

Clemson Analysis Prelim Solutions

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Winter 2010

1. (a) *Proof.* If E is bounded, E is pre-compact since \mathbb{R} is finite (one) dimensional. If $f(E)$ is unbounded, then there exists $\{x_n\} \subseteq E$ such that $f(x_n) \rightarrow \infty$. Since E is precompact $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ with limit $x \in \mathbb{R}$. Then, since f is continuous,

$$f(x) = \lim_{k \rightarrow \infty} f(x_{n_k}) = \infty$$

However, since f maps \mathbb{R} to \mathbb{R} , $f(x)$ cannot be ∞ . □

- (b) *Proof.* Since f is uniformly continuous, there exists $\delta > 0$ such that whenever $|x - y| < \delta$,

$$|f(x) - f(y)| < 1$$

Since E bounded, it can be covered by finitely many balls of radius δ , say $\{B(x_i, \delta)\}_{i=1}^N$. Then,

$$f(E) = \cup_{i=1}^N f(B(x_i, \delta))$$

Fix i , for any $f(y) \in f(B(x_i, \delta))$,

$$|f(y) - f(x_i)| \leq 1$$

So $f(B(x_i, \delta))$ is bounded. Then, a finite union of bounded sets is also bounded. □

Counterexample: $E = (0, 1)$ and $f(x) = 1/x$. $f(E) = (1, \infty)$.

3. (a) *Proof.* Recall the Bessel inequality for any orthonormal set $\{e_n\}$ in an inner product space, X . For and $f \in X$,

$$\sum |\langle f, e_n \rangle|^2 \leq \|f\|^2$$

In particular, $\langle f, e_n \rangle \rightarrow 0$ as $n \rightarrow \infty$ for any $f \in X$. Now, since

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{2\pi}} \right\}$$

form an orthonormal set in $C[-\pi, \pi]$, we have

$$\int_{-\pi}^{\pi} \sin(2nx) f(x) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any $f \in C[-\pi, \pi]$. Then,

$$\int_{-\pi}^{\pi} \sin^2(nx) f(x) dx = \frac{1}{2} \int_{-\pi}^{\pi} f(x) dx - \frac{1}{2} \int_{-\pi}^{\pi} \sin(2nx) f(x) dx \rightarrow \frac{1}{2} \int_{-\pi}^{\pi} f(x) dx$$

□

(b) *Proof.* For any $f \in C[-\pi, \pi]$, $n \in \mathbb{N}$,

$$\begin{aligned} \left| \int_{-\pi}^{\pi} \frac{x^n}{\pi^n} f(x) dx \right|^2 &\leq \int_{-\pi}^{\pi} \frac{x^{2n}}{\pi^{2n}} dx \int_{-\pi}^{\pi} |f(x)|^2 dx \\ &= \frac{\pi^{2n+1} - (-\pi)^{2n+1}}{(2n+1)\pi^{2n}} \|f\|_{L^2}^2 \\ &= \frac{2\pi}{2n+1} \|f\|_{L^2}^2 \end{aligned}$$

which goes to 0 as $n \rightarrow \infty$. □

9. (a) There exists $\varepsilon_n \searrow 0$ such that

$$\mu\{|f - g| \geq \varepsilon_n\} \leq \varepsilon_n$$

Then,

$$\begin{aligned} \mu\{f \neq g\} &= \mu\{|f - g| > 0\} = \mu\left(\bigcup_n \{|f - g| \geq \varepsilon_n\}\right) \\ &= \lim_{n \rightarrow \infty} \mu\{|f - g| \leq \varepsilon_n\} \leq \lim_{n \rightarrow \infty} \varepsilon_n = 0 \end{aligned}$$

(b) We only need to show the triangle inequality. Let $t, s > 0$, f, g, h measurable functions.

If $|f - h| \leq t$ and $|g - h| \leq s$, then

$$|f - g| \leq |f - h| + |g - h| \leq t + s$$

Thus, $\{|f - h| \leq t\} \cap \{|g - h| \leq s\} \subset \{|f - g| \leq t + s\}$. Then, taking complements, we have

$$\{|f - h| > t\} \cup \{|g - h| > s\} \supset \{|f - g| > t + s\}$$

Therefore, $\mu\{|f - g| > t + s\} \leq \mu\{|f - h| > t\} + \mu\{|g - h| > s\}$. Let $\delta > 0$. There exists $\varepsilon_1, \varepsilon_2$ such that

$$\mu\{|f - h| > \varepsilon_1\} < \varepsilon_1 \text{ and } \varepsilon_1 < \rho(f, h) + \delta/2$$

and similarly for ε_2 and $|g - h|$. Therefore,

$$\mu\{|f - g| > \varepsilon_1 + \varepsilon_2\} \leq \mu\{|f - h| > \varepsilon_1\} + \mu\{|g - h| > \varepsilon_2\} < \varepsilon_1 + \varepsilon_2$$

So,

$$\begin{aligned} \rho(f, g) &= \inf\{\varepsilon : \mu\{|f - g| > \varepsilon\} < \varepsilon\} \\ &\leq \varepsilon_1 + \varepsilon_2 \\ &\leq \rho(f, h) + \rho(g, h) + \delta \end{aligned}$$

for any $\delta > 0$. This proves the Triangle Inequality.

Summer 2010

3. (a)

$$\|Tf\|_\infty = \sup_{x \in [0,1]} |x^2 f(x)| \leq \sup_{x \in [0,1]} |f(x)| = \|f\|_\infty$$

(b) For $f \equiv 1$, $\|Tf\| = 1$ and $\|f\| = 1$.

(c) By triangle inequality, $\|(I+T)f\|_\infty \leq \|f\|_\infty + \|Tf\|_\infty \leq 2\|f\|$. So, we only need to show $\|I+T\| = 2$. Again, this follows from taking $f \equiv 1$.

$$\|(I+T)f\| = \sup_{x \in [0,1]} |1+x^2| = 2$$

4. (a) We will actually show more, namely that

$$\|x\|_q \leq \|x\|_p \quad \text{for } 1 \leq p < q < \infty$$

Proof. Let $x \in \ell^p$. Define

$$y = \frac{x}{\|x\|_p}$$

Then, $|y_i| \leq 1$ for every $i \in \mathbb{N}$. This implies $|y_i|^q \leq |y_i|^p$ for all i . Therefore,

$$\|y\|_q^q = \sum |y_i|^q \leq \sum |y_i|^p = \sum \frac{|x_i|^p}{\|x\|_p^p} = \frac{\sum |x_i|^p}{\sum |x_i|^p} = 1$$

so $\|y\|_q \leq 1$. But this implies

$$\frac{\|x\|_q}{\|x\|_p} = \left\| \frac{x}{\|x\|_p} \right\|_q = \|y\|_q \leq 1$$

□

5. (a) Let $f \in L^q$, $1 \leq p < q < \infty$. we will apply holder inequality with exponent $q/p > 1$.

$$\int |f|^p \leq \left(\int |f|^q \right)^{p/q} \left(\int 1 \right)^{1-p/q} = \left(\int |f|^q \right)^{p/q} (\mu(X))^{1-p/q} < \infty$$

(b) Take

$$f(x) = \begin{cases} 0 & x = 0 \\ x^{-1/2} & 0 < |x| \leq 1 \\ 0 & |x| < 1 \end{cases}$$

$$g(x) = \begin{cases} x^{-1} & |x| \geq 1 \\ 1 & |x| < 1 \end{cases}$$

6. There are many options for f_n :

$$n^2\chi_{[0,1/n]} \quad n\chi_{[n,n+1]} \quad \chi_{[n,2n]}$$

Then, just take

$$g_n = \frac{1}{n}f_n$$

and

$$h_n = (-1)^n g_n$$

7.

Winter 2012

Summer 2012

4. (a) We will show K is totally bounded. Let $\epsilon > 0$. For $x, y \in K$,

$$d_S(x, y) = \sum_i \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)} \leq \sum_i \frac{2}{2^i} < \infty$$

So, there exists N such that

$$\sum_{i=N+1}^{\infty} \frac{1}{2^{i-1}} < \epsilon/2$$

Consider the set $M = \{x \in K : x_i = 0, i > N\} \subseteq K$. M is compact since it is finite dimensional and bounded. So, there is an $\epsilon/2$ net for M which will be an ϵ net for K .

- (b) False. Consider the sequence $\{e_n\} \subseteq K$, which is entirely zero except the n -th entry. For $n \neq m$,

$$d_{\infty}(e_n, e_m) = 1$$

so this sequence cannot have a convergent subsequence.

5. (a) Define $g_n = \sum_{k=1}^n f_k$. Since f_k are non-negative, $\{g_n\}$ is monotone. Moreover,

$$g_n \nearrow \sum_{k=1}^{\infty} f_k$$

Then, by Monotone Convergence Theorem,

$$\sum_{k=1}^{\infty} \int f_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int f_k = \lim_{n \rightarrow \infty} \int g_n = \int \sum_{k=1}^{\infty} f_k$$

- (b) By part (a),

$$\infty > \sum \int |f_k - f| = \int \sum |f_k - f|$$

Since $\sum |f_k - f|$ is integrable, it is finite almost everywhere. Let $E \subseteq \mathbb{R}$ be the set of measure zero where it may not be finite. Let $x \notin E$. Let $\epsilon > 0$. There exists N such that

$$|f_n(x) - f(x)| \leq \sum_{k=N}^{\infty} |f_k(x) - f(x)| < \epsilon$$

for all $n \geq N$.

7. See Winter 15 #9

Winter 2013

8. See Summer 13 #5

Summer 2013

3. (a) B is complete. Let $\{f_n\} \subseteq C[0, 1]$ be a Cauchy sequence in ρ_∞ . Since $(C[0, 1], \rho_\infty)$ is a complete metric space, there exists $f \in C[0, 1]$ such that $f_n \rightarrow f$ in ρ_∞ . Now, we claim that $f \in B$. For any $\epsilon > 0$ there exists N such that

$$\rho(f, f_n) < \epsilon \quad \forall n \geq N$$

Then,

$$\sup_{0 \leq t \leq 1} |f(t)| = \rho(f, 0) \leq \rho(f, f_n) + \rho(f_n, 0) < \epsilon + 1$$

but $\epsilon > 0$ was arbitrary so

$$\sup_{0 \leq t \leq 1} |f(t)| \leq 1$$

- (b) Consider the spike functions, $\{f_n\}$. For $n \neq m$,

$$\rho(f_n, f_m) = 1$$

so there cannot be a convergent subsequence.

4. (a) Let $f \in L^2(\mu)$. Then, using the Cauchy-Schwarz inequality, we compute

$$\begin{aligned} \|Af\|_{L^2(\mu)}^2 &= \int_X \left(\int_X K(x, y) f(y) d\mu(y) \right)^2 d\mu(x) \\ &\leq \int_X \left(\int_X K(x, y)^2 d\mu(y) \right) \left(\int_X f(y)^2 d\mu(y) \right) d\mu(x) \\ &= \left(\int_X f(y)^2 d\mu(y) \right) \int_X \left(\int_X K(x, y)^2 d\mu(y) \right) d\mu(x) = \|f\|_{L^2(\mu)}^2 \|K\|_{L^2(\mu \times \mu)}^2 \end{aligned}$$

Therefore $\|A\| \leq \|K\|_{L^2(\mu \times \mu)}$.

- (b) First we note that the correspondence $K \mapsto A$ is linear due to the linearity of the integral. So, it suffices to prove the following: Let $K \in L^2(\mu \times \mu)$ such that for any $f \in L^2(\mu)$,

$$\int_X K(x, y) f(y) d\mu(y) = 0$$

for almost every $x \in X$ (w.r.t μ). Then, $K = 0$ a.e. To prove this, suppose there exists $E \subseteq X \times X$ such that E has positive $\mu \times \mu$ measure in the sense that

$$(\mu \times \mu)(E) := \int_X \int_X \mathbf{1}_E(x, y) d\mu(x) d\mu(y) > 0$$

So we define the measure $\mu \times \mu$ on the cylinder $X \times X$ in this way.

5. ($a \Rightarrow b$) Let $x = \hat{m} + e$ where \hat{m} is the closest point to x in M . Let $y \in M$ non-zero. For any $t \in \mathbb{C}$, $ty \in M$ so

$$\|x - \hat{m}\|^2 \leq \|e - ty\|^2 = \|e\|^2 - 2\operatorname{Re}\langle e, ty \rangle + \|ty\|^2$$

which implies

$$\operatorname{Re} \bar{t} \langle e, y \rangle \leq |t|^2 \|y\|^2$$

Take $t = \overline{\langle e, y \rangle} \|y\|^{-2}$. Then we have

$$\frac{|\langle e, y \rangle|^2}{\|y\|^2} \leq \frac{|\langle e, y \rangle|^2}{2\|y\|^2}$$

therefore $\langle e, y \rangle = 0$.

($b \Rightarrow a$) Let $x = \hat{m} + e$ for $\hat{m} \in M$ and $e \in M^\perp$. For any $y \in M$,

$$\|x - y\|^2 = \|e - (y - \hat{m})\|^2 = \|e\|^2 - 2\operatorname{Re}\langle e, y - \hat{m} \rangle + \|y - \hat{m}\|^2 = \|e\|^2 + \|y\|^2 \geq \|e\|^2 = \|x - \hat{m}\|^2$$

Suppose \tilde{m} is another closest point. Set $d = d(x, M)$. By the parallelogram identity,

$$\|\tilde{m} - \hat{m}\|^2 = \|x - \tilde{m} - (x - \hat{m})\|^2 = 2d^2 + 2d^2 - \|2x - 2(\tilde{m} + \hat{m})\|^2 \leq 4d^2 - 4d^2 = 0$$

6.

$$f_n = n \mathbf{1}_{[0, 1/n]}$$

7. (a) We will show that for $h \geq 0$ measurable, $\int h = 0 \implies h = 0$ a.e. Indeed, consider $A_n = \{n^{-1} > h \geq (n+1)^{-1}\}$ for each $n \in \mathbb{N}$ and $A_0 = \{h \geq 1\}$. Then, for any $n \in \mathbb{N} \cup \{0\}$,

$$(n+1)^{-1} \mu(A_n) \leq \int_{A_n} h \, d\mu \leq \int_{\mathbb{R}} h \, d\mu = 0$$

Therefore $\mu(A_n) = 0$. So, $\{h \neq 0\} = A = \cup A_n$ which has measure zero.

