Project 1

Double Pendulum

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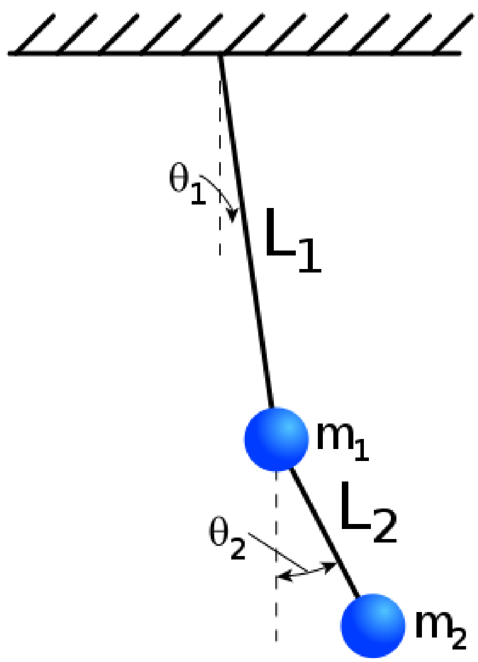


Wednesday, March 1st, 2023, 11:59 PM

1 Introduction

A double pendulum as depicted above is considered. Where the initial conditions are as follows: $\theta_1 = \theta_2 = \pi/2$ are the angular positions, $m_1 = m_2 = 1$kg are the masses, $\ell_1 = \ell_2 = \ell = 0.40$m is the length, and $\omega_1 = \omega_2$ are the angular velocities. The goal of this project is to numerically model the positions and energies of this double pendulum for one hundred seconds to determine: the amount of total energy lost in the system; how our step sizes dt can affect the total energy lost; and how the system changes depending on θ.

1.1 Equations

The equations to describe the motion are as follows:

$$\frac{d\theta_1}{dt} = \omega_1$$

$$\frac{d\theta_2}{dt} = \omega_2$$

$$\frac{d\omega_1}{dt} = -\frac{\omega_1^2 \sin(2\theta_1 - 2\theta_2) + 2\omega_2^2 \sin(\theta_1 - \theta_2) + (g/\ell)[\sin(\theta_1 - 2\theta_2) + 3\sin\theta_1]}{3 - \cos(2\theta_1 - 2\theta_2)}$$

$$\frac{d\omega_2}{dt} = \frac{4\omega_1^2 \sin(\theta_1 - \theta_2) + \omega_2^2 \sin(2\theta_1 - 2\theta_2) + 2(g/\ell)[\sin(2\theta_1 - \theta_2) - \sin\theta_2]}{3 - \cos(2\theta_1 - 2\theta_2)}$$

The equations to describe the total energy of the system along with a derivation of total energy in terms of θ_1, θ_2, ω_1, and ω_2 with constants g, ℓ, and m, are as follows:

$$E = T + V$$

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$

$$x_1 = \ell \sin(\theta_1) \qquad y_1 = \ell \cos(\theta_1)$$

$$x_2 = \ell \sin(\theta_1) + \ell \sin(\theta_2) \qquad y_2 = \ell \cos(\theta_1) + \ell \cos(\theta_2)$$

$$\dot{x}_1 = \ell \cos\theta_1 \dot{\theta}_1 \qquad \dot{y}_1 = \ell \sin\theta_1 \dot{\theta}_1$$

$$\dot{x}_2 = \ell \cos(\theta_1 \dot{\theta}_1) + \ell \cos\theta_2 \dot{\theta}_2 \qquad \dot{y}_2 = \ell \sin\theta_1 \dot{\theta}_1 + \ell \sin\theta_2 \dot{\theta}_2$$

$$T = \frac{1}{2} ml^2 (2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2$$

