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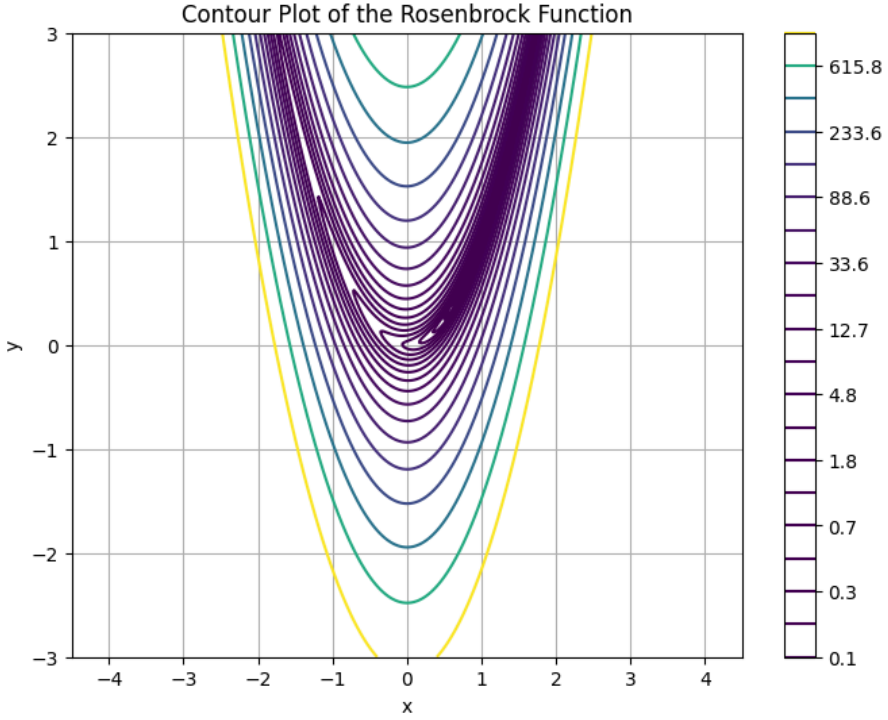
Homework 7

Problem 1: Comparing Optimization Algorithms

1(a): Contour Plot of the Objective Function

```
import numpy as np
import matplotlib.pyplot as plt

# Function and gradient definition based on the Rosenbrock function:
# f(x, y) = (1 - x)^2 + 100*(y - x^2)^2

def f(X):
    x, y = X[0], X[1]
    return (1 - x)**2 + 100 * (y - x**2)**2

x_vals = np.linspace(-4.5, 4.5, 400)
y_vals = np.linspace(-3.0, 3.0, 400)
X, Y = np.meshgrid(x_vals, y_vals)
Z = np.array([[f([x, y]) for x in x_vals] for y in y_vals])

plt.figure(figsize=(8, 6))
cp = plt.contour(X, Y, Z, levels=np.logspace(-1, 3, 20), cmap='viridis')
plt.colorbar(cp)
plt.title('Contour Plot of the Rosenbrock Function')
plt.xlabel('x')
plt.ylabel('y')
plt.grid(True)
plt.show()
```



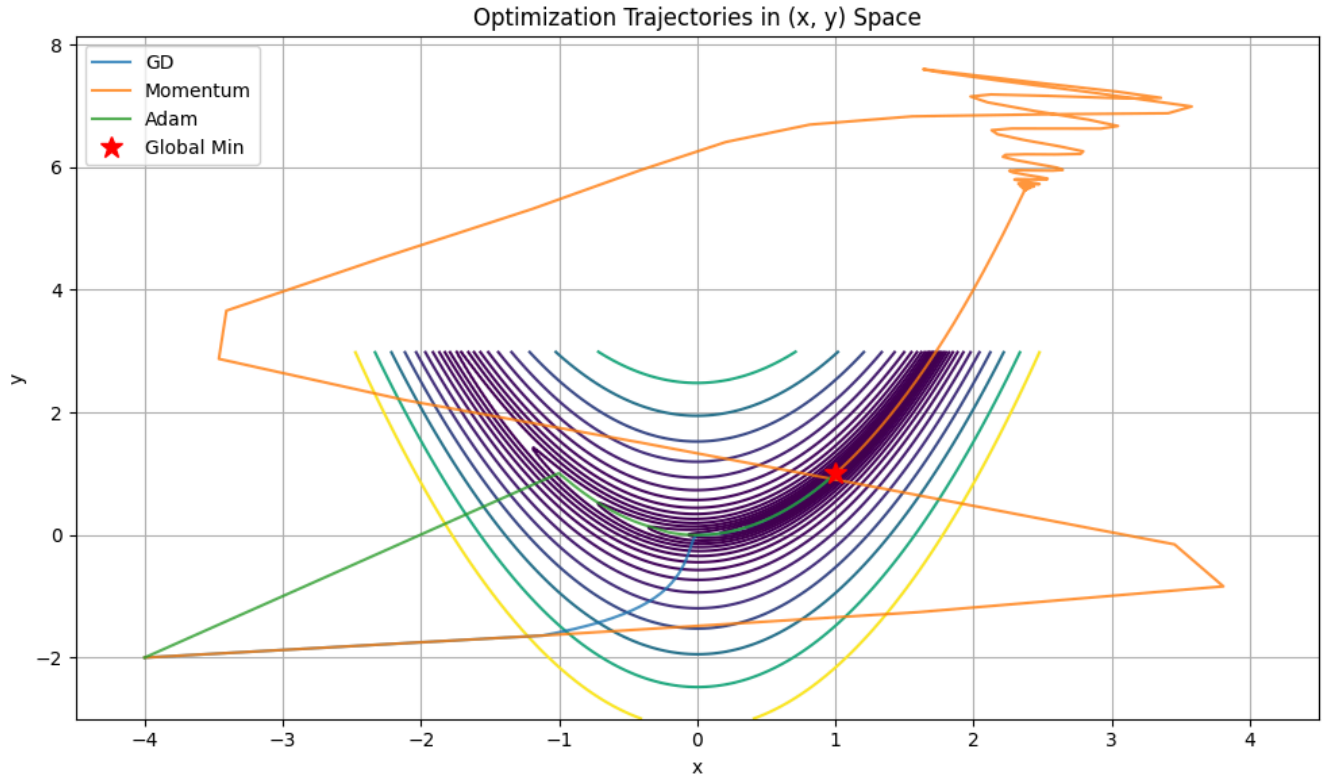
1(b): Optimization Algorithms: Setup for Gradient Descent Comparison

