Astronomical Formulæ for Calculators

FOURTH EDITION
Enlarged & Revised

Jean Meeus

$$k = \frac{1 + \cos i}{2}$$

$$\begin{aligned}\beta = &+5^\circ.1282 \sin F \\ &+0^\circ.2806 \sin (F + M') \\ &+0^\circ.2777 \sin (M' - F) \\ &+0^\circ.1732 \sin (2D - F)\end{aligned}$$

Foreword by
Roger W. Sinnott
of **Sky and Telescope** Magazine

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Foreword

Astronomy and calculating techniques have enjoyed a long and fruitful association. Over 150 years ago, so the story goes, young Charles Babbage at the University of Cambridge in England became infuriated at the number of errors he found in some astronomical tables. He muttered that he wished they had been produced by steam machinery, instead of by humans, and his friend John Herschel replied, "It is quite possible!"

This apparently offhand remark launched Babbage's lifelong obsession to design and build various calculating engines – the forerunners of today's computers. In the early 1950's, soon after the first "electronic brains" were constructed, their earliest applications included the wholesale calculation of tables and ephemerides for the astronomical almanacs.

The arrival of pocket calculators and home computers in the 1970's marked yet another advance. Logarithmic and trigonometric tables became outmoded almost overnight. Working alone at home or on a ship at sea, anyone could now perform calculations for navigation, practical astronomy, or celestial mechanics, without access to preprinted tables or a giant computer. So novel was the approach that virtually no book existed that was devoted, purely from scratch, to the art of astronomical calculation.

The present work fills this need brilliantly. Any serious amateur or student of astronomy will find it a superb companion to the introductory texts of the classroom. All too often, the modern educational trend is to present important concepts vaguely. A teacher might say, for example, that nutation is a "nodding" of the Earth's axis, and then try to explain the cause. How much more meaningful it is, and how rare in any classroom book, to find in Chapter 15 the actual formulae describing this effect!

The motions of the Moon and most planets were analyzed in great detail around the turn of the present century. The results are generally locked away in dust-laden observatory annals, to be consulted only at special libraries. One can look up, for instance, E.W. Brown's lifework on the Moon, published in 1919. To find the Moon's position at a particular date and time to full precision requires evaluating over 1,650 trigonometric terms. The modern texts on celestial mechanics, understandably, omit these concrete formulae and dwell on theory, leaving the practical-minded person in the dark. However, it is in fact much easier to evaluate the main terms on a pocket calculator than it is to understand the theory.

Jean Meeus has done an important service by consulting these classic treatises on the Moon and planets, selecting the main perturbations from among a multitude of smaller effects, and presenting them concisely here. In doing this, he has carefully included the so-called secular terms, which are small or nearly constant at the present time, but which grow sizable a few centuries in the past and future. Accordingly, the formulae in this book (unlike those often found elsewhere) may safely be used in serious historical investigations.

Worthwhile discoveries, too laborious a few years ago, probably await anyone who cares to spend pleasant evenings with a pocket calculator. Consider the faint star that Galileo plotted near Jupiter's satellites on December 28, 1612. The world did not know until 1980 that this "star" was actually the planet Neptune. Yet a discovery of this kind could easily be made by some reader of the present book! All the needed formulae are here – the positions of Jupiter and Neptune are deduced by the methods of Chapters 22 – 25, and the configuration of Jupiter's satellites, as drawn by the great Italian astronomer on that evening in 1612, are easily confirmed by the method of Chapter 36.

This book includes ingenious methods of the author's own devising, such as the simple formulae for the times of the equinoxes and solstices in Chapter 20. His many other books and articles, exhibiting astronomical calculations of all sorts, have won him high esteem in professional circles over the years. In 1981 the International Astronomical Union announced the naming of asteroid 2213 Meeus in his honor. The first edition of **Astronomical Formulae for Calculators** was eagerly snapped up in a few months. Now that it is available again in a revised and expanded form, it is sure to become a classic.

Roger W. Sinnott
Sky and Telescope Magazine

PREFACE

TO THE FIRST (BELGIAN) EDITION

With the spectacular rise of the pocket calculating machines and the even spectacular fall of their prices in recent years, these wonderful machines are now within reach of everyone. The number of amateur astronomers possessing such calculating machines nearly equals now the number of amateur astronomers themselves. The number of the latter who own a programmable calculating machine is already impressive too and always growing. It is mainly for this last category of interested persons that this book is intended.

Anyone who endeavours to make astronomical calculations has to be very familiar with the essential astronomical conceptions and rules and he must have sufficient knowledge of elementary mathematical techniques. As a matter of course he must have a perfect command of his calculating machine, knowing all possibilities it offers the competent user. However, all these necessities don't suffice. Creating useful, successful and beautiful programs requires much practice. Experience is the mother of all science. This general truth is certainly valid for the art of programming. Only by experience and practice one can learn the innumerable tricks and dodges that are so useful and often essential in a good program.

Astronomical Formulae for Calculators intends to be a guide for the amateur astronomer who wants to do calculations. Before I specify briefly the aims and contents of the book, let me outline first what it is not.

This book is not a general textbook on Astronomy. Elementary astronomical knowledge is taken for granted. For instance, no definitions are given of right ascension and declination, ecliptic, precession, magnitude, etc., but all these notions are used continually throughout the book. Only exceptionally a definition will be given. Nor is this book a textbook on mathematics or a manual for programmable pocket calculators. As I said, the reader is assumed to be able to use his machine appropriately.

