

# Clemson Analysis Prelim Solutions

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## Winter 2010

1. (a) *Proof.* If  $E$  is bounded,  $E$  is pre-compact since  $\mathbb{R}$  is finite (one) dimensional. If  $f(E)$  is unbounded, then there exists  $\{x_n\} \subseteq E$  such that  $f(x_n) \rightarrow \infty$ . Since  $E$  is precompact  $\{x_n\}$  has a convergent subsequence, say  $\{x_{n_k}\}$  with limit  $x \in \mathbb{R}$ . Then, since  $f$  is continuous,

$$f(x) = \lim_{k \rightarrow \infty} f(x_{n_k}) = \infty$$

However, since  $f$  maps  $\mathbb{R}$  to  $\mathbb{R}$ ,  $f(x)$  cannot be  $\infty$ . □

- (b) *Proof.* Since  $f$  is uniformly continuous, there exists  $\delta > 0$  such that whenever  $|x - y| < \delta$ ,

$$|f(x) - f(y)| < 1$$

Since  $E$  bounded, it can be covered by finitely many balls of radius  $\delta$ , say  $\{B(x_i, \delta)\}_{i=1}^N$ . Then,

$$f(E) = \cup_{i=1}^N f(B(x_i, \delta))$$

Fix  $i$ , for any  $f(y) \in f(B(x_i, \delta))$ ,

$$|f(y) - f(x_i)| \leq 1$$

So  $f(B(x_i, \delta))$  is bounded. Then, a finite union of bounded sets is also bounded. □

Counterexample:  $E = (0, 1)$  and  $f(x) = 1/x$ .  $f(E) = (1, \infty)$ .

3. (a) *Proof.* Recall the Bessel inequality for any orthonormal set  $\{e_n\}$  in an inner product space,  $X$ . For and  $f \in X$ ,

$$\sum |\langle f, e_n \rangle|^2 \leq \|f\|^2$$

In particular,  $\langle f, e_n \rangle \rightarrow 0$  as  $n \rightarrow \infty$  for any  $f \in X$ . Now, since

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{2\pi}} \right\}$$

form an orthonormal set in  $C[-\pi, \pi]$ , we have

$$\int_{-\pi}^{\pi} \sin(2nx) f(x) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any  $f \in C[-\pi, \pi]$ . Then,

$$\int_{-\pi}^{\pi} \sin^2(nx) f(x) dx = \frac{1}{2} \int_{-\pi}^{\pi} f(x) dx - \frac{1}{2} \int_{-\pi}^{\pi} \sin(2nx) f(x) dx \rightarrow \frac{1}{2} \int_{-\pi}^{\pi} f(x) dx$$

□

(b) *Proof.* For any  $f \in C[-\pi, \pi]$ ,  $n \in \mathbb{N}$ ,

$$\begin{aligned} \left| \int_{-\pi}^{\pi} \frac{x^n}{\pi^n} f(x) dx \right|^2 &\leq \int_{-\pi}^{\pi} \frac{x^{2n}}{\pi^{2n}} dx \int_{-\pi}^{\pi} |f(x)|^2 dx \\ &= \frac{\pi^{2n+1} - (-\pi)^{2n+1}}{(2n+1)\pi^{2n}} \|f\|_{L^2}^2 \\ &= \frac{2\pi}{2n+1} \|f\|_{L^2}^2 \end{aligned}$$

which goes to 0 as  $n \rightarrow \infty$ . □

9. (a) There exists  $\varepsilon_n \searrow 0$  such that

$$\mu\{|f - g| \geq \varepsilon_n\} \leq \varepsilon_n$$

Then,

$$\begin{aligned} \mu\{f \neq g\} &= \mu\{|f - g| > 0\} = \mu\left(\bigcup_n \{|f - g| \geq \varepsilon_n\}\right) \\ &= \lim_{n \rightarrow \infty} \mu\{|f - g| \leq \varepsilon_n\} \leq \lim_{n \rightarrow \infty} \varepsilon_n = 0 \end{aligned}$$

(b) We only need to show the triangle inequality. Let  $t, s > 0$ ,  $f, g, h$  measurable functions.

