Homework 10 — PHYS373 2021

Domingues, Douglas

Due Apr 29, 2021

The Convolution

1. In class we showed, by direct integration, that $t^2 * t = \frac{t^4}{12}$, and checked that $\mathcal{L}\left\{t^2\right\}\mathcal{L}\left\{t\right\} = \mathcal{L}\left\{t^2 * t\right\} = \mathcal{L}\left\{\frac{t^4}{12}\right\}$. Show, by direct integration, that $t * t^2$ is also $\frac{t^4}{12}$. This shows one specific example of the general fact that $f * g = g * f$.

   Solution: We want to compute $t * t^2$. By the definition of the convolution:

$$t * t^2 = \int_0^t \tau(t-\tau)^2 \, \mathrm{d}\tau$$

$$= \int_0^t \tau(t^2 - 2t\tau + \tau^2) \, \mathrm{d}\tau$$

$$= \int_0^t (\tau t^2 - 2t\tau^2 + \tau^3) \, \mathrm{d}\tau$$

$$= [\frac{\tau^2}{2}t^2 - 2t\frac{\tau^3}{3} + \frac{\tau^4}{4}]_0^t$$

$$= \frac{t^4}{2}\frac{-2t^4}{3} + \frac{t^4}{4} = \frac{6-8+3}{12}t^4 = \frac{t^4}{12}$$

2. Suppose a garbage truck in a small town collects some material at a rate $g(t)$ (g for garbage; maybe it's tons/week, or something like that). Let's assume that each piece of that material decays according to the function $d(t)$ (d for decay). That is, if you start with 1 ton at time $t = 0$ and don't add any, at a later time you have $d(t)$ tons of that material (d is typically a decreasing function, and $d(0) = 1$ because after no time the material hasn't decayed at all).

   Let's assume the garbage dump starts empty and the garbage truck arrives at the dump once a week and deposits $g(t)\Delta t$ tons of material (Δt=1 week). After the zeroeth week the total in the dump is $(g(0)\Delta t)d(0)$. After the first week, the total is composed of what was already in the dump, but now it's had one week to decay $((g(0)\Delta t)d(1)$; plus the new material is added, $(g(1)\Delta t)d(0)$.

   (a) How much material is in the dump after the truck deposit's the second week's haul?

   Solution: $(g(2)\Delta t)d(0) + (g(1)\Delta t)d(1) + (g(0)\Delta t)d(2)$

   (b) How much after the third week's haul?

Solution: $(g(3)\Delta t)d(0) + (g(2)\Delta t)d(1) + (g(1)\Delta t)d(2) + (g(0)\Delta t)d(3)$

(c) You don't have to write anything for this part, but convince yourself that after n weeks the dump contains

$$\text{total material} = \sum_{w=0}^{n} g(w)d(n-w)\Delta t \tag{1}$$

(you can use this to check you answer to the previous two parts).

Solution: Makes sense. Here's the sum for week 3:

$$\text{total material} = \sum_{w=0}^{3} g(w)d(3-w)\Delta t$$
$$= (g(0)\Delta t)d(3) + (g(1)\Delta t)d(2) + (g(2)\Delta t)d(1) + (g(3)\Delta t)d(0)$$

Suppose instead of a small town with a garbage truck, your dump services the New York Department of Sanitation. So, instead of garbage arriving once a week, garbage arrives at the dump continuously at a rate $g(t)$. Convince yourself (you don't have to write anything as an answer) that the Riemann sum from the previous part goes to

$$\int_0^t g(\tau)d(t-\tau)\ \mathrm{d}\tau \tag{2}$$

where we took the limit $\Delta t \to 0$ and changed w (named to indicate weeks) to τ to indicate a continuous time.

(d) Suppose you work for a normal dump and it collects styrofoam, at a rate $g(t)$. To a good approximation, styrofoam never decays, $d(t) = 1$, independent of t. How much styrofoam is in the dump at time t? (Leave your answer as an integral. Does the answer make sense?)

