Homework 4

Exercise 1 (0.1 point). Master Theorem (Sect. 4.3 of Cormen book) is valid not only for integers but for any real positive values of parameter d. Let us consider the following its version:

Theorem 1. *If complexity $L(n)$ can be expressed by the recurrence relation*

$$L(n) = aL(n/b) + cn^d,$$

where $a \geq 1$ and $b > 1$ are integers, and $c, d \geq 0$, then $L(n)$ can be bounded asymptotically as follows:

$$L(n) = \begin{cases} \mathrm{O}(n^d), & \text{if } a < b^d, \\ \mathrm{O}(n^d \log_b n), & \text{if } a = b^d, \\ \mathrm{O}(n^{\log_b a}), & \text{if } a > b^d. \end{cases}$$

Apply Master Theorem and find asymptotics for the given recurrence relation.

Example 1. Multiplying square matrices by Strassen method we have $L(n) = 7L(n/2) + cn^2$. It is easy to see that $a = 7$, $b = 2$ and $d = 2$. Since $7 > 2^2$, we obtain $L(n) = \mathrm{O}(n^{\log_2 7}) \approx \mathrm{O}(n^{2.8074})$.

Problems

1. (a) $L(n) = 2L(n/2) + 4n$, (b) $L(n) = L(n/3) + 2n\sqrt{n}$;
2. (a) $L(n) = 6L(n/2) + 12n$, (b) $L(n) = L(n/4) + 8\sqrt[3]{n}$;
3. (a) $L(n) = 2L(n/3) + n^2$, (b) $L(n) = 3L(n/2) + 100$;
4. (a) $L(n) = 3L(n/3) + 9n$, (b) $L(n) = 2L(n/2) + 16$;
5. (a) $L(n) = L(n/2) + 10n$, (b) $L(n) = 6L(n/3) + 18$;
6. (a) $L(n) = 4L(n/2) + 4n\sqrt{n}$, (b) $L(n) = 2L(n/3) + 6n$;
7. (a) $L(n) = L(n/2) + \sqrt[3]{n}$, (b) $L(n) = 7L(n/3) + 14n$;
8. (a) $L(n) = 8L(n/3) + 16n$, (b) $L(n) = L(n/2) + 8$;
9. (a) $L(n) = 7L(n/3) + n^2$, (b) $L(n) = 5L(n/4) + 20n$;
10. (a) $L(n) = 7L(n/2) + 28n$, (b) $L(n) = L(n/3) + 27$;
11. (a) $L(n) = 5L(n/2) + n^2$, (b) $L(n) = 3L(n/3) + 18n^2$;
12. (a) $L(n) = 5L(n/3) + 10n^2$, (b) $L(n) = 8L(n/2) + 16n$;
13. (a) $L(n) = 2L(n/4) + 8$, (b) $L(n) = 3L(n/2) + 9n^2$;
14. (a) $L(n) = 2L(n/3) + 6n^3$, (b) $L(n) = 4L(n/2) + 16$;

15. (a) $L(n) = 6L(n/2) + 12n^2$, (b) $L(n) = L(n/4) + 12$;
16. (a) $L(n) = L(n/3) + 6\sqrt{n}$, (b) $L(n) = 4L(n/2) + 10n$;
17. (a) $L(n) = 5L(n/2) + 10n$, (b) $L(n) = 7L(n/4) + 28n^2$;
18. (a) $L(n) = 8L(n/2) + 16n^2$, (b) $L(n) = 4L(n/4) + 8n$;
19. (a) $L(n) = 2L(n/2) + 6\sqrt{n}$, (b) $L(n) = L(n/3) + 9n$;
20. (a) $L(n) = 4L(n/2) + 8n^2$, (b) $L(n) = 3L(n/3) + 3n^3$;
21. (a) $L(n) = 4L(n/3) + 16n$, (b) $L(n) = 2L(n/2) + 4n^2$;
22. (a) $L(n) = 6L(n/4) + 12n^2$, (b) $L(n) = 2L(n/3) + 7$;
23. (a) $L(n) = 8L(n/2) + 16n^3$, (b) $L(n) = 5L(n/2) + 10$;
24. (a) $L(n) = 3L(n/3) + 3\sqrt{n}$, (b) $L(n) = 2L(n/4) + 6n$;
25. (a) $L(n) = 2L(n/4) + 5\sqrt{n}$, (b) $L(n) = 4L(n/3) + 9$;
26. (a) $L(n) = 3L(n/2) + 17n$, (b) $L(n) = 2L(n/8) + 16\sqrt[3]{n}$;
27. (a) $L(n) = 4L(n/4) + 61n^2$, (b) $L(n) = 3L(n/2) + 6\sqrt{n}$;
28. (a) $L(n) = 3L(n/3) + 9$, (b) $L(n) = L(n/2) + 17n^2$;
29. (a) $L(n) = L(n/2) + 2n^3$, (b) $L(n) = L(n/4) + 7n/4$;
30. (a) $L(n) = 8L(n/4) + 2n\sqrt{n}$, (b) $L(n) = 6L(n/2) + 33$;

