PALEY–WIENER THEOREM FOR PROBABILISTIC FRAMES

DONGWEI CHEN

Abstract. The Paley–Wiener Theorem is a classical result about the stability of basis in Banach spaces claiming that if a sequence is close to a basis, then this sequence is a basis. Similar results are also extended to frames in Hilbert spaces. As the extension of finite frames for $\mathbb{R}^d$, probabilistic frames are probability measures on $\mathbb{R}^d$ with finite second moments and the support of which span $\mathbb{R}^d$. This paper generalizes the Paley–Wiener theorem to the probabilistic frame setting. We claim that if a probability measure is close to a probabilistic frame, then this probability measure is also a probabilistic frame.

Contents

1. Introduction 1
2. Mathematical Preliminaries 4
2.1. Probabilistic Frames and Optimal Transport 4
2.2. Invertibility of Linear Operators on Banach Spaces 6
3. Paley-Wiener Theorem for Probabilistic Frames 7
4. Perturbations Including Probabilistic Dual Frames 12
Acknowledgement 18
References 18

1. Introduction

First proposed by Paley and Wiener in [1], the Wiener theorem is a classical result about the stability and perturbation analysis of basis in a Hilbert space, which claims that if a sequence is close to an orthonormal basis in a Hilbert space, then this sequence also forms a basis. However, Boas in [2] noticed that Paley and Wiener's proof still holds in a Banach space:

Theorem 1.1 (Theorem 1 in [2], Theorem 10 in [3]). *Let $\{x_i\}_{i=1}^{\infty}$ be a basis for a Banach space $\mathcal{X}$ and suppose $\{y_i\}_{i=1}^{\infty}$ is a sequence of elements of $\mathcal{X}$ such that*

$$\Big\|\sum_{i=1}^{n} c_i(x_i - y_i)\Big\| \leq \lambda \Big\|\sum_{i=1}^{n} c_i x_i\Big\|$$

for some constant $0 \leq \lambda < 1$, and all choices of scalars $c_1, \dots, c_n (n = 1, 2, 3, \dots)$. Then $\{y_i\}_{i=1}^{\infty}$ is a basis for the Banach space $\mathcal{X}$ and is equivalent to $\{x_i\}_{i=1}^{\infty}$ [1].

[1]Equivalence of basis $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ for the Banach space $\mathcal{X}$ means that there exists a bounded and invertible operator T on $\mathcal{X}$ such that $Tx_i = y_i$, for any i.

Since then, many variations of this stability theorem have been generalized to study the perturbation theory of basis in a Banach space [4, p. 84-109], entire functions of exponential type [3, p. 85], and frames [5–23]. For a more complete treatment of frame perturbation theory, see [24, Chapter 22] for more information.

As the extension of orthonormal basis, frames were first introduced by Duffin and Schaeffer in the context of nonharmonic analysis [25] and have been applied in pure and applied mathematics, for instance, the Kadison-Singer problem [26], time-frequency analysis [27], wavelet analysis [28], coding theory [29], and sampling theory [30]. Recall that a sequence $\{f_i\}_{i=1}^{\infty}$ in a separable Hilbert space $\mathscr{H}$ is said to be a frame for $\mathscr{H}$ if there exist $0 < A \le B < \infty$ such that for any $f \in \mathscr{H}$,

$$A\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

A and B are called lower and upper frame bounds. Furthermore, $\{f_i\}_{i=1}^{\infty}$ is said to be a tight frame if $A = B$ and Parseval if $A = B = 1$. Suppose $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathscr{H}$ with bounds $0 < A \le B < \infty$. A frame $\{g_i\}_{i=1}^{\infty}$ for $\mathscr{H}$ is said to be a dual frame of $\{f_i\}_{i=1}^{\infty}$ if for any $f \in \mathscr{H}$,

$$f = \sum_{i=1}^{\infty} \langle f, g_i\rangle f_i = \sum_{i=1}^{\infty} \langle f, f_i\rangle g_i.$$

An example of dual frames for $\{f_i\}_{i=1}^{\infty}$ is its canonical dual frame $\{S^{-1}f_i\}_{i=1}^{\infty}$ with frame bounds $0 < \frac{1}{B} \le \frac{1}{A} < \infty$, where S is the frame operator of $\{f_i\}_{i=1}^{\infty}$:

$$S : \mathscr{H} \to \mathscr{H}, \quad S(f) = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i.$$

For interested readers, we refer to [24, 31–34] for more details about frames.

Christensen first generalized Theorem 1.1 to study the stability of frames in Hilbert spaces by the following theorem:

Theorem 1.2 (Theorem 1 in [5]). *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for a Hilbert space $\mathscr{H}$ with bounds $0 < A \le B < \infty$. Let $\{g_i\}_{i=1}^{\infty}$ be a sequence in $\mathscr{H}$ and assume that there exist constants $\lambda, \delta \ge 0$ such that $\lambda + \frac{\delta}{\sqrt{A}} < 1$ and*

$$\Big\|\sum_{i=1}^{n} c_i(f_i - g_i)\Big\| \le \lambda\Big\|\sum_{i=1}^{n} c_i f_i\Big\| + \delta\Big[\sum_{i=1}^{n}|c_i|^2\Big]^{1/2}$$

for all scalars $c_1, \dots, c_n (n = 1, 2, 3, \dots)$. Then $\{g_i\}_{i=1}^{\infty}$ is a frame for $\mathscr{H}$ with bounds

$$A\Big(1 - (\lambda + \frac{\delta}{\sqrt{A}})\Big)^2 \text{ and } B\Big(1 + \lambda + \frac{\delta}{\sqrt{B}}\Big)^2.$$

Later, Casazza and Christensen improved Theorem 1.2 by adding one more term related to the sequence $\{g_i\}_{i=1}^{\infty}$ on the right-hand side of the inequality:

Theorem 1.3 (Theorem 2 in [8]). *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for a Hilbert space $\mathscr{H}$ with bounds $0 < A \le B < \infty$. Let $\{g_i\}_{i=1}^{\infty}$ be a sequence in $\mathscr{H}$ and assume that there exist constants $\lambda_1, \lambda_2, \delta \ge 0$ such that $max(\lambda_1 + \frac{\delta}{\sqrt{A}}, \lambda_2) < 1$ and*

$$\Big\|\sum_{i=1}^{n} c_i(f_i - g_i)\Big\| \le \lambda_1\Big\|\sum_{i=1}^{n} c_i f_i\Big\| + \lambda_2\Big\|\sum_{i=1}^{n} c_i g_i\Big\| + \delta\Big[\sum_{i=1}^{n}|c_i|^2\Big]^{1/2}$$

for all scalars $c_1, \dots, c_n (n = 1, 2, 3, \dots)$. *Then* $\{g_i\}_{i=1}^{\infty}$ *is a frame for* $\mathcal{H}$ *with bounds*

$$A\Big(1 - \frac{\lambda_1 + \lambda_2 + \frac{\delta}{\sqrt{A}}}{1+\lambda_2}\Big)^2 \text{ and } B\Big(1 + \frac{\lambda_1 + \lambda_2 + \frac{\delta}{\sqrt{B}}}{1-\lambda_2}\Big)^2.$$

From then on, Paley-Wiener type theorems have been studied for many mathematical objects, such as Banach frames [9], frames containing Riesz basis [10], frame sequence [11], sequences with reconstruction properties in a Banach space [13], von Neumann–Schatten dual frames, [15], Operator represented frames [16], g–frames [12, 18, 23], continuous frames on quaternionic Hilbert spaces [19], approximately dual frames [21], frames for metric spaces [22], Hilbert–Schmidt frames and sequences [17, 20]. Especially in [14], they introduced dual frames in the perturbation condition, which differs from previous conditions to preserve Hilbert frames:

Theorem 1.4 (Theorem 2.1 in [14])**.** *Let* $\{f_i\}_{i=1}^{\infty}$ *be a frame for the Hilbert space* $\mathcal{H}$ *with bounds* $0 < A \le B < \infty$, *and let* $\{h_i\}_{i=1}^{\infty}$ *denote a dual frame of* $\{f_i\}_{i=1}^{\infty}$ *with upper frame bound* $0 < D < \infty$. *Suppose* $\{g_i\}_{i=1}^{\infty}$ *is a sequence in* $\mathcal{H}$ *such that*

$$\alpha := \sum_{i=1}^{\infty} \|f_i - g_i\|^2 < \infty,\ \beta := \sum_{i=1}^{\infty} \|f_i - g_i\| \|h_i\| < 1.$$

Then $\{g_i\}_{i=1}^{\infty}$ *is a frame in* $\mathcal{H}$ *with bounds* $\frac{(1-\beta)^2}{D}$ *and* $B(1 + \sqrt{\frac{\alpha}{B}})$.