Solution:

$$\int_0^t g(\tau)d(t-\tau)\ \mathrm{d}\tau$$

Since $d = 1 \forall \tau$

$$\int_0^t g(\tau)\ \mathrm{d}\tau$$

Yes, the answer makes sense. Afterall, the amount of styrofoam in the dump is what was put there, or the accumulation of what was continuously deposited there over time.

3. Suppose you work for the Department of Energy and run a nuclear waste storage facility in a deep, geologically stable underground salt cave. The facility opens in 1990 and collects waste material at a constant rate and is sealed in 2030, at which point the facility is closed and no new waste may be accepted; $g(t) = gu_{1990,2030}(t)$ (g has units of mass/time, say kg/year).

The radioactive material decays with a halflife h; $d = e^{-t\ln 2/h}$. We can show that the amount of radioactive material in the repository is given by

$$g * d = \frac{gh}{\ln 2}\left[u(t-1990)\left(1-e^{-(t-1990)\ln 2/h}\right) - u(t-2030)\left(1-e^{-(t-2030)\ln 2/h}\right)\right]. \tag{3}$$

in two ways.

First, we can evaluate a very tricky integral as follows: We convolve

$$\begin{aligned}
g * d &= \int_0^t g(\tau) e^{-(t-\tau)\ln 2/h}\,\mathrm{d}\tau \\
&= \int_0^t g u_{1990,2030}(\tau) e^{-(t-\tau)\ln 2/h}\,\mathrm{d}\tau \\
&= g \int_0^t \left((u(\tau-1990) - u(\tau-2030)\right) e^{-(t-\tau)\ln 2/h}\,\mathrm{d}\tau \\
&= g e^{-t\ln 2/h} \int_0^t (u(\tau-1990) - u(\tau-2030) e^{\tau\ln 2/h}\,\mathrm{d}\tau \\
&= g e^{-t\ln 2/h} \left[\int_0^t u(\tau-1990) e^{\tau\ln 2/h}\,\mathrm{d}\tau - \int_0^t u(\tau-2030) e^{\tau\ln 2/h}\,\mathrm{d}\tau\right]
\end{aligned}$$

Now we need to think. The first integral is zero if $t < 1990$, because the integrand is zero there. So that piece should be proprotional to $u(t-1990)$. Similarly the second integral is proportional to $u(t-2030)$.

$$\begin{aligned}
g * d &= g e^{-t\ln 2/h} \left[u(t-1990)\int_{1990}^t e^{\tau\ln 2/h}\,\mathrm{d}\tau - u(t-2030)\int_{2030}^t e^{\tau\ln 2/h}\,\mathrm{d}\tau\right] \\
&= g e^{-t\ln 2/h} \left[u(t-1990)\left(\frac{e^{t\ln 2/h} - e^{1990\ln 2/h}}{\ln 2/h}\right) - u(t-2030)\left(\frac{e^{t\ln 2/h} - e^{2030\ln 2/h}}{\ln 2/h}\right)\right] \\
&= \frac{gh}{\ln 2}\left[u(t-1990)\left(1 - e^{-(t-1990)\ln 2/h}\right) - u(t-2030)\left(1 - e^{-(t-2030)\ln 2/h}\right)\right]
\end{aligned}$$

Arrive at this same conclusion by computing the convolution via the Laplace transform: $g * d = \mathcal{L}^{-1}\{\mathcal{L}\{g\}\,\mathcal{L}\{d\}\}$. (You'll need to perform partial fractions and use some of the rules we've shown in class; you can look them up in the table in Boas.)

