

Clemson Analysis Prelim Solutions

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Contents

Winter 2010	2
Summer 2010	4
Winter 2012	6
Summer 2012	7
Winter 2013	8
Summer 2013	9
Winter 2014	14
Summer 2014	15
Winter 2015	16
Summer 2015	20
Winter 2016	25
Summer 2016	29
Winter 2017	33
Summer 2017	38
Winter 2018	40
Summer 2018	43

Winter 2010

1. (a) *Proof.* If E is bounded, E is pre-compact since \mathbb{R} is finite (one) dimensional. If $f(E)$ is unbounded, then there exists $\{x_n\} \subseteq E$ such that $f(x_n) \rightarrow \infty$. Since E is precompact $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ with limit $x \in \mathbb{R}$. Then, since f is continuous,

$$f(x) = \lim_{k \rightarrow \infty} f(x_{n_k}) = \infty$$

However, since f maps \mathbb{R} to \mathbb{R} , $f(x)$ cannot be ∞ . □

- (b) *Proof.* Since f is uniformly continuous, there exists $\delta > 0$ such that whenever $|x - y| < \delta$,

$$|f(x) - f(y)| < 1$$

Since E bounded, it can be covered by finitely many balls of radius δ , say $\{B(x_i, \delta)\}_{i=1}^N$. Then,

$$f(E) = \cup_{i=1}^N f(B(x_i, \delta))$$

Fix i , for any $f(y) \in f(B(x_i, \delta))$,

$$|f(y) - f(x_i)| \leq 1$$

So $f(B(x_i, \delta))$ is bounded. Then, a finite union of bounded sets is also bounded. □

Counterexample: $E = (0, 1)$ and $f(x) = 1/x$. $f(E) = (1, \infty)$.

3. (a) *Proof.* Recall the Bessel inequality for any orthonormal set $\{e_n\}$ in an inner product space, X . For and $f \in X$,

$$\sum |\langle f, e_n \rangle|^2 \leq \|f\|^2$$

In particular, $\langle f, e_n \rangle \rightarrow 0$ as $n \rightarrow \infty$ for any $f \in X$. Now, since

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{2\pi}} \right\}$$

form an orthonormal set in $C[-\pi, \pi]$, we have

$$\int_{-\pi}^{\pi} \sin(2nx) f(x) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any $f \in C[-\pi, \pi]$. Then,

$$\int_{-\pi}^{\pi} \sin^2(nx) f(x) dx = \frac{1}{2} \int_{-\pi}^{\pi} f(x) dx - \frac{1}{2} \int_{-\pi}^{\pi} \sin(2nx) f(x) dx \rightarrow \frac{1}{2} \int_{-\pi}^{\pi} f(x) dx$$

□

(b) *Proof.* For any $f \in C[-\pi, \pi]$, $n \in \mathbb{N}$,

$$\begin{aligned} \left| \int_{-\pi}^{\pi} \frac{x^n}{\pi^n} f(x) dx \right|^2 &\leq \int_{-\pi}^{\pi} \frac{x^{2n}}{\pi^{2n}} dx \int_{-\pi}^{\pi} |f(x)|^2 dx \\ &= \frac{\pi^{2n+1} - (-\pi)^{2n+1}}{(2n+1)\pi^{2n}} \|f\|_{L^2}^2 \\ &= \frac{2\pi}{2n+1} \|f\|_{L^2}^2 \end{aligned}$$

which goes to 0 as $n \rightarrow \infty$. □

9. (a) There exists $\varepsilon_n \searrow 0$ such that

$$\mu\{|f - g| \geq \varepsilon_n\} \leq \varepsilon_n$$

Then,

$$\begin{aligned} \mu\{f \neq g\} &= \mu\{|f - g| > 0\} = \mu\left(\bigcup_n \{|f - g| \geq \varepsilon_n\}\right) \\ &= \lim_{n \rightarrow \infty} \mu\{|f - g| \leq \varepsilon_n\} \leq \lim_{n \rightarrow \infty} \varepsilon_n = 0 \end{aligned}$$

(b) We only need to show the triangle inequality. Let $t, s > 0$, f, g, h measurable functions.

If $|f - h| \leq t$ and $|g - h| \leq s$, then

$$|f - g| \leq |f - h| + |g - h| \leq t + s$$

Thus, $\{|f - h| \leq t\} \cap \{|g - h| \leq s\} \subset \{|f - g| \leq t + s\}$. Then, taking complements, we have

$$\{|f - h| > t\} \cup \{|g - h| > s\} \supset \{|f - g| > t + s\}$$

Therefore, $\mu\{|f - g| > t + s\} \leq \mu\{|f - h| > t\} + \mu\{|g - h| > s\}$. Let $\delta > 0$. There exists $\varepsilon_1, \varepsilon_2$ such that

$$\mu\{|f - h| > \varepsilon_1\} < \varepsilon_1 \text{ and } \varepsilon_1 < \rho(f, h) + \delta/2$$

and similarly for ε_2 and $|g - h|$. Therefore,

$$\mu\{|f - g| > \varepsilon_1 + \varepsilon_2\} \leq \mu\{|f - h| > \varepsilon_1\} + \mu\{|g - h| > \varepsilon_2\} < \varepsilon_1 + \varepsilon_2$$

So,

$$\begin{aligned} \rho(f, g) &= \inf\{\varepsilon : \mu\{|f - g| > \varepsilon\} < \varepsilon\} \\ &\leq \varepsilon_1 + \varepsilon_2 \\ &\leq \rho(f, h) + \rho(g, h) + \delta \end{aligned}$$

for any $\delta > 0$. This proves the Triangle Inequality.

Summer 2010

3. (a)

$$\|Tf\|_\infty = \sup_{x \in [0,1]} |x^2 f(x)| \leq \sup_{x \in [0,1]} |f(x)| = \|f\|_\infty$$

(b) For $f \equiv 1$, $\|Tf\| = 1$ and $\|f\| = 1$.

(c) By triangle inequality, $\|(I+T)f\|_\infty \leq \|f\|_\infty + \|Tf\|_\infty \leq 2\|f\|$. So, we only need to show $\|I+T\| = 2$. Again, this follows from taking $f \equiv 1$.

$$\|(I+T)f\| = \sup_{x \in [0,1]} |1+x^2| = 2$$

4. (a) We will actually show more, namely that

$$\|x\|_q \leq \|x\|_p \quad \text{for } 1 \leq p < q < \infty$$

Proof. Let $x \in \ell^p$. Define

$$y = \frac{x}{\|x\|_p}$$

Then, $|y_i| \leq 1$ for every $i \in \mathbb{N}$. This implies $|y_i|^q \leq |y_i|^p$ for all i . Therefore,

$$\|y\|_q^q = \sum |y_i|^q \leq \sum |y_i|^p = \sum \frac{|x_i|^p}{\|x\|_p^p} = \frac{\sum |x_i|^p}{\sum |x_i|^p} = 1$$

so $\|y\|_q \leq 1$. But this implies

$$\frac{\|x\|_q}{\|x\|_p} = \left\| \frac{x}{\|x\|_p} \right\|_q = \|y\|_q \leq 1$$

□

5. (a) Let $f \in L^q$, $1 \leq p < q < \infty$. we will apply holder inequality with exponent $q/p > 1$.

$$\int |f|^p \leq \left(\int |f|^q \right)^{p/q} \left(\int 1 \right)^{1-p/q} = \left(\int |f|^q \right)^{p/q} (\mu(X))^{1-p/q} < \infty$$

(b) Take

$$f(x) = \begin{cases} 0 & x = 0 \\ x^{-1/2} & 0 < |x| \leq 1 \\ 0 & |x| < 1 \end{cases}$$

$$g(x) = \begin{cases} x^{-1} & |x| \geq 1 \\ 1 & |x| < 1 \end{cases}$$

6. There are many options for f_n :

$$n^2\chi_{[0,1/n]} \quad n\chi_{[n,n+1]} \quad \chi_{[n,2n]}$$

Then, just take

$$g_n = \frac{1}{n}f_n$$

and

$$h_n = (-1)^n g_n$$

7.

Winter 2012

Summer 2012

4. (a) We will show K is totally bounded. Let $\epsilon > 0$. For $x, y \in K$,

$$d_S(x, y) = \sum_i \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)} \leq \sum_i \frac{2}{2^i} < \infty$$

So, there exists N such that

$$\sum_{i=N+1}^{\infty} \frac{1}{2^{i-1}} < \epsilon/2$$

Consider the set $M = \{x \in K : x_i = 0, i > N\} \subseteq K$. M is compact since it is finite dimensional and bounded. So, there is an $\epsilon/2$ net for M which will be an ϵ net for K .

- (b) False. Consider the sequence $\{e_n\} \subseteq K$, which is entirely zero except the n -th entry. For $n \neq m$,

$$d_{\infty}(e_n, e_m) = 1$$

so this sequence cannot have a convergent subsequence.

5. (a) Define $g_n = \sum_{k=1}^n f_k$. Since f_k are non-negative, $\{g_n\}$ is monotone. Moreover,

$$g_n \nearrow \sum_{k=1}^{\infty} f_k$$

Then, by Monotone Convergence Theorem,

$$\sum_{k=1}^{\infty} \int f_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int f_k = \lim_{n \rightarrow \infty} \int g_n = \int \sum_{k=1}^{\infty} f_k$$

- (b) By part (a),

$$\infty > \sum \int |f_k - f| = \int \sum |f_k - f|$$

Since $\sum |f_k - f|$ is integrable, it is finite almost everywhere. Let $E \subseteq \mathbb{R}$ be the set of measure zero where it may not be finite. Let $x \notin E$. Let $\epsilon > 0$. There exists N such that

$$|f_n(x) - f(x)| \leq \sum_{k=N}^{\infty} |f_k(x) - f(x)| < \epsilon$$

for all $n \geq N$.

7. See Winter 15 #9

Winter 2013

8. See Summer 13 #5

Summer 2013

3. (a) B is complete. Let $\{f_n\} \subseteq C[0, 1]$ be a Cauchy sequence in ρ_∞ . Since $(C[0, 1], \rho_\infty)$ is a complete metric space, there exists $f \in C[0, 1]$ such that $f_n \rightarrow f$ in ρ_∞ . Now, we claim that $f \in B$. For any $\epsilon > 0$ there exists N such that

$$\rho(f, f_n) < \epsilon \quad \forall n \geq N$$

Then,

$$\sup_{0 \leq t \leq 1} |f(t)| = \rho(f, 0) \leq \rho(f, f_n) + \rho(f_n, 0) < \epsilon + 1$$

but $\epsilon > 0$ was arbitrary so

$$\sup_{0 \leq t \leq 1} |f(t)| \leq 1$$

- (b) Consider the spike functions, $\{f_n\}$. For $n \neq m$,

$$\rho(f_n, f_m) = 1$$

so there cannot be a convergent subsequence.

4. (a) Let $f \in L^2(\mu)$. Then, using the Cauchy-Schwarz inequality, we compute

$$\begin{aligned} \|Af\|_{L^2(\mu)}^2 &= \int_X \left(\int_X K(x, y) f(y) d\mu(y) \right)^2 d\mu(x) \\ &\leq \int_X \left(\int_X K(x, y)^2 d\mu(y) \right) \left(\int_X f(y)^2 d\mu(y) \right) d\mu(x) \\ &= \left(\int_X f(y)^2 d\mu(y) \right) \int_X \left(\int_X K(x, y)^2 d\mu(y) \right) d\mu(x) = \|f\|_{L^2(\mu)}^2 \|K\|_{L^2(\mu \times \mu)}^2 \end{aligned}$$

Therefore $\|A\| \leq \|K\|_{L^2(\mu \times \mu)}$.

- (b) First we note that the correspondence $K \mapsto A$ is linear due to the linearity of the integral. So, it suffices to prove the following: Let $K \in L^2(\mu \times \mu)$ such that for any $f \in L^2(\mu)$,

$$\int_X K(x, y) f(y) d\mu(y) = 0$$

for almost every $x \in X$ (w.r.t μ). Then, $K = 0$ a.e. To prove this, suppose there exists $E \subseteq X \times X$ such that E has positive $\mu \times \mu$ measure in the sense that

$$(\mu \times \mu)(E) := \int_X \int_X \mathbf{1}_E(x, y) d\mu(x) d\mu(y) > 0$$

So we define the measure $\mu \times \mu$ on the cylinder $X \times X$ in this way.

