A REMARK ON THE C-MONOTONICITY OF OPTIMAL TRANSPORT WITH CAPACITY CONSTRAINTS

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Abstract. This paper studies the geometry of the optimizer for optimal transport with capacity constraints problem (capacity OT). We introduce the c-capacity monotonicity for capacity OT, which is a generalization of c-cyclical monotonicity in optimal transport. We show that the optimizer of capacity OT is c-capacity monotone.

1. Introduction

Considering the capacity limitations of transport tools, Korman and Mccann in [1] propose the optimal transport with capacity constraints problem where the transport coupling density is bounded above. suppose X and Y are compact subsets in $\mathbb{R}^d$ and $0 \leq \overline{h} \in L^\infty(X \times Y)$. Let $f(x)dx$ be a probability measure on X and $g(y)dy$ a probability measure on Y. Suppose $\Pi(f,g)^{\overline{h}}$ is the set of all transport couplings with marginals $f(x)dx$ and $g(y)dy$ and bounded above by $\overline{h}$ a.e., which is given by

$$\Pi(f,g)^{\overline{h}} = \Big\{ h(x,y)dxdy : 0 \leq h \in L^1[X \times Y], \int_X h(x,y)dy = f(x), \\ \int_Y h(x,y)dx = g(y), h(x,y) \leq \overline{h}(x,y) \ a.e. \Big\}.$$

Then optimal transportation problem with capacity constraints is given by

$$\inf_{h \in \Pi(f,g)^{\overline{h}}} \int_{X \times Y} c(x,y)h(x,y)dxdy,$$

where the cost function $c(x,y)$ is continuous and bounded on $X \times Y$. Korman and Mccann show the existence and uniqueness of the optimizer in [1, 2], and prove the duality in [3, 4]:

$$\inf_{h \in \Pi(f,g)^{\overline{h}}} \int_{X \times Y} c(x,y)h(x,y)dxdy = \sup_{(\phi,\psi,w) \in \Phi_c} \Big\{ \int_X \phi(x)f(x)dx + \int_Y \psi(y)g(y)dy \\ - \int_{X \times Y} w(x,y)\overline{h}(x,y)dxdy \Big\},$$

where Φ_c is defined as follows:

$$\Phi_c := \Big\{ (\phi,\psi,w) : \phi(x) + \psi(y) \leq c(x,y) + w(x,y) \text{ where } \phi \in L^1(f(x)dx), \\ \psi \in L^1(g(y)dy), 0 \leq w \in L^1(\overline{h}(x,y)dxdy) \Big\}.$$

However, the study on c-cyclical monotonicity of capacity OT is missing in the above work. Recall that a set $\Lambda \subset X \times Y$ is said to be c-cyclically monotone if for any finite sequence $\{(x_i, y_i)\}_{i=1}^N \subset \Lambda$, the following holds

$$\sum_{i=1}^N c(x_i, y_i) \leq \sum_{i=1}^N c(x_{i+1}, y_i),$$

where $x_{N+1} \equiv x_1$. Furthermore, a measure is said to be c-cyclically monotone if it is concentrated on a c-cyclically monotone set. C-cyclical monotonicity is useful since it relates the geometry of a transport coupling to the optimality of the minimizer.

Theorem 1.1 (Theorem 2.4.3 in [5]). *Let μ and ν be probability measures on $\mathbb{R}^d$. Suppose the cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous. Then $\pi \in \Pi(\mu, \nu)$ is the optimal transport coupling if and only if the support of π is c-cyclically monotone.*

The optimizer of optimal transport with capacity constraints is usually different from the transport problem without capacity constraints, which means that the optimizer of constrained transport problem is not c-cyclically monotone. Therefore, it is natural to ask whether we could generalize the definition of c-cyclical monotonicity to transport with capacity constraints setting.

Indeed, similar work has been done recently. Zaev considered optimal transport with linear constraints problem and introduced (c, W)–monotonicity in [6]. He claimed that the optimizer of linearly constrained transport problem must be (c, W)–monotone. Later, Mathias Beiglböck and Claus Griessler studied the Generalized Moment Problem and introduced c–monotonicity in [7]. They claimed that if the optimizer of Generalized Moment Problem exists, it must be c–monotone.

In this paper, we generalize the definition of c-cyclical monotonicity for optimal transport with the capacity constraints problem, called c-capacity monotonicity, and claim that the optimizer of capacity OT is c-capacity monotone.