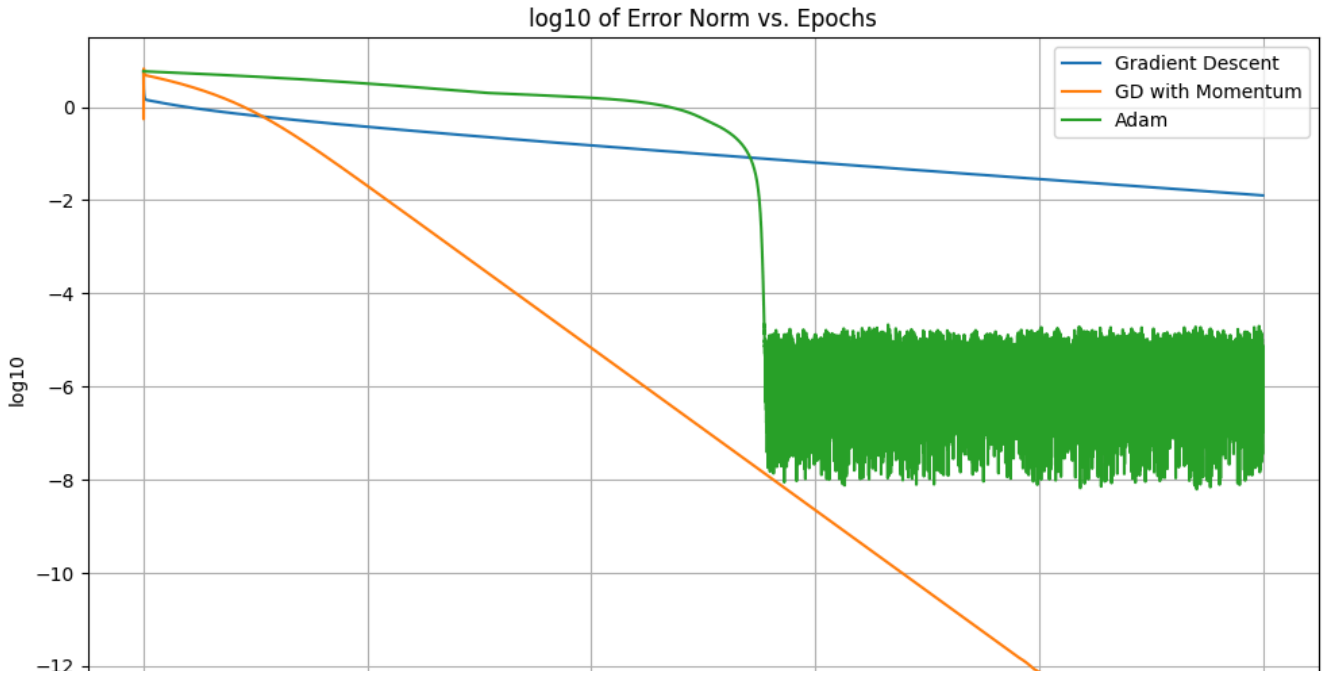
```
# Run each algorithm starting from (-4, -2), eta = 1e-4, and n_epochs = 100000
# Plot their trajectories and log10 of distance to minimum at (1, 1)

# Gradient of the function
def grad(X):
    x, y = X[0], X[1]
    dfdx = -2 * (1 - x) - 400 * x * (y - x**2)
    dfdy = 200 * (y - x**2)
    return np.array([dfdx, dfdy])

def gd(grad, init, n_epochs=1000, eta=10**-4):
    params = np.array(init)
    param_traj = np.zeros([n_epochs+1, 2])
    param_traj[0,] = init
    v = 0;
    for j in range(n_epochs):
        v = eta * (np.array(grad(params)))
        params = params - v
        param_traj[j+1,] = params
    return param_traj

# --- 1(b) Trajectory Visualization and Log Error Plot ---

```

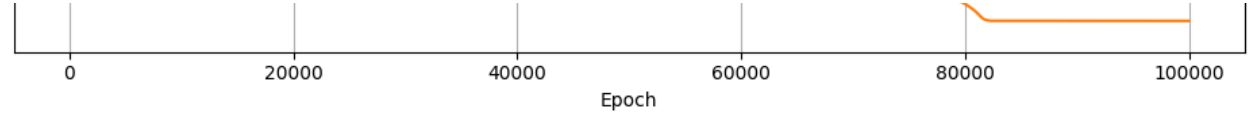
```
def gd_with_mom(grad, init, n_epochs=5000, eta=10**-4, beta=0.9,gamma=0.9):
    params=np.array(init) # Start with initial condition
    param_traj=np.zeros([n_epochs+1,2]) # Save the entire trajecotry
    param_traj[0,]=init # Also save the initial condition to the trajectory

    v=0 # Starting with 0 momentum

    # Epochs is borrowing term from machine learning
    # Here it means timestep

    for j in range(n_epochs):
        v=gamma*v+(np.array(grad(params))) # Compute v
        params=params-eta*v  # Update the location
        param_traj[j+1,]=params # Save the trajectory
    return param_traj


def adams(grad, init, n_epochs=5000, eta=10**-4, gamma=0.9, beta=0.99, epsilon=10**-8):
    params = np.array(init)
    param_traj = np.zeros([n_epochs+1, 2])
    param_traj[0,] = init
    v = 0
    grad_sq = 0
    for j in range(n_epochs):
        g = np.array(grad(params))
        v = gamma * v + (1 - gamma) * g
        grad_sq = beta * grad_sq + (1 - beta) * g * g
        v_hat = v / (1 - gamma**(j + 1))
        grad_sq_hat = grad_sq / (1 - beta**(j + 1))
        params = params - eta * np.divide(v_hat, np.sqrt(grad_sq_hat + epsilon))
        param_traj[j+1,] = params
    return param_traj

# Plot the trajectories of each algorithm and the  base 10 of the error rate...
init_point = [-4, -2]
n_epochs = 100000
eta = 1e-4

traj_gd = gd(grad, init_point, n_epochs=n_epochs, eta=eta)
traj_mom = gd_with_mom(grad, init_point, n_epochs=n_epochs, eta=eta)
traj_adam = adams(grad, init_point, n_epochs=n_epochs, eta=eta)

# Plot optimization trajectories in the (x, y) plane
plt.figure(figsize=(10, 6))
plt.contour(X, Y, Z, levels=np.logspace(-1, 3, 20), cmap='viridis')
plt.plot(traj_gd[:, 0], traj_gd[:, 1], label='GD', alpha=0.8)
plt.plot(traj_mom[:, 0], traj_mom[:, 1], label='Momentum', alpha=0.8)
plt.plot(traj_adam[:, 0], traj_adam[:, 1], label='Adam', alpha=0.8)
plt.plot(1, 1, 'r*', markersize=12, label='Global Min')
plt.title('Optimization Trajectories in (x, y) Space')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()