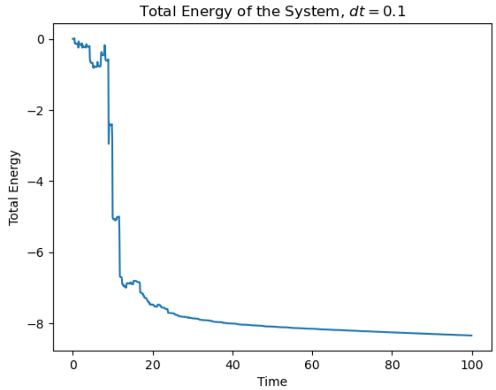
$$V = mgy_1 + mgy_2 = -mgl(2\cos\theta_1 + \cos\theta_2)$$

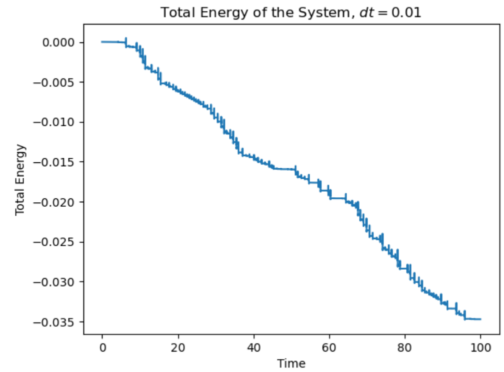
Total Energy is then:

$$E = \left[\frac{1}{2} m\ell^2 (2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2\right] + \left[-mgl(2\cos\theta_1 + \cos\theta_2)\right]$$

2 Numerical Methods & Results

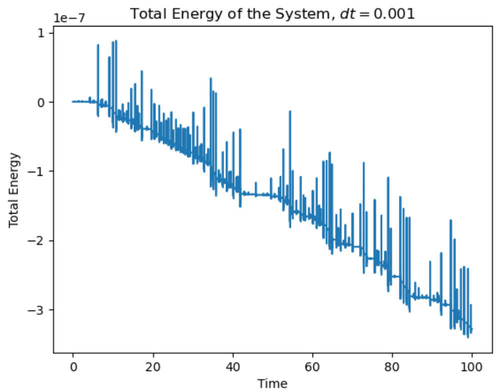
For this project, the tools at disposal were Euler, Euler-Cromer, And Runge-Kutta. Euler is less accurate and less stable than Runge-Kutta. Utilizing Euler would only provide results that were first-order accurate, while a second-order Runge-Kutta is second-order accurate. The good news with this is that we can extend Runge-Kutta to achieve higher accuracy. The best bang-for-buck order of accuracy for Runge-Kutta is fourth-order, which is the method implemented. However, while this method is superior to Euler it still provides some issues. If the step size incorporated is not small enough, then there will be a considerable amount of drift in the results for the total energy, as seen in figure 1 below.

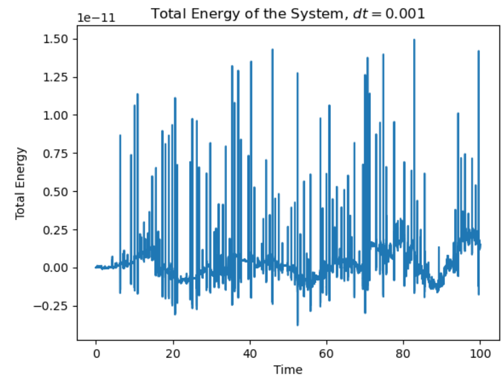




(a) $E(100s) \approx -8.347$, $dt = 0.1$

(b) $E(100s) \approx -0.03472$, $dt = 0.01$



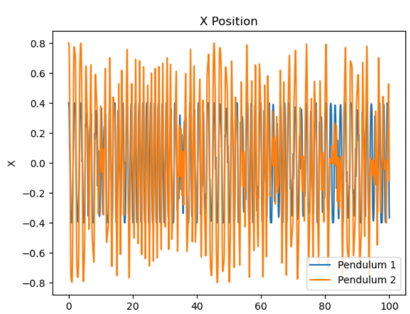


(c) $E(100s) \approx -3.2835E-07$, $dt = 0.001$

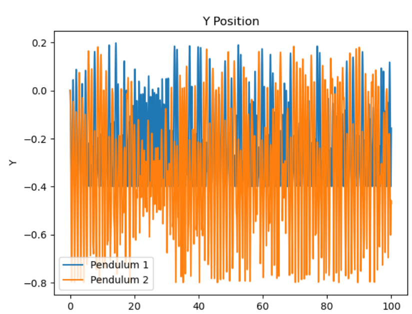
(d) $E(100s) \approx 1.50E-12$, $dt = 0.0001$

Figure 1: How the total energy changes with dt.

