Clemson Analysis Prelim Solutions

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Contents

Winter 2010

1. (a) *Proof.* If E is bounded, E is pre-compact since $\mathbb{R}$ is finite (one) dimensional. If $f(E)$ is unbounded, then there exists $\{x_n\} \subseteq E$ such that $f(x_n) \to \infty$. Since E is precompact $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ with limit $x \in \mathbb{R}$. Then, since f is continuous,

$$f(x) = \lim_{k\to\infty} f(x_{n_k}) = \infty$$

However, since f maps $\mathbb{R}$ to $\mathbb{R}$, $f(x)$ cannot be ∞. □

(b) *Proof.* Since f is uniformly continuous, there exists $\delta > 0$ such that whenever $|x - y| < \delta$,

$$|f(x) - f(y)| < 1$$

Since E bounded, it can be covered by finitely many balls of radius δ, say $\{B(x_i, \delta)\}_{i=1}^N$. Then,

$$f(E) = \cup_{i=1}^N f(B(x_i, \delta))$$

Fix i, for any $f(y) \in f(B(x_i, \delta))$,

$$|f(y) - f(x_i)| \leq 1$$

So $f(B(x_i, \delta))$ is bounded. Then, a finite union of bounded sets is also bounded. □

Counterexample: $E = (0, 1)$ and $f(x) = 1/x$. $f(E) = (1, \infty)$.

3. (a) *Proof.* Recall the Bessel inequality for any orthonormal set $\{e_n\}$ in an inner product space, X. For and $f \in X$,

$$\sum |\langle f, e_n\rangle|^2 \leq \|f\|^2$$

In particular, $\langle f, e_n\rangle \to 0$ as $n \to \infty$ for any $f \in X$. Now, since

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{2\pi}}\right\}$$

form an orthonormal set in $C[-\pi, \pi]$, we have

$$\int_{-\pi}^{\pi} \sin(2nx) f(x)\, dx \to 0 \quad \text{as } n \to \infty$$

for any $f \in C[-\pi, \pi]$. Then,

$$\int_{-\pi}^{\pi} \sin^2(nx) f(x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} f(x)\, dx - \frac{1}{2}\int_{-\pi}^{\pi} \sin(2nx) f(x)\, dx \to \frac{1}{2}\int_{-\pi}^{\pi} f(x)\, dx$$

□

(b) *Proof.* For any $f \in C[-\pi, \pi]$, $n \in \mathbb{N}$,

$$\begin{aligned}
\left| \int_{-\pi}^{\pi} \frac{x^n}{\pi^n} f(x)\, dx \right|^2 &\le \int_{-\pi}^{\pi} \frac{x^{2n}}{\pi^{2n}}\, dx \int_{-\pi}^{\pi} |f(x)|^2\, dx \\
&= \frac{\pi^{2n+1} - (-\pi)^{2n+1}}{(2n+1)\pi^{2n}} \|f\|_{L^2}^2 \\
&= \frac{2\pi}{2n+1} \|f\|_{L^2}^2
\end{aligned}$$

which goes to 0 as $n \to \infty$. $\square$

9. (a) There exists $\varepsilon_n \searrow 0$ such that

$$\mu\{|f-g| \ge \varepsilon_n\} \le \varepsilon_n$$

Then,

$$\begin{aligned}
\mu\{f \ne g\} = \mu\{|f-g| > 0\} &= \mu\left(\bigcup_n \{|f-g| \ge \varepsilon_n\} \right) \\
&= \lim_{n\to\infty} \mu\{|f-g| \le \varepsilon_n\} \le \lim_{n\to\infty} \varepsilon_n = 0
\end{aligned}$$

(b) We only need to show the triangle inequality. Let $t, s > 0$, f, g, h measurable functions.
If $|f-h| \le t$ and $|g-h| \le s$, then

$$|f-g| \le |f-h| + |g-h| \le t+s$$

Thus, $\{|f-h| \le t\} \cap \{|g-h| \le s\} \subset \{|f-g| \le t+s\}$. Then, taking complements, we have

$$\{|f-h| > t\} \cup \{|g-h| > s\} \supset \{|f-g| > t+s\}$$

Therefore, $\mu\{|f-g| > t+s\} \le \mu\{|f-h| > t\} + \mu\{|g-h| > s\}$. Let $\delta > 0$. There exists $\varepsilon_1, \varepsilon_2$ such that

$$\mu\{|f-h| > \varepsilon_1\} < \varepsilon_1 \text{ and } \varepsilon_1 < \rho(f,h) + \delta/2$$

and similarly for ε_2 and $|g-h|$. Therefore,

$$\mu\{|f-g| > \varepsilon_1 + \varepsilon_2\} \le \mu\{|f-h| > \varepsilon_1\} + \mu\{|g-h| > \varepsilon_2\} < \varepsilon_1 + \varepsilon_2$$

So,

$$\begin{aligned}
\rho(f,g) &= \inf\{\varepsilon : \mu\{|f-g| > \varepsilon\} < \varepsilon\} \\
&\le \varepsilon_1 + \varepsilon_2 \\
&\le \rho(f,h) + \rho(g,h) + \delta
\end{aligned}$$

for any $\delta > 0$. This proves the Triangle Ineqaulity.

Summer 2010

3. (a)
$$\|Tf\|_\infty = \sup_{x\in[0,1]} |x^2 f(x)| \le \sup_{x\in[0,1]} |f(x)| = \|f\|_\infty$$

   (b) For $f \equiv 1$, $\|Tf\| = 1$ and $\|f\| = 1$.

   (c) By triangle inequality, $\|(I+T)f\|_\infty \le \|f\|_\infty + \|Tf\|_\infty \le 2\|f\|$. So, we only need to show $\|I+T\| = 2$. Again, this follows from taking $f \equiv 1$.
$$\|(I+T)f\| = \sup_{x\in[0,1]} |1+x^2| = 2$$

4. (a) We will actually show more, namely that
$$\|x\|_q \le \|x\|_p \quad \text{for } 1 \le p < q < \infty$$

   Proof. Let $x \in \ell^p$. Define
$$y = \frac{x}{\|x\|_p}$$
   Then, $|y_i| \le 1$ for every $i \in \mathbb{N}$. This implies $|y_i|^q \le |y_i|^p$ for all i. Therefore,
$$\|y\|_q^q = \sum |y_i|^q \le \sum |y_i|^p = \sum \frac{|x_i|^p}{\|x\|_p^p} = \frac{\sum |x_i|^p}{\sum |x_i|^p} = 1$$
   so $\|y\|_q \le 1$. But this implies
$$\frac{\|x\|_q}{\|x\|_p} = \left\| \frac{x}{\|x\|_p} \right\|_q = \|y\|_q \le 1$$
   $\square$

5. (a) Let $f \in L^q$, $1 \le p < q < \infty$. we will apply holder inequality with exponent $q/p > 1$.
$$\int |f|^p \le \left(\int |f|^q\right)^{p/q} \left(\int 1\right)^{1-p/q} = \left(\int |f|^q\right)^{p/q} (\mu(X))^{1-p/q} < \infty$$

   (b) Take
$$f(x) = \begin{cases} 0 & x = 0 \\ x^{-1/2} & 0 < |x| \le 1 \\ 0 & |x| < 1 \end{cases}$$
$$g(x) = \begin{cases} x^{-1} & |x| \ge 1 \\ 1 & |x| < 1 \end{cases}$$

6. There are many options for f_n:

$$n^2\chi_{[0,1/n]} \quad n\chi_{[n,n+1]} \quad \chi_{[n,2n]}$$

Then, just take

$$g_n = \frac{1}{n} f_n$$

and

$$h_n = (-1)^n g_n$$

7.

Winter 2012

Summer 2012

4. (a) We will show K is totally bounded. Let $\epsilon > 0$. For $x, y \in K$,

$$d_S(x,y) = \sum_i \frac{|x_i - y_i|}{2_i(1 + |x_i - y_i|)} \leq \sum_i \frac{2}{2^i} < \infty$$

So, there exists N such that

$$\sum_{i=N+1}^{\infty} \frac{1}{2^{i-1}} < \epsilon/2$$

Consider the set $M = \{x \in K : x_i = 0, i > N\} \subseteq K$. M is compact since it is finite dimensional and bounded. So, there is an $\epsilon/2$ net for M which will be an ϵ net for K.

(b) False. Consider the sequence $\{e_n\} \subseteq K$, which is entirely zero except the n-th entry. For $n \neq m$,

$$d_\infty(e_n, e_m) = 1$$

so this sequence cannot have a convergent subsequence.

5. (a) Define $g_n = \sum_{k=1}^n f_k$. Since f_k are non-negative, $\{g_n\}$ is monotone. Moreover,

$$g_n \nearrow \sum_{k=1}^{\infty} f_k$$

Then, by Monotone Convergence Theorem,

$$\sum_{k=1}^{\infty} \int f_k = \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k = \lim_{n\to\infty} \int g_n = \int \sum_{k=1}^{\infty} f_k$$

(b) By part (a),

$$\infty > \sum \int |f_k - f| = \int \sum |f_k - f|$$

Since $\sum |f_k - f|$ is integrable, it is finite almost everywhere. Let $E \subseteq \mathbb{R}$ be the set of measure zero where it may not be finite. Let $x \notin E$. Let $\epsilon > 0$. There exists N such that

$$|f_n(x) - f(x)| \leq \sum_{k=N}^{\infty} |f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$.

7. See Winter 15 #9

Winter 2013

8. See Summer 13 #5

Summer 2013

3. (a) B is complete. Let $\{f_n\} \subseteq C[0,1]$ be a Cauchy sequence in ρ_∞. Since $(C[0,1], \rho_\infty)$ is a complete metric space, there exists $f \in C[0,1]$ such that $f_n \to f$ in ρ_∞. Now, we claim that $f \in B$. For any $\epsilon > 0$ there exists N such that

$$\rho(f, f_n) < \epsilon \quad \forall n \geq N$$

Then,

$$\sup_{0 \leq t \leq 1} |f(t)| = \rho(f, 0) \leq \rho(f, f_n) + \rho(f_n, 0) < \epsilon + 1$$

but $\epsilon > 0$ was arbitrary so

$$\sup_{0 \leq t \leq 1} |f(t)| \leq 1$$

(b) Consider the spike functions, $\{f_n\}$. For $n \neq m$,

$$\rho(f_n, f_m) = 1$$

so there cannot be a convergent subsequence.

4. (a) Let $f \in L^2(\mu)$. Then, using the Cauchy-Schwarz inequality, we compute

$$||Af||^2_{L^2(\mu)} = \int_X \left(\int_X K(x,y) f(y) \, d\mu(y) \right)^2 d\mu(x)$$

$$\leq \int_X \left(\int_X K(x,y)^2 \, d\mu(y) \right) \left(\int_X f(y)^2 \, d\mu(y) \right) d\mu(x)$$

$$= \left(\int_X f(y)^2 \, d\mu(y) \right) \int_X \left(\int_X K(x,y)^2 \, d\mu(y) \right) d\mu(x) = ||f||^2_{L^2(\mu)} ||K||^2_{L^2(\mu \times \mu)}$$

Therefore $||A|| \leq ||K||_{L^2(\mu \times \mu)}$.

(b) First we note that the correspondence $K \mapsto A$ is linear due to the linearity of the integral. So, it suffices to prove the following: Let $K \in L^2(\mu \times \mu)$ such that for any $f \in L^2(\mu)$,

$$\int_X K(x,y) f(y) \, d\mu(y) = 0$$

for almost every $x \in X$ (w.r.t μ). Then, $K = 0$ a.e. To prove this, suppose there exists $E \subseteq X \times X$ such that E has positive $\mu \times \mu$ measure in the sense that

$$(\mu \times \mu)(E) := \int_X \int_X \mathbf{1}_E(x,y) \, d\mu(x) \, d\mu(y) > 0$$

So we define the measure $\mu \times \mu$ on the cylinder $X \times X$ in this way.

5. ($a \Rightarrow b$) Let $x = \hat{m} + e$ where $\hat{m}$ is the closest point to x is M. Let $y \in M$ non-zero. For any $t \in \mathbb{C}$, $ty \in M$ so

$$\|x - \hat{m}\|^2 \leq \|e - ty\|^2 = \|e\|^2 - 2\,\mathrm{Re}\langle e, ty\rangle + \|ty\|^2$$

which implies

$$\mathrm{Re}\,\bar{t}\langle e, y\rangle \leq |t|^2\|y\|^2$$

Take $t = \overline{\langle e, y\rangle}\|y\|^{-2}$. Then we have

$$\frac{|\langle e, y\rangle|^2}{\|y\|^2} \leq \frac{|\langle e, y\rangle|^2}{2\|y\|^2}$$

therefore $\langle e, y\rangle = 0$.