```

```
## Compute log10 error over time relative to the global minimum (1, 1)
optimum = np.array([1.0, 1.0])
log_err_gd = np.log10(np.linalg.norm(traj_gd - optimum, axis=1))
log_err_mom = np.log10(np.linalg.norm(traj_mom - optimum, axis=1))
log_err_adam = np.log10(np.linalg.norm(traj_adam - optimum, axis=1))

## Plot the log10 error decay
plt.figure(figsize=(10, 6))
plt.plot(log_err_gd, label='Gradient Descent')
plt.plot(log_err_mom, label='GD with Momentum')
plt.plot(log_err_adam, label='Adam')
plt.title('log10 of Error Norm vs. Epochs')
plt.xlabel('Epoch')
plt.ylabel('log10')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()

# --- Report final point and function value (minimum) found by each method ---
print("Final point (Gradient Descent):", traj_gd[-1])
print("Final function value:", f(traj_gd[-1]))

print("Final point (GD with Momentum):", traj_mom[-1])
print("Final function value:", f(traj_mom[-1]))

print("Final point (Adam):", traj_adam[-1])
print("Final function value:", f(traj_adam[-1]))
```

[4]


Optimization Trajectories in (x, y) Space
GD
Momentum
Adam
Global Min
x
y


log10 of Error Norm vs. Epochs
Gradient Descent
GD with Momentum
Adam
log10




```
Final point (Gradient Descent): [0.99433318 0.98867571]
Final function value: 3.216465854863436e-05
Final point (GD with Momentum): [1. 1.]
Final function value: 2.0850579801122868e-26
Final point (Adam): [0.99999479 1.00000519]
Final function value: 2.4411637529060323e-08
```

Discussion: What do you notice about the performance of the different algorithms, both in terms of convergence speed and ultimate accuracy?

Gradient Descent (GD)

- **Convergence Speed:** Very slow and linear. Error decreases steadily but slowly over 100,000 iterations.
- **Final Accuracy:** Final point: `[0.99433318, 0.98867571]`, function value: `3.216465854863436e-05`.

Gradient Descent with Momentum

- **Convergence Speed:** Very fast. Achieves rapid convergence due to momentum accumulation.
- **Final Accuracy:** Final point: `[1. 1.]`, function value: `2.0850579801122868e-26` (close to machine precision).

Adam

- **Convergence Speed:** Fast, especially in the middle phase. However, it shows high-frequency fluctuations near the end.
- **Final Accuracy:** Final point: `[0.99999479, 1.00000519]`, function value: `2.4411637529060323e-08`.

Overall Observations

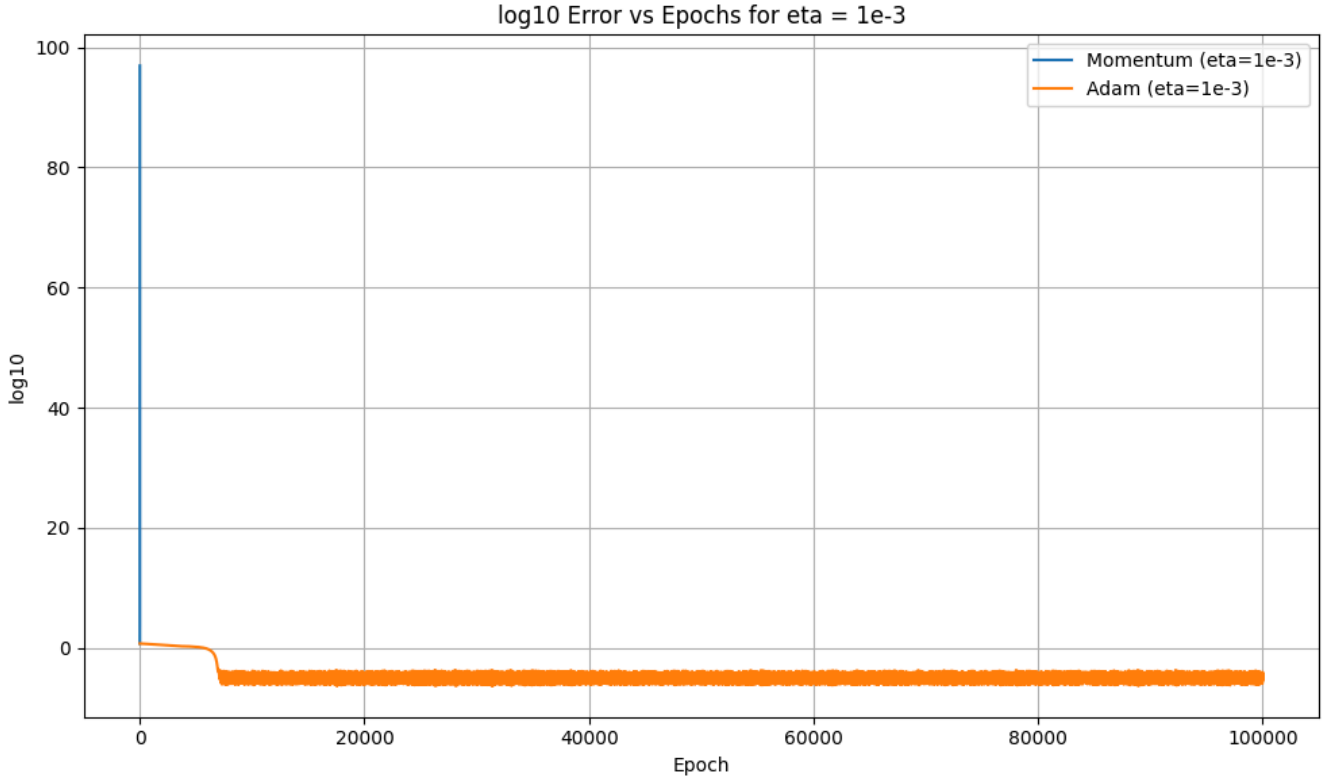
- GD is robust but slow.
- Momentum significantly accelerates convergence and achieves highest precision.
- Adam is efficient and smooth at first but slightly unstable near the minimum due to adaptive learning rate dynamics.

