Tremaux trees and planarity

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Trémaux Trees and Planarity [1]

Hubert de Fraysseix [2]

Centre d'Analyse et de Mathématiques Sociales – CNRS, UMR 8557
École des Hautes Études en Sciences Sociales
Paris, France

Abstract

We present two characterizations of planarity based on Trémaux trees (i.e. DFS trees). From the last one, we deduce a simple and efficient planarity test algorithm which is eventually a new implementation of the Hopcroft Tarjan planarity algorithm. We finally recall a theorem on "cotree critical non-planar graphs" which very much simplify the search of a Kuratowski subdivision in a non-planar graph.

1 Introduction

It is well known since the publications in 1973-74 by J. Hopcroft and R.E. Tarjan [14] that the time complexity of the problem of graph planarity is linear in the number of edges. In the late seventies, we produced the so called Left-Right algorithm which has been recognized as the fastest among the implemented ones by the comparative tests performed by graph specialists [1]. In 1982 [9] we gave a characterization of planarity in terms of Trémaux trees which ease the justification the Left-Right algorithm, although a formal

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[1] This or has been initiated with Pierre Rosenstiehl and then developed with Patrice Ossona de Mendez.
[2] Email:hf@ehess.fr

proof of its rightness remains rather tricky. Two years ago while working on
the correctness of the Left-Right algorithm we were puzzle to realize that we
did not use our simpler characterization of planarity published in 1985 [12].
Looking at it closely, we introduced a third characterization of planarity [7]
— again based on Trémaux trees — which checks planarity considering at
most Δm configurations instead of a cubic number. That characterization
suggests an algorithm which straightforwardly computes the embedding of
the graph if it is planar. It is quite simpler than the Left-Right algorithm
and even more efficient. That characterization will also simplify our linear
time algorithm to extract a Kuratowski subdivision in a non planar graph.
Actually this new algorithm can be depicted as a fast non recursive version of
the Hopcroft-Tarjan one.

After presenting properties of Trémaux trees related to planarity and the
main ideas of our new planarity algorithm, we shall recall some newer results
[6] which very much simplify the search of a Kuratowski subdivision in a non-
planar graph.

2 Trémaux Trees

Depth-first search (DFS) is a fundamental graph searching technique known
since the 19th century see for intance Luca's report on Trémaux's work [15] or
Tarry's publication of Trémaux's algorithm [17]) and popularized by Hopcroft
and Tarjan [13,16] in the seventies. The structure of DFS enables efficient
algorithms for many other graph problems [2]. Performing a DFS on a graph
defines a spanning tree with specific properties (also known as a *Trémaux tree*)
and an embedding of it as a rooted planar tree, the edges going out of a vertex
being circularly ordered according to the discover order of the DFS).

A rooted spanning tree $\mathcal{T}$ of a graph $G = (V, E)$ defines a partition of the
edge set of G into two classes, the set of *tree edges* T and the set of *cotree-
edges* $E \setminus T$. It also defines a partial order $\preceq$ on V: $x \preceq y$ if the tree path
linking y to the root of $\mathcal{T}$ includes x. The rooted tree $\mathcal{T}$ is a *Trémaux tree* if
every cotree edge is incident to two comparable vertices (with respect to $\preceq$).
A Trémaux tree $\mathcal{T}$ defines an orientation of the edges of the graph: an edge
$\{x, y\}$ (with $x \prec y$) is oriented from x to y (upwards) if it is a tree edge and
from y to x (downwards) if it is a cotree edge. Cotree edges of a Trémaux tree
are called *back edges*. We will denote by $\omega^+(v)$ the set of the edges incident
to a vertex v and going out of v. When $\mathcal{T}$ is a Trémaux tree, the partial order
$\preceq$ is extended to $V \cup E$ (or to G for short) as follows: for any edge $e = (x, y)$
oriented from x to y, put $x \prec e$ and if $x \prec y$ (that is: if e is a tree-edge) also
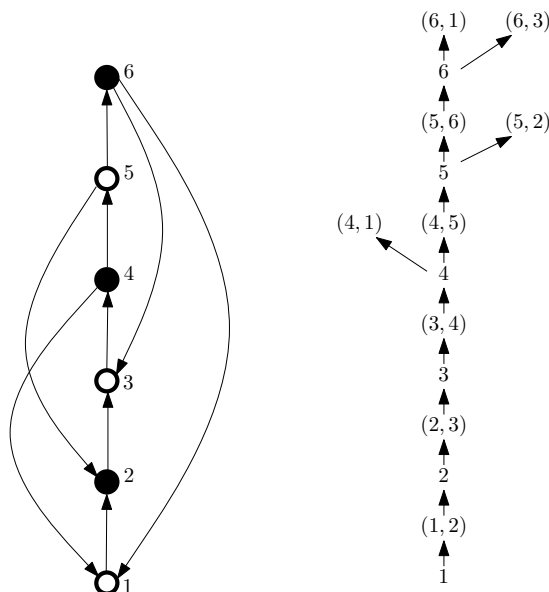
put $e \prec y$.