($b \Rightarrow a$) Let $x = \hat{m} + e$ for $\hat{m} \in M$ and $e \in M^\perp$. For any $y \in M$,

$$\|x-y\|^2 = \|e-(y-\hat{m})\|^2 = \|e\|^2 - 2\,\mathrm{Re}\langle e, y-\hat{m}\rangle + \|y-\hat{m}\|^2 = \|e\|^2 + \|y\|^2 \geq \|e\|^2 = \|x-\hat{m}\|^2$$

Suppose $\tilde{m}$ is another closest point. Set $d = d(x, M)$. By the parallelogram identity,

$$\|\tilde{m} - \hat{m}\|^2 = \|x - \tilde{m} - (x - \hat{m})\|^2 = 2d^2 + 2d^2 - \|2x - 2(\tilde{m} + \hat{m})\|^2 \leq 4d^2 - 4d^2 = 0$$

6.

$$f_n = n\mathbf{1}_{[0,1/n]}$$

7. (a) We will show that for $h \geq 0$ measurable, $\int h = 0 \implies h = 0$ a.e. Indeed, consider $A_n = \{n^{-1} > h \geq (n+1)^{-1}\}$ for each $n \in \mathbb{N}$ and $A_0 = \{h \geq 1\}$. Then, for any $n \in \mathbb{N} \cup \{0\}$,

$$(n+1)^{-1}\mu(A_n) \leq \int_{A_n} h\,d\mu \leq \int_{\mathbb{R}} h\,d\mu = 0$$

Therefore $\mu(A_n) = 0$. So, $\{h \neq 0\} = A = \cup A_n$ which has measure zero.

Now, we apply this to the problem by taking $h = g - f$. Then, $h \geq 0$ and

$$\int h = \int g - f = \int g - \int f = 0$$

Then by the above lemma, $h = 0$ so $f = g$ a.e.

(b) If f and g are continuous, then being equal almost everywhere will imply they are equal everywhere. Indeed, suppose there exists $x_0 \in \mathbb{R}$ such that $f(x_0) < g(x_0)$. Then, $f - g$ is continuous around x_0 so there exists $\epsilon > 0$ such that

$$f(x) < g(x) \quad \text{for } |x - x_0| < \epsilon$$

However, $\lambda(\{|x - x_0| < \epsilon\}) = \epsilon$ so $f \neq g$ on a set of measure ϵ which contradicts $f = g$ a.e.

(c) INCOMPLETE

8. (a) First, $\mathcal{M}$ is clearly a linear space since linear combinations of finite signed measures are still finite signedmeasure. N ow we show that total variation is a norm on $\mathcal{M}$. If μ has total variation 0, this means

$$\mu_+(X) = \mu_-(X) = 0$$

so X is a null set of both μ_+ and μ_-. Thus for every $E \subseteq X$, $\mu(E) = \mu_+(E) - \mu_-(E) = 0-0 = 0$. Thus μ is the zero measure. By definition, $|\alpha\mu| = \alpha\mu_+ + \alpha\mu_- = \alpha|\mu|$. Finally, to check the triangle inequality, let $\mu, \lambda \in \mathcal{M}$. Let $A \cup B = X$ be a Hahn decomposition of X with respect to the signed measure $(\mu + \lambda)$. Then,

$$(\mu + \lambda)_+(X) = \mu(A) + \lambda(A) \leq \mu_+(A) + \lambda_+(A) \leq \mu_+(X) + \lambda_+(X)$$

and

$$(\mu + \lambda)_-(X) = -\mu(B) - \lambda(B) \leq \mu_-(B) + \lambda_-(B) \leq \mu_-(X) + \lambda_-(X)$$

Therefore

$$|\mu + \lambda|(X) = (\mu + \lambda)_+(X) + (\mu + \lambda)_-(X) \leq \mu_+(X) + \lambda_+(X) + \mu_-(X) + \lambda_-(X)$$

$$= |\mu|(X) + |\lambda|(X)$$

(b) Let μ be a σ-finite measure. Clearly $\mathcal{L}_\nu = \{\mu \in \mathcal{M} : \mu << \nu\}$ is a linear subspace since if $\lambda, \mu << \nu$, and $\nu(E) = 0$, then

$$\alpha\lambda(E) + \beta\mu(E) = 0$$

for any scalars α, β. We note the crucial property of this subspace. If $\mu << \nu$, then the null sets of ν are also null sets of μ_+ and μ_-. Indeed, let $E \subset X$ such that $\nu(E) = 0$. Let $A \cup B$ be a Hahn decomposition for μ. Then,

$$\nu(A \cap E) \leq \nu(E) = 0 \quad \nu(B \cap E) \leq \nu(E) = 0$$

so $\nu(A \cap E) = \nu(B \cap E) = 0$. Therefore

$$\mu_+(E) = \mu(A \cap E) = 0 \quad \mu_-(E) = \mu(B \cap E) = 0$$

Now, let $\{\mu_n\} \subseteq \mathcal{L}_\nu$ converge to μ in the total variation norm. Then, let $E \subset X$ such that $\nu(E) = 0$. Then, $|\mu_n|(E) = 0$. By the reverse triangle inequality (a consequence of the triangle inequaity for $||\cdot||$ shown above)

$$|\mu|(E) = |\,|\mu|(E) - |\mu_n|(E)\,| \leq |\mu - \mu_n|(E) \leq |\mu - \mu_n|(X) = ||\mu - \mu_n|| \to 0$$

so $\mu(E) = 0$ and $\mu \in \mathcal{L}_\nu$.

(c) Let $f \in L^1(X, \mathcal{F}, \nu)$. Then,

$$\mu(A) = \int_A f \, d\nu$$

defines a signed measure for $A \subseteq X$. We only need to check that this pairing is isometric and onto. Surjectivity follows from the Radon-Nikodyn theorem which states that if $\rho << \lambda$, then there exists λ-measurable g such that

$$\rho = g\,d\lambda$$

Then, to check the norms are preserved, we first show that the Hahn decomposition of μ corresponds to the positive and negative parts of f. Indeed, let $A = \{f \geq 0\}$. Then, for any $E \subseteq A$,

$$\mu(E) = \int_E f\,d\nu \geq 0$$

Similarly, for $B = \{f < 0\}$, $F \subseteq B$,

$$\mu(F) = \int_F f\,d\nu \leq 0$$

So $A \cup B$ is a Hahn decomposition for μ. Therefore,

$$\int_X |f|\,d\nu = \int_X f^+ + f^-\,d\nu = \int_X f^+\,d\nu + \int_X f^-\,d\nu = \int_A f^+\,d\nu + \int_B f^-\,d\nu$$

$$= \mu_+(A) + \mu_-(B) = \mu_+(X) + \mu_-(X) = |\mu|(X)$$

9.

We have shown many times before that if $\sum \lambda(E_n) < \infty$, then $\lambda(\limsup_n E_n) = 0$. Set

$$E_n = [r_n - 2^{-n-1}, r_n + 2^{-n-1}]$$

Then,

$$\sum_n \lambda(E_n) = \sum_n 2^{-n} < \infty$$

Set $E = \limsup_{n\to\infty} E_n$. Then, $\lambda(E) = 0$. So, for any $x \notin E$, we have that there exists $k \in \mathbb{N}$ such that $x \notin E_n$ for all $n \geq k$. Therefore, $f(x)$ is only nonzero for finintely many indices so the sum must converge at x.

(b) Set

$$X_n = \left(\bigcup_{k\neq n} E_k\right)^c$$

Then, $\mathbb{R} = \cup X_n$ and, $\mu(X_n) \leq 1$ since $f_k = 0$ on X_n for $n \neq k$. Indeed,

$$\mu(X_n) = \int_{X_n} \sum f_k\,d\lambda = \int_{X_n} f_n \leq \int f_n = 1$$

(c) To show $\mu << \lambda$, let $E \subset \mathbb{R}$ such that $\lambda(E) = 0$. Then, integration over a set of measure zero is also zero so $\mu(E) = 0$.

(d) Without loss of generality, we can just show that each open ball has infinite measure since every open set contains an open ball. Let $B(x,\epsilon) \subseteq \mathbb{R}$. Then, there exists a subsequence of $\{r_n\}$ such that $\{r_{n_k}\} \subseteq B(x,\epsilon/2)$. Moreover, since the radii of E_{n_k} are decreasing, there exists N such that $E_{n_k} \subseteq B(x,\epsilon)$ for all $k \geq N$. Thus,

$$\mu(B(x,\epsilon)) = \int_{B(x,\epsilon)} \sum f_n \, d\lambda \geq \int_{B(x,\epsilon)} \sum_{k=N}^{\infty} f_{n_k} \, d\lambda = \sum_{k=N}^{\infty} \int_{B(x,\epsilon)} f_{n_k} \geq \sum_{k=N}^{\infty} \int_{E_{n_k}} f_{n_k} = \infty$$

Winter 2014

8. See Summer 13 #9 (a)
9. See Summer 13 #7
10. See Summer 13 #9

Summer 2014

1. (a) *Proof.* Suppose f is discontinuous at some $x \in (0,1)$. There there exists $\epsilon > 0$ such that $\forall \delta > 0$ there exists $y \in B(x, \delta)$ such that

$$|f(x) - f(y)| \geq \epsilon$$

However, consider the compact interval $[x + \gamma, x - \gamma] \subseteq (0,1)$ for some $\gamma > 0$. Then, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \epsilon/3$$

for all $z \in [x - \gamma, x + \gamma]$, $n \geq N$. □

(b) False. Let

$$f_n(x) = \frac{1}{x} + \frac{1}{n} \qquad f(x) = \frac{1}{x}$$

Then, $f_n \to f$ uniformly on $(0,1)$ but f is not uniformly continuous.

(c) *Proof.* □

9. See Summer 13 #7

Winter 2015

1. *Proof.* For $u = 1 + n^2x^2$, $du = 2n^2x\,dx$,

$$||f_n - 0||_1 = \int_0^1 \frac{nx}{1+n^2x^2}\,dx = \int_1^2 \frac{du}{2nu} = \frac{1}{2n}[\ln(2) - \ln(1)] \to 0$$

as $n \to \infty$. Therefore $f_n \to 0$ in $L^1[0,1]$. Now, since $||\cdot||_\infty$ and $||\cdot||_{\text{sup}}$ coincide on continuous functions,

$$||f_n - 0||_\infty = \sup_{x\in[0,1]} |f_n(x)| \geq f_n(n^{-3/2}) = \frac{n^{-1/2}}{1+n^{-1}} \to \infty$$

as $n \to \infty$. So $f_n \not\to 0$ in $L^\infty[0,1]$. $\square$

2. *Proof.* Let f be convex. Let $x_n \searrow x$. Then, define $t_n \in [0,1]$ by

$$(1-t_n)1 + t_nx = x_n$$

Notice that $t_n \to 1$ as $n \to \infty$. Then,

$$f(x_n) = f(t_nx + (1-t_n)1) \leq t_nf(x) + (1-t_n)f(1)$$

also define $s_n \in [0,1]$ such that

$$(1-s_n)(-1) + s_nx_n = x$$

then $s_n \to 1$ as $n \to \infty$ so

$$f(x) = f(s_nx_n + (1-s_n)(-1)) \leq s_nf(x_n) + (1-s_n)f(-1)$$

Combining this, we get

$$f(x) \leq s_nf(x_n) + (1-s_n)f(-1) \leq s_nt_nf(x) + s_n(1-t_n)f(1) + (1-s_n)f(-1)$$

Notice that the RHS converges to $f(x)$ as $n \to \infty$ so by the Squeeze theorem

$$\lim_{n\to\infty} f(x_n) = f(x)$$

So f is right continuous. To show left continuity, we follow the same steps but modify t_n and s_n so they are convex combiniations with the opposite endpoints. Therefore f is continuous. $\square$

3. *Proof.* For each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$d(x_n, f(x_n)) < \frac{1}{n}$$

Since X is compact, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$ with limit x. Then,

$$d(x, f(x)) \leq d(x, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) + d(f(x_{n_k}), f(x)) \to 0$$

as $k \to \infty$ by construction of x_{n_k} and since f is continuous. Thus $f(x) = x$. $\square$

4. (a) *Proof.* Let $\{y_n\} \subseteq Y$ be Cauchy. There exists $\{x_n\} \subseteq X$ such that $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is Cauchy and therefore convergent to some $x \in X$. Then,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = f(x) \in Y$$

$\square$

(b) False. Let $X = (0,1)$, $Y = \mathbb{R}$. Let $d_Y = d_X = |(\cdot) - (\cdot)|$. Let $f(x) = 1/x$. Then, clearly

$$|x_1 - x_2| \leq \left| \frac{x_1}{x_1 x_2} - \frac{x_2}{x_1 x_2} \right| = |f(x_2) - f(x_1)|$$

so the inequality holds. Additionally, Y is complete but X is not.