1(c) Comparison with Higher Learning Rate: ADAM and GD with Momentum using eta = 1e-3

```
# Compare ADAM and GD with Momentum using eta = 1e-3
eta_c = 1e-3

traj_mom_c = gd_with_mom(grad, init_point, n_epochs=n_epochs, eta=eta_c)
traj_adam_c = adams(grad, init_point, n_epochs=n_epochs, eta=eta_c)

# Compute log10 error over time relative to (1,1)
log_err_mom_c = np.log10(np.linalg.norm(traj_mom_c - optimum, axis=1))
log_err_adam_c = np.log10(np.linalg.norm(traj_adam_c - optimum, axis=1))

# Plot the log10 error decay
plt.figure(figsize=(10, 6))
plt.plot(log_err_mom_c, label='Momentum (eta=1e-3)')
plt.plot(log_err_adam_c, label='Adam (eta=1e-3)')
plt.title('log10 Error vs Epochs for eta = 1e-3')
plt.xlabel('Epoch')
```

```
plt.ylabel('log10')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()

# Print final results
print("Final point (GD with Momentum, eta=1e-3):", traj_mom_c[-1])
print("Final function value:", f(traj_mom_c[-1]))

print("Final point (Adam, eta=1e-3):", traj_adam_c[-1])
print("Final function value:", f(traj_adam_c[-1]))
```

```
<ipython-input-2-082c0dc1750b>:7: RuntimeWarning: overflow encountered in scalar power
  dfdx = -2 * (1 - x) - 400 * x * (y - x**2)
<ipython-input-2-082c0dc1750b>:8: RuntimeWarning: overflow encountered in scalar power
  dfdy = 200 * (y - x**2)
<ipython-input-2-082c0dc1750b>:7: RuntimeWarning: invalid value encountered in scalar subtract
  dfdx = -2 * (1 - x) - 400 * x * (y - x**2)
<ipython-input-2-082c0dc1750b>:8: RuntimeWarning: invalid value encountered in scalar subtract
  dfdy = 200 * (y - x**2)
/usr/local/lib/python3.11/dist-packages/numpy/linalg/_linalg.py:2772: RuntimeWarning: overflow encountered in multiply
  s = (x.conj() * x).real
```




```
Final point (GD with Momentum, eta=1e-3): [nan nan]
Final function value: nan
Final point (Adam, eta=1e-3): [1.00001807 0.99998026]
Final function value: 3.1256313541136063e-07
```

Discussion: Performance at High Learning Rate (eta = 1e-3)

Gradient Descent with Momentum:

- Failed to converge. Encountered numerical overflow and returned NaN (Not a Number) for both coordinates and function value.
- This is visible in the early spike in the error plot and invalid gradient operations.

Adam:

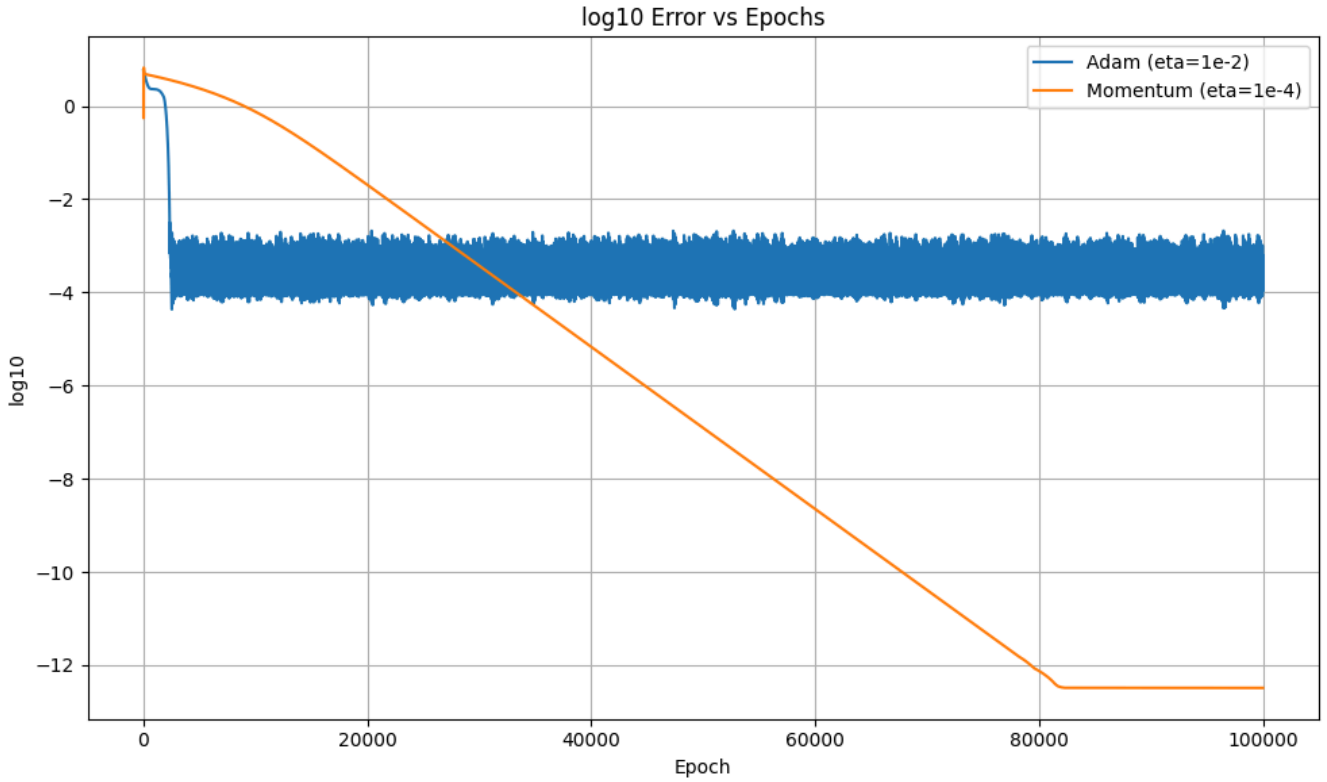
- Successfully converged despite the high learning rate.
- Final point: [1.00001807, 0.99998026], function value: 3.1256313541136063e-07
- Slightly less precise than with smaller eta, but remained stable.

Observation:

- Adam handled the high learning rate robustly, while momentum-based gradient descent became unstable and unusable.
- Thus, **Adam worked better with eta = 1e-3**.

1(d) Trade-Off Comparison: ADAM (eta=1e-2) vs Momentum (eta=1e-4)

```
# Setup for different learning rates
eta_adam_d = 1e-2
eta_mom_d = 1e-4

# Run both optimizers
traj_adam_d = adams(grad, init_point, n_epochs=n_epochs, eta=eta_adam_d)
traj_mom_d = gd_with_mom(grad, init_point, n_epochs=n_epochs, eta=eta_mom_d)

# Compute error over time
log_err_adam_d = np.log10(np.linalg.norm(traj_adam_d - optimum, axis=1))
log_err_mom_d = np.log10(np.linalg.norm(traj_mom_d - optimum, axis=1))

# Plot
plt.figure(figsize=(10, 6))
plt.plot(log_err_adam_d, label='Adam (eta=1e-2)')
plt.plot(log_err_mom_d, label='Momentum (eta=1e-4)')
plt.title('log10 Error vs Epochs')
plt.xlabel('Epoch')
plt.ylabel('log10')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()

# Final point & function values
print("Final point (Adam, eta=1e-2):", traj_adam_d[-1])
print("Final function value:", f(traj_adam_d[-1]))

print("Final point (GD with Momentum, eta=1e-4):", traj_mom_d[-1])
print("Final function value:", f(traj_mom_d[-1]))
```




```
Final point (Adam, eta=1e-2): [1.00005344 0.99980312]
Final function value: 9.230096545900223e-06
Final point (GD with Momentum, eta=1e-4): [1. 1.]
Final function value: 2.0850579801122868e-26
```

Discussion: Trade-Offs Between Adam (eta=1e-2) and Momentum (eta=1e-4)

Comparing ADAM with a high learning rate (eta=1e-2) against gradient descent with momentum using a smaller eta=1e-4

ADAM (eta=1e-2):

- Converges quickly at first but oscillates heavily near the minimum.
- Final point: [1.00005344, 0.99980312], function value: 9.230096545900223e-06
- Less accurate and noisy due to aggressive step size.