5. ($a \Rightarrow b$) Let $x = \hat{m} + e$ where \hat{m} is the closest point to x in M . Let $y \in M$ non-zero. For any $t \in \mathbb{C}$, $ty \in M$ so

$$\|x - \hat{m}\|^2 \leq \|e - ty\|^2 = \|e\|^2 - 2\operatorname{Re}\langle e, ty \rangle + \|ty\|^2$$

which implies

$$\operatorname{Re} \bar{t} \langle e, y \rangle \leq |t|^2 \|y\|^2$$

Take $t = \overline{\langle e, y \rangle} \|y\|^{-2}$. Then we have

$$\frac{|\langle e, y \rangle|^2}{\|y\|^2} \leq \frac{|\langle e, y \rangle|^2}{2\|y\|^2}$$

therefore $\langle e, y \rangle = 0$.

($b \Rightarrow a$) Let $x = \hat{m} + e$ for $\hat{m} \in M$ and $e \in M^\perp$. For any $y \in M$,

$$\|x - y\|^2 = \|e - (y - \hat{m})\|^2 = \|e\|^2 - 2\operatorname{Re}\langle e, y - \hat{m} \rangle + \|y - \hat{m}\|^2 = \|e\|^2 + \|y\|^2 \geq \|e\|^2 = \|x - \hat{m}\|^2$$

Suppose \tilde{m} is another closest point. Set $d = d(x, M)$. By the parallelogram identity,

$$\|\tilde{m} - \hat{m}\|^2 = \|x - \tilde{m} - (x - \hat{m})\|^2 = 2d^2 + 2d^2 - \|2x - 2(\tilde{m} + \hat{m})\|^2 \leq 4d^2 - 4d^2 = 0$$

6.

$$f_n = n \mathbf{1}_{[0, 1/n]}$$

7. (a) We will show that for $h \geq 0$ measurable, $\int h = 0 \implies h = 0$ a.e. Indeed, consider $A_n = \{n^{-1} > h \geq (n+1)^{-1}\}$ for each $n \in \mathbb{N}$ and $A_0 = \{h \geq 1\}$. Then, for any $n \in \mathbb{N} \cup \{0\}$,

$$(n+1)^{-1} \mu(A_n) \leq \int_{A_n} h \, d\mu \leq \int_{\mathbb{R}} h \, d\mu = 0$$

Therefore $\mu(A_n) = 0$. So, $\{h \neq 0\} = A = \cup A_n$ which has measure zero.

Now, we apply this to the problem by taking $h = g - f$. Then, $h \geq 0$ and

$$\int h = \int g - f = \int g - \int f = 0$$

Then by the above lemma, $h = 0$ so $f = g$ a.e.

- (b) If f and g are continuous, then being equal almost everywhere will imply they are equal everywhere. Indeed, suppose there exists $x_0 \in \mathbb{R}$ such that $f(x_0) < g(x_0)$. Then, $f - g$ is continuous around x_0 so there exists $\epsilon > 0$ such that

$$f(x) < g(x) \quad \text{for } |x - x_0| < \epsilon$$

However, $\lambda(\{|x - x_0| < \epsilon\}) = \epsilon$ so $f \neq g$ on a set of measure ϵ which contradicts $f = g$ a.e.

(c) INCOMPLETE

8. (a) First, \mathcal{M} is clearly a linear space since linear combinations of finite signed measures are still finite signed measures. Now we show that total variation is a norm on \mathcal{M} . If μ has total variation 0, this means

$$\mu_+(X) = \mu_-(X) = 0$$

so X is a null set of both μ_+ and μ_- . Thus for every $E \subseteq X$, $\mu(E) = \mu_+(E) - \mu_-(E) = 0 - 0 = 0$. Thus μ is the zero measure. By definition, $|\alpha\mu| = \alpha\mu_+ + \alpha\mu_- = \alpha|\mu|$. Finally, to check the triangle inequality, let $\mu, \lambda \in \mathcal{M}$. Let $A \cup B = X$ be a Hahn decomposition of X with respect to the signed measure $(\mu + \lambda)$. Then,

$$(\mu + \lambda)_+(X) = \mu(A) + \lambda(A) \leq \mu_+(A) + \lambda_+(A) \leq \mu_+(X) + \lambda_+(X)$$

and

$$(\mu + \lambda)_-(X) = -\mu(B) - \lambda(B) \leq \mu_-(B) + \lambda_-(B) \leq \mu_-(X) + \lambda_-(X)$$

Therefore

$$\begin{aligned} |\mu + \lambda|(X) &= (\mu + \lambda)_+(X) + (\mu + \lambda)_-(X) \leq \mu_+(X) + \lambda_+(X) + \mu_-(X) + \lambda_-(X) \\ &= |\mu|(X) + |\lambda|(X) \end{aligned}$$

- (b) Let μ be a σ -finite measure. Clearly $\mathcal{L}_\nu = \{\mu \in \mathcal{M} : \mu \ll \nu\}$ is a linear subspace since if $\lambda, \mu \ll \nu$, and $\nu(E) = 0$, then

$$\alpha\lambda(E) + \beta\mu(E) = 0$$

for any scalars α, β . We note the crucial property of this subspace. If $\mu \ll \nu$, then the null sets of ν are also null sets of μ_+ and μ_- . Indeed, let $E \subset X$ such that $\nu(E) = 0$. Let $A \cup B$ be a Hahn decomposition for μ . Then,

$$\nu(A \cap E) \leq \nu(E) = 0 \quad \nu(B \cap E) \leq \nu(E) = 0$$

so $\nu(A \cap E) = \nu(B \cap E) = 0$. Therefore

$$\mu_+(E) = \mu(A \cap E) = 0 \quad \mu_-(E) = \mu(B \cap E) = 0$$

Now, let $\{\mu_n\} \subseteq \mathcal{L}_\nu$ converge to μ in the total variation norm. Then, let $E \subset X$ such that $\nu(E) = 0$. Then, $|\mu_n|(E) = 0$. By the reverse triangle inequality (a consequence of the triangle inequality for $\|\cdot\|$ shown above)

$$|\mu|(E) = ||\mu|(E) - |\mu_n|(E)|| \leq |\mu - \mu_n|(E) \leq |\mu - \mu_n|(X) = \|\mu - \mu_n\| \rightarrow 0$$

so $\mu(E) = 0$ and $\mu \in \mathcal{L}_\nu$.

- (c) Let $f \in L^1(X, \mathcal{F}, \nu)$. Then,

$$\mu(A) = \int_A f d\nu$$

defines a signed measure for $A \subseteq X$. We only need to check that this pairing is isometric and onto. Surjectivity follows from the Radon-Nikodym theorem which states that if $\rho \ll \lambda$, then there exists λ -measurable g such that

$$\rho = g d\lambda$$

Then, to check the norms are preserved, we first show that the Hahn decomposition of μ corresponds to the positive and negative parts of f . Indeed, let $A = \{f \geq 0\}$. Then, for any $E \subseteq A$,

$$\mu(E) = \int_E f d\nu \geq 0$$

Similarly, for $B = \{f < 0\}$, $F \subseteq B$,

$$\mu(F) = \int_F f d\nu \leq 0$$

So $A \cup B$ is a Hahn decomposition for μ . Therefore,

$$\begin{aligned} \int_X |f| d\nu &= \int_X f^+ + f^- d\nu = \int_X f^+ d\nu + \int_X f^- d\nu = \int_A f^+ d\nu + \int_B f^- d\nu \\ &= \mu_+(A) + \mu_-(B) = \mu_+(X) + \mu_-(X) = |\mu|(X) \end{aligned}$$

9.

We have shown many times before that if $\sum \lambda(E_n) < \infty$, then $\lambda(\limsup_n E_n) = 0$. Set

$$E_n = [r_n - 2^{-n-1}, r_n + 2^{-n-1}]$$

Then,

$$\sum_n \lambda(E_n) = \sum_n 2^{-n} < \infty$$

Set $E = \limsup_{n \rightarrow \infty} E_n$. Then, $\lambda(E) = 0$. So, for any $x \notin E$, we have that there exists $k \in \mathbb{N}$ such that $x \notin E_n$ for all $n \geq k$. Therefore, $f(x)$ is only nonzero for finitely many indices so the sum must converge at x .

(b) Set

$$X_n = \left(\bigcup_{k \neq n} E_k \right)^c$$

Then, $\mathbb{R} = \bigcup X_n$ and, $\mu(X_n) \leq 1$ since $f_k = 0$ on X_n for $n \neq k$. Indeed,

$$\mu(X_n) = \int_{X_n} \sum f_k d\lambda = \int_{X_n} f_n \leq \int f_n = 1$$

(c) To show $\mu \ll \lambda$, let $E \subset \mathbb{R}$ such that $\lambda(E) = 0$. Then, integration over a set of measure zero is also zero so $\mu(E) = 0$.

- (d) Without loss of generality, we can just show that each open ball has infinite measure since every open set contains an open ball. Let $B(x, \epsilon) \subseteq \mathbb{R}$. Then, there exists a subsequence of $\{r_n\}$ such that $\{r_{n_k}\} \subseteq B(x, \epsilon/2)$. Moreover, since the radii of E_{n_k} are decreasing, there exists N such that $E_{n_k} \subseteq B(x, \epsilon)$ for all $k \geq N$. Thus,

$$\mu(B(x, \epsilon)) = \int_{B(x, \epsilon)} \sum f_n d\lambda \geq \int_{B(x, \epsilon)} \sum_{k=N}^{\infty} f_{n_k} d\lambda = \sum_{k=N}^{\infty} \int_{B(x, \epsilon)} f_{n_k} \geq \sum_{k=N}^{\infty} \int_{E_{n_k}} f_{n_k} = \infty$$

Winter 2014

8. See Summer 13 #9 (a)
9. See Summer 13 #7
10. See Summer 13 #9

Summer 2014

1. (a) *Proof.* Suppose f is discontinuous at some $x \in (0, 1)$. Then there exists $\epsilon > 0$ such that $\forall \delta > 0$ there exists $y \in B(x, \delta)$ such that

$$|f(x) - f(y)| \geq \epsilon$$

However, consider the compact interval $[x - \gamma, x + \gamma] \subseteq (0, 1)$ for some $\gamma > 0$. Then, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \epsilon/3$$

for all $z \in [x - \gamma, x + \gamma]$, $n \geq N$. □

- (b) False. Let

$$f_n(x) = \frac{1}{x} + \frac{1}{n} \quad f(x) = \frac{1}{x}$$

Then, $f_n \rightarrow f$ uniformly on $(0, 1)$ but f is not uniformly continuous.

- (c) *Proof.* □

9. See Summer 13 #7

Winter 2015

1. *Proof.* For $u = 1 + n^2x^2$, $du = 2n^2x dx$,

$$\|f_n - 0\|_1 = \int_0^1 \frac{nx}{1 + n^2x^2} dx = \int_1^2 \frac{du}{2nu} = \frac{1}{2n} [\ln(2) - \ln(1)] \rightarrow 0$$

as $n \rightarrow \infty$. Therefore $f_n \rightarrow 0$ in $L^1[0, 1]$. Now, since $\|\cdot\|_\infty$ and $\|\cdot\|_{\sup}$ coincide on continuous functions,

$$\|f_n - 0\|_\infty = \sup_{x \in [0, 1]} |f_n(x)| \geq f_n(n^{-3/2}) = \frac{n^{-1/2}}{1 + n^{-1}} \rightarrow \infty$$

as $n \rightarrow \infty$. So $f_n \not\rightarrow 0$ in $L^\infty[0, 1]$. □

2. *Proof.* Let f be convex. Let $x_n \searrow x$. Then, define $t_n \in [0, 1]$ by

$$(1 - t_n)1 + t_nx = x_n$$

Notice that $t_n \rightarrow 1$ as $n \rightarrow \infty$. Then,

$$f(x_n) = f(t_nx + (1 - t_n)1) \leq t_nf(x) + (1 - t_n)f(1)$$

also define $s_n \in [0, 1]$ such that

$$(1 - s_n)(-1) + s_nx_n = x$$

then $s_n \rightarrow 1$ as $n \rightarrow \infty$ so

$$f(x) = f(s_nx_n + (1 - s_n)(-1)) \leq s_nf(x_n) + (1 - s_n)f(-1)$$

Combining this, we get

$$f(x) \leq s_nf(x_n) + (1 - s_n)f(-1) \leq s_nt_nf(x) + s_n(1 - t_n)f(1) + (1 - s_n)f(-1)$$

Notice that the RHS converges to $f(x)$ as $n \rightarrow \infty$ so by the Squeeze theorem

$$\lim_{n \rightarrow \infty} f(x_n) = f(x)$$

So f is right continuous. To show left continuity, we follow the same steps but modify t_n and s_n so they are convex combinations with the opposite endpoints. Therefore f is continuous. □

3. *Proof.* For each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$d(x_n, f(x_n)) < \frac{1}{n}$$

Since X is compact, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$ with limit x . Then,

$$d(x, f(x)) \leq d(x, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) + d(f(x_{n_k}), f(x)) \rightarrow 0$$

as $k \rightarrow \infty$ by construction of x_{n_k} and since f is continuous. Thus $f(x) = x$. □

4. (a) *Proof.* Let $\{y_n\} \subseteq Y$ be Cauchy. There exists $\{x_n\} \subseteq X$ such that $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is Cauchy and therefore convergent to some $x \in X$. Then,

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} f(x_n) = f(x) \in Y$$

□

- (b) False. Let $X = (0, 1)$, $Y = \mathbb{R}$. Let $d_Y = d_X = |(\cdot) - (\cdot)|$. Let $f(x) = 1/x$. Then, clearly

$$|x_1 - x_2| \leq \left| \frac{x_1}{x_1 x_2} - \frac{x_2}{x_1 x_2} \right| = |f(x_2) - f(x_1)|$$

so the inequality holds. Additionally, Y is complete but X is not.