5. *Proof.*

$$||S(a)||_2 = \sqrt{\sum_{n=1}^{\infty} s_n^2 a_n^2} \leq ||s||_\infty ||a||_2$$

For each $k \in \mathbb{N}$, there exists $s_{n_k} \in s$ such that

$$|s_{n_k}| > ||s||_\infty - \frac{1}{k}$$

Then, consider $e_{n_k} = (0, \ldots, 0 \overset{n_k}{1}, 0, \ldots) \in \ell^2$. $||e_{n_k}||_2 = 1$ so

$$||S(e_{n_k})||_2 = |s_{n_k}| > ||s||_\infty - \frac{1}{k}$$

for all $k \in \mathbb{N}$ thus

$$||S|| = ||s||_\infty$$

$\square$

6. *Proof.* First, notice that T is bounded below:

$$||x||^2 \leq \langle Tx, x \rangle \leq ||Tx|| \cdot ||x||$$

so, $||Tx|| \geq ||x||$ for all $x \in \mathcal{H}$. Now, we show one-to-one. Let $x \in \mathcal{H}$ such that $Tx = 0$. Then,

$$0 = ||Tx|| \geq ||x||_{\mathcal{H}} \geq 0$$

so $x = 0$. Next, we show T has a closed range. Let $x_n \in \mathcal{H}$ such that $Tx_n \to y$ for some $y \in \mathcal{H}$. Then,

$$||Tx_n - Tx_m|| \geq ||x_n - x_m||$$

for all $n, m \in \mathbb{N}$. So, $\{x_n\}$ is Cauchy. Thus, there exists $x \in \mathcal{H}$ such that $x_n \to x$. Since T is bounded,

$$y = \lim_{n\to\infty} Tx_n = Tx$$

so $y \in \text{Ran}T$. Finally, we show T is onto. For $w \in (\text{Ran}T)^\perp$

$$\langle Tv, w \rangle = 0$$

for all $v \in \mathcal{H}$. In particular, for $v = w$,

$$0 = \langle Tw, w\rangle \geq ||w|| \geq 0$$

which implies $w = 0$. Thus, $(\text{Ran}T)^{\perp} = \{0\}$ so $\text{Ran}T = \overline{\text{Ran}T} = \mathcal{H}$. We have T is one-to-one and onto therefore is it invertible so $Tx = y$ has a unique solution for every $y \in \mathcal{H}$. □

7. Solution by Hao Chen and Walton Green (4/18)

Proof. We will prove the contrapositive of the statement. Suppose $\{E_k\}_{k=1}^n$ are Borel subsets of $[0, 1]$ such that

$$\lambda\left(\bigcap_{k=1}^{n} E_k\right) = 0$$

Then, we have that

$$1 = \lambda([0,1]) = \lambda\left[\left(\bigcap_{k=1}^{n} E_k\right)^c\right] = \lambda\left(\bigcup_{k=1}^{n} E_k^c\right)$$

Therefore,

$$n = \sum_{k=1}^{n} \lambda([0,1]) = \sum_{k=1}^{n} \lambda(E_k) + \lambda(E_k^c) \geq \sum_{k=1}^{n} \lambda(E_k) + \lambda\left(\bigcup_{k=1}^{n} E_k^c\right) = \sum_{k=1}^{n} \lambda(E_k) + 1$$

so $\sum_{k=1}^n \lambda(E_k) \leq n - 1$. □

8. Let $\{q_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ be an enumeration of the rational numbers. Then, let

$$U := \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$$

So,

$$\lambda(U) \leq \sum_{n=1}^{\infty} \frac{2}{n^2} = 2 < \infty$$

and $U \subseteq \mathbb{R}$ is open. Now, notice that $\bar{U} = \mathbb{R}$ since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q} \subseteq U$. Then,

$$\lambda(\partial U) = \lambda(\bar{U} \backslash U) = \infty$$

9. *Proof.* ($\Rightarrow$) let $\lambda(E) = M > 0$. Let $f_n \xrightarrow{\lambda} 0$. Then, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\inf\{c > 0 : \lambda\{|f_n| > c\} < c\} < \epsilon$$

for all $n \geq N$ which implies

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

Now, we will use the fact that

$$x \mapsto \frac{x}{x+1}$$

is monotone increasing and ≤ 1.

$$\begin{aligned}
\int_E \frac{|f_n|}{1+|f_n|} &= \int_{E\cap\{|f_n|>\epsilon\}} + \int_{E\cap\{|f_n|<\epsilon\}} \frac{|f_n|}{1+|f_n|} \\
&\leq \int_{E\cap\{|f_n|>\epsilon\}} 1 + \int_E \frac{\epsilon}{1+\epsilon} \\
&\leq \lambda\{|f_n| > \epsilon\} + \lambda(E)\left(\frac{\epsilon}{1+\epsilon}\right) \\
&< \epsilon + M\left(\frac{\epsilon}{1+\epsilon}\right) \to 0
\end{aligned}$$

as $\epsilon \to 0$.
($\Leftarrow$) Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$\frac{\epsilon^2}{1+\epsilon} > \int_E \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n|>\epsilon\}} \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n|>\epsilon\}} \frac{\epsilon}{1+\epsilon} = \lambda\{|f_n| > \epsilon\}\frac{\epsilon}{1+\epsilon}$$

so

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

for all $n \geq N$. Thus,

$$||f_n||_\lambda = \inf\{c > 0 : \lambda\{|f_n| > c\} < c\} < \epsilon$$

$\square$

10. *Proof.* Define

$$F_i := \bigcup_{n=i}^{\infty} E_n$$

for each $i \in \mathbb{N}$. Notice that F_i are reverse nested (i.e. $F_{i+1} \subseteq F_i$ therefore $F_i^c \subseteq F_{i+1}^c$) Then,

$$\mu(F_i) = \mu\left(\bigcup_{n=i}^{\infty} E_n\right) \leq \sum_{n=i}^{\infty} \mu(E_n) \to 0$$

as $i \to \infty$. Now,

$$\begin{aligned}
\mu\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n\right) &= \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \mu\left(F_1 \cap \bigcap_{k=2}^{\infty} F_k\right) = \mu\left(F_1 \backslash \bigcup_{k=2}^{\infty} F_k^c\right) \\
&= \mu(F_1) - \mu\left(\bigcup_{k=2}^{\infty} F_k^c\right) = \mu(F_1) - \lim_{k\to\infty}\mu(F_k^c) \\
&= \lim_{k\to\infty}\mu(F_1 \backslash F_k^c) = \lim_{k\to\infty}\mu(F_1 \cap F_k) \\
&= \lim_{k\to\infty}\mu(F_k) = 0
\end{aligned}$$

$\square$

Summer 2015

1. (a) *Proof.* Let $\epsilon > 0$, pick $N \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all $n \geq N$. This can be done since $\sum M_n < \infty$. Now, for all $x \in \mathbb{R}$,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_n(x)| \leq \sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all $n \geq N$. Therefore

$$\sum_{k=1}^{\infty} f_k(x)$$

is uniformly convergent. □

(b) Define

$$f_n(x) := \begin{cases} \dfrac{1}{n} & n \leq x < n+1 \\ 0 & \text{otherwise} \end{cases} \qquad \forall n\ in \mathbb{N}$$

Then, clearly $\sum f_n(x)$ is convergent pointwise and

$$\sum_{n=1}^{\infty} ||f_n||_{\infty} \leq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Now we need to show this convergence is actually uniform. Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \epsilon$. Then, for all $x \in \mathbb{R}$,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{n} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \frac{1}{n+1} \leq \frac{1}{N} < \epsilon$$

for all $n \geq N$.

2. *Proof.* First we show $(A^{\perp})^{\perp}$ is a closed subspace containing A. Clearly $A \subset (A^{\perp})^{\perp}$. Let $x, y \in (A^{\perp})^{\perp}$ and $a, b \in \mathbb{C}$. Then,

$$\langle ax + by | z \rangle = a\langle x|z\rangle + b\langle y|z\rangle = 0 + 0 = 0 \quad \forall z \in A^{\perp}$$

Let $\{x_n\}_{n=1}^{\infty} \subset (A^{\perp})^{\perp}$ such that $x_n \to x$.

$$\langle x|z\rangle = \lim_{n\to\infty} \langle x_n|z\rangle = \lim_{n\to\infty} 0 = 0 \quad \forall z \in A^{\perp}$$

So, we have shown $\overline{\text{span}} A \subset (A^{\perp})^{\perp}$. Now, let $x \in (A^{\perp})^{\perp}$. Then,

$$d(x, \overline{\text{span}} A) = \sup_{\substack{y \in (\overline{\text{span}} A))^{\perp}, \\ ||y|| \leq 1}} |\langle x|y\rangle| = \sup_{y \in A^{\perp}, ||y|| \leq 1} |\langle x|y\rangle| = 0$$

So $x \in \overline{\text{span}} A$ since it is closed. □

3. (a) *Proof.* (i) Clearly, $d_s(A,A) = 0$. Now, suppose $d_s(A,B) = 0$. Then for all $\epsilon > 0$ and $x \in A$,

$$d(x,B) \leq \epsilon$$

thus $d(x,B) = 0$ so $x \in B$ since B is closed. Thus $A \subseteq B$. Likewise $B \subseteq A$. so $A = B$.

(ii) Clearly $d_s(A,B) = d_s(B,A)$.

(iii) Let $C \subseteq X$ be closed. Let $\epsilon_1 > 0$ be such that $A_{\epsilon_1} \subset C$ and $C_{\epsilon_1} \subseteq A$. Let $\epsilon_2 > 0$ such that $B_{\epsilon_2} \subset C$ and $C_{\epsilon_2} \subseteq B$. Then,

$$A_{\epsilon_1+\epsilon_2} \subset C_{\epsilon_2} \subset B \quad \text{and} \quad B_{\epsilon_1+\epsilon_2} \subset C_{\epsilon_1} \subset A$$

So, $d_s(A,B) \leq \epsilon_1 + \epsilon_2$ for all such ϵ_1, ϵ_2. Thuerefore,

$$d_s(A,B) \leq \inf\{\epsilon_1\} + \inf\{\epsilon_2\} = d_s(A,C) + d_s(C,B)$$

□

(b) If the sets are not closed, then the first property of the metric fails. $d_s(A,A) = 0$ but $d_s(A,B) = 0$ does not necessarily $A = B$. Consider $X = \mathbb{R}$ and $A = [0,1]$ and $B = (0,1)$. $d_s(A,B) = 0$ but $A \neq B$.

4. (a) *Proof.* First, we show T is bounded:

$$\begin{aligned} ||Tf||_\infty &= \sup_{t\in[0,1]} \left| \int_0^t sf(s)\,ds \right| \leq \sup_{t\in[0,1]} \int_0^t s|f(s)|\,ds \\ &\leq ||f||_\infty \sup_{t\in[0,1]} \int_0^t s\,ds \leq ||f||_\infty \int_0^1 s\,ds = \frac{1}{2}||f||_\infty \end{aligned}$$

so $||T|| \leq \frac{1}{2}$. Let $f, g \in C[0,1]$ and $a, b \in \mathbb{R}$. Then,

$$T(af+bg)(t) = \int_0^t s(af+bg)(s)\,ds = a\int_0^t sf(s)\,ds + b\int_0^t sg(f)\,ds = a(Tf)(t) + b(Tg)(t)$$

so T is linear. □

(b) *Proof.* Let $f(t) = 1$ for all $t \in [0,1]$. Then, $||f||_\infty = 1$ and

$$||Tf||_\infty = \sup_{t\in[0,1]} \left| \int_0^t s\,ds \right| = \sup_{t\in[0,1]} \frac{t^2}{2} = \frac{1}{2}$$

so $||T|| = \frac{1}{2}$. □

5. (a) *Proof.* For every $n \in \mathbb{N}$ there exists $E_n \subseteq X$ such that $\mu(E_n) < \frac{1}{n^2}$ and $f_k \to f$ uniformly on $X \backslash E_n$. Let

$$E := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

Then, $\mu(E) = 0$ (For proof see Winter 15 #10) since

$$\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Now, for $x \notin E$, there exists k such that $x \notin \bigcup_{n=k}^{\infty} E_n$ so $x \notin E_n$ for all $n \geq k$ (However we only need it to hold for a single set, E_k. So, since $x \in E_k^c$,

$$f_n(x) \to f(x)$$

as $n \to \infty$. Therefore $f_n \to f$ pointwise a.e. $\square$

(b) *Proof.* Let $\epsilon > 0$. Then, there exists some E_ϵ such that $\mu(E_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on E_ϵ. Moreover, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$, $x \in E_\epsilon^c$. Then,

$$\mu\{|f_n - f| > \epsilon\} \leq \mu(E_\epsilon) < \epsilon$$

so

$$||f_n - f||_\mu = \inf\{c > 0 : \mu\{|f_n - f| > c\} < c\} < \epsilon$$

therefore $f_n \to f$ in measure. $\square$

6. *Proof.* Let $E \subset [a, b]$ be Borel measurable with $\lambda(E) > 0$. Let $\{q_n\}$ be an enumeration of the rational numbers in the interval $[0, 1]$. Set

$$F = \bigcup_n (E + q_n)$$

If $\{E + q_n\}$ are all disjoint, then, $\lambda(F) = \sum_{n=1}^{\infty} \lambda(E + q_n) = \sum_{n=1}^{\infty} \lambda(E) = \infty$ since $\lambda(E) > 0$. But this is a contradiction since $F \subseteq [a, b+1]$ which has finite Lebesgue measure. Thus there exists $x \in (E + q_n) \cap (E + q_m)$ for some n and m not equal (so $q_n \neq q_m$). Then, there exists $y, z \in E$ such that

$$y + q_n = x = z + q_m$$

so $y - z = q_m - q_n \in \mathbb{Q} \backslash \{0\}$. $\square$

7. (a) False. Consider the following function with a "spike" at every natural number, $n \geq 2$.

