T x)$

$\therefore \nabla^2 f(x) = g''(a^T x)\, a a^T$

2. (a) $A = z z^T$

$A^T = (z z^T)^T = z z^T = A$

for $x \in R^n$ $x \neq \vec{0}$

$x^T A x = x^T z z^T x = (z^T x)^T z^T x$

$= \| z^T x \|_2^2 \geq 0$

$\therefore$ A is positive semidefinite.

(b) $A = z z^T$

~~let $Ax = 0$. $x \in R^n$~~

~~$z z^T x = 0$~~

~~$x^T z z^T x = 0$~~

~~$\| z^T x \|_2^2 = 0$.~~

~~hence z is non-zero n-vector~~

~~x is non-zero n-vector.~~

~~$\therefore z^T z x = 0$.~~

~~for $x \in R^n$ $x \neq 0$.~~

~~$x^T A x = x^T z z^T x = (z^T x)^T z^T x$~~

~~$= \| z^T x \|_2^2$~~

let $Ax = 0$. $x \in R^n$. $x \neq 0$.

$z z^T x = \vec{0} \Rightarrow x^T z z^T x = x^T \cdot \vec{0} = 0$

$(z^T x)^T z^T x = 0$.

$\therefore z^T x = 0$.

when $z^T x = 0$ $z z^T x = \cancel{z(0)}\ 0 \cdot z = \vec{0}$

hence $z z^T x = \vec{0} \Leftrightarrow z^T x = 0$.

$\therefore$ the null-space of A

$X(A) = R^n - \mathrm{Span}(z)$

$\mathrm{rank}(A) = 1$.

(c) $x^T B A B^T x = (B^T x)^T A (B^T x)$

let $y = B^T x \in R^n$.

~~for~~ $\forall z \in R^n$ except $\vec{0}$

$z^T A z \geq 0$.

$\vec{0}^T A \vec{0} = 0$.

$\therefore \forall z \in R^n$ $z^T A z \geq 0$

$\therefore y^T A y \geq 0$ so BAB^T is PSD

3\. a) Proof: $AT = \cancel{TA} T\Lambda T^{-1} T = T\Lambda$

$= (t^{(1)}, t^{(2)}, \cdots t^{(n)}) \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$

$= (\lambda_1 t^{(1)}, \lambda_2 t^{(2)}, \cdots \lambda_n t^{(n)})$

i.e. $\cancel{A} A t^{(i)} = \lambda_i t^{(i)}$

So that the ~~eigenvald~~ eigenvalue/eigenvector pairs of A are $(t^{(i)}, \lambda_i)$

b) $\cancel{A = U\Lambda U^T}$ $A = U\Lambda U^T$

Hence U is orthogonal.

$UU^T = I \quad U^{-1} = U^T$

$\cancel{AU = UA} \quad U^T A U = \Lambda$

according to a),

$\forall i \; (u^{(i)}, \lambda_i)$ is one eigenvector/eigenvalue pair of A.

c). $U^{(i)T} A U^{(i)} = U^{(i)T} (\lambda_i U^{(i)})$

$= \lambda_i U^{(i)T} \cdot U^{(i)}$

$\because$ A is PSD.

$\therefore U^{(i)T} A U^{(i)} \geq 0$

$\therefore \lambda_i(A) U^{(i)T} \cdot U^{(i)} \geq 0$.

$\because U^{(i)T} \cdot U^{(i)} = \| U^{(i)} \|_2^2 \geq 0$.

$\therefore \lambda_i(A) \geq 0$.

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