Fig. 1. The partial order $\prec$ defined by a Trémaux tree of $K_{3,3}$ (cover relation correspond to bottom-up arrows as usual).

Notice that in the partial order $\prec$, all elements α and β of G have an unique greatest lower bound (meet) $\alpha \wedge \beta$.

When $\alpha \preceq \beta$, the unique chain (of $\prec$) with minimum α and maximum β which is maximal (with respect to set-inclusion) is denoted by $[\alpha\,;\beta]$.

It will be helpful to introduce a notation for the minimal element of the interval $]\alpha\,;\beta]$:

Definition 2.1 For $x \prec e$, where $x \in V$ and $e \in E$ we define

$$\mathrm{stem}(x, e) = \min\,]x\,;e] \tag{1}$$

This means that f is the first edge in the unique chain of $\prec$ with minimum x and maximum e.

Notice that in this definition, as in the remaining of the paper, intervals and "min" will always be related to the partial order $\prec$ defined by the considered Trémaux tree.

Definition 2.2 Although usually defined on vertices in the literature, we de-

fine here the low as a mapping from E to V:

$$\text{low}(e) = \begin{cases} \min\left\{v \in V : \exists (u,v) \in E, \ (u,v) \succeq e\right\} & \text{if } e \text{ is a tree edge,} \\ y & \text{if } e \text{ is a cotree edge.} \end{cases}$$

Definition 2.3 The *fringe* Fringe(e) of an edge $e = (x,y)$ is defined by:

$$\text{Fringe}(e) = \{f \in E \setminus T : f \succeq e \text{ and } \text{low}(f) \prec x\}$$

Definition 2.4 Le e be a tree edge

- If Fringe(e) = $\emptyset$, the edge e is a *block edge* (dotted in the picture);
- otherwise, if all the edges in Fringe(e) have the same low, the edge e is a *thin edge* (light in the picture);
- otherwise, the edge e is a *thick edge* (fat in the picture).
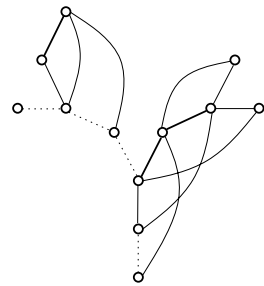
Definition 2.5 A *TT-precedence order* $\prec^\star$ is a partial order on E such that, for any $v \in V$ and any $e, f \in \omega^+(v)$:

- if $\text{low}(e) \prec \text{low}(f)$ then $e \prec^\star f$,
- if $\text{low}(e) = \text{low}(f)$, f is a thick tree edge but e is not, then $e \prec^\star f$.

3 Trémaux trees and planarity

Planarity has been related to Trémaux trees by de Fraysseix and Rosenstiehl in a series of articles [12,11,8]. One of these characterizations is based on a the existence of a special bipartition of the low angles of the cotree edges into left ones and right ones. The constraints that the bipartition has to fulfill is encoded into two relations, namely the $\mathcal{T}$-alike and $\mathcal{T}$-opposite relations.