$$f(x) := \begin{cases} \text{lin} \nearrow & n \leq x \leq n + \dfrac{1}{n^3} \\ n & x = n + \dfrac{1}{n^3} \\ \text{lin} \searrow & n + \dfrac{1}{n^3} \leq x \leq n + \dfrac{2}{n^3} \\ 0 & \text{else} \end{cases}$$

Notice that

$$\int_{\mathbb{R}} f(x)\,dx = \sum_{n=2}^{\infty} \frac{1}{2}\cdot\frac{2}{n^3}\cdot n = \sum_{n=2}^{\infty}\frac{1}{n^2} < \infty$$

but

$$\limsup_{x\to\infty} |f(x)| = \infty$$

(b) *Proof.* Let f be integrable and differentiable and let $D > 0$ such that $|f'(x)| \leq D$ for all $x \in \mathbb{R}$. Fix $x \in \mathbb{R}$. Using the mean-value theorem, for all $y \in \mathbb{R}$ such that

$$|x-y| \leq \frac{f(x)}{D},$$

we know that

$$f(y) \geq f(x) - |x-y|D$$

Suppose without loss of generality that

$$\limsup_{x\to\infty} f(x) = M$$

for some $M > 0$. Then for all $n \in \mathbb{N}$, there exists $x_n \geq n$ such that

$$f(x_n) \geq \frac{M}{2}$$

Then,

$$\int_{\mathbb{R}} f(x)\,dx \geq \sum_{n=1}^{\infty} \frac{1}{2}\cdot\min\left\{\frac{f(x_n)}{D}, 1\right\}\cdot f(x_n) \geq \sum_{n=1}^{\infty}\frac{1}{8}\cdot\min\left\{\frac{M}{D},1\right\}\cdot M = \infty$$

which contradicts the fact that f is integrable. □

8. *Proof.* (a $\implies$ b)

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^{\infty}\int_{F_k} 2^k\,dm \leq \sum_{k=-\infty}^{\infty}\int_{F_k} f\,dm = \int_{\mathbb{R}} f\,dm$$

(b $\implies$ c) First, notice that $E_k \cup F_{k-1} = E_{k-1}$ and the union is disjoint therefore

$$m(F_k) = m(E_{k+1}) - m(E_k)$$

Mutliply by 2^k and sum from $-N$ to N we have

$$\sum_{k=-N}^{N} 2^k m(F_k) = \sum_{k=-N}^{N} 2^k m(E_{k+1}) - \sum_{k=-N}^{N} 2^k m(E_k) = \frac{1}{2}\sum_{k=-N}^{N} 2^{k+1} k m(E_{k+1}) - \sum_{k=-N}^{N} 2^k m(E_k)$$

$$= -\frac{1}{2}\sum_{k=-N+1}^{N-1} 2^k m(E_k) + 2^N m(E_{N+1}) - 2^{-N} m(E_{-N})$$

The final two terms can be bounded by $\int f$: $2^N m(E_N) \leq \int_{E_N} f\, dm \leq \int_{\mathbb{R}} f\, dm < \infty$. Therefore,for any N,

$$\sum_{k=-(N-1)}^{N-1} 2^k m(E_k) \leq -2 \sum_{k=-\infty}^{\infty} 2^k m(F_k) + 4 \int_{\mathbb{R}} f\, dm < \infty$$

(c $\implies$ a) Notice that since f is non-negative, $\mathbb{R} = \{f = 0\} \cup E_k$.

$$\int f\, dm = \sum_{k=-\infty}^{\infty} \int_{F_k} f\, dm \leq \sum_{k=-\infty}^{\infty} \int_{F_k} 2^{k+1}\, dm = 2 \sum_{k=-\infty}^{\infty} m(F_k) \leq 2 \sum_{k=-\infty}^{\infty} m(E_k)$$

$\square$

Winter 2016

1. Recall
$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$
So for $x = 1$,
$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

2. *Proof.* Since ℓ^2 is a Hilbert space, A being dense in ℓ^2 is equivalent to
$$A^\perp = \{0\}$$
Let $x = (x_1, x_2, \dots) \in A^\perp$. Then, $\langle x, a \rangle_{\ell^2} = 0$ for all $a \in A$. Notice that $a = e^{(k)} = (0, \dots, 0, \overset{k^{th}}{1}, 0, \dots)$ is in A for any $k \in \mathbb{N}$.
$$0 = \langle x, e^{(k)} \rangle = \sum_{i=1}^{\infty} x_i e_i^{(k)} = x_k$$
for $k \in \mathbb{N}$. Therefore $x = 0$. Now we show the same thing for A^c. Let $y \in (A^c)^\perp$. Also, define $f^{(k)} = e^{(k)} - e^{(k+1)} \in A^c$. Then,
$$0 = \langle y, f^{(k)} \rangle = y_k - y_{k+1}$$
So $y_k = y_{k+1}$ for all $k \in \mathbb{N}$. Thus y is a constant sequence. The only constant sequence in ℓ^2 is the zero sequence therefore $y = 0$. □

3. *Proof.* First we show subspace. Let $x_1 + y_1, x_2 + y_2 \in X + Y$ and $a, b \in \mathbb{R}$. Then,
$$a(x_1 + y_1) + b(x_2 + y_2) = (ax_1 + bx_2) + (ay_1 + by_2) \in X + Y$$
Now we show closure. Let $\{(x_n + y_n)\}_{n=1}^{\infty}$ be a sequence in $X + Y$ with limit z. This sequence is also Cauchy. So, using the fact that $X \perp Y$,
$$\begin{aligned}
||(x_n + y_n) - (x_m + y_m)||^2 &= ||(x_n - x_m) + (y_n - y_m)||^2 \\
&= \langle (x_n - x_m) + (y_n - y_m), (x_n - x_m) + (y_n - y_m) \rangle \\
&= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (x_n - x_m), (y_n - y_m) \rangle \\
&\quad + \langle (y_n - y_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\
&= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\
&= ||x_n - x_m||^2 + ||y_n - y_m||^2
\end{aligned}$$
and thus $\{x_n\}$ and $\{y_n\}$ are both Cauchy. Since $\mathcal{H}$ is a Hilbert space, they are convergent to some x, y respectively. Since X, Y are closed, $x \in X$ and $y \in Y$. Then,
$$z = \lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = x + y \in X + Y$$
Therefore $X + Y$ is closed. □

4. *Proof.* Since Y is a Banach space, $\mathcal{B}(X,Y)$ is also a Banach space. In a Banach space, any absolutely convergent series is convergent. Since $||T|| < 1$,

$$\sum_{n=0}^{\infty} ||T||^n < \infty$$

So

$$\sum_{n=0}^{\infty} T^n \in \mathcal{B}(X,Y)$$

□

5. (a) *Proof.* First, to show T is well-defined we need to show $T\xi$ is continuous for a fixed ξ. This follows from the fact that for $n > m$,

$$\left|\left|\sum_{k=0}^{n} a_k\xi_k x^k - \sum_{k=0}^{m} a_k\xi_k x^k\right|\right|_\infty = \sup_{x\in[0,1]}\left|\sum_{k=m+1}^{n} a_k\xi_k x^k\right| \le ||a||_\infty \sum_{k=m+1}^{n} |\xi_k| \to 0$$

as $n, m \to \infty$ since $\xi \in \ell^1$. Thus, this sequence of partial sums is Cauchy in $||\cdot||_\infty$. Since $(C[0,1], ||\cdot||_\infty)$ is a Banach space, it's limit, $T\xi \in C[0,1]$. To show linearity, let $\xi, \zeta \in \ell^1$ and $\alpha, \beta \in \mathbb{R}$.

$$T(\alpha\xi + \beta\zeta)(x) = \sum_{k=0}^{\infty} a_k(\alpha\xi_k + \beta\zeta_k)x^k = \alpha\sum_{k=0}^{\infty} a_k\xi_k x^k + \beta\sum_{k=0}^{\infty} a_k\zeta_k x^k$$

$$= \alpha T(\xi)(x) + \beta T(\zeta)(x)$$

□

(b) *Proof.*

$$||T(\xi)||_\infty = \sup_{x\in[0,1]} |T(\xi)(x)| = \sup_{x\in[0,1]}\left|\sum_{k=0}^{\infty} a_k\xi_k x^k\right| \le ||a||_\infty \sum_{k=0}^{\infty} |\xi_k| = ||a||_\infty \cdot ||\xi||_1$$

So $||T|| \le ||a||_\infty$. We claim this is actually the norm. For $\epsilon > 0$ there exists $a_n \in a$ such that

$$|a_n| > ||a||_\infty - \epsilon$$

Pick $\xi^{(n)} = (0, \ldots, 0, \overset{n^{th}}{1}, 0, \ldots) \in \ell^1$. Then,

$$||T\xi^{(n)}||_\infty = \sup_{x\in[0,1]}\left|\sum_{k=0}^{\infty} a_k\xi_k^{(n)} x^k\right| = \sup_{x\in[0,1]} |a_n x^n| = |a_n| > ||a||_\infty - \epsilon$$

Since there exists such $\xi^{(n)}$ for all $\epsilon > 0$, $||T|| = ||a||_\infty$. □

6. (i) LDCT cannot be applied to f_n since any k which bounds every f_n above, must be greater than 1 everywhere thus $\int_{\mathbb{R}} k = \infty$.

(ii) LDCT cannot be applied to g_n since any k which bounds every g_n above, must be greater than $1/x$ everywhere thus $\int_{\mathbb{R}} k \geq \int_{\mathbb{R}} 1/x = \infty$.