If  $|f - h| \leq t$  and  $|g - h| \leq s$ , then

$$|f - g| \leq |f - h| + |g - h| \leq t + s$$

Thus,  $\{|f - h| \leq t\} \cap \{|g - h| \leq s\} \subset \{|f - g| \leq t + s\}$ . Then, taking complements, we have

$$\{|f - h| > t\} \cup \{|g - h| > s\} \supset \{|f - g| > t + s\}$$

Therefore,  $\mu\{|f - g| > t + s\} \leq \mu\{|f - h| > t\} + \mu\{|g - h| > s\}$ . Let  $\delta > 0$ . There exists  $\varepsilon_1, \varepsilon_2$  such that

$$\mu\{|f - h| > \varepsilon_1\} < \varepsilon_1 \text{ and } \varepsilon_1 < \rho(f, h) + \delta/2$$

and similarly for  $\varepsilon_2$  and  $|g - h|$ . Therefore,

$$\mu\{|f - g| > \varepsilon_1 + \varepsilon_2\} \leq \mu\{|f - h| > \varepsilon_1\} + \mu\{|g - h| > \varepsilon_2\} < \varepsilon_1 + \varepsilon_2$$

So,

$$\begin{aligned} \rho(f, g) &= \inf\{\varepsilon : \mu\{|f - g| > \varepsilon\} < \varepsilon\} \\ &\leq \varepsilon_1 + \varepsilon_2 \\ &\leq \rho(f, h) + \rho(g, h) + \delta \end{aligned}$$

for any  $\delta > 0$ . This proves the Triangle Inequality.

## Summer 2010

3. (a)

$$\|Tf\|_\infty = \sup_{x \in [0,1]} |x^2 f(x)| \leq \sup_{x \in [0,1]} |f(x)| = \|f\|_\infty$$

(b) For  $f \equiv 1$ ,  $\|Tf\| = 1$  and  $\|f\| = 1$ .

(c) By triangle inequality,  $\|(I+T)f\|_\infty \leq \|f\|_\infty + \|Tf\|_\infty \leq 2\|f\|$ . So, we only need to show  $\|I+T\| = 2$ . Again, this follows from taking  $f \equiv 1$ .

$$\|(I+T)f\| = \sup_{x \in [0,1]} |1+x^2| = 2$$

4. (a) We will actually show more, namely that

$$\|x\|_q \leq \|x\|_p \quad \text{for } 1 \leq p < q < \infty$$

*Proof.* Let  $x \in \ell^p$ . Define

$$y = \frac{x}{\|x\|_p}$$

Then,  $|y_i| \leq 1$  for every  $i \in \mathbb{N}$ . This implies  $|y_i|^q \leq |y_i|^p$  for all  $i$ . Therefore,

$$\|y\|_q^q = \sum |y_i|^q \leq \sum |y_i|^p = \sum \frac{|x_i|^p}{\|x\|_p^p} = \frac{\sum |x_i|^p}{\sum |x_i|^p} = 1$$

so  $\|y\|_q \leq 1$ . But this implies

$$\frac{\|x\|_q}{\|x\|_p} = \left\| \frac{x}{\|x\|_p} \right\|_q = \|y\|_q \leq 1$$

□

5. (a) Let  $f \in L^q$ ,  $1 \leq p < q < \infty$ . we will apply holder inequality with exponent  $q/p > 1$ .

$$\int |f|^p \leq \left( \int |f|^q \right)^{p/q} \left( \int 1 \right)^{1-p/q} = \left( \int |f|^q \right)^{p/q} (\mu(X))^{1-p/q} < \infty$$

(b) Take

$$f(x) = \begin{cases} 0 & x = 0 \\ x^{-1/2} & 0 < |x| \leq 1 \\ 0 & |x| < 1 \end{cases}$$

$$g(x) = \begin{cases} x^{-1} & |x| \geq 1 \\ 1 & |x| < 1 \end{cases}$$

6. There are many options for  $f_n$ :

$$n^2\chi_{[0,1/n]} \quad n\chi_{[n,n+1]} \quad \chi_{[n,2n]}$$

Then, just take

$$g_n = \frac{1}{n}f_n$$

and

$$h_n = (-1)^n g_n$$

7.