Solution: Basically we need to evaluate $g * d = \mathcal{L}^{-1}\{\mathcal{L}\{g\}\,\mathcal{L}\{d\}\}$. Let me do it piece by piece:

$$\mathcal{L}\{g\}(k) = \mathcal{L}\{g u_{1990,2030}\} = g\frac{e^{-1990k} - e^{-2030k}}{k}$$

$$\mathcal{L}\{d\}(k) = \mathcal{L}\left\{e^{-t\ln 2/h}\right\} = \frac{1}{k + \ln 2/h}$$

$$= \mathcal{L}\{g\}\,\mathcal{L}\{d\}(k) = g\frac{e^{-1990k} - e^{-2030k}}{k}\frac{1}{k+\ln 2/h} = g\frac{e^{-1990k} - e^{-2030k}}{k^2 + k\ln 2/h}$$

Using partial fractions,

$$\frac{1}{k(k+\ln 2/h)} = \frac{A}{k} + \frac{B}{k+\ln 2/h}$$

$$1 = (k+\ln 2/h)A + kB = (\ln 2/h)A + (A+B)k$$

$$\Rightarrow A + B = 0,\ A\ln 2/h = 1$$

$$\Rightarrow A = \frac{1}{\ln 2/h},\ B = -A$$

Thus,

$$\begin{aligned}
g\frac{e^{-1990k} - e^{-2030k}}{k^2 + k\ln 2/h} &= \frac{gh}{\ln 2}(e^{-1990k} - e^{-2030k})\left(\frac{1}{k} - \frac{1}{(k+\ln 2/h)}\right) \\
&= \frac{gh}{\ln 2}\left[e^{-1990k}\left(\frac{1}{k} - \frac{1}{(k+\ln 2/h)}\right) - (e^{-2030k})\left(\frac{1}{k} - \frac{1}{(k+\ln 2/h)}\right)\right]
\end{aligned}$$

Using the following results from the table

$$g(t-a)u(t-a) \xrightarrow{\mathcal{L}} e^{-pa}G(k), \; e^{-at} \xrightarrow{\mathcal{L}} \frac{1}{k+a}$$

we get

$$\xrightarrow{\mathcal{L}^{-1}} \frac{gh}{\ln 2}\left[u(t-1990)\left(1-e^{-(t-1990)\ln 2/h}\right) - u(t-2030)\left(1-e^{-(t-2030)\ln 2/h}\right)\right].$$

4. **This question does not constitute financial advice!**[1] At age A you start maximizing your contributions to your 401(k), an individual retirement saving account that invests in the stock market. Each year until you retire at age R you deposit into your account the maximum allowed by law, $M = \$19,500$ per year, and you never make a withdrawal[2]. So, your deposit rate is $d(t) = Mu_{A,R}(t)$ Suppose your 401(k) grows according to the historical market return, 10% per year. In other words, $g(t) = e^{rt}$ where $r \approx 0.095$ is set by $g(1) = 1.1$. Accounting for your deposits and the growth in your portfolio, the total balance b in your account as a function of time is $b(t) = d * g$.

   (a) Argue, using the solution to the nuclear dumping question, or by using Laplace transforms, that

$$b(t) = d * g = \frac{M}{r}\left[u(t-A)\left(e^{r(t-A)}-1\right) - u(t-R)\left(e^{r(t-R)}-1\right)\right] \tag{4}$$

Solution: The deposits d are made from $t = A$ until $t = R$ - in other words, from age A until retirement - at a rate of M per year. So $d(t) = Mu_{A,R}$. As said, the growth $g(t) = e^{rt}$. Let's now compute $d * g$:

$$\begin{aligned}
&= \int_0^t d(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_0^t Mu_{A,R}e^{r(t-\tau)}\,\mathrm{d}\tau \\
&= Me^{rt}\int_0^t (u(\tau-A)-u(\tau-R))e^{-r\tau}\,\mathrm{d}\tau \\
&= Me^{rt}\left[\int_0^t u(\tau-A)e^{-r\tau}\,\mathrm{d}\tau - \int_0^t u(\tau-R)e^{-r\tau}\,\mathrm{d}\tau\right]
\end{aligned}$$