GD with Momentum (eta=1e-4):

- Converges more slowly but steadily, with minimal noise.
- Final point: [1.0, 1.0], function value: 2.0850579801122868e-26 (machine precision)

Trade-Off Summary:

- **Adam** offers speed and tolerance to noise at the cost of final precision.
- **Momentum** delivers higher accuracy but requires more iterations and a carefully tuned smaller learning rate.

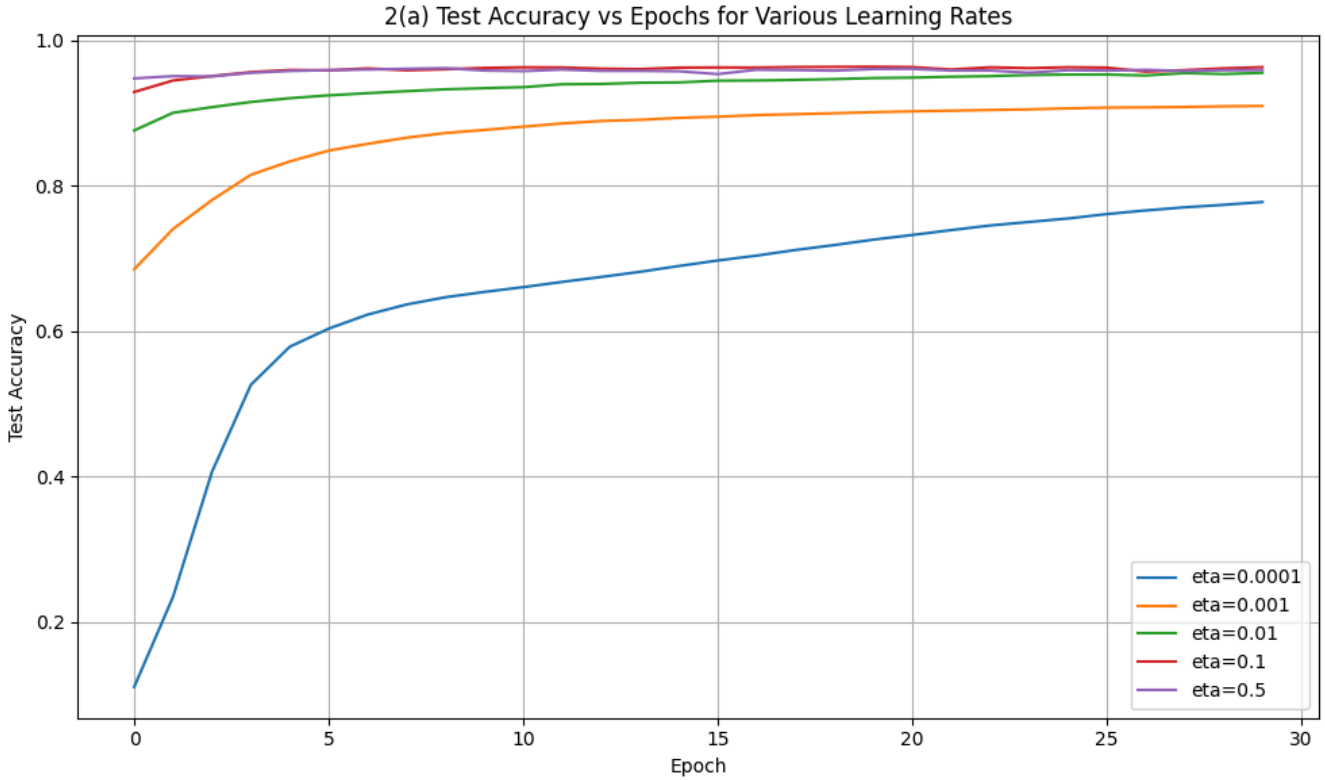
Problem 2: Shallow Nets and MNIST

2(a) Validates Default Training and Explores Learning Rate Effects

```
# Setup: Dataset and Network Initialization

import torch
import torch.nn as nn
import torch.nn.functional as F
import torch.optim as optim
from torch.utils.data import DataLoader
from torchvision import datasets, transforms

# Define transformation: convert to tensor and flatten 28x28 -> 784
transform = transforms.Compose([
    transforms.ToTensor(),
    transforms.Lambda(lambda x: x.view(-1))  # flatten
])

# Load MNIST dataset
train_dataset = datasets.MNIST('data/', train=True, download=True, transform=transform)
test_dataset = datasets.MNIST('data/', train=False, transform=transform)

train_loader = DataLoader(train_dataset, batch_size=10, shuffle=True)
test_loader = DataLoader(test_dataset, batch_size=10, shuffle=False)

# Network Class Definition (From Helper Notebook)

class Network(nn.Module):
    def __init__(self, sizes):
        super(Network, self).__init__()
        self.sizes = sizes
        self.num_layers = len(sizes)
        self.layers = nn.ModuleList()
        for i in range(self.num_layers - 1):
            layer = nn.Linear(sizes[i], sizes[i+1])
            nn.init.xavier_normal_(layer.weight)  # suited for shallow/sigmoid nets

            nn.init.zeros_(layer.bias) # initialize the bias to 0
            self.layers.append(layer)

    # Forward is the method that calculates the value of the neural network. Basically we recursively apply the activations in each layer
    def forward(self, x):
        for layer in self.layers[:-1]:
            x = F.sigmoid(layer(x))  # default is sigmoid for shallow nets
        x = self.layers[-1](x)  # last layer: logits
        return x

# Train Function

def train(network, train_loader, epochs, eta, test_data):
    optimizer = optim.SGD(network.parameters(), lr=eta)
    loss_fn = nn.CrossEntropyLoss()
```

```
    test_accuracies = []

    for epoch in range(epochs):
        network.train()
        for images, labels in train_loader:
            output = network(images)
            loss = loss_fn(output, labels)
            optimizer.zero_grad()
            loss.backward()
            optimizer.step()

        ## Evaluate on test set
        network.eval()
        correct = 0
        total = 0

        ## We have the "with torch.no_grad()" line for efficiency purposes
        with torch.no_grad():
            for images, labels in test_data:
                output = network(images)
                preds = output.argmax(dim=1)
                correct += (preds == labels).sum().item()
                total += labels.size(0)
        accuracy = correct / total
        test_accuracies.append(accuracy)

    return test_accuracies

# Sweep Over Different Learning Rates
etas = [0.0001, 0.001, 0.01, 0.1, 0.5]
final_accuracies = []
all_curves = {}

for eta in etas:
    net = Network([784, 30, 10])
    acc_curve = train(net, train_loader, epochs=30, eta=eta, test_data=test_loader)
    all_curves[eta] = acc_curve
    final_accuracies.append((eta, acc_curve[-1]))

# Plot test accuracy curves
import matplotlib.pyplot as plt
plt.figure(figsize=(10, 6))
for eta, curve in all_curves.items():
    plt.plot(curve, label=f'eta={eta}')
plt.xlabel('Epoch')
plt.ylabel('Test Accuracy')
plt.title('2(a) Test Accuracy vs Epochs for Various Learning Rates')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()

# Print best learning rate and its final accuracy
best_eta = max(final_accuracies, key=lambda x: x[1])
print("Best learning rate:", best_eta[0])
print("Final test accuracy:", best_eta[1])
```

```
100%|██████████| 9.91M/9.91M [00:01<00:00, 5.10MB/s]
100%|██████████| 28.9k/28.9k [00:00<00:00, 135kB/s]
100%|██████████| 1.65M/1.65M [00:06<00:00, 247kB/s]
100%|██████████| 4.54k/4.54k [00:00<00:00, 8.50MB/s]
```




```
Best learning rate: 0.1
Final test accuracy: 0.9635
```