5. *Proof.*

$$\|S(a)\|_2 = \sqrt{\sum_{n=1}^{\infty} s_n^2 a_n^2} \leq \|s\|_{\infty} \|a\|_2$$

For each $k \in \mathbb{N}$, there exists $s_{n_k} \in s$ such that

$$|s_{n_k}| > \|s\|_{\infty} - \frac{1}{k}$$

Then, consider $e_{n_k} = (0, \dots, 0 \overset{n_k}{1}, 0, \dots) \in \ell^2$. $\|e_{n_k}\|_2 = 1$ so

$$\|S(e_{n_k})\|_2 = |s_{n_k}| > \|s\|_{\infty} - \frac{1}{k}$$

for all $k \in \mathbb{N}$ thus

$$\|S\| = \|s\|_{\infty}$$

□

6. *Proof.* First, notice that T is bounded below:

$$\|x\|^2 \leq \langle Tx, x \rangle \leq \|Tx\| \cdot \|x\|$$

so, $\|Tx\| \geq \|x\|$ for all $x \in \mathcal{H}$. Now, we show one-to-one. Let $x \in \mathcal{H}$ such that $Tx = 0$. Then,

$$0 = \|Tx\| \geq \|x\|_{\mathcal{H}} \geq 0$$

so $x = 0$. Next, we show T has a closed range. Let $x_n \in \mathcal{H}$ such that $Tx_n \rightarrow y$ for some $y \in \mathcal{H}$. Then,

$$\|Tx_n - Tx_m\| \geq \|x_n - x_m\|$$

for all $n, m \in \mathbb{N}$. So, $\{x_n\}$ is Cauchy. Thus, there exists $x \in \mathcal{H}$ such that $x_n \rightarrow x$. Since T is bounded,

$$y = \lim_{n \rightarrow \infty} Tx_n = Tx$$

so $y \in \text{Ran} T$. Finally, we show T is onto. For $w \in (\text{Ran} T)^{\perp}$

$$\langle Tv, w \rangle = 0$$

for all $v \in \mathcal{H}$. In particular, for $v = w$,

$$0 = \langle Tw, w \rangle \geq \|w\|^2 \geq 0$$

which implies $w = 0$. Thus, $(\text{Ran} T)^\perp = \{0\}$ so $\text{Ran} T = \overline{\text{Ran} T} = \mathcal{H}$. We have T is one-to-one and onto therefore is it invertible so $Tx = y$ has a unique solution for every $y \in \mathcal{H}$. \square

7. Solution by Hao Chen and Walton Green (4/18)

Proof. We will prove the contrapositive of the statement. Suppose $\{E_k\}_{k=1}^n$ are Borel subsets of $[0, 1]$ such that

$$\lambda\left(\bigcap_{k=1}^n E_k\right) = 0$$

Then, we have that

$$1 = \lambda([0, 1]) = \lambda\left[\left(\bigcap_{k=1}^n E_k\right)^c\right] = \lambda\left(\bigcup_{k=1}^n E_k^c\right)$$

Therefore,

$$n = \sum_{k=1}^n \lambda([0, 1]) = \sum_{k=1}^n \lambda(E_k) + \lambda(E_k^c) \geq \sum_{k=1}^n \lambda(E_k) + \lambda\left(\bigcup_{k=1}^n E_k^c\right) = \sum_{k=1}^n \lambda(E_k) + 1$$

$$\text{so } \sum_{k=1}^n \lambda(E_k) \leq n - 1. \quad \square$$

8. Let $\{q_n\}_{n=1}^\infty \subseteq \mathbb{R}$ be an enumeration of the rational numbers. Then, let

$$U := \bigcup_{n=1}^\infty \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$$

So,

$$\lambda(U) \leq \sum_{n=1}^\infty \frac{2}{n^2} = 2 < \infty$$

and $U \subseteq \mathbb{R}$ is open. Now, notice that $\bar{U} = \mathbb{R}$ since \mathbb{Q} is dense in \mathbb{R} and $\mathbb{Q} \subseteq U$. Then,

$$\lambda(\partial U) = \lambda(\bar{U} \setminus U) = \infty$$

9. *Proof.* (\Rightarrow) let $\lambda(E) = M > 0$. Let $f_n \xrightarrow{\lambda} 0$. Then, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\inf \{c > 0 : \lambda\{|f_n| > c\} < \epsilon\} < \epsilon$$

for all $n \geq N$ which implies

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

Now, we will use the fact that

$$x \mapsto \frac{x}{x+1}$$

is monotone increasing and ≤ 1 .

$$\begin{aligned} \int_E \frac{|f_n|}{1+|f_n|} &= \int_{E \cap \{|f_n| > \epsilon\}} \frac{|f_n|}{1+|f_n|} + \int_{E \cap \{|f_n| < \epsilon\}} \frac{|f_n|}{1+|f_n|} \\ &\leq \int_{E \cap \{|f_n| > \epsilon\}} 1 + \int_E \frac{\epsilon}{1+\epsilon} \\ &\leq \lambda\{|f_n| > \epsilon\} + \lambda(E) \left(\frac{\epsilon}{1+\epsilon} \right) \\ &< \epsilon + M \left(\frac{\epsilon}{1+\epsilon} \right) \rightarrow 0 \end{aligned}$$

as $\epsilon \rightarrow 0$.

(\Leftarrow) Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$\frac{\epsilon^2}{1+\epsilon} > \int_E \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n| > \epsilon\}} \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n| > \epsilon\}} \frac{\epsilon}{1+\epsilon} = \lambda\{|f_n| > \epsilon\} \frac{\epsilon}{1+\epsilon}$$

so

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

for all $n \geq N$. Thus,

$$\|f_n\|_\lambda = \inf\{c > 0 : \lambda\{|f_n| > c\} < c\} < \epsilon$$

□

10. *Proof.* Define

$$F_i := \bigcup_{n=i}^{\infty} E_n$$

for each $i \in \mathbb{N}$. Notice that F_i are reverse nested (i.e. $F_{i+1} \subseteq F_i$ therefore $F_i^c \subseteq F_{i+1}^c$)
Then,

$$\mu(F_i) = \mu\left(\bigcup_{n=i}^{\infty} E_n\right) \leq \sum_{n=i}^{\infty} \mu(E_n) \rightarrow 0$$

as $i \rightarrow \infty$. Now,

$$\begin{aligned} \mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n\right) &= \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \mu\left(F_1 \cap \bigcap_{k=2}^{\infty} F_k\right) = \mu\left(F_1 \setminus \bigcup_{k=2}^{\infty} F_k^c\right) \\ &= \mu(F_1) - \mu\left(\bigcup_{k=2}^{\infty} F_k^c\right) = \mu(F_1) - \lim_{k \rightarrow \infty} \mu(F_k^c) \\ &= \lim_{k \rightarrow \infty} \mu(F_1 \setminus F_k^c) = \lim_{k \rightarrow \infty} \mu(F_1 \cap F_k) \\ &= \lim_{k \rightarrow \infty} \mu(F_k) = 0 \end{aligned}$$

□

Summer 2015

1. (a) *Proof.* Let $\epsilon > 0$, pick $N \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all $n \geq N$. This can be done since $\sum M_n < \infty$. Now, for all $x \in \mathbb{R}$,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all $n \geq N$. Therefore

$$\sum_{k=1}^{\infty} f_k(x)$$

is uniformly convergent. □

- (b) Define

$$f_n(x) := \begin{cases} \frac{1}{n} & n \leq x < n+1 \\ 0 & \text{otherwise} \end{cases} \quad \forall n \in \mathbb{N}$$

Then, clearly $\sum f_n(x)$ is convergent pointwise and

$$\sum_{n=1}^{\infty} \|f_n\|_{\infty} \leq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Now we need to show this convergence is actually uniform. Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then, for all $x \in \mathbb{R}$,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \frac{1}{n+1} \leq \frac{1}{N} < \epsilon$$

for all $n \geq N$.

2. *Proof.* First we show $(A^{\perp})^{\perp}$ is a closed subspace containing A . Clearly $A \subset (A^{\perp})^{\perp}$. Let $x, y \in (A^{\perp})^{\perp}$ and $a, b \in \mathbb{C}$. Then,

$$\langle ax + by | z \rangle = a \langle x | z \rangle + b \langle y | z \rangle = 0 + 0 = 0 \quad \forall z \in A^{\perp}$$

Let $\{x_n\}_{n=1}^{\infty} \subset (A^{\perp})^{\perp}$ such that $x_n \rightarrow x$.

$$\langle x | z \rangle = \lim_{n \rightarrow \infty} \langle x_n | z \rangle = \lim_{n \rightarrow \infty} 0 = 0 \quad \forall z \in A^{\perp}$$

So, we have shown $\overline{\text{span} A} \subset (A^{\perp})^{\perp}$. Now, let $x \in (A^{\perp})^{\perp}$. Then,

$$d(x, \overline{\text{span} A}) = \sup_{\substack{y \in (\overline{\text{span} A})^{\perp} \\ \|y\| \leq 1}} |\langle x | y \rangle| = \sup_{y \in A^{\perp}, \|y\| \leq 1} |\langle x | y \rangle| = 0$$

So $x \in \overline{\text{span} A}$ since it is closed. □

3. (a) *Proof.* (i) Clearly, $d_s(A, A) = 0$. Now, suppose $d_s(A, B) = 0$. Then for all $\epsilon > 0$ and $x \in A$,

$$d(x, B) \leq \epsilon$$

thus $d(x, B) = 0$ so $x \in B$ since B is closed. Thus $A \subseteq B$. Likewise $B \subseteq A$. so $A = B$.

(ii) Clearly $d_s(A, B) = d_s(B, A)$.

(iii) Let $C \subseteq X$ be closed. Let $\epsilon_1 > 0$ be such that $A_{\epsilon_1} \subset C$ and $C_{\epsilon_1} \subseteq A$. Let $\epsilon_2 > 0$ such that $B_{\epsilon_2} \subset C$ and $C_{\epsilon_2} \subseteq B$. Then,

$$A_{\epsilon_1 + \epsilon_2} \subset C_{\epsilon_2} \subset B \quad \text{and} \quad B_{\epsilon_1 + \epsilon_2} \subset C_{\epsilon_1} \subset A$$

So, $d_s(A, B) \leq \epsilon_1 + \epsilon_2$ for all such ϵ_1, ϵ_2 . Therefore,

$$d_s(A, B) \leq \inf\{\epsilon_1\} + \inf\{\epsilon_2\} = d_s(A, C) + d_s(C, B)$$

□

(b) If the sets are not closed, then the first property of the metric fails. $d_s(A, A) = 0$ but $d_s(A, B) = 0$ does not necessarily $A = B$. Consider $X = \mathbb{R}$ and $A = [0, 1]$ and $B = (0, 1)$. $d_s(A, B) = 0$ but $A \neq B$.