(iii) LDCT can be applied since for $k = 1/x^2$, $|h_n| \leq k$ and

$$\int_{\mathbb{R}} \frac{1}{x^2}\, dx < \infty$$

7. *Proof.* Let $E \subset \mathbb{R}$, $\epsilon \in (0,1)$. Set $\delta = m^*(E)(1/\epsilon - 1) > 0$. By definition of outer measure, there exists an open set $G \supset E$ such that $m^*(E)+\delta > m^*(G) = m(G)$. Then,

$$\epsilon m(G) < \epsilon(m^*(E) + \delta) = \epsilon m^*(E)(1 + 1/\epsilon - 1) = m^*(E)$$

Moreover, since G is open, it can be written as a countable, disjoint union of open intervals, say I_k. Then,

$$\sum_k \epsilon m(I_k) = \epsilon m(G) < m^*(E) = m^*(E \cap G) \leq \sum_k m^*(E \cap I_k)$$

Therefore, at least one term in the left hand sum must be smaller than one term in the right sand sum, i.e. there exists k such that $\epsilon m^*(I_k) = \epsilon m(I_k) < m^*(E \cap I_k)$. $\square$

8. *Proof.* Define $A_n := \{x \in [0,1] : n+1 > |f(x)| \geq n\}$

$$\sum_{n=1}^{\infty} n\lambda(A_n) = \sum_{n=1}^{\infty} \int_{A_n} n\, dx \leq \sum_{n=1}^{\infty} \int_{A_n} f(x)\, dx = \int_0^1 f(x) < \infty$$

So,

$$\lim_{n\to\infty} n\lambda(\{x \in [0,1] : |f(x)| \geq n\}) = \lim_{n\to\infty} n\lambda\left(\bigcup_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} n \sum_{k=n}^{\infty} \lambda(A_k)$$

$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} k\lambda(A_k) = 0$$

$\square$

9. (a) *Proof.* Let $f(x) > 0$ for $x \in [0,1]$ and $E \subseteq [0,1]$ such that $\lambda(E) > 0$. Suppose $\int_E f\, d\lambda = 0$. Then

$$f(x) = 0$$

for almost every $x \in E$. However, since $\lambda(E) > 0$ there exists $x \in E$ such that $f(x) = 0$ which is a contradiction. $\square$

(b) First we prove the following fact: $\mu(\limsup E_n) \geq \limsup \mu(E_n)$. Indeed, set $F_k = \cup_{n=k}^{\infty} E_n$. F_k are decreasing.

$$\mu(\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} E_n) = \mu(\cap_{k=1}^{\infty} F_k) = \inf_k \mu(F_k) = \inf_k \mu(\cup_{n=k}^{\infty} E_n) \geq \inf_k \sup_{n\geq k} \mu(E_n)$$

Proof. Fix $\epsilon \in (0,1]$. Suppose $\inf\limits_{\lambda(E)\geq\epsilon} \displaystyle\int_E f\,d\lambda = 0$. Then, for each n there exists E_n with $\lambda(E_n) \geq \epsilon$ and

$$\int_{E_n} f\,d\lambda < \frac{1}{n^2}$$

Then, consider $E = \limsup E_n = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} E_n$. By the fact above, $\mu(E) \geq \epsilon$. By part (a), this means $\int_E f\,d\lambda > 0$. However,

$$\int_E f\,d\lambda = \int_{\cap_{k=1}^{\infty}\cup_{n=k}^{\infty} E_n} f\,d\lambda \leq \int_{\cup_{n=k}^{\infty} E_n} f\,d\lambda \leq \sum_{n=k}^{\infty} \int_{E_n} f\,d\lambda \leq \sum_{n=k}^{\infty} \frac{1}{n^2}$$

for any k. Therefore $\int_E f\,d\lambda = 0$ which is a contradiction. $\square$

Summer 2016

1. (a) *Proof.* We show that f_n does not converge uniformly on the half-open interval $[0, 1)$. The pointwise limit is clearly $f(x) = 0$ for $x \in [0, 1)$. If $\{f_n\}$ converges uniformly, then it must converge to f, the pointwise limit. Let $\epsilon > 0$. For any $n \in \mathbb{N}$ there exists $x \in (0, 1]$ such that

$$1 > x > \left(\frac{\epsilon}{1-\epsilon}\right)^{1/n}$$

Then,

$$|f(x)| > \epsilon$$

so $\{f_n\}$ does not converge uniformly on $[0, 1)$ therefore it does converge uniformly on $[0, 1]$. $\square$

(b) *Proof.* Notice that

$$f_n(x) \leq 1$$

for $x \in [0, 1]$. Since $\displaystyle\int_0^1 1\,dx < \infty$, by Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0$$

$\square$

2. False. Counterexample:
Consider $\{x^{(n)}\}_{n=1}^{\infty} \subset X$ where

$$x^{(n)} := (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{n}, 0, 0, \ldots) \in X$$

Then, $\{x^{(n)}\}_{n=1}^{\infty}$ is Cauchy: For $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \epsilon$. So, for all $n, m \geq N$ $(n > m)$,

$$d(x^{(n)}, x^{(m)}) = \sup_{i\in\mathbb{N}} \left|x_i^{(n)} - x_i^{(m)}\right| = \frac{1}{m} < \frac{1}{N} < \epsilon$$

However, $x_n \to (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ which is not in X.

3. *Proof.* Let $\{y_n\}_{n=1}^{\infty} \subset K$. Let $\{y_{n_k}\}_k$ denote the set of distinct elements of $\{y_n\}_{n=1}^{\infty}$. If $\{y_{n_k}\}_k$ is finite, then there exists some $m \in \mathbb{N}$ such that y_m occurs infintely many times in $\{y_n\}_{n=1}^{\infty}$ thus the constant sequence $\{y_m\}$ is a convergent subsequence of $\{y_n\}_{n=1}^{\infty}$. On the other hand if $\{y_{n_k}\}_k$ is infinte, then

$$\{y_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=0}^{\infty}$$

is a subsequence of a convergent sequence so it is itself covergent to $\lim_{n\to\infty} x_n = x_0 \in K$. $\square$

4. *Proof.* First, notice

$$\begin{aligned}(T-S)^3 &= (T^2 - ST - TS + S^2)(T-S)\\ &= (T - 2ST + S)(T-S)\\ &= (T^2 - 2ST^2 + ST - ST + 2S^2T - S^2)\\ &= (T - 2ST + ST - ST + 2ST - S)\\ &= (T-S)\end{aligned}$$

Then, by Cauchy-Schwarz for the operator norm,

$$||T-S|| = ||(T-S)^3|| \leq ||T-S||^3$$

Therefore

$$1 \leq ||T-S||^2$$

and

$$||T-S|| \geq 1$$

□

5. *Proof.* Let $n, m \in \mathbb{N}$. Without loss of generality, let $n > m$. First,

$$||x_m||^2 = \langle x_n, x_m \rangle \leq ||x_n|| \cdot ||x_m||$$

so $\{||x_n||\}_{n=1}^{\infty}$ is monotone decreasing. Moreover it is bounded below by 0 so it is convergent to some $K \in \mathbb{R}$. Moreover, since

$$\lim_{n\to\infty} ||x_n||^2 = \left(\lim_{n\to\infty} ||x_n||\right)^2 = K^2$$

$\{||x_n||^2\}_{n=1}^{\infty}$ is convergent and therefore Cauchy. Then, for $n > m$,

$$\begin{aligned}||x_n - x_m||^2 &= \langle x_n - x_m, x_n - x_m \rangle\\ &= ||x_n||^2 - \langle x_n, x_m \rangle - \langle x_m, x_n \rangle + ||x_m||^2\\ &= ||x_n||^2 - ||x_m||^2 - ||x_m||^2 + ||x_m||^2\\ &= ||x_n||^2 - ||x_m||^2\\ &= \big|||x_n||^2 - ||x_m||^2\big| \to 0\end{aligned}$$

So $\{x_n\}_{n=1}^{\infty}$ is Cauchy and therefore convergent since $\mathcal{H}$ is a Hilbert space. □

6. *Proof.* Let $A = (0,1)$ and $B = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then,

$$d(A,B) = \lambda(A \Delta B) = \lambda(\{\tfrac{1}{2}\}) = 0$$

but $A \neq B$. Thus the first property of a metric $d(A,B) = 0 \implies A = B$ fails. □

7. *Proof.* ($\Leftarrow$) Fix $\epsilon > 0$. Then, there exists an open set $\mathcal{O} \supseteq A$ such that

$$\lambda(\mathcal{O} \backslash A) < \epsilon$$

Thus $\mathcal{O} \backslash A \in \mathcal{L}$, the σ-algebra of Lebesgue-measurable sets. Moreover, since $\mathcal{O}$ is open, it is also Lebesgue measurable. Thus,

$$A = \mathcal{O} \backslash (\mathcal{O} \backslash A) \in \mathcal{L}$$

since $\mathcal{L}$ is closed under set-minus.
($\Rightarrow$) Let A be Lebesgue measurable. Then,

$$\lambda(A) = \lambda^*(A) = \inf_{A \subseteq \bigcup_n I_n} \sum_{n=1}^{\infty} \lambda(I_n)$$

where $I_n = [a_n, b_n)$. Now, let $\epsilon > 0$. By definition of inf, there exists $\{I_n\}_{n=1}^{\infty}$ such that

$$\lambda(A) + \frac{\epsilon}{2} > \sum_{n=1}^{\infty} \lambda(I_n) \quad \text{and} \quad A \subseteq \bigcup_{n=1}^{\infty} I_n$$

Now, define

$$J_n = \left(a_n, b_n + \frac{\epsilon}{2^{n+1}}\right)$$

Then, $I_n \subset J_n$ for all i and for $\mathcal{O} := \bigcup\limits_{n=1}^{\infty} J_n$

$$\lambda(\mathcal{O} \backslash A) = \lambda(\mathcal{O}) - \lambda(A) \leq \sum_{n=1}^{\infty} \lambda(J_n) - \lambda(A) = \sum_{n=1}^{\infty} \left(\lambda(I_n) + \frac{\epsilon}{2^{n+1}}\right) - \lambda(A) < \epsilon$$

□

8. (a) *Proof.* Proof by contraposition. Suppose $\lambda(E_n) = 0$ for all $n \in \mathbb{N}$. Then,

$$\lambda\left(\{x \in I : f(x) > 0\}\right) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \lambda(E_n) = 0$$

since $\{E_n\}$ are nested. □

(b) *Proof.* Suppose the assumption holds and that $\lambda\left(\{x \in I : f(x) > 0\}\right) > 0$. Then, by part (a), there exists some $n \in \mathbb{N}$ such that $\lambda(E_n) > 0$. Since the measure of E_n is positive, it contains infintely many points. Now, pick $x_1, \ldots, x_{n \cdot M} \in E_n$, then,

$$f(x_1) + \cdots + f(x_{n \cdot M}) > \frac{1}{n} + \cdots + \frac{1}{n} = Mn\left(\frac{1}{n}\right) = M$$

which is a contradiction. □

9. INCOMPLETE

Proof. ($\Rightarrow$) By the Triangle Inequality,

$$||f_n||_1 \leq ||f_n - f||_1 + ||f||_1$$

and

$$||f||_1 \leq ||f - f_n||_1 + ||f_n||_1$$

therefore

$$\Big| ||f_n||_1 - ||f||_1 \Big| \leq ||f_n - f||_1 \to 0$$

as $n \to \infty$.
($\Leftarrow$) $\square$

10. *Proof.* Applying Holder's Inequality,

$$\sum_{n=0}^{\infty} \int_n^{n+1} f(x)\,dx \leq \sum_{n=0}^{\infty} \left(\int_n^{n+1} f(x)^2\,dx \right)^{1/2} \left(\int_n^{n+1} 1^2\,dx \right)^{1/2}$$

$$\leq \left(\int_{\mathbb{R}} f(x)^2\,dx \right)^{1/2} = ||f||_{L^2(\mathbb{R})} < \infty$$

therefore

$$\lim_{n\to\infty} \int_n^{n+1} f(x)\,dx = 0$$

$\square$

Winter 2017

1. Notice that
$$\sum \frac{\sin(nx)}{n}$$
is the Fourier series of the function $x \mapsto \dfrac{\pi - x}{2}$. Indeed,
$$\frac{\pi}{2}\int_{-\pi}^{\pi} \sin(nx)\,dx - \frac{1}{2}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{x\cos(nx)}{2n}\Big|_{x=-\pi}^{\pi} - \frac{1}{2n}\int_{-\pi}^{\pi}\cos(nx)\,dx =$$

2. (a) *Proof.* (i) First, notice that
$$-M|x-y| \leq f(x) - f(y) \leq M|x-y|$$
therefore $|f(x)| \leq |f(y)| + M|x-y|$ for all $x, y \in \mathbb{R}$. Therefore,
$$\begin{aligned} 0 \leq d(f,g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x\in[-n,n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x\in[-n,n]} |f(x)| + |g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x\in[-n,n]} |f(0)| + M|x| + |g(0)| + M|x| \\ &\leq \sum_{n=1}^{\infty} \frac{f(0)+g(0)+2Mn}{2^n} < \infty \end{aligned}$$
so $d(f,g)$ is well-defined and non-negative.

(ii) Clearly $d(f,f) = 0$. Assume $d(f,g) = 0$. Then $\sup_{x\in[-n,n]} |f(x) - g(x)| = 0$ for all n thus $f(x) = g(x)$ on $\mathbb{R}$.

(iii) Clearly $d(f,g) = d(g,f)$.