Now, we apply this to the problem by taking $h = g - f$. Then, $h \geq 0$ and

$$\int h = \int g - f = \int g - \int f = 0$$

Then by the above lemma, $h = 0$ so $f = g$ a.e.

- (b) If f and g are continuous, then being equal almost everywhere will imply they are equal everywhere. Indeed, suppose there exists $x_0 \in \mathbb{R}$ such that $f(x_0) < g(x_0)$. Then, $f - g$ is continuous around x_0 so there exists $\epsilon > 0$ such that

$$f(x) < g(x) \quad \text{for } |x - x_0| < \epsilon$$

However, $\lambda(\{|x - x_0| < \epsilon\}) = \epsilon$ so $f \neq g$ on a set of measure ϵ which contradicts $f = g$ a.e.

(c) INCOMPLETE

8. (a) First, \mathcal{M} is clearly a linear space since linear combinations of finite signed measures are still finite signed measures. Now we show that total variation is a norm on \mathcal{M} . If μ has total variation 0, this means

$$\mu_+(X) = \mu_-(X) = 0$$

so X is a null set of both μ_+ and μ_- . Thus for every $E \subseteq X$, $\mu(E) = \mu_+(E) - \mu_-(E) = 0 - 0 = 0$. Thus μ is the zero measure. By definition, $|\alpha\mu| = \alpha\mu_+ + \alpha\mu_- = \alpha|\mu|$. Finally, to check the triangle inequality, let $\mu, \lambda \in \mathcal{M}$. Let $A \cup B = X$ be a Hahn decomposition of X with respect to the signed measure $(\mu + \lambda)$. Then,

$$(\mu + \lambda)_+(X) = \mu(A) + \lambda(A) \leq \mu_+(A) + \lambda_+(A) \leq \mu_+(X) + \lambda_+(X)$$

and

$$(\mu + \lambda)_-(X) = -\mu(B) - \lambda(B) \leq \mu_-(B) + \lambda_-(B) \leq \mu_-(X) + \lambda_-(X)$$

Therefore

$$\begin{aligned} |\mu + \lambda|(X) &= (\mu + \lambda)_+(X) + (\mu + \lambda)_-(X) \leq \mu_+(X) + \lambda_+(X) + \mu_-(X) + \lambda_-(X) \\ &= |\mu|(X) + |\lambda|(X) \end{aligned}$$

- (b) Let μ be a σ -finite measure. Clearly $\mathcal{L}_\nu = \{\mu \in \mathcal{M} : \mu \ll \nu\}$ is a linear subspace since if $\lambda, \mu \ll \nu$, and $\nu(E) = 0$, then

$$\alpha\lambda(E) + \beta\mu(E) = 0$$

for any scalars α, β . We note the crucial property of this subspace. If $\mu \ll \nu$, then the null sets of ν are also null sets of μ_+ and μ_- . Indeed, let $E \subset X$ such that $\nu(E) = 0$. Let $A \cup B$ be a Hahn decomposition for μ . Then,

$$\nu(A \cap E) \leq \nu(E) = 0 \quad \nu(B \cap E) \leq \nu(E) = 0$$

so $\nu(A \cap E) = \nu(B \cap E) = 0$. Therefore

$$\mu_+(E) = \mu(A \cap E) = 0 \quad \mu_-(E) = \mu(B \cap E) = 0$$

Now, let $\{\mu_n\} \subseteq \mathcal{L}_\nu$ converge to μ in the total variation norm. Then, let $E \subset X$ such that $\nu(E) = 0$. Then, $|\mu_n|(E) = 0$. By the reverse triangle inequality (a consequence of the triangle inequality for $\|\cdot\|$ shown above)

$$|\mu|(E) = ||\mu|(E) - |\mu_n|(E)|| \leq |\mu - \mu_n|(E) \leq |\mu - \mu_n|(X) = \|\mu - \mu_n\| \rightarrow 0$$

so $\mu(E) = 0$ and $\mu \in \mathcal{L}_\nu$.

- (c) Let $f \in L^1(X, \mathcal{F}, \nu)$. Then,

$$\mu(A) = \int_A f d\nu$$

defines a signed measure for $A \subseteq X$. We only need to check that this pairing is isometric and onto. Surjectivity follows from the Radon-Nikodym theorem which states that if $\rho \ll \lambda$, then there exists λ -measurable g such that

$$\rho = g d\lambda$$

Then, to check the norms are preserved, we first show that the Hahn decomposition of μ corresponds to the positive and negative parts of f . Indeed, let $A = \{f \geq 0\}$. Then, for any $E \subseteq A$,

$$\mu(E) = \int_E f d\nu \geq 0$$

Similarly, for $B = \{f < 0\}$, $F \subseteq B$,

$$\mu(F) = \int_F f d\nu \leq 0$$

So $A \cup B$ is a Hahn decomposition for μ . Therefore,

$$\begin{aligned} \int_X |f| d\nu &= \int_X f^+ + f^- d\nu = \int_X f^+ d\nu + \int_X f^- d\nu = \int_A f^+ d\nu + \int_B f^- d\nu \\ &= \mu_+(A) + \mu_-(B) = \mu_+(X) + \mu_-(X) = |\mu|(X) \end{aligned}$$

9.

We have shown many times before that if $\sum \lambda(E_n) < \infty$, then $\lambda(\limsup_n E_n) = 0$. Set

$$E_n = [r_n - 2^{-n-1}, r_n + 2^{-n-1}]$$

Then,

$$\sum_n \lambda(E_n) = \sum_n 2^{-n} < \infty$$

Set $E = \limsup_{n \rightarrow \infty} E_n$. Then, $\lambda(E) = 0$. So, for any $x \notin E$, we have that there exists $k \in \mathbb{N}$ such that $x \notin E_n$ for all $n \geq k$. Therefore, $f(x)$ is only nonzero for finitely many indices so the sum must converge at x .

(b) Set

$$X_n = \left(\bigcup_{k \neq n} E_k \right)^c$$

Then, $\mathbb{R} = \cup X_n$ and, $\mu(X_n) \leq 1$ since $f_k = 0$ on X_n for $n \neq k$. Indeed,

$$\mu(X_n) = \int_{X_n} \sum f_k d\lambda = \int_{X_n} f_n \leq \int f_n = 1$$

(c) To show $\mu \ll \lambda$, let $E \subset \mathbb{R}$ such that $\lambda(E) = 0$. Then, integration over a set of measure zero is also zero so $\mu(E) = 0$.

- (d) Without loss of generality, we can just show that each open ball has infinite measure since every open set contains an open ball. Let $B(x, \epsilon) \subseteq \mathbb{R}$. Then, there exists a subsequence of $\{r_n\}$ such that $\{r_{n_k}\} \subseteq B(x, \epsilon/2)$. Moreover, since the radii of E_{n_k} are decreasing, there exists N such that $E_{n_k} \subseteq B(x, \epsilon)$ for all $k \geq N$. Thus,

$$\mu(B(x, \epsilon)) = \int_{B(x, \epsilon)} \sum f_n d\lambda \geq \int_{B(x, \epsilon)} \sum_{k=N}^{\infty} f_{n_k} d\lambda = \sum_{k=N}^{\infty} \int_{B(x, \epsilon)} f_{n_k} \geq \sum_{k=N}^{\infty} \int_{E_{n_k}} f_{n_k} = \infty$$

Winter 2014

8. See Summer 13 #9 (a)
9. See Summer 13 #7
10. See Summer 13 #9

Summer 2014

1. (a) *Proof.* Suppose f is discontinuous at some $x \in (0, 1)$. There there exists $\epsilon > 0$ such that $\forall \delta > 0$ there exists $y \in B(x, \delta)$ such that

$$|f(x) - f(y)| \geq \epsilon$$

However, consider the compact interval $[x - \gamma, x + \gamma] \subseteq (0, 1)$ for some $\gamma > 0$. Then, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \epsilon/3$$

for all $z \in [x - \gamma, x + \gamma]$, $n \geq N$. □

- (b) False. Let

$$f_n(x) = \frac{1}{x} + \frac{1}{n} \quad f(x) = \frac{1}{x}$$

Then, $f_n \rightarrow f$ uniformly on $(0, 1)$ but f is not uniformly continuous.

- (c) *Proof.* □

9. See Summer 13 #7

Winter 2015

1. *Proof.* For $u = 1 + n^2x^2$, $du = 2n^2x dx$,

$$\|f_n - 0\|_1 = \int_0^1 \frac{nx}{1 + n^2x^2} dx = \int_1^2 \frac{du}{2nu} = \frac{1}{2n} [\ln(2) - \ln(1)] \rightarrow 0$$

as $n \rightarrow \infty$. Therefore $f_n \rightarrow 0$ in $L^1[0, 1]$. Now, since $\|\cdot\|_\infty$ and $\|\cdot\|_{\sup}$ coincide on continuous functions,

$$\|f_n - 0\|_\infty = \sup_{x \in [0, 1]} |f_n(x)| \geq f_n(n^{-3/2}) = \frac{n^{-1/2}}{1 + n^{-1}} \rightarrow \infty$$

as $n \rightarrow \infty$. So $f_n \not\rightarrow 0$ in $L^\infty[0, 1]$. □

2. *Proof.* Let f be convex. Let $x_n \searrow x$. Then, define $t_n \in [0, 1]$ by

$$(1 - t_n)1 + t_nx = x_n$$

Notice that $t_n \rightarrow 1$ as $n \rightarrow \infty$. Then,

$$f(x_n) = f(t_nx + (1 - t_n)1) \leq t_nf(x) + (1 - t_n)f(1)$$

also define $s_n \in [0, 1]$ such that

$$(1 - s_n)(-1) + s_nx_n = x$$

then $s_n \rightarrow 1$ as $n \rightarrow \infty$ so

$$f(x) = f(s_nx_n + (1 - s_n)(-1)) \leq s_nf(x_n) + (1 - s_n)f(-1)$$

Combining this, we get

$$f(x) \leq s_nf(x_n) + (1 - s_n)f(-1) \leq s_nt_nf(x) + s_n(1 - t_n)f(1) + (1 - s_n)f(-1)$$

Notice that the RHS converges to $f(x)$ as $n \rightarrow \infty$ so by the Squeeze theorem

$$\lim_{n \rightarrow \infty} f(x_n) = f(x)$$

So f is right continuous. To show left continuity, we follow the same steps but modify t_n and s_n so they are convex combinations with the opposite endpoints. Therefore f is continuous. □

3. *Proof.* For each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$d(x_n, f(x_n)) < \frac{1}{n}$$

Since X is compact, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$ with limit x . Then,

$$d(x, f(x)) \leq d(x, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) + d(f(x_{n_k}), f(x)) \rightarrow 0$$

as $k \rightarrow \infty$ by construction of x_{n_k} and since f is continuous. Thus $f(x) = x$. □

4. (a) *Proof.* Let $\{y_n\} \subseteq Y$ be Cauchy. There exists $\{x_n\} \subseteq X$ such that $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is Cauchy and therefore convergent to some $x \in X$. Then,

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} f(x_n) = f(x) \in Y$$

□

- (b) False. Let $X = (0, 1)$, $Y = \mathbb{R}$. Let $d_Y = d_X = |(\cdot) - (\cdot)|$. Let $f(x) = 1/x$. Then, clearly

$$|x_1 - x_2| \leq \left| \frac{x_1}{x_1 x_2} - \frac{x_2}{x_1 x_2} \right| = |f(x_2) - f(x_1)|$$

so the inequality holds. Additionally, Y is complete but X is not.

5. *Proof.*

$$\|S(a)\|_2 = \sqrt{\sum_{n=1}^{\infty} s_n^2 a_n^2} \leq \|s\|_{\infty} \|a\|_2$$

For each $k \in \mathbb{N}$, there exists $s_{n_k} \in s$ such that

$$|s_{n_k}| > \|s\|_{\infty} - \frac{1}{k}$$

Then, consider $e_{n_k} = (0, \dots, 0 \overset{n_k}{1}, 0, \dots) \in \ell^2$. $\|e_{n_k}\|_2 = 1$ so

$$\|S(e_{n_k})\|_2 = |s_{n_k}| > \|s\|_{\infty} - \frac{1}{k}$$

for all $k \in \mathbb{N}$ thus

$$\|S\| = \|s\|_{\infty}$$

□

6. *Proof.* First, notice that T is bounded below:

$$\|x\|^2 \leq \langle Tx, x \rangle \leq \|Tx\| \cdot \|x\|$$

so, $\|Tx\| \geq \|x\|$ for all $x \in \mathcal{H}$. Now, we show one-to-one. Let $x \in \mathcal{H}$ such that $Tx = 0$. Then,

$$0 = \|Tx\| \geq \|x\|_{\mathcal{H}} \geq 0$$

so $x = 0$. Next, we show T has a closed range. Let $x_n \in \mathcal{H}$ such that $Tx_n \rightarrow y$ for some $y \in \mathcal{H}$. Then,

$$\|Tx_n - Tx_m\| \geq \|x_n - x_m\|$$

for all $n, m \in \mathbb{N}$. So, $\{x_n\}$ is Cauchy. Thus, there exists $x \in \mathcal{H}$ such that $x_n \rightarrow x$. Since T is bounded,

$$y = \lim_{n \rightarrow \infty} Tx_n = Tx$$

so $y \in \text{Ran} T$. Finally, we show T is onto. For $w \in (\text{Ran} T)^{\perp}$

$$\langle Tv, w \rangle = 0$$

for all $v \in \mathcal{H}$. In particular, for $v = w$,

$$0 = \langle Tw, w \rangle \geq \|w\|^2 \geq 0$$

which implies $w = 0$. Thus, $(\text{Ran} T)^\perp = \{0\}$ so $\text{Ran} T = \overline{\text{Ran} T} = \mathcal{H}$. We have T is one-to-one and onto therefore is it invertible so $Tx = y$ has a unique solution for every $y \in \mathcal{H}$. \square

7. Solution by Hao Chen and Walton Green (4/18)

Proof. We will prove the contrapositive of the statement. Suppose $\{E_k\}_{k=1}^n$ are Borel subsets of $[0, 1]$ such that

$$\lambda\left(\bigcap_{k=1}^n E_k\right) = 0$$

Then, we have that

$$1 = \lambda([0, 1]) = \lambda\left[\left(\bigcap_{k=1}^n E_k\right)^c\right] = \lambda\left(\bigcup_{k=1}^n E_k^c\right)$$