4. (a) *Proof.* First, we show T is bounded:

$$\begin{aligned} \|Tf\|_\infty &= \sup_{t \in [0, 1]} \left| \int_0^t s f(s) ds \right| \leq \sup_{t \in [0, 1]} \int_0^t s |f(s)| ds \\ &\leq \|f\|_\infty \sup_{t \in [0, 1]} \int_0^t s ds \leq \|f\|_\infty \int_0^1 s ds = \frac{1}{2} \|f\|_\infty \end{aligned}$$

so $\|T\| \leq \frac{1}{2}$. Let $f, g \in C[0, 1]$ and $a, b \in \mathbb{R}$. Then,

$$T(af + bg)(t) = \int_0^t s(af + bg)(s) ds = a \int_0^t s f(s) ds + b \int_0^t s g(s) ds = a(Tf)(t) + b(Tg)(t)$$

so T is linear. □

(b) *Proof.* Let $f(t) = 1$ for all $t \in [0, 1]$. Then, $\|f\|_\infty = 1$ and

$$\|Tf\|_\infty = \sup_{t \in [0, 1]} \left| \int_0^t s ds \right| = \sup_{t \in [0, 1]} \frac{t^2}{2} = \frac{1}{2}$$

so $\|T\| = \frac{1}{2}$. □

5. (a) *Proof.* For every $n \in \mathbb{N}$ there exists $E_n \subseteq X$ such that $\mu(E_n) < \frac{1}{n^2}$ and $f_k \rightarrow f$ uniformly on $X \setminus E_n$. Let

$$E := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

Then, $\mu(E) = 0$ (For proof see Winter 15 #10) since

$$\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Now, for $x \notin E$, there exists k such that $x \notin \bigcup_{n=k}^{\infty} E_n$ so $x \notin E_n$ for all $n \geq k$ (However we only need it to hold for a single set, E_k . So, since $x \in E_k^c$,

$$f_n(x) \rightarrow f(x)$$

as $n \rightarrow \infty$. Therefore $f_n \rightarrow f$ pointwise a.e. □

- (b) *Proof.* Let $\epsilon > 0$. Then, there exists some E_ϵ such that $\mu(E_\epsilon) < \epsilon$ and $f_n \rightarrow f$ uniformly on E_ϵ . Moreover, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$, $x \in E_\epsilon^c$. Then,

$$\mu\{|f_n - f| > \epsilon\} \leq \mu(E_\epsilon) < \epsilon$$

so

$$\|f_n - f\|_\mu = \inf\{c > 0 : \mu\{|f_n - f| > c\} < c\} < \epsilon$$

therefore $f_n \rightarrow f$ in measure. □

6. *Proof.* Let $E \subset [a, b]$ be Borel measurable with $\lambda(E) > 0$. Let $\{q_n\}$ be an enumeration of the rational numbers in the interval $[0, 1]$. Set

$$F = \bigcup_n (E + q_n)$$

If $\{E + q_n\}$ are all disjoint, then, $\lambda(F) = \sum_{n=1}^{\infty} \lambda(E + q_n) = \sum_{n=1}^{\infty} \lambda(E) = \infty$ since $\lambda(E) > 0$. But this is a contradiction since $F \subseteq [a, b + 1]$ which has finite Lebesgue measure. Thus there exists $x \in (E + q_n) \cap (E + q_m)$ for some n and m not equal (so $q_n \neq q_m$). Then, there exists $y, z \in E$ such that

$$y + q_n = x = z + q_m$$

so $y - z = q_m - q_n \in \mathbb{Q} \setminus \{0\}$. □

7. (a) False. Consider the following function with a “spike” at every natural number, $n \geq 2$.

$$f(x) := \begin{cases} \nearrow & n \leq x \leq n + \frac{1}{n^3} \\ n & x = n + \frac{1}{n^3} \\ \searrow & n + \frac{1}{n^3} \leq x \leq n + \frac{2}{n^3} \\ 0 & \text{else} \end{cases}$$

Notice that

$$\int_{\mathbb{R}} f(x) dx = \sum_{n=2}^{\infty} \frac{1}{2} \cdot \frac{2}{n^3} \cdot n = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$$

but

$$\limsup_{x \rightarrow \infty} |f(x)| = \infty$$

- (b) *Proof.* Let f be integrable and differentiable and let $D > 0$ such that $|f'(x)| \leq D$ for all $x \in \mathbb{R}$. Fix $x \in \mathbb{R}$. Using the mean-value theorem, for all $y \in \mathbb{R}$ such that

$$|x - y| \leq \frac{f(x)}{D},$$

we know that

$$f(y) \geq f(x) - |x - y|D$$

Suppose without loss of generality that

$$\limsup_{x \rightarrow \infty} f(x) = M$$

for some $M > 0$. Then for all $n \in \mathbb{N}$, there exists $x_n \geq n$ such that

$$f(x_n) \geq \frac{M}{2}$$

Then,

$$\int_{\mathbb{R}} f(x) dx \geq \sum_{n=1}^{\infty} \frac{1}{2} \cdot \min \left\{ \frac{f(x_n)}{D}, 1 \right\} \cdot f(x_n) \geq \sum_{n=1}^{\infty} \frac{1}{8} \cdot \min \left\{ \frac{M}{D}, 1 \right\} \cdot M = \infty$$

which contradicts the fact that f is integrable. □

8. *Proof.* (a \implies b)

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^{\infty} \int_{F_k} 2^k dm \leq \sum_{k=-\infty}^{\infty} \int_{F_k} f dm = \int_{\mathbb{R}} f dm$$

(b \implies c) First, notice that $E_k \cup F_{k-1} = E_{k-1}$ and the union is disjoint therefore

$$m(F_k) = m(E_{k+1}) - m(E_k)$$

Mutlply by 2^k and sum from $-N$ to N we have

$$\begin{aligned} \sum_{k=-N}^N 2^k m(F_k) &= \sum_{k=-N}^N 2^k m(E_{k+1}) - \sum_{k=-N}^N 2^k m(E_k) = \frac{1}{2} \sum_{k=-N}^N 2^{k+1} m(E_{k+1}) - \sum_{k=-N}^N 2^k m(E_k) \\ &= -\frac{1}{2} \sum_{k=-N+1}^{N-1} 2^k m(E_k) + 2^N m(E_{N+1}) - 2^{-N} m(E_{-N}) \end{aligned}$$

The final two terms can be bounded by $\int f$: $2^N m(E_N) \leq \int_{E_N} f \, dm \leq \int_{\mathbb{R}} f \, dm < \infty$.
Therefore, for any N ,

$$\sum_{k=-(N-1)}^{N-1} 2^k m(E_k) \leq -2 \sum_{k=-\infty}^{\infty} 2^k m(F_k) + 4 \int_{\mathbb{R}} f \, dm < \infty$$

(c \implies a) Notice that since f is non-negative, $\mathbb{R} = \{f = 0\} \cup E_k$.

$$\int f \, dm = \sum_{k=-\infty}^{\infty} \int_{F_k} f \, dm \leq \sum_{k=-\infty}^{\infty} \int_{F_k} 2^{k+1} \, dm = 2 \sum_{k=-\infty}^{\infty} m(F_k) \leq 2 \sum_{k=-\infty}^{\infty} m(E_k)$$

□

Winter 2016

1. Recall

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

So for $x = 1$,

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

2. *Proof.* Since ℓ^2 is a Hilbert space, A being dense in ℓ^2 is equivalent to

$$A^\perp = \{0\}$$

Let $x = (x_1, x_2, \dots) \in A^\perp$. Then, $\langle x, a \rangle_{\ell^2} = 0$ for all $a \in A$. Notice that $a = e^{(k)} = (0, \dots, 0, \underset{k^{th}}{1}, 0, \dots)$ is in A for any $k \in \mathbb{N}$.

$$0 = \langle x, e^{(k)} \rangle = \sum_{i=1}^{\infty} x_i e_i^{(k)} = x_k$$

for $k \in \mathbb{N}$. Therefore $x = 0$. Now we show the same thing for A^c . Let $y \in (A^c)^\perp$. Also, define $f^{(k)} = e^{(k)} - e^{(k+1)} \in A^c$. Then,

$$0 = \langle y, f^{(k)} \rangle = y_k - y_{k+1}$$

So $y_k = y_{k+1}$ for all $k \in \mathbb{N}$. Thus y is a constant sequence. The only constant sequence in ℓ^2 is the zero sequence therefore $y = 0$. \square

3. *Proof.* First we show subspace. Let $x_1 + y_1, x_2 + y_2 \in X + Y$ and $a, b \in \mathbb{R}$. Then,

$$a(x_1 + y_1) + b(x_2 + y_2) = (ax_1 + bx_2) + (ay_1 + by_2) \in X + Y$$

Now we show closure. Let $\{(x_n + y_n)\}_{n=1}^{\infty}$ be a sequence in $X + Y$ with limit z . This sequence is also Cauchy. So, using the fact that $X \perp Y$,

$$\begin{aligned} \|(x_n + y_n) - (x_m + y_m)\|^2 &= \|(x_n - x_m) + (y_n - y_m)\|^2 \\ &= \langle (x_n - x_m) + (y_n - y_m), (x_n - x_m) + (y_n - y_m) \rangle \\ &= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (x_n - x_m), (y_n - y_m) \rangle \\ &\quad + \langle (y_n - y_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\ &= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\ &= \|x_n - x_m\|^2 + \|y_n - y_m\|^2 \end{aligned}$$

and thus $\{x_n\}$ and $\{y_n\}$ are both Cauchy. Since \mathcal{H} is a Hilbert space, they are convergent to some x, y respectively. Since X, Y are closed, $x \in X$ and $y \in Y$. Then,

$$z = \lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n = x + y \in X + Y$$

Therefore $X + Y$ is closed. \square

4. *Proof.* Since Y is a Banach space, $\mathcal{B}(X, Y)$ is also a Banach space. In a Banach space, any absolutely convergent series is convergent. Since $\|T\| < 1$,

$$\sum_{n=0}^{\infty} \|T\|^n < \infty$$

So

$$\sum_{n=0}^{\infty} T^n \in \mathcal{B}(X, Y)$$

□

5. (a) *Proof.* First, to show T is well-defined we need to show $T\xi$ is continuous for a fixed ξ . This follows from the fact that for $n > m$,

$$\left\| \sum_{k=0}^n a_k \xi_k x^k - \sum_{k=0}^m a_k \xi_k x^k \right\|_{\infty} = \sup_{x \in [0,1]} \left| \sum_{k=m+1}^n a_k \xi_k x^k \right| \leq \|a\|_{\infty} \sum_{k=m+1}^n |\xi_k| \rightarrow 0$$

as $n, m \rightarrow \infty$ since $\xi \in \ell^1$. Thus, this sequence of partial sums is Cauchy in $\|\cdot\|_{\infty}$. Since $(C[0, 1], \|\cdot\|_{\infty})$ is a Banach space, it's limit, $T\xi \in C[0, 1]$. To show linearity, let $\xi, \zeta \in \ell^1$ and $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} T(\alpha\xi + \beta\zeta)(x) &= \sum_{k=0}^{\infty} a_k (\alpha\xi_k + \beta\zeta_k) x^k = \alpha \sum_{k=0}^{\infty} a_k \xi_k x^k + \beta \sum_{k=0}^{\infty} a_k \zeta_k x^k \\ &= \alpha T(\xi)(x) + \beta T(\zeta)(x) \end{aligned}$$

□

- (b) *Proof.*

$$\|T(\xi)\|_{\infty} = \sup_{x \in [0,1]} |T(\xi)(x)| = \sup_{x \in [0,1]} \left| \sum_{k=0}^{\infty} a_k \xi_k x^k \right| \leq \|a\|_{\infty} \sum_{k=0}^{\infty} |\xi_k| = \|a\|_{\infty} \cdot \|\xi\|_1$$

So $\|T\| \leq \|a\|_{\infty}$. We claim this is actually the norm. For $\epsilon > 0$ there exists $a_n \in a$ such that

$$|a_n| > \|a\|_{\infty} - \epsilon$$

Pick $\xi^{(n)} = (0, \dots, 0, \overset{n^{th}}{1}, 0, \dots) \in \ell^1$. Then,

$$\|T\xi^{(n)}\|_{\infty} = \sup_{x \in [0,1]} \left| \sum_{k=0}^{\infty} a_k \xi_k^{(n)} x^k \right| = \sup_{x \in [0,1]} |a_n x^n| = |a_n| > \|a\|_{\infty} - \epsilon$$

Since there exists such $\xi^{(n)}$ for all $\epsilon > 0$, $\|T\| = \|a\|_{\infty}$. □

6. (i) LDCT cannot be applied to f_n since any k which bounds every f_n above, must be greater than 1 everywhere thus $\int_{\mathbb{R}} k = \infty$.