Discussion: Evaluation of Learning Rate Sweep

We trained a shallow neural network (784 → 30 → 10) using stochastic gradient descent (SGD) across a range of learning rates (etas = [0.0001, 0.001, 0.01, 0.1, 0.5]). Each configuration was run for 30 epochs on the MNIST dataset, and test accuracy was tracked.

We Observe:

- **Low learning rates** (`η = 0.0001` and `0.001`) resulted in slow learning and lower accuracy.
- **Moderate learning rates** (`η = 0.01` and `0.1`) achieved rapid convergence and high final accuracy.
- **High learning rate** (`η = 0.5`) showed signs of instability (oscillations), although still high in accuracy.

Best Result:

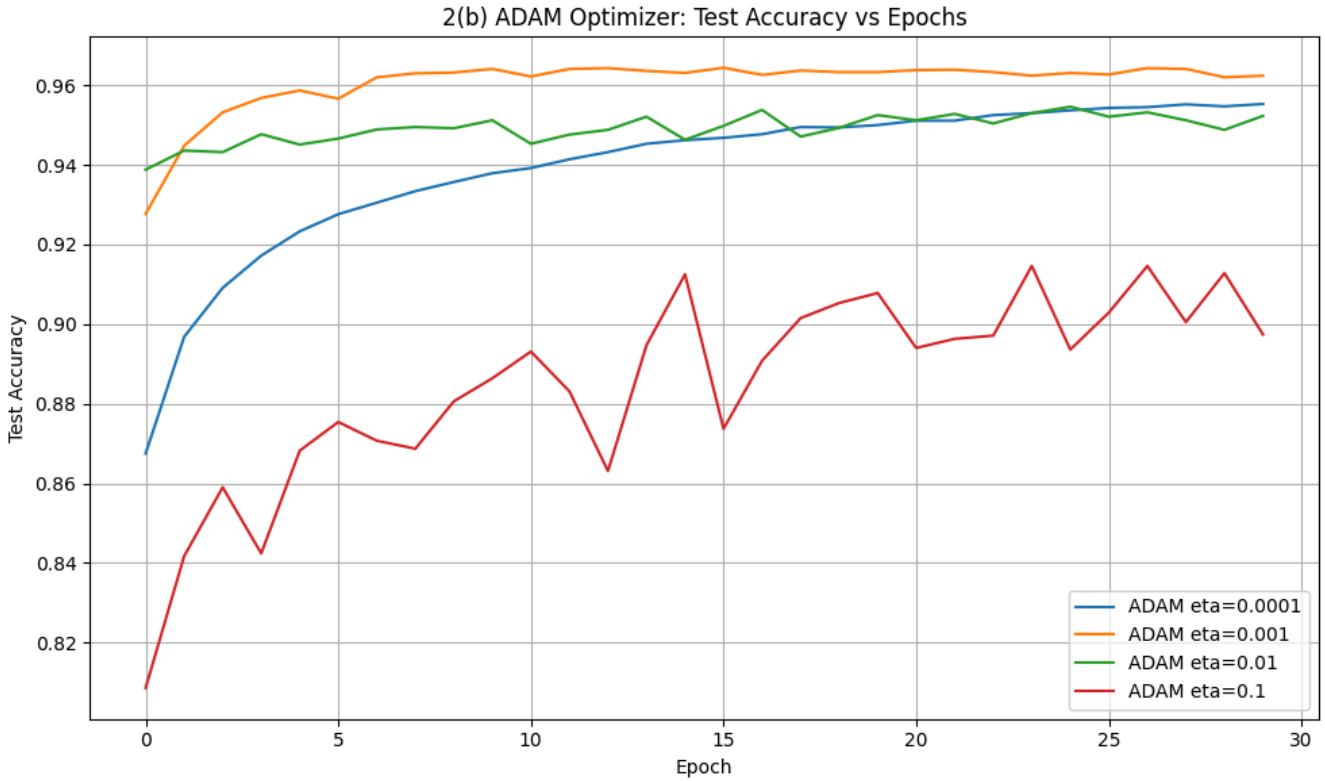
- **Best learning rate:** `η = 0.1`
- **Final test accuracy: 0.9635**

This suggests that for sigmoid-activated shallow networks on MNIST using SGD, a learning rate around `0.1` provides the best balance between convergence speed and stability.

2(b) Comparison with ADAM Optimizer

```
# This block tests ADAM optimizer using similar network and training setup

adam_etas = [0.0001, 0.001, 0.01, 0.1]
adam_final_accuracies = []
adam_curves = {}

def train_adam(network, train_loader, epochs, eta, test_data):
    optimizer = optim.Adam(network.parameters(), lr=eta)
    loss_fn = nn.CrossEntropyLoss()
    test_accuracies = []

    for epoch in range(epochs):
        network.train()
        for images, labels in train_loader:
            output = network(images)
            loss = loss_fn(output, labels)
            optimizer.zero_grad()
            loss.backward()
            optimizer.step()

        network.eval()
        correct = 0
        total = 0
        with torch.no_grad():
            for images, labels in test_data:
                output = network(images)
                preds = output.argmax(dim=1)
                correct += (preds == labels).sum().item()
                total += labels.size(0)
        accuracy = correct / total
        test_accuracies.append(accuracy)

    return test_accuracies

# Run experiments for different ADAM learning rates
for eta in adam_etas:
    net = Network([784, 30, 10])
    acc_curve = train_adam(net, train_loader, epochs=30, eta=eta, test_data=test_loader)
    adam_curves[eta] = acc_curve
    adam_final_accuracies.append((eta, acc_curve[-1]))

# Plot ADAM test accuracy curves
plt.figure(figsize=(10, 6))
for eta, curve in adam_curves.items():
    plt.plot(curve, label=f'ADAM eta={eta}')
plt.xlabel('Epoch')
plt.ylabel('Test Accuracy')
plt.title('2(b) ADAM Optimizer: Test Accuracy vs Epochs')
plt.legend()
plt.grid(True)
```

```
plt.tight_layout()
plt.show()

# Print best learning rate and accuracy for ADAM
best_adam_eta = max(adam_final_accuracies, key=lambda x: x[1])
print("Best ADAM learning rate:", best_adam_eta[0])
print("Final test accuracy:", best_adam_eta[1])
```




```
Best ADAM learning rate: 0.001
Final test accuracy: 0.9624
```