In this paper, we generalize the frame perturbation theory to probabilistic frames for $\mathbb{R}^d$ that are probability measures on $\mathbb{R}^d$ satisfying frame-like condition: A probability measure μ on $\mathbb{R}^d$ is said to be a probabilistic frame if there exist $0 < A \le B < \infty$ such that for any $x \in \mathbb{R}^d$,

$$A\|x\|^2 \le \int_{\mathbb{R}^d} |\langle x, y\rangle|^2 d\mu(y) \le B\|x\|^2.$$

μ is said to be a tight probabilistic frame if $A = B$ and Parseval if $A = B = 1$.

It is worthy to note that by taking $\mu_f := \sum_{i=1}^{N} \frac{1}{N}\delta_{y_i} \in \mathcal{P}(\mathbb{R}^d)$, the (finite) frame condition for $\mathbb{R}^d$ is equivalent to

$$\frac{A}{N}\|x\|^2 \le \int_{\mathbb{R}^d} |\langle x, y\rangle|^2 d\mu_f(y) \le \frac{B}{N}\|x\|^2, \text{ for any } x \in \mathbb{R}^d.$$

Probabilistic frames for $\mathbb{R}^d$ were first introduced by Martin Ehler in [35], then studied in directional statistics [36], minimization of p–frame potential [37], and further reviewed in [38]. C.G. Wickman generalized the definition of dual frames, analysis operator, and synthesis operator to probabilistic frames [39–41], and used gradient flows to study the probabilistic p-frame potential [42]. Minimizing problems about probabilistic frames under Wasserstein metric from optimal transport, like finding the closest Parseval probabilistic frame, are open and active topics [43–46]. Some equalities and inequalities for probabilistic frames are also studied in [47]. For interested readers, we refer to Section 2 for detailed introductions to probabilistic frames, optimal transport (and invertibility of linear operators on Banach spaces).

This paper is organized as follows. In Section 2, we describe the mathematical preliminaries. In Section 3, we generalize Paley-Wiener theorem to probabilistic frames. First, we generalize Theorem 1.2 to Theorem 3.1 by using integration and continuous functions with compact support as test functions. Then in Lemma 3.2, we consider a special case that is inspired by a particular case in the perturbation condition with $\lambda = 0, \delta = \sqrt{R}$ in Theorem 3.1. Furthermore, we add one more term related to the probability measure on the right-hand side of the inequality, which generalizes Theorem 1.3 and Theorem 3.1 to Theorem 3.4.

In Section 4, we give a sufficient perturbation condition by including the probabilistic dual frames, which generalizes Theorem 1.4 to Theorem 4.1. Then in Corollary 4.2, we consider a particular situation where the probabilistic dual frame is given by the canonical probabilistic dual frame. Nevertheless, the perturbation condition is given by a product measure $\mu \otimes \eta$. As stated in Proposition 4.3, we claim that the perturbation condition in Corollary 4.2 is also true if we use any transport coupling $\gamma \in \Gamma(\mu, \eta)$. In the end, we use a different reconstruction formula given by the canonical probabilistic Parseval frame and further generalize Corollary 4.2 and Proposition 4.3 to Proposition 4.4.

2. Mathematical Preliminaries

This section introduces probabilistic frames and the invertibility of (bounded) linear operators on Banach spaces. We also briefly introduce optimal transport and Wasserstein distance that is often used to quantify the distance between two probabilistic frames.

2.1. Probabilistic Frames and Optimal Transport.

Let $\mathscr{P}(\mathbb{R}^d)$ denote the set of Borel probability measures on $\mathbb{R}^d$, $\|\cdot\|$ the Euclidean norm, and $\mathscr{P}_2(\mathbb{R}^d)$ the set of probability measure $\mu \in \mathscr{P}(\mathbb{R}^d)$ with finite second moments $M_2(\mu)$, i.e.,

$$M_2(\mu) := \int_{\mathbb{R}^d} \|x\|^2 d\mu(x) < +\infty.$$

Let $B_r(x)$ be the open ball centered at x with radius $r > 0$. The support of $\mu \in \mathscr{P}(\mathbb{R}^d)$ is defined by

$$supp(\mu) = \Big\{x \in \mathbb{R}^d : \text{for any } r > 0, \mu(B_r(x)) > 0\Big\}.$$

Before going further, we need to introduce the pushforward of a probability measure by a measure map.

Definition 2.1 (Pushforward)**.** Let $M, N > 0$ and $\mu \in \mathscr{P}(\mathbb{R}^M)$. The pushforward of μ by a measurable map $f : \mathbb{R}^M \to \mathbb{R}^N$ is a probability measure in $\mathbb{R}^N$, which is denoted by $f_\# \mu$ and

$$f_\#\mu(E) := (\mu \circ f^{-1})(E) = \mu\big(f^{-1}(E)\big), \text{for any Borel set } E \subset \mathbb{R}^N.$$

Furthermore, we have the change-of-variables formula:

$$\int_{\mathbb{R}^N} g(y) d(f_\#\mu)(y) = \int_{\mathbb{R}^M} g(f(x)) d\mu(x),$$

where g is measurable such that $g \in L^1(\mathbb{R}^N, f_\#\mu)$ and $g \circ f \in L^1(\mathbb{R}^M, \mu)$.

We have the following definitions for probabilistic frames and frame operators.

Definition 2.2 (Probabilistic Frame)**.** $\mu \in \mathscr{P}(\mathbb{R}^d)$ is said to be a probabilistic frame if there exist $0 < A \leq B < \infty$ such that for any $x \in \mathbb{R}^d$,

$$A\|x\|^2 \leq \int_{\mathbb{R}^d} \langle x, y \rangle^2 \, d\mu(y) \leq B\|x\|^2.$$

μ is said to be a tight probabilistic frame if $A = B$ and Parseval if $A = B = 1$. And μ is said to be a Bessel probability measure if only the upper bound holds.

By Cauchy-Schwartz inequality, it is easy to show that if $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, then μ is a Bessel probability measure with bound $M_2(\mu)$.

Definition 2.3 (Frame Operator)**.** Let μ be a probabilistic frame. The frame operator S_μ for μ is defined by

$$S_\mu := \int_{\mathbb{R}^d} yy^T d\mu(y).$$

Note that S_μ is a $d \times d$ matrix and $S_\mu > 0$ means S_μ is positive definite. $\|S_\mu\|_2$ is used to denote the 2-matrix norm of S_μ. Let the frame bounds of μ be $0 < A \leq B < \infty$ and $S_\mu > 0$. Since S_μ is symmetric and each eigenvalue of S_μ is within $[A, B]$, then

$$A \leq \|S_\mu\|_2 \leq B, \ \frac{1}{B} \leq \|S_\mu^{-1}\|_2 \leq \frac{1}{A}, \ \frac{1}{\sqrt{B}} \leq \|S_\mu^{-1/2}\|_2 \leq \frac{1}{\sqrt{A}}.$$

We also have the following characterization for probabilistic frames.