What this book intends is to lend a helping hand to every amateur astronomer with mathematical interests and to give him much practical information, advice and examples. About forty topics in the field of calendar problems, celestial phenomena and celestial mechanics are dealt with, and also a few astronomy oriented mathematical techniques, as interpolation and linear regression. For all these cases there is an outline of the problem, its meaning and its signification. The formulae describing the problem in mathematical terms are given and treated at some lenght so as to enable the reader to use them for making his own programs. Many numerical examples are then offered to illustrate the subject and the applications of the formulae.

No programs are given. The reasons are clear. A program is useful only for one type of calculating machine. For instance, a program for a HP-67 cannot be used on a TI-59, and even not on a HP-65. Every calculator thus must learn to create his own programs. There is the added circumstance that the precise contents of a program usually depend on the specific goals of the computation, that are impossible to anticipate always by the author.

The writing of a program to solve some astronomical problem sometimes will require a study of more than one chapter of this book. For instance, in order to create a program for the calculation of the Sun's altitude for a given time of a given date at a given place, one must first convert the date and time to Julian Date (Chapter 3), then calculate the Sun's longitude for that time (Chapter 18), its right ascension (Chapter 8), the sidereal time (Chapter 7), and finally the required altitude of the Sun (Chapter 8).

It is clear that not all topics of mathematical astronomy could have been dealt with in this book. So nothing is said about orbit determination, occultations of stars by the Moon, the calculation of the longitude of the central meridian of Mars and Jupiter for a given instant, meteor astronomy, eclipsing binaries, etc. However, a hasty look on the Table of Contents convinces there is enough fascinating material in this fourth monograph on astronomy and astrophysics edited by *Urania* and *VVS*, to keep every amateur busy for years to come.

G. Bodifée

PREFACE TO THE SECOND EDITION

In this second edition several misprints and errors have been corrected. The principal change in the new edition is the addition of some material, such as a method for the calculation of the date of Easter in the Julian Calendar, the principal periodic terms in the motion of Uranus and Neptune, and a formula for the direct calculation of the equation of the center from the orbital eccentricity and the mean anomaly.

The chapter on the positions of the satellites of Jupiter has been improved, while a new chapter has been added, namely concerning the calculation of the longitude of the central meridian of Jupiter, and of the planetocentric latitude of the center of its disk.

Since the first edition of this book (November 1978), many types of microcomputers became available at reasonable prices, and many amateur astronomers have purchased such a powerful machine. With a microcomputer it's possible to write much more sophisticated and longer programs, and to perform much more accurate calculations, than with a pocket calculator. For instance, the position of the Sun can be calculated with full accuracy, the times of a solar eclipse can be found with a precision better than one second, and so on.

Nevertheless, the present book remains intended principally for the owner of a pocket calculator. Detailed methods and formulae, designed for programs on a microcomputer, are wholly outside the scope of this book : they would require a much larger work than this one.

Jean Meeus

April 1982

PREFACE TO THE FOURTH EDITION

The only change in this edition is the addition of a chapter on the calculation of the heliocentric position of Pluto.

Jean Meeus

October 1988

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Some Symbols and Abbreviations

e	Eccentricity (of an orbit)
h	Altitude above the horizon
r	Radius vector, or distance of a body to the Sun, in AU
v	True anomaly
A	Azimuth
H	Hour angle
M	Mean anomaly
R	Distance from Sun to Earth, in AU
α	Right ascension
δ	Declination
ε	Obliquity of the ecliptic
θ	Sidereal time
θ_0	Sidereal time at Greenwich
π	Parallax
ϕ	Geographical latitude
ϕ'	Geocentric latitude
Δ	Distance to the Earth, in AU
ΔT	Difference ET − UT
$\Delta\varepsilon$	Nutation in obliquity
$\Delta\psi$	Nutation in longitude
$\odot$	Geocentric longitude of the Sun
AU	Astronomical unit
ET	Ephemeris Time
UT	Universal Time
JD	Julian Day
INT	Integer part of
A.E.	Astronomical Ephemeris
IAUC	International Astronomical Union Circular

Following a general astronomical practice (see for instance the *A.E.*), small superior symbols are placed immediately above the decimal point, not after the last decimal. For instance, $28\overset{\circ}{.}5793$ means 28.5793 degrees.

Moreover, note carefully the difference between hours with decimals, and hours - minutes - seconds. For example, $1\overset{h}{.}30$ is *not* 1 hour and 30 minutes, but 1.30 hours, that is 1 hour and 30 hundredths of an hour, or 1^h18^m.

The author wishes to express his gratitude to Mr. G. Bodifée, for his valuable advice and assistance.

1

Hints and Tips

To explain how to calculate or to program on a calculating machine is out of the scope of this book. The reader should, instead, study carefully his instructions manual. However, even then writing good programs cannot be learned in the lapse of one day. It is an art which can be acquired only progressively. Only by practice one can learn to write better and shorter programs.

In this first Chapter, we will give some practical hints and tips, which may be of general interest.

Accuracy

The accuracy of a calculation depends on its aims. If one only wants to know whether an occultation by the Moon will be visible in some countries, an accuracy of 100 kilometers in the northern or southern limit of the region of visibility is probably sufficient ; however, if one wants to organize an expedition to observe a grazing occultation by the Moon, the limit has to be calculated with an accuracy better than 1 kilometer.

If one wants to calculate the position of a planet with the goal of obtaining the moments of rise or setting, an accuracy of 0.01 degree is sufficient. But if the position of the planet is needed to calculate the occultation of a star by the planet, an accuracy of better than 1″ will be necessary because of the small size of the planet's disk.