- (ii) LDCT cannot be applied to g_n since any k which bounds every g_n above, must be greater than $1/x$ everywhere thus $\int_{\mathbb{R}} k \geq \int_{\mathbb{R}} 1/x = \infty$.
- (iii) LDCT can be applied since for $k = 1/x^2$, $|h_n| \leq k$ and

$$\int_{\mathbb{R}} \frac{1}{x^2} dx < \infty$$

7. *Proof.* Let $E \subset \mathbb{R}$, $\epsilon \in (0, 1)$. Set $\delta = m^*(E)(1/\epsilon - 1) > 0$. By definition of outer measure, there exists an open set $G \supset E$ such that $m^*(E) + \delta > m^*(G) = m(G)$. Then,

$$\epsilon m(G) < \epsilon(m^*(E) + \delta) = \epsilon m^*(E)(1 + 1/\epsilon - 1) = m^*(E)$$

Moreover, since G is open, it can be written as a countable, disjoint union of open intervals, say I_k . Then,

$$\sum_k \epsilon m(I_k) = \epsilon m(G) < m^*(E) = m^*(E \cap G) \leq \sum_k m^*(E \cap I_k)$$

Therefore, at least one term in the left hand sum must be smaller than one term in the right hand sum, i.e. there exists k such that $\epsilon m(I_k) < m^*(E \cap I_k)$. \square

8. *Proof.* Define $A_n := \{x \in [0, 1] : n + 1 > |f(x)| \geq n\}$

$$\sum_{n=1}^{\infty} n \lambda(A_n) = \sum_{n=1}^{\infty} \int_{A_n} n dx \leq \sum_{n=1}^{\infty} \int_{A_n} f(x) dx = \int_0^1 f(x) < \infty$$

So,

$$\begin{aligned} \lim_{n \rightarrow \infty} n \lambda(\{x \in [0, 1] : |f(x)| \geq n\}) &= \lim_{n \rightarrow \infty} n \lambda\left(\bigcup_{k=n}^{\infty} A_k\right) = \lim_{n \rightarrow \infty} n \sum_{k=n}^{\infty} \lambda(A_k) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} k \lambda(A_k) = 0 \end{aligned}$$

\square

9. (a) *Proof.* Let $f(x) > 0$ for $x \in [0, 1]$ and $E \subseteq [0, 1]$ such that $\lambda(E) > 0$. Suppose $\int_E f d\lambda = 0$. Then

$$f(x) = 0$$

for almost every $x \in E$. However, since $\lambda(E) > 0$ there exists $x \in E$ such that $f(x) > 0$ which is a contradiction. \square

- (b) First we prove the following fact: $\mu(\limsup E_n) \geq \limsup \mu(E_n)$. Indeed, set $F_k = \bigcup_{n=k}^{\infty} E_n$. F_k are decreasing.

$$\mu(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n) = \mu(\bigcap_{k=1}^{\infty} F_k) = \inf_k \mu(F_k) = \inf_k \mu(\bigcup_{n=k}^{\infty} E_n) \geq \inf_k \sup_{n \geq k} \mu(E_n)$$

Proof. Fix $\epsilon \in (0, 1]$. Suppose $\inf_{\lambda(E) \geq \epsilon} \int_E f d\lambda = 0$. Then, for each n there exists E_n with $\lambda(E_n) \geq \epsilon$ and

$$\int_{E_n} f d\lambda < \frac{1}{n^2}$$

Then, consider $E = \limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. By the fact above, $\mu(E) \geq \epsilon$. By part (a), this means $\int_E f d\lambda > 0$. However,

$$\int_E f d\lambda = \int_{\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n} f d\lambda \leq \int_{\bigcup_{n=k}^{\infty} E_n} f d\lambda \leq \sum_{n=k}^{\infty} \int_{E_n} f d\lambda \leq \sum_{n=k}^{\infty} \frac{1}{n^2}$$

for any k . Therefore $\int_E f d\lambda = 0$ which is a contradiction. □

Summer 2016

1. (a) *Proof.* We show that f_n does not converge uniformly on the half-open interval $[0, 1)$. The pointwise limit is clearly $f(x) = 0$ for $x \in [0, 1)$. If $\{f_n\}$ converges uniformly, then it must converge to f , the pointwise limit. Let $\epsilon > 0$. For any $n \in \mathbb{N}$ there exists $x \in (0, 1]$ such that

$$1 > x > \left(\frac{\epsilon}{1 - \epsilon} \right)^{1/n}$$

Then,

$$|f(x)| > \epsilon$$

so $\{f_n\}$ does not converge uniformly on $[0, 1)$ therefore it does converge uniformly on $[0, 1]$. \square

- (b) *Proof.* Notice that

$$f_n(x) \leq 1$$

for $x \in [0, 1]$. Since $\int_0^1 1 \, dx < \infty$, by Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \rightarrow \infty} f_n(x) \, dx = \int_0^1 0 \, dx = 0$$

\square

2. False. Counterexample:

Consider $\{x^{(n)}\}_{n=1}^\infty \subset X$ where

$$x^{(n)} := (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots) \in X$$

Then, $\{x^{(n)}\}_{n=1}^\infty$ is Cauchy: For $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. So, for all $n, m \geq N$ ($n > m$),

$$d(x^{(n)}, x^{(m)}) = \sup_{i \in \mathbb{N}} |x_i^{(n)} - x_i^{(m)}| = \frac{1}{m} < \frac{1}{N} < \epsilon$$

However, $x_n \rightarrow (1, \frac{1}{2}, \frac{1}{3}, \dots)$ which is not in X .

3. *Proof.* Let $\{y_n\}_{n=1}^\infty \subset K$. Let $\{y_{n_k}\}_k$ denote the set of distinct elements of $\{y_n\}_{n=1}^\infty$. If $\{y_{n_k}\}_k$ is finite, then there exists some $m \in \mathbb{N}$ such that y_m occurs infinitely many times in $\{y_n\}_{n=1}^\infty$ thus the constant sequence $\{y_m\}$ is a convergent subsequence of $\{y_n\}_{n=1}^\infty$. On the other hand if $\{y_{n_k}\}_k$ is infinite, then

$$\{y_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=0}^\infty$$

is a subsequence of a convergent sequence so it is itself convergent to $\lim_{n \rightarrow \infty} x_n = x_0 \in K$. \square

4. *Proof.* First, notice

$$\begin{aligned}
(T - S)^3 &= (T^2 - ST - TS + S^2)(T - S) \\
&= (T - 2ST + S)(T - S) \\
&= (T^2 - 2ST^2 + ST - ST + 2S^2T - S^2) \\
&= (T - 2ST + ST - ST + 2ST - S) \\
&= (T - S)
\end{aligned}$$

Then, by Cauchy-Schwarz for the operator norm,

$$\|T - S\| = \|(T - S)^3\| \leq \|T - S\|^3$$

Therefore

$$1 \leq \|T - S\|^2$$

and

$$\|T - S\| \geq 1$$

□

5. *Proof.* Let $n, m \in \mathbb{N}$. Without loss of generality, let $n > m$. First,

$$\|x_m\|^2 = \langle x_n, x_m \rangle \leq \|x_n\| \cdot \|x_m\|$$

so $\{\|x_n\|\}_{n=1}^\infty$ is monotone decreasing. Moreover it is bounded below by 0 so it is convergent to some $K \in \mathbb{R}$. Moreover, since

$$\lim_{n \rightarrow \infty} \|x_n\|^2 = \left(\lim_{n \rightarrow \infty} \|x_n\| \right)^2 = K^2$$

$\{\|x_n\|^2\}_{n=1}^\infty$ is convergent and therefore Cauchy. Then, for $n > m$,

$$\begin{aligned}
\|x_n - x_m\|^2 &= \langle x_n - x_m, x_n - x_m \rangle \\
&= \|x_n\|^2 - \langle x_n, x_m \rangle - \langle x_m, x_n \rangle + \|x_m\|^2 \\
&= \|x_n\|^2 - \|x_m\|^2 - \|x_m\|^2 + \|x_m\|^2 \\
&= \|x_n\|^2 - \|x_m\|^2 \\
&= |\|x_n\|^2 - \|x_m\|^2| \rightarrow 0
\end{aligned}$$

So $\{x_n\}_{n=1}^\infty$ is Cauchy and therefore convergent since \mathcal{H} is a Hilbert space. □

6. *Proof.* Let $A = (0, 1)$ and $B = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then,

$$d(A, B) = \lambda(A \Delta B) = \lambda(\{\frac{1}{2}\}) = 0$$

but $A \neq B$. Thus the first property of a metric $d(A, B) = 0 \implies A = B$ fails. □

7. *Proof.* (\Leftarrow) Fix $\epsilon > 0$. Then, there exists an open set $\mathcal{O} \supseteq A$ such that

$$\lambda(\mathcal{O} \setminus A) < \epsilon$$

Thus $\mathcal{O} \setminus A \in \mathcal{L}$, the σ -algebra of Lebesgue-measurable sets. Moreover, since \mathcal{O} is open, it is also Lebesgue measurable. Thus,

$$A = \mathcal{O} \setminus (\mathcal{O} \setminus A) \in \mathcal{L}$$

since \mathcal{L} is closed under set-minus.

(\Rightarrow) Let A be Lebesgue measurable. Then,

$$\lambda(A) = \lambda^*(A) = \inf_{A \subseteq \bigcup_n I_n} \sum_{n=1}^{\infty} \lambda(I_n)$$

where $I_n = [a_n, b_n)$. Now, let $\epsilon > 0$. By definition of inf, there exists $\{I_n\}_{n=1}^{\infty}$ such that

$$\lambda(A) + \frac{\epsilon}{2} > \sum_{n=1}^{\infty} \lambda(I_n) \quad \text{and} \quad A \subseteq \bigcup_{n=1}^{\infty} I_n$$

Now, define

$$J_n = \left(a_n, b_n + \frac{\epsilon}{2^{n+1}} \right)$$

Then, $I_n \subset J_n$ for all i and for $\mathcal{O} := \bigcup_{n=1}^{\infty} J_n$

$$\lambda(\mathcal{O} \setminus A) = \lambda(\mathcal{O}) - \lambda(A) \leq \sum_{n=1}^{\infty} \lambda(J_n) - \lambda(A) = \sum_{n=1}^{\infty} \left(\lambda(I_n) + \frac{\epsilon}{2^{n+1}} \right) - \lambda(A) < \epsilon$$

□

8. (a) *Proof.* Proof by contraposition. Suppose $\lambda(E_n) = 0$ for all $n \in \mathbb{N}$. Then,

$$\lambda(\{x \in I : f(x) > 0\}) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} \lambda(E_n) = 0$$

since $\{E_n\}$ are nested. □

(b) *Proof.* Suppose the assumption holds and that $\lambda(\{x \in I : f(x) > 0\}) > 0$. Then, by part (a), there exists some $n \in \mathbb{N}$ such that $\lambda(E_n) > 0$. Since the measure of E_n is positive, it contains infinitely many points. Now, pick $x_1, \dots, x_{n \cdot M} \in E_n$, then,

$$f(x_1) + \dots + f(x_{n \cdot M}) > \frac{1}{n} + \dots + \frac{1}{n} = Mn \left(\frac{1}{n} \right) = M$$

which is a contradiction. □

9. INCOMPLETE

Proof. (\Rightarrow) By the Triangle Inequality,

$$\|f_n\|_1 \leq \|f_n - f\|_1 + \|f\|_1$$

and

$$\|f\|_1 \leq \|f - f_n\|_1 + \|f_n\|_1$$

therefore

$$|\|f_n\|_1 - \|f\|_1| \leq \|f_n - f\|_1 \rightarrow 0$$

as $n \rightarrow \infty$.

(\Leftarrow)

□

10. *Proof.* Applying Holder's Inequality,

$$\begin{aligned} \sum_{n=0}^{\infty} \int_n^{n+1} f(x) dx &\leq \sum_{n=0}^{\infty} \left(\int_n^{n+1} f(x)^2 dx \right)^{1/2} \left(\int_n^{n+1} 1^2 dx \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} f(x)^2 dx \right)^{1/2} = \|f\|_{L^2(\mathbb{R})} < \infty \end{aligned}$$

therefore

$$\lim_{n \rightarrow \infty} \int_n^{n+1} f(x) dx = 0$$

□

Winter 2017

1. Notice that

$$\sum \frac{\sin(nx)}{n}$$

is the Fourier series of the function $x \mapsto \frac{\pi - x}{2}$. Indeed,

$$\frac{\pi}{2} \int_{-\pi}^{\pi} \sin(nx) dx - \frac{1}{2} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{x \cos(nx)}{2n} \Big|_{x=-\pi}^{\pi} - \frac{1}{2n} \int_{-\pi}^{\pi} \cos(nx) dx =$$

2. (a) *Proof.* (i) First, notice that

$$-M|x - y| \leq f(x) - f(y) \leq M|x - y|$$

therefore $|f(x)| \leq |f(y)| + M|x - y|$ for all $x, y \in \mathbb{R}$. Therefore,

$$\begin{aligned} 0 \leq d(f, g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x)| + |g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(0)| + M|x| + |g(0)| + M|x| \\ &\leq \sum_{n=1}^{\infty} \frac{f(0) + g(0) + 2Mn}{2^n} < \infty \end{aligned}$$

so $d(f, g)$ is well-defined and non-negative.

(ii) Clearly $d(f, f) = 0$. Assume $d(f, g) = 0$. Then $\sup_{x \in [-n, n]} |f(x) - g(x)| = 0$ for all n thus $f(x) = g(x)$ on \mathbb{R} .