2. c-capacity monotonicity

The generalization is based on the duality of optimal transport with capacity constraints problem. Recall that the duality is given by

$$\inf_{h \in \Pi(f,g)^{\overline{h}}} \int_{X \times Y} c(x,y)h(x,y)dxdy = \sup_{(\phi,\psi,w)\in\Phi_c} \Big\{ \int_X \phi(x)f(x)dx + \int_Y \psi(y)g(y)dy - \int_{X\times Y} w(x,y)\overline{h}(x,y)dxdy \Big\}.$$

Note that in the duality, $w(x, y) \geq 0$, which means $w(x, y)$ is the additional cost due to capacity constraints. Since the cost function $c(x, y)$ is continuous and bounded on $X \times Y$, then

$$\sup_{(\phi,\psi,w)\in\Phi_c} \Big\{ \int_X \phi(x)f(x)dx + \int_Y \psi(y)g(y)dy - \int_{X\times Y} w(x,y)\overline{h}(x,y)dxdy \Big\} < +\infty.$$

Therefore, there exists a maximizing sequence. Suppose $(\phi_k(x), \psi_k(y), w_k(x, y))_{k=1}^\infty \subset \Phi_c$ is a maximizing sequence and let $f_\# \mu$ be the pushforward of μ by a measurable map f. Then we have the following definition for competitors.

Definition 2.1. Given a probability measure γ on the set $\Omega \subset X \times Y$, the competitor of γ is said to be a probability measure γ' on Ω such that for the additional cost $\{w_k(x, y)\}_{k=1}^\infty$ in the maximizing sequence of the duality for capacity OT,

$$\int_\Omega w_k(x,y)d\gamma(x,y) = \int_\Omega w_k(x,y)d\gamma'(x,y), \ \forall\ k \in \mathbb{N}^+. \tag{2.1}$$

$$P_{1\#}\gamma = P_{1\#}\gamma', P_{2\#}\gamma = P_{2\#}\gamma'. \tag{2.2}$$

where P_1 and P_2 are projections on x and y coordinates. And if γ is supported on finitely many points of Ω, the competitor of γ is also finitely supported on Ω.

Definition 2.2. A set Γ is said to be c-capacity monotone if each finite measure γ, which is supported on finitely many points on Γ, is cost-minimizing amongst its competitors, i.e., if γ' is a competitor of γ, then

$$\int_\Gamma c(x,y)d\gamma(x,y) \leq \int_\Gamma c(x,y)d\gamma'(x,y).$$

Furthermore, a probability measure γ is said to be c-capacity monotone if it is concentrated on a c-capacity monotone set.

In the definition of competitors of the measure γ, Equation (2.1) means that for a family of additional cost $\{w_k(x,y)\}_{k=1}^\infty$ (maximizing sequence in the duality), γ and its competitors share the same additional transportation cost due to capacity constraints. Equation (2.2) means that γ and its competitors have the same mass and marginals. Note that if for each k, $w_k(x,y) = 0$ (no capacity constraints), c-capacity monotonicity will become c-cyclical monotonicity.

Based on the above definitions, we claim the optimizer of capacity OT is c-capacity monotone.

Theorem 2.3. *If $\gamma^* = h^*(x,y)dxdy$ is the optimizer of optimal transport with capacity constraints problem, then γ^* is c-capacity monotone.*

Proof. Let $(\phi_k(x), \psi_k(y), w_k(x,y))_{k=1}^\infty \subset \Phi_c$ be the maximizing sequence in the duality of capacity OT, and for each k, let

$$c_k(x,y) := c(x,y) + w_k(x,y) - \phi_k(x) - \psi_k(y).$$

Then $c_k(x,y) \geq 0$ and

$$\begin{aligned}\int_{X\times Y} c_k(x,y)d\gamma^*(x,y) = &\int_{X\times Y} c(x,y)d\gamma^*(x,y) + \int_{X\times Y} w_k(x,y)d\gamma^*(x,y) \\ &- \int_X \phi_k(x)f(x)dx - \int_Y \psi_k(y)g(y)dy.\end{aligned}$$

Since $w_k(x,y) \geq 0$ and $0 \leq h^*(x,y) \leq \overline{h}(x,y)$ a.e., then

$$\int_{X\times Y} w_k(x,y)d\gamma^*(x,y) = \int_{X\times Y} w_k(x,y)h^*(x,y)dxdy \leq \int_{X\times Y} w_k(x,y)\overline{h}(x,y)dxdy.$$