To obtain a better accuracy it is sometimes necessary to use another method of calculation, not just to keep more decimals in the result of an approximate calculation. For example, if one has to know the position of Mars with an accuracy of 0.1 degree, it suffices to use an unperturbed elliptical orbit (Keplerian motion) although secular perturbations of the orbit are to be taken into account eventually. However, if the position of Mars is to be known with a precision of 10″ or better, perturbations due to the other planets have to be calculated and the program will be a much longer one.

So the calculator, who knows his formulae and the desired accuracy in a given problem, must himself considerate which terms, if any, may be omitted in order to keep the program handsome and as short as possible. For instance, the geometric mean longitude of the Sun, referred to the mean equinox of the date, is given by

$$L = 279°41'48''.04 + 129\,602\,768''.13\,T + 1''.089\,T^2$$

where T is in Julian centuries of 36525 ephemeris days from the epoch 1900 January 0.5 ET. In this expression the last term (secular acceleration of the Sun) is smaller than 1″ if $|T| < 0.96$, that is between the years 1804 and 1996. If an accuracy of 1″ is sufficient, the term in T^2 may thus be dropped for any instant in that period. But for the year −100 we have $T = -20$, so that the last term becomes 436″, which is larger than 0.1 degree.

Rounding

Rounding should be made where it is necessary. Do not retain meaningless decimals in your result. Some "feeling" and sufficient astronomical knowledge are necessary here. For instance, it would be completely irrelevant to give the illuminated fraction of the Moon's disk with an accuracy of 0.000 000 001.

If one calculates by hand and not with a program, the rounding should be performed *after* the whole calculation has been made.

Example : Calculate 1.4 + 1.4 to the nearest integer. If we first round the given numbers, we obtain 1 + 1 = 2. In fact, 1.4 + 1.4 = 2.8, which is to be rounded to 3.

Rounding should be made to the nearest value. For instance, 15.88 is to be rounded to 15.9 or to 16, not to 15. However, calendar dates and years are exceptions. For example, March 15.88 denotes an instant belonging to March 15 ; thus, if we read that an event occurs on March 15.88, it takes place on March 15, not on March 16. Similarly, 1977.69 denotes an event occurring in the year 1977, not 1978.

Trigonometric functions of large angles

Large angles frequently appear in astronomical calculations. In Example 18.a we find that on 1978 November 12.0 the Sun's mean longitude is 28670.77554 degrees. Even larger angles are found for rapidly moving objects, such as the Moon or the bright satellites of Jupiter.

According to the type of the machine, it may be necessary or desirable to reduce the angles to the range 0 − 360 degrees. Some calculating machines give incorrect values for the trigonometric functions of large angles. For instance,

the HP-55 gives	sin 360000030° =	0.499 481 3556
TI-52		0.499 998 1862
Casio fx 2200		Error

while the HP-67 gives the correct value 0.500 000 0000.

Angle modes

The calculating machines do not calculate directly the trigonometric functions of an angle which is given in degrees, minutes and seconds. Before performing the trigonometric function, the angle should be converted to degrees and *decimals*. Thus, to calculate the cosine of 23°26′49″, first convert the angle to 23.44694444 degrees, and *then* press the key COS.

Similarly, angles should be converted from degrees, minutes and seconds to degrees and decimals, before they can be interpolated. For instance, it is impossible to apply an interpolation formula directly to the values

5°03′45″
5°34′22″
6°17′09″

Right ascensions

Right ascensions are generally expressed in hours, minutes and seconds. If the trigonometric function of a right ascension must be calculated, it is thus necessary to convert that right ascension to degrees. Remember that one hour corresponds to 15 degrees.

Example 1.a : Calculate $\tan \alpha$, where $\alpha = 9^h 14^m 55^s\!.8$.

We first convert α to hours and decimals :

$$9^h 14^m 55^s\!.8 = 9.248\,833\,333 \text{ hours}.$$

Then, by multiplying by 15,

$$\alpha = 138^\circ\!.73250$$

whence $\tan \alpha = -0.877\,517$.

The correct quadrant

When the sine, the cosine or the tangent of an angle is known, the angle itself can be obtained by pressing the corresponding key : arc sin, arc cos, or arc tan, sometimes written as $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$. — The latter are, in fact, incorrect designations, for x^{-1} is the same as $1/x$. But $\cos^{-1} x$ is (incorrectly) used to designate the inverse function, and *not* $1/\cos x$.

In this case, on most pocket calculating machines, arc sin and arc tan give an angle lying between −90 and +90 degrees, while arc cos gives a value between 0 and +180 degrees.

In some cases, the result obtained in this way may not be in the correct quadrant. Each problem must be examined separately. For instance, formulae (8.4) and (25.15) give the sine of the declination. The instruction arc sin will then give the declination always in the correct quadrant, because all declinations lie between −90 and +90 degrees.

This is also the case for the angular separation whose cosine is given by formula (9.1). Indeed, any angular separation lies between the values 0° and 180°, and this is precisely what the operation arc cos gives.

When the tangent of an angle is given, for example by means of formulae (8.1), (8.3) and (18.3), the angle may be obtained directly in the correct quadrant by using a trick : the rectangular/polar transformation applied to the numerator and the denominator of the fraction in the right-hand member of the formula, as explained in Chapter 8 and at some other places in this book.

