Faculty of Mathematics and Physics, Charles University

# Mathematical problems of prisoners
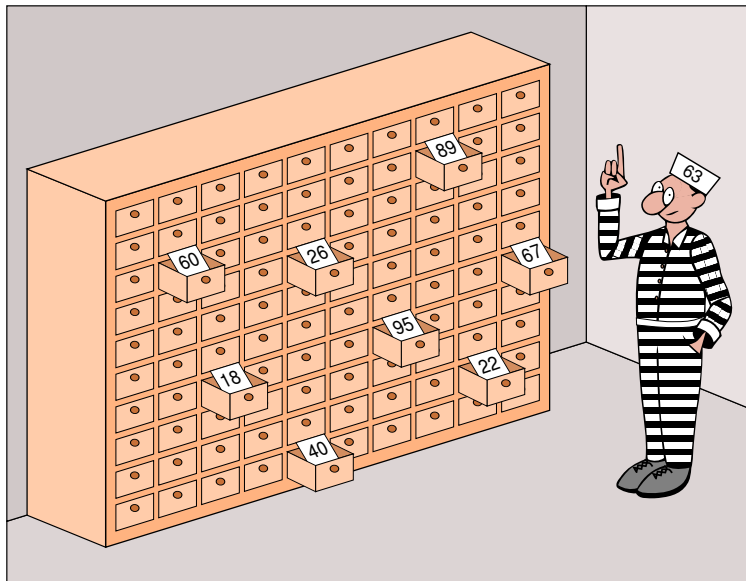
February 7, 2015　　　　　　　　　　　　　Ondrej Škopek

# 100 prisoners problem [2]

- 100 prisoners, numbered 1-100
- 100 drawers, numbered 1-100
- Every prisoner opens $\leq 50$ drawers
- No information exchange allowed during play
- Everyone is pardoned, if everyone finds their number
- No one is pardoned, if at least one prisoner fails

# 100 prisoners problem [6]

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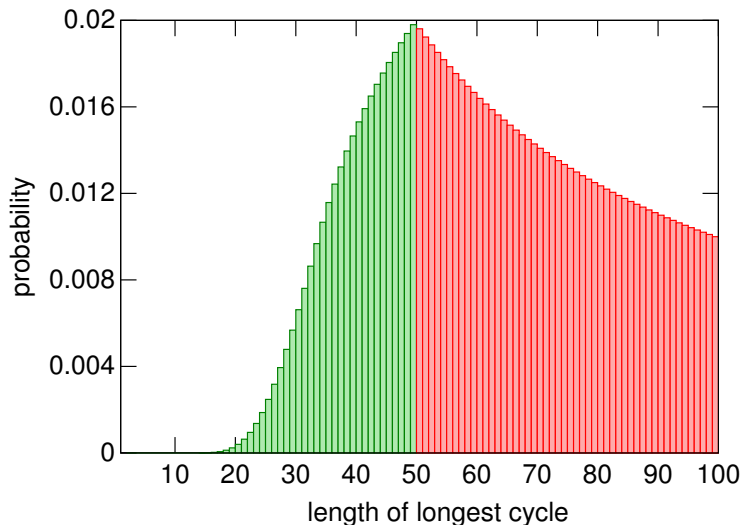- But. . . why?

# Example 1 [5]
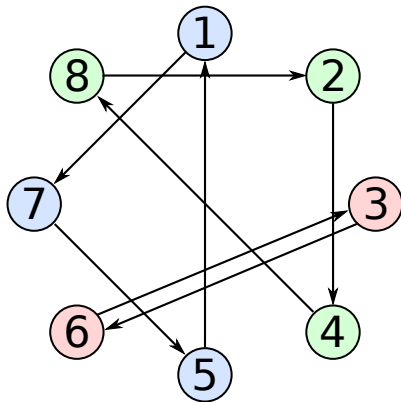


| number of drawer | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| number of prisoner | 7 | 4 | 6 | 8 | 1 | 3 | 5 | 2 |

# Example 2 [4]
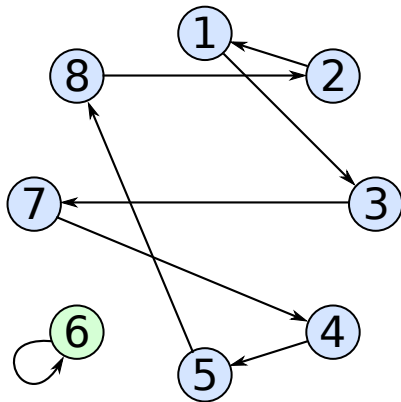


| number of drawer | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| number of prisoner | 3 | 1 | 7 | 5 | 8 | 6 | 4 | 2 |

# Probability distribution of the length of the longest cycle of a random permutation [9]

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*Monty Hall problem [7]*

- Three doors
- Monty knows, behind which door the car is, but can't tell
- Monty tells the player what is behind one of the remaining (not picked) doors
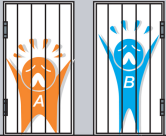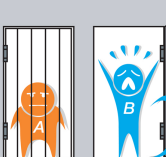- Which doors are hiding the car with a higher probability?
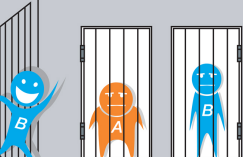
# Decision tree [3]
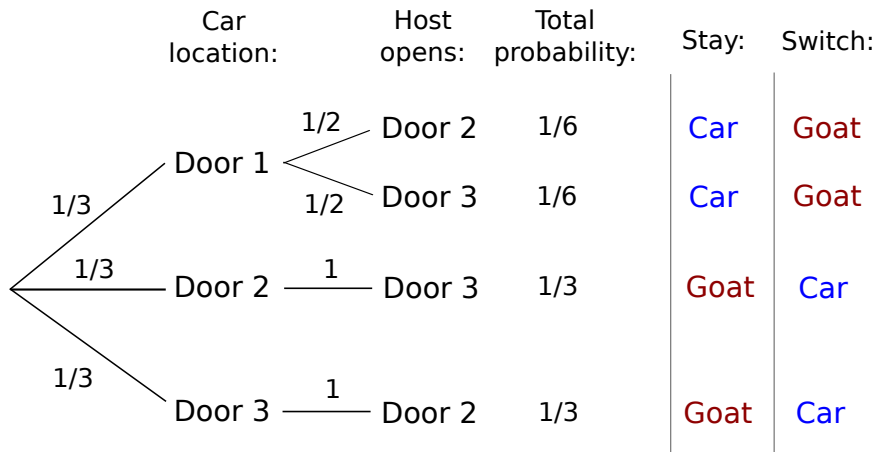
# Prisoner's dilemma [8]

- Prisoners can't communicate
- If $A$ and $B$ each betray the other, each of them serves 5 years in prison
- If $A$ betrays $B$ but $B$ remains silent, $A$ will be set free and $B$ will serve 20 years in prison (and vice versa)
- If $A$ and $B$ both remain silent, both of them will only serve 1 year in prison (on the lesser charge)
- How will they (probably) act?

# Illustration [1]



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- Welcome to Game Theory:
  `https://www.coursera.org/course/welcomegametheory`

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- Programming for Everybody (Python):
  https://www.coursera.org/course/pythonlearn

# Thank you for your attention

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📄 Wikipedia. *Three Prisoners problem.* 2015. URL: http://en.wikipedia.org/wiki/Three_Prisoners_problem.