Exercise 2 (0.1 point). Let us consider Knapsack problem (section Dynamic programming; hardcopy was given in autumn semester). NAME is an array of thing names, SIZE of thing sizes and VALUE of thing values. The capacity of knapsack is M. Apply dynamic programming and find optimal knapsack fulfilment.

Problems

1. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 6, 8, 9, 13\}$, $M = 22$.
2. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{5, 6, 8, 9, 13\}$, $M = 19$.
3. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 8, 9\}$, VALUE $= \{4, 6, 8, 11, 13\}$, $M = 19$.
4. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{5, 6, 7, 10, 13\}$, $M = 20$.

5. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 6, 8, 10, 12\}$, $M = 19$.
6. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 5, 8, 10, 13\}$, $M = 21$.
7. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{5, 7, 8, 11, 14\}$, $M = 21$.
8. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 6, 10, 12, 13\}$, $M = 19$.
9. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 7, 8, 10, 13\}$, $M = 21$.
10. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{5, 6, 9, 11, 16\}$, $M = 19$.
11. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{5, 7, 9, 12, 15\}$, $M = 19$.
12. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 6, 8, 12, 14\}$, $M = 20$.
13. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 5, 8, 10, 13\}$, $M = 19$.
14. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{5, 7, 9, 11, 14\}$, $M = 21$.
15. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 6, 10, 12, 13\}$, $M = 19$.
16. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 7, 8, 10, 13\}$, $M = 21$.
17. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{5, 7, 9, 11, 14\}$, $M = 18$.
18. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{5, 7, 9, 12, 15\}$, $M = 19$.
19. NAME $= \{A, B, C, D, E\}$, SIZE $= \{3, 4, 5, 7, 9\}$, VALUE $= \{4, 6, 9, 12, 15\}$, $M = 21$.
20. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 5, 7\}$, VALUE $= \{4, 6, 8, 10\}$, $M = 26$.
21. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 5, 7\}$, VALUE $= \{4, 6, 8, 11\}$, $M = 26$.

22. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 5, 7\}$, VALUE $= \{4, 5, 8, 10\}$, $M = 27$.

23. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 5, 7\}$, VALUE $= \{4, 6, 9, 12\}$, $M = 27$.

24. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 5, 7\}$, VALUE $= \{4, 6, 9, 11\}$, $M = 24$.

25. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 5, 7\}$, VALUE $= \{5, 7, 9, 11\}$, $M = 27$.

26. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 5, 7\}$, VALUE $= \{5, 7, 9, 12\}$, $M = 26$.

27. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 7, 8\}$, VALUE $= \{5, 6, 10, 13\}$, $M = 26$.

28. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 7, 8\}$, VALUE $= \{5, 7, 11, 13\}$, $M = 25$.

29. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 7, 8\}$, VALUE $= \{5, 7, 10, 13\}$, $M = 23$.