By a similar argument to the previous question,

$$\begin{aligned}
&= Me^{rt}\left[u(t-A)\int_A^t e^{-r\tau}\,\mathrm{d}\tau - u(t-R)\int_R^t e^{-r\tau}\,\mathrm{d}\tau\right] \\
&= Me^{rt}\left[u(t-A)\frac{e^{-rt}-e^{-rA}}{-r} - u(t-R)\frac{e^{-rt}-e^{-rR}}{-r}\right] \\
&= \frac{M}{-r}\left[u(t-A)(1-e^{-r(A-t)}) - u(t-R)(1-e^{-r(R-t)})\right] \\
&= \frac{M}{r}\left[u(t-A)(e^{r(t-A)}-1) - u(t-R)(e^{r(t-R)}-1)\right]
\end{aligned}$$

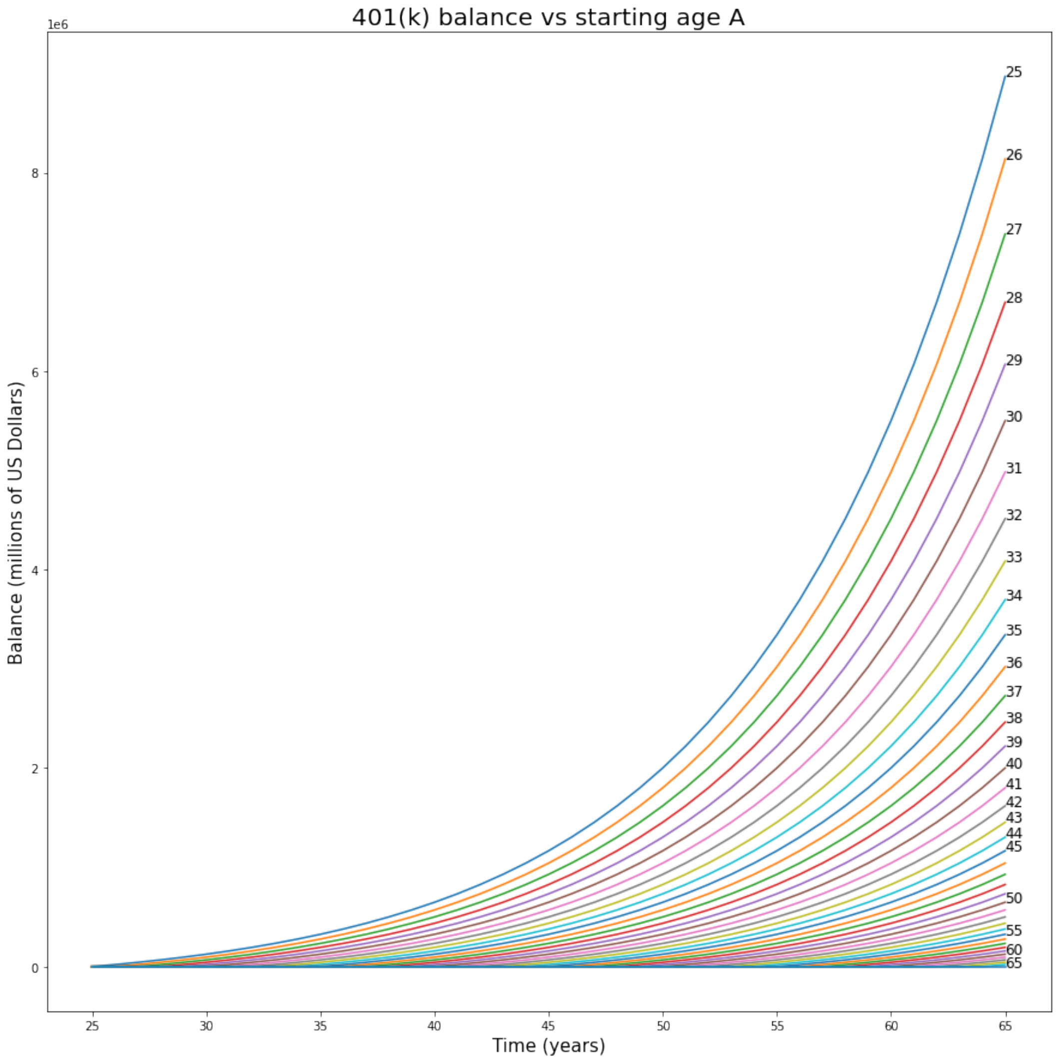
Q.E.D.