Discussion: ADAM Optimizer Evaluation and Comparison with Stochastic Gradient Descent (SGD) in 2(a)

We evaluated the ADAM optimizer across several learning rates using the same shallow neural network (784 → 30 → 10) on the MNIST dataset and compared it to the results obtained in Part 2(a) using SGD.

Observations with ADAM:

- **η = 0.001** provided the best performance, reaching **0.9624** test accuracy.
- **η = 0.0001** converged more slowly but stably.
- **η = 0.01** achieved high accuracy but slightly less consistent than 0.001.
- **η = 0.1** caused instability and lower accuracy due to overly aggressive updates.

Comparison with SGD (from 2a):

Optimizer	Best η	Final Accuracy
SGD	0.1	**0.9635**
ADAM	0.001	0.9624

- **SGD with η=0.1** reached the highest final test accuracy overall (0.9635).
- **ADAM with η=0.001** was more stable across training and easier to tune.
- ADAM showed better initial convergence at lower η but plateaued slightly earlier.

Conclusion:

While **SGD slightly outperformed ADAM in final accuracy**, ADAM showed strong early learning performance and required less tuning effort.

2(c) Hyperparameter and Architecture Tuning

In this part we explores how to improve performance. We vary hidden layer size, optimizer settings, and regularization.

```
# We vary hidden layer size, optimizer settings, and regularization.
custom_configs = [
    {"hidden": 50, "optimizer": "adam", "eta": 0.001, "weight_decay": 1e-5},
    {"hidden": 100, "optimizer": "adam", "eta": 0.001, "weight_decay": 1e-4},
    {"hidden": 50, "optimizer": "sgd",  "eta": 0.1,   "momentum": 0.8},
    {"hidden": 100, "optimizer": "sgd",  "eta": 0.1,   "momentum": 0.9}
]

results_2c = []

def train_custom(network, optimizer, train_loader, epochs, test_data):
    loss_fn = nn.CrossEntropyLoss()
    test_accuracies = []
    for epoch in range(epochs):
        network.train()
        for images, labels in train_loader:
            output = network(images)
            loss = loss_fn(output, labels)
            optimizer.zero_grad()
            loss.backward()
            optimizer.step()

        network.eval()
        correct = 0
        total = 0
        with torch.no_grad():
            for images, labels in test_data:
                output = network(images)
                preds = output.argmax(dim=1)
                correct += (preds == labels).sum().item()
                total += labels.size(0)
        accuracy = correct / total
        test_accuracies.append(accuracy)

    return test_accuracies

for config in custom_configs:
    net = Network([784, config["hidden"], 10])
    if config["optimizer"] == "adam":
```

```
        optimizer = optim.Adam(net.parameters(), lr=config["eta"], weight_decay=config["weight_decay"])
    else:
        optimizer = optim.SGD(net.parameters(), lr=config["eta"], momentum=config["momentum"])

    acc_curve = train_custom(net, optimizer, train_loader, epochs=30, test_data=test_loader)
    results_2c.append((config, acc_curve[-1]))

# Report top performing configuration
best_config = max(results_2c, key=lambda x: x[1])
print("Best custom configuration (2c):")
print("Hidden units:", best_config[0]["hidden"])
print("Optimizer:", best_config[0]["optimizer"])
print("Learning rate:", best_config[0]["eta"])
print("Final accuracy:", best_config[1])
```

```
Best custom configuration (2c):
Hidden units: 100
Optimizer: sgd
Learning rate: 0.1
Final accuracy: 0.9801
```

Discussion: Custom Network & Hyperparameter Tuning

In this one, we explored various combinations of hidden layer size, optimizer choice, learning rate, and regularization to improve performance.

Configurations Tested:

- **Hidden units**: 50 and 100
- **Optimizers**: SGD with momentum, and ADAM with weight decay
- **Learning rates**: 0.1 (SGD), 0.001 (ADAM)

Best Configuration:

- **Hidden units**: 100
- **Optimizer**: `sgd`
- **Learning rate**: `0.1`
- **Final test accuracy**: **0.9801**

Interpretation:

- Increasing the number of hidden units to 100 provided better representational power.
- A learning rate of 0.1 with SGD led to rapid convergence and high generalization.
- Among all configurations tested, this combination yielded the best accuracy (0.9801.

Conclusion:

With careful tuning of both the **network architecture** and **training hyperparameters**, we achieved a final test accuracy of **0.9801**, the best observed in this set of experiments.