(iii) Clearly $d(f, g) = d(g, f)$.

(iv)

$$\begin{aligned} d(f, g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} (|f(x) - h(x)| + |h(x) - g(x)|) \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \left(\sup_{x \in [-n, n]} |f(x) - h(x)| + \sup_{x \in [-n, n]} |h(x) - g(x)| \right) \\ &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - h(x)| + \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |h(x) - g(x)| \\ &= d(f, h) + d(h, g) \end{aligned}$$

□

(b) *Proof.* Let $\{f_k\}_{k=1}^\infty \subseteq \mathcal{L}$ be Cauchy in d . Fix $x \in \mathbb{R}$. Then, $x \in [-N, N]$ for some $N \in \mathbb{N}$. For any $k, \ell \in \mathbb{N}$,

$$|f_k(x) - f_\ell(x)| \leq 2^N 2^{-N} \sup_{x \in [-N, N]} |f_k(x) - f_\ell(x)| \leq 2^N d(f_k, f_\ell) \rightarrow 0$$

as $k, \ell \rightarrow \infty$. Therefore $\{f_k(x)\}_{k=1}^\infty \subseteq \mathbb{R}$ is Cauchy for each x and therefore convergent since \mathbb{R} is complete. Then define

$$f(x) := \lim_{k \rightarrow \infty} f_k(x)$$

First, we show $f \in \mathcal{L}$. Fix $x, y \in \mathbb{R}$. For $\epsilon > 0$ there exists $n_1, n_2 \in \mathbb{N}$ such that

$$|f_k(x) - f(x)| < \epsilon \quad |f_\ell(y) - f(y)| < \epsilon \quad \forall k > n_1 \ell > n_2$$

Then, for $n = \max\{n_1, n_2\}$,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 2\epsilon + M|x - y|$$

so $|f(x) - f(y)| \leq M|x - y|$ and $f \in \mathcal{L}$. Now we will show $f_k \rightarrow f$ in d . Let $\epsilon > 0$. Since $\{f_k\}$ is Cauchy in d , $\{d(f_k, 0)\}$ is uniformly bounded, i.e. there exist $C > 0$ such that $d(f_k, 0) \leq C$ for all k . Indeed, there exists N such that $d(f_k, f_j) < 1$ for $j, k \geq N$. Thus,

$$d(f_k, 0) \leq d(f_k, f_N) + d(f_N, 0) \leq 1 + d(f_N, 0)$$

So $d(f_k, 0) \leq \max_{j=1, \dots, N} \{1 + d(f_j, 0)\}$ for all k . Thus,

$$d(f_k, f) \leq C + d(f, 0)$$

for all k so there exists N such that

$$\sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| < \epsilon/2$$

for all k . Moreover, since $f_k(x) \rightarrow f(x)$ for each $x \in [-N, N]$, $f_k \rightarrow f$ uniformly on $[-N, N]$ since it is closed and bounded. Therefore we can take k large enough so that

$$\sum_{n=1}^N 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| < N 2^{-N} \sup_{x \in [-N, N]} |f_k(x) - f(x)| < \epsilon/2$$

Then,

$$d(f_k, f) = \sum_{n=1}^N + \sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| \leq \epsilon$$

for k large enough. □

3. *Proof.* Let $f \in C^1[0, 1]$.

$$|\varphi_0(f)| = |f'(0)| \leq \sup_{x \in [0, 1]} |f'(x)| \leq \|f\|$$

So $\|\varphi_0\| \leq 1$. We will show $\|\varphi_0\| = 1$. Consider the sequence defined

$$f_n(x) := \frac{\sin(nx)}{n}$$

Notice $\|f_n\| = 1/n + 1$ and $|\varphi_0(f_n)| = 1$. Thus,

$$1 \geq \|\varphi_0\| = \sup_{f \neq 0} \frac{|\varphi_0(f)|}{\|f\|} \geq \sup_{n \in \mathbb{N}} \frac{|\varphi_0(f_n)|}{\|f_n\|} = \sup_{n \in \mathbb{N}} \frac{1}{1 + 1/n} = 1$$

so $\|\varphi_0\| = 1$. □

4. (a) *Proof.* Let $x \in \ell^2$. Then for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |x_k|^2 < \epsilon$$

Then, for $y = (x_1, x_2, \dots, x_N, 0, 0, \dots) \in Y$,

$$\|x - y\|_2^2 = \sum_{k=1}^{\infty} |x_k - y_k|^2 = \sum_{k=N+1}^{\infty} |x_k - 0|^2 < \epsilon$$

so Y is dense in ℓ^2 . □

(b) *Proof.* By Cauchy-Schwarz,

$$\left| \sum_{k=1}^n x_k \right| \leq \left(\sum_{k=1}^n |1|^2 \right)^{1/2} \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} = \sqrt{n} \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2}$$

Moreover, if $x \in \ell^2$, then $\sum_{k=1}^{\infty} |x_k|^2$ converges so we can bound the final term by $\|x\|_2$. □

(c) *Proof.* Let $x \in \ell^2$, $\epsilon > 0$. By part (a) there exists $y \in Y$ such that $\|x - y\|_2 \leq \epsilon/2$. Then,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left| \sum y_n \right| = 0$$

since the second term is bounded and the first is decreasing to 0. So, there exists N such that

$$\frac{1}{\sqrt{n}} \left| \sum y_n \right| < \epsilon/2 \quad \text{for } n \geq N$$

By triangle inequality for $|\cdot|$ and part(b),

$$\frac{1}{\sqrt{n}} \left| \sum x_n \right| \leq \frac{1}{\sqrt{n}} \left| \sum x_n - y_n \right| + \frac{1}{\sqrt{n}} \left| \sum y_n \right| < \|x - y\|_2 + \epsilon/2 < \epsilon$$

for $n \geq N$ so

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left| \sum x_n \right| = 0$$

□

5. *Proof.* First notice that

$$0 \leq \int_0^\infty \frac{x}{1+x^3} dx = \int_0^1 + \int_1^\infty \frac{x}{1+x^3} dx \leq \int_0^1 1 dx + \int_1^\infty \frac{1}{x^2} dx < \infty$$

Then, notice that $\frac{x}{1+x^n} \leq \frac{x}{1+x^{n+1}}$ for $x \in (0, 1)$. Therefore, by monotone convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{x}{1+x^n} dx = 0$$

since $x/(1+x^n) \rightarrow 0$ pointwise on $(0, 1)$. Moreover,

$$\int_1^\infty \frac{x}{1+x^n} dx$$

is monotone decreasing and bounded below by zero therefore

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{x}{1+x^n} dx = \lim_{n \rightarrow \infty} \int_0^1 + \int_1^\infty \frac{x}{1+x^n} dx$$

exists. Moreover,

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{x}{1+x^n} dx = \int_1^\infty \lim_{n \rightarrow \infty} \frac{x}{1+x^n} dx = 0$$

by the Lebesgue dominated convergence theorem since

$$\frac{x}{1+x^n} \leq \frac{x}{x^n} = x^{1-n}$$

which is integrable on $(1, \infty)$ for $n \geq 3$. □

6. (a) *Proof.* Set $f(x) = \mathbf{1}_{\liminf_n A_n}$.

(i) $f(x) = 1 \iff x \in \cup_k \cap_{n=k}^\infty A_n$. So, there exists k such that $x \in A_n$ ($\mathbf{1}_{A_n}(x) = 1$) for all $n \geq k$. So, $\lim_{n \rightarrow \infty} \mathbf{1}_{A_n}(x) = 1$ (so \liminf is also 1).

(ii) Suppose $f(x) = 0$. For each k there exists $n \geq k$ such that $x \notin A_n$ ($\mathbf{1}_{A_n}(x) = 0$) so $\liminf_n \mathbf{1}_{A_n} = 0$. □

(b) By Fatou's Lemma,

$$\mu(\liminf_n A_n) = \int_X f d\mu = \int_X \liminf_n \mathbf{1}_{A_n} d\mu \leq \liminf_n \int_X \mathbf{1}_{A_n} d\mu = \liminf_n \mu(A_n)$$

7. *Proof.* Define $f = \sup_N \sum_{n=1}^N f_n$. f is a measurable function, moreover, since f_n are non-negative, $\sum f_n \nearrow f$. So, by Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f = \sum \int f_n \leq \sum \frac{1}{n^2} < \infty$$

So f is non-negative and integrable. We claim this implies $f < \infty$ a.e. If not, then there exists E with $\lambda(E) > 0$ and $f = \infty$ on E . Then,

$$\int_{\mathbb{R}} f \geq \int_E f = \infty$$

so $f < \infty$ a.e. □

8. (a) *Proof.* By Hölder's Inequality,

$$\left| \int f_n d\mu - \int f d\mu \right| \leq \int |f - f_n| d\mu \leq \|f - f_n\|_{\infty} \int d\mu = \|f - f_n\|_{\infty} \mu(X) \rightarrow 0$$

as $n \rightarrow \infty$. □

Summer 2017

5. Hao Chen

Proof. To show the orthonormal set $\{f_n\}$ is an orthonormal basis we will show that $\{f_n\}^\perp = \{0\}$. If not, then there exists $x \neq 0$ such that $\langle x, f_n \rangle = 0$ for all n . However, by Parseval's identity and the Cauchy-Schwarz inequality,

$$\|x\|^2 = \sum |\langle x, e_n \rangle|^2 = \sum |\langle x, e_n - f_n \rangle|^2 \leq \sum \|x\|^2 \|e_n - f_n\|^2 < \|x\|^2$$

but this is a contradiction so $\{f_n\}^\perp = \{0\}$. \square

A more complicated proof by Walton:

Proof. Let $c = \sum \|e_n - f_n\|^2 < 1$. Define $T : \mathcal{H} \rightarrow \mathcal{H}$ by sending $x = \sum \langle x, e_n \rangle e_n \mapsto \sum \langle x, e_n \rangle f_n$. The second sum converges by the Bessel inequality. Now, by the Cauchy-Schwarz inequality and Parseval's identity,

$$\|(I - T)x\|^2 = \left\| \sum \langle x, e_n \rangle (e_n - f_n) \right\|^2 \leq \sum |\langle x, e_n \rangle|^2 \sum \|e_n - f_n\|^2 = c \|x\|^2$$

So, $\|T - I\| \leq \sqrt{c} < 1$. We claim that this means T is invertible. Indeed, set

$$S = \sum_{n=0}^{\infty} (I - T)^n$$

The sum is absolutely convergent since $\|I - T\| < 1$ so S is bounded linear operator since $\mathcal{L}(\mathcal{H})$ is a Banach space. Moreover,

$$S - TS, S - ST = \sum_{n=1}^{\infty} (I - T)^n = S - (I - T)^0 = S - I$$

so $S = T^{-1}$. Now, let $y \in \mathcal{H}$. Then, there exists x ($T^{-1}y$) such that $Tx = y$. Therefore,

$$y = \sum \langle x, e_n \rangle f_n \tag{1}$$

and therefore $\overline{\text{span}}\{f_n\} = \mathcal{H}$. \square

Remark: This actually holds if $\sum \|e_n - f_n\|^2 < \infty$.

6. Define $A_n = \{f \geq 1/n\}$. Since $A_n \subseteq A_{n+1}$,

$$0 < \mu(\{f > 0\}) = \mu\left(\bigcup_n A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n)$$

Therefore there exists n such that $\mu(A_n) > 0$. Then,

$$\int f \geq \int_{A_n} f \geq \frac{1}{n} \mu(A_n) > 0$$

7. (a) We first show that if f is integrable, the $\mu(E_n) \rightarrow 0$ implies $\int_{E_n} f \rightarrow 0$. Since f is integrable, for $A_n = \{n-1 \leq |f| \leq n\}$,

$$\infty > \int |f| \geq \sum (n-1)\mu(A_n)$$

Given $\epsilon > 0$ there exists N such that $\sum_{n=N}^{\infty} (n-1)\mu(A_n) < \epsilon/2$. Also, we can find M such that

$$\mu(E_n) < \epsilon/(2N) \quad \forall n \geq M$$

Then, for $n \geq M$,

$$\begin{aligned} \left| \int_{E_n} f \right| &\leq \int_{E_n} |f| = \int_{E_n \cap \{|f| \leq N\}} |f| + \int_{E_n \cap \{|f| > N\}} |f| \\ &\leq N\mu(E_n) + \sum_{k=N}^{\infty} (k-1)\mu(A_k) \leq N\epsilon/2N + \epsilon/2 = \epsilon \end{aligned}$$

Now we can prove the statement. Let $a, b \in \mathbb{R}$. Then there exists $\{a_n\}, \{b_n\} \subseteq \mathbb{Q}$ such that

$$a_n \rightarrow a \quad b_n \rightarrow b$$

Then,

$$\int_a^b f = \int_a^{a_n} f + \int_{a_n}^{b_n} f + \int_{b_n}^b f$$

The middle term is zero by assumption and applying the above lemma, the first and third terms go to 0.