[1] Seriously, this is a cartoon. I'm making all sorts of simplifying assumptions! For example, it's false that the market grows 10% a year; that's a rough historical average that doesn't necessarily indicated future growth. Talk to someone who knows more about this than a physics professor when planning for your retirement. Still—the lessons of this problem: 'in general the earlier you start saving for retirement, the better' and 'those first few years can make an enormous difference later down the line' are true.

[2] Two things: 1) the IRS increases M now and then to keep up with inflation, which we're ignoring in this question; it's a few percent a year which is really nontrivial! 2) Retirement accounts generally punish withdrawals before retirement age.

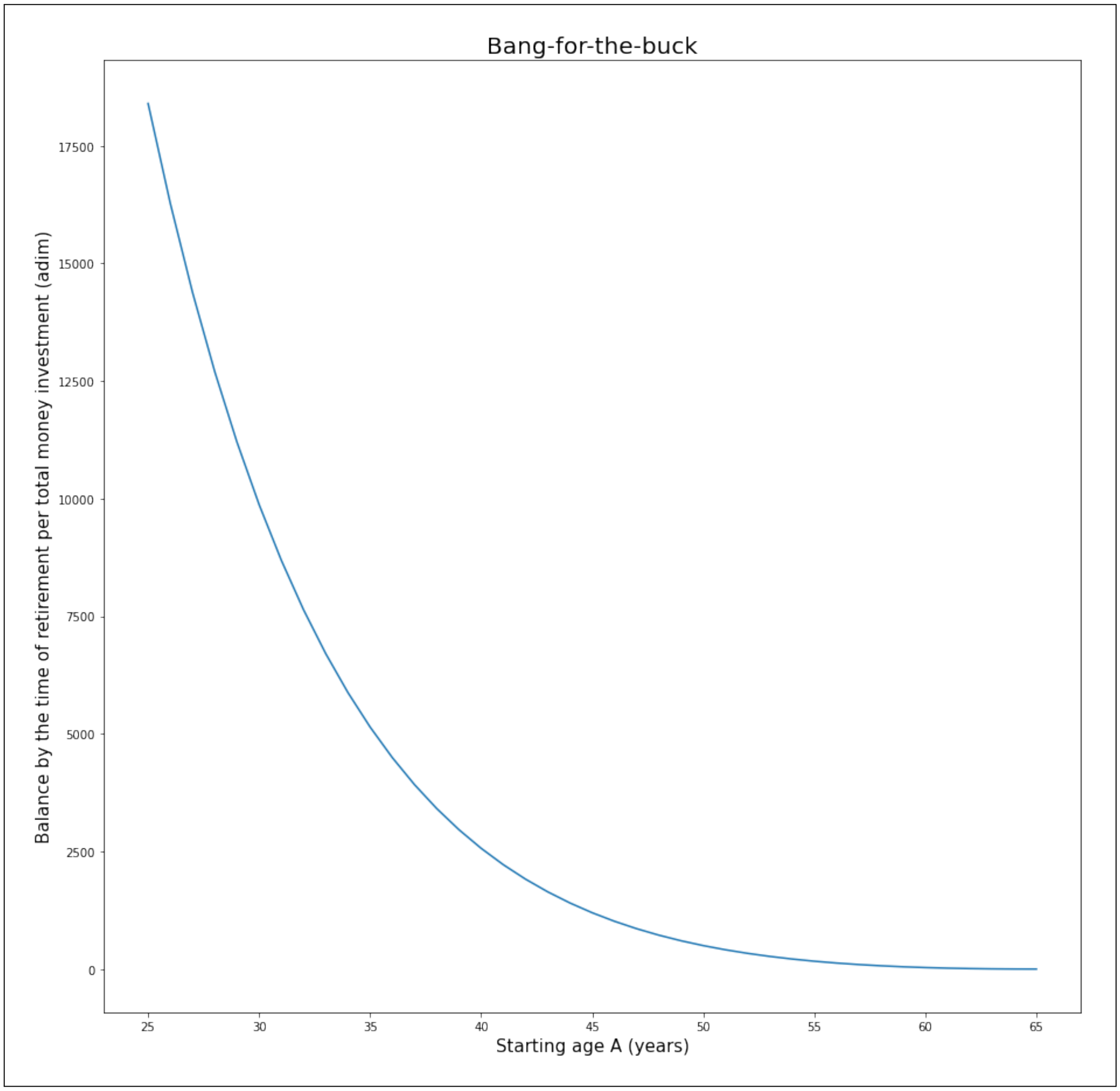
(b) Suppose you retire at $R = 65$. Plot how much money you will have in your 401(k) at $t = R = 65$ as a function of A from 25 to 65, the year you start making contributions.