Therefore,

$$\begin{aligned}\int_{X\times Y} c_k(x,y)d\gamma^*(x,y) \leq &\int_{X\times Y} c(x,y)d\gamma^*(x,y) + \int_{X\times Y} w_k(x,y)\overline{h}(x,y)dxdy \\ &- \int_X \phi_k(x)f(x)dx - \int_Y \psi_k(y)g(y)dy.\end{aligned}$$

Since $(\phi_k(x), \psi_k(y), w_k(x,y))_{k=1}^\infty \subset \Phi_c$ is the maximizing sequence in the duality and $\gamma^* = h^*(x,y)dxdy$ is the optimizer, then

$$\begin{aligned}\int_{X\times Y} c(x,y)d\gamma^*(x,y) = \lim_{k\to\infty} \Big\{ &\int_X \phi_k(x)f(x)dx + \int_Y \psi_k(y)g(y)dy \\ &- \int_{X\times Y} w_k(x,y)\overline{h}(x,y)dxdy\Big\}.\end{aligned}$$

Thus as $k \to \infty$,

$$0 \leq \lim_{k\to\infty} \int_{X\times Y} c_k(x,y) d\gamma^*(x,y) \leq 0.$$

Therefore, by Fatou's lemma, we have

$$0 \leq \int_{X\times Y} \liminf_{k\to\infty} c_k d\gamma^* \leq \liminf_{k\to\infty} \int_{X\times Y} c_k d\gamma^* = 0.$$

That is to say,

$$\int_{X\times Y} \liminf_{k\to\infty} c_k(x,y) d\gamma^*(x,y) = 0.$$

Since $\liminf_{k\to\infty} c_k \geq 0$, then we can find a set $\Gamma \subset X \times Y$ with $\gamma^*(\Gamma) = 1$ and $\liminf_{k\to\infty} c_k = 0$ on Γ. Hence we can find a subsequence c_{k_j} such that $c_{kj} \to 0$ on Γ. Now we are going to show Γ is c-capacity monotone. Let $S = \{(x_i, y_i)\}_{i=1}^N \subset \Gamma$ and β_s is a probability measure with support S. Let α be the competitor of β_s. Since $(\phi_{k_j}(x), \psi_{k_j}(y), w_{k_j}(x,y))_{j=1}^{\infty} \subset \Phi_c$, then for each j,

$$\int_S c(x,y)d\alpha(x,y) \geq \int_S \phi_{k_j}(x)d\alpha(x,y) + \int_S \psi_{k_j}(y)d\alpha(x,y) - \int_S w_{k_j}(x,y)d\alpha(x,y).$$

By the definition of competitors, we have

$$\int_S w_{k_j}(x,y)d\alpha(x,y) = \int_S w_{k_j}(x,y)d\beta_s(x,y),$$

$$\int_S \phi_{k_j}(x)d\alpha(x,y) = \int_S \phi_{k_j}(x)d\beta_s(x,y),$$

$$\int_S \psi_{k_j}(y)d\alpha(x,y) = \int_S \psi_{k_j}(y)d\beta_s(x,y).$$

Thus for each j, we get

$$\begin{aligned}\int_S c(x,y)d\alpha(x,y) &\geq \int_S \phi_{k_j}(x) + \psi_{k_j}(y) - w_{k_j}(x,y)\ d\beta_s(x,y)\\ &= \int_S c(x,y) - c_{k_j}(x,y)\ d\beta_s(x,y).\end{aligned}$$

As $j \to \infty$, we have

$$\int_S c(x,y)\ d\beta_s(x,y) \leq \int_S c(x,y)d\alpha(x,y).$$

The last step follows from the fact that $c_{k_j} \geq 0$ and $c - c_{k_j} \leq c$. Since $c_{k_j}(x,y) \to 0$ on $S \subset \Gamma$ and $c(x,y)$ is bounded, then by dominated convergence theorem, we get

$$\int_S c(x,y) - c_{k_j}(x,y)\ d\beta_s(x,y) \to \int_S c(x,y)\ d\beta_s(x,y).$$

Therefore, any probability measure β_s finitely supported on Γ is cost-minimizing among its competitors. Hence Γ is a c-capacity monotone set. Since $\gamma^*(\Gamma) = 1$, then γ^* is c-capacity monotone. This completes the proof. $\square$

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