Therefore,

$$n = \sum_{k=1}^n \lambda([0, 1]) = \sum_{k=1}^n \lambda(E_k) + \lambda(E_k^c) \geq \sum_{k=1}^n \lambda(E_k) + \lambda\left(\bigcup_{k=1}^n E_k^c\right) = \sum_{k=1}^n \lambda(E_k) + 1$$

$$\text{so } \sum_{k=1}^n \lambda(E_k) \leq n - 1. \quad \square$$

8. Let $\{q_n\}_{n=1}^\infty \subseteq \mathbb{R}$ be an enumeration of the rational numbers. Then, let

$$U := \bigcup_{n=1}^\infty \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$$

So,

$$\lambda(U) \leq \sum_{n=1}^\infty \frac{2}{n^2} = 2 < \infty$$

and $U \subseteq \mathbb{R}$ is open. Now, notice that $\bar{U} = \mathbb{R}$ since \mathbb{Q} is dense in \mathbb{R} and $\mathbb{Q} \subseteq U$. Then,

$$\lambda(\partial U) = \lambda(\bar{U} \setminus U) = \infty$$

9. *Proof.* (\Rightarrow) let $\lambda(E) = M > 0$. Let $f_n \xrightarrow{\lambda} 0$. Then, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\inf \{c > 0 : \lambda\{|f_n| > c\} < \epsilon\} < \epsilon$$

for all $n \geq N$ which implies

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

Now, we will use the fact that

$$x \mapsto \frac{x}{x+1}$$

is monotone increasing and ≤ 1 .

$$\begin{aligned} \int_E \frac{|f_n|}{1+|f_n|} &= \int_{E \cap \{|f_n| > \epsilon\}} \frac{|f_n|}{1+|f_n|} + \int_{E \cap \{|f_n| < \epsilon\}} \frac{|f_n|}{1+|f_n|} \\ &\leq \int_{E \cap \{|f_n| > \epsilon\}} 1 + \int_E \frac{\epsilon}{1+\epsilon} \\ &\leq \lambda\{|f_n| > \epsilon\} + \lambda(E) \left(\frac{\epsilon}{1+\epsilon} \right) \\ &< \epsilon + M \left(\frac{\epsilon}{1+\epsilon} \right) \rightarrow 0 \end{aligned}$$

as $\epsilon \rightarrow 0$.

(\Leftarrow) Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$\frac{\epsilon^2}{1+\epsilon} > \int_E \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n| > \epsilon\}} \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n| > \epsilon\}} \frac{\epsilon}{1+\epsilon} = \lambda\{|f_n| > \epsilon\} \frac{\epsilon}{1+\epsilon}$$

so

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

for all $n \geq N$. Thus,

$$\|f_n\|_\lambda = \inf\{c > 0 : \lambda\{|f_n| > c\} < c\} < \epsilon$$

□

10. *Proof.* Define

$$F_i := \bigcup_{n=i}^{\infty} E_n$$

for each $i \in \mathbb{N}$. Notice that F_i are reverse nested (i.e. $F_{i+1} \subseteq F_i$ therefore $F_i^c \subseteq F_{i+1}^c$)
Then,

$$\mu(F_i) = \mu\left(\bigcup_{n=i}^{\infty} E_n\right) \leq \sum_{n=i}^{\infty} \mu(E_n) \rightarrow 0$$

as $i \rightarrow \infty$. Now,

$$\begin{aligned} \mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n\right) &= \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \mu\left(F_1 \cap \bigcap_{k=2}^{\infty} F_k\right) = \mu\left(F_1 \setminus \bigcup_{k=2}^{\infty} F_k^c\right) \\ &= \mu(F_1) - \mu\left(\bigcup_{k=2}^{\infty} F_k^c\right) = \mu(F_1) - \lim_{k \rightarrow \infty} \mu(F_k^c) \\ &= \lim_{k \rightarrow \infty} \mu(F_1 \setminus F_k^c) = \lim_{k \rightarrow \infty} \mu(F_1 \cap F_k) \\ &= \lim_{k \rightarrow \infty} \mu(F_k) = 0 \end{aligned}$$

□

Summer 2015

1. (a) *Proof.* Let $\epsilon > 0$, pick $N \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all $n \geq N$. This can be done since $\sum M_n < \infty$. Now, for all $x \in \mathbb{R}$,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all $n \geq N$. Therefore

$$\sum_{k=1}^{\infty} f_k(x)$$

is uniformly convergent. □

- (b) Define

$$f_n(x) := \begin{cases} \frac{1}{n} & n \leq x < n+1 \\ 0 & \text{otherwise} \end{cases} \quad \forall n \in \mathbb{N}$$

Then, clearly $\sum f_n(x)$ is convergent pointwise and

$$\sum_{n=1}^{\infty} \|f_n\|_{\infty} \leq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Now we need to show this convergence is actually uniform. Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then, for all $x \in \mathbb{R}$,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \frac{1}{n+1} \leq \frac{1}{N} < \epsilon$$

for all $n \geq N$.

2. *Proof.* First we show $(A^{\perp})^{\perp}$ is a closed subspace containing A . Clearly $A \subset (A^{\perp})^{\perp}$. Let $x, y \in (A^{\perp})^{\perp}$ and $a, b \in \mathbb{C}$. Then,

$$\langle ax + by | z \rangle = a \langle x | z \rangle + b \langle y | z \rangle = 0 + 0 = 0 \quad \forall z \in A^{\perp}$$

Let $\{x_n\}_{n=1}^{\infty} \subset (A^{\perp})^{\perp}$ such that $x_n \rightarrow x$.

$$\langle x | z \rangle = \lim_{n \rightarrow \infty} \langle x_n | z \rangle = \lim_{n \rightarrow \infty} 0 = 0 \quad \forall z \in A^{\perp}$$

So, we have shown $\overline{\text{span} A} \subset (A^{\perp})^{\perp}$. Now, let $x \in (A^{\perp})^{\perp}$. Then,

$$d(x, \overline{\text{span} A}) = \sup_{\substack{y \in (\overline{\text{span} A})^{\perp} \\ \|y\| \leq 1}} |\langle x | y \rangle| = \sup_{y \in A^{\perp}, \|y\| \leq 1} |\langle x | y \rangle| = 0$$

So $x \in \overline{\text{span} A}$ since it is closed. □

3. (a) *Proof.* (i) Clearly, $d_s(A, A) = 0$. Now, suppose $d_s(A, B) = 0$. Then for all $\epsilon > 0$ and $x \in A$,

$$d(x, B) \leq \epsilon$$

thus $d(x, B) = 0$ so $x \in B$ since B is closed. Thus $A \subseteq B$. Likewise $B \subseteq A$. so $A = B$.

(ii) Clearly $d_s(A, B) = d_s(B, A)$.

(iii) Let $C \subseteq X$ be closed. Let $\epsilon_1 > 0$ be such that $A_{\epsilon_1} \subset C$ and $C_{\epsilon_1} \subseteq A$. Let $\epsilon_2 > 0$ such that $B_{\epsilon_2} \subset C$ and $C_{\epsilon_2} \subseteq B$. Then,

$$A_{\epsilon_1 + \epsilon_2} \subset C_{\epsilon_2} \subset B \quad \text{and} \quad B_{\epsilon_1 + \epsilon_2} \subset C_{\epsilon_1} \subset A$$

So, $d_s(A, B) \leq \epsilon_1 + \epsilon_2$ for all such ϵ_1, ϵ_2 . Therefore,

$$d_s(A, B) \leq \inf\{\epsilon_1\} + \inf\{\epsilon_2\} = d_s(A, C) + d_s(C, B)$$

□

- (b) If the sets are not closed, then the first property of the metric fails. $d_s(A, A) = 0$ but $d_s(A, B) = 0$ does not necessarily $A = B$. Consider $X = \mathbb{R}$ and $A = [0, 1]$ and $B = (0, 1)$. $d_s(A, B) = 0$ but $A \neq B$.

4. (a) *Proof.* First, we show T is bounded:

$$\begin{aligned} \|Tf\|_\infty &= \sup_{t \in [0, 1]} \left| \int_0^t s f(s) ds \right| \leq \sup_{t \in [0, 1]} \int_0^t s |f(s)| ds \\ &\leq \|f\|_\infty \sup_{t \in [0, 1]} \int_0^t s ds \leq \|f\|_\infty \int_0^1 s ds = \frac{1}{2} \|f\|_\infty \end{aligned}$$

so $\|T\| \leq \frac{1}{2}$. Let $f, g \in C[0, 1]$ and $a, b \in \mathbb{R}$. Then,

$$T(af + bg)(t) = \int_0^t s(af + bg)(s) ds = a \int_0^t s f(s) ds + b \int_0^t s g(s) ds = a(Tf)(t) + b(Tg)(t)$$

so T is linear. □

- (b) *Proof.* Let $f(t) = 1$ for all $t \in [0, 1]$. Then, $\|f\|_\infty = 1$ and

$$\|Tf\|_\infty = \sup_{t \in [0, 1]} \left| \int_0^t s ds \right| = \sup_{t \in [0, 1]} \frac{t^2}{2} = \frac{1}{2}$$

so $\|T\| = \frac{1}{2}$. □

5. (a) *Proof.* For every $n \in \mathbb{N}$ there exists $E_n \subseteq X$ such that $\mu(E_n) < \frac{1}{n^2}$ and $f_k \rightarrow f$ uniformly on $X \setminus E_n$. Let

$$E := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

Then, $\mu(E) = 0$ (For proof see Winter 15 #10) since

$$\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Now, for $x \notin E$, there exists k such that $x \notin \bigcup_{n=k}^{\infty} E_n$ so $x \notin E_n$ for all $n \geq k$ (However we only need it to hold for a single set, E_k . So, since $x \in E_k^c$,

$$f_n(x) \rightarrow f(x)$$

as $n \rightarrow \infty$. Therefore $f_n \rightarrow f$ pointwise a.e. □

- (b) *Proof.* Let $\epsilon > 0$. Then, there exists some E_ϵ such that $\mu(E_\epsilon) < \epsilon$ and $f_n \rightarrow f$ uniformly on E_ϵ . Moreover, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$, $x \in E_\epsilon^c$. Then,

$$\mu\{|f_n - f| > \epsilon\} \leq \mu(E_\epsilon) < \epsilon$$

so

$$\|f_n - f\|_\mu = \inf\{c > 0 : \mu\{|f_n - f| > c\} < c\} < \epsilon$$

therefore $f_n \rightarrow f$ in measure. □

6. *Proof.* Let $E \subset [a, b]$ be Borel measurable with $\lambda(E) > 0$. Let $\{q_n\}$ be an enumeration of the rational numbers in the interval $[0, 1]$. Set

$$F = \bigcup_n (E + q_n)$$

If $\{E + q_n\}$ are all disjoint, then, $\lambda(F) = \sum_{n=1}^{\infty} \lambda(E + q_n) = \sum_{n=1}^{\infty} \lambda(E) = \infty$ since $\lambda(E) > 0$. But this is a contradiction since $F \subseteq [a, b + 1]$ which has finite Lebesgue measure. Thus there exists $x \in (E + q_n) \cap (E + q_m)$ for some n and m not equal (so $q_n \neq q_m$). Then, there exists $y, z \in E$ such that

$$y + q_n = x = z + q_m$$

so $y - z = q_m - q_n \in \mathbb{Q} \setminus \{0\}$. □

7. (a) False. Consider the following function with a “spike” at every natural number, $n \geq 2$.

$$f(x) := \begin{cases} \text{lin} \nearrow & n \leq x \leq n + \frac{1}{n^3} \\ n & x = n + \frac{1}{n^3} \\ \text{lin} \searrow & n + \frac{1}{n^3} \leq x \leq n + \frac{2}{n^3} \\ 0 & \text{else} \end{cases}$$

Notice that

$$\int_{\mathbb{R}} f(x) dx = \sum_{n=2}^{\infty} \frac{1}{2} \cdot \frac{2}{n^3} \cdot n = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$$

but

$$\limsup_{x \rightarrow \infty} |f(x)| = \infty$$

- (b) *Proof.* Let f be integrable and differentiable and let $D > 0$ such that $|f'(x)| \leq D$ for all $x \in \mathbb{R}$. Fix $x \in \mathbb{R}$. Using the mean-value theorem, for all $y \in \mathbb{R}$ such that

$$|x - y| \leq \frac{f(x)}{D},$$

we know that

$$f(y) \geq f(x) - |x - y|D$$

Suppose without loss of generality that

$$\limsup_{x \rightarrow \infty} f(x) = M$$

for some $M > 0$. Then for all $n \in \mathbb{N}$, there exists $x_n \geq n$ such that

$$f(x_n) \geq \frac{M}{2}$$

Then,

$$\int_{\mathbb{R}} f(x) dx \geq \sum_{n=1}^{\infty} \frac{1}{2} \cdot \min \left\{ \frac{f(x_n)}{D}, 1 \right\} \cdot f(x_n) \geq \sum_{n=1}^{\infty} \frac{1}{8} \cdot \min \left\{ \frac{M}{D}, 1 \right\} \cdot M = \infty$$

which contradicts the fact that f is integrable. □

8. *Proof.* (a \implies b)

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^{\infty} \int_{F_k} 2^k dm \leq \sum_{k=-\infty}^{\infty} \int_{F_k} f dm = \int_{\mathbb{R}} f dm$$

(b \implies c) First, notice that $E_k \cup F_{k-1} = E_{k-1}$ and the union is disjoint therefore

$$m(F_k) = m(E_{k+1}) - m(E_k)$$

Multiply by 2^k and sum from $-N$ to N we have

$$\begin{aligned} \sum_{k=-N}^N 2^k m(F_k) &= \sum_{k=-N}^N 2^k m(E_{k+1}) - \sum_{k=-N}^N 2^k m(E_k) = \frac{1}{2} \sum_{k=-N}^N 2^{k+1} m(E_{k+1}) - \sum_{k=-N}^N 2^k m(E_k) \\ &= -\frac{1}{2} \sum_{k=-N+1}^{N-1} 2^k m(E_k) + 2^N m(E_{N+1}) - 2^{-N} m(E_{-N}) \end{aligned}$$