Solution:



(c) Obviously, the total quantity of money you have deposited at time t is $M(t-A)$. Plot $b(R)/M(R-A)$ as a function of A the age you start saving, assuming you retire at $R = 65$. This ratio is your 'bang-for-the-buck'.

Solution:



(d) According to this naive model, how much more money will you save if you start saving at $A = 25$ than if you start at $A = 30$, assuming $R = 65$?

Solution: Using the algorithm (annex),

$$b(t = 65, A = 25) - b(t = 65, A = 30) = \textbf{US\$}\ 3,469,396.04.$$

Reality check: the average American—especially the average 25-year-old—doesn't typically have a job where they can easily deposit the maximum M into their 401(k)—it'd be too large a percentage of their take-home pay. So what happens more realistically is that the deposit rate d starts out low and grows as you advance in your career. Don't feel stressed out if you don't get such a lucrative job at 25; the point of this question is to prepare you to understand the gist of the computation you must do to roughly understand your future; you can plug in more realistic functions!

Solving ODEs with Laplace Transforms (using convolutions!)

5. Boas 8.12.5

Solution: "Obtain (12.6) by using the convolution integral to solve (12.1)."
(12.1) is given by $y'' + \omega^2 y = f(t)$, $y_0 = y_0' = 0$. Taking the laplace transform,

$$\begin{aligned}
\xrightarrow{\mathcal{L}} k^2 Y - k y_0 - y_0' + \omega^2 Y &= F(k) \\
(k^2 + \omega^2) Y &= F(k) \\
Y = \frac{F(k)}{(k^2+\omega^2)} &= F(k)\omega^{-1}\frac{\omega}{(k^2+\omega^2)} \\
\xrightarrow{\mathcal{L}^{-1}} y = f(t) * \omega^{-1}\sin\omega t &= \frac{f(t) * \sin\omega t}{\omega} \\
&= \int_0^t \frac{1}{\omega}\sin\omega(t-\tau) f(\tau)\,\mathrm{d}\tau
\end{aligned}$$

which is equation (12.6).

6. Consider a system goverened by the ODE

$$\ddot{y} + 2b\dot{y} + \omega_0^2 y = f(t) \tag{5}$$

which starts with $\dot{y}(0) = \dot{y}_0$ and $y(0) = y_0$.

(a) Show that the Laplace transform of the ODE leads to

$$Y = \frac{F}{s^2 + 2bs + \omega_0^2} + \frac{s y_0}{s^2 + 2bs + \omega_0^2} + \frac{2b y_0 + \dot{y}_0}{s^2 + 2bs + \omega_0^2} \tag{6}$$

where $Y = \mathcal{L}\{y\}$ and $F = \mathcal{L}\{f\}$.