We don't give here the formal definition of $\mathcal{T}$-alike and $\mathcal{T}$-opposite relations in terms of $\prec$, but simply recall the characterization given in [12]:

Theorem 3.1 *Let G be a graph with Trémaux tree $\mathcal{T}$. Then G is planar if and only if there exists a partition of the cotree edges of G into two classes so that any two edges belong to a same class if they are $\mathcal{T}$-alike and any two edges belong to different classes if they are $\mathcal{T}$-opposite.*
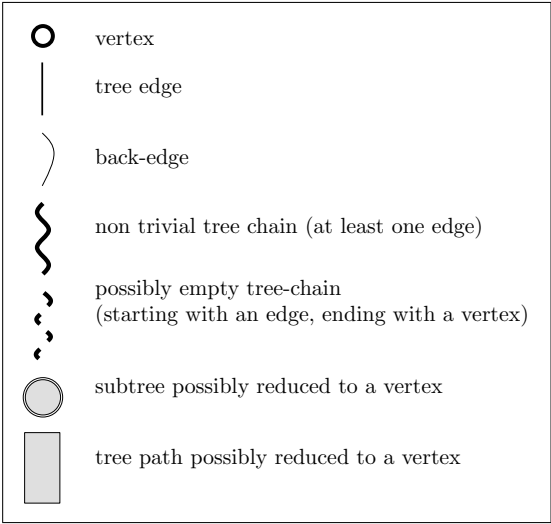
Fig. 2. Symbols used in the figures

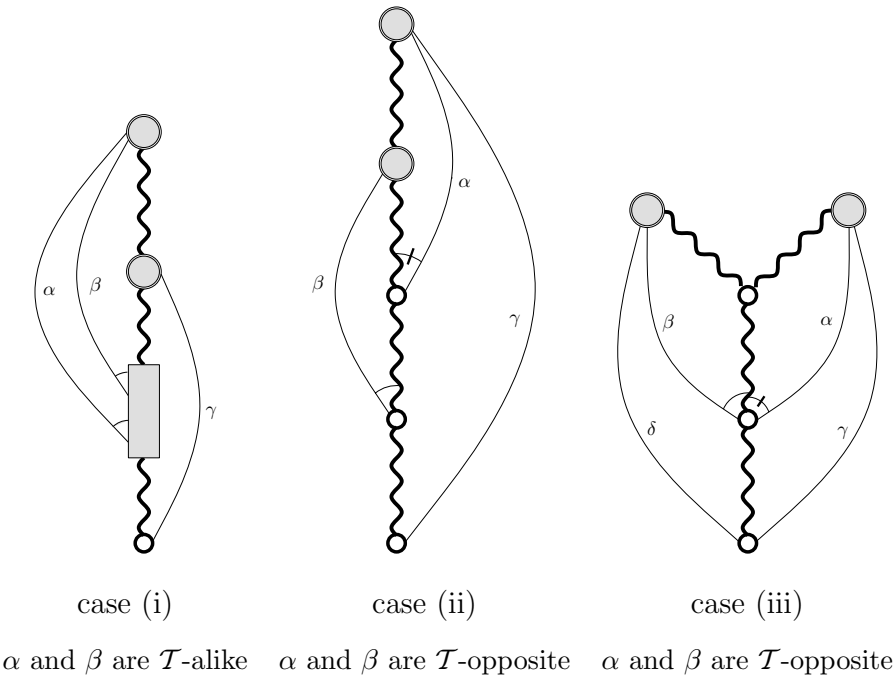| case (i)	case (ii)	case (iii)
α and β are $\mathcal{T}$-alike	α and β are $\mathcal{T}$-opposite	α and β are $\mathcal{T}$-opposite

Fig. 3. Definition of $\mathcal{T}$-alike and $\mathcal{T}$-opposite relations

Instead of working with this characterization, we introduce an equivalent characterization based on a single configuration.

Definition 3.2 Let v be a vertex and let $e_1, e_2 \in \omega^+(v)$.
    The *interlaced set* Interlaced(e_1, e_2) is defined by:

$$\text{Interlaced}(e_1, e_2) = \{f \in \text{Fringe}(e_1) : \text{low}(f) \succ \text{low}(e_2)\}$$

Definition 3.3 Given a graph G and a Trémaux tree $\mathcal{T}$ of G, a coloring $\lambda : E \setminus T \to \{-1, 1\}$ is an *F-coloring* if, for every vertex v and any edges $e_1, e_2 \in \omega^+(v)$, Interlaced(e_1, e_2) and Interlaced(e_2, e_1) are monochromatic and colored differently.