The final two terms can be bounded by $\int f$: $2^N m(E_N) \leq \int_{E_N} f \, dm \leq \int_{\mathbb{R}} f \, dm < \infty$.
Therefore, for any N ,

$$\sum_{k=-(N-1)}^{N-1} 2^k m(E_k) \leq -2 \sum_{k=-\infty}^{\infty} 2^k m(F_k) + 4 \int_{\mathbb{R}} f \, dm < \infty$$

(c \implies a) Notice that since f is non-negative, $\mathbb{R} = \{f = 0\} \cup E_k$.

$$\int f \, dm = \sum_{k=-\infty}^{\infty} \int_{F_k} f \, dm \leq \sum_{k=-\infty}^{\infty} \int_{F_k} 2^{k+1} \, dm = 2 \sum_{k=-\infty}^{\infty} m(F_k) \leq 2 \sum_{k=-\infty}^{\infty} m(E_k)$$

□

Winter 2016

1. Recall

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

So for $x = 1$,

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

2. *Proof.* Since ℓ^2 is a Hilbert space, A being dense in ℓ^2 is equivalent to

$$A^\perp = \{0\}$$

Let $x = (x_1, x_2, \dots) \in A^\perp$. Then, $\langle x, a \rangle_{\ell^2} = 0$ for all $a \in A$. Notice that $a = e^{(k)} = (0, \dots, 0, \underset{k^{th}}{1}, 0, \dots)$ is in A for any $k \in \mathbb{N}$.

$$0 = \langle x, e^{(k)} \rangle = \sum_{i=1}^{\infty} x_i e_i^{(k)} = x_k$$

for $k \in \mathbb{N}$. Therefore $x = 0$. Now we show the same thing for A^c . Let $y \in (A^c)^\perp$. Also, define $f^{(k)} = e^{(k)} - e^{(k+1)} \in A^c$. Then,

$$0 = \langle y, f^{(k)} \rangle = y_k - y_{k+1}$$

So $y_k = y_{k+1}$ for all $k \in \mathbb{N}$. Thus y is a constant sequence. The only constant sequence in ℓ^2 is the zero sequence therefore $y = 0$. \square

3. *Proof.* First we show subspace. Let $x_1 + y_1, x_2 + y_2 \in X + Y$ and $a, b \in \mathbb{R}$. Then,

$$a(x_1 + y_1) + b(x_2 + y_2) = (ax_1 + bx_2) + (ay_1 + by_2) \in X + Y$$

Now we show closure. Let $\{(x_n + y_n)\}_{n=1}^{\infty}$ be a sequence in $X + Y$ with limit z . This sequence is also Cauchy. So, using the fact that $X \perp Y$,

$$\begin{aligned} \|(x_n + y_n) - (x_m + y_m)\|^2 &= \|(x_n - x_m) + (y_n - y_m)\|^2 \\ &= \langle (x_n - x_m) + (y_n - y_m), (x_n - x_m) + (y_n - y_m) \rangle \\ &= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (x_n - x_m), (y_n - y_m) \rangle \\ &\quad + \langle (y_n - y_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\ &= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\ &= \|x_n - x_m\|^2 + \|y_n - y_m\|^2 \end{aligned}$$

and thus $\{x_n\}$ and $\{y_n\}$ are both Cauchy. Since \mathcal{H} is a Hilbert space, they are convergent to some x, y respectively. Since X, Y are closed, $x \in X$ and $y \in Y$. Then,

$$z = \lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n = x + y \in X + Y$$

Therefore $X + Y$ is closed. \square

4. *Proof.* Since Y is a Banach space, $\mathcal{B}(X, Y)$ is also a Banach space. In a Banach space, any absolutely convergent series is convergent. Since $\|T\| < 1$,

$$\sum_{n=0}^{\infty} \|T\|^n < \infty$$

So

$$\sum_{n=0}^{\infty} T^n \in \mathcal{B}(X, Y)$$

□

5. (a) *Proof.* First, to show T is well-defined we need to show $T\xi$ is continuous for a fixed ξ . This follows from the fact that for $n > m$,

$$\left\| \sum_{k=0}^n a_k \xi_k x^k - \sum_{k=0}^m a_k \xi_k x^k \right\|_{\infty} = \sup_{x \in [0,1]} \left| \sum_{k=m+1}^n a_k \xi_k x^k \right| \leq \|a\|_{\infty} \sum_{k=m+1}^n |\xi_k| \rightarrow 0$$

as $n, m \rightarrow \infty$ since $\xi \in \ell^1$. Thus, this sequence of partial sums is Cauchy in $\|\cdot\|_{\infty}$. Since $(C[0, 1], \|\cdot\|_{\infty})$ is a Banach space, it's limit, $T\xi \in C[0, 1]$. To show linearity, let $\xi, \zeta \in \ell^1$ and $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} T(\alpha\xi + \beta\zeta)(x) &= \sum_{k=0}^{\infty} a_k (\alpha\xi_k + \beta\zeta_k) x^k = \alpha \sum_{k=0}^{\infty} a_k \xi_k x^k + \beta \sum_{k=0}^{\infty} a_k \zeta_k x^k \\ &= \alpha T(\xi)(x) + \beta T(\zeta)(x) \end{aligned}$$

□

- (b) *Proof.*

$$\|T(\xi)\|_{\infty} = \sup_{x \in [0,1]} |T(\xi)(x)| = \sup_{x \in [0,1]} \left| \sum_{k=0}^{\infty} a_k \xi_k x^k \right| \leq \|a\|_{\infty} \sum_{k=0}^{\infty} |\xi_k| = \|a\|_{\infty} \cdot \|\xi\|_1$$

So $\|T\| \leq \|a\|_{\infty}$. We claim this is actually the norm. For $\epsilon > 0$ there exists $a_n \in a$ such that

$$|a_n| > \|a\|_{\infty} - \epsilon$$

Pick $\xi^{(n)} = (0, \dots, 0, \overset{n^{th}}{1}, 0, \dots) \in \ell^1$. Then,

$$\|T\xi^{(n)}\|_{\infty} = \sup_{x \in [0,1]} \left| \sum_{k=0}^{\infty} a_k \xi_k^{(n)} x^k \right| = \sup_{x \in [0,1]} |a_n x^n| = |a_n| > \|a\|_{\infty} - \epsilon$$

Since there exists such $\xi^{(n)}$ for all $\epsilon > 0$, $\|T\| = \|a\|_{\infty}$. □

6. (i) LDCT cannot be applied to f_n since any k which bounds every f_n above, must be greater than 1 everywhere thus $\int_{\mathbb{R}} k = \infty$.

- (ii) LDCT cannot be applied to g_n since any k which bounds every g_n above, must be greater than $1/x$ everywhere thus $\int_{\mathbb{R}} k \geq \int_{\mathbb{R}} 1/x = \infty$.
- (iii) LDCT can be applied since for $k = 1/x^2$, $|h_n| \leq k$ and

$$\int_{\mathbb{R}} \frac{1}{x^2} dx < \infty$$

7. *Proof.* Let $E \subset \mathbb{R}$, $\epsilon \in (0, 1)$. Set $\delta = m^*(E)(1/\epsilon - 1) > 0$. By definition of outer measure, there exists an open set $G \supset E$ such that $m^*(E) + \delta > m^*(G) = m(G)$. Then,

$$\epsilon m(G) < \epsilon(m^*(E) + \delta) = \epsilon m^*(E)(1 + 1/\epsilon - 1) = m^*(E)$$

Moreover, since G is open, it can be written as a countable, disjoint union of open intervals, say I_k . Then,

$$\sum_k \epsilon m(I_k) = \epsilon m(G) < m^*(E) = m^*(E \cap G) \leq \sum_k m^*(E \cap I_k)$$

Therefore, at least one term in the left hand sum must be smaller than one term in the right hand sum, i.e. there exists k such that $\epsilon m^*(I_k) = \epsilon m(I_k) < m^*(E \cap I_k)$. \square

8. *Proof.* Define $A_n := \{x \in [0, 1] : n + 1 > |f(x)| \geq n\}$

$$\sum_{n=1}^{\infty} n \lambda(A_n) = \sum_{n=1}^{\infty} \int_{A_n} n dx \leq \sum_{n=1}^{\infty} \int_{A_n} f(x) dx = \int_0^1 f(x) < \infty$$

So,

$$\begin{aligned} \lim_{n \rightarrow \infty} n \lambda(\{x \in [0, 1] : |f(x)| \geq n\}) &= \lim_{n \rightarrow \infty} n \lambda\left(\bigcup_{k=n}^{\infty} A_k\right) = \lim_{n \rightarrow \infty} n \sum_{k=n}^{\infty} \lambda(A_k) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} k \lambda(A_k) = 0 \end{aligned}$$

\square

9. (a) *Proof.* Let $f(x) > 0$ for $x \in [0, 1]$ and $E \subseteq [0, 1]$ such that $\lambda(E) > 0$. Suppose $\int_E f d\lambda = 0$. Then

$$f(x) = 0$$

for almost every $x \in E$. However, since $\lambda(E) > 0$ there exists $x \in E$ such that $f(x) > 0$ which is a contradiction. \square

- (b) First we prove the following fact: $\mu(\limsup E_n) \geq \limsup \mu(E_n)$. Indeed, set $F_k = \bigcup_{n=k}^{\infty} E_n$. F_k are decreasing.

$$\mu(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n) = \mu(\bigcap_{k=1}^{\infty} F_k) = \inf_k \mu(F_k) = \inf_k \mu(\bigcup_{n=k}^{\infty} E_n) \geq \inf_k \sup_{n \geq k} \mu(E_n)$$

Proof. Fix $\epsilon \in (0, 1]$. Suppose $\inf_{\lambda(E) \geq \epsilon} \int_E f d\lambda = 0$. Then, for each n there exists E_n with $\lambda(E_n) \geq \epsilon$ and

$$\int_{E_n} f d\lambda < \frac{1}{n^2}$$

Then, consider $E = \limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. By the fact above, $\mu(E) \geq \epsilon$. By part (a), this means $\int_E f d\lambda > 0$. However,

$$\int_E f d\lambda = \int_{\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n} f d\lambda \leq \int_{\bigcup_{n=k}^{\infty} E_n} f d\lambda \leq \sum_{n=k}^{\infty} \int_{E_n} f d\lambda \leq \sum_{n=k}^{\infty} \frac{1}{n^2}$$

for any k . Therefore $\int_E f d\lambda = 0$ which is a contradiction. □

Summer 2016

1. (a) *Proof.* We show that f_n does not converge uniformly on the half-open interval $[0, 1)$. The pointwise limit is clearly $f(x) = 0$ for $x \in [0, 1)$. If $\{f_n\}$ converges uniformly, then it must converge to f , the pointwise limit. Let $\epsilon > 0$. For any $n \in \mathbb{N}$ there exists $x \in (0, 1]$ such that

$$1 > x > \left(\frac{\epsilon}{1 - \epsilon} \right)^{1/n}$$

Then,

$$|f(x)| > \epsilon$$

so $\{f_n\}$ does not converge uniformly on $[0, 1)$ therefore it does converge uniformly on $[0, 1]$. \square

- (b) *Proof.* Notice that

$$f_n(x) \leq 1$$

for $x \in [0, 1]$. Since $\int_0^1 1 \, dx < \infty$, by Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \rightarrow \infty} f_n(x) \, dx = \int_0^1 0 \, dx = 0$$

\square

2. False. Counterexample:

Consider $\{x^{(n)}\}_{n=1}^\infty \subset X$ where

$$x^{(n)} := (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots) \in X$$

Then, $\{x^{(n)}\}_{n=1}^\infty$ is Cauchy: For $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. So, for all $n, m \geq N$ ($n > m$),

$$d(x^{(n)}, x^{(m)}) = \sup_{i \in \mathbb{N}} |x_i^{(n)} - x_i^{(m)}| = \frac{1}{m} < \frac{1}{N} < \epsilon$$

However, $x_n \rightarrow (1, \frac{1}{2}, \frac{1}{3}, \dots)$ which is not in X .

3. *Proof.* Let $\{y_n\}_{n=1}^\infty \subset K$. Let $\{y_{n_k}\}_k$ denote the set of distinct elements of $\{y_n\}_{n=1}^\infty$. If $\{y_{n_k}\}_k$ is finite, then there exists some $m \in \mathbb{N}$ such that y_m occurs infinitely many times in $\{y_n\}_{n=1}^\infty$ thus the constant sequence $\{y_m\}$ is a convergent subsequence of $\{y_n\}_{n=1}^\infty$. On the other hand if $\{y_{n_k}\}_k$ is infinite, then

$$\{y_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=0}^\infty$$

is a subsequence of a convergent sequence so it is itself convergent to $\lim_{n \rightarrow \infty} x_n = x_0 \in K$. \square

4. *Proof.* First, notice

$$\begin{aligned}
(T - S)^3 &= (T^2 - ST - TS + S^2)(T - S) \\
&= (T - 2ST + S)(T - S) \\
&= (T^2 - 2ST^2 + ST - ST + 2S^2T - S^2) \\
&= (T - 2ST + ST - ST + 2ST - S) \\
&= (T - S)
\end{aligned}$$

Then, by Cauchy-Schwarz for the operator norm,

$$\|T - S\| = \|(T - S)^3\| \leq \|T - S\|^3$$

Therefore

$$1 \leq \|T - S\|^2$$

and

$$\|T - S\| \geq 1$$

□

5. *Proof.* Let $n, m \in \mathbb{N}$. Without loss of generality, let $n > m$. First,

$$\|x_m\|^2 = \langle x_n, x_m \rangle \leq \|x_n\| \cdot \|x_m\|$$

so $\{\|x_n\|\}_{n=1}^\infty$ is monotone decreasing. Moreover it is bounded below by 0 so it is convergent to some $K \in \mathbb{R}$. Moreover, since

$$\lim_{n \rightarrow \infty} \|x_n\|^2 = \left(\lim_{n \rightarrow \infty} \|x_n\| \right)^2 = K^2$$

$\{\|x_n\|^2\}_{n=1}^\infty$ is convergent and therefore Cauchy. Then, for $n > m$,

$$\begin{aligned}
\|x_n - x_m\|^2 &= \langle x_n - x_m, x_n - x_m \rangle \\
&= \|x_n\|^2 - \langle x_n, x_m \rangle - \langle x_m, x_n \rangle + \|x_m\|^2 \\
&= \|x_n\|^2 - \|x_m\|^2 - \|x_m\|^2 + \|x_m\|^2 \\
&= \|x_n\|^2 - \|x_m\|^2 \\
&= \left| \|x_n\|^2 - \|x_m\|^2 \right| \rightarrow 0
\end{aligned}$$

So $\{x_n\}_{n=1}^\infty$ is Cauchy and therefore convergent since \mathcal{H} is a Hilbert space. □

6. *Proof.* Let $A = (0, 1)$ and $B = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then,

$$d(A, B) = \lambda(A \Delta B) = \lambda(\{\frac{1}{2}\}) = 0$$

but $A \neq B$. Thus the first property of a metric $d(A, B) = 0 \implies A = B$ fails. □

7. *Proof.* (\Leftarrow) Fix $\epsilon > 0$. Then, there exists an open set $\mathcal{O} \supseteq A$ such that

$$\lambda(\mathcal{O} \setminus A) < \epsilon$$

Thus $\mathcal{O} \setminus A \in \mathcal{L}$, the σ -algebra of Lebesgue-measurable sets. Moreover, since \mathcal{O} is open, it is also Lebesgue measurable. Thus,

$$A = \mathcal{O} \setminus (\mathcal{O} \setminus A) \in \mathcal{L}$$

since \mathcal{L} is closed under set-minus.

