Homework 11 — PHYS373 2021

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Linear Spaces

1. Suppose we have an inner product space, and two vectors in that space $|\, u \,\rangle$ and $|\, v \,\rangle$.

   (a) Suppose $\langle\, u \,|\, v \,\rangle = 0$, and let $|\, u+v \,\rangle = |\, u \,\rangle + |\, v \,\rangle$. Prove that $\langle\, u+v \,|\, u+v \,\rangle = \langle\, u \,|\, u \,\rangle + \langle\, v \,|\, v \,\rangle$. This is the *Pythagorean theorem.*

   Solution: Using linearity of inner products, we get

$$\langle\, u+v \,|\, u+v \,\rangle$$

$$= \langle\, u \,|\, u+v \,\rangle + \langle\, v \,|\, u+v \,\rangle$$
$$= \langle\, u \,|\, u \,\rangle + \langle\, u \,|\, v \,\rangle + \langle\, v \,|\, u \,\rangle + \langle\, v \,|\, v \,\rangle$$

   Since $|\, u \,\rangle$ and $|\, v \,\rangle$ are orthogonal, we have

$$= \langle\, u \,|\, u \,\rangle + 0 + 0 + \langle\, v \,|\, v \,\rangle = \langle\, u \,|\, u \,\rangle + \langle\, v \,|\, v \,\rangle$$

   (b) Let the vector $|\, \mu \,\rangle = |\, u \,\rangle - \mu |\, v \,\rangle$ for two generic vectors $|\, u \,\rangle$ and $|\, v \,\rangle$ (not necessarily orthogonal) and any complex number μ. Since $\langle\, \mu \,|\, \mu \,\rangle$ is the inner product of a vector with itself, it is at least 0; $\langle\, \mu \,|\, \mu \,\rangle \geq 0$. Show that this implies $\langle\, u \,|\, u \,\rangle - \mu \langle\, u \,|\, v \,\rangle - \mu^* \langle\, v \,|\, u \,\rangle + |\mu|^2 \langle\, v \,|\, v \,\rangle \geq 0$ for any μ at all whatsoever.

   Solution:
   Since $|\, \mu \,\rangle = |\, u \,\rangle - \mu |\, v \,\rangle$, $\langle\, \mu \,| = \langle\, u \,| - \mu^* \langle\, v \,|$
   $\langle\, \mu \,|\, \mu \,\rangle \geq 0$ for any μ, so

$$(\langle\, u \,| - \mu^* \langle\, v \,|)\,(|\, u \,\rangle - \mu |\, v \,\rangle)$$
$$= \langle\, u \,|\, u \,\rangle - \mu \langle\, u \,|\, v \,\rangle - \mu^* \langle\, v \,|\, u \,\rangle + \mu^* \mu \langle\, v \,|\, v \,\rangle$$
$$= \langle\, u \,|\, u \,\rangle - \mu \langle\, u \,|\, v \,\rangle - \mu^* \langle\, v \,|\, u \,\rangle + |\mu|^2 \langle\, v \,|\, v \,\rangle \geq 0$$

   (c) By a tricky choice of $\mu = \langle\, v \,|\, u \,\rangle / \langle\, v \,|\, v \,\rangle$ (which is *some* number as long as $\langle\, v \,|\, v \,\rangle \neq 0$!) show that one finds $0 \leq |\langle\, u \,|\, v \,\rangle|^2 \leq \langle\, u \,|\, u \,\rangle \langle\, v \,|\, v \,\rangle$. This is called *Schwarz's Inequality.*

   Solution: From the last part,

$$\langle\, u \,|\, u \,\rangle - \mu \langle\, u \,|\, v \,\rangle - \mu^* \langle\, v \,|\, u \,\rangle + |\mu|^2 \langle\, v \,|\, v \,\rangle \geq 0$$