Powers of time

Some quantities are calculated by means of a formula containing powers of the time (T, T^2, T^3, ...). It is important to note that such polynomial expressions are valid only for values of T which are not too large. For instance, the formula

$$e = 0.016\,751\,04 - 0.000\,0418\,T - 0.000\,000\,126\,T^2$$

given in Chapter 18 for the eccentricity of the Earth's orbit, is valid only for several centuries before and after the year 1900, and *not* for millions of years ! For instance, for $T = 1000$, the above-mentioned formula gives $e = -0.151 < 0$, an absurd result.

The same is true for instance for formula (18.4), which would give the completely invalid results $\varepsilon = 0°$ for $T = -383$, and $\varepsilon = 90°$ for $T = +527$.

One should further carefully note the difference between periodic terms, which remain small throughout the centuries, and secular terms (terms in T^2, T^3, ...) which rapidly increase with time.

In formula (32.1), for instance, the last term is a periodic one which always lies between −0.00033 and +0.00033. On the other hand, the term $+0.000\,1178\,T^2$, which is very small when T is very small, becomes increasingly important for larger values of $|T|$. For $T = \pm 10$, that term takes the value +0.01178, which is large in comparison to the above-mentioned periodic term. Thus, for large values of T it is meaningless to take into account small periodic terms if secular terms are dropped.

To shorten a program

To make a program as short as possible is not always an art for art's sake, but sometimes a necessity as long as the memory capacities of the calculating machine have their limits.

There exist many tricks to make programs shorter, even for simple calculations. For instance, if one wants to calculate the polynomial

$$Ax^4 + Bx^3 + Cx^2 + Dx + E$$

with A, B, C, D and E constants, and x a variable. Now, one may program the machine directly to calculate this polynomial term after term and adding all terms, so that for each given x the machine obtains the value of the polynomial. However, instead of calculating all the powers of x, it appears to be wiser to write the polynomial as follows :

$$\left[((Ax + B)x + C)x + D\right]x + E$$

In this expression all power functions have disappeared and only additions and multiplications are to be performed. The program will be shorter now. If we use for instance a HP-67 machine and store the constants A to E in the registers 1 to 5, the programs for the calculation will in each case be as follows.

First version	*Second version*
STO A	STO A
4	RCL 1
y^x	×
RCL 1	RCL 2
×	+
RCL A	RCL A
3	×
y^x	RCL 3
RCL 2	+
×	RCL A
+	×
RCL A	RCL 4
x^2	+
RCL 3	RCL A
×	×
+	RCL 5
RCL A	+
RCL 4	
×	
+	
RCL 5	
+	

Thus, by using this simple trick, one has saved five steps, a gain of 23 % in this short program !

2

INTERPOLATION

The astronomical almanacs or other publications contain numerical tables giving some quantities y for *equidistant* values of an argument x. For example, y is the right ascension of the Sun, and the values x are the different days of the year at 0^h ET.

Interpolation is the process of finding values for instants, quantities, etc., intermediate to those given in a table.

In this Chapter we will consider two cases : interpolation from three or from five tabular values. In both cases we will also show how an extremum or a zero of the function can be found. The case of only two tabular values will not be considered here, for in that case the interpolation can but be linear, and this will give no difficulty at all.

Three tabular values

Three tabular values y_1, y_2, y_3 of the function y are given, corresponding to the values x_1, x_2, x_3 of the argument x. Let us form the table of differences

$$\begin{array}{cccc} x_1 & y_1 & & \\ & & a & \\ x_2 & y_2 & & c \\ & & b & \\ x_3 & y_3 & & \end{array} \tag{2.1}$$

where $a = y_2 - y_1$ and $b = y_3 - y_2$ are called the *first differences*. The *second* difference c is equal to $b - a$, that is

$$c = y_1 + y_3 - 2y_2$$

Generally, the differences of the successive orders are gradually smaller. Interpolation from three tabular values is permitted when the second differences are almost constant in that part of the table, that is when the third differences are almost zero. Let us consider, for instance, the distance of Mars to the Earth from 4 to 8 August 1969, at 0^h ET. The values are given in astronomical units, and the differences are in units of the sixth decimal :

August	4	0.659441			
			+5090		
	5	0.664531		+30	
			+5120		−1
	6	0.669651		+29	
			+5149		0
	7	0.674800		+29	
			+5178		
	8	0.679978			

Since the third differences are almost zero, we may interpolate from only three tabular values.

The central value y_2 must be choosen in such a manner that it is that value of y that is closest to the required value.

For example, if from the table above we must deduce the value of the function for August 6 at 22^h14^m, then y_2 is the value for August 7.00. In that case, we should consider the tabular values for August 6, 7 and 8, namely the table

$$\begin{array}{lll} \text{August } 6 & y_1 = 0.669651 \\ \phantom{\text{August }} 7 & y_2 = 0.674800 \\ \phantom{\text{August }} 8 & y_3 = 0.679978 \end{array} \tag{2.2}$$

and the differences are

$$\begin{array}{ll} a = +0.005149 & \\ & c = +0.000029 \\ b = +0.005178 & \end{array}$$

Let n be the interpolation interval. That is, if the value y of the function is required for the value x of the argument, we have $n = x - x_2$ in units of the tabular interval. The number n is positive if $x > x_2$, that is for a value "later" than x_2 , or from x_2 towards the bottom of the table. If x precedes x_2 , then $n < 0$.

If y_2 has been correctly choosen, then n will be between −0.5 and +0.5, although the following formulae will also give correct results for all values of n between −1 and +1.

The interpolation formula is

$$y = y_2 + \frac{n}{2}(a + b + nc) \qquad (2.3)$$

Example 2.a : From the table (2.2), calculate the distance of Mars to the Earth on 1969 August 7 at 4^h21^m ET.