(\Rightarrow) Let A be Lebesgue measurable. Then,

$$\lambda(A) = \lambda^*(A) = \inf_{A \subseteq \bigcup_n I_n} \sum_{n=1}^{\infty} \lambda(I_n)$$

where $I_n = [a_n, b_n)$. Now, let $\epsilon > 0$. By definition of inf, there exists $\{I_n\}_{n=1}^{\infty}$ such that

$$\lambda(A) + \frac{\epsilon}{2} > \sum_{n=1}^{\infty} \lambda(I_n) \quad \text{and} \quad A \subseteq \bigcup_{n=1}^{\infty} I_n$$

Now, define

$$J_n = \left(a_n, b_n + \frac{\epsilon}{2^{n+1}} \right)$$

Then, $I_n \subset J_n$ for all i and for $\mathcal{O} := \bigcup_{n=1}^{\infty} J_n$

$$\lambda(\mathcal{O} \setminus A) = \lambda(\mathcal{O}) - \lambda(A) \leq \sum_{n=1}^{\infty} \lambda(J_n) - \lambda(A) = \sum_{n=1}^{\infty} \left(\lambda(I_n) + \frac{\epsilon}{2^{n+1}} \right) - \lambda(A) < \epsilon$$

□

8. (a) *Proof.* Proof by contraposition. Suppose $\lambda(E_n) = 0$ for all $n \in \mathbb{N}$. Then,

$$\lambda(\{x \in I : f(x) > 0\}) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} \lambda(E_n) = 0$$

since $\{E_n\}$ are nested. □

(b) *Proof.* Suppose the assumption holds and that $\lambda(\{x \in I : f(x) > 0\}) > 0$. Then, by part (a), there exists some $n \in \mathbb{N}$ such that $\lambda(E_n) > 0$. Since the measure of E_n is positive, it contains infinitely many points. Now, pick $x_1, \dots, x_{n \cdot M} \in E_n$, then,

$$f(x_1) + \dots + f(x_{n \cdot M}) > \frac{1}{n} + \dots + \frac{1}{n} = Mn \left(\frac{1}{n} \right) = M$$

which is a contradiction. □

9. INCOMPLETE

Proof. (\Rightarrow) By the Triangle Inequality,

$$\|f_n\|_1 \leq \|f_n - f\|_1 + \|f\|_1$$

and

$$\|f\|_1 \leq \|f - f_n\|_1 + \|f_n\|_1$$

therefore

$$|\|f_n\|_1 - \|f\|_1| \leq \|f_n - f\|_1 \rightarrow 0$$

as $n \rightarrow \infty$.

(\Leftarrow)

□

10. *Proof.* Applying Holder's Inequality,

$$\begin{aligned} \sum_{n=0}^{\infty} \int_n^{n+1} f(x) dx &\leq \sum_{n=0}^{\infty} \left(\int_n^{n+1} f(x)^2 dx \right)^{1/2} \left(\int_n^{n+1} 1^2 dx \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} f(x)^2 dx \right)^{1/2} = \|f\|_{L^2(\mathbb{R})} < \infty \end{aligned}$$

therefore

$$\lim_{n \rightarrow \infty} \int_n^{n+1} f(x) dx = 0$$

□

Winter 2017

1. Notice that

$$\sum \frac{\sin(nx)}{n}$$

is the Fourier series of the function $x \mapsto \frac{\pi - x}{2}$. Indeed,

$$\frac{\pi}{2} \int_{-\pi}^{\pi} \sin(nx) dx - \frac{1}{2} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{x \cos(nx)}{2n} \Big|_{x=-\pi}^{\pi} - \frac{1}{2n} \int_{-\pi}^{\pi} \cos(nx) dx =$$

2. (a) *Proof.* (i) First, notice that

$$-M|x - y| \leq f(x) - f(y) \leq M|x - y|$$

therefore $|f(x)| \leq |f(y)| + M|x - y|$ for all $x, y \in \mathbb{R}$. Therefore,

$$\begin{aligned} 0 \leq d(f, g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x)| + |g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(0)| + M|x| + |g(0)| + M|x| \\ &\leq \sum_{n=1}^{\infty} \frac{f(0) + g(0) + 2Mn}{2^n} < \infty \end{aligned}$$

so $d(f, g)$ is well-defined and non-negative.

(ii) Clearly $d(f, f) = 0$. Assume $d(f, g) = 0$. Then $\sup_{x \in [-n, n]} |f(x) - g(x)| = 0$ for all n thus $f(x) = g(x)$ on \mathbb{R} .

(iii) Clearly $d(f, g) = d(g, f)$.

(iv)

$$\begin{aligned} d(f, g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} (|f(x) - h(x)| + |h(x) - g(x)|) \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \left(\sup_{x \in [-n, n]} |f(x) - h(x)| + \sup_{x \in [-n, n]} |h(x) - g(x)| \right) \\ &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - h(x)| + \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |h(x) - g(x)| \\ &= d(f, h) + d(h, g) \end{aligned}$$

□

(b) *Proof.* Let $\{f_k\}_{k=1}^\infty \subseteq \mathcal{L}$ be Cauchy in d . Fix $x \in \mathbb{R}$. Then, $x \in [-N, N]$ for some $N \in \mathbb{N}$. For any $k, \ell \in \mathbb{N}$,

$$|f_k(x) - f_\ell(x)| \leq 2^N 2^{-N} \sup_{x \in [-N, N]} |f_k(x) - f_\ell(x)| \leq 2^N d(f_k, f_\ell) \rightarrow 0$$

as $k, \ell \rightarrow \infty$. Therefore $\{f_k(x)\}_{k=1}^\infty \subseteq \mathbb{R}$ is Cauchy for each x and therefore convergent since \mathbb{R} is complete. Then define

$$f(x) := \lim_{k \rightarrow \infty} f_k(x)$$

First, we show $f \in \mathcal{L}$. Fix $x, y \in \mathbb{R}$. For $\epsilon > 0$ there exists $n_1, n_2 \in \mathbb{N}$ such that

$$|f_k(x) - f(x)| < \epsilon \quad |f_\ell(y) - f(y)| < \epsilon \quad \forall k > n_1 \ell > n_2$$

Then, for $n = \max\{n_1, n_2\}$,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 2\epsilon + M|x - y|$$

so $|f(x) - f(y)| \leq M|x - y|$ and $f \in \mathcal{L}$. Now we will show $f_k \rightarrow f$ in d . Let $\epsilon > 0$. Since $\{f_k\}$ is Cauchy in d , $\{d(f_k, 0)\}$ is uniformly bounded, i.e. there exist $C > 0$ such that $d(f_k, 0) \leq C$ for all k . Indeed, there exists N such that $d(f_k, f_j) < 1$ for $j, k \geq N$. Thus,

$$d(f_k, 0) \leq d(f_k, f_N) + d(f_N, 0) \leq 1 + d(f_N, 0)$$

So $d(f_k, 0) \leq \max_{j=1, \dots, N} \{1 + d(f_j, 0)\}$ for all k . Thus,

$$d(f_k, f) \leq C + d(f, 0)$$

for all k so there exists N such that

$$\sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| < \epsilon/2$$

for all k . Moreover, since $f_k(x) \rightarrow f(x)$ for each $x \in [-N, N]$, $f_k \rightarrow f$ uniformly on $[-N, N]$ since it is closed and bounded. Therefore we can take k large enough so that

$$\sum_{n=1}^N 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| < N 2^{-N} \sup_{x \in [-N, N]} |f_k(x) - f(x)| < \epsilon/2$$

Then,

$$d(f_k, f) = \sum_{n=1}^N 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| + \sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| \leq \epsilon$$

for k large enough. □

3. *Proof.* Let $f \in C^1[0, 1]$.

$$|\varphi_0(f)| = |f'(0)| \leq \sup_{x \in [0, 1]} |f'(x)| \leq \|f\|$$

So $\|\varphi_0\| \leq 1$. We will show $\|\varphi_0\| = 1$. Consider the sequence defined

$$f_n(x) := \frac{\sin(nx)}{n}$$

Notice $\|f_n\| = 1/n + 1$ and $|\varphi_0(f_n)| = 1$. Thus,

$$1 \geq \|\varphi_0\| = \sup_{f \neq 0} \frac{|\varphi_0(f)|}{\|f\|} \geq \sup_{n \in \mathbb{N}} \frac{|\varphi_0(f_n)|}{\|f_n\|} = \sup_{n \in \mathbb{N}} \frac{1}{1 + 1/n} = 1$$

so $\|\varphi_0\| = 1$. □

4. (a) *Proof.* Let $x \in \ell^2$. Then for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |x_k|^2 < \epsilon$$

Then, for $y = (x_1, x_2, \dots, x_N, 0, 0, \dots) \in Y$,

$$\|x - y\|_2^2 = \sum_{k=1}^{\infty} |x_k - y_k|^2 = \sum_{k=N+1}^{\infty} |x_k|^2 < \epsilon$$

so Y is dense in ℓ^2 . □

(b) *Proof.* By Cauchy-Schwarz,

$$\left| \sum_{k=1}^n x_k \right| \leq \left(\sum_{k=1}^n |1|^2 \right)^{1/2} \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} = \sqrt{n} \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2}$$

Moreover, if $x \in \ell^2$, then $\sum_{k=1}^{\infty} |x_k|^2$ converges so we can bound the final term by $\|x\|_2$. □

(c) *Proof.* Let $x \in \ell^2$, $\epsilon > 0$. By part (a) there exists $y \in Y$ such that $\|x - y\|_2 \leq \epsilon/2$. Then,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left| \sum y_n \right| = 0$$

since the second term is bounded and the first is decreasing to 0. So, there exists N such that

$$\frac{1}{\sqrt{n}} \left| \sum y_n \right| < \epsilon/2 \quad \text{for } n \geq N$$

By triangle inequality for $|\cdot|$ and part(b),

$$\frac{1}{\sqrt{n}} \left| \sum x_n \right| \leq \frac{1}{\sqrt{n}} \left| \sum x_n - y_n \right| + \frac{1}{\sqrt{n}} \left| \sum y_n \right| < \|x - y\|_2 + \epsilon/2 < \epsilon$$

for $n \geq N$ so

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left| \sum x_n \right| = 0$$

□

5. *Proof.* First notice that

$$0 \leq \int_0^\infty \frac{x}{1+x^3} dx = \int_0^1 + \int_1^\infty \frac{x}{1+x^3} dx \leq \int_0^1 1 dx + \int_1^\infty \frac{1}{x^2} dx < \infty$$

Then, notice that $\frac{x}{1+x^n} \leq \frac{x}{1+x^{n+1}}$ for $x \in (0, 1)$. Therefore, by monotone convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{x}{1+x^n} dx = 0$$

since $x/(1+x^n) \rightarrow 0$ pointwise on $(0, 1)$. Moreover,

$$\int_1^\infty \frac{x}{1+x^n} dx$$

is monotone decreasing and bounded below by zero therefore

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{x}{1+x^n} dx = \lim_{n \rightarrow \infty} \int_0^1 + \int_1^\infty \frac{x}{1+x^n} dx$$

exists. Moreover,

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{x}{1+x^n} dx = \int_1^\infty \lim_{n \rightarrow \infty} \frac{x}{1+x^n} dx = 0$$

by the Lebesgue dominated convergence theorem since

$$\frac{x}{1+x^n} \leq \frac{x}{x^n} = x^{1-n}$$

which is integrable on $(1, \infty)$ for $n \geq 3$. □

6. (a) *Proof.* Set $f(x) = \mathbf{1}_{\liminf_n A_n}$.

(i) $f(x) = 1 \iff x \in \cup_k \cap_{n=k}^\infty A_n$. So, there exists k such that $x \in A_n$ ($\mathbf{1}_{A_n}(x) = 1$) for all $n \geq k$. So, $\lim_{n \rightarrow \infty} \mathbf{1}_{A_n}(x) = 1$ (so \liminf is also 1).