Using $\mu = \langle\, v \mid u \,\rangle / \langle\, v \mid v \,\rangle$,

$$\langle\, u \mid u \,\rangle - \frac{\langle\, v \mid u \,\rangle}{\langle\, v \mid v \,\rangle} \langle\, u \mid v \,\rangle - \frac{\langle\, v \mid u \,\rangle^*}{\langle\, v \mid v \,\rangle} \langle\, v \mid u \,\rangle + \frac{|\langle\, v \mid u \,\rangle|^2}{\langle\, v \mid v \,\rangle^2} \langle\, v \mid v \,\rangle \geq 0$$

Also, since $\langle\, v \mid u \,\rangle \langle\, v \mid u \,\rangle^* = \langle\, v \mid u \,\rangle^* \langle\, v \mid u \,\rangle = |\langle\, v \mid u \,\rangle|^2$, and $\langle\, u \mid v \,\rangle = \langle\, v \mid u \,\rangle^*$,

$$\langle\, u \mid u \,\rangle - \frac{|\langle\, v \mid u \,\rangle|^2}{\langle\, v \mid v \,\rangle} - \frac{|\langle\, v \mid u \,\rangle|^2}{\langle\, v \mid v \,\rangle} + \frac{|\langle\, v \mid u \,\rangle|^2}{\langle\, v \mid v \,\rangle} \geq 0$$

$$\langle\, u \mid u \,\rangle - \frac{|\langle\, v \mid u \,\rangle|^2}{\langle\, v \mid v \,\rangle} \geq 0$$

$$\langle\, u \mid u \,\rangle \geq \frac{|\langle\, v \mid u \,\rangle|^2}{\langle\, v \mid v \,\rangle}$$

$$\langle\, u \mid u \,\rangle \langle\, v \mid v \,\rangle \geq |\langle\, v \mid u \,\rangle|^2$$

If you want to think in terms of familiar geometric language, $|\langle\, u \mid v \,\rangle|^2$ is like the (magnitude) square of the dot product of two vectors. That's got to be at least zero. Since $\vec{a} \cdot \vec{b} = ab\cos\theta$ where θ is the angle between them, the dot product is at most ab (when $\theta = 0$—the vectors are parallel).

(d) The zero vector is the vector whose length is zero. It gets a lot of special treatment. For example, a lot of times we don't write it as a ket, we just write 0, so $|\, u \,\rangle = |\, u \,\rangle + 0$. Rarely, you might find texts that write just $|\ \rangle$ (with nothing inside), 'the null ket'.[1] When we say its length is zero we mean $\langle\ |\ \rangle = 0$.

Use the inequality proved in part (c) to show that the inner product of the zero vector with *any* vector is zero. Note that you cannot let $|\, v \,\rangle$ be the null ket because $\langle\, v \mid v \,\rangle$ is in the denominator for μ—you've got to let $|\, u \,\rangle$ be the null ket.

Solution:
Let $|\, u \,\rangle = |\ \rangle$ and $|\, v \,\rangle$ is an arbitrary vector. By the Schwarz inequality,

$$\langle\, u \mid u \,\rangle \langle\, v \mid v \,\rangle \geq |\langle\, v \mid u \,\rangle|^2 \geq 0$$

$$\langle\ |\ \rangle \langle\, v \mid v \,\rangle \geq |\langle\, v \mid\ \rangle|^2 \geq 0$$

Clearly $\langle\ |\ \rangle = 0$, thus

$$0 \geq |\langle\, v \mid\ \rangle|^2 \geq 0$$

$$\Rightarrow |\langle\, v \mid\ \rangle|^2 = 0 \Rightarrow \langle\, v \mid\ \rangle = 0$$

This is what *justifies* writing the null ket as just 0 (without the | funny brackets ⟩) in the first place! Otherwise, we'd be adding a vector to a number which. . . ?

