Tutorial for homework 2

Exercise 1 (0.2 point). (a) Fill in the table indicating by plus (respectively minus) that we can write (respectively cannot write) $A = C(B)$, where A and B are functions and C is a notation from set $\{O, o, \Omega, \omega, \Theta\}$.

A	B	O	o	Ω	ω	Θ
$f(n)$	$g(n)$					
$g(n)$	$f(n)$					

In some cases l'Hopital's rule may be helpful as well as formulas

$$(a^x)' = a^x \ln a \quad \text{and} \quad (\log_a x)' = \frac{1}{x \ln a}.$$

(b) Find maximal n values such that functions f and g satisfy inequalities $f(n) \leq 1000000$ and $g(n) \leq 1000000$, i.e., indicate numbers N_f and N_g such that $f(N_f) \leq 1000000$ and $g(N_g) \leq 1000000$, but $f(N_f + 1) > 1000000$ and $g(N_g + 1) > 1000000$.

See Section 3.1 in Cormen book.

Example 1. Let functions

$$f(n) = 10n^2 \quad \text{and} \quad g(n) = n^2 + \log_2^2 n^{10}$$

be given.

(a) We have to compare growth order of functions f and g. It is easy to show by mathematical induction that $n < 2^n$ which is equivalent to $\log_2 n < n$, so $\log_2^2 n < n^2$ (where $n > 0$). Since $\log_2^2 n^{10} = (10 \log_2 n)^2$, second function satisfies double inequality

$$n^2 \leq g(n) < 101n^2.$$

Therefore, we have

$$\frac{1}{10} f(n) \leq g(n) < 11 f(n).$$

So $g(n) = O(f(n))$ (since $g(n) \leq c_1 f(n)$ $\forall n > 0$, where $c_1 = 11$) and $f(n) = O(g(n))$ (since $f(n) \leq c_2 g(n)$ $\forall n > 0$, where $c_2 = 10$). These two upper bounds show that both functions have the same growth order: $f(n) = \Theta(g(n))$ and $g(n) = \Theta(f(n))$. According to Ω definition these bounds also give that $f(n) = \Omega(g(n))$ and $g(n) = \Omega(f(n))$.

The ratio of functions f and g is bounded both from above and below:

$$\frac{1}{c_1} \leq \frac{f(n)}{g(n)} \leq c_2,$$

so it cannot tend neither to zero nor to ∞. Therefore, it cannot be $f(n) = o(g(n))$ and $f(n) = \omega(g(n))$. Analogically, it cannot be $g(n) = o(f(n))$ and $g(n) = \omega(f(n))$.

Finally, we have the following answer:

		O	o	Ω	ω	Θ
$10n^2$	$n^2 + \log_2^2 n^{10}$	+	−	+	−	+
$n^2 + \log_2^2 n^{10}$	$10n^2$	+	−	+	−	+

(b) From inequality $10n^2 < 1000000$ we have $n < \sqrt{100000}$. By means of calculator we obtain $N_f = 316$. Using computer or calculator, the inequality $n^2 + \log_2^2 n^{10}$ may be solved by "trial and error" method. It is easy to show that $N_g = 995$, since $g(995) < 1000000$, but $g(996) > 1000000$.

Example 2. Let functions

$$f(n) = \left(\frac{n}{10\log_2 n}\right)^2 \quad \text{and} \quad g(n) = n\sqrt{n}$$

be given.

(a) We have to compare growth order of functions f and g. Let us denote $k = \sqrt{n}$. Then we can find an asymptotics of the ratio of given functions:

$$\begin{aligned}
\lim_{n\to\infty} \frac{g(n)}{f(n)} &= \lim_{n\to\infty} \frac{n\sqrt{n}}{\frac{n^2}{100\log_2^2 n}} = \lim_{n\to\infty} \frac{100\log_2^2 n}{\sqrt{n}} \\
&= \lim_{k\to\infty} \frac{400\log_2^2 k}{k} = \lim_{k\to\infty} \frac{800\log_2 k \cdot \frac{1}{k\ln 2}}{1} = \lim_{k\to\infty} \frac{800\log_2 k}{k\ln 2} \\
&= \lim_{k\to\infty} \frac{800 \cdot \frac{1}{k\ln 2}}{\ln 2} = \lim_{k\to\infty} \frac{800}{k\ln^2 2} = 0.
\end{aligned}$$

We proved that $g(n) = o(f(n))$ and $f(n) = \omega(g(n))$. From the last relations we obtain $g(n) = O(f(n))$ and $f(n) = \Omega(g(n))$. Clearly that functions f and g cannot have the same growth order: $f(n) \neq \Theta(g(n))$. Using similar arguments we can obtain the remaining relations and fill in the table:

		O	o	Ω	ω	Θ
$(\frac{n}{10\log_2 n})^2$	$n\sqrt{n}$	−	−	+	+	−
$n\sqrt{n}$	$(\frac{n}{10\log_2 n})^2$	+	+	−	−	−

(b) It remains to find N_f and N_g. Inequality $(\frac{n}{10\log_2 n})^2 < 1000000$ is equivalent to inequality $(\frac{n}{\log_2 n}) < 10000$, which can be solved by "trial and error" method by means of calculator. We obtain that $N_f = 174095$, since $f(174095) < 1000000$, but $f(174096) > 1000000$. The solution for second function is even more easy. From inequality $n^{3/2} \leq 10^6$ we have $n \leq 10^4$. So, $N_g = 10000$. It is interesting that although second function grows more slowly but for ”practical” n values the values of the first function are smaller. E.g., $f(10^{10})$ still is less than $g(10^{10})$, but $f(10^{11})$ is already more than $g(10^{11})$.