Proposition 2.4 (Theorem 12.1 in [38], Proposition 3.1 in [43])**.** *Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. Then the following holds:*

(1) *μ is a probabilistic frame $\Leftrightarrow$ $S_\mu > 0$ (positive definite) $\Leftrightarrow$ $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $span\{supp(\mu)\} = \mathbb{R}^d$.*
(2) *μ is a tight probabilistic frame with bound $A > 0 \Leftrightarrow S_\mu = AI_{d\times d}$ where $I_{d\times d}$ is the $d \times d$ identity matrix. Furthermore, if μ is tight with bound $A > 0$,*
$$f = \frac{1}{A}\int_{\mathbb{R}^d} \langle f, x\rangle\, x \ d\mu(x), \ \forall f \in \mathbb{R}^d.$$
(3) *μ is a Parseval probabilistic frame $\Leftrightarrow S_\mu = I_{d\times d}$.*

An obvious metric to quantify the distance between two probabilistic frames μ and ν is 2-Wasserstein metric $W_2(\mu, \nu)$ in optimal transport, which is given by

$$W_2^2(\mu,\nu) = \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x - y\|^2 d\gamma(x, y),$$

where $\Gamma(\mu, \nu)$ is the set of transport couplings on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν

$$\Gamma(\mu,\nu) = \Big\{\gamma \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d) : P_{1\#}\gamma = \mu, \ P_{2\#}\gamma = \nu\Big\},$$

and P_1 and P_2 are the projections on x and y, i.e., for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, $P_1(x, y) = x, P_2(x, y) = y$.

Similarly, one could define the probabilistic dual frames for a given frame.

Definition 2.5 (Definition 3.1, Theorem 3.6 and 3.7 in [39])**.** Let μ be a probabilistic frame for $\mathbb{R}^d$. The set of transport duals for μ is defined as

$$D_\mu := \Big\{\nu \in \mathscr{P}_2(\mathbb{R}^d) : \exists\ \gamma \in \Gamma(\mu,\nu) \text{ with } \int_{\mathbb{R}^d\times\mathbb{R}^d} xy^T d\gamma(x, y) = I_{d\times d}\Big\}.$$

Furthermore, D_μ is not empty and a compact subset of $\mathscr{P}_2(\mathbb{R}^d)$ with respect to the weak topology.

Proposition 2.6 (Proposition 3.2 in [39])**.** *Let μ be a probabilistic frame for $\mathbb{R}^d$ and take $\nu \in D_\mu$. Then ν is also a probabilistic frame.*

Therefore, the following definition is well-defined.

Definition 2.7 (Probabilistic Dual Frame)**.** Let μ be a probabilistic frame for $\mathbb{R}^d$. $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ is called a probabilistic dual frame of μ with respect to $\gamma \in \Gamma(\mu, \nu)$ if

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} xy^T d\gamma(x, y) = I_{d \times d}.$$

Remark. *Let μ be a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \leq B < \infty$. ${S_\mu^{-1}}_{\#}\mu$ is said to be the canonical probabilistic dual frame of μ since $\nu := {S_\mu^{-1}}_{\#}\mu$ is the transport dual of μ with respect to $\gamma := (Id, S_\mu^{-1})_{\#}\mu \in \Gamma(\mu, \nu)$, and the frame bounds are $0 < \frac{1}{B} \leq \frac{1}{A} < \infty$.*

If μ is a probabilistic frame, there are two canonical probabilistic frames related to μ: the canonical Parseval frame ${S_\mu^{-1/2}}_{\#}\mu$ and the canonical dual frame ${S_\mu^{-1}}_{\#}\mu$. Therefore, we have the following reconstruction formulas: for any $f \in \mathbb{R}^d$,

$$f = \int_{\mathbb{R}^d} \left\langle f, S_\mu^{-1/2} x \right\rangle S_\mu^{-1/2} x \, d\mu(x) = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1/2} f, x \right\rangle S_\mu^{-1/2} x \, d\mu(x),$$

$$f = \int_{\mathbb{R}^d} \left\langle f, S_\mu^{-1} x \right\rangle x \, d\mu(x) = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1} f, x \right\rangle x \, d\mu(x).$$

To show the perturbation theory that includes probabilistic dual frames, we need the following gluing lemma that "glues" two transport couplings together. Similarly, P_1, P_2, P_{12}, P_{23} are projections, i.e., for any $(x, y, z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$,

$$P_1(x, y, z) = x, \ P_2(x, y, z) = y, \ P_{12}(x, y, z) = (x, y), \ P_{23}(x, y, z) = (y, z).$$

Lemma 2.8 (Gluing Lemma [48, pp.59])**.** *Let $\mu_1, \mu_2, \mu_3 \in \mathscr{P}_2(\mathbb{R}^d)$. Suppose $\gamma^{12} \in \Gamma(\mu_1, \mu_2)$ and $\gamma^{23} \in \Gamma(\mu_2, \mu_3)$ such that $P_{2\#}\gamma^{12} = P_{1\#}\gamma^{23} = \mu_2$. Then there exists $\gamma^{123} \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ such that $P_{12\#}\gamma^{123} = \gamma^{12}$ and $P_{23\#}\gamma^{123} = \gamma^{23}$.*

With Gluing Lemma, we claim the following proposition without proof.

Proposition 2.9. *Let μ be a probabilistic frame for $\mathbb{R}^d$ and ν a probabilistic dual frame of μ with respect to $\gamma_{12} \in \Gamma(\mu, \nu)$. Suppose $\eta \in \mathscr{P}_2(\mathbb{R}^d)$ and $\gamma_{23} \in \Gamma(\nu, \eta)$, then there exists $\pi \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ such that*

$$P_{12\#}\pi = \gamma_{12}, \ P_{23\#}\pi = \gamma_{23}.$$

For a complete introduction to probabilistic frames and optimal transport, we refer to [38] and [48] for more details.

2.2. **Invertibility of Linear Operators on Banach Spaces.** This subsection gives a brief introduction to the invertibility of linear operators on Banach spaces, which is used many times in this paper. It is well-known that a bound linear operator U on a Banach space $\mathcal{X}$ is invertible if $\|I - U\| < 1$ where I is identity operator in $\mathcal{X}$, and

$$\|U^{-1}\| \leq \frac{1}{1 - \|I - U\|}.$$

Casazza and Christensen further generalized this result in the following lemma:

Lemma 2.10 (Lemma 1 in [8])**.** *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and $U : \mathcal{X} \to \mathcal{X}$ a linear operator on $\mathcal{X}$. If there exist $\lambda_1, \lambda_2 \in [0, 1)$ such that for any $x \in \mathcal{X}$,*

$$\|Ux - x\| \leq \lambda_1 \|x\| + \lambda_2 \|Ux\|.$$

Then U is bounded invertible, and for any $x \in \mathcal{X}$,

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \le \|Ux\| \le \frac{1+\lambda_1}{1-\lambda_2}\|x\|, \; \frac{1-\lambda_2}{1+\lambda_1}\|x\| \le \|U^{-1}x\| \le \frac{1+\lambda_2}{1-\lambda_1}\|x\|.$$

We also have the following extension result for linear operators on Banach spaces:

Corollary 2.11 (Remark of Corollary 1 in [8]). *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Let $U : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator, $\mathcal{X}_0$ a dense subspace of $\mathcal{X}$, and $V : \mathcal{X} \to \mathcal{Y}$ a linear mapping. If for any $x \in \mathcal{X}_0$,*

$$\|Ux - Vx\| \le \lambda_1\|Ux| + \lambda_2\|Vx\| + \delta\|x\|,$$

where $\lambda_1, \lambda_2, \delta \in [0, 1)$. Then V has a unique extension to a bounded linear operator (of the same norm) from $\mathcal{X}$ to $\mathcal{Y}$, and the extension still satisfies the inequality.

For the mathematical proof of the above lemma and corollary, we refer to Casazza and Christensen's paper [8] for more details.

3. Paley-Wiener Theorem for Probabilistic Frames

In this section, we generalize the Paley-Wiener theorem to probabilistic frames. We first generalize Theorem 1.2 to Theorem 3.1. Then by using Casazza and Christensen's criteria for the invertibility of linear operators in Lemma 2.10, we generalize Theorem 1.3 to Theorem 3.4. Recall that $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ means that ν has finite second moment , i.e.,

$$M_2(\nu) := \int_{\mathbb{R}^d} \|x\|^2 d\nu(x) < +\infty.$$

And if $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, ν is a Bessel probability measure with bound $M_2(\nu)$. Let $C_c(\mathbb{R}^d)$ be the set of continuous functions on $\mathbb{R}^d$ with compact support. Then we have the first perturbation theorem about probabilistic frames.