(ii) Suppose $f(x) = 0$. For each k there exists $n \geq k$ such that $x \notin A_n$ ($\mathbf{1}_{A_n}(x) = 0$) so $\liminf_n \mathbf{1}_{A_n} = 0$. □

(b) By Fatou's Lemma,

$$\mu(\liminf_n A_n) = \int_X f d\mu = \int_X \liminf_n \mathbf{1}_{A_n} d\mu \leq \liminf_n \int_X \mathbf{1}_{A_n} d\mu = \liminf_n \mu(A_n)$$

7. *Proof.* Define $f = \sup_N \sum_{n=1}^N f_n$. f is a measurable function, moreover, since f_n are non-negative, $\sum f_n \nearrow f$. So, by Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f = \sum \int f_n \leq \sum \frac{1}{n^2} < \infty$$

So f is non-negative and integrable. We claim this implies $f < \infty$ a.e. If not, then there exists E with $\lambda(E) > 0$ and $f = \infty$ on E . Then,

$$\int_{\mathbb{R}} f \geq \int_E f = \infty$$

so $f < \infty$ a.e. □

8. (a) *Proof.* By Hölder's Inequality,

$$\left| \int f_n d\mu - \int f d\mu \right| \leq \int |f - f_n| d\mu \leq \|f - f_n\|_{\infty} \int d\mu = \|f - f_n\|_{\infty} \mu(X) \rightarrow 0$$

as $n \rightarrow \infty$. □

Summer 2017

5. Hao Chen

Proof. To show the orthonormal set $\{f_n\}$ is an orthonormal basis we will show that $\{f_n\}^\perp = \{0\}$. If not, then there exists $x \neq 0$ such that $\langle x, f_n \rangle = 0$ for all n . However, by Parseval's identity and the Cauchy-Schwarz inequality,

$$\|x\|^2 = \sum |\langle x, e_n \rangle|^2 = \sum |\langle x, e_n - f_n \rangle|^2 \leq \sum \|x\|^2 \|e_n - f_n\|^2 < \|x\|^2$$

but this is a contradiction so $\{f_n\}^\perp = \{0\}$. \square

A more complicated proof by Walton:

Proof. Let $c = \sum \|e_n - f_n\|^2 < 1$. Define $T : \mathcal{H} \rightarrow \mathcal{H}$ by sending $x = \sum \langle x, e_n \rangle e_n \mapsto \sum \langle x, e_n \rangle f_n$. The second sum converges by the Bessel inequality. Now, by the Cauchy-Schwarz inequality and Parseval's identity,

$$\|(I - T)x\|^2 = \left\| \sum \langle x, e_n \rangle (e_n - f_n) \right\|^2 \leq \sum |\langle x, e_n \rangle|^2 \sum \|e_n - f_n\|^2 = c \|x\|^2$$

So, $\|T - I\| \leq \sqrt{c} < 1$. We claim that this means T is invertible. Indeed, set

$$S = \sum_{n=0}^{\infty} (I - T)^n$$

The sum is absolutely convergent since $\|I - T\| < 1$ so S is bounded linear operator since $\mathcal{L}(\mathcal{H})$ is a Banach space. Moreover,

$$S - TS, S - ST = \sum_{n=1}^{\infty} (I - T)^n = S - (I - T)^0 = S - I$$

so $S = T^{-1}$. Now, let $y \in \mathcal{H}$. Then, there exists x ($T^{-1}y$) such that $Tx = y$. Therefore,

$$y = \sum \langle x, e_n \rangle f_n \tag{1}$$

and therefore $\overline{\text{span}}\{f_n\} = \mathcal{H}$. \square

Remark: This actually holds if $\sum \|e_n - f_n\|^2 < \infty$.

6. Define $A_n = \{f \geq 1/n\}$. Since $A_n \subseteq A_{n+1}$,

$$0 < \mu(\{f > 0\}) = \mu\left(\bigcup_n A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n)$$

Therefore there exists n such that $\mu(A_n) > 0$. Then,

$$\int f \geq \int_{A_n} f \geq \frac{1}{n} \mu(A_n) > 0$$

7. (a) We first show that if f is integrable, the $\mu(E_n) \rightarrow 0$ implies $\int_{E_n} f \rightarrow 0$. Since f is integrable, for $A_n = \{n-1 \leq |f| \leq n\}$,

$$\infty > \int |f| \geq \sum (n-1)\mu(A_n)$$

Given $\epsilon > 0$ there exists N such that $\sum_{n=N}^{\infty} (n-1)\mu(A_n) < \epsilon/2$. Also, we can find M such that

$$\mu(E_n) < \epsilon/(2N) \quad \forall n \geq M$$

Then, for $n \geq M$,

$$\begin{aligned} \left| \int_{E_n} f \right| &\leq \int_{E_n} |f| = \int_{E_n \cap \{|f| \leq N\}} |f| + \int_{E_n \cap \{|f| > N\}} |f| \\ &\leq N\mu(E_n) + \sum_{k=N}^{\infty} (k-1)\mu(A_k) \leq N\epsilon/2N + \epsilon/2 = \epsilon \end{aligned}$$

Now we can prove the statement. Let $a, b \in \mathbb{R}$. Then there exists $\{a_n\}, \{b_n\} \subseteq \mathbb{Q}$ such that

$$a_n \rightarrow a \quad b_n \rightarrow b$$

Then,

$$\int_a^b f = \int_a^{a_n} f + \int_{a_n}^{b_n} f + \int_{b_n}^b f$$

The middle term is zero by assumption and applying the above lemma, the first and third terms go to 0.

(b) INCOMPLETE

8. This is a special case of Winter 15 #10.

Winter 2018 - Yiran Zhu

1. Let $\mathcal{C}(0, 1)$ be the collection of all continuous functions on $(0, 1)$ which is a unit *open* interval in \mathbb{R} . Suppose $\{f_n\}_{n=1}^\infty \subset \mathcal{C}(0, 1)$ converges uniformly to f on $(0, 1)$, i.e.,

$$\|f_n - f\|_\infty = \sup_{t \in (0, 1)} |f_n(t) - f(t)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Can we say $f \in \mathcal{C}(0, 1)$? Prove or disprove.

Proof. Fix $x \in (0, 1)$. Observe that $\forall y \in (0, 1)$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq 2\|f_n - f\| + |f_n(x) - f_n(y)| \end{aligned}$$

For $\epsilon > 0$, there exists $N \geq 1$ such that $\|f_N - f\| < \epsilon/3$. Furthermore, since f_N is continuous, there exists $\delta > 0$ such that $\forall |x - y| < \delta$, $|f_N(x) - f_N(y)| < \epsilon/3$.

$$\forall |x - y| < \delta, \quad |f(x) - f(y)| \leq 2\|f_N - f\| + |f_N(x) - f_N(y)| < \epsilon$$

Therefore f is continuous at x . Since x can be a arbitrary number in $(0, 1)$, $f \in \mathcal{C}(0, 1)$. \square

2. Easy to show. $\|f\|_\infty \leq \|f - f_n\|_\infty + \|f_n\|_\infty \leq \epsilon + M < \infty$.
3. If $x \in Y^\perp$, then $\|x - y\|^2 = \|x\|^2 + \|y\|^2 \geq \|x\|^2$. Conversely, since Y is a closed subspace, $H = Y \oplus Y^\perp$. There exists $x' \in Y$ and $x^\perp \in Y^\perp$ such that $x = x' + x^\perp$. Then

$$\|x^\perp\|^2 = \|x - x'\|^2 = \|x\|^2 = \|x' + x^\perp\|^2 = \|x'\|^2 + \|x^\perp\|^2$$

Therefore, $\|x'\|^2 = 0 \Rightarrow x' = 0 \Rightarrow x = x^\perp \in Y^\perp$.

4. Given a Cauchy sequence $\{T_n\}_n \subseteq \mathcal{B}(X, Y)$ with respect to operator norm, we need to construct an operator T such that $T_n \rightarrow T$. Note that, for all $x \in X$, $\{T_n(x)\}_n$ is a Cauchy sequence in Y and thus convergent to some point in Y . We denote this extreme point by T_x . Our mapping is $T : x \rightarrow T_x$. Then one can easily check T is linear and also bounded so $T \in \mathcal{B}(X, Y)$. Finally, $T_n \rightarrow T$ in operator norm.
5. (a) $\|T\| = 1$. This norm cannot be attained but can be approached by e_n as $n \rightarrow \infty$.

$$\|T(x)\| = \left\| \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x_n \right\| \leq \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) |x_n| \leq \|x\|_1$$

- (b) Suppose there exists x with $\|x\| \leq 1$ such that $|T(x)| = \|T\| = 1 \geq \|x\|$. From above inequality, we essentially have

$$\|x_1\| = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) |x_n| = \|x\|_1 - \sum_{n=1}^{\infty} \frac{|x_n|}{n}$$

Hence

$$\sum_{n=1}^{\infty} \frac{|x_n|}{n} = 0 \Rightarrow \forall n \geq 1, x_n = 0 \Rightarrow |T(x)| = 0 \neq 1$$

6. Prove by contradiction. Suppose there is a such measure $(H, \mathcal{M}, \lambda)$ for Hilbert Space H . Let $r = 2$ and $x = 0$,

$$0 < \lambda(B_2(0)) < \infty$$

Since $\dim H = \infty$, there is an orthonormal sequence $\{x_n\}_n \subseteq B_2(0)$. For all x_n , we claim that $B_{1/2}(x_n) \subseteq B_2(0)$. Indeed,

$$\forall y \in B_{1/2}(x_n), \|y - 0\| \leq \|y - x_n\| + \|x_n\| \leq 1/2 + 1 = 3/2 < 2 \Rightarrow y \in B_2(0)$$

Observe that $\|x_n - x_m\|^2 = \langle x_n - x_m, x_n - x_m \rangle = \|x_n\|^2 + \|x_m\|^2 = 2$ for all $n \neq m$. Moreover, if $n \neq m$, then we can check that $B_{1/2}(x_n) \cap B_{1/2}(x_m) = \emptyset$ as follows:

$$\forall y \in B_{1/2}(x_n), \|y_n - x_m\| \geq \|x_n - x_m\| - \|y - x_n\| = \sqrt{2} - \frac{1}{2} > \frac{1}{2} \Rightarrow y \notin B_{1/2}(x_m)$$

By assumption that measure of balls is invariant under translation, we have

$$\forall n \geq 1, \lambda(B_{1/2}(x_n)) = \lambda(B_{1/2}(x_1))$$

Note that $\cup_{n=1}^{\infty} B_{1/2}(x_n)$, the union of disjoint balls, is a subset of $B_2(0)$.

$$\lambda(B_2(0)) \geq \lambda\left(\cup_{n=1}^{\infty} B_{1/2}(x_n)\right) = \sum_{n=1}^{\infty} \lambda(B_{1/2}(x_n)) = \sum_{n=1}^{\infty} \lambda(B_{1/2}(x_1))$$

Therefore, $\lambda(B_{1/2}(x_1)) = 0$. However, we assume that measure of a ball is greater than 0.

7. Let (X, \mathcal{M}, μ) be a measure space and $f \in L^1(X, \mathcal{M}, \mu)$. Then $\{x : f(x) \neq 0\}$ is σ -finite with respect to μ .

Proof. Let $E_n = \{x : |f(x)| \geq 1/n\}$ and then $\{x : f(x) \neq 0\} = \cup_{n=1}^{\infty} E_n$. From Chebyshev's Inequality,

$$\frac{\mu(E_n)}{n} \leq \int_X |f| d\mu < \infty \Rightarrow \mu(E_n) < \infty$$

Therefore, $\{x : f(x) \neq 0\}$ is σ -finite. □

8. (a) Observe that

$$\bigcup_{n=1}^{\infty} E_n = \{x \in I : f(x) > 0\} \Rightarrow \lambda(\{x \in I : f(x) > 0\}) \leq \sum_{n=1}^{\infty} \lambda(E_n)$$

So $\lambda(\{x \in I : f(x) > 0\}) > 0$ implies that there $\lambda(E_n) > 0$ for some n .

- (b) This is also obvious. We show that $\lambda(E_n) = 0$ for all $n \geq 1$. Then the inequality derived in part (a) asserts that $\lambda(\{x \in I : f(x) > 0\}) = 0$. Suppose $\lambda(E_n) > 0$ for some n , then we pick a finite set $\{x_1, \dots, x_m\} \subseteq E_n$ where $m = 2Mn$.

$$\sum_{n=1}^m f(x_n) \geq \sum_{n=1}^m \frac{1}{n} = 2M > M$$

However, by assumption, $\sum_{n=1}^m f(x_n) \leq M$. Therefore, $\lambda(E_n) = 0$ holds for all $n \geq 1$.

9. This is a direct application of *Monotone Convergence Theorem*. Let $h_m(x) = \sum_{n=1}^m f(x+n)$.

$$0 \leq h_1(x) \leq h_2(x) \leq \dots \leq h_m(x) \leq h_{m+1}(x) \leq \dots; \quad \lim_{m \rightarrow \infty} h_m(x) = \sum_{n=1}^{\infty} f(x+n) = g(x)$$

Monotone Convergence Theorem says

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}} h_m(x) d\mu = \int_{\mathbb{R}} \lim_{m \rightarrow \infty} h_m(x) d\mu = \int_{\mathbb{R}} g(x) d\mu$$

Let's compute the left hand side,

$$\int_{\mathbb{R}} h_m(x) d\mu = \int_{\mathbb{R}} \sum_{n=1}^m f(x+n) d\mu = \sum_{n=1}^m \int_{\mathbb{R}} f(x+n) d\mu = m \int_{\mathbb{R}} f(x) d\mu$$

Recall that Lebesgue measure is invariant under translation. Combine above two equations,

$$\lim_{m \rightarrow \infty} m \int_{\mathbb{R}} f(x) d\mu = \int_{\mathbb{R}} g(x) d\mu < \infty \Rightarrow \int_{\mathbb{R}} f(x) d\mu = 0$$

Since $f(x)$ is nonnegative, $\int_{\mathbb{R}} f d\mu = 0$ is equivalent to $f = 0$ a.e.

10. (a) This is a immediate result of Cauchy-Schwartz Inequality.

$$\left(\int_B f d\mu \right)^2 = \left(\int_X f \chi_B d\mu \right)^2 \leq \left(\int_X f^2 d\mu \right) \left(\int_X \chi_B^2 d\mu \right) = \mu_B \int_X f^2 d\mu$$

- (b) Let $f = \sum_{k=1}^n \chi_{A_k}$ where χ_{A_k} is a characteristic function of measurable set A_k . Furthermore, let $B = \cup_{k=1}^n A_k$. This inequality holds directly from part (a).

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1. Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence with $a_n \rightarrow 0, n \rightarrow \infty$. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a_n = 0$$

Proof. For $\epsilon > 0$, there exists a $M \geq 1$ such that $\forall n \geq M, |a_n| \leq \epsilon/2$. For $N \geq M+1$, we have

$$\begin{aligned} \left| \frac{1}{N} \sum_{n=1}^N a_n \right| &= \left| \frac{1}{N} \sum_{n=1}^M a_n + \frac{1}{N} \sum_{n=M+1}^N a_n \right| \leq \frac{1}{N} \left| \sum_{n=1}^M a_n \right| + \frac{1}{N} \sum_{n=M+1}^N |a_n| \\ &\leq \frac{1}{N} \left| \sum_{n=1}^M a_n \right| + \left(\frac{N-M}{N} \right) \frac{\epsilon}{2} \leq \frac{1}{N} \left| \sum_{n=1}^M a_n \right| + \frac{\epsilon}{2} \end{aligned}$$

Let \widehat{N} be an integer greater than $2 \left| \sum_{n=1}^M a_n \right| / \epsilon$ and $\widehat{M} = \max\{\widehat{N}, M+1\}$

$$\forall N \geq \widehat{M}, \quad \left| \frac{1}{N} \sum_{n=1}^N a_n \right| < \epsilon$$

Equivalently, $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a_n = 0$. □

2. Let X be a normed linear space and $\emptyset \neq Y \subset X$ be a subset with the property that $X \setminus Y$ is a linear subspace. Show that Y is dense in X .