Solution:

$$\begin{aligned}
\xrightarrow{\mathcal{L}} s^2 Y - s y_0 - \dot{y_0} + 2b(sY - y_0) + \omega^2 Y &= F(s) \\
(s^2 + 2bs + \omega^2) Y &= F(s) + s y_0 + \dot{y_0} + 2b y_0 \\
Y = \frac{F}{s^2 + 2bs + \omega^2} + \frac{s y_0}{s^2 + 2bs + \omega^2} &+ \frac{2b y_0 + \dot{y_0}}{s^2 + 2bs + \omega^2}
\end{aligned}$$

Let's stop and take this this form in for a second. The first piece is the same no matter what the initial conditions are—it therefore corresponds to a particular solution. The other two pieces have constants in them fixed by the initial conditions—they are part of the complementary function (remember? the solution to the associated homogeneous equation?)

(b) Let's first consider $F = 0$ with the above initial conditions so that we get the appropriate complementary function. Assume were are not in the critically damped case. Show that

$$y_c = \frac{1}{2\sqrt{b^2 - \omega_0^2}}\left[(\dot{y}_0 - y_0 r_-)\, e^{r_+ t} - (\dot{y}_0 - y_0 r_+)\, e^{r_- t}\right] u(t) \qquad r_\pm = -b \pm \sqrt{b^2 - \omega_0^2} \tag{7}$$

where $r_\pm$ are the charactersitic roots of the equation (remember?!). Notice that this solution has the property that $y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$ it satisfied the ODE for all $t > 0$; where we expect Laplace transform methods to work.

Solution: Suppose that $F = 0$ and $b^2 \neq \omega^2$. Also, $s^2 + 2bs + \omega^2 = (s - r_-)(s - r_+)$ where $r = -b \pm \sqrt{b^2 - \omega_0^2}$.

$$Y_c = \frac{sy_0}{(s - r_-)(s - r_+)} + \frac{2by_0 + \dot{y}_0}{(s - r_-)(s - r_+)}$$

$$Y_c = y_0 \frac{s}{(s - r_-)(s - r_+)} + (2by_0 + \dot{y}_0) \frac{1}{(s - r_-)(s - r_+)}$$

Using L7 and L8 from Boas' Laplace table,

$$y_c = y_0 \left[\frac{(-r_-)e^{r_- t} - (-r_+)e^{r_+ t}}{(-r_-) - (-r_+)} \right] + (2by_0 + \dot{y}_0) \left[\frac{e^{r_- t} - e^{r_+ t}}{(-r_+) - (-r_-)} \right]$$

$$y_c = y_0 \left[\frac{r_- e^{r_- t} - r_+ e^{r_+ t}}{r_- - r_+} \right] + (2by_0 + \dot{y}_0) \left[\frac{e^{r_- t} - e^{r_+ t}}{r_- - r_+} \right]$$

$$y_c = \frac{1}{r_- - r_+} \left[e^{r_- t} (y_0 r_- + 2by_0 + \dot{y}_0) - e^{r_+ t} (y_0 r_+ + 2by_0 + \dot{y}_0) \right]$$

$$y_c = \frac{1}{r_- - r_+} \left[e^{r_- t} (y_0(r_- + 2b) + \dot{y}_0) - e^{r_+ t} (y_0(r_+ + 2b) + \dot{y}_0) \right]$$

There's a subtlety here that $r_+ + 2b = -r_-$ and $r_- + 2b = -r_+$.
Also, $r_- - r_+ = -2\sqrt{b^2 - \omega_0^2}$. And thus,

$$y_c = \frac{-1}{2\sqrt{b^2 - \omega_0^2}} \left[e^{r_- t} (y_0(-r_+) + \dot{y}_0) - e^{r_+ t} (y_0(-r_-) + \dot{y}_0) \right]$$

$$y_c = \frac{1}{2\sqrt{b^2 - \omega_0^2}} \left[e^{r_+ t} (\dot{y}_0 - y_0 r_-) - e^{r_- t} (\dot{y}_0 - y_0 r_+) \right] u(t)$$

The $u(t)$ was added as a convention; since the Laplace erases the function for $t < 0$, we can't have any information on that domain after we take the inverse laplace.

(c) Show that

$$W(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2bs + \omega_0^2} \right\} = \frac{1}{2\sqrt{b^2 - \omega_0^2}} \left[e^{r_+ t} - e^{r_- t} \right] u(t), \tag{8}$$

again assuming we're not in the critical case. (You can probably re-use some of the partial-fractions from the previous part, if that's how you did it.)