(b) INCOMPLETE

8. This is a special case of Winter 15 #10.

Winter 2018

Analysis Prelim Solution - Winter 2018

Yiran Zhu — Clemson - Math

1. Let $\mathcal{C}(0, 1)$ be the collection of all continuous functions on $(0, 1)$ which is a unit *open* interval in \mathbb{R} . Suppose $\{f_n\}_{n=1}^\infty \subset \mathcal{C}(0, 1)$ converges uniformly to f on $(0, 1)$, i.e.,

$$\|f_n - f\|_\infty = \sup_{t \in (0, 1)} |f_n(t) - f(t)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Can we say $f \in \mathcal{C}(0, 1)$? Prove or disprove.

Proof. Fix $x \in (0, 1)$. Observe that $\forall y \in (0, 1)$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq 2\|f_n - f\| + |f_n(x) - f_n(y)| \end{aligned}$$

For $\epsilon > 0$, there exists $N \geq 1$ such that $\|f_N - f\| < \epsilon/3$. Furthermore, since f_N is continuous, there exists $\delta > 0$ such that $\forall |x - y| < \delta$, $|f_N(x) - f_N(y)| < \epsilon/3$.

$$\forall |x - y| < \delta, \quad |f(x) - f(y)| \leq 2\|f_N - f\| + |f_N(x) - f_N(y)| < \epsilon$$

Therefore f is continuous at x . Since x can be an arbitrary number in $(0, 1)$, $f \in \mathcal{C}(0, 1)$. \square

2. Easy to show. $\|f\|_\infty \leq \|f - f_n\|_\infty + \|f_n\|_\infty \leq \epsilon + M < \infty$.
3. If $x \in Y^\perp$, then $\|x - y\|^2 = \|x\|^2 + \|y\|^2 \geq \|x\|^2$. Conversely, since Y is a closed subspace, $H = Y \oplus Y^\perp$. There exists $x' \in Y$ and $x^\perp \in Y^\perp$ such that $x = x' + x^\perp$. Then

$$\|x^\perp\|^2 = \|x - x'\|^2 = \|x\|^2 = \|x' + x^\perp\|^2 = \|x'\|^2 + \|x^\perp\|^2$$

Therefore, $\|x'\|^2 = 0 \Rightarrow x' = 0 \Rightarrow x = x^\perp \in Y^\perp$.

4. Given a Cauchy sequence $\{T_n\}_n \subseteq \mathcal{B}(X, Y)$ with respect to norm of operators, we need to construct an operator T such that $T_n \rightarrow T$. Note that, for all $x \in X$, $\{T_n(x)\}_n$ is a Cauchy sequence in Y and thus convergent to some point in Y . We denote this extreme point by T_x . Our mapping is $T : x \rightarrow T_x$. Then one can easily check T is linear and also bounded so $T \in \mathcal{B}(X, Y)$. Finally, $T_n \rightarrow T$ in operator norm.
5. (a) $\|T\| = 1$. This norm cannot be attained but can be approached by e_n when $n \rightarrow \infty$.

$$\|T(x)\| = \left| \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x_n \right| \leq \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) |x_n| \leq \|x\|_1$$

- (b) Suppose there exists x with $\|x\| \leq 1$ such that $|T(x)| = \|T\| = 1 \geq \|x\|$. From above inequality, we essentially have

$$\|x_1\| = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) |x_n| = \|x\|_1 - \sum_{n=1}^{\infty} \frac{|x_n|}{n}$$

Hence

$$\sum_{n=1}^{\infty} \frac{|x_n|}{n} = 0 \Rightarrow \forall n \geq 1, x_n = 0 \Rightarrow |T(x)| = 0 \neq 1$$

6. Prove by contradiction. Suppose there is a such measure $(H, \mathcal{M}, \lambda)$ for Hilbert Space H . Let $r = 2$ and $x = 0$,

$$0 < \lambda(B_2(0)) < \infty$$

Since $\dim H = \infty$, there is an orthonormal sequence $\{x_n\}_n \subseteq B_2(0)$. For all x_n , we claim that $B_1(x_n) \subseteq B_2(0)$. Indeed,

$$\forall y \in B_{1/2}(x_n), \|y - 0\| \leq \|y - x_n\| + \|x_n\| \leq 1/2 + 1 = 3/2 < 2 \Rightarrow y \in B_2(0)$$

Observe that $\|x_n - x_m\|^2 = \langle x_n - x_m, x_n - x_m \rangle = \|x_n\|^2 + \|x_m\|^2 = 2$ for all $n \neq m$. Moreover, if $n \neq m$, then we can check that $B_{1/2}(x_n) \cap B_{1/2}(x_m) = \emptyset$ as follows:

$$\forall y \in B_{1/2}(x_n), \|y_n - x_m\| \geq \|x_n - x_m\| - \|y - x_n\| = \sqrt{2} - \frac{1}{2} > \frac{1}{2} \Rightarrow y \notin B_{1/2}(x_m)$$

By assumption that measure of balls is invariant under translation, we have

$$\forall n \geq 1, \lambda(B_{1/2}(x_n)) = \lambda(B_{1/2}(x_1))$$

Note that $\cup_{n=1}^{\infty} B_{1/2}(x_n)$, the union of disjoint balls, is a subset of $B_2(0)$.

$$\lambda(B_2(0)) \geq \lambda\left(\cup_{n=1}^{\infty} B_{1/2}(x_n)\right) = \sum_{n=1}^{\infty} \lambda(B_{1/2}(x_n)) = \sum_{n=1}^{\infty} \lambda(B_{1/2}(x_1))$$

Therefore, $\lambda(B_{1/2}(x_1)) = 0$. However, we assume that measure of a ball is greater than 0.

7. Let (X, \mathcal{M}, μ) be a measure space and $f \in L^1(X, \mathcal{M}, \mu)$. Then $\{x : f(x) \neq 0\}$ is σ -finite with respect to μ .

Proof. Let $E_n = \{x : |f(x)| \geq 1/n\}$ and then $\{x : f(x) \neq 0\} = \cup_{n=1}^{\infty} E_n$. From Chebyshev's Inequality,

$$\frac{\mu(E_n)}{n} \leq \int_X |f| d\mu < \infty \Rightarrow \mu(E_n) < \infty$$

Therefore, $\{x : f(x) \neq 0\}$ is σ -finite. □

8. (a) Observe that

$$\bigcup_{n=1}^{\infty} E_n = \{x \in I : f(x) > 0\} \Rightarrow \lambda(\{x \in I : f(x) > 0\}) \leq \sum_{n=1}^{\infty} \lambda(E_n)$$

So $\lambda(\{x \in I : f(x) > 0\}) > 0$ implies that there $\lambda(E_n) > 0$ for some n .

- (b) This is also obvious. We show that $\lambda(E_n) = 0$ for all $n \geq 1$. Then the inequality derived in part (a) asserts that $\lambda(\{x \in I : f(x) > 0\}) = 0$. Suppose $\lambda(E_n) > 0$ for some n , then we pick a finite set $\{x_1, \dots, x_m\} \subseteq E_n$ where $m = 2Mn$.

$$\sum_{n=1}^m f(x_n) \geq \sum_{n=1}^m \frac{1}{n} = 2M > M$$

However, by assumption, $\sum_{n=1}^m f(x_n) \leq M$. Therefore, $\lambda(E_n) = 0$ holds for all $n \geq 1$.

9. This is a direct application of *Monotone Convergence Theorem*. Let $h_m(x) = \sum_{n=1}^m f(x+n)$.

$$0 \leq h_1(x) \leq h_2(x) \leq \dots \leq h_m(x) \leq h_{m+1}(x) \leq \dots; \quad \lim_{m \rightarrow \infty} h_m(x) = \sum_{n=1}^{\infty} f(x+n) = g(x)$$

Monotone Convergence Theorem says

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}} h_m(x) d\mu = \int_{\mathbb{R}} \lim_{m \rightarrow \infty} h_m(x) d\mu = \int_{\mathbb{R}} g(x) d\mu$$

Let's compute the left hand side,

$$\int_{\mathbb{R}} h_m(x) d\mu = \int_{\mathbb{R}} \sum_{n=1}^m f(x+n) d\mu = \sum_{n=1}^m \int_{\mathbb{R}} f(x+n) d\mu = m \int_{\mathbb{R}} f(x) d\mu$$

Recall that Lebesgue measure is invariant under translation. Combine above two equations,

$$\lim_{m \rightarrow \infty} m \int_{\mathbb{R}} f(x) d\mu = \int_{\mathbb{R}} g(x) d\mu < \infty \Rightarrow \int_{\mathbb{R}} f(x) d\mu = 0$$

Since $f(x)$ is nonnegative, $\int_{\mathbb{R}} f d\mu = 0$ is equivalent to $f = 0$ a.e.

10. (a) This is a immediate result of Cauchy-Schwartz Inequality.

$$\left(\int_B f d\mu \right)^2 = \left(\int_X f \chi_B d\mu \right)^2 \leq \left(\int_X f^2 d\mu \right) \left(\int_X \chi_B^2 d\mu \right) = \mu_B \int_X f^2 d\mu$$

- (b) Let $f = \sum_{k=1}^n \chi_{A_k}$ where χ_{A_k} is a characteristic function of measurable set A_k . Furthermore, let $B = \cup_{k=1}^n A_k$. This inequality holds directly from part (a).

Summer 2018

Analysis Prelim Solution - 2018 Summer

Yiran Zhu — Clemson - Math

1. Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence with $a_n \rightarrow 0, n \rightarrow \infty$. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a_n = 0$$

Proof. For $\epsilon > 0$, there exists a $M \geq 1$ such that $\forall n \geq M, |a_n| \leq \epsilon/2$. For $N \geq M+1$, we have

$$\begin{aligned} \left| \frac{1}{N} \sum_{n=1}^N a_n \right| &= \left| \frac{1}{N} \sum_{n=1}^M a_n + \frac{1}{N} \sum_{n=M+1}^N a_n \right| \leq \frac{1}{N} \left| \sum_{n=1}^M a_n \right| + \frac{1}{N} \sum_{n=M+1}^N |a_n| \\ &\leq \frac{1}{N} \left| \sum_{n=1}^M a_n \right| + \left(\frac{N-M}{N} \right) \frac{\epsilon}{2} \leq \frac{1}{N} \left| \sum_{n=1}^M a_n \right| + \frac{\epsilon}{2} \end{aligned}$$

Let \widehat{N} be an integer greater than $2 \left| \sum_{n=1}^M a_n \right| / \epsilon$ and $\widehat{M} = \max\{\widehat{N}, M+1\}$

$$\forall N \geq \widehat{M}, \quad \left| \frac{1}{N} \sum_{n=1}^N a_n \right| < \epsilon$$

Equivalently, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a_n = 0$. □

2. Let X be a normed linear space and $\emptyset \neq Y \subset X$ be a subset with the property that $X \setminus Y$ is a linear subspace. Show that Y is dense in X .

Proof. Suppose Y is not dense in X . Then there exists a point $z \in X \setminus Y$ and a number $r > 0$ such that $B(z, r) \cap Y = \emptyset$. Equivalently, $B(z, r) \subseteq X \setminus Y$. Then we will show this implies $Y = \emptyset$. Pick $x \in X$ and let $d = \|x - z\|$. Then

$$r > \left\| \frac{r(x - z)}{2d} \right\| = \left\| \frac{rx - (r - 2d)z}{2d} - z \right\| \Rightarrow a := \frac{rx - (r - 2d)z}{2d} \in B(z, r) \subseteq X \setminus Y$$

Since $X \setminus Y$ is a subspace, we have $x = (2da + (r - 2d)z)/r \in X \setminus Y$. Note that x is arbitrarily picked from X . Therefore, $X \subseteq X \setminus Y \subseteq X \Rightarrow Y = \emptyset$. By contradiction, Y is dense in X . □

3. Define $d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ by $d(x, y) = |f(x) - f(y)|$ where f is defined as

$$f(x) = \frac{x}{1 + |x|}, \forall x \in \mathbb{R}$$

Show that d is a metric on \mathbb{R} and determine if (\mathbb{R}, d) is complete.

Proof. (i) Positive-definite: $d(x, y) = |f(x) - f(y)| \geq 0$ and $d(x, y) = 0 \Leftrightarrow f(x) = f(y)$. Note that

$$f(x) = f(y) \Leftrightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|} \Leftrightarrow x(1+|y|) = y(1+|x|)$$

Suppose $x < 0$, then $y < 0$ and $x(1+|y|) = y(1+|x|) \Rightarrow x = y$. Similarly, if $x \geq 0$, then $y \geq 0$ and $x(1+|y|) = y(1+|x|) \Rightarrow x = y$. In a word, $d(x, y) = 0 \Leftrightarrow x = y$.