**Winter 2012**

## Summer 2012

4. (a) We will show  $K$  is totally bounded. Let  $\epsilon > 0$ . For  $x, y \in K$ ,

$$d_S(x, y) = \sum_i \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)} \leq \sum_i \frac{2}{2^i} < \infty$$

So, there exists  $N$  such that

$$\sum_{i=N+1}^{\infty} \frac{1}{2^{i-1}} < \epsilon/2$$

Consider the set  $M = \{x \in K : x_i = 0, i > N\} \subseteq K$ .  $M$  is compact since it is finite dimensional and bounded. So, there is an  $\epsilon/2$  net for  $M$  which will be an  $\epsilon$  net for  $K$ .

- (b) False. Consider the sequence  $\{e_n\} \subseteq K$ , which is entirely zero except the  $n$ -th entry. For  $n \neq m$ ,

$$d_{\infty}(e_n, e_m) = 1$$

so this sequence cannot have a convergent subsequence.

5. (a) Define  $g_n = \sum_{k=1}^n f_k$ . Since  $f_k$  are non-negative,  $\{g_n\}$  is monotone. Moreover,

$$g_n \nearrow \sum_{k=1}^{\infty} f_k$$

Then, by Monotone Convergence Theorem,

$$\sum_{k=1}^{\infty} \int f_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int f_k = \lim_{n \rightarrow \infty} \int g_n = \int \sum_{k=1}^{\infty} f_k$$

- (b) By part (a),

$$\infty > \sum \int |f_k - f| = \int \sum |f_k - f|$$

Since  $\sum |f_k - f|$  is integrable, it is finite almost everywhere. Let  $E \subseteq \mathbb{R}$  be the set of measure zero where it may not be finite. Let  $x \notin E$ . Let  $\epsilon > 0$ . There exists  $N$  such that

$$|f_n(x) - f(x)| \leq \sum_{k=N}^{\infty} |f_k(x) - f(x)| < \epsilon$$

for all  $n \geq N$ .

7. See Winter 15 #9

## Winter 2013

8. See Summer 13 #5



## Summer 2013

3. (a)  $B$  is complete. Let  $\{f_n\} \subseteq C[0, 1]$  be a Cauchy sequence in  $\rho_\infty$ . Since  $(C[0, 1], \rho_\infty)$  is a complete metric space, there exists  $f \in C[0, 1]$  such that  $f_n \rightarrow f$  in  $\rho_\infty$ . Now, we claim that  $f \in B$ . For any  $\epsilon > 0$  there exists  $N$  such that

$$\rho(f, f_n) < \epsilon \quad \forall n \geq N$$

Then,

$$\sup_{0 \leq t \leq 1} |f(t)| = \rho(f, 0) \leq \rho(f, f_n) + \rho(f_n, 0) < \epsilon + 1$$

but  $\epsilon > 0$  was arbitrary so

$$\sup_{0 \leq t \leq 1} |f(t)| \leq 1$$

- (b) Consider the spike functions,  $\{f_n\}$ . For  $n \neq m$ ,

$$\rho(f_n, f_m) = 1$$

so there cannot be a convergent subsequence.

4. (a) Let  $f \in L^2(\mu)$ . Then, using the Cauchy-Schwarz inequality, we compute

$$\begin{aligned} \|Af\|_{L^2(\mu)}^2 &= \int_X \left( \int_X K(x, y) f(y) d\mu(y) \right)^2 d\mu(x) \\ &\leq \int_X \left( \int_X K(x, y)^2 d\mu(y) \right) \left( \int_X f(y)^2 d\mu(y) \right) d\mu(x) \\ &= \left( \int_X f(y)^2 d\mu(y) \right) \int_X \left( \int_X K(x, y)^2 d\mu(y) \right) d\mu(x) = \|f\|_{L^2(\mu)}^2 \|K\|_{L^2(\mu \times \mu)}^2 \end{aligned}$$

Therefore  $\|A\| \leq \|K\|_{L^2(\mu \times \mu)}$ .