2. Consider a four-dimensional inner product space with orthonormal kets $|\, 1 \,\rangle$, $|\, 2 \,\rangle$, $|\, 3 \,\rangle$, and $|\, 4 \,\rangle$ which satisfy

$$\langle\, m \mid n \,\rangle = \delta_{mn} \qquad \text{(that's 16 equations, one for each combination of } m, n \in \{1, 2, 3, 4\}) \tag{1}$$

[1] But! We *do not* usually write the zero vector as $|\, 0 \,\rangle$. In quantum mechanics, for example, we often write $|\, 0 \,\rangle$ for the ground state—which is a state just like any other generic state, in that its norm isn't zero

Suppose we have two vectors $|\, f \,\rangle$ and $|\, g \,\rangle$ given in that basis in terms of their (complex) components f_i and g_i,

$$|\, f \,\rangle = \sum_{i=1}^{4} f_i \,|\, i \,\rangle \qquad\qquad |\, g \,\rangle = \sum_{i=1}^{4} g_i \,|\, i \,\rangle \tag{2}$$

(a) Show that $\langle\, f \mid g \,\rangle = \sum_{i=1}^{4} f_i^* g_i$.

Solution: Given $|\, f \,\rangle$, $\langle\, f \,| = \sum_{i=1}^{4} f_i^* \,\langle\, i \,|$

$$\begin{aligned}
\langle\, f \mid g \,\rangle &= \sum_{i=1}^{4} f_i^* \,\langle\, i \,|\sum_{j=1}^{4} g_j \,|\, j \,\rangle \\
&= \sum_{i=1}^{4}\sum_{j=1}^{4} f_i^* \,\langle\, i \,|\, g_j \,|\, j \,\rangle \\
&= \sum_{i=1}^{4}\sum_{j=1}^{4} f_i^* g_j \,\langle\, i \mid j \,\rangle \\
&= \sum_{i=1}^{4}\sum_{j=1}^{4} f_i^* g_j \delta_{ij} \\
&= \sum_{i=1}^{4} f_i^* g_i
\end{aligned}$$

(b) Suppose we restricted ourselves to the space of vectors where the third and fourth components were always equal. In other words, vectors of the form $|\, g \,\rangle = g_1 \,|\, 1 \,\rangle + g_2 \,|\, 2 \,\rangle + g_3(|\, 3 \,\rangle + |\, 4 \,\rangle)$. This subspace is 3-dimensional, in the sense that you've only got to specify 3 numbers (components) to determine any vector in it. However, they don't have the 'usual' inner product! Show that for these vectors the inner product can be written $\langle\, f \mid g \,\rangle = \sum_{i=1}^{4} f_i^* g_i = \sum_{i=1}^{3} f_i^* g_i w_i$ where $w_1 = 1$, $w_2 = 1$, and $w_3 = 2$.

Solution: $f_3^* = f_4^*$, so $f_3^* + f_4^* = 2f_3^*$, and the same is valid for g, so

$$\begin{aligned}
\langle\, f \mid g \,\rangle &= f_1^* g_1 + f_2^* g_2 + f_3^* g_3 + f_4^* g_4 \\
&= f_1^* g_1 + f_2^* g_2 + 2 f_3^* g_3 = \sum_{i=1}^{3} f_i^* g_i w_i
\end{aligned}$$

where $w_1 = 1$, $w_2 = 1$, and $w_3 = 2$.

(c) The general construction $\langle\, f \mid g \,\rangle = \sum_i f_i^* g_i w_i$ is called a *weighted* inner product; w_i is called the *weight* of component i. Show that if all the weights are real and positive, this inner product satisfies

1. $\langle\, f \mid g \,\rangle = \langle\, g \mid f \,\rangle^*$ (complex conjugation property)
2. $\langle\, f \mid f \,\rangle \geq 0$ (positive-semidefiniteness)
3. $\langle\, f \mid f \,\rangle = 0$ if and only if all the components of f are 0 (uniqueness of the zero vector).
4. $\langle\, f \mid g + h \,\rangle = \langle\, f \mid g \,\rangle + \langle\, f \mid h \,\rangle$ (linearity; $|\, g + h \,\rangle$ means $|\, g \,\rangle + |\, h \,\rangle$)
5. $\langle\, f \mid \alpha g \,\rangle = \alpha \,\langle\, f \mid g \,\rangle$ (linearity; $|\, \alpha g \,\rangle$ means $\alpha \,|\, g \,\rangle$)

(Notice that linearity in the bra, $\langle\, f + g \mid h \,\rangle = \langle\, f \mid h \,\rangle + \langle\, g \mid h \,\rangle$ and $\langle\, \alpha f \mid g \,\rangle = \alpha^* \langle\, f \mid g \,\rangle$ follow from combining the complex conjugation property with the linearity properties you showed for the ket.)