We have $4^h21^m = 4.35$ hours and, since the tabular interval is 1 day or 24 hours, we have $n = 4.35/24 = 0.18125$.

Formula (2.3) then gives $y = 0.675\,736$, the required value.

If the tabulated function reaches an *extremum* (that is, a maximum or a minimum value), this extremum can be found as follows. Let us again form the difference table (2.1) for the appropriate part of the ephemeris. The extreme value of the function then is

$$y_m = y_2 - \frac{(a + b)^2}{8c}$$

and the corresponding value of the argument x is given by

$$n_m = -\frac{a + b}{2c}$$

in units of the tabular interval, and again measured from the central value x_2.

Example 2.b : Calculate the time of passage of Mars through the perihelion of its orbit in January 1966, and the value of Mars' radius vector at that instant.

From the *Astronomical Ephemeris* we take the following values for the distance Sun - Mars :

1966 January	11.0	1.381 701
	15.0	1.381 502
	19.0	1.381 535

The differences are $a = -0.000199$, $b = +0.000033$, $c = +0.000232$

from which we deduce

$$y_m = 1.381\,487 \quad \text{and} \quad n_m = +0.35776$$

The least distance from Mars to the Sun was thus 1.381 487 AU. The corresponding time is found by multiplying 4 days (the tabular interval) by +0.35776. This gives 1.43104 day, or 1 day and 10 hours later than the central time, or 1966 January 16 at 10^h.

The value of the argument x for which the function y becomes zero can be found by again forming the difference table (2.1) for the appropriate part of the ephemeris. The interpolation interval corresponding to a zero of the function is then given by

$$n_o = \frac{-2y_2}{a + b + cn_o} \qquad (2.4)$$

Equation (2.4) can be solved by first putting $n_o = 0$ in the second member. Then the formula gives an approximate value for n_o. This value is then used to calculate the right hand side again, which gives a still better value for n_o. This process, called *iteration* (Latin : *iterare* = to repeat), can be continued until the value found for n_o does not longer vary, to the precision of the calculating machine.

Example 2.c : The *A.E.* gives the following values for the declination of Mercury :

1973 February	26.0	−0° 28′ 13″.4
	27.0	+0 06 46.3
	28.0	+0 38 23.2

Calculate when the planet's declination was zero.

We firstly convert the tabulated values into seconds of a degree, and then form the differences :

$$\begin{array}{lll} y_1 = -1693.4 & & \\ & a = +2099.7 & \\ y_2 = +\ 406.3 & & c = -202.8 \\ & b = +1896.9 & \\ y_3 = +2303.2 & & \end{array}$$

Formula (2.4) then becomes

$$n_o = \frac{-812.6}{+3996.6 - 202.8\, n_o}$$

Putting $n_o = 0$ in the second member, we find $n_o = -0.20332$. Repeating the calculation, we find successively -0.20125 and -0.20127. Thus $n_o = -0.20127$ and therefore, the tabular interval being one day, Mercury crossed the celestial equator on

$$\begin{aligned} \text{1973 February } 27.0 - 0.20127 &= \text{February } 26.79873 \\ &= \text{February 26 at } 19^h10^m \text{ ET.} \end{aligned}$$

Five tabular values

When the third differences may not be neglected, more than three tabular values must be used. Taking five consecutive tabular values, y_1 to y_5, we form, as before, the difference table

$$\begin{array}{cccccc}
 & y_1 & & & & \\
 & & A & & & \\
 & y_2 & & E & & \\
 & & B & & H & \\
n & y_3 & & F & & K \\
 & & C & & J & \\
 & y_4 & & G & & \\
 & & D & & & \\
 & y_5 & & & &
\end{array}$$

where $A = y_2 - y_1$, $H = F - E$, etc. If n is the interpolation interval, measured from the central value y_3 towards y_4 in units of the tabular interval, we have the interpolation formula

$$y = y_3 + \frac{n}{2}(B + C) + \frac{n^2}{2}F + \frac{n(n^2 - 1)}{12}(H + J) + \frac{n^2(n^2 - 1)}{24}K$$

which may also be written (2.5)

$$y = y_3 + n\left(\frac{B + C}{2} - \frac{H + J}{12}\right) + n^2\left(\frac{F}{2} - \frac{K}{24}\right) + n^3\left(\frac{H + J}{12}\right) + n^4\left(\frac{K}{24}\right)$$

Example 2.d : The *A.E.* gives the following values for the Moon's horizontal parallax :

1979 December	9.0	54′45″.5099
	9.5	54 34.4060
	10.0	54 25.6303
	10.5	54 19.3253
	11.0	54 15.5940

The differences (in ″) are

$A = -11.1039$			
	$E = +2.3282$		
$B = -\ 8.7757$		$H = +0.1425$	
	$F = +2.4707$		$K = -0.0395$
$C = -\ 6.3050$		$J = +0.1030$	
	$G = +2.5737$		
$D = -\ 3.7313$			

We see that the third differences may not be neglected, unless an accuracy of about 0″.1 is sufficient.

Let us now calculate the Moon's parallax on December 10 at 3^h20^m ET. The tabular interval being 12 hours, we find

$$n = +0.277\,7778\ .$$

Formula (2.5) then gives

$$y = 54'25''.6303 - 2''.0043 = 54'23''.6260.$$

The interpolation interval n_m corresponding to an extremum of the function may be obtained by solving the equation

$$n_m = \frac{6B + 6C - H - J + 3n_m^2\,(H + J) + 2n_m^3 K}{K - 12F} \tag{2.6}$$

As before, this may be performed by iteration, firstly putting $n_m = 0$ in the second member. Once n_m is found, the corresponding value of the function can be calculated by means of formula (2.5).