30. NAME $= \{A, B, C, D\}$, SIZE $= \{3, 4, 7, 8\}$, VALUE $= \{4, 6, 10, 13\}$, $M = 26$.

Exercise 3 (0.2 point). Apply Branch&Bound method to find an optimal tour for Travelling Salesman problem when asymmetric matrix of distances between 5 cities is given. See page 60 of http://uosis.mif.vu.lt/ valdas/ALGORITMAI/Vadovelis/algoritmu_analize.pdf or http://faculty.cs.byu.edu/ringger/Winter2006-CS312/lectures/lecture28-tspbandb2.pdf

Problems

1.
```
 0  11  17  16   1
13   0  15  24  24
 9   4   0   6  29
21   6   7   0  25
29   9  15  28   0
```

2.
```
 0  19  14   4  29
21   0  22   9  26
28  30   0   3   8
11  27   9   0   2
18  21  26  10   0
```

3.

0	5	10	7	10
23	0	9	20	10
25	29	0	2	22
20	25	5	0	29
8	28	7	21	0

4.

0	19	29	27	29
4	0	10	23	6
5	7	0	12	24
28	22	29	0	19
5	17	18	5	0

5.

0	29	7	1	5
9	0	6	27	5
30	2	0	11	10
29	18	1	0	1
7	12	24	30	0

6.

0	4	5	3	6
24	0	8	1	30
30	5	0	5	22
4	7	12	0	27
19	12	16	27	0

7.

0	28	27	1	30
9	0	18	12	22
2	1	0	20	24
8	12	10	0	8
6	21	8	24	0

8.

0	26	30	20	21
22	0	15	11	16
30	21	0	29	7
20	21	20	0	22
17	25	10	3	0

9.

0	11	25	22	27
7	0	27	29	11
5	2	0	20	2
20	3	22	0	22
1	13	26	11	0

10.

0	12	29	17	24
13	0	25	14	20
3	11	0	26	1
14	23	19	0	4
27	20	2	15	0

11.

0	8	14	27	12
7	0	10	14	12
10	26	0	24	15
27	6	27	0	30
10	16	23	2	0

12.

0	15	17	17	18
18	0	19	9	17
16	25	0	24	7
5	26	5	0	3
15	19	13	1	0

13.

0	3	8	24	17
7	0	28	14	9
2	14	0	30	12
19	2	27	0	2
17	12	20	16	0

14.

0	5	2	12	23
5	0	12	28	12
6	16	0	21	14
3	29	13	0	2
25	4	13	5	0

15.

0	11	12	16	29
24	0	28	3	12
24	15	0	23	22
12	12	12	0	21
8	23	28	19	0

16.

0	20	2	4	24
18	0	12	15	9
29	15	0	29	6
8	23	29	0	21
16	4	15	10	0

17.

0	30	6	29	13
26	0	2	14	9
27	4	0	17	9
22	22	3	0	23
23	27	20	15	0

18.

0	12	25	22	29
22	0	14	20	19
23	12	0	21	24
20	22	30	0	1
25	6	11	20	0

19.

0	2	3	30	4
6	0	10	21	21
16	13	0	8	9
7	6	30	0	20
17	2	17	29	0

20.

0	15	13	10	14
8	0	13	25	1
20	8	0	30	9
10	22	27	0	21
15	26	21	30	0

21.

0	1	27	17	30
6	0	23	24	7
30	23	0	19	28
11	1	21	0	20
5	4	30	2	0

22.

0	8	30	17	16
14	0	5	6	27
24	22	0	18	19
9	22	25	0	2
21	19	2	3	0

23.

0	20	14	8	13
25	0	3	1	6
19	15	0	12	16
26	21	17	0	22
4	4	8	18	0

24.

0	1	21	27	5
30	0	18	23	23
27	29	0	20	10
15	2	27	0	14
28	9	15	6	0

25.

0	11	10	12	27
11	0	18	26	19
8	3	0	26	5
30	8	23	0	5
28	28	4	8	0

26.

0	20	23	8	19
19	0	2	12	30
11	15	0	15	11
7	15	24	0	9
14	3	7	19	0

27.

0	14	28	20	6
9	0	25	24	2
18	30	0	2	10
25	19	10	0	25
29	29	17	26	0

28.

0	30	7	17	14
28	0	10	28	18
10	20	0	24	4
7	24	20	0	10
11	14	20	23	0

29.

0	3	25	10	15
29	0	4	18	13
1	21	0	5	8
4	30	7	0	30
13	12	3	10	0

30.

0	25	23	19	10
14	0	9	2	1
10	10	0	13	14
1	1	13	0	21
5	11	27	5	0