- (b) First we note that the correspondence  $K \mapsto A$  is linear due to the linearity of the integral. So, it suffices to prove the following: Let  $K \in L^2(\mu \times \mu)$  such that for any  $f \in L^2(\mu)$ ,

$$\int_X K(x, y) f(y) d\mu(y) = 0$$

for almost every  $x \in X$  (w.r.t  $\mu$ ). Then,  $K = 0$  a.e. To prove this, suppose there exists  $E \subseteq X \times X$  such that  $E$  has positive  $\mu \times \mu$  measure in the sense that

$$(\mu \times \mu)(E) := \int_X \int_X \mathbf{1}_E(x, y) d\mu(x) d\mu(y) > 0$$

So we define the measure  $\mu \times \mu$  on the cylinder  $X \times X$  in this way.

5. ( $a \Rightarrow b$ ) Let  $x = \hat{m} + e$  where  $\hat{m}$  is the closest point to  $x$  in  $M$ . Let  $y \in M$  non-zero. For any  $t \in \mathbb{C}$ ,  $ty \in M$  so

$$\|x - \hat{m}\|^2 \leq \|e - ty\|^2 = \|e\|^2 - 2\operatorname{Re}\langle e, ty \rangle + \|ty\|^2$$

which implies

$$\operatorname{Re} \bar{t} \langle e, y \rangle \leq |t|^2 \|y\|^2$$

Take  $t = \overline{\langle e, y \rangle} \|y\|^{-2}$ . Then we have

$$\frac{|\langle e, y \rangle|^2}{\|y\|^2} \leq \frac{|\langle e, y \rangle|^2}{2\|y\|^2}$$

therefore  $\langle e, y \rangle = 0$ .

( $b \Rightarrow a$ ) Let  $x = \hat{m} + e$  for  $\hat{m} \in M$  and  $e \in M^\perp$ . For any  $y \in M$ ,

$$\|x - y\|^2 = \|e - (y - \hat{m})\|^2 = \|e\|^2 - 2\operatorname{Re}\langle e, y - \hat{m} \rangle + \|y - \hat{m}\|^2 = \|e\|^2 + \|y\|^2 \geq \|e\|^2 = \|x - \hat{m}\|^2$$

Suppose  $\tilde{m}$  is another closest point. Set  $d = d(x, M)$ . By the parallelogram identity,

$$\|\tilde{m} - \hat{m}\|^2 = \|x - \tilde{m} - (x - \hat{m})\|^2 = 2d^2 + 2d^2 - \|2x - 2(\tilde{m} + \hat{m})\|^2 \leq 4d^2 - 4d^2 = 0$$

6.

$$f_n = n \mathbf{1}_{[0, 1/n]}$$

7. (a) We will show that for  $h \geq 0$  measurable,  $\int h = 0 \implies h = 0$  a.e. Indeed, consider  $A_n = \{n^{-1} > h \geq (n+1)^{-1}\}$  for each  $n \in \mathbb{N}$  and  $A_0 = \{h \geq 1\}$ . Then, for any  $n \in \mathbb{N} \cup \{0\}$ ,

$$(n+1)^{-1} \mu(A_n) \leq \int_{A_n} h \, d\mu \leq \int_{\mathbb{R}} h \, d\mu = 0$$

Therefore  $\mu(A_n) = 0$ . So,  $\{h \neq 0\} = A = \cup A_n$  which has measure zero.

Now, we apply this to the problem by taking  $h = g - f$ . Then,  $h \geq 0$  and

$$\int h = \int g - f = \int g - \int f = 0$$

Then by the above lemma,  $h = 0$  so  $f = g$  a.e.

- (b) If  $f$  and  $g$  are continuous, then being equal almost everywhere will imply they are equal everywhere. Indeed, suppose there exists  $x_0 \in \mathbb{R}$  such that  $f(x_0) < g(x_0)$ . Then,  $f - g$  is continuous around  $x_0$  so there exists  $\epsilon > 0$  such that

$$f(x) < g(x) \quad \text{for } |x - x_0| < \epsilon$$

However,  $\lambda(\{|x - x_0| < \epsilon\}) = \epsilon$  so  $f \neq g$  on a set of measure  $\epsilon$  which contradicts  $f = g$  a.e.