Problem 3: Deep Nets: Overcoming Gradients

3(a) Setup: Deep Network and Gradient Ratio Measurement

```
import torch
import torch.nn as nn
import torch.nn.functional as F
import torch.optim as optim
from torch.utils.data import DataLoader
from torchvision import datasets, transforms

# Load MNIST (same transform as Problem 2)
transform = transforms.Compose([
    transforms.ToTensor(),
    transforms.Lambda(lambda x: x.view(-1))
])

train_dataset = datasets.MNIST('data/', train=True, download=True, transform=transform)
test_dataset = datasets.MNIST('data/', train=False, transform=transform)

train_loader = DataLoader(train_dataset, batch_size=10, shuffle=True)

# Deep Network with Gradient Tracking (sigmoid + xavier)

class DeepNetwork(nn.Module):
    def __init__(self, sizes):
        super(DeepNetwork, self).__init__()
        self.sizes = sizes
        self.num_layers = len(sizes)
        self.layers = nn.ModuleList()
        for i in range(self.num_layers - 1):
            layer = nn.Linear(sizes[i], sizes[i+1])
            nn.init.xavier_normal_(layer.weight)
            nn.init.zeros_(layer.bias)
            self.layers.append(layer)

    def forward(self, x):
        self.activations = []
        for i, layer in enumerate(self.layers[:-1]):
            x = F.sigmoid(layer(x))
            self.activations.append(x)
        x = self.layers[-1](x)  # no softmax needed for CrossEntropyLoss
        return x

# Function to compute and return gradient ratio

def compute_gradient_ratio(model):
    grad_norm_input = model.layers[0].weight.grad.norm().item()
    grad_norm_output = model.layers[-1].weight.grad.norm().item()
    ratio = grad_norm_input / grad_norm_output if grad_norm_output != 0 else float('inf')
    return grad_norm_input, grad_norm_output, ratio

# Initialize deep net and train for 1 epoch to observe gradient vanishing
net = DeepNetwork([784, 30, 30, 30, 30, 30, 30, 30, 30, 10])
optimizer = optim.SGD(net.parameters(), lr=0.01)
loss_fn = nn.CrossEntropyLoss()

for images, labels in train_loader:
    output = net(images)
    loss = loss_fn(output, labels)
```

```
    optimizer.zero_grad()
    loss.backward()
    optimizer.step()
    break  # Just one batch for gradient observation

# Gradient stats
grad_in, grad_out, ratio = compute_gradient_ratio(net)
print("3(a) Gradient Norm of Input Layer:", grad_in)
print("3(a) Gradient Norm of Output Layer:", grad_out)
print("3(a) Gradient Ratio (Input/Output):", ratio)
```

```
3(a) Gradient Norm of Input Layer: 4.663690197048709e-05
3(a) Gradient Norm of Output Layer: 1.0840625762939453
3(a) Gradient Ratio (Input/Output): 4.3020488844770725e-05
```

Observing Vanishing Gradients in Deep Sigmoid Network

In this part, we constructed a deep feedforward neural network with 9 hidden layers, all using **sigmoid activation** functions and **Xavier initialization**. We trained it on a single batch of MNIST data and calculated the gradient norms at the input and output layers.

Observed Values:

- **Input Layer Gradient Norm:** `4.66e-05`
- **Output Layer Gradient Norm:** `1.08`
- **Gradient Ratio (Input/Output):** `4.30e-05`

Interpretation:

- The input layer's gradients are **orders of magnitude smaller** than the output layer's.
- This confirms the presence of **vanishing gradients**, which makes it difficult for early layers to learn meaningful features during backpropagation.

3(b) ReLU Activation + Kaiming Initialization

Replacing sigmoid with ReLU and xavier with kaiming initialization.

```
# Replace sigmoid with ReLU and xavier with kaiming initialization.

class DeepNetworkReLU(nn.Module):
    def __init__(self, sizes):
        super(DeepNetworkReLU, self).__init__()
        self.sizes = sizes
        self.num_layers = len(sizes)
        self.layers = nn.ModuleList()
        for i in range(self.num_layers - 1):
            layer = nn.Linear(sizes[i], sizes[i+1])
            nn.init.kaiming_uniform_(layer.weight, nonlinearity='relu')
            nn.init.zeros_(layer.bias)
            self.layers.append(layer)

    def forward(self, x):
        self.activations = []
        for i, layer in enumerate(self.layers[:-1]):
```

```
            x = F.relu(layer(x))
            self.activations.append(x)
        x = self.layers[-1](x)
        return x

# Initialize deep ReLU network
net_relu = DeepNetworkReLU([784, 30, 30, 30, 30, 30, 30, 30, 30, 10])
optimizer_relu = optim.SGD(net_relu.parameters(), lr=0.01)

# Train on one batch to observe gradient behavior
for images, labels in train_loader:
    output = net_relu(images)
    loss = loss_fn(output, labels)
    optimizer_relu.zero_grad()
    loss.backward()
    optimizer_relu.step()
    break

# Compute and print gradient stats for ReLU
grad_in_relu, grad_out_relu, ratio_relu = compute_gradient_ratio(net_relu)
print("3(b) Gradient Norm of Input Layer (ReLU):", grad_in_relu)
print("3(b) Gradient Norm of Output Layer (ReLU):", grad_out_relu)
print("3(b) Gradient Ratio (Input/Output, ReLU):", ratio_relu)
```

```
3(b) Gradient Norm of Input Layer (ReLU): 4.0457539558410645
3(b) Gradient Norm of Output Layer (ReLU): 0.8337193131446838
3(b) Gradient Ratio (Input/Output, ReLU): 4.852657113796479
```

Discussion: ReLU Activation and Kaiming Initialization

To resolve the vanishing gradient problem observed in Part 3(a), we replaced sigmoid activations with **ReLU**, and used **Kaiming uniform initialization**, which is well-suited for ReLU-based deep networks.

Observed Gradient Norms:

- **Input Layer Gradient Norm:** `4.05`
- **Output Layer Gradient Norm:** `0.83`
- **Gradient Ratio (Input/Output):** `4.85`

Interpretation (Gradient Flow):

- The input layer gradient is significantly stronger than the output layer's, indicating **no vanishing** and healthy gradient propagation.
- This is a major improvement over 3(a), where the input gradient was nearly 5 orders of magnitude smaller than the output.

Conclusion:

Switching to **ReLU + Kaiming** resolved vanishing gradients and unlocked the potential for training deeper networks.

3(c) Deep Network Improvement Experiment

Experimenting with deeper networks or decreasing hidden layer sizes layer-by-layer

```
class DeepCustomNet(nn.Module):
    def __init__(self, sizes):
        super(DeepCustomNet, self).__init__()
```

```
        super(DeepCustomNet, self).__init__()
        self.sizes = sizes
        self.num_layers = len(sizes)
        self.layers = nn.ModuleList()
        for i in range(self.num_layers - 1):
            layer = nn.Linear(sizes[i], sizes[i+1])
            nn.init.kaiming_uniform_(layer.weight, nonlinearity='relu')
            nn.init.zeros_(layer.bias)
            self.layers.append(layer)

    def forward(self, x):
        for i, layer in enumerate(self.layers[:-1]):
            x = F.relu(layer(x))
        return self.layers[-1](x)

# Try an architecture that tapers from 784 → 512 → 256 → 128 → 64 → 10
net_custom = DeepCustomNet([784, 512, 256, 128, 64, 10])
optimizer_custom = optim.Adam(net_custom.parameters(), lr=0.001, weight_decay=1e-4)
loss_fn = nn.CrossEntropyLoss()

def evaluate_accuracy(model):
    model.eval()
    correct = 0
    total = 0
    with torch.no_grad():
        for images, labels in DataLoader(test_dataset, batch_size=64):
            output = model(images)
            preds = output.argmax(dim=1)
            correct += (preds == labels).sum().item()
            total += labels.size(0)
    return correct / total

# Train for a few epochs
epochs = 5
for epoch in range(epochs):
    net_custom.train()
    for images, labels in train_loader:
        output = net_custom(images)
        loss = loss_fn(output, labels)
        optimizer_custom.zero_grad()
        loss.backward()
        optimizer_custom.step()

```