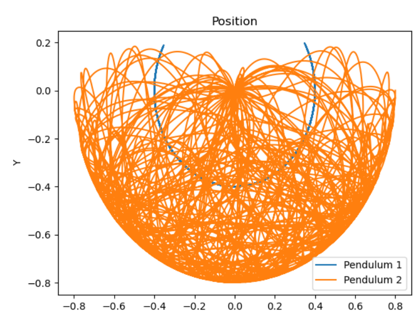
From this, the total energy of the system is losing less than 10^{-5} J when $dt = 0.001$ or smaller. In terms of the positions of the double pendulum, using the predefined initial conditions and a dt of 0.001, the plots are also seen below.



(a) x(t)



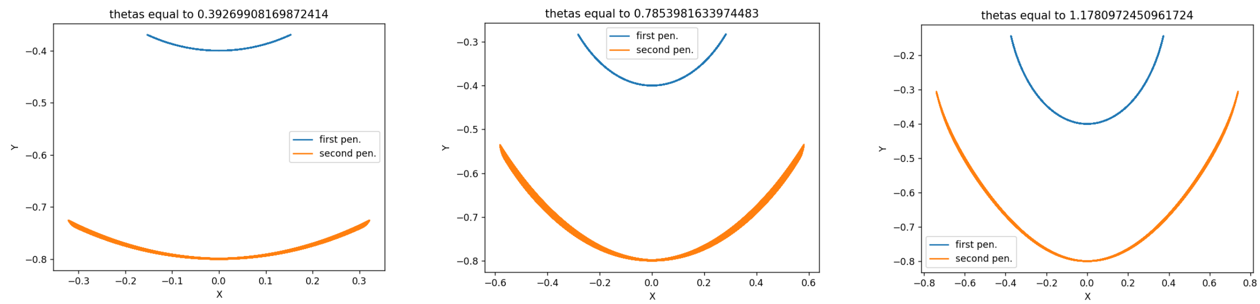
(b) y(t)



(c) y(x)

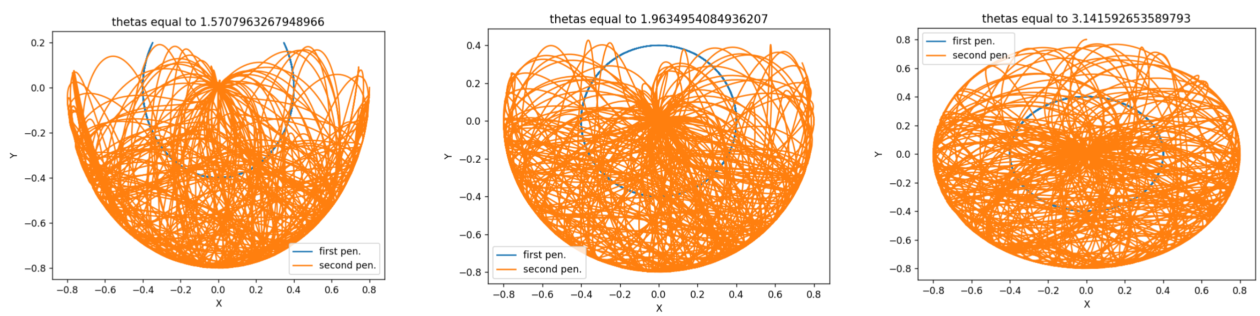
Figure 2: Positions of Pendulums

However, these positions are all from those initial conditions stated earlier. If the thetas are altered, the system will become more or less chaotic. To observe how the initial angles affect just how chaotic the system becomes the following plots have been made.