(c) INCOMPLETE

8. (a) First,  $\mathcal{M}$  is clearly a linear space since linear combinations of finite signed measures are still finite signed measures. Now we show that total variation is a norm on  $\mathcal{M}$ . If  $\mu$  has total variation 0, this means

$$\mu_+(X) = \mu_-(X) = 0$$

so  $X$  is a null set of both  $\mu_+$  and  $\mu_-$ . Thus for every  $E \subseteq X$ ,  $\mu(E) = \mu_+(E) - \mu_-(E) = 0 - 0 = 0$ . Thus  $\mu$  is the zero measure. By definition,  $|\alpha\mu| = \alpha\mu_+ + \alpha\mu_- = \alpha|\mu|$ . Finally, to check the triangle inequality, let  $\mu, \lambda \in \mathcal{M}$ . Let  $A \cup B = X$  be a Hahn decomposition of  $X$  with respect to the signed measure  $(\mu + \lambda)$ . Then,

$$(\mu + \lambda)_+(X) = \mu(A) + \lambda(A) \leq \mu_+(A) + \lambda_+(A) \leq \mu_+(X) + \lambda_+(X)$$

and

$$(\mu + \lambda)_-(X) = -\mu(B) - \lambda(B) \leq \mu_-(B) + \lambda_-(B) \leq \mu_-(X) + \lambda_-(X)$$

Therefore

$$\begin{aligned} |\mu + \lambda|(X) &= (\mu + \lambda)_+(X) + (\mu + \lambda)_-(X) \leq \mu_+(X) + \lambda_+(X) + \mu_-(X) + \lambda_-(X) \\ &= |\mu|(X) + |\lambda|(X) \end{aligned}$$

- (b) Let  $\mu$  be a  $\sigma$ -finite measure. Clearly  $\mathcal{L}_\nu = \{\mu \in \mathcal{M} : \mu \ll \nu\}$  is a linear subspace since if  $\lambda, \mu \ll \nu$ , and  $\nu(E) = 0$ , then

$$\alpha\lambda(E) + \beta\mu(E) = 0$$

for any scalars  $\alpha, \beta$ . We note the crucial property of this subspace. If  $\mu \ll \nu$ , then the null sets of  $\nu$  are also null sets of  $\mu_+$  and  $\mu_-$ . Indeed, let  $E \subset X$  such that  $\nu(E) = 0$ . Let  $A \cup B$  be a Hahn decomposition for  $\mu$ . Then,

$$\nu(A \cap E) \leq \nu(E) = 0 \quad \nu(B \cap E) \leq \nu(E) = 0$$

so  $\nu(A \cap E) = \nu(B \cap E) = 0$ . Therefore

$$\mu_+(E) = \mu(A \cap E) = 0 \quad \mu_-(E) = \mu(B \cap E) = 0$$

Now, let  $\{\mu_n\} \subseteq \mathcal{L}_\nu$  converge to  $\mu$  in the total variation norm. Then, let  $E \subset X$  such that  $\nu(E) = 0$ . Then,  $|\mu_n|(E) = 0$ . By the reverse triangle inequality (a consequence of the triangle inequality for  $\|\cdot\|$  shown above)

$$|\mu|(E) = ||\mu|(E) - |\mu_n|(E)|| \leq |\mu - \mu_n|(E) \leq |\mu - \mu_n|(X) = \|\mu - \mu_n\| \rightarrow 0$$

so  $\mu(E) = 0$  and  $\mu \in \mathcal{L}_\nu$ .