Solution:

1. $\langle\, g \mid f \,\rangle = \sum_i g_i^* f_i w_i$, so $\langle\, g \mid f \,\rangle^* = \sum_i g_i f_i^* w_i^*$ But since w_i is real, $w_i = w_i^*$ and
$$\langle\, g \mid f \,\rangle^* = \sum_i g_i f_i^* w_i^* = \sum_i g_i f_i^* w_i = \langle\, f \mid g \,\rangle$$

2. $$\langle\, f \mid f \,\rangle = \sum_i f_i^* f_i w_i = \sum_i |f_i|\, w_i \geq 0$$
since w_i is positive, this is a sum of all positive terms.

3. If $f = 0$, $\langle\, f \mid f \,\rangle = 0$ is clear per 1(d). (multiplying zeros by weights wouldn't change anything)
Now suppose $\langle\, f \mid f \,\rangle = 0$, thus
$$\langle\, f \mid f \,\rangle = \sum_i |f_i|\, w_i = 0$$
Since $w_i > 0$, $|f_i| = 0$, for all i. Therefore $f = 0$.

4. $$\langle\, f \mid g + h \,\rangle = \sum_i f_i^*(g_i + h_i) w_i = \sum_i f_i^* g_i w_i + \sum_i f_i^* h_i w_i = \langle\, f \mid g \,\rangle + \langle\, f \mid h \,\rangle$$

5. $$\langle\, f \mid \alpha g \,\rangle = \sum_i f_i^* \alpha g_i w_i = \alpha \sum_i f_i^* g_i w_i = \alpha \langle\, f \mid g \,\rangle$$

(d) Which property/properties in part (c) wouldn't be true if the weights w were allowed to be complex?

Solution: 1, since we use the fact that $w_i = w_i^*$.

(e) Which property/properties in part (c) wouldn't be true if the weights w were allowed to be negative?

Solution: Considering non-zero negatives, 2, since we use that $w_i > 0$ to ensure it's a sum of positive terms.

(f) Which property/properties in part (c) wouldn't be true if the weights were all real but some of the weights were allowed to be zero?

Solution: 3. If a weight is zero, it can 'kill' a non-zero term of a vector. A vector with a non-zero term isn't the null vector, but this way it could have a 0 product.

(g) You don't have to write anything, but convince yourself that for functions, the *weighted inner product* $\langle\, f \mid g \,\rangle = \int f^*(x) g(x)\; w(x)\; \mathrm{d}x$ for some finite or infinite interval of x has all the nice part-(c)-properties as long as the *weight function* $w(x)$ is a real, positive function.

3. Suppose you have a complete orthogonal (but ortho*normal*) basis of kets $\{|\, i\, \rangle\}$, $\langle\, i \mid j\, \rangle = n_i^2 \delta_{ij}$ for all i and j, where n_i is the *normalization* if $|\, i\, \rangle$; n_i is a positive, real number. Since it's a complete basis we can write any vector $|\, f\, \rangle = \sum_i f_i \,|\, i\, \rangle$ in that basis.

   (a) Show that $f_j = \langle\, j \mid f\, \rangle / \langle\, j \mid j\, \rangle$.

   Solution:

$$\frac{\langle\, j \mid f\, \rangle}{\langle\, j \mid j\, \rangle} = \frac{\langle\, j \,|\sum_i f_i \,|\, i\, \rangle}{n_j^2} = \sum_i f_i \frac{\langle\, j \mid i\, \rangle}{n_j^2} = \sum_i f_i \frac{n_j^2 \delta j i}{n_j^2} = \sum_i f_i \delta j i = f_j$$

   (b) Suppose you have another vector $|\, g\, \rangle$ (defined like $|\, f\, \rangle$, but with components g_i in the basis we're discussing). Show that $\langle\, f \mid g\, \rangle = \sum_i f_i^* g_i w_i$ where $w_i = n_i^2$. *In other words, a weighted inner product comes up in a basis where the kets aren't necessarily normalized to 1.*