Theorem 3.1. *Let μ be a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \le B < \infty$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$. Suppose there exist $\lambda, \delta \ge 0$ such that $\lambda + \frac{\delta}{\sqrt{A}} < 1$ and*

$$\Big\| \int_{\mathbb{R}^d} w(x)x d\mu(x) - \int_{\mathbb{R}^d} w(y)y d\nu(y) \Big\| \le \lambda \Big\| \int_{\mathbb{R}^d} w(x)x d\mu(x) \Big\| + \delta\|w\|_{L^2(\mu)}$$

for all $w \in C_c(\mathbb{R}^d)$. Then ν is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{A^2\Big(1 - (\lambda + \frac{\delta}{\sqrt{A}})\Big)^2}{M_2(\nu)} \text{ and } M_2(\nu).$$

Proof. Since $\nu \in \mathcal{P}_2(\mathbb{R}^d)$, then ν is a Bessel probabilistic measure with bound $M_2(\nu) = \int_{\mathbb{R}^d} \|y\|^2 d\nu(y)$. Now let us get the lower frame bound. Let $U : L^2(\mu) \to \mathbb{R}^d$ be the (synthesis) operator for probabilistic frame μ, which is defined by

$$U(w) := \int_{\mathbb{R}^d} w(x)x d\mu(x).$$

Then U is bounded linear and $\|U\| \le M_2(\mu)$. Similarly, we define another linear operator $T : L^2(\mu) \to \mathbb{R}^d$ by

$$T(w) := \int_{\mathbb{R}^d} w(y)y d\nu(y).$$

Note that T is well-defined. By definition, for any $w \in C_c(\mathbb{R}^d)$, we have

$$\Big\|U(w) - T(w)\Big\| \leq \lambda \Big\|U(w)\Big\| + \delta \|w\|_{L^2(\mu)}.$$

Since $C_c(\mathbb{R}^d)$ is dense in $L^2(\mu)$, then by Corollary 2.11, we know that T could be extended uniquely to a bounded linear operator that is still denoted by T, and for any $w \in L^2(\mu)$,

$$\Big\|U(w) - T(w)\Big\| \leq \lambda \Big\|U(w)\Big\| + \delta \|w\|_{L^2(\mu)}. \tag{3.1}$$

Therefore, for any $w \in L^2(\mu)$,

$$\|T(w)\| \leq \|U(w)\| + \|U(w) - T(w)\| \leq ((\lambda+1)\|U\| + \delta)\|w\|_{L^2(\mu)} < +\infty.$$

Thus T is well-defined and

$$\|T\| \leq (\lambda+1)\|U\| + \delta < +\infty.$$

Now let us define $U^+ : \mathbb{R}^d \to L^2(\mu)$ by

$$(U^+x)(\cdot) := (U^*(UU^*)^{-1}x)(\cdot) = (U^*(S_\mu^{-1}x))(\cdot) = \left\langle S_\mu^{-1}x, \cdot \right\rangle \in L^2(\mu).$$

where U^* is the adjoint operator of U and $UU^* = S_\mu$. Then

$$\|U^+x\|^2_{L^2(\mu)} = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}x, y\right\rangle^2 d\mu(y) = \int_{\mathbb{R}^d} \left\langle x, S_\mu^{-1}y\right\rangle^2 d\mu(y),$$

where the last equality is obtained since S_μ^{-1} is self-adjoint. Since ${S_\mu^{-1}}_{\#}\mu$ is the probabilistic dual frame with bounds $\frac{1}{B}$ and $\frac{1}{A}$, then

$$\|U^+x\|^2_{L^2(\mu)} = \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d({S_\mu^{-1}}_{\#}\mu)(y) \leq \frac{1}{A}\|x\|^2.$$

Replacing w in Equation (3.1) by U^+x leads to

$$\Big\|x - T(U^+x)\Big\| \leq \lambda\|x\| + \delta\|U^+x\|_{L^2(\mu)} \leq (\lambda + \frac{\delta}{\sqrt{A}})\|x\|.$$

Therefore, $\|I - TU^+\| \leq \lambda + \frac{\delta}{\sqrt{A}} < 1$. Then TU^+ is invertible and

$$\|(TU^+)^{-1}\| \leq \frac{1}{1 - (\lambda + \frac{\delta}{\sqrt{A}})}.$$

Note that any $x \in \mathbb{R}^d$ could be written as

$$x = TU^+(TU^+)^{-1}x = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}(TU^+)^{-1}x, y\right\rangle y\, d\nu(y).$$

Therefore,

$$\begin{aligned}
\|x\|^4 = \langle x, x\rangle^2 &= \Big| \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}(TU^+)^{-1}x, y\right\rangle \langle x, y\rangle \, d\nu(y)\Big|^2 \\
&= \left\langle \left\langle S_\mu^{-1}(TU^+)^{-1}x, \cdot\right\rangle, \langle x, \cdot\rangle\right\rangle^2_{L^2(\nu)} \\
&\leq \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}(TU^+)^{-1}x, y\right\rangle^2 d\nu(y) \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\nu(y) \\
&\leq \|S_\mu^{-1}(TU^+)^{-1}x\|^2 \int_{\mathbb{R}^d} \|y\|^2 d\nu(y) \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\nu(y),
\end{aligned}$$

where the last two inequalities come from Cauchy–Schwarz inequality and the second equality is obtained since $\langle x, \cdot\rangle \in L^2(\nu)$ and $\left\langle S_\mu^{-1}(TU^+)^{-1}x, \cdot\right\rangle \in L^2(\nu)$. Let

$\|S_\mu^{-1}\|_2$ be the 2-matrix norm of S_μ^{-1}. Since S_μ^{-1} is symmetric, then $\|S_\mu^{-1}\|_2$ is the largest eigenvalue of S_μ^{-1}. Therefore, $\|S_\mu^{-1}\|_2 \le \frac{1}{A}$. Thus

$$\begin{aligned}\|x\|^4 &\le \|S_\mu^{-1}\|_2^2 \; \|(TU^+)^{-1}\|^2 \; \|x\|^2 \int_{\mathbb{R}^d} \|y\|^2 d\nu(y) \; \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\nu(y) \\ &\le \frac{1}{A^2}\Big(\frac{1}{1-(\lambda+\frac{\delta}{\sqrt{A}})}\Big)^2 \|x\|^2 \int_{\mathbb{R}^d} \|y\|^2 d\nu(y) \; \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\nu(y).\end{aligned}$$

Thus for any $x \in \mathbb{R}^d$,

$$\frac{A^2(1-(\lambda+\frac{\delta}{\sqrt{A}}))^2}{\int_{\mathbb{R}^d}\|y\|^2 d\nu(y)}\|x\|^2 \le \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\nu(y) \le \int_{\mathbb{R}^d} \|y\|^2 d\nu(y) \; \|x\|^2.$$

That is to say, ν is a probabilistic frame with bounds

$$\frac{A^2\Big(1-(\lambda+\frac{\delta}{\sqrt{A}})\Big)^2}{M_2(\nu)} \text{ and } M_2(\nu).$$

where $M_2(\nu) := \int_{\mathbb{R}^d} \|y\|^2 d\nu(y)$. ☐

The following lemma is inspired by a particular case in the condition with $\lambda = 0, \delta = \sqrt{R}$ in Theorem 3.1, just formulated in "adjoint" form of the "synthesis" operator of sighed measure "$\mu - \nu$". However, it is not accurate to definite the "adjoint". An easier way to prove is to apply the definition of probabilistic frames.

Lemma 3.2 (Sweetie's Lemma). *Let μ be a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \le B < \infty$ and $\nu \in \mathscr{P}(\mathbb{R}^d)$. Suppose there exists a constant R where $0 < R < A$, such that for any $x \in \mathbb{R}^d$,*

$$\Big| \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu(y) - \int_{\mathbb{R}^d} \langle x, z\rangle^2 \, d\nu(z) \Big| \le R\|x\|^2,$$

or equivalently, for any $x \in \mathbb{S}^{d-1}$(unit sphere in $\mathbb{R}^d$),

$$\Big| \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu(y) - \int_{\mathbb{R}^d} \langle x, z\rangle^2 \, d\nu(z) \Big| \le R.$$

Then ν is a probabilistic frame for $\mathbb{R}^d$ with bounds $A-R$ and $B+R$.