Solution:

$$\frac{1}{s^2 + 2bs + \omega_0^2} = \frac{1}{(s - r_-)(s - r_+)} = \frac{A}{s - r_-} + \frac{B}{s - r_+}$$

$$1 = A(s - r_+) + B(s - r_-) = s(A + B) - Ar_+ - Br_-$$

$$\Rightarrow B = -A \Rightarrow -Ar_+ - Br_- = A(r_- - r_+) = 1$$

$$\Rightarrow A = \frac{-1}{2\sqrt{b^2 - \omega_0^2}},\ B = \frac{1}{2\sqrt{b^2 - \omega_0^2}}$$

Thus,

$$\frac{1}{s^2 + 2bs + \omega_0^2} = \frac{1}{2\sqrt{b^2 - \omega_0^2}} \left[\frac{1}{s - r_+} - \frac{1}{s - r_-} \right]$$

Taking the inverse laplace (L2 from the table),

$$W(t) = \frac{1}{2\sqrt{b^2 - \omega_0^2}} \left[e^{r_+ t} - e^{r_- t}\right] u(t)$$

(d) Now you know the general solution to the ODE for any $f(t)$—just convolve $W * f$ to find the particular solution and add the complementary solution. Try it for a delta-function impulse at time $t_i > 0$ $f = f_i \delta(t - t_i)$. Show that the particular solution

$$y_p = W * f = \frac{e^{r_+ -(t-t_i)} - e^{r_-(t-t_i)}}{r_+ - r_-} f_i u(t - t_i) \tag{9}$$

by directly evaluating the convolution integral $W * f = \int_0^t \mathrm{d}\tau\, W(\tau) f(t - \tau)$. You might want to consider two cases: $t < t_i$ and $t > t_i$ separately, as shown for the question on nuclear dumping. Notice that the particular solution only 'kicks in' once the system receives the impulse—that makes physical sense! Also—what is the behavior? It's exactly what you expect—decaying exponentially (there are decaying oscillations if we're in the underdamped case).

Solution: So $f = f_i \delta(t - t_i)$. Let's compute the above convolution:

$$\begin{aligned} y_p = W * f &= \int_0^t \frac{1}{2\sqrt{b^2 - \omega_0^2}} \left[e^{r_+ \tau} - e^{r_- \tau}\right] u(\tau) f_i \delta(\tau - (t - t_i))\, \mathrm{d}\tau \\ &= \frac{f_i}{2\sqrt{b^2 - \omega_0^2}} \left[e^{r_+(t-t_i)} - e^{r_-(t-t_i)}\right] u(t - t_i) \\ &= \frac{e^{r_+ -(t-t_i)} - e^{r_-(t-t_i)}}{r_+ - r_-} f_i u(t - t_i) \end{aligned}$$

In class we mentioned that the δ is the identity element for the convolution operation, $\delta * g = g * \delta = g$ for any g. We also mentioned that convolution is associative, $(a * b) * c = a * (b * c)$, because the Laplace transform of both sides is ABC. Suppose we have some other signal, not $f(t) = \delta(t)$ (setting $t_i = 0$ and $f_i = 1$) but $g(t)$. The particular solution given f is $f * W = \delta * W = y_p$. Since $g = g * \delta$ the particular solution for g as input is $(g * \delta) * W = g * (\delta * W) = g * y_p$ where y_p is what we just found (with $t_i = 0$ and $f_i = 1$ plugged in—which is W itself!). The whole thing hangs together!

As Always

7. How long did this problem set take you? About 7 hours.

Optional Practice

8. Boas 8.10.1
9. Boas 8.10.2
10. Boas 8.10.3-12 are good practice for using the table or for applying partial fractions decomposition.
11. Boas 8.10.13

12. Boas 8.10.14,15 for practice solving differential equations.

13. Boas 8.10.17; you can also try it with $+a^2$ instead of $-a^2$.