Proof. Suppose Y is not dense in X . Then there exists a point $z \in X \setminus Y$ and a number $r > 0$ such that $B(z, r) \cap Y = \emptyset$. Equivalently, $B(z, r) \subseteq X \setminus Y$. Then we will show this implies $Y = \emptyset$. Pick $x \in X$ and let $d = \|x - z\|$. Then

$$r > \left\| \frac{r(x - z)}{2d} \right\| = \left\| \frac{rx - (r - 2d)z}{2d} - z \right\| \Rightarrow a := \frac{rx - (r - 2d)z}{2d} \in B(z, r) \subseteq X \setminus Y$$

Since $X \setminus Y$ is a subspace, we have $x = (2da + (r - 2d)z)/r \in X \setminus Y$. Note that x is arbitrarily picked from X . Therefore, $X \subseteq X \setminus Y \subseteq X \Rightarrow Y = \emptyset$. By contradiction, Y is dense in X . □

3. Define $d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ by $d(x, y) = |f(x) - f(y)|$ where f is defined as

$$f(x) = \frac{x}{1 + |x|}, \forall x \in \mathbb{R}$$

Show that d is a metric on \mathbb{R} and determine if (\mathbb{R}, d) is complete.

Proof. (i) Positive-definite: $d(x, y) = |f(x) - f(y)| \geq 0$ and $d(x, y) = 0 \Leftrightarrow f(x) = f(y)$. Note that

$$f(x) = f(y) \Leftrightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|} \Leftrightarrow x(1+|y|) = y(1+|x|)$$

Suppose $x < 0$, then $y < 0$ and $x(1+|y|) = y(1+|x|) \Rightarrow x = y$. Similarly, if $x \geq 0$, then $y \geq 0$ and $x(1+|y|) = y(1+|x|) \Rightarrow x = y$. In a word, $d(x, y) = 0 \Leftrightarrow x = y$.

(ii) Symmetric: $d(x, y) = |f(x) - f(y)| = |f(y) - f(x)| = d(y, x)$

(iii) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ follows from

$$|f(x) - f(z)| = |f(x) - f(y) + f(y) - f(z)| \leq |f(x) - f(y)| + |f(y) - f(z)|$$

So d is a metric. However, (\mathbb{R}, d) is not complete. Consider sequence $\{x_n\}_n$ with $x_n = n$.

$$d(x_n, x_{n+m}) = \left| \frac{m}{(1+n)(1+n+m)} \right| \leq \frac{1}{n+1}, \quad \forall n \geq 1, \forall m \geq 0$$

Therefore, $\{x_n\}_n$ is Cauchy. It's obvious that $\{x_n\}_n$ does not converge in \mathbb{R} . \square

4. Let H be a Hilbert space and Y_1, Y_2 be two closed linear subspaces in H . Denote P_1 and P_2 as the orthogonal projections onto Y_1 and Y_2 , respectively. Show that $\|P_1 - P_2\| \leq 1$.

Proof. Observe that $(2P - I)^2 = 4P^2 + I - 4P = I$ holds for all projection P . In particular, if P is orthogonal, then, for all $h \in H$,

$$\|(2P - I)h\|^2 = \langle (2P - I)h, (2P - I)h \rangle = \langle h, (2P - I)^2 h \rangle = \langle h, h \rangle = \|h\|^2$$

Therefore, $\|2P - I\| = 1 \Rightarrow \|P - \frac{1}{2}I\| = \frac{1}{2}$.

$$\|P_1 - P_2\| \leq \left\| P_1 - \frac{1}{2}I \right\| + \left\| P_2 - \frac{1}{2}I \right\| \leq 1$$

\square

5. Assume $C[0, 1]$ is equipped with the supremum norm and let $T_n : C[0, 1] \rightarrow C[0, 1]$ be defined by

$$T_n(f) = f\left(x^{1+\frac{1}{n}}\right), \quad \forall n \in \mathbb{N}$$

(a) Show that $T_n(f) \rightarrow f, n \rightarrow \infty, \forall f \in C[0, 1]$

Proof. Fix $f \in C[0, 1]$. Since $[0, 1]$ is compact, f is also uniformly continuous on $[0, 1]$, i.e. for $\epsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, y \in [0, 1] \text{ s.t } |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Let's give an estimation for $g_n(x) := \left| x^{1+\frac{1}{n}} - x \right| = x \left(1 - x^{\frac{1}{n}} \right)$. Obviously, $g_n(x)$ is continuous on $[0, 1]$ and $g_n(0) = g_n(1) = 0$. To find the maximum value of $g_n(x)$, we let

$$g'_n(x) = \left(1 + \frac{1}{n} \right) x^{\frac{1}{n}} - 1 = 0 \Rightarrow \sup_{x \in [0, 1]} g_n(x) = \left(\frac{n}{n+1} \right)^n \frac{1}{n+1} \leq \frac{1}{n+1}$$

Pick $N \in \mathbb{Z}^+$ such that $\frac{1}{N+1} < \delta$, then

$$\forall n \geq N, \forall x \in [0, 1] \quad \left| x^{1+\frac{1}{n}} - x \right| < \frac{1}{N+1} < \delta \Rightarrow \|T_n(f) - f\|_\infty < \epsilon$$

Therefore, $T_n(f) \rightarrow f$ as $n \rightarrow \infty$. □

(b) For each $n \in N$, find $\|T_n - I\|$.

Proof. For $f \in C[0, 1]$,

$$\|(T_n - I)f\| = \|T_n(f) - f\|_\infty = \sup_{x \in [0, 1]} |f(x^{1+\frac{1}{n}}) - f(x)| \leq 2\|f\|_\infty$$

So $\|T_n - I\| \leq 2$. Let $x_0 = \frac{1}{2}$ and $x_1 = \left(\frac{1}{2}\right)^{1+\frac{1}{n}} \in (0, x_0)$. Construct a function f as follows

$$f(x) = \begin{cases} -1 & x \in [0, x_1) \\ -1 + \frac{2(x - x_1)}{x_0 - x_1} & x \in [x_1, x_0] \\ 1 & x \in (x_0, 1] \end{cases}$$

Note that $f(x_0) = 1$ and $f(x_1) = -1$. So f is continuous and $\|f\|_\infty = 1$.

$$|(T_n(f) - f)(x_0)| = |f(x_1) - f(x_0)| = 2 = 2\|f\|_\infty$$

As shown before, $\|T_n(f) - f\|_\infty \leq 2\|f\|_\infty$. Thus

$$\|T_n(f) - f\|_\infty = 2\|f\|_\infty \Rightarrow \|T_n - I\| = 2$$

□

6. Assume that λ is the Lebesgue measure on the real line and f a Lebesgue integrable function on the real line. Show that

$$F(x) := \int_{-\infty}^x f d\lambda$$

is uniformly continuous.

Proof. Let $A_n = \{x \in X \mid |f(x)| \geq n\}$. Then, Dominated Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \int_{A_n} |f| d\lambda = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f| \chi_{A_n} d\lambda = 0$$

For $\epsilon > 0$, there exists $N \geq 1$ such that

$$\int_{A_N} |f| d\lambda < \frac{\epsilon}{2}$$

Then

$$\forall x, y \in \mathbb{R}, \quad |F(x) - F(y)| = \left| \int_{-\infty}^x f d\lambda - \int_{-\infty}^y f d\lambda \right| = \left| \int_y^x f d\lambda \right| \leq \int_y^x |f| d\lambda$$

Observe that, if $|x - y| < \frac{\epsilon}{2N}$, then

$$\int_y^x |f| d\lambda = \int_{[x,y] \cap A_N} |f| d\lambda + \int_{[x,y] \setminus A_N} |f| d\lambda \leq \int_{A_N} |f| d\lambda + N|x - y| < \epsilon$$

□

7. Let (X, \mathcal{M}, μ) be a measure space and $\{A_n\}_n$ be a sequence of sets in \mathcal{M} . Recall that $\limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

- (a) Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$

Proof. Observe that

$$\mu \left(\limsup_{n \rightarrow \infty} A_n \right) \leq \mu \left(\bigcup_{k=n}^{\infty} A_k \right) \leq \sum_{k=n}^{\infty} \mu(A_k), \quad \forall n \geq 1$$

Since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, for $\epsilon > 0$, there exists $N \geq 1$ such that

$$\sum_{k=N}^{\infty} \mu(A_k) < \epsilon \Rightarrow \mu \left(\limsup_{n \rightarrow \infty} A_n \right) \leq \epsilon$$

Let $\epsilon \rightarrow 0$, we derive $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$.

□

- (b) Is the converse true? If yes, prove it. If no, give a counter-example.

Proof. Converse is not true. Consider $A_n = [0, 1/n]$. Then $\bigcup_{k=n}^{\infty} A_k = A_n = [0, 1/n]$.

$$\limsup_{n \rightarrow \infty} A_n = \lim_{N \rightarrow \infty} \bigcap_{n=1}^N \bigcup_{k=n}^{\infty} A_k = \lim_{N \rightarrow \infty} \left[0, \frac{1}{N} \right] = \{0\}$$

Therefore,

$$\mu \left(\limsup_{n \rightarrow \infty} A_n \right) = 0$$

However, $\mu(A_n) = \frac{1}{n}$ for all $n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

□

8. Let (X, \mathcal{M}, μ) be a finite measure space. Prove that a monotone increasing sequence of measurable functions $f_n : X \rightarrow \mathbb{R}$ converges in measure if and only if it converges pointwise a.e..

Proof. (i) Suppose f_n converges to f pointwise a.e.: By Egorov theorem, for each $\epsilon > 0$, there exists a measurable set E with measure $\mu(E) < \epsilon$ such that f_n converges uniformly to f on $X \setminus E$. In other words, for each $\delta > 0$, there exists $N \geq 1$ such that

$$\forall n \geq N, \quad |f_n(x) - f(x)| < \delta, \quad \forall x \in X \setminus E$$

Consequently,

$$\forall n \geq N, \quad A_n := \{x \in X \mid |f_n(x) - f(x)| \geq \delta\} \subseteq E \Rightarrow \mu(A_n) < \epsilon$$

Therefore, f_n converges to f in measure.

- (ii) Suppose f_n converges to f in measure: There exists a subsequence $\{f_{n_k}\}_k$ converges to f pointwise a.e.. Let E be the zero-measure set that $\{f_{n_k}\}_k$ does not converge to f . Then fix $x \in X \setminus E$, for each $\epsilon > 0$, there exists $N \geq 1$ such that

$$\forall k \geq N, \quad |f_{n_k}(x) - f(x)| < \epsilon$$

Since $\{f_n\}_n$ is monotone increasing, we have

$$\forall n \geq n_N, \quad |f_n(x) - f(x)| = f(x) - f_n(x) \leq f(x) - f_{n_N}(x) = |f_{n_N}(x) - f(x)| < \epsilon$$

Note that $\{f_{n_k}\}_k$ is also monotone increasing.

□

9. Suppose that g is a non-negative Borel measurable function on \mathbb{R} with $\int_{\mathbb{R}} g d\lambda = 1$ where λ denotes Lebesgue measure on \mathbb{R} . For $k \in \mathbb{N}$ set $g_k(x) = kg(kx)$. Let f be a bounded continuous function. Prove that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} g_k f d\lambda = f(0)$$

Proof. Suppose $\sup_{x \in \mathbb{R}} |f(x)| = M$ and define $h_k(x) = g(x)f(x/k)$. Observe that

$$\int_{\mathbb{R}} g_k f d\lambda = \int_{\mathbb{R}} kg(kx)f(x)d\lambda = \int_{\mathbb{R}} g(x)f(x/k)d\lambda = \int_{\mathbb{R}} h_k d\lambda$$

In order to apply Dominated Convergence Theorem, we need to show h_k is uniformly bounded by a integrable function. Indeed,

$$\forall k \geq 1, \quad |h_k(x)| = |g(x)f(x/k)| \leq M g(x) \quad \text{and} \quad \int_{\mathbb{R}} M g d\lambda = M < \infty$$

By DCT,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} h_k d\lambda = \int_{\mathbb{R}} \lim_{k \rightarrow \infty} h_k d\lambda = \int_{\mathbb{R}} f(0) g d\lambda = f(0)$$

□

10. Let λ be the Lebesgue measure on $(0, 1)$. Suppose the $f_n : (0, 1) \rightarrow [0, \infty)$ is a sequence of Borel measurable functions such that $\int_{(0,1)} f_n d\lambda = 1$ for all $n \geq 1$ and $\lim_{n \rightarrow \infty} f_n(x) = x$ for all $x \in (0, 1)$.

(a) Give an example of such a sequence.

Proof.

$$f_n(x) = \begin{cases} (n+1)(1-nx) & x \in \left(0, \frac{1}{n}\right) \\ \frac{n}{n-1} \left(x - \frac{1}{n}\right) & x \in \left[\frac{1}{n}, 1\right) \end{cases}$$

Then

$$\int_{(0,1)} f_n d\lambda = \int_{(0, \frac{1}{n})} (n+1)(1-nx) d\lambda + \int_{[\frac{1}{n}, 1)} \frac{n}{n-1} \left(x - \frac{1}{n}\right) d\lambda = \frac{n+1}{2n} + \frac{n-1}{2n} = 1$$

Fix $x \in (0, 1)$, there exists $N \geq 1$ such that $x > \frac{1}{N}$, then

$$\forall n \geq N, \quad f_n(x) - x = \frac{n}{n-1} \left(x - \frac{1}{n}\right) - x = \frac{x-1}{n-1}$$

So $f_n(x) \rightarrow x$ as $n \rightarrow \infty$.

□

- (b) Show that one can find $n \geq 1$ and $x \in (0, 1)$ such that $f_n(x)\sqrt{x} \geq 2018$

Proof. Prove by contradiction. Suppose not, then

$$\forall n \geq 1, \quad f_n(x) \leq \frac{2018}{\sqrt{x}}$$

Note that

$$\int_{(0,1)} \frac{2018}{\sqrt{x}} d\lambda = 4036 < \infty$$

By Dominated Convergence Theorem,

$$1 = \lim_{n \rightarrow \infty} \int_{(0,1)} f_n d\lambda = \int_{(0,1)} \lim_{n \rightarrow \infty} f_n d\lambda = \int_{(0,1)} x d\lambda = \frac{1}{2}$$

Above equality cannot be true.