Proof. For any $x \in \mathbb{R}^d$, we have

$$-R\|x\|^2 \le \int_{\mathbb{R}^d} \langle x, z\rangle^2 \, d\nu(z) - \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu(y) \le R\|x\|^2.$$

Therefore,

$$\int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu(y) - R\|x\|^2 \le \int_{\mathbb{R}^d} \langle x, z\rangle^2 \, d\nu(z) \le \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu(y) + R\|x\|^2.$$

Since μ is a probabilistic frame for $\mathbb{R}^d$ with bounds A and B, then

$$A\|x\|^2 \le \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu(y) \le B\|x\|^2.$$

Therefore, for any $x \in \mathbb{R}^d$,

$$(A-R)\|x\|^2 \le \int_{\mathbb{R}^d} \langle x, z\rangle^2 \, d\nu(z) \le (B+R)\|x\|^2.$$

That is to say, ν is a probabilistic frame for $\mathbb{R}^d$ with bounds $A-R$ and $B+R$. ☐

Furthermore, Lemma 3.2 could be improved to any coupling $\gamma \in \Gamma(\mu,\nu)$ with marginals μ and ν.

Corollary 3.3. *Let μ be a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \leq B < \infty$, $\nu \in \mathscr{P}(\mathbb{R}^d)$, and $\gamma \in \Gamma(\mu,\nu)$. Suppose there exists R where $0 < R < A$, such that for any $x \in \mathbb{R}^d$,*

$$\Big| \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle x, y\rangle^2 - \langle x, z\rangle^2 \, d\gamma(y,z) \Big| \leq R\|x\|^2,$$

or

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \Big| \langle x, y\rangle^2 - \langle x, z\rangle^2 \Big| d\gamma(y,z) \leq R\|x\|^2.$$

Then ν is a probabilistic frame for $\mathbb{R}^d$ with bounds $A - R$ and $B + R$.

Proof. Since $\gamma \in \Gamma(\mu,\nu)$, then

$$\begin{aligned}\Big| \int_{\mathbb{R}^d} \langle x, y\rangle^2 \, d\mu(y) - \int_{\mathbb{R}^d} \langle x, z\rangle^2 \, d\nu(z) \Big| &= \Big| \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle x, y\rangle^2 - \langle x, z\rangle^2 \, d\gamma(y,z) \Big| \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \Big| \langle x, y\rangle^2 - \langle x, z\rangle^2 \Big| d\gamma(y,z).\end{aligned}$$

□

Remark. *Indeed, the test function in Lemma 3.2 could be improved to continuous functions $C(\mathbb{R}^d)$. Suppose μ is a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \leq B < \infty$ and $\nu \in \mathscr{P}(\mathbb{R}^d)$. If there exists $0 \leq R < A$ such that*

$$\sup_{w \in C(\mathbb{R}^d)} \Big| \int_{\mathbb{R}^d} w(y) d\mu(y) - \int_{\mathbb{R}^d} w(y) d\nu(y) \Big| \leq R,$$

then v is a probabilistic frame for $\mathbb{R}^d$ with bounds $A - R$ and $B + R$. The proof is clear by taking the test functions to be $w_x(y) = \left\langle \frac{x}{\|x\|}, y \right\rangle^2$ where x is nonzero.

Recall that $C_c(\mathbb{R}^d)$ is the set of continuous functions on $\mathbb{R}^d$ with compact support and $M_2(\nu) := \int_{\mathbb{R}^d} \|y\|^2 d\nu(y)$ the second moment of the probability measure ν. By adding one more term related to the probability measure ν on the right-hand side of the inequality in Theorem 3.1, we get a more general perturbation result that corresponds to Paley-wiener theorem for frames in Theorem 1.3.

Theorem 3.4. *Let μ be a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \leq B < \infty$ and $\nu \in \mathscr{P}_2(\mathbb{R}^d)$. If there exist $\lambda_1, \lambda_2, \delta \geq 0$ such that $max(\lambda_1 + \frac{\delta}{\sqrt{A}}, \lambda_2) < 1$ and*

$$\begin{aligned}&\Big\| \int_{\mathbb{R}^d} w(x) x d\mu(x) - \int_{\mathbb{R}^d} w(y) y d\nu(y) \Big\| \\ &\qquad \leq \lambda_1 \Big\| \int_{\mathbb{R}^d} w(x) x d\mu(x) \Big\| + \lambda_2 \Big\| \int_{\mathbb{R}^d} w(y) y d\nu(y) \Big\| + \delta \|w\|_{L^2(\mu)}\end{aligned}$$

for all $w \in C_c(\mathbb{R}^d)$. Then ν is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{A^2(1 - (\lambda_1 + \frac{\delta}{\sqrt{A}}))^2}{(1+\lambda_2)^2 M_2(\nu)} \text{ and } M_2(\nu)$$

Proof. Since $\nu \in \mathscr{P}_2(\mathbb{R}^d)$, then ν is Bessel with bound $M_2(\nu) := \int_{\mathbb{R}^d} \|y\|^2 d\nu(y)$. Now let us get the lower frame bound. Similarly, let us define $U : L^2(\mu) \to \mathbb{R}^d$ and $T : L^2(\mu) \to \mathbb{R}^d$ in the following way

$$U(w) := \int_{\mathbb{R}^d} w(x) x d\mu(x),$$

$$T(w) := \int_{\mathbb{R}^d} w(y) y d\nu(y).$$

Then U is bounded linear and $\|U\| \leq M_2(\mu)$. Furthermore, T is well-defined. Since $C_c(\mathbb{R}^d)$ is dense in $L^2(\mu)$, then by Corollary 2.11, we know that T could be extended uniquely to a bounded linear operator that is still denoted by T, and for any $w \in L^2(\mu)$,

$$\Big\|U(w) - T(w)\Big\| \leq \lambda_1 \Big\|U(w)\Big\| + \lambda_2 \Big\|T(w)\Big\| + \delta \|w\|_{L^2(\mu)}. \tag{3.2}$$

Therefore, for any $w \in L^2(\mu)$,

$$\|T(w)\| \leq \|U(w)\| + \|U(w) - T(w)\| \leq ((\lambda_1 + 1)\|U\| + \delta)\|w\|_{L^2(\mu)} + \lambda_2 \|T(w)\|.$$

Thus T is well-defined and

$$\|T\| \leq \frac{(\lambda_1 + 1)\|U\| + \delta}{1 - \lambda_2} < +\infty.$$

Similarly, let us define $U^+ : \mathbb{R}^d \to L^2(\mu)$ by

$$(U^+ x)(\cdot) := (U^*(UU^*)^{-1}x)(\cdot) = (U^*(S_\mu^{-1}x))(\cdot) = \left\langle S_\mu^{-1}x, \cdot \right\rangle \in L^2(\mu).$$

where U^* is the adjoint operator of U and $UU^* = S_\mu$. Then

$$\begin{aligned} \|U^+ x\|^2_{L^2(\mu)} &= \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}x, y \right\rangle^2 d\mu(y) = \int_{\mathbb{R}^d} \left\langle x, S_\mu^{-1}y \right\rangle^2 d\mu(y) \\ &= \int_{\mathbb{R}^d} \left\langle x, y \right\rangle^2 d({S_\mu^{-1}}_{\#}\mu)(y) \leq \frac{1}{A}\|x\|^2. \end{aligned}$$

Replacing w in Equation (3.2) by $U^+ x$ leads to

$$\begin{aligned} \Big\|x - T(U^+ x)\Big\| &\leq \lambda_1 \|x\| + \lambda_2 \|T(U^+ x)\Big\| + \delta \|U^+ x\|_{L^2(\mu)} \\ &\leq (\lambda_1 + \frac{\delta}{\sqrt{A}})\|x\| + \lambda_2 \|T(U^+ x)\Big\|. \end{aligned}$$