- (c) Let  $f \in L^1(X, \mathcal{F}, \nu)$ . Then,

$$\mu(A) = \int_A f d\nu$$

defines a signed measure for  $A \subseteq X$ . We only need to check that this pairing is isometric and onto. Surjectivity follows from the Radon-Nikodym theorem which states that if  $\rho \ll \lambda$ , then there exists  $\lambda$ -measurable  $g$  such that

$$\rho = g d\lambda$$

Then, to check the norms are preserved, we first show that the Hahn decomposition of  $\mu$  corresponds to the positive and negative parts of  $f$ . Indeed, let  $A = \{f \geq 0\}$ . Then, for any  $E \subseteq A$ ,

$$\mu(E) = \int_E f d\nu \geq 0$$

Similarly, for  $B = \{f < 0\}$ ,  $F \subseteq B$ ,

$$\mu(F) = \int_F f d\nu \leq 0$$

So  $A \cup B$  is a Hahn decomposition for  $\mu$ . Therefore,

$$\begin{aligned} \int_X |f| d\nu &= \int_X f^+ + f^- d\nu = \int_X f^+ d\nu + \int_X f^- d\nu = \int_A f^+ d\nu + \int_B f^- d\nu \\ &= \mu_+(A) + \mu_-(B) = \mu_+(X) + \mu_-(X) = |\mu|(X) \end{aligned}$$

9.

We have shown many times before that if  $\sum \lambda(E_n) < \infty$ , then  $\lambda(\limsup_n E_n) = 0$ . Set

$$E_n = [r_n - 2^{-n-1}, r_n + 2^{-n-1}]$$

Then,

$$\sum_n \lambda(E_n) = \sum_n 2^{-n} < \infty$$

Set  $E = \limsup_{n \rightarrow \infty} E_n$ . Then,  $\lambda(E) = 0$ . So, for any  $x \notin E$ , we have that there exists  $k \in \mathbb{N}$  such that  $x \notin E_n$  for all  $n \geq k$ . Therefore,  $f(x)$  is only nonzero for finitely many indices so the sum must converge at  $x$ .

(b) Set

$$X_n = \left( \bigcup_{k \neq n} E_k \right)^c$$

Then,  $\mathbb{R} = \cup X_n$  and,  $\mu(X_n) \leq 1$  since  $f_k = 0$  on  $X_n$  for  $n \neq k$ . Indeed,

$$\mu(X_n) = \int_{X_n} \sum f_k d\lambda = \int_{X_n} f_n \leq \int f_n = 1$$

(c) To show  $\mu \ll \lambda$ , let  $E \subset \mathbb{R}$  such that  $\lambda(E) = 0$ . Then, integration over a set of measure zero is also zero so  $\mu(E) = 0$ .

- (d) Without loss of generality, we can just show that each open ball has infinite measure since every open set contains an open ball. Let  $B(x, \epsilon) \subseteq \mathbb{R}$ . Then, there exists a subsequence of  $\{r_n\}$  such that  $\{r_{n_k}\} \subseteq B(x, \epsilon/2)$ . Moreover, since the radii of  $E_{n_k}$  are decreasing, there exists  $N$  such that  $E_{n_k} \subseteq B(x, \epsilon)$  for all  $k \geq N$ . Thus,

$$\mu(B(x, \epsilon)) = \int_{B(x, \epsilon)} \sum f_n d\lambda \geq \int_{B(x, \epsilon)} \sum_{k=N}^{\infty} f_{n_k} d\lambda = \sum_{k=N}^{\infty} \int_{B(x, \epsilon)} f_{n_k} \geq \sum_{k=N}^{\infty} \int_{E_{n_k}} f_{n_k} = \infty$$

## Winter 2014

8. See Summer 13 #9 (a)
9. See Summer 13 #7
10. See Summer 13 #9

## Summer 2014

1. (a) *Proof.* Suppose  $f$  is discontinuous at some  $x \in (0, 1)$ . Then there exists  $\epsilon > 0$  such that  $\forall \delta > 0$  there exists  $y \in B(x, \delta)$  such that

$$|f(x) - f(y)| \geq \epsilon$$

However, consider the compact interval  $[x - \gamma, x + \gamma] \subseteq (0, 1)$  for some  $\gamma > 0$ . Then, there exists  $N \in \mathbb{N}$  such that

$$|f_n(z) - f(z)| < \epsilon/3$$

for all  $z \in [x - \gamma, x + \gamma]$ ,  $n \geq N$ . □

- (b) False. Let

$$f_n(x) = \frac{1}{x} + \frac{1}{n} \quad f(x) = \frac{1}{x}$$

Then,  $f_n \rightarrow f$  uniformly on  $(0, 1)$  but  $f$  is not uniformly continuous.