□

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1. Prove by contradiction. Suppose not. There exists $\epsilon > 0$ such that, for all $N > 0$, $\exists n > N$ s.t. $\|x_n - x_0\| \geq \epsilon$. Therefore, we can pick a subsequence $\{x'_i\}_i$ such that

$$\forall i \in \mathbb{N}, \|x'_i - x_0\| \geq \epsilon$$

By assumption, $\{x'_i\}_i$ contains a subsequence $\{x''_j\}_j$ converging to x_0 . However, by construction, there is no element in $\{x''_j\}_j$ such that $\|x''_j - x_0\| < \epsilon$.

2. $P([0, 1], d)$ is not a complete metric space. Firstly, d is essentially a metric which can be proved easily. However, $P([0, 1], d)$ is not complete. By Taylor's expansion,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Note that e^x is not a polynomial. Indeed, suppose $e^x = \sum_{n=0}^k a_n x^n$. Take $(k+1)$ -th derivative of both sides, we have $e^x = 0$, which is impossible.

Let $f_m = \sum_{n=0}^m \frac{x^n}{n!}$. We will prove $f_m \rightarrow e^x$ w.r.t. d as $m \rightarrow \infty$.

$$d(e^x, f_m) = \int_0^1 |e^x - f_m| dx = \int_0^1 \left| \sum_{n=m+1}^{\infty} \frac{x^n}{n!} \right| dx = \sum_{n=m+1}^{\infty} \frac{1}{(n+1)!}$$

Observe that

$$\sum_{n=m+1}^{\infty} \frac{1}{(n+1)!} < \frac{1}{(m+1)!} + \frac{1}{m!} \sum_{n=m+1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{(m+1)!} + \frac{1}{m!} \sum_{n=m+1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Note that

$$\sum_{n=m+1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{m+1} \Rightarrow \sum_{n=m+1}^{\infty} \frac{1}{(n+1)!} < \frac{2}{(m+1)!}$$

Hence, with $m \rightarrow \infty$, $d(e^x, f_m) \rightarrow 0$. To show $P([0, 1], d)$ is not complete, we need to show sequence $\{f_m\}_m$ is Cauchy. For $\epsilon > 0$, there exist $N \geq 0$ such that $\forall m \geq N$, $d(e^x, f_m) < \epsilon/2$. By triangle inequality,

$$\forall n, m \geq N, d(f_m, f_n) \leq d(f_m, e^x) + d(e^x, f_n) < \epsilon$$

3. Prove by contradiction. Suppose X is separable and let Y be a dense countable subset of X .

$$A \subseteq X \subseteq \bigcup_{y \in Y} B\left(y, \frac{\epsilon}{3}\right)$$

We show that, for each $y \in Y$, $A \cap B(y, \epsilon/3)$ contains at most one element from A . Suppose $A \cap B(y, \epsilon/3)$ contains two distinct elements $a, b \in A$. Then

$$d(a, b) \leq d(a, y) + d(y, b) \leq 2\epsilon/3 < \epsilon$$

This is impossible. Therefore, A contains at most as many elements as Y . In other words, A is countable. (Contradiction!)

4. First, there exists a sequence $\{(a_n, b_n)\}_n$ such that $d(A, B) = \lim_{n \rightarrow \infty} d(a_n, b_n)$. Since A is compact, there exists a convergent subsequence of $\{a_n\}_n$. Denote this subsequence as $\{a_{n,1}\}_n$ and let $a^* \in A$ be its limit point. Note that $\{b_{n,1}\}_n$ is a sequence in B and B is also compact. So there exists a convergent subsequence $\{b_{n,2}\}_n$ and we denote its limit point as $b^* \in B$. Then we will prove $d(A, B) = d(a^*, b^*)$.

$$d(A, B) = \lim_{n \rightarrow \infty} d(a_n, b_n) = \lim_{n \rightarrow \infty} d(a_{n,2}, b_{n,2})$$

For $\epsilon > 0$, there exist $N \geq 0$ such that

$$\forall n \geq N, \quad d(a_{n,2}, b_{n,2}) - d(A, B) < \epsilon$$

Note that

$$d(a^*, b^*) \leq d(a^*, a_{n,2}) + d(a_{n,2}, b_{n,2}) + d(b_{n,2}, b^*)$$

Combine above two inequalities to claim

$$\forall n \geq N, d(a^*, b^*) - d(A, B) < d(a^*, a_{n,2}) + d(b_{n,2}, b^*) + \epsilon$$

Let $n \rightarrow \infty$, we conclude $d(a^*, b^*) - d(A, B) < \epsilon$. Since ϵ can be arbitrarily small, $d(a^*, b^*) \leq d(A, B)$. Recall that, by definition, $d(A, B) \leq d(a^*, b^*) \Rightarrow d(A, B) = d(a^*, b^*)$.

5. By definition, T is orthogonal $\Leftrightarrow \langle x, Ty \rangle = \langle Tx, y \rangle$ for all $x, y \in H$.

- (a) \rightarrow (b): for all $x \in H$, $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^2x \rangle = \langle x, Tx \rangle \leq \|x\| \|Tx\|$. Therefore, $\|Tx\| \leq \|x\| \Rightarrow \|T\| \leq 1$. Since T is nonzero, there exists $x \in H$ such that $Tx \neq 0$.

$$\|T(Tx)\| = \|Tx\| \Rightarrow \|T\| \geq 1$$

Combine all results, we conclude $\|T\| = 1$.

- (b) \rightarrow (c): We first show that $\ker(T) \subseteq T(H)^\perp$ by contradiction. Suppose not, there exists $x, y \in H$ such that $T(x) = 0$ but $\langle x, Ty \rangle \neq 0$. Without loss of generality, we may assume $\operatorname{Re}\langle x, T(y) \rangle > 0$. Otherwise, we replace y by one of $\{-y, iy, -iy\}$. Then, for $\alpha > 0$,

$$\|\alpha x - Ty\|^2 = \alpha^2 \|x\|^2 + \|Ty\|^2 - 2\alpha \operatorname{Re}\langle x, T(y) \rangle := f(\alpha)$$

Let $\alpha^* := \operatorname{Re}\langle x, Ty \rangle / (\|x\|^2) > 0$. We have $f(\alpha^*) = \|Ty\|^2 - \alpha^* \operatorname{Re}\langle x, Ty \rangle < \|Ty\|^2$. However,

$$\|Ty\|^2 = \|T(\alpha^* x - Ty)\|^2 \leq \|T\|^2 \|\alpha^* x - Ty\|^2 = \|\alpha^* x - Ty\|^2 = f(\alpha^*)$$

This contradiction leads to that $\ker(T) \subseteq T(H)^\perp$. Conversely, we need to show $T(H)^\perp \subseteq \ker(T)$. For $y \in T(H)^\perp$, we observe that $y - Ty \in \ker(T) \subseteq T(H)^\perp$, so

$$\langle y - Ty, Ty \rangle = 0 \Rightarrow \langle y, Ty \rangle - \langle Ty, Ty \rangle = 0$$

By choice of y , we have $\langle y, Ty \rangle = 0$. Hence, the right equation above essentially shows that $\langle Ty, Ty \rangle = 0 \Rightarrow Ty = 0 \Rightarrow y \in \ker(T)$.

- (c) \rightarrow (a): For all $x, y \in H$, $x - Tx, y - Ty \in \ker(T) = (T(H))^\perp$. Then

$$\langle x - Tx, y - Ty \rangle = \langle x, y - Ty \rangle + \langle Tx, y - Ty \rangle = \langle x, y - Ty \rangle = \langle x, y \rangle - \langle x, Ty \rangle$$

On the other side,

$$\langle x - Tx, y - Ty \rangle = \langle x - Tx, y \rangle - \langle x - Tx, Ty \rangle = \langle x - Tx, y \rangle = \langle x, y \rangle - \langle Tx, y \rangle$$

After canceling $\langle x, y \rangle$, we derive $\langle x, Ty \rangle = \langle Tx, y \rangle$.

6. (a) Omitted.

(b) Observe that

$$\forall n \in \mathbb{N}, \quad \frac{e^{-x}}{1 + (x/n)} \leq e^{-x} \text{ on } (0, \infty)$$

And $\int_0^\infty e^{-x} = 1$. By dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{e^{-x}}{1 + (x/n)} dx = \int_0^\infty \lim_{n \rightarrow \infty} \frac{e^{-x}}{1 + (x/n)} dx = \int_0^\infty e^{-x} dx = 1$$

7. Define $A_k = \{x \in E : f(x) \geq 1/k\}$ for each positive integer k . Since f is measurable, so is A_k . Let $Z = \{x \in E : f(x) = 0\}$. Follows from problem assumption, $m(Z) = 0$. By sub-additivity of measure, together with fact that $E_n \setminus Z = \bigcup_{k=1}^\infty (E_n \cap A_k)$.

$$\forall n \in \mathbb{N}, \quad m(E_n) = m(E_n \setminus Z) = m\left(\bigcup_{k=1}^\infty (E_n \cap A_k)\right) \leq m(E) < \infty$$

For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$m(E_n) = m\left(\bigcup_{k=1}^\infty (E_n \cap A_k)\right) \leq m\left(\bigcup_{k=1}^N (E_n \cap A_k)\right) + \epsilon \leq \sum_{k=1}^N m(E_n \cap A_k) + \epsilon$$

We will prove that, for each fixed k , $\lim_{n \rightarrow \infty} m(E_n \cap A_k) = 0$.

$$\int_{E_n} f(x) dx = \int_{E_n \setminus Z} f(x) dx \geq \int_{E_n \cap A_k} f(x) dx \geq \frac{m(E_n \cap A_k)}{k}$$

Let $n \rightarrow \infty$, the left hand side goes to 0 and hence $\lim_{n \rightarrow \infty} m(E_n \cap A_k) = 0$. As a consequence,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^N m(E_n \cap A_k) = \sum_{k=1}^N \lim_{n \rightarrow \infty} m(E_n \cap A_k) = 0 \Rightarrow \lim_{n \rightarrow \infty} m(E_n) \leq \epsilon$$

As ϵ can be arbitrarily small, $\lim_{n \rightarrow \infty} m(E_n) = 0$.

8. By Cauchy inequality, we have

$$|L| := \left| \int_0^1 (f(x) - \cos(2\pi x))(f(x) - \sin(2\pi x)) dx \right| \leq \sqrt{L_1 L_2} = \frac{1}{9}$$

where $L_1 = \int_0^1 |f(x) - \cos(2\pi x)|^2 dx$ and $L_2 = \int_0^1 |f(x) - \sin(2\pi x)|^2 dx$. Since $L_1 = L_2 = 1/9$, we have

$$\int_0^1 f(x) \cos(2\pi x) dx = \int_0^1 f(x) \sin(2\pi x) dx$$

This equality allows us to compute L directly.

$$\begin{aligned} L &= \int_0^1 (f^2(x) + f(x) \cos(2\pi x) + f(x) \sin(2\pi x) + \cos(2\pi x) \sin(2\pi x)) dx \\ &= \int_0^1 (f^2(x) + 2f(x) \cos(2\pi x)) dx = L_1 - \int_0^1 \cos^2(2\pi x) dx = L_1 - \frac{1}{2} \\ &= -\frac{7}{18} \end{aligned}$$

However, by the inequality given at the beginning, $|L| \leq 1/9$. This contradiction leads to that there is no such function $f(x)$ satisfying $L_1 = L_2 = 1/9$.

9. By substitution $y = \sqrt{x}$, we convert integral to following form

$$\int_0^1 n\sqrt{x}f(nx)dx = \int_0^1 ny^2f(ny^2)dy$$

Since $|xf(x)| \rightarrow 0$ as $x \rightarrow \infty$. For $\epsilon > 0$, there exists a number $N_\epsilon > 0$ such that $\forall x \geq N_\epsilon$, $|xf(x)| < \epsilon$. Now, for each fixed $n \geq N_\epsilon$, we construct a set $A_n = \{y \in [0, 1] : ny^2 \geq N_\epsilon\} = [\sqrt{N_\epsilon/n}, 1]$ and $B_n = [0, 1] \setminus A_n = [0, \sqrt{N_\epsilon/n}]$. Then

$$\left| \int_0^1 ny^2f(ny^2)dy \right| = \left| \int_{\sqrt{N_\epsilon/n}}^1 ny^2f(ny^2)dy + \int_0^{\sqrt{N_\epsilon/n}} ny^2f(ny^2)dy \right| \leq \epsilon \left(1 - \sqrt{N_\epsilon/n}\right) + M\sqrt{N_\epsilon/n}$$

where M is the upper bound for $f(x)$ on $[0, 1]$. We conclude that

$$\forall n \geq M^2 N_\epsilon / \epsilon^2, \quad \left| \int_0^1 ny^2f(ny^2)dy \right| \leq \epsilon \left(1 - \sqrt{N_\epsilon/n}\right) + M\sqrt{N_\epsilon/n} \leq \epsilon + M\sqrt{N_\epsilon/n} \leq 2\epsilon$$

Since ϵ is arbitrarily picked,

$$\lim_{n \rightarrow \infty} \left| \int_0^1 ny^2f(ny^2)dy \right| = 0 \Rightarrow \lim_{n \rightarrow \infty} \int_0^1 n\sqrt{x}f(nx)dx = 0$$

10. Observe that $\limsup_{n \rightarrow \infty} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \subseteq \bigcup_{m=k}^{\infty} A_m$ for each fixed k . Hence

$$\forall k \geq 1, \quad \mu \left(\limsup_{n \rightarrow \infty} A_n \right) \leq \mu \left(\bigcup_{m=k}^{\infty} A_m \right) \leq \sum_{m=k}^{\infty} \mu(A_m)$$

Since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, for $\epsilon > 0$, there exists $N \geq 0$ such that

$$\forall m \geq N, \quad \sum_{n=m}^{\infty} \mu(A_n) < \epsilon \Rightarrow \mu \left(\limsup_{n \rightarrow \infty} A_n \right) < \epsilon$$

As ϵ can be arbitrarily small, $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$.