Since $\max(\lambda_1 + \frac{\delta}{\sqrt{A}}, \lambda_2) < 1$, by Lemma 2.10, we know that TU^+ is invertible, and

$$\|(TU^+)^{-1}\| \leq \frac{1 + \lambda_2}{1 - (\lambda_1 + \frac{\delta}{\sqrt{A}})}.$$

Similarly, any $x \in \mathbb{R}^d$ could be written as

$$x = TU^+(TU^+)^{-1}x = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}(TU^+)^{-1}x, y \right\rangle y d\nu(y).$$

Therefore,

$$\begin{aligned}
\|x\|^4 = \langle x,x\rangle^2 &= \Big| \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}(TU^+)^{-1}x, y\right\rangle \langle x,y\rangle \, d\nu(y)\Big|^2 \\
&\le \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}(TU^+)^{-1}x, y\right\rangle^2 d\nu(y) \ \int_{\mathbb{R}^d} \langle x,y\rangle^2 \, d\nu(y) \\
&\le \|S_\mu^{-1}\|_2^2 \ \|(TU^+)^{-1}\|^2 \ \|x\|^2 \int_{\mathbb{R}^d} \|y\|^2 d\nu(y) \ \int_{\mathbb{R}^d} \langle x,y\rangle^2 \, d\nu(y) \\
&\le \frac{1}{A^2}\Big(\frac{1+\lambda_2}{1-(\lambda_1+\frac{\delta}{\sqrt{A}})}\Big)^2 \|x\|^2 \int_{\mathbb{R}^d} \|y\|^2 d\nu(y) \ \int_{\mathbb{R}^d} \langle x,y\rangle^2 \, d\nu(y),
\end{aligned}$$

where $\|S_\mu^{-1}\|_2$ is the 2-matrix norm of S_μ^{-1} and $\|S_\mu^{-1}\|_2 \le \frac{1}{A}$. Thus for any $x \in \mathbb{R}^d$,

$$\frac{A^2(1-(\lambda_1+\frac{\delta}{\sqrt{A}}))^2}{(1+\lambda_2)^2\int_{\mathbb{R}^d}\|y\|^2 d\nu(y)}\|x\|^2 \le \int_{\mathbb{R}^d} \langle x,y\rangle^2 \, d\nu(y) \le \int_{\mathbb{R}^d} \|y\|^2 d\nu(y) \ \|x\|^2.$$

That is to say, ν is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{A^2(1-(\lambda_1+\frac{\delta}{\sqrt{A}}))^2}{(1+\lambda_2)^2 M_2(\nu)} \text{ and } M_2(\nu)$$

where $M_2(\nu) := \int_{\mathbb{R}^d}\|y\|^2 d\nu(y)$. ∎

Remark. *Since $\frac{\delta}{\sqrt{A}} \le \frac{\sqrt{B}\delta}{A}$, the condition $\lambda+\frac{\delta}{\sqrt{A}} < 1$ in Theorem 3.1 and max$(\lambda_1+\frac{\delta}{\sqrt{A}}, \lambda_2) < 1$ in Theorem 3.4 could be replaced by $\lambda + \frac{\sqrt{B}\delta}{A} < 1$ and max$(\lambda_1 + \frac{\sqrt{B}\delta}{A}, \lambda_2) < 1$, respectively. In this case, the lower frame bounds for ν are*

$$\frac{A^2\Big(1-(\lambda+\frac{\sqrt{B}\delta}{A})\Big)^2}{M_2(\nu)} \ \textit{and} \ \frac{A^2\Big(1-(\lambda_1+\frac{\sqrt{B}\delta}{A})\Big)^2}{(1+\lambda_2)^2 M_2(\nu)}$$

This is due to another way to get $\|U^+x\|_{L^2(\mu)}$:

$$\|U^+x\|^2_{L^2(\mu)} = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}x, y\right\rangle^2 d\mu(y) \le B\|S_\mu^{-1}x\|^2 \le B\|S_\mu^{-1}\|_2^2\|x\|^2 \le \frac{B}{A^2}\|x\|^2.$$

4. Perturbations Including Probabilistic Dual Frames

Let $\{f_i\}_{i=1}^\infty$ be a frame for the Hilbert space $\mathscr{H}$ with bounds $0 < A \le B < \infty$. Recall that a frame $\{h_i\}_{i=1}^\infty$ for $\mathscr{H}$ is a dual frame of $\{f_i\}_{i=1}^\infty$ if for any $f \in \mathscr{H}$,

$$f = \sum_{i=1}^\infty \langle f, h_i\rangle f_i = \sum_{i=1}^\infty \langle f, f_i\rangle h_i.$$

Suppose the upper frame bound for $\{h_i\}_{i=1}^\infty$ is D and $\{g_i\}_{i=1}^\infty$ is a sequence in $\mathscr{H}$ such that

$$\alpha := \sum_{i=1}^\infty \|f_i - g_i\|^2 < \infty, \ \ \beta := \sum_{i=1}^\infty \|f_i - g_i\|\|h_i\| < 1.$$

Then by Theorem 1.4, $\{g_i\}_{i=1}^\infty$ is a frame in $\mathscr{H}$ with bounds $\frac{(1-\beta)^2}{D}, B(1+\sqrt{\frac{\alpha}{B}})$.

In this section, we generalize the above result to the probabilistic frames setting: we give a sufficient perturbation condition where the probabilistic dual frames are used, which is similar to Theorem 1.4 without the quadratic close condition $\alpha < \infty$.

Let μ be a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \le B < \infty$. Recall that ν is said to be a probabilistic dual frame of μ with respect to $\gamma_{12} \in \Gamma(\mu, \nu)$ if

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} xy^T d\gamma_{12}(x, y) = I_{d\times d}.$$

Furthermore, suppose $\eta \in \mathscr{P}_2(\mathbb{R}^d)$ and $\gamma_{23} \in \Gamma(\nu, \eta)$. Then by Lemma 2.8(gluing lemma) and Proposition 2.9, there exists $\pi \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ such that

$$P_{12\#}\pi = \gamma_{12}, \;\; P_{23\#}\pi = \gamma_{23}.$$

Now we are ready to state the main perturbation theorem about probabilistic frames where the probabilistic dual frame is included.

Theorem 4.1. *Let μ be a probabilistic frame for $\mathbb{R}^d$ and ν the probabilistic dual frame of μ with respect to $\gamma_{12} \in \Gamma(\mu, \nu)$. Let $\eta \in \mathscr{P}_2(\mathbb{R}^d)$, $\gamma_{23} \in \Gamma(\nu, \eta)$, and $\pi \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ be the coupling with marginals γ_{12} and γ_{23} obtained by Gluing Lemma. Suppose*

$$\sigma := \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \|x - z\| \|y\| d\pi(x, y, z) < 1,$$

then η is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{(1-\sigma)^2}{M_2(\nu)} \;\; \text{and } M_2(\eta).$$

And if the upper frame bound for ν is $0 < D < \infty$, then the frame bounds for η are

$$\frac{(1-\sigma)^2}{D} \;\; \text{and } M_2(\eta).$$

Proof. Since $\eta \in \mathscr{P}_2(\mathbb{R}^d)$, then η is Bessel with bound $M_2(\eta) := \int_{\mathbb{R}^d} \|z\|^2 d\eta(z) < \infty$. Next let us show the lower frame bound. Define a linear operator $L : \mathbb{R}^d \to \mathbb{R}^d$ by

$$L(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle f, y\rangle \, z \; d\gamma_{23}(y, z), \;\; \text{for any } f \in \mathbb{R}^d.$$

Since ν is the probabilistic dual frame of μ with respect to $\gamma_{12} \in \Gamma(\mu, \nu)$, then

$$f = \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle f, y\rangle \, x \; d\gamma_{12}(x, y), \;\; \text{for any } f \in \mathbb{R}^d.$$