It is easily checked that a coloring is an F-coloring if and only if two $\mathcal{T}$-alike cotree edges are colored the same and two $\mathcal{T}$-opposite cotree edges are colored differently.
    In a planar drawing, an F-coloring is defined by the partition of the cotree edges on two sets, the edges f having their low incidence on the left (resp. the right) of the tree edge stem$(\text{low}(f), f)$. The following lemma is straightforward and does not deserve a proof:

Lemma 3.4 *Let G be a planar graph with Trémaux tree $\mathcal{T}$. Then G has an F-coloring.*   □

Definition 3.5 An F-coloring $\lambda : E \setminus T \to \{-1, 1\}$ is *strong* if, for any $v \in V$ the low set L(v) is monochromatic, where

$$\text{L}(v) = \{f \in E \setminus T : f \succ v \text{ and } \forall g \in E \setminus T, g \succ v \Rightarrow \text{low}(g) \succeq \text{low}(f)\}.$$

Lemma 3.6 *If G has an F-coloring then it has a strong F-coloring.*

Definition 3.7 Let $\lambda : E \setminus T \to \{-1, 1\}$ be a strong F-coloring. We define the coloring $\widehat{\lambda} : E \to \{-1, 1\}$ by:

$$\widehat{\lambda}(e) = \begin{cases} \lambda(e), & \text{if } e \text{ is a cotree edge} \\ \lambda(f), & \text{if } e \text{ is a tree edge and } f \in \text{Fringe}(e) \text{ with maximal low}(f) \\ 1, & \text{if } e \text{ is a tree edge and Fringe}(e) = \emptyset \end{cases}$$

Notice that if e is a tree edge, any two edges f_1, f_2 with the same low have the same value of λ as λ is assumed to be a strong F-coloring.

Lemma 3.8 *Let G be a graph, let $\mathcal{T}$ be a Trémaux tree of G, let λ be a strong F-coloring and let $\widehat{\lambda}$ be the associated mapping. We first define the circular order of the outgoing edges at a vertex v as follows:*
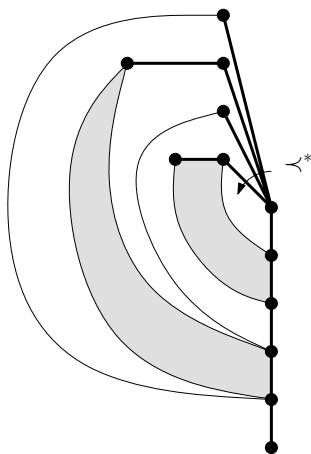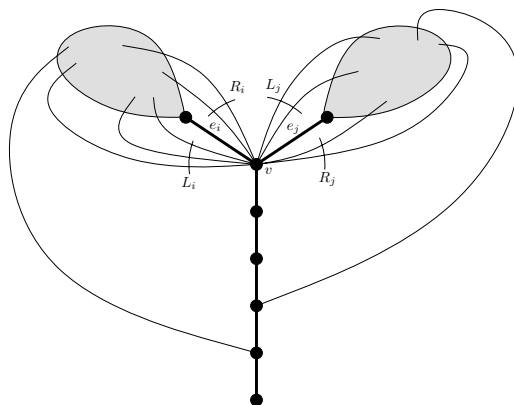
Fig. 4. Circular Order of Outgoing Edges with $\widehat{\lambda} = -1$

Fig. 5. Circular Order of Incoming Edges

Let $e_1 \succ^\star e_2 \succ^\star \cdots \succ^\star e_p$ be the edges in $\omega^+(v)$ with $\widehat{\lambda}(e_i) = -1$ (see Fig. 4) and let $e_{p+1} \prec^\star e_{p+2} \prec^\star \cdots \prec^\star e_q$ be the edges in $\omega^+(v)$ with $\widehat{\lambda}(e_i) = 1$. In the circular order around v one finds the outgoing edges in the following order $e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_q$.