(iv)
$$\begin{aligned} d(f,g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x\in[-n,n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x\in[-n,n]} (|f(x) - h(x)| + |h(x) - g(x)|) \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \left(\sup_{x\in[-n,n]} |f(x) - h(x)| + \sup_{x\in[-n,n]} |h(x) - g(x)| \right) \\ &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x\in[-n,n]} |f(x) - h(x)| + \sum_{n=1}^{\infty} 2^{-n} \sup_{x\in[-n,n]} |h(x) - g(x)| \\ &= d(f,h) + d(h,g) \end{aligned}$$

$\square$

(b) *Proof.* Let $\{f_k\}_{k=1}^{\infty} \subseteq \mathcal{L}$ be Cauchy in d. Fix $x \in \mathbb{R}$. Then, $x \in [-N, N]$ for some $N \in \mathbb{N}$. For any $k, \ell \in \mathbb{N}$,

$$|f_k(x) - f_\ell(x)| \leq 2^N 2^{-N} \sup_{x \in [-N,N]} |f_k(x) - f_\ell(x)| \leq 2^N d(f_k, f_\ell) \to 0$$

as $k, \ell \to \infty$. Therefore $\{f_k(x)\}_{k=1}^{\infty} \subseteq \mathbb{R}$ is Cauchy for each x and therefore convergent since $\mathbb{R}$ is complete. Then define

$$f(x) := \lim_{k \to \infty} f_k(x)$$

First, we show $f \in \mathcal{L}$. Fix $x, y \in \mathbb{R}$. For $\epsilon > 0$ there exists $n_1, n_2 \in \mathbb{N}$ such that

$$|f_k(x) - f(x)| < \epsilon \qquad |f_\ell(y) - f(y)| < \epsilon \quad \forall\, k > n_1\, \ell > n_2$$

Then, for $n = \max\{n_1, n_2\}$,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 2\epsilon + M|x - y|$$

so $|f(x) - f(y)| \leq M|x - y|$ and $f \in \mathcal{L}$. Now we will show $f_k \to f$ in d. Let $\epsilon > 0$. Since $\{f_k\}$ is Cauchy in d. $\{d(f_k, 0)\}$ is uniformly bounded, i.e. there exist $C > 0$ such that $d(f_k, 0) \leq C$ for all k. Indeed, there exists N such that $d(f_k, f_j) < 1$ for $j, k \geq N$. Thus,

$$d(f_k, 0) \leq d(f_k, f_N) + d(f_N, 0) \leq 1 + d(f_N, 0)$$

So $d(f_k.0) \leq \max_{j=1,\ldots,N}\{1 + d(f_j, 0)\}$ for all k. Thus,

$$d(f_k, f) \leq C + d(f, 0)$$

for all k so there exists N such that

$$\sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n,n]} |f_k(x) - f(x)| < \epsilon/2$$

for all k. Moreover, since $f_k(x) \to f(x)$ for each $x \in [-N, N]$, $f_k \to f$ uniformly on $[-N, N]$ since it is closed and bounded. Therefore we can take k large enough so that

$$\sum_{n=1}^{N} 2^{-n} \sup_{x \in [-n,n]} |f_k(x) - f(x)| < N 2^{-N} \sup_{x \in [-N,N]} |f_k(x) - f(x)| < \epsilon/2$$

Then,

$$d(f_k, f) = \sum_{n=1}^{N} + \sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n,n]} |f_k(x) - f(x)| \leq \epsilon$$

for k large enough. $\square$

3. *Proof.* Let $f \in C^1[0,1]$.

$$|\varphi_0(f)| = |f'(0)| \leq \sup_{x\in[0,1]} |f'(x)| \leq ||f||$$

So $||\varphi_0|| \leq 1$. We will show $||\varphi_0|| = 1$. Consider the sequence defined

$$f_n(x) := \frac{\sin(nx)}{n}$$

Notice $||f_n|| = 1/n + 1$ and $|\varphi_0(f_n)| = 1$. Thus,

$$1 \geq ||\varphi_0|| = \sup_{f\neq 0} \frac{|\varphi_0(f)|}{||f||} \geq \sup_{n\in\mathbb{N}} \frac{|\varphi_0(f_n)|}{||f_n||} = \sup_{n\in\mathbb{N}} \frac{1}{1+1/n} = 1$$

so $||\varphi_0|| = 1$. ☐

4. (a) *Proof.* Let $x \in \ell^2$. Then for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |x_k|^2 < \epsilon$$

Then, for $y = (x_1, x_2, \ldots, x_N, 0, 0, \ldots) \in Y$,

$$||x-y||_2^2 = \sum_{k=1}^{\infty} |x_k - y_k|^2 = \sum_{k=N+1}^{\infty} |x_k - 0|^2 < \epsilon$$

so Y is dense in ℓ^2. ☐

(b) *Proof.* By Cauchy-Schwarz,

$$\left|\sum_{k=1}^{n} x_k\right| \leq \left(\sum_{k=1}^{n} |1|^2\right)^{1/2} \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} = \sqrt{n}\left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2}$$

Moreover, if $x \in \ell^2$, then $\sum_{k=1}^{\infty} |x_k|^2$ converges so we can bound the final term by $||x||_2$. ☐

(c) *Proof.* Let $x \in \ell^2$, $\epsilon > 0$. By part (a) there exists $y \in Y$ such that $\|x-y\|_2 \leq \epsilon/2$. Then,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left|\sum y_n\right| = 0$$

since the second term is bounded and the first is decreasing to 0. So, there exists N such that

$$\frac{1}{\sqrt{n}} \left|\sum y_n\right| < \epsilon/2 \quad \text{for } n \geq N$$

By triangle inequality for $|\cdot|$ and part(b),

$$\frac{1}{\sqrt{n}} \left|\sum x_n\right| \leq \frac{1}{\sqrt{n}} \left|\sum x_n - y_n\right| + \frac{1}{\sqrt{n}} \left|\sum y_n\right| < \|x-y\|_2 + \epsilon/2 < \epsilon$$

for $n \geq N$ so

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left|\sum x_n\right| = 0$$

☐

5. *Proof.* First notice that

$$0 \leq \int_0^\infty \frac{x}{1+x^3}\,dx = \int_0^1 + \int_1^\infty \frac{x}{1+x^3}\,dx \leq \int_0^1 1\,dx + \int_1^\infty \frac{1}{x^2}\,dx < \infty$$

Then, notice that $\frac{x}{1+x^n} \leq \frac{x}{1+x^{n+1}}$ for $x \in (0,1)$. Therefore, by monotone convergence theorem,

$$\lim_{n\to\infty} \int_0^1 \frac{x}{1+x^n}\,dx = 0$$

since $x/(1+x^n) \to 0$ pointwise on $(0,1)$. Moreover,

$$\int_1^\infty \frac{x}{1+x^n}\,dx$$

is monotone decreasing and bounded below by zero therefore

$$\lim_{n\to\infty} \int_0^\infty \frac{x}{1+x^n}\,dx = \lim_{n\to\infty} \int_0^1 + \int_1^\infty \frac{x}{1+x^n}\,dx$$

exists. Moreover,

$$\lim_{n\to\infty} \int_0^\infty \frac{x}{1+x^n}\,dx = \int_1^\infty \lim_{n\to\infty} \frac{x}{1+x^n}\,dx = 0$$

by the Lebesgue dominated convergence theorem since

$$\frac{x}{1+x^n} \leq \frac{x}{x^n} = x^{1-n}$$

which is integrable on $(1,\infty)$ for $n \geq 3$. $\square$

6. (a) *Proof.* Set $f(x) = \mathbf{1}_{\liminf_n A_n}$.

   (i) $f(x) = 1 \iff x \in \cup_k \cap_{n=k}^\infty A_n$. So, there exists k such that $x \in A_n$ ($\mathbf{1}_{A_n}(x) = 1$) for all $n \geq k$. So, $\lim_{n\to\infty} \mathbf{1}_{A_n}(x) = 1$ (so lim inf is also 1).

   (ii) Suppose $f(x) = 0$. For each k there exists $n \geq k$ such that $x \notin A_n$ ($\mathbf{1}_{A_n}(x) = 0$) so $\liminf_n \mathbf{1}_{A_n} = 0$.

   $\square$

   (b) By Fatou's Lemma,

$$\mu(\liminf_n A_n) = \int_X f\,d\mu = \int_X \liminf_n \mathbf{1}_{A_n}\,d\mu \leq \liminf_n \int_X \mathbf{1}_{A_n}\,d\mu = \liminf_n \mu(A_n)$$

7. *Proof.* Define $f = \sup_N \sum_{n=1}^N f_n$. f is a measurable function, moreover, since f_n are non-negative, $\sum f_n \nearrow f$. So, by Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f = \sum \int f_n \leq \sum \frac{1}{n^2} < \infty$$

So f is non-negative and integrable. We claim this implies $f < \infty$ a.e. If not, then there exists E with $\lambda(E) > 0$ and $f = \infty$ on E. Then,

$$\int_{\mathbb{R}} f \geq \int_E f = \infty$$

so $f < \infty$ a.e. $\square$

8. (a) *Proof.* By Hölder's Inequality,

$$\left|\int f_n \, d\mu - \int f \, d\mu\right| \leq \int |f - f_n| \, d\mu \leq \|f - f_n\|_\infty \int \, d\mu = \|f - f_n\|_\infty \mu(X) \to 0$$

as $n \to \infty$. $\square$

Summer 2017

5. Hao Chen

   Proof. To show the orthonormal set $\{f_n\}$ is an orthonormal basis we will show that $\{f_n\}^\perp = \{0\}$. If not, then there exists $x \neq 0$ such that $\langle x, f_n \rangle = 0$ for all n. However, by Parseval's identity and the Cauchy-Schwarz inequality,

   $$\|x\|^2 = \sum |\langle x, e_n \rangle|^2 = \sum |\langle x, e_n - f_n \rangle|^2 \leq \sum \|x\|^2 \|e_n - f_n\|^2 < \|x\|^2$$

   but this is a contradiction so $\{f_n\}^\perp = \{0\}$. $\square$

   A more complicated proof by Walton:

   Proof. Let $c = \sum \|e_n - f_n\|^2 < 1$. Define $T : \mathcal{H} \to \mathcal{H}$ by sending $x = \sum \langle x, e_n \rangle e_n \mapsto \sum \langle x, e_n \rangle f_n$. The second sum converges by the Bessel inequality. Now, by the Cauchy-schwarz inequality and Parseval's identity,

   $$\|(I - T)x\|^2 = \left\| \sum \langle x, e_n \rangle (e_n - f_n) \right\|^2 \leq \sum |\langle x, e_n \rangle|^2 \sum \|e_n - f_n\|^2 = c\|x\|^2$$

   So, $\|T - I\| \leq \sqrt{c} < 1$. We claim that this means T is invertible. Indeed, set

   $$S = \sum_{n=0}^{\infty} (I - T)^n$$

   The sum is absolutely convergent since $\|I - T\| < 1$ so S is bounded linear operator since $\mathcal{L}(\mathcal{H})$ is a Banach space. Moreover,

   $$S - TS, S - ST = \sum_{n=1}^{\infty} (I - T)^n = S - (I - T)^0 = S - I$$

   so $S = T^{-1}$. Now, let $y \in \mathcal{H}$. Then, there exists x $(T^{-1}y)$ such that $Tx = y$. Therefore,

   $$y = \sum \langle x, e_n \rangle f_n \tag{1}$$

   and therefore $\overline{\text{span}}\{f_n\} = \mathcal{H}$. $\square$

   Remark: This acutally holds if $\sum \|e_n - f_n\|^2 < \infty$.

6. Define $A_n = \{f \geq 1/n\}$. Since $A_n \subseteq A_{n+1}$,

   $$0 < \mu(\{f > 0\}) = \mu\left(\bigcup_n A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

   Therefore there exists n such that $\mu(A_n) > 0$. Then,

   $$\int f \geq \int_{A_n} f \geq \frac{1}{n}\mu(A_n) > 0$$

7. (a) We first show that if f is integrable, the $\mu(E_n) \to 0$ implies $\int_{E_n} f \to 0$. Since f is integrable, for $A_n = \{n-1 \leq |f| \leq n\}$,

$$\infty > \int |f| \geq \sum (n-1)\mu(A_n)$$

Given $\epsilon > 0$ there exists N such that $\sum_{n=N}^{\infty}(n-1)\mu(A_n) < \epsilon/2$. Also, we can find M such that

$$\mu(E_n) < \epsilon/(2N) \quad \forall\, n \geq M$$

Then, for $n \geq M$,

$$\left|\int_{E_n} f\right| \leq \int_{E_n} |f| = \int_{E_n \cap \{f \leq N\}} + \int_{E_n \cap \{f > N\}} |f|$$

$$\leq N\mu(E_n) + \sum_{k=N}^{\infty}(k-1)\mu(A_k) \leq N\epsilon/2N + \epsilon/2 = \epsilon$$

Now we can prove the statement. Let $a, b \in \mathbb{R}$. Then there exists $\{a_n\}, \{b_n\} \subseteq \mathbb{Q}$ such that

$$a_n \to a \quad b_n \to b$$

Then,

$$\int_a^b f = \int_a^{a_n} f + \int_{a_n}^{b_n} f + \int_{b_n}^{b} f$$

The middle term is zero by assumption and applying the above lemma, the first and third terms go to 0.