Therefore,

$$\begin{aligned}
\|f - L(f)\| &= \Big\| \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle f, y\rangle \, x \; d\gamma_{12}(x, y) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle f, y\rangle \, z \; d\gamma_{23}(y, z) \Big\| \\
&= \Big\| \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \langle f, y\rangle \, (x - z) \; d\pi(x, y, z) \Big\| \\
&\le \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \|y\| \|x - z\| d\pi(x, y, z) \; \|f\| = \sigma \|f\|.
\end{aligned}$$

Thus $L : \mathbb{R}^d \to \mathbb{R}^d$ is invertible and $\|L^{-1}\| \le \frac{1}{1-\sigma}$. Note that for any $f \in \mathbb{R}^d$,

$$f = LL^{-1}(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left\langle L^{-1} f, y\right\rangle z \; d\gamma_{23}(y, z).$$

Therefore,

$$\begin{aligned}
\|f\|^4 = \langle f, f\rangle^2 &= \Big| \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle L^{-1}f, y\right\rangle \langle f, z\rangle \ d\gamma_{23}(y,z)\Big|^2 \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle L^{-1}f, y\right\rangle^2 d\gamma_{23}(y,z) \ \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle f, z\rangle^2 d\gamma_{23}(y,z) \\
&= \int_{\mathbb{R}^d} \left\langle L^{-1}f, y\right\rangle^2 d\nu(y) \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z) \\
&\leq \|L^{-1}\|^2 \ \|f\|^2 \int_{\mathbb{R}^d} \|y\|^2 d\nu(y) \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z) \\
&\leq \frac{1}{(1-\sigma)^2} \int_{\mathbb{R}^d} \|y\|^2 d\nu(y) \ \|f\|^2 \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z).
\end{aligned}$$

where the first inequality is due to Cauchy Schwarz inequality and the second equality comes from $\gamma_{23} \in \Gamma(\nu, \eta)$. Thus for any $f \in \mathbb{R}^d$,

$$\frac{(1-\sigma)^2}{\int_{\mathbb{R}^d} \|y\|^2 d\nu(y)} \|f\|^2 \leq \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z) \leq \int_{\mathbb{R}^d} \|z\|^2 d\eta(z) \ \|f\|^2.$$

Therefore, η is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{(1-\sigma)^2}{M_2(\nu)} \text{ and } M_2(\eta).$$

If the upper frame bound for the probabilistic dual frame ν is $0 < D < \infty$, then

$$\begin{aligned}
\|f\|^4 \leq \int_{\mathbb{R}^d} \left\langle L^{-1}f, y\right\rangle^2 d\nu(y) \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z) &\leq D\|L^{-1}f\|^2 \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z) \\
&\leq \frac{D}{(1-\sigma)^2} \ \|f\|^2 \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z).
\end{aligned}$$

In this case, the frame bounds for η are $\frac{(1-\sigma)^2}{D}$ and $M_2(\eta)$. □

If the probabilistic dual frame of μ is given by the canonical probabilistic dual frame $S_\mu^{-1}{}_{\#}\mu$, we will have the following corollary.

Corollary 4.2. *Let μ be a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \leq B < \infty$, and $\eta \in \mathscr{P}_2(\mathbb{R}^d)$. If*

$$\hat{\sigma} := \int_{\mathbb{R}^d\times\mathbb{R}^d} \|S_\mu^{-1}x\| \ \|x - z\| d\mu(x) d\eta(z) < 1,$$

or

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \|x\| \ \|x - z\| d\mu(x) d\eta(z) < A,$$

then η is a probabilistic frame for $\mathbb{R}^d$ with bounds $A(1-\hat{\sigma})^2$ and $M_2(\eta)$.

Proof. In the previous theorem, let $S_\mu^{-1}{}_{\#}\mu$ be the canonical probabilistic dual frame of μ with respect to $\gamma_{12} := (Id, S_\mu^{-1})_{\#}\mu \in \Gamma(\mu, S_\mu^{-1}{}_{\#}\mu)$. Let γ_{23} be the product measure $\gamma_{23} := S_\mu^{-1}{}_{\#}\mu \otimes \eta \in \Gamma(S_\mu^{-1}{}_{\#}\mu, \eta)$. Then by the disintegration theorem and gluing lemma, the transport coupling with marginals γ_{12} and γ_{23} is given by $\pi := \gamma_{12} \otimes \eta \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. Thus

$$\hat{\sigma} := \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x - z\| \|S_\mu^{-1}x\| d\mu(x) d\eta(z) = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \|x - z\| \|y\| d\pi(x,y,z).$$

Since $\hat{\sigma} < 1$ and the upper bound of ${S_\mu^{-1}}_{\#}\mu$ is $\frac{1}{A}$, then by Theorem 4.1, η is a probabilistic frame for $\mathbb{R}^d$ with bounds $A(1-\hat{\sigma})^2$ and $M_2(\eta)$.

Since $\|S_\mu^{-1}x\| \leq \|S_\mu^{-1}\|_2\|x\| \leq \frac{1}{A}\|x\|$ where $\|S_\mu^{-1}\|_2$ is the 2-matrix norm of S_μ^{-1} and $\|S_\mu^{-1}\|_2 \leq \frac{1}{A}$, then

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-z\|\|x\|d\mu(x)d\eta(z) < A$$

implies

$$\hat{\sigma} := \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-z\|\|S_\mu^{-1}x\|d\mu(x)d\eta(z) < 1.$$

□

Indeed, the condition in Corollary 4.2 could be generalized to any coupling $\gamma \in \Gamma(\mu,\eta)$ with marginal μ and η.

Proposition 4.3. *Let μ be a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \leq B < \infty$ and $\eta \in \mathscr{P}_2(\mathbb{R}^d)$. Let $\gamma \in \Gamma(\mu,\eta)$ be any coupling with marginal μ and η. Suppose*

$$\epsilon := \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x\|\ \|x-z\|d\gamma(x,z) < A,$$

then η is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{(A-\epsilon)^2}{B} \text{ and } M_2(\eta).$$

Furthermore, if

$$\chi := \int_{\mathbb{R}^d\times\mathbb{R}^d} \|S_\mu^{-1}x\|\ \|x-z\|d\gamma(x,z) < 1,$$

then η is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{A^2(1-\chi)^2}{B} \text{ and } M_2(\eta).$$

Proof. Since $\eta \in \mathscr{P}_2(\mathbb{R}^d)$, then η is Bessel with bound $M(\eta) := \int_{\mathbb{R}^d}\|z\|^2d\eta(z) < \infty$. Next let us show the lower frame bound. Since ${S_\mu^{-1}}_{\#}\mu$ is the canonical probabilistic dual frame of μ with respect to $(Id, S_\mu^{-1})_{\#}\mu \in \Gamma(\mu, {S_\mu^{-1}}_{\#}\mu)$, then

$$f = \int_{\mathbb{R}^d} \left\langle f, S_\mu^{-1}x\right\rangle x\ d\mu(x) = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}f, x\right\rangle x\ d\mu(x), \text{ for any } f \in \mathbb{R}^d.$$

Define a linear operator $L : \mathbb{R}^d \to \mathbb{R}^d$ by

$$L(f) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle S_\mu^{-1}f, x\right\rangle z\ d\gamma(x,z), \text{ for any } f \in \mathbb{R}^d.$$

Therefore,

$$\begin{aligned}
\|f - L(f)\| &= \Big\| \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle S_\mu^{-1}f, x\right\rangle x\ d\mu(x) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle S_\mu^{-1}f, x\right\rangle z\ d\gamma(x,z)\Big\| \\
&= \Big\| \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle S_\mu^{-1}f, x\right\rangle (x-z)\ d\gamma(x,z)\Big\| \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x\|\|x-z\|d\gamma(x,z)\ \|S_\mu^{-1}f\| \\
&\leq \epsilon\ \|S_\mu^{-1}\|_2\ \|f\| \leq \frac{\epsilon}{A}\|f\|.
\end{aligned}$$

Thus $L : \mathbb{R}^d \to \mathbb{R}^d$ is invertible and

$$\|L^{-1}\| \leq \frac{1}{1-\frac{\epsilon}{A}}.$$

Then for any $f \in \mathbb{R}^d$,

$$f = LL^{-1}(f) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle S_\mu^{-1}L^{-1}f, x\right\rangle z \; d\gamma(x,z).$$

