

Faculty of Mathematics and Physics, Charles University

Mathematical problems of prisoners

February 2, 2015

Ondrej Škopek

100 prisoners problem [2]

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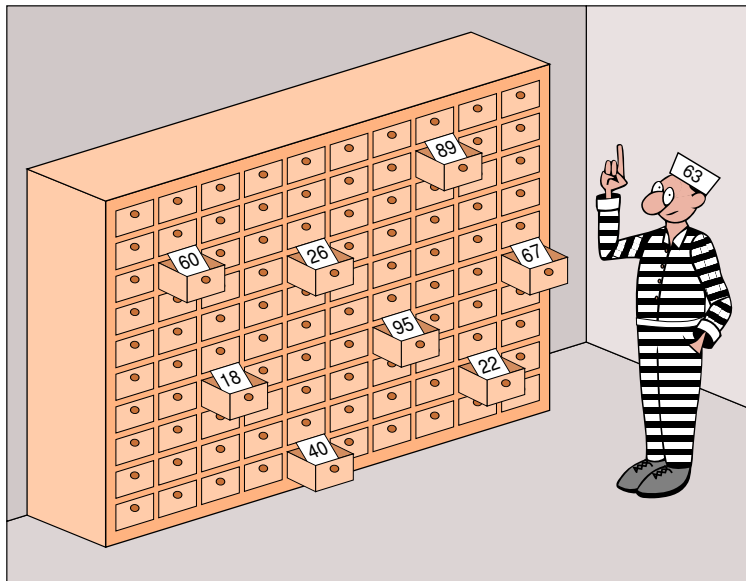
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- **No one** is pardoned, if at least one prisoner fails

100 prisoners problem [6]



Strategy?

- “Inteligent” prisoner: “We each open 50 drawers at random, there is no better strategy.”

Strategy?

- [illegible]

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- Survival probability: $\left(\frac{1}{2}\right)^{100} = 0.0000000000000000000000000000007889 \approx 8 \cdot 10^{-31}$
- Mathematician prisoner: “We each open the drawer with our number and continue to open that drawer, which has the number we found in the previous drawer.”

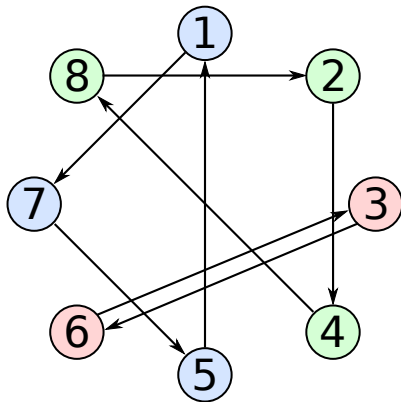
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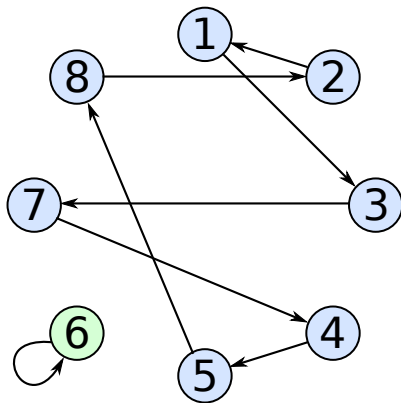
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- Survival probability: > 30% !!!
- But... why?

Example 1 [5]



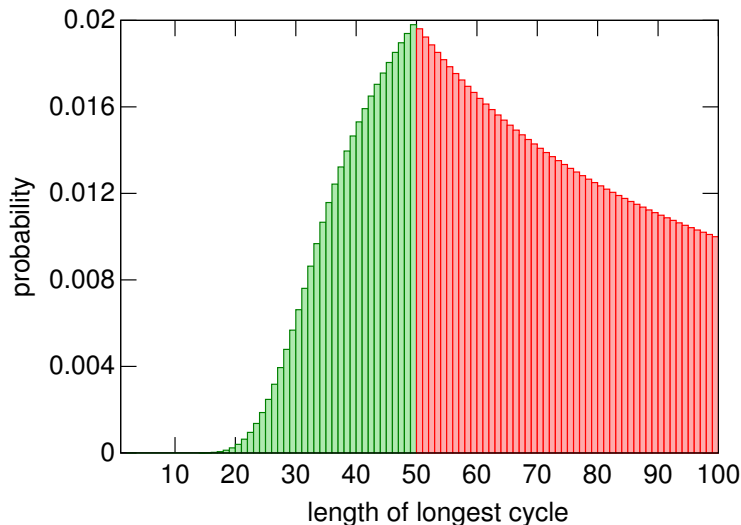
number of drawer	1	2	3	4	5	6	7	8
number of prisoner	7	4	6	8	1	3	5	2

Example 2 [4]



number of drawer	1	2	3	4	5	6	7	8
number of prisoner	3	1	7	5	8	6	4	2

Probability distribution of the length of the longest cycle of a random permutation [9]



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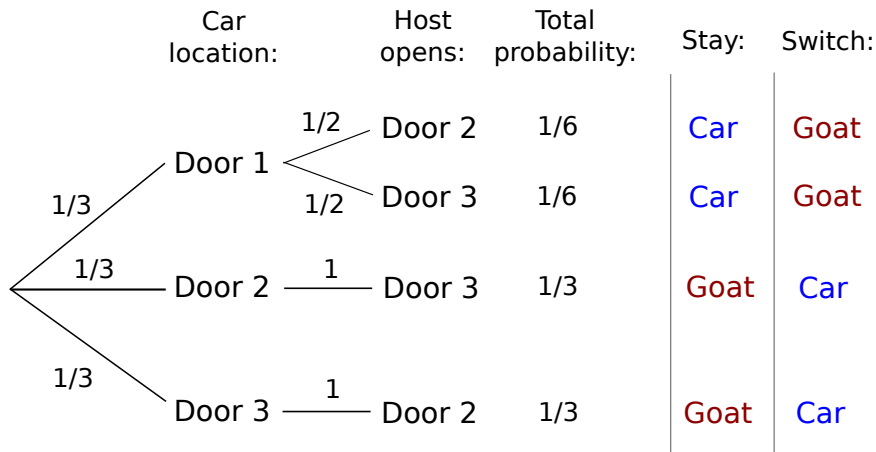
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Monty Hall problem [7]

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- Monty knows, behind which door the car is, but can't tell
- Monty tells the player what is behind one of the remaining (not picked) doors
- Which doors are hiding the car with a higher probability?

Decision tree [3]



Prisoner's dilemma [8]

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Prisoner's dilemma [8]




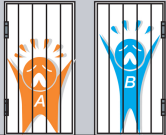


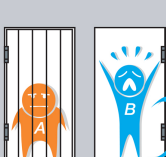
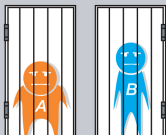
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- How will they (probably) act?

Illustration [1]

Prisoners' dilemma

Prisoners' dilemma		prisoner B			
		confess 		remain silent 	
prisoner A	confess 	 5 years 5 years	 0 year 20 years		
	remain silent 	 20 years 0 year	 1 year 1 year		

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Thank you for your attention

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Ondrej Škopek
<oskopek@matfyz.cz>

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