Exercise 2 (0.2 point). (a) Sort given functions f_1, f_2, f_3, f_4, f_5 in the increasing (nondecrerasing) order of their growth (each function should be O(next function)). Additionally indicate the functions that have the same growth order (each function is Θ of another function).

(b) Sort in the increasing (nondecreasing) order the values $f_1(n), f_2(n), f_3(n), f_4(n), f_5(n)$ for $n = 16$.

(c) Sort in the increasing (nondecreasing) order the values $f_1(n), f_2(n), f_3(n), f_4(n), f_5(n)$ for $n = 2^{16} = 65536$.

Example 3. Let us consider the functions $f_1(n) = 10n + \log_2^2(8^n)$, $f_2(n) = 100n \log_2 n$, $f_3(n) = 10n\sqrt{n}$, $f_4(n) = 2^{\sqrt{n}}$ and $f_5(n) = n^{\log_4 8}$.

(a) Firstly we rearrange functions f_1 and f_5:

$$f_1(n) = 10n + \log_2^2(2^{3n}) = 10n + (3n)^2 = 9n^2 + 10n,$$
$$f_5(n) = n^{\frac{\log_2 8}{\log_2 4}} = n^{3/2} = n\sqrt{n}.$$

Now let us sort the functions in the increasing order of their growth:

$$f_2(n) = 100n \log_2 n, \quad f_5(n) = n\sqrt{n}, \quad f_3(n) = 10n\sqrt{n}, \quad f_1(n) = 9n^2+10n, \quad f_4(n) = 2^{\sqrt{n}}.$$

Indeed:

$$\lim_{n\to\infty} \frac{100n \log_2 n}{n\sqrt{n}} = 100 \lim_{n\to\infty} \frac{\log_2 n}{\sqrt{n}} = 100 \lim_{n\to\infty} \frac{\frac{1}{n \ln 2}}{\frac{1}{2}\frac{1}{\sqrt{n}}} = 100 \lim_{n\to\infty} \frac{2}{\sqrt{n}\ln 2} = 0,$$

$$n\sqrt{n} = \Theta(10n\sqrt{n}), \quad \text{since } n\sqrt{n} = 0.1 \cdot 10n\sqrt{n},$$

$$\lim_{n\to\infty} \frac{10n\sqrt{n}}{9n^2 + 10n} = \lim_{n\to\infty} \frac{10\sqrt{n}}{9n + 10} = \lim_{n\to\infty} \frac{10}{9\sqrt{n} + \frac{10}{\sqrt{n}}} = 0$$

and finally

$$\lim_{n\to\infty} \frac{9n^2 + 10n}{2^{\sqrt{n}}} = \lim_{k\to\infty} \frac{9k^4 + 10k^2}{2^k} = \lim_{k\to\infty} \frac{36k^3 + 20k}{2^k \ln 2} = \lim_{k\to\infty} \frac{108k^2 + 20}{2^k \ln^2 2}$$
$$= \lim_{k\to\infty} \frac{216k}{2^k \ln^3 2} = \lim_{k\to\infty} \frac{216}{2^k \ln^4 2} = 0.$$

(b) After inserting $n = 16$, we have $f_1(16) = 2464$, $f_2(16) = 6400$, $f_3(16) = 640$, $f_4(16) = 16$ and $f_5(16) = 64$. So, we obtain the following ordering:

$$f_4(16) < f_5(16) < f_3(16) < f_1(16) < f_2(16).$$

(c) After inserting $n = 2^{16}$, we have $f_1(2^{16}) = 9 \cdot 2^{32} + 10 \cdot 2^{16}$, $f_2(2^{16}) = 100 \cdot 2^{16} \cdot 16 = 100 \cdot 2^{20}$, $f_3(2^{16}) = 10 \cdot 2^{24} = 160 \cdot 2^{20}$, $f_4(2^{16}) = 2^{256}$ ir $f_5(2^{16}) = 2^{24} = 16 \cdot 2^{20}$. So, we obtain the following ordering:

$$f_5(2^{16}) < f_2(2^{16}) < f_3(2^{16}) < f_1(2^{16}) < f_4(2^{16}).$$

Answer. Sorted order of given functions is the following: (a) 25314; (b) 45312; (c) 52314.