- (c) *Proof.* □

9. See Summer 13 #7

## Winter 2015

1. *Proof.* For  $u = 1 + n^2x^2$ ,  $du = 2n^2x dx$ ,

$$\|f_n - 0\|_1 = \int_0^1 \frac{nx}{1 + n^2x^2} dx = \int_1^2 \frac{du}{2nu} = \frac{1}{2n} [\ln(2) - \ln(1)] \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore  $f_n \rightarrow 0$  in  $L^1[0, 1]$ . Now, since  $\|\cdot\|_\infty$  and  $\|\cdot\|_{\sup}$  coincide on continuous functions,

$$\|f_n - 0\|_\infty = \sup_{x \in [0, 1]} |f_n(x)| \geq f_n(n^{-3/2}) = \frac{n^{-1/2}}{1 + n^{-1}} \rightarrow \infty$$

as  $n \rightarrow \infty$ . So  $f_n \not\rightarrow 0$  in  $L^\infty[0, 1]$ . □

2. *Proof.* Let  $f$  be convex. Let  $x_n \searrow x$ . Then, define  $t_n \in [0, 1]$  by

$$(1 - t_n)1 + t_nx = x_n$$

Notice that  $t_n \rightarrow 1$  as  $n \rightarrow \infty$ . Then,

$$f(x_n) = f(t_nx + (1 - t_n)1) \leq t_nf(x) + (1 - t_n)f(1)$$

also define  $s_n \in [0, 1]$  such that

$$(1 - s_n)(-1) + s_nx_n = x$$

then  $s_n \rightarrow 1$  as  $n \rightarrow \infty$  so

$$f(x) = f(s_nx_n + (1 - s_n)(-1)) \leq s_nf(x_n) + (1 - s_n)f(-1)$$

Combining this, we get

$$f(x) \leq s_nf(x_n) + (1 - s_n)f(-1) \leq s_nt_nf(x) + s_n(1 - t_n)f(1) + (1 - s_n)f(-1)$$

Notice that the RHS converges to  $f(x)$  as  $n \rightarrow \infty$  so by the Squeeze theorem

$$\lim_{n \rightarrow \infty} f(x_n) = f(x)$$

So  $f$  is right continuous. To show left continuity, we follow the same steps but modify  $t_n$  and  $s_n$  so they are convex combinations with the opposite endpoints. Therefore  $f$  is continuous. □

3. *Proof.* For each  $n \in \mathbb{N}$  there exists  $x_n \in X$  such that

$$d(x_n, f(x_n)) < \frac{1}{n}$$

Since  $X$  is compact, there exists a convergent subsequence  $\{x_{n_k}\}_{k=1}^\infty$  with limit  $x$ . Then,

$$d(x, f(x)) \leq d(x, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) + d(f(x_{n_k}), f(x)) \rightarrow 0$$

as  $k \rightarrow \infty$  by construction of  $x_{n_k}$  and since  $f$  is continuous. Thus  $f(x) = x$ . □



4. (a) *Proof.* Let  $\{y_n\} \subseteq Y$  be Cauchy. There exists  $\{x_n\} \subseteq X$  such that  $f(x_n) = y_n$  for all  $n \in \mathbb{N}$ . Then  $\{x_n\}$  is Cauchy and therefore convergent to some  $x \in X$ . Then,

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} f(x_n) = f(x) \in Y$$

□

- (b) False. Let  $X = (0, 1)$ ,  $Y = \mathbb{R}$ . Let  $d_Y = d_X = |(\cdot) - (\cdot)|$ . Let  $f(x) = 1/x$ . Then, clearly

$$|x_1 - x_2| \leq \left| \frac{x_1}{x_1 x_2} - \frac{x_2}{x_1 x_2} \right| = |f(x_2) - f(x_1)|$$

so the inequality holds. Additionally,  $Y$  is complete but  $X$  is not.