Therefore,

$$\begin{aligned}
\|f\|^4 = \langle f, f\rangle^2 &= \Big| \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle S_\mu^{-1}L^{-1}f, x\right\rangle \langle f, z\rangle \; d\gamma(x,z)\Big|^2 \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle S_\mu^{-1}L^{-1}f, x\right\rangle^2 d\gamma(x,z) \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle f, y\rangle^2 d\gamma(x,z) \\
&= \int_{\mathbb{R}^d} \left\langle S_\mu^{-1}L^{-1}f, x\right\rangle^2 d\mu(x) \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z) \\
&\leq B\|S_\mu^{-1}\|_2^2 \; \|L^{-1}\|^2 \; \|f\|^2 \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z) \\
&\leq \frac{B}{A^2(1-\frac{\epsilon}{A})^2} \; \|f\|^2 \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z),
\end{aligned}$$

where the first inequality is due to Cauchy-Schwarz inequality. Thus for any $f \in \mathbb{R}^d$,

$$\frac{A^2(1-\frac{\epsilon}{A})^2}{B}\|f\|^2 \leq \int_{\mathbb{R}^d} \langle f, z\rangle^2 d\eta(z) \leq \|f\|^2 \int_{\mathbb{R}^d} \|z\|^2 d\eta(z).$$

Therefore, η is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{(A-\epsilon)^2}{B} \text{ and } M_2(\eta).$$

Furthermore, if

$$\chi := \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-z\|\|S_\mu^{-1}x\| d\gamma(x,z) < 1,$$

then

$$\|f - L(f)\| = \Big\| \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle f, S_\mu^{-1}x\right\rangle (x-z) \; d\gamma(x,z)\Big\| \leq \chi \; \|f\|.$$

Therefore, $\|I - L\| \leq \chi < 1$ implies L is invertible and $\|L^{-1}\| \leq \frac{1}{1-\chi}$. Similarly, we conclude that η is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{A^2(1-\chi)^2}{B} \text{ and } M_2(\eta).$$

□

Remark. $\frac{(A-\epsilon)^2}{B}$ *is a smaller lower frame bounds than* $\frac{A^2(1-\chi)^2}{B}$*, since*

$$\begin{aligned}
\frac{(A-\epsilon)^2}{B} &= \frac{A^2}{B}\Big(1 - \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-z\|\frac{\|x\|}{A} d\gamma(x,z)\Big)^2 \\
&\leq \frac{A^2}{B}\Big(1 - \int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-z\|\|S_\mu^{-1}x\| d\gamma(x,z)\Big)^2 = \frac{A^2(1-\chi)^2}{B}.
\end{aligned}$$

The key step in the proof of Proposition 4.3 is to use the canonical probabilistic dual frame to give a constructive formula for f and show the invertibility of linear operator L. Another way to construct f is to use the canonical Parseval probabilistic frame ${S_\mu^{-1/2}}_{\#}\mu$, i.e., for any $f \in \mathbb{R}^d$,

$$f = \int_{\mathbb{R}^d} \left\langle f, S_\mu^{-1/2} x \right\rangle S_\mu^{-1/2} x \; d\mu(x) = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1/2} f, x \right\rangle S_\mu^{-1/2} x \; d\mu(x).$$

According to this reconstruction formula, we have the last proposition of this paper.

Proposition 4.4. *Let μ be a probabilistic frame for $\mathbb{R}^d$ with bounds $0 < A \leq B < \infty$ and $\eta \in \mathscr{P}_2(\mathbb{R}^d)$. Let $\gamma \in \Gamma(\mu, \eta)$ be any coupling with marginal μ and η. Suppose*

$$\tau := \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x\| \|S_\mu^{-1/2} x - z\| d\gamma(x, z) < \sqrt{A},$$

then η is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{(\sqrt{A} - \tau)^2}{B} \text{ and } M_2(\eta).$$

Proof. Since $\eta \in \mathscr{P}_2(\mathbb{R}^d)$, then η is Bessel with bound $M_2(\eta) := \int_{\mathbb{R}^d} \|z\|^2 d\eta(z) < \infty$. Next let us show the lower frame bound. Since ${S_\mu^{-1/2}}_{\#}\mu$ is the canonical Parseval probabilistic frame of μ, then

$$f = \int_{\mathbb{R}^d} \left\langle S_\mu^{-1/2} f, x \right\rangle S_\mu^{-1/2} x \; d\mu(x), \text{ for any } f \in \mathbb{R}^d.$$

Define a linear operator $L : \mathbb{R}^d \to \mathbb{R}^d$ by

$$L(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left\langle S_\mu^{-1/2} f, x \right\rangle z \; d\gamma(x, z), \text{ for any } f \in \mathbb{R}^d.$$

Therefore,

$$\begin{aligned} \|f - L(f)\| &= \left\| \int_{\mathbb{R}^d} \left\langle S_\mu^{-1/2} f, x \right\rangle S_\mu^{-1/2} x \; d\mu(x) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \left\langle S_\mu^{-1/2} f, x \right\rangle z \; d\gamma(x, z) \right\| \\ &= \left\| \int_{\mathbb{R}^d \times \mathbb{R}^d} \left\langle S_\mu^{-1/2} f, x \right\rangle (S_\mu^{-1/2} x - z) \; d\gamma(x, z) \right\| \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x\| \|S_\mu^{-1/2} x - z\| d\gamma(x, z) \; \|S_\mu^{-1/2} f\| \\ &\leq \tau \; \|S_\mu^{-1/2}\|_2 \; \|f\| \leq \frac{\tau}{\sqrt{A}} \|f\|. \end{aligned}$$

Thus $L : \mathbb{R}^d \to \mathbb{R}^d$ is invertible and

$$\|L^{-1}\| \leq \frac{1}{1 - \frac{\tau}{\sqrt{A}}}.$$

Then for any $f \in \mathbb{R}^d$,

$$f = LL^{-1}(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left\langle S_\mu^{-1/2} L^{-1} f, x \right\rangle z \; d\gamma(x, z).$$

Therefore,

$$\begin{aligned}
\|f\|^4 = \langle f, f\rangle^2 &= \Big| \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle S_\mu^{-1/2} L^{-1} f, x \right\rangle \langle f, z\rangle \ \, d\gamma(x,z) \Big|^2 \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle S_\mu^{-1/2} L^{-1} f, x \right\rangle^2 d\gamma(x,z) \ \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle f, z\rangle^2 \, d\gamma(x,z) \\
&= \int_{\mathbb{R}^d} \left\langle S_\mu^{-1/2} L^{-1} f, x \right\rangle^2 d\mu(x) \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 \, d\eta(z) \\
&\leq B \ \|S_\mu^{-1/2}\|_2^2 \ \|L^{-1}\|^2 \ \|f\|^2 \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 \, d\eta(z) \\
&\leq \frac{B}{A(1-\frac{\tau}{\sqrt{A}})^2} \ \|f\|^2 \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 \, d\eta(z),
\end{aligned}$$

where the first inequality is due to Cauchy-Schwarz inequality, the second inequality is because μ is a probabilistic frame with upper bound B, and the last inequality comes from $\|S_\mu^{-1/2}\|_2 \leq \frac{1}{\sqrt{A}}$. Thus for any $f \in \mathbb{R}^d$,

$$\frac{A(1-\frac{\tau}{\sqrt{A}})^2}{B}\|f\|^2 \leq \ \int_{\mathbb{R}^d} \langle f, z\rangle^2 \, d\eta(z) \leq \|f\|^2 \int_{\mathbb{R}^d} \|z\|^2 d\eta(z).$$

Therefore, η is a probabilistic frame for $\mathbb{R}^d$ with bounds

$$\frac{(\sqrt{A}-\tau)^2}{B} \text{ and } M_2(\eta).$$

$\square$

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School of Mathematical and Statistical Sciences, Clemson University, 105 Sikes Hall, Clemson, SC, USA, 29634.

Email address: dongwec@g.clemson.edu;