Finally, the interpolation interval n_o corresponding to a zero of the function may be found from

$$n_o = \frac{-24y_3 + n_o^2\,(K - 12F) - 2n_o^3\,(H + J) - n_o^4 K}{2\,(6B + 6C - H - J)} \tag{2.7}$$

where, again, n_o can be found by iteration, starting from putting $n_o = 0$ in the second member.

Note that the quantities $(6B + 6C - H - J)$, $(K - 12F)$, and $(H + J)$ appear in both formulae (2.6) and (2.7). Consequently, it may be useful to calculate these quantities in a subroutine which will be used in both cases.

Exercise. - From the following values of the heliocentric latitude of Mercury, find the instant when the latitude is zero, by using formula (2.7).

1979 May 25.0 ET	−1°16′00″.5
26.0	−0 33 01.3
27.0	+0 11 12.0
28.0	+0 56 03.3
29.0	+1 40 52.2

Answer : Mercury reaches the ascending node of its orbit for $n_o = -0.251\,360$, that is on 1979 May 26 at 17^h58^m ET.

Important remarks

1. Interpolation cannot be performed on complex quantities directly. These quantities should be converted, in advance, into a single suitable unit. For instance, angles expressed in degrees, minutes and seconds should be expressed either in degrees and decimals, or in seconds.

Thus, for instance, 12°44′03″.7 should be written either as 12°.73436, or as 45843″.7.

2. *Interpolating times and right ascensions.* - We draw attention on the fact that the time and the right ascension jump to zero when the value of 24 hours is reached. This should be taken into account when interpolation is performed on tabulated values. Suppose, for example, that we wish to calculate the right ascension of Mercury for the instant 1979 April 16.2743 ET, using three tabulated values. We find in the *A.E.* :

1979 April 15.0	$\alpha = 23^h56^m09^s.20$
16.0	23 58 46.63
17.0	0 01 36.80

Not only is it necessary to convert these values to hours and decimals, but the last value should be written as $24^h\,01^m\,36^s.80$, otherwise the machine will consider that, from April 16.0 to 17.0, the value of α *decreases* from 23^h58^m... to 0^h01^m....

We find a similar situation in some other cases. For instance, here is the longitude of the central meridian of the Sun for a few dates :

1979 December	25.0	37°.39
	26.0	24.22
	27.0	11.05
	28.0	357.88

It is evident that the variation is −13.17 degrees per day. Thus, one should *not* interpolate directly between 11.05 and 357.88. Either the first value should be written as 371.05, or the second one should be considered as being −2.12.

3

Julian Day and Calendar Date

In this Chapter we will give a method for converting a date in the Julian or Gregorian calendars into the corresponding Julian Day number (JD), and vice versa.

General remarks

The Julian Day begins at Greenwich mean *noon*, that is at 12^h Universal Time (or 12^h Ephemeris Time, and in that case the expression Julian Ephemeris Day is generally used). For example, 1977 April 26.4 = JD 2443 259.9.

In the methods described below, the Gregorian calendar reform is taken into account. Thus, the day following 1582 October 4 is 1582 October 15.

The "B.C." years are counted astronomically. Thus, the year before the year +1 is the year zero, and the year preceding the latter is the year −1.

We will indicate by INT(x) the integer part of x, that is the integer which precedes its decimal point. For example :

INT (7/4) = 1	INT (5.9999) = 5
INT (8/4) = 2	INT (−4.98) = −4
INT (5.02) = 5	

Calculation of the JD

A date may be entered in the machine as consecutive numbers, for instance the year first, then the month number, and finally the day with decimals. Thus, 1976 August 22.09 can be entered by entering successively the numbers 1976, 8 and 22.09.

However, it may be more interesting to enter a date as one single number, namely as *YYYY.MMDDdd*, where *YYYY* is the year, *MM* the month, and *DDdd* the day of the month with decimals. In that case, the month number should always be written as a two-digit number, and a decimal point must separate *YYYY* from *MM*. For example, 1976 August 22.09 should then be entered as 1976.082209. The program must then start with a procedure separating the numbers *YYYY*, *MM* and *DD.dd* and storing them in suitable registers. For example, for 1976 August 22.09, the number 1976.082209 is given to the machine, which stores *YYYY* = 1976 in one register, *MM* = 8 in a second one, and *DD.dd* = 22.09 in a third register.

In what follows, we will suppose that this separation has been performed.

If *MM* is greater than 2, take

$$y = YYYY \qquad \text{and} \qquad m = MM\ ;$$

if *MM* = 1 or 2, take

$$y = YYYY - 1 \qquad \text{and} \qquad m = MM + 12.$$

If the number *YYYY.MMDDdd* is equal or larger than 1582.1015 (that is, in the Gregorian calendar), calculate

$$A = \text{INT}\left(\frac{y}{100}\right) \qquad\qquad B = 2 - A + \text{INT}\left(\frac{A}{4}\right)$$

If *YYYY.MMDDdd* < 1582.1015, it is not necessary to calculate *A* and *B*.

The required Julian Day is then

$$\text{JD} = \text{INT}\,(365.25\,y) + \text{INT}\,\big(30.6001\,(m+1)\big) + DD.dd + 1720\,994.5 \qquad (3.1)$$

and, to this result, add the quantity *B* *if* the date is in the Gregorian calendar.