(b) INCOMPLETE

8. This is a special case of Winter 15 #10.

Winter 2018

Analysis Prelim Solution - Winter 2018

Yiran Zhu — Clemson - Math

1. Let $\mathcal{C}(0,1)$ be the collection of all continuous functions on $(0,1)$ which is a unit *open* interval in $\mathbb{R}$. Suppose $\{f_n\}_{n=1}^{\infty} \subset \mathcal{C}(0,1)$ converges uniformly to f on $(0,1)$, i.e.,

$$\|f_n - f\|_\infty = \sup_{t\in(0,1)} |f_n(t) - f(t)| \to 0 \text{ as } n \to \infty.$$

Can we say $f \in \mathcal{C}(0,1)$? Prove or disprove.

Proof. Fix $x \in (0,1)$. Observe that $\forall y \in (0,1)$,

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq 2\|f_n - f\| + |f_n(x) - f_n(y)|\end{aligned}$$

For $\epsilon > 0$, there exists $N \geq 1$ such that $\|f_N - f\| < \epsilon/3$. Furthermore, since f_N is continuous, there exists $\delta > 0$ such that $\forall |x - y| < \delta,\ \ |f_N(x) - f_N(y)| < \epsilon/3$.

$$\forall |x-y| < \delta, \ \ |f(x) - f(y)| \leq 2\|f_N - f\| + |f_N(x) - f_N(y)| < \epsilon$$

Therefore f is continuous at x. Since x can be a arbitrary number in $(0,1)$, $f \in \mathcal{C}(0,1)$. □

2. Easy to show. $\|f\|_\infty \leq \|f - f_n\|_\infty + \|f_n\|_\infty \leq \epsilon + M < \infty$.

3. If $x \in Y^\perp$, then $\|x - y\|^2 = \|x\|^2 + \|y\|^2 \geq \|x\|^2$. Conversely, since Y is a closed subspace, $H = Y \oplus Y^\perp$. There exists $x' \in Y$ and $x^\perp \in Y^\perp$ such that $x = x' + x^\perp$. Then

$$\|x^\perp\|^2 = \|x - x'\|^2 = \|x\|^2 = \|x' + x^\perp\|^2 = \|x'\|^2 + \|x^\perp\|^2$$

Therefore, $\|x'\|^2 = 0 \Rightarrow x' = 0 \Rightarrow x = x^\perp \in Y^\perp$.

4. Given a Cauchy sequence $\{T_n\}_n \subseteq \mathcal{B}(X,Y)$ with respect to operator norm, we need to construct an operator T such that $T_n \to T$. Note that, for all $x \in X$, $\{T_n(x)\}_n$ is a Cauchy sequence in Y and thus convergent to some point in Y. We denote this extreme point by T_x. Our mapping is $T : x \to T_x$. Then one can easily check T is linear and also bounded so $T \in \mathcal{B}(X,Y)$. Finally, $T_n \to T$ in operator norm.

5. (a) $\|T\| = 1$. This norm cannot be attained but can be approached by e_n as $n \to \infty$.

$$\|T(x)\| = \left|\sum_{n=1}^{\infty}\left(1 - \frac{1}{n}\right)x_n\right| \leq \sum_{n=1}^{\infty}\left(1 - \frac{1}{n}\right)|x_n| \leq \|x\|_1$$

(b) Suppose there exists x with $\|x\| \leq 1$ such that $|T(x)| = \|T\| = 1 \geq \|x\|$. From above inequality, we essentially have

$$\|x_1\| = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) |x_n| = \|x\|_1 - \sum_{n=1}^{\infty} \frac{|x_n|}{n}$$

Hence

$$\sum_{n=1}^{\infty} \frac{|x_n|}{n} = 0 \Rightarrow \forall n \geq 1,\ x_n = 0 \Rightarrow |T(x)| = 0 \neq 1$$

6. Prove by contradiction. Suppose there is a such measure $(H, \mathcal{M}, \lambda)$ for Hilbert Space H. Let $r = 2$ and $x = 0$,

$$0 < \lambda\left(B_2(0)\right) < \infty$$

Since $\dim H = \infty$, there is an orthonormal sequence $\{x_n\}_n \subseteq B_2(0)$. For all x_n, we claim that $B_{1/2}(x_n) \subseteq B_2(0)$. Indeed,

$$\forall y \in B_{1/2}(x_n), \|y - 0\| \leq \|y - x_n\| + \|x_n\| \leq 1/2 + 1 = 3/2 < 2 \Rightarrow y \in B_2(0)$$

Observe that $\|x_n - x_m\|^2 = \langle x_n - x_m, x_n - x_m \rangle = \|x_n\|^2 + \|x_m\|^2 = 2$ for all $n \neq m$. Moreover, if $n \neq m$, then we can check that $B_{1/2}(x_n) \cap B_{1/2}(x_m) = \emptyset$ as follows:

$$\forall y \in B_{1/2}(x_n), \|y_n - x_m\| \geq \|x_n - x_m\| - \|y - x_n\| = \sqrt{2} - \frac{1}{2} > \frac{1}{2} \Rightarrow y \notin B_{1/2}(x_m)$$

By assumption that measure of balls is invariant under translation, we have

$$\forall n \geq 1,\ \lambda(B_{1/2}(x_n)) = \lambda(B_{1/2}(x_1))$$

Note that $\cup_{n=1}^{\infty} B_{1/2}(x_n)$, the union of disjoint balls, is a subset of $B_2(0)$.

$$\lambda(B_2(0)) \geq \lambda\left(\cup_{n=1}^{\infty} B_{1/2}(x_n)\right) = \sum_{n=1}^{\infty} \lambda(B_{1/2}(x_n)) = \sum_{n=1}^{\infty} \lambda(B_{1/2}(x_1))$$

Therefore, $\lambda(B_{1/2}(x_1)) = 0$. However, we assume that measure of a ball is greater than 0.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^1(X, \mathcal{M}, \mu)$. Then $\{x : f(x) \neq 0\}$ is σ-finite with respect to μ.

Proof. Let $E_n = \{x : |f(x)| \geq 1/n\}$ and then $\{x : f(x) \neq 0\} = \cup_{n=1}^{\infty} E_n$. From Chebyshev's Inequality,

$$\frac{\mu(E_n)}{n} \leq \int_X |f| d\mu < \infty \ \Rightarrow \ \mu(E_n) < \infty$$

Therefore, $\{x : f(x) \neq 0\}$ is σ-finite. □

8. (a) Observe that

$$\bigcup_{n=1}^{\infty} E_n = \{x \in I : f(x) > 0\} \Rightarrow \lambda(\{x \in I : f(x) > 0\}) \leq \sum_{n=1}^{\infty} \lambda(E_n)$$

So $\lambda(\{x \in I : f(x) > 0\}) > 0$ implies that there $\lambda(E_n) > 0$ for some n.

(b) This is also obvious. We show that $\lambda(E_n) = 0$ for all $n \geq 1$. Then the inequality derived in part (a) asserts that $\lambda(\{x \in I : f(x) > 0\}) = 0$. Suppose $\lambda(E_n) > 0$ for some n, then we pick a finite set $\{x_1, \ldots, x_m\} \subseteq E_n$ where $m = 2Mn$.

$$\sum_{n=1}^{m} f(x_n) \geq \sum_{n=1}^{m} \frac{1}{n} = 2M > M$$

However, by assumption, $\sum_{n=1}^{m} f(x_n) \leq M$. Therefore, $\lambda(E_n) = 0$ holds for all $n \geq 1$.

9. This is a direct application of *Monotone Convergence Theorem.* Let $h_m(x) = \sum_{n=1}^{m} f(x+n)$.

$$0 \leq h_1(x) \leq h_2(x) \leq \ldots \leq h_m(x) \leq h_{m+1}(x) \leq \ldots; \quad \lim_{m\to\infty} h_m(x) = \sum_{n=1}^{\infty} f(x+n) = g(x)$$

Monotone Convergence Theorem says

$$\lim_{m\to\infty} \int_{\mathbb{R}} h_m(x) d\mu = \int_{\mathbb{R}} \lim_{m\to\infty} h_m(x) d\mu = \int_{\mathbb{R}} g(x) d\mu$$

Let's compute the left hand side,

$$\int_{\mathbb{R}} h_m(x) d\mu = \int_{\mathbb{R}} \sum_{n=1}^{m} f(x+n) d\mu = \sum_{n=1}^{m} \int_{\mathbb{R}} f(x+n) d\mu = m \int_{\mathbb{R}} f(x) d\mu$$

Recall that Lebesgue measure is invariant under translation. Combine above two equations,

$$\lim_{m\to\infty} m \int_{\mathbb{R}} f(x) d\mu = \int_{\mathbb{R}} g(x) d\mu < \infty \Rightarrow \int_{\mathbb{R}} f(x) d\mu = 0$$

Since $f(x)$ is nonnegative, $\int_{\mathbb{R}} f d\mu = 0$ is equivalent to $f = 0$ a.e.

10. (a) This is a immediate result of Cauchy-Schwartz Inequality.

$$\left(\int_B f d\mu\right)^2 = \left(\int_X f\chi_B d\mu\right)^2 \leq \left(\int_X f^2 d\mu\right)\left(\int_X \chi_B^2 d\mu\right) = \mu_B \int_X f^2 d\mu$$

(b) Let $f = \sum_{k=1}^{n} \chi_{A_k}$ where χ_{A_k} is a characteristic function of measurable set A_k. Furthermore, let $B = \cup_{k=1}^{n} A_k$. This inequality holds directly from part (a).

Summer 2018

Analysis Prelim Solution - 2018 Summer

Yiran Zhu — Clemson - Math

1. Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence with $a_n \to 0$, $n \to \infty$. Prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n = 0$$

Proof. For $\epsilon > 0$, there exists a $M \geq 1$ such that $\forall n \geq M,\ \ |a_n| \leq \epsilon/2$. For $N \geq M+1$, we have

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} a_n\right| &= \left|\frac{1}{N}\sum_{n=1}^{M} a_n + \frac{1}{N}\sum_{n=M+1}^{N} a_n\right| \leq \frac{1}{N}\left|\sum_{n=1}^{M} a_n\right| + \frac{1}{N}\sum_{n=M+1}^{N} |a_n| \\ &\leq \frac{1}{N}\left|\sum_{n=1}^{M} a_n\right| + \left(\frac{N-M}{N}\right)\frac{\epsilon}{2} \leq \frac{1}{N}\left|\sum_{n=1}^{M} a_n\right| + \frac{\epsilon}{2}\end{aligned}$$

Let $\widehat{N}$ be an integer greater than $2\left|\sum_{n=1}^{M} a_n\right|/\epsilon$ and $\widehat{M} = \max\{\widehat{N}, M+1\}$

$$\forall N \geq \widehat{M}, \quad \left|\frac{1}{N}\sum_{n=1}^{N} a_n\right| < \epsilon$$

Equivalently, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n = 0$. □

2. Let X be a normed linear space and $\emptyset \neq Y \subset X$ be a subset with the property that $X\backslash Y$ is a linear subspace. Show that Y is dense in X.

Proof. Suppose Y is not dense in X. Then there exists a point $z \in X\backslash Y$ and a number $r > 0$ such that $B(z,r) \cap Y = \emptyset$. Equivalently, $B(z,r) \subseteq X\backslash Y$. Then we will show this implies $Y = \emptyset$. Pick $x \in X$ and let $d = \|x - z\|$. Then

$$r > \left\|\frac{r(x-z)}{2d}\right\| = \left\|\frac{rx-(r-2d)z}{2d} - z\right\| \ \Rightarrow\ a := \frac{rx-(r-2d)z}{2d} \in B(z,r) \subseteq X\backslash Y$$

Since $X\backslash Y$ is a subspace, we have $x = (2da + (r-2d)z)/r \in X\backslash Y$. Note that x is arbitrarily picked from X. Therefore, $X \subseteq X\backslash Y \subseteq X \ \Rightarrow\ Y = \emptyset$. By contradiction, Y is dense in X. □

3. Define $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$ by $d(x,y) = |f(x) - f(y)|$ where f is defined as

$$f(x) = \frac{x}{1+|x|}\ ,\forall x \in \mathbb{R}$$

Show that d is a metric on $\mathbb{R}$ and determine if $(\mathbb{R}, d)$ is complete.

Proof. (i) Positive-definite: $d(x, y) = |f(x) - f(y)| \geq 0$ and $d(x, y) = 0 \Leftrightarrow f(x) = f(y)$ Note that

$$f(x) = f(y) \Leftrightarrow \frac{x}{1 + |x|} = \frac{y}{1 + |y|} \Leftrightarrow x(1 + |y|) = y(1 + |x|)$$

Suppose $x < 0$, then $y < 0$ and $x(1+|y|) = y(1+|x|) \Rightarrow x = y$. Similarly, if $x \geq 0$, then $y \geq 0$ and $x(1 + |y|) = y(1 + |x|) \Rightarrow x = y$. In a word, $d(x, y) = 0 \Leftrightarrow x = y$.