(ii) Symmetric: $d(x, y) = |f(x) - f(y)| = |f(y) - f(x)| = d(y, x)$

(iii) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ follows from

$$|f(x) - f(z)| = |f(x) - f(y) + f(y) - f(z)| \leq |f(x) - f(y)| + |f(y) - f(z)|$$

So d is a metric. However, (\mathbb{R}, d) is not complete. Consider sequence $\{x_n\}_n$ with $x_n = n$.

$$d(x_n, x_{n+m}) = \left| \frac{m}{(1+n)(1+n+m)} \right| \leq \frac{1}{n+1}, \quad \forall n \geq 1, \forall m \geq 0$$

Therefore, $\{x_n\}_n$ is Cauchy. It's obvious that $\{x_n\}_n$ does not converge in \mathbb{R} . \square

4. Let H be a Hilbert space and Y_1, Y_2 be two closed linear subspaces in H . Denote P_1 and P_2 as the orthogonal projections onto Y_1 and Y_2 , respectively. Show that $\|P_1 - P_2\| \leq 1$.

Proof. Observe that $(2P - I)^2 = 4P^2 + I - 4P = I$ holds for all projection P . In particular, if P is orthogonal, then, for all $h \in H$,

$$\|(2P - I)h\|^2 = \langle (2P - I)h, (2P - I)h \rangle = \langle h, (2P - I)^2 h \rangle = \langle h, h \rangle = \|h\|^2$$

Therefore, $\|2P - I\| = 1 \Rightarrow \|P - \frac{1}{2}I\| = \frac{1}{2}$.

$$\|P_1 - P_2\| \leq \left\| P_1 - \frac{1}{2}I \right\| + \left\| P_2 - \frac{1}{2}I \right\| \leq 1$$

\square

5. Assume $C[0, 1]$ is equipped with the supremum norm and let $T_n : C[0, 1] \rightarrow C[0, 1]$ be defined by

$$T_n(f) = f\left(x^{1+\frac{1}{n}}\right), \quad \forall n \in \mathbb{N}$$

(a) Show that $T_n(f) \rightarrow f, n \rightarrow \infty, \forall f \in C[0, 1]$

Proof. Fix $f \in C[0, 1]$. Since $[0, 1]$ is compact, f is also uniformly continuous on $[0, 1]$, i.e. for $\epsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, y \in [0, 1] \text{ s.t } |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Let's give an estimation for $g_n(x) := \left| x^{1+\frac{1}{n}} - x \right| = x \left(1 - x^{\frac{1}{n}} \right)$. Obviously, $g_n(x)$ is continuous on $[0, 1]$ and $g_n(0) = g_n(1) = 0$. To find the maximum value of $g_n(x)$, we let

$$g'_n(x) = \left(1 + \frac{1}{n} \right) x^{\frac{1}{n}} - 1 = 0 \Rightarrow \sup_{x \in [0, 1]} g_n(x) = \left(\frac{n}{n+1} \right)^n \frac{1}{n+1} \leq \frac{1}{n+1}$$

Pick $N \in \mathbb{Z}^+$ such that $\frac{1}{N+1} < \delta$, then

$$\forall n \geq N, \forall x \in [0, 1] \quad \left| x^{1+\frac{1}{n}} - x \right| < \frac{1}{N+1} < \delta \Rightarrow \|T_n(f) - f\|_\infty < \epsilon$$

Therefore, $T_n(f) \rightarrow f$ as $n \rightarrow \infty$. □

(b) For each $n \in N$, find $\|T_n - I\|$.

Proof. For $f \in C[0, 1]$,

$$\|(T_n - I)f\| = \|T_n(f) - f\|_\infty = \sup_{x \in [0, 1]} |f(x^{1+\frac{1}{n}}) - f(x)| \leq 2\|f\|_\infty$$

So $\|T_n - I\| \leq 2$. Let $x_0 = \frac{1}{2}$ and $x_1 = \left(\frac{1}{2}\right)^{1+\frac{1}{n}} \in (0, x_0)$. Construct a function f as follows

$$f(x) = \begin{cases} -1 & x \in [0, x_1) \\ -1 + \frac{2(x - x_1)}{x_0 - x_1} & x \in [x_1, x_0] \\ 1 & x \in (x_0, 1] \end{cases}$$

Note that $f(x_0) = 1$ and $f(x_1) = -1$. So f is continuous and $\|f\|_\infty = 1$.

$$|(T_n(f) - f)(x_0)| = |f(x_1) - f(x_0)| = 2 = 2\|f\|_\infty$$

As shown before, $\|T_n(f) - f\|_\infty \leq 2\|f\|_\infty$. Thus

$$\|T_n(f) - f\|_\infty = 2\|f\|_\infty \Rightarrow \|T_n - I\| = 2$$

□

6. Assume that λ is the Lebesgue measure on the real line and f a Lebesgue integrable function on the real line. Show that

$$F(x) := \int_{-\infty}^x f d\lambda$$

is uniformly continuous.

Proof. Let $A_n = \{x \in X \mid |f(x)| \geq n\}$. Then, Dominated Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \int_{A_n} |f| d\lambda = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f| \chi_{A_n} d\lambda = 0$$

For $\epsilon > 0$, there exists $N \geq 1$ such that

$$\int_{A_N} |f| d\lambda < \frac{\epsilon}{2}$$

Then

$$\forall x, y \in \mathbb{R}, \quad |F(x) - F(y)| = \left| \int_{-\infty}^x f d\lambda - \int_{-\infty}^y f d\lambda \right| = \left| \int_y^x f d\lambda \right| \leq \int_y^x |f| d\lambda$$

Observe that, if $|x - y| < \frac{\epsilon}{2N}$, then

$$\int_y^x |f| d\lambda = \int_{[x,y] \cap A_N} |f| d\lambda + \int_{[x,y] \setminus A_N} |f| d\lambda \leq \int_{A_N} |f| d\lambda + N|x - y| < \epsilon$$

□

7. Let (X, \mathcal{M}, μ) be a measure space and $\{A_n\}_n$ be a sequence of sets in \mathcal{M} . Recall that $\limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

- (a) Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$

Proof. Observe that

$$\mu \left(\limsup_{n \rightarrow \infty} A_n \right) \leq \mu \left(\bigcup_{k=n}^{\infty} A_k \right) \leq \sum_{k=n}^{\infty} \mu(A_k), \quad \forall n \geq 1$$

Since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, for $\epsilon > 0$, there exists $N \geq 1$ such that

$$\sum_{k=N}^{\infty} \mu(A_k) < \epsilon \Rightarrow \mu \left(\limsup_{n \rightarrow \infty} A_n \right) \leq \epsilon$$

Let $\epsilon \rightarrow 0$, we derive $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$.

□

- (b) Is the converse true? If yes, prove it. If no, give a counter-example.

Proof. Converse is not true. Consider $A_n = [0, 1/n]$. Then $\bigcup_{k=n}^{\infty} A_k = A_n = [0, 1/n]$.

$$\limsup_{n \rightarrow \infty} A_n = \lim_{N \rightarrow \infty} \bigcap_{n=1}^N \bigcup_{k=n}^{\infty} A_k = \lim_{N \rightarrow \infty} \left[0, \frac{1}{N} \right] = \{0\}$$

Therefore,

$$\mu \left(\limsup_{n \rightarrow \infty} A_n \right) = 0$$

However, $\mu(A_n) = \frac{1}{n}$ for all $n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

□

8. Let (X, \mathcal{M}, μ) be a finite measure space. Prove that a monotone increasing sequence of measurable functions $f_n : X \rightarrow \mathbb{R}$ converges in measure if and only if it converges pointwise a.e..

Proof. (i) Suppose f_n converges to f pointwise a.e.: By Egorov theorem, for each $\epsilon > 0$, there exists a measurable set E with measure $\mu(E) < \epsilon$ such that f_n converges uniformly to f on $X \setminus E$. In other words, for each $\delta > 0$, there exists $N \geq 1$ such that

$$\forall n \geq N, \quad |f_n(x) - f(x)| < \delta, \quad \forall x \in X \setminus E$$

Consequently,

$$\forall n \geq N, \quad A_n := \{x \in X \mid |f_n(x) - f(x)| \geq \delta\} \subseteq E \Rightarrow \mu(A_n) < \epsilon$$

Therefore, f_n converges to f in measure.

- (ii) Suppose f_n converges to f in measure: There exists a subsequence $\{f_{n_k}\}_k$ converges to f pointwise a.e.. Let E be the zero-measure set that $\{f_{n_k}\}_k$ does not converge to f . Then fix $x \in X \setminus E$, for each $\epsilon > 0$, there exists $N \geq 1$ such that

$$\forall k \geq N, \quad |f_{n_k}(x) - f(x)| < \epsilon$$

Since $\{f_n\}_n$ is monotone increasing, we have

$$\forall n \geq n_N, \quad |f_n(x) - f(x)| = f(x) - f_n(x) \leq f(x) - f_{n_N}(x) = |f_{n_N}(x) - f(x)| < \epsilon$$

Note that $\{f_{n_k}\}_k$ is also monotone increasing.

□

9. Suppose that g is a non-negative Borel measurable function on \mathbb{R} with $\int_{\mathbb{R}} g d\lambda = 1$ where λ denotes Lebesgue measure on \mathbb{R} . For $k \in \mathbb{N}$ set $g_k(x) = kg(kx)$. Let f be a bounded continuous function. Prove that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} g_k f d\lambda = f(0)$$

Proof. Suppose $\sup_{x \in \mathbb{R}} |f(x)| = M$ and define $h_k(x) = g(x)f(x/k)$. Observe that

$$\int_{\mathbb{R}} g_k f d\lambda = \int_{\mathbb{R}} kg(kx)f(x)d\lambda = \int_{\mathbb{R}} g(x)f(x/k)d\lambda = \int_{\mathbb{R}} h_k d\lambda$$

In order to apply Dominated Convergence Theorem, we need to show h_k is uniformly bounded by a integrable function. Indeed,

$$\forall k \geq 1, \quad |h_k(x)| = |g(x)f(x/k)| \leq M g(x) \quad \text{and} \quad \int_{\mathbb{R}} M g d\lambda = M < \infty$$

By DCT,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} h_k d\lambda = \int_{\mathbb{R}} \lim_{k \rightarrow \infty} h_k d\lambda = \int_{\mathbb{R}} f(0) g d\lambda = f(0)$$

□

10. Let λ be the Lebesgue measure on $(0, 1)$. Suppose the $f_n : (0, 1) \rightarrow [0, \infty)$ is a sequence of Borel measurable functions such that $\int_{(0,1)} f_n d\lambda = 1$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} f_n(x) = x$ for all $x \in (0, 1)$.

(a) Give an example of such a sequence.

Proof.

$$f_n(x) = \begin{cases} (n+1)(1-nx) & x \in \left(0, \frac{1}{n}\right) \\ \frac{n}{n-1} \left(x - \frac{1}{n}\right) & x \in \left[\frac{1}{n}, 1\right) \end{cases}$$

Then

$$\int_{(0,1)} f_n d\lambda = \int_{(0, \frac{1}{n})} (n+1)(1-nx) d\lambda + \int_{[\frac{1}{n}, 1)} \frac{n}{n-1} \left(x - \frac{1}{n}\right) d\lambda = \frac{n+1}{2n} + \frac{n-1}{2n} = 1$$

Fix $x \in (0, 1)$, there exists $N \geq 1$ such that $x > \frac{1}{N}$, then

$$\forall n \geq N, \quad f_n(x) - x = \frac{n}{n-1} \left(x - \frac{1}{n}\right) - x = \frac{x-1}{n-1}$$

So $f_n(x) \rightarrow x$ as $n \rightarrow \infty$.

□

- (b) Show that one can find $n \geq 1$ and $x \in (0, 1)$ such that $f_n(x)\sqrt{x} \geq 2018$

Proof. Prove by contradiction. Suppose not, then

$$\forall n \geq 1, \quad f_n(x) \leq \frac{2018}{\sqrt{x}}$$

Note that

$$\int_{(0,1)} \frac{2018}{\sqrt{x}} d\lambda = 4036 < \infty$$

By Dominated Convergence Theorem,

$$1 = \lim_{n \rightarrow \infty} \int_{(0,1)} f_n d\lambda = \int_{(0,1)} \lim_{n \rightarrow \infty} f_n d\lambda = \int_{(0,1)} x d\lambda = \frac{1}{2}$$

Above equality cannot be true.

□