Then, once the circular order of outgoing edges is computed for all vertices, we insert the incoming edges as follows:

In the circular order around v one finds the incoming tree edge (if $v \neq r$) and then $L_1, e_1, R_1, L_2, e_2, R_2, \ldots, L_q, e_q, R_q$ where L_i (resp. R_i) is the set of incoming cotree edges $f = (x, v)$ such that $\lambda(f) = -1$ (resp. $\lambda(f) = 1$) and the tree-path linking r to x includes e_i (see Fig 5). For $e_i, e_j \in L_k$ (resp. for $e_i, e_j \in R_k$), one finds e_i before (resp. after) e_j in the circular order if

stem$(e_i \wedge e_j, e_i)$ *is before* stem$(e_i \wedge e_j, e_j)$ *in the circular order of outgoing edges at vertex* $e_i \wedge e_j$.

Then these circular orders define a planar embedding of G.

Sketch of the proof. In a drawing where the tree edges cross no other edges, only two kinds of crossing could occur, which are considered successively, each leading to a contradiction.
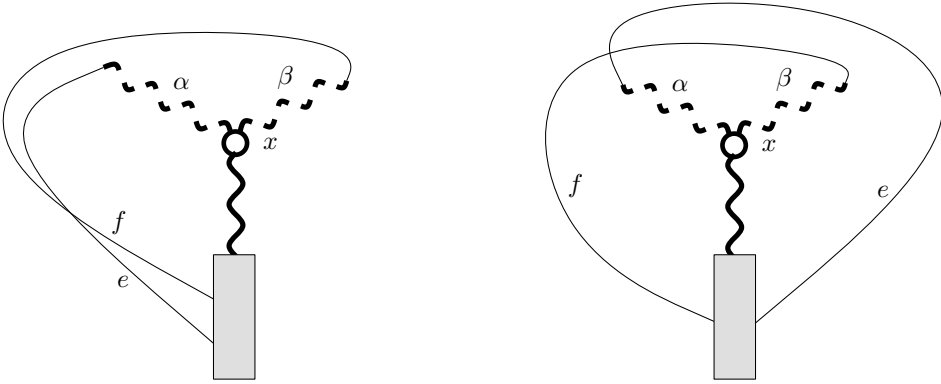
From these lemmas follows directly:

Theorem 3.9 *Let* G *be a graph, let* $\mathcal{T}$ *be a Trémaux tree of* G. *The following conditions are equivalent:*

(i) G *is planar,*

(ii) G *admits an F-coloring,*

(iii) G *admits a strong F-coloring.*

Moreover, if G *is planar, any strong F-coloring* λ *defines a planar embedding of* G *in which a cotree edge e has its lowest incidence to the left of the tree if* $\lambda(e) = -1$ *and to the right of the tree if* $\lambda(e) = 1$.

3.1   *Outline of a Planarity Testing Algorithm*

Let G be a graph of size m. The three steps are performed in $O(m)$-time. The first step is composed of a preliminary DFS on G and the computation of the low function and the status of the edges (block/thin/thick). The second step is the computation of a TT-precedence order, which may be efficiently performed using a bucket sort. We now examine the last step of the algorithm, which tests the planarity of the graph.

We shall consider some data structure DS responsible for maintaining a set of bicoloration constraints on a set of cotree edges. We assign to each edge of the graph e such a data structure DS(e). These structures are initialized as follows: DS(e) is empty if e is a tree edge and includes e (with no bicoloration constraints) if e is a cotree edge. We say that all the cotree edges have been *processed* and that the tree edges are still *unprocessed*.

- While there exists a vertex v, different from the root, such that all the edges in $\omega^+(v)$ have been processed. Let $e = (u, v)$ be the tree edge entering v. Let $e_1 \prec^\star e_2 \prec^\star \cdots \prec^\star e_k$ be the edges in $\omega^+(v)$ ($k \geq 1$). We do the following:
  - · Initialize DS(e) with DS(e_1).
  - · For $i : 2 \to k$, *Merge* DS(e_i) into DS(e), that is: add to DS(e) the edges in DS(e_i) and add the F-coloring constraints corresponding to the pairs of edges e_j, e_i with $j < i$ (notice that all the concerned cotree edges belong to DS(e)). If some constraint may not be satisfied, the graph is declared non-planar.
  - · Remove from all of the DS(e) every cotree edge with lower incidence u.
  - · We declare that edge e has been processed.
- As all the edges have been processed, we declare that the graph is planar.