Example 3.a : Calculate the JD corresponding to 1957 October 4.81, the time of launch of Sputnik 1.

Because *MM* = 10 is greater than 2, we have $y = 1957$ and $m = 10$.

Because 1957.100481 > 1582.1015, the date is in the Gregorian calendar, and we calculate

$$A = \text{INT}\left(\frac{1957}{100}\right) = \text{INT}\,(19.57) = 19$$

$$B = 2 - 19 + \text{INT}\left(\frac{19}{4}\right) = 2 - 19 + 4 = -13$$

$$\text{JD} = \text{INT}(365.25 \times 1957) + \text{INT}(30.6001 \times 11) + 4.81 + 1720\,994.5 - 13$$

$$\text{JD} = 2436\,116.31$$

Example 3.b : Calculate the JD corresponding to January 27 at 12^h of the year 333.

Because $MM = 1$, we have

$$y = 333 - 1 = 332 \qquad \text{and} \qquad m = 1 + 12 = 13.$$

The number *YYYY.MMDDdd* = 333.01275 being less than 1582.1015, the date is in the Julian calendar, and the quantities A and B are not needed.

$$\text{JD} = \text{INT}(365.25 \times 332) + \text{INT}(30.6001 \times 14) + 27.5 + 1720\,994.5$$

$$\text{JD} = 1842\,713.0$$

Note. - Your program will not work for negative years. One reason is that, if you enter the date as *YYYY.MMDDdd* preceded by a minus sign, the machine will read *MM* and *DD.dd* as negative numbers. For example, if May 28.63 of the year -584 is entered as -584.052863, the machine will correctly deduce *YYYY* = -584, but will find *MM* = -5 and *DD.dd* = -28.63 instead of the correct values +5 and +28.63.

You may make your program valid for negative years by correcting it as follows.

1. After *YYYY* has been deduced (with proper sign) from the number *YYYY.MMDDdd*, take the absolute value of *.MMDDdd* before calculating *MM* and *DD.dd* ;

2. If $y < 0$, replace, in formula (3.1),

$$\text{INT}(365.25\,y) \quad \text{by} \quad \text{INT}(365.25\,y - 0.75).$$

As an exercise, try your corrected program on -584 May 28.63. The result should be JD = 1507 900.13. But check whether your program is still valid for positive years !

Calculation of the Calendar Date from the JD

The following method is valid for positive as well as for negative years, but not for negative Julian Day numbers.

Add 0.5 to the JD, and let Z be the integer part, and F the fractional (decimal) part of the result.

If $Z < 2299\,161$, take $A = Z$.

If Z is equal to or larger than 2299 161, calculate

$$\alpha = \mathrm{INT}\left(\frac{Z - 1867\,216.25}{36524.25}\right)$$

$$A = Z + 1 + \alpha - \mathrm{INT}\left(\frac{\alpha}{4}\right)$$

Then calculate

$$B = A + 1524$$

$$C = \mathrm{INT}\left(\frac{B - 122.1}{365.25}\right)$$

$$D = \mathrm{INT}\,(365.25\,C)$$

$$E = \mathrm{INT}\left(\frac{B - D}{30.6001}\right)$$

The day of the month (with decimals) is then

$$B - D - \mathrm{INT}\,(30.6001\,E) + F$$

The month number m is $\begin{array}{lll} E - 1 & \text{if} & E < 13.5 \\ E - 13 & \text{if} & E > 13.5 \end{array}$

The year is $\begin{array}{lll} C - 4716 & \text{if} & m > 2.5 \\ C - 4715 & \text{if} & m < 2.5 \end{array}$

Example 3.c : Calculate the calendar date corresponding to JD 2436 116.31.

$2436\,116.31 + 0.5 = 2436\,116.81$,

thus $Z = 2436\,116$ and $F = 0.81$

Because $Z > 2299\,161$, we have

$$\alpha = \mathrm{INT}\left(\frac{2436\,116 - 1867\,216.25}{36524.25}\right) = 15$$

$$A = 2436\,116 + 1 + 15 - \mathrm{INT}\left(\frac{15}{4}\right) = 2436\,129$$

Then we find

$B = 2437\,653$, $C = 6673$, $D = 2437\,313$,
$E = 11$,

day of month = 4.81
month $m = E - 1 = 10$ (because $E < 13.5$)
year $= C - 4716 = 1957$ (because $m > 2.5$)

Thus, the required date is 1957 October 4.81

Exercises : Calculate the calendar dates corresponding to JD = 1842 713.0 and to JD = 1507 900.13 .

(Answers : 333 January 27.5 and -584 May 28.63)

Time interval in days

The number of days between two calendar dates can be found by calculating the difference between their corresponding Julian Days.

Example 3.d : The periodic comet Halley passed through perihelion on 1835 November 16 and on 1910 April 20. What is the time interval between these two passages ?

1835 November 16.0 corresponds to JD 2391 598.5
1910 April 20.0 corresponds to JD 2418 781.5

The difference is 27 183 days.

Exercise : Find the date exactly 10 000 days after 1954 June 30.
(Answer : 1981 November 15)

Day of Week

The day of the week corresponding to a given date can be obtained as follows. Compute the JD for that date at 0^h, add 1.5, and divide the result by 7. The remainder of this division will indicate the weekday, as follows : if the remainder is 0, it is a Sunday, 1 a Monday, 2 a Tuesday, 3 a Wednesday, 4 a Thursday, 5 a Friday, 6 a Saturday.

Example 3.e : Find the weekday of 1954 June 30.