(a) $\theta \approx 0.39$ (b) $\theta \approx 0.78$ (c) $\theta \approx 1.18$

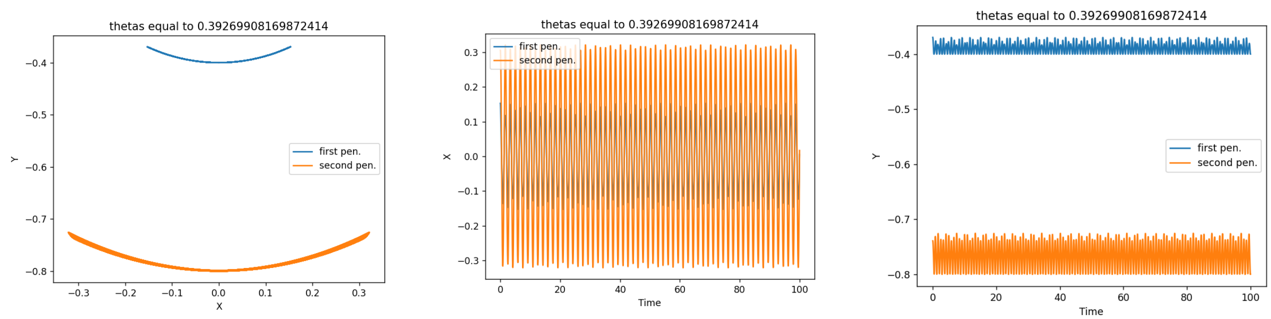
Figure 3: Linear non-chaotic systems when theta is low in terms of y(x).



(a) $\theta \approx 1.57$ (b) $\theta \approx 1.96$ (c) $\theta \approx 3.14$

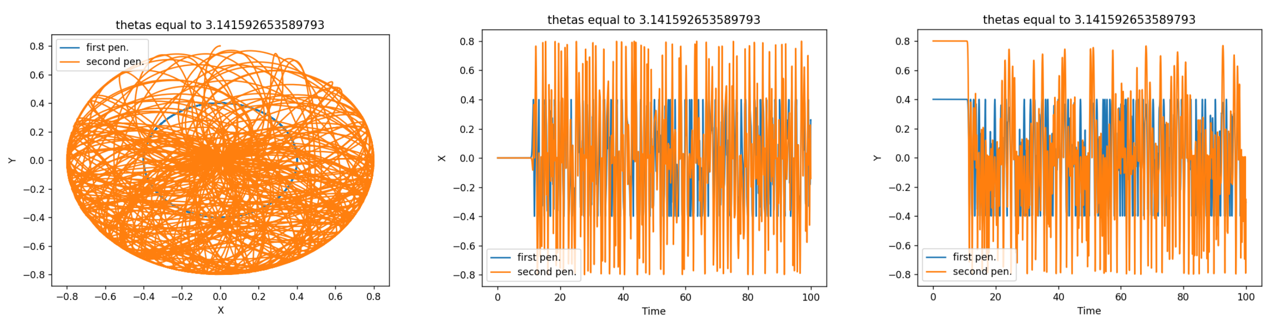
Figure 4: Chaotic systems when theta is high in terms of y(x).

As for specific plots of x(t) and y(t) for both the linear and chaotic case the following plots on the next page are helpful.



(a) $y(x)$ (b) $x(t)$ (c) $y(t)$

Figure 5: Plots of a Linear system for $\theta \approx 0.39$.



(a) $y(x)$ (b) $x(t)$ (c) $y(t)$

Figure 6: Plots of a chaotic system when theta is $\theta \approx 3.14$.

Something interesting to note from the last chaotic set of plots is that the system is constant for a short amount of time, but then rapidly becomes chaotic. In general, when theta is low the system will act in a more predictive manner, but when theta is high the system is far more difficult to predict without numerical tools. The numerical tools must also be stable and accurate otherwise large amounts of drift will occur in the data much like what occurred in the total energy calculations for large steps of fourth-order Runge-Kutta. In the future, if this project wanted to be revisited a question that could be answered is the differences in time and space complexity of Runge-Kutta and other potential methods like the leap-frog method.