4   Trémaux trees and Kuratowski subdivisions

In this last section we present a characterisation of Trémaux-cotree critical graph which is at the basis of our program in Pigale [5] which exhibit Kuratowski subdivisions in non-planar graphs. As the proofs are rather long and technical, we won't even sketch them here, and we refer the reader to [10][3][4][6].

A Trémaux-cotree critical graph is a simple graph of minimum degree 3 having a Trémaux tree, such that any cotree edge is *critical*, in the sense that its deletion would lead to a planar graph. A first study of Trémaux-cotree critical appeared in [10], in which it is proved that a Trémaux-cotree critical graph either is isomorphic to K_5 or includes a subdivision of $K_{3,3}$ and no subdivision of K_5. We shall now recall stronger results that we published later in [6].

A Möbius pseudo-ladder is a natural extension of Möbius ladders allowing triangles. This may be formalized by the following definition.

Definition 4.1 A *Möbius pseudo-ladder* is a non planar simple graph, which is the union of a polygon $(v_1, \ldots, v_n)$ and chords (called *bars*) such that any
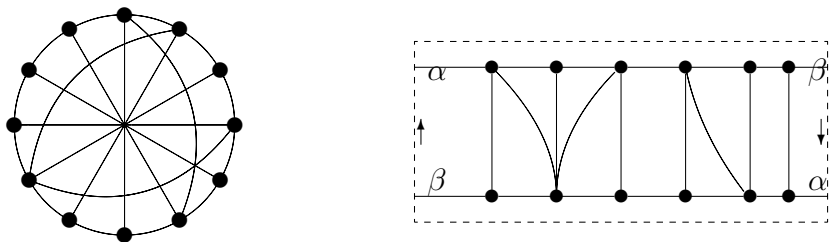
Fig. 6. A Möbius pseudo-ladder on the plane.

two non adjacent bars are interlaced (recall two non adjacent edges $\{v_i, v_j\}$ and $\{v_k, v_l\}$ are *interlaced* if, in circular order, one finds exactly one of $\{v_k, v_l\}$ between v_i and v_j).

This definition means that a Möbius pseudo-ladder may be drawn in the plane as a polygon and internal chords such that any two non adjacent chords cross (see Figure 6). Notice that $K_{3,3}$ and K_5 are both Möbius pseudo-ladders.

Theorem 4.2 *Any Trémaux-cotree critical graph is a Möbius pseudo-ladder.*

The following refined theorem gives right away a trivial algorithm to exhibit a Kuratowski subdivision in a Trémaux-cotree critical graph.

Theorem 4.3 *A simple graph is Trémaux-cotree critical if and only if it is a Möbius pseudo ladder which non-critical edges belong to some Hamiltonian path.*

Moreover, if G is Trémaux-cotree critical according to a Trémaux tree $\mathcal{T}$ and G has at least 9 vertices, then $\mathcal{T}$ is a chain and G is the union of a cycle of critical edges and pairwise interlaced non critical chords (see Figure 7).
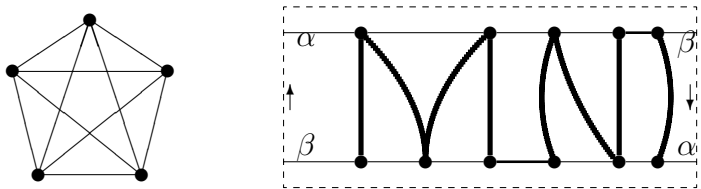

Fig. 7. The Trémaux-cotree critical graphs are either K_5 or a Möbius pseudo-ladders having all its non critical edges (thickest) included in a single path.

The algorithm first computes the set critical edges of G. That for, we use the property that a tree edge is critical if and only if it belongs to a fundamental cycle of length 4 of some cotree edge to which it is not adjacent.

Then, three pairwise non-adjacent non-critical edges are found to complete a Kuratowski subdivision of G isomorphic to $K_{3,3}$.

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