(ii) Symmetric: $d(x, y) = |f(x) - f(y)| = |f(y) - f(x)| = d(y, x)$

(iii) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ follows from

$$|f(x) - f(z)| = |f(x) - f(y) + f(y) - f(z)| \leq |f(x) - f(y)| + |f(y) - f(z)|$$

So d is a metric. However, $(\mathbb{R}, d)$ is not complete. Consider sequence $\{x_n\}_n$ with $x_n = n$.

$$d(x_n, x_{n+m}) = \left| \frac{m}{(1 + n)(1 + n + m)} \right| \leq \frac{1}{n + 1}, \quad \forall n \geq 1, \ \forall m \geq 0$$

Therefore, $\{x_n\}_n$ is Cauchy. It's obvious that $\{x_n\}_n$ does not converge in $\mathbb{R}$. □

4. Let H be a Hilbert space and Y_1, Y_2 be two closed linear subspaces in H. Denote P_1 and P_2 as the orthogonal projections onto Y_1 and Y_2, respectively. Show that $\|P_1 - P_2\| \leq 1$.

Proof. Observe that $(2P - I)^2 = 4P^2 + I - 4P = I$ holds for all projection P. In particular, if P is orthogonal, then, for all $h \in H$,

$$\|(2P - I)h\|^2 = \langle (2P - I)h, (2P - I)h \rangle = \langle h, (2P - I)^2 h \rangle = \langle h, h \rangle = \|h\|^2$$

Therefore, $\|2P - I\| = 1 \Rightarrow \|P - \frac{1}{2}I\| = \frac{1}{2}$.

$$\|P_1 - P_2\| \leq \left\| P_1 - \frac{1}{2}I \right\| + \left\| P_2 - \frac{1}{2}I \right\| \leq 1$$

□

5. Assume $C[0, 1]$ is equipped with the supremum norm and let $T_n : C[0, 1] \to C[0, 1]$ be defined by

$$T_n(f) = f\left(x^{1+\frac{1}{n}}\right), \quad \forall n \in \mathbb{N}$$

(a) Show that $T_n(f) \to f, n \to \infty, \forall f \in C[0, 1]$

Proof. Fix $f \in C[0, 1]$. Since $[0, 1]$ is compact, f is also uniformly continuous on $[0, 1]$, i.e. for $\epsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, y \in [0, 1] \ s.t \ |x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon$$

Let's give an estimation for $g_n(x) := \left|x^{1+\frac{1}{n}} - x\right| = x\left(1 - x^{\frac{1}{n}}\right)$. Obviously, $g_n(x)$ is continuous on $[0,1]$ and $g_n(0) = g_n(1) = 0$. To find the maximum value of $g_n(x)$. we let

$$g_n'(x) = \left(1 + \frac{1}{n}\right) x^{\frac{1}{n}} - 1 = 0 \;\Rightarrow\; \sup_{x\in[0,1]} g_n(x) = \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \leq \frac{1}{n+1}$$

Pick $N \in \mathbb{Z}^+$ such that $\frac{1}{N+1} < \delta$, then

$$\forall n \geq N,\ \forall x \in [0,1] \quad \left|x^{1+\frac{1}{n}} - x\right| < \frac{1}{N+1} < \delta \;\Rightarrow\; \|T_n(f) - f\|_\infty < \epsilon$$

Therefore, $T_n(f) \to f$ as $n \to \infty$. □

(b) For each $n \in N$, find $\|T_n - I\|$.

Proof. For $f \in C[0,1]$,

$$\|(T_n - I)f\| = \|T_n(f) - f\|_\infty = \sup_{x\in[0,1]} |f(x^{1+\frac{1}{n}}) - f(x)| \leq 2\|f\|_\infty$$

So $\|T_n - I\| \leq 2$. Let $x_0 = \frac{1}{2}$ and $x_1 = \left(\frac{1}{2}\right)^{1+\frac{1}{n}} \in (0, x_0)$. Construct a function f as follows

$$f(x) = \begin{cases} -1 & x \in [0, x_1) \\ -1 + \dfrac{2(x - x_1)}{x_0 - x_1} & x \in [x_1, x_0] \\ 1 & x \in (x_0, 1] \end{cases}$$

Note that $f(x_0) = 1$ and $f(x_1) = -1$. So f is continuous and $\|f\|_\infty = 1$.

$$|(T_n(f) - f)(x_0)| = |f(x_1) - f(x_0)| = 2 = 2\|f\|_\infty$$

As shown before, $\|T_n(f) - f\|_\infty \leq 2\|f\|_\infty$. Thus

$$\|T_n(f) - f\|_\infty = 2\|f\|_\infty \;\Rightarrow\; \|T_n - I\| = 2$$

□

6. Assume that λ is the Lebesgue measure on the real line and f a Lebesgue integrable function on the real line. Show that

$$F(x) := \int_{-\infty}^{x} f d\lambda$$

is uniformly continuous.

Proof. Let $A_n = \{x \in X \mid |f(x)| \geq n\}$. Then, Dominated Convergence Theorem gives

$$\lim_{n\to\infty} \int_{A_n} |f| d\lambda = \lim_{n\to\infty} \int_{-\infty}^{\infty} |f| \chi_{A_n} d\lambda = 0$$

For $\epsilon > 0$, there exists $N \geq 1$ such that

$$\int_{A_N} |f| d\lambda < \frac{\epsilon}{2}$$

Then

$$\forall x, y \in \mathbb{R}, \quad |F(x) - F(y)| = \left| \int_{-\infty}^{x} f d\lambda - \int_{-\infty}^{y} f d\lambda \right| = \left| \int_{y}^{x} f d\lambda \right| \leq \int_{y}^{x} |f| \, d\lambda$$

Observe that, if $|x - y| < \frac{\epsilon}{2N}$, then

$$\int_{y}^{x} |f| \, d\lambda = \int_{[x,y]\cap A_N} |f| \, d\lambda + \int_{[x,y]\setminus A_N} |f| d\lambda \leq \int_{A_N} |f| d\lambda + N|x - y| < \epsilon$$

□

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{A_n\}_n$ be a sequence of sets in $\mathcal{M}$. Recall that $\limsup_{n\to\infty} A_n := \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$.

   (a) Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup_{n\to\infty} A_n) = 0$

   Proof. Observe that

   $$\mu\left(\limsup_{n\to\infty} A_n\right) \leq \mu\left(\cup_{k=n}^{\infty} A_k\right) \leq \sum_{k=n}^{\infty} \mu(A_k), \quad \forall n \geq 1$$

   Since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, for $\epsilon > 0$, there exists $N \geq 1$ such that

   $$\sum_{k=N}^{\infty} \mu(A_k) < \epsilon \;\Rightarrow\; \mu\left(\limsup_{n\to\infty} A_n\right) \leq \epsilon$$

   Let $\epsilon \to 0$, we derive $\mu\left(\limsup_{n\to\infty} A_n\right) = 0$. □

   (b) Is the converse true? If yes, prove it. If no, give a counter-example.

   Proof. Converse is not true. Consider $A_n = [0, 1/n]$. Then $\cup_{k=n}^{\infty} A_k = A_n = [0, 1/n]$.

   $$\limsup_{n\to\infty} A_n = \lim_{N\to\infty} \cap_{n=1}^{N} \cup_{k=n}^{\infty} A_k = \lim_{N\to\infty} \left[0, \frac{1}{N}\right] = \{0\}$$

   Therefore,

   $$\mu\left(\limsup_{n\to\infty} A_n\right) = 0$$

However, $\mu(A_n) = \frac{1}{n}$ for all $n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

□

8. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Prove that a monotone increasing sequence of measurable functions $f_n : X \to \mathbb{R}$ converges in measure if and only if it converges pointwise a.e..

Proof. (i) Suppose f_n converges to f poinwise a.e.: By Egorov theorem, for each $\epsilon > 0$, there exists a measurable set E with measure $\mu(E) < \epsilon$ such that f_n converges uniformly to f on $X \backslash E$. In other words, for each $\delta > 0$, there exists $N \geq 1$ such that

$$\forall n \geq N, \quad |f_n(x) - f(x)| < \delta, \ \forall x \in X \backslash E$$

Consequently,

$$\forall n \geq N, \quad A_n := \{x \in X \mid |f_n(x) - f(x)| \geq \delta\} \subseteq E \ \Rightarrow \ \mu(A_n) < \epsilon$$

Therefore, f_n converges to f in measure.

(ii) Suppose f_n converges to f in measure: There exists a subsequence $\{f_{n_k}\}_k$ converges to f pointwise a.e.. Let E be the zero-measure set that $\{f_{n_k}\}_k$ does not converge to f. Then fix $x \in X \backslash E$, for each $\epsilon > 0$, there exists $N \geq 1$ such that

$$\forall k \geq N, \quad |f_{n_k}(x) - f(x)| < \epsilon$$

Since $\{f_n\}_n$ is monotone increasing, we have

$$\forall n \geq n_N, \quad |f_n(x) - f(x)| = f(x) - f_n(x) \leq f(x) - f_{n_N}(x) = |f_{n_k}(x) - f(x)| < \epsilon$$

Note that $\{f_{n_k}\}_k$ is also monotone increasing.

□

9. Suppose that g is a non-negative Borel measurable function on $\mathbb{R}$ with $\int_{\mathbb{R}} g d\lambda = 1$ where λ denotes Lebesgue measure on $\mathbb{R}$. For $k \in \mathbb{N}$ set $g_k(x) = kg(kx)$. Let f be a bounded continuous function. Prove that

$$\lim_{k \to \infty} \int_{\mathbb{R}} g_k f d\lambda = f(0)$$

Proof. Suppose $\sup_{x \in \mathbb{R}} |f(x)| = M$ and define $h_k(x) = g(x) f(x/k)$. Observe that

$$\int_{\mathbb{R}} g_k f d\lambda = \int_{\mathbb{R}} kg(kx) f(x) d\lambda = \int_{\mathbb{R}} g(x) f(x/k) d\lambda = \int_{\mathbb{R}} h_k d\lambda$$

In order to apply Dominated Convergence Theorem, we need to show h_k is uniformly bounded by a integrable function. Indeed,

$$\forall k \geq 1, \quad |h_k(x)| = |g(x)f(x/k)| \leq Mg(x) \text{ and } \int_{\mathbb{R}} Mgd\lambda = M < \infty$$

By DCT,

$$\lim_{k\to\infty} \int_{\mathbb{R}} h_k d\lambda = \int_{\mathbb{R}} \lim_{k\to\infty} h_k d\lambda = \int_{\mathbb{R}} f(0)gd\lambda = f(0)$$

□

10. Let λ be the Lebesgue measure on $(0,1)$. Suppose the $f_n : (0,1) \to [0,\infty)$ is a sequence of Borel measurable functions such that $\int_{(0,1)} f_n d\lambda = 1$ for all $n \geq 1$ and $\lim_{n\to\infty} f_n(x) = x$ for all $x \in (0,1)$.

    (a) Give an example of such a sequence.

    Proof.

$$f_n(x) = \begin{cases} (n+1)(1-nx) & x \in \left(0, \frac{1}{n}\right) \\ \frac{n}{n-1}\left(x - \frac{1}{n}\right) & x \in \left[\frac{1}{n}, 1\right) \end{cases}$$

    Then

$$\int_{(0,1)} f_n d\lambda = \int_{\left(0,\frac{1}{n}\right)} (n+1)(1-nx)\, d\lambda + \int_{\left[\frac{1}{n},1\right)} \frac{n}{n-1}\left(x - \frac{1}{n}\right) d\lambda = \frac{n+1}{2n} + \frac{n-1}{2n} = 1$$

    Fix $x \in (0,1)$, there exists $N \geq 1$ such that $x > \frac{1}{N}$, then

$$\forall n \geq N, \quad f_n(x) - x = \frac{n}{n-1}\left(x - \frac{1}{n}\right) - x = \frac{x-1}{n-1}$$

    So $f_n(x) \to x$ as $n \to \infty$. □

    (b) Show that one can find $n \geq 1$ and $x \in (0,1)$ such that $f_n(x)\sqrt{x} \geq 2018$

    Proof. Prove by contradiction. Suppose not, then

$$\forall n \geq 1, \quad f_n(x) \leq \frac{2018}{\sqrt{x}}$$

    Note that

$$\int_{(0,1)} \frac{2018}{\sqrt{x}} d\lambda = 4036 < \infty$$

    By Dominated Convergence Theorem,

$$1 = \lim_{n\to\infty} \int_{(0,1)} f_n d\lambda = \int_{(0,1)} \lim_{n\to\infty} f_n d\lambda = \int_{(0,1)} x d\lambda = \frac{1}{2}$$

    Above equality cannot be true. □