1954 June 30.0 corresponds to JD 2434 923.5
2434 923.5 + 1.5 = 2434 925
The remainder after division by 7 is 3. Thus it was a Wednesday.

Day of the Year

The number N of a day in the year can be computed as follows.

For common years :

$$N = \text{INT}\left(\frac{275\,M}{9}\right) - 2\ \text{INT}\left(\frac{M+9}{12}\right) + D - 30$$

For leap (bissextile) years :

$$N = \text{INT}\left(\frac{275\,M}{9}\right) - \text{INT}\left(\frac{M+9}{12}\right) + D - 30$$

where M is the month number, and D is the day of the month.

N takes integer values, from 1 on January 1 to 365 (or 366 in leap years) on December 31.

Example 3.f : 1978 November 14.

Common year, $M = 11$, $D = 14$.
One finds $N = 318$.

Example 3.g : 1980 April 22.

Leap year, $M = 4$, $D = 22$.
One finds $N = 113$.

Let us now consider the reverse problem : the day number N in the year is known, and we wish to find the corresponding date, namely the month number M and the day D of that month. This can be performed as follows.

Let $A = 1889$ in the case of a common year, $A = 1523$ in the case of a leap year. Then calculate

$$B = \text{INT}\left(\frac{N + A - 122.1}{365.25}\right)$$

$C = N + A - \mathrm{INT}(365.25B)$

$E = \mathrm{INT}(C/30.6001)$

$M = E - 1$ if $E < 13.5$, $M = E - 13$ if $E > 13.5$

$D = C - \mathrm{INT}(30.6001\,E)$

Example 3.h : Common year, $N = 222$.

One finds successively :
$A = 1889$, $B = 5$, $C = 285$, $E = 9$, $M = 9 - 1 = 8$, $D = 10$.
Hence, the date is August 10.

Important note on the Integer Part, INT

The microcomputers and several pocket calculators have the INT ("Integer Part") function. It is important to note that these functions differ with negative numbers.

On the microcomputers, the INT function is defined as follows : INT (x) is the greatest integer less than or equal to x. In that case, we have for instance INT (-7.83) = -8, because -7 is indeed larger than -7.83.

On the pocket calculators, INT is really the integer part of the *written* number, that is the part of the number that precedes the decimal point ; for instance, INT (-7.83) = -7.

The HP-85 microcomputer has not only the INT function, but also an IP ("integer part") function which is identical to the INT one of the pocket calculators. Hence, on the HP-85, we have

INT (-7.83) = -8 and IP (-7.83) = -7 .

Hence, take care when using the INT function ; it depends on your calculating machine ! The INT function used on the preceding pages is identical to the INT function of the pocket calculators.

4

Date of Easter

The method described below has been given by Spencer Jones in his book *General Astronomy* (pages 73-74 of the edition of 1922). It has been published again in the *Journal of the British Astronomical Association*, Vol. 88, page 91 (December 1977) where it is said that it was devised in 1876 and appeared in Butcher's *Ecclesiastical Calendar*.

Unlike the formula given by Gauss, this method has no exception and is valid for all years in the *Gregorian calendar*, that is from the year 1583 on. The procedure for determining the date of Easter is as follows :

Divide	by	Quotient	Remainder
the year x	19	–	a
the year x	100	b	c
b	4	d	e
$b + 8$	25	f	–
$b - f + 1$	3	g	–
$19a + b - d - g + 15$	30	–	h
c	4	i	k
$32 + 2e + 2i - h - k$	7	–	l
$a + 11h + 22l$	451	m	–
$h + l - 7m + 114$	31	n	p

Then n = number of the month (3 = March, 4 = April),
$p+1$ = day of that month upon which Easter Sunday falls.

Try to have your result displayed in one of the following formats :

DD.M (day.month), for instance 26.3 = 26 March ;

M.DD (month.day), for instance 3.26 = March 26 ;

YYYY.MMDD (year.month day), for instance 1978.0326 = 1978 March 26.

The month and the day of the month may also be displayed successively as integer numbers, but the formats above have the advantage that the complete date is read at a glance.

The calculation of the remainder of a division must be programmed carefully. Suppose that the remainder of the division of 34 by 30 should be found. On the HP-67 machine, we find

$$34/30 = 1.133\,333\,333$$

the fractional part of which is 0.133 333 333. When multiplied by 30, this gives 3.999 999 990. This result differs from 4, the correct value, and may give a wrong date for Easter at the end of the calculation.

On the HP-67, the correct value of the remainder may be found by using the instructions

```
DSP 0
f RND
```

On other machines, it may be necessary to use another trick.

If you have enough program steps, you might add some tests at the beginning of your program. For instance, write your program in such a way that *"Error"* appears if the year is not an integer number.

Try your program on the following years :

1978 → March 26	1954 → April 18
1979 → April 15	2000 → April 23
1980 → April 6	1983.6 → Error

The extreme dates of Easter are March 22 (as in 1818 and 2285) and April 25 (as in 1886, 1943, 2038).

Julian Easter

In the Julian Calendar, the date of Easter can be found by means of the following method :

Divide	by	Quotient	Remainder
the year x	4	–	a
the year x	7	–	b
the year x	19	–	c
$19c + 15$	30	–	d
$2a + 4b - d + 34$	7	–	e
$d + e + 114$	31	f	g

Then f = number of the month (3 = March, 4 = April),
$g + 1$ = day of that month upon which Easter Sunday falls.

The date of the *Julian* Easter has a periodicity of 532 years. For instance, we find April 12 for the years 179, 711 and 1243.