   Solution:

$$\begin{aligned}
\langle\, f \mid g\, \rangle &= \sum_i f_i^* \,\langle\, i \,|\sum_j g_j \,|\, j\, \rangle \\
&= \sum_i \sum_j f_i^* \,\langle\, i \,|\, g_j \,|\, j\, \rangle \\
&= \sum_i \sum_j f_i^* g_j \,\langle\, i \mid j\, \rangle \\
&= \sum_i \sum_j f_i^* g_j n_i^2 \delta_{ij} = \sum_i f_i^* g_i n_i^2
\end{aligned}$$

Operators

4. In HW07Q9 and HW09Q1 we discussed the *translation operator*. We said $T(\Delta)f(x) = f(x-\Delta)$. Suppose that $|\, f \,\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x f(x) \,|\, x \,\rangle$. Consider the operator

$$T(\Delta) = \int_{-\infty}^{+\infty} \mathrm{d}x \,|\, x+\Delta \,\rangle \;\langle\, x \,| \tag{3}$$

which replaces any $|\, x \,\rangle$ with the ket $|\, x+\Delta \,\rangle$ instead. Show that when we apply the operator $T(\Delta)$ to $|\, f \,\rangle$ we get

$$T(\Delta) \,|\, f \,\rangle = \int_{-\infty}^{+\infty} f(x-\Delta) \,|\, x \,\rangle \,. \tag{4}$$

Solution:

5. The differentiation operator D is the[2] *limit*,

$$D = \lim_{\epsilon \to 0} \frac{T(0) - T(\epsilon)}{\epsilon}; \tag{5}$$

where what we *mean* is: first apply the operator $T(0)$ and $T(\epsilon)$, then take the limit $\epsilon \to 0$. Show that when applied to $|\, f \,\rangle = \int \mathrm{d}x f(x) \,|\, x \,\rangle$,

$$D \,|\, f \,\rangle = \int \mathrm{d}x f'(x) \,|\, x \,\rangle \tag{6}$$

which we can call $|\, f' \,\rangle$. (Use the result from the previous question!; you can assume the limit commutes with any integration and the necessary limits exist)

Solution:

6. Consider the space of complex-valued functions on $x = (-\infty, +\infty)$. The *Laplacian* L is a linear operator given by

$$L = D^2 \tag{7}$$

where D is the differentiation operator from the previous question. (D^2 means: first apply D and then apply D again.)

(a) Suppose we have a vector $|\, f \,\rangle = \int \mathrm{d}x f(x) \,|\, x \,\rangle$. Show that $L \,|\, f \,\rangle = |\, f'' \,\rangle = \int \mathrm{d}x f''(x) \,|\, x \,\rangle$. (*Use the result from the previous problem!*)

Solution:

(b) Recall that the *Fourier basis* is given by the vectors

$$|\, k \,\rangle = \int \mathrm{d}x \; e^{ikx} \,|\, x \,\rangle \qquad\qquad \langle\, p \,|\, k \,\rangle = 2\pi\delta(p-k) \tag{8}$$

for all $k \in \mathbb{R}$. Show that $L \,|\, k \,\rangle = -k^2 \,|\, k \,\rangle$. (Hint: use the result from the previous part!) (In other words, the *Fourier basis* is made up of eigenstates of the Laplacian; the eigenvalue corresponding to $|\, k \,\rangle$ is $-k^2$. Since all the eigenvalues are real, the Laplacian is a Hermitian operator.)

[2] You could take a symmetrical definition $\lim_{\epsilon \to 0}[T(-\epsilon) - T(+\epsilon)]/2\epsilon$ or some other sensible definition instead; they all agree in the $\epsilon \to 0$ limit.

Solution:

(c) Check that in the Fourier basis we may write

$$L = \int \frac{\mathrm{d}k}{2\pi} \mid k \,\rangle \, (-k^2) \, \langle\, k \mid \tag{9}$$

by applying this operator to $\mid p\,\rangle$ (a vector in the Fourier basis) to get $L \mid p\,\rangle = -p^2 \mid p\,\rangle$. (Hint: it will be easier to stay in the Fourier basis and use the orthogonality relation $\langle\, k \mid p\,\rangle = 2\pi\delta(p-k)$.)

Solution:

Prep Work for Spherical Coordinates

7. Consider 3-dimensional space with coordinates x, y, and z. We can describe any point by its 3 cartesian coordinates (x, y, z), but can also describe any point by its *spherical coordinates* (r, θ, ϕ), as shown in the figure below. (Another figure is available in Boas 5.4 by equation 4.5.) While x, y, and z can take any value $(-\infty, +\infty)$, and each point has only one set of coordinates in that basis, we have to be more careful with the r, θ, and ϕ coorinates. First of all, ϕ is 2π-periodic—if you increase ϕ by 2π you get back to the same point. So ϕ can go from 0 to 2π (or from $-\pi$ to $+\pi$, or any other convenient choice—functions are always 2π-periodic in ϕ).

Next, θ can go from 0 to π; you can describe a point which has a θ that's more than π by increasing ϕ by π and using a θ that's less than π. In other words, a point $(r, \theta > \pi, \phi)$ can also be described by $(r, 2\pi - \theta, \phi + \pi)$. Finally, r can go from 0 to ∞; a point with coordinates $(-r, \theta, \phi)$ can be described by $(+r, \pi - \theta, \phi + \pi)$ instead.

(You can either do part a using geometry and then part b via algebra, or part b using geometry and then part a via algebra, whichever you find easier! Maybe do both from geometry. Up to you!)

(a) Use the normal rules of euclidean geometry to show

$$r^2 = x^2 + y^2 + z^2 \qquad \tan\phi = \frac{y}{x} \qquad \cos\theta = \frac{z}{r} \tag{10}$$

Solution:

(b) Show that these relationships may also be solved for x, y, and z to give

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{11}$$

Solution:

(c) When we compute integrals over space using cartesian coordinates it's easy; $\int \mathrm{d}V = \int \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$. But if we integrate over space it's not just $\int \mathrm{d}V \neq \int \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi$. For one thing, the units are wrong! $\mathrm{d}V$ is supposed to have dimensions of length^3 but since θ and ϕ are angles they don't have dimension (or, if you like, their dimensions are radians; certainly not length, anyway!). So, what went wrong? We need to include the *Jacobian*.

The Jacobian is given by

$$J = \det\begin{pmatrix} \partial_r x & \partial_r y & \partial_r z \\ \partial_\theta x & \partial_\theta y & \partial_\theta z \\ \partial_\phi x & \partial_\phi y & \partial_\phi z \end{pmatrix} \tag{12}$$

Show that $J = r^2 \sin\theta$. (See Boas 5.4 eq (4.14) for help.)

Solution:

Therefore, if we integrate some function f over some region of space $\int \mathrm{d}V f$ we can either day the integral is $\int \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z f(x, y, z)$ or $\int r^2\ \mathrm{d}r \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi f(r, \theta, \phi)$ over the same region of space (but written in different coordinates). We usually say $f(r, \theta, \phi)$ doesn't mean plug in r where you see x, θ for y, and ϕ for z—instead it means you should plug in x, y, and z in terms of the r, θ, and ϕ variables, given by (11).

(d) Sometimes it's convenient to think in terms of $u = \cos\theta$. What is $\mathrm{d}u$ in terms of θ and $\mathrm{d}\theta$? If θ goes from 0 to π, what values does u take? Rewrite $\int_0^\pi \mathrm{d}\theta \sin\theta f(\cos\theta)$ as an integral over u.

Solution:

As Always

8. How long did this problem set take you?

Optional Practice

9. Boas 3.10.7 and 3.10.8
10. Boas 3.10.10 (the triangle inequality; the notation $||f||$ means $\sqrt{\langle f \mid f \rangle}$, see 3.10 eqn 10.2.)
11. Find the Jacobian for *cylindrical coordinates*, $(r, \phi, z) = (\sqrt{x^2 + y^2}, \arctan y/x, z)$.