5. *Proof.*

$$\|S(a)\|_2 = \sqrt{\sum_{n=1}^{\infty} s_n^2 a_n^2} \leq \|s\|_{\infty} \|a\|_2$$

For each  $k \in \mathbb{N}$ , there exists  $s_{n_k} \in s$  such that

$$|s_{n_k}| > \|s\|_{\infty} - \frac{1}{k}$$

Then, consider  $e_{n_k} = (0, \dots, 0 \overset{n_k}{1}, 0, \dots) \in \ell^2$ .  $\|e_{n_k}\|_2 = 1$  so

$$\|S(e_{n_k})\|_2 = |s_{n_k}| > \|s\|_{\infty} - \frac{1}{k}$$

for all  $k \in \mathbb{N}$  thus

$$\|S\| = \|s\|_{\infty}$$

□

6. *Proof.* First, notice that  $T$  is bounded below:

$$\|x\|^2 \leq \langle Tx, x \rangle \leq \|Tx\| \cdot \|x\|$$

so,  $\|Tx\| \geq \|x\|$  for all  $x \in \mathcal{H}$ . Now, we show one-to-one. Let  $x \in \mathcal{H}$  such that  $Tx = 0$ . Then,

$$0 = \|Tx\| \geq \|x\|_{\mathcal{H}} \geq 0$$

so  $x = 0$ . Next, we show  $T$  has a closed range. Let  $x_n \in \mathcal{H}$  such that  $Tx_n \rightarrow y$  for some  $y \in \mathcal{H}$ . Then,

$$\|Tx_n - Tx_m\| \geq \|x_n - x_m\|$$

for all  $n, m \in \mathbb{N}$ . So,  $\{x_n\}$  is Cauchy. Thus, there exists  $x \in \mathcal{H}$  such that  $x_n \rightarrow x$ . Since  $T$  is bounded,

$$y = \lim_{n \rightarrow \infty} Tx_n = Tx$$

so  $y \in \text{Ran} T$ . Finally, we show  $T$  is onto. For  $w \in (\text{Ran} T)^{\perp}$

$$\langle Tv, w \rangle = 0$$

for all  $v \in \mathcal{H}$ . In particular, for  $v = w$ ,

$$0 = \langle Tw, w \rangle \geq \|w\|^2 \geq 0$$

which implies  $w = 0$ . Thus,  $(\text{Ran} T)^\perp = \{0\}$  so  $\text{Ran} T = \overline{\text{Ran} T} = \mathcal{H}$ . We have  $T$  is one-to-one and onto therefore is it invertible so  $Tx = y$  has a unique solution for every  $y \in \mathcal{H}$ .  $\square$

7. Solution by Hao Chen and Walton Green (4/18)

*Proof.* We will prove the contrapositive of the statement. Suppose  $\{E_k\}_{k=1}^n$  are Borel subsets of  $[0, 1]$  such that

$$\lambda\left(\bigcap_{k=1}^n E_k\right) = 0$$

Then, we have that

$$1 = \lambda([0, 1]) = \lambda\left[\left(\bigcap_{k=1}^n E_k\right)^c\right] = \lambda\left(\bigcup_{k=1}^n E_k^c\right)$$

Therefore,

$$n = \sum_{k=1}^n \lambda([0, 1]) = \sum_{k=1}^n \lambda(E_k) + \lambda(E_k^c) \geq \sum_{k=1}^n \lambda(E_k) + \lambda\left(\bigcup_{k=1}^n E_k^c\right) = \sum_{k=1}^n \lambda(E_k) + 1$$

$$\text{so } \sum_{k=1}^n \lambda(E_k) \leq n - 1. \quad \square$$

8. Let  $\{q_n\}_{n=1}^\infty \subseteq \mathbb{R}$  be an enumeration of the rational numbers. Then, let

$$U := \bigcup_{n=1}^\infty \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$$

So,

$$\lambda(U) \leq \sum_{n=1}^\infty \frac{2}{n^2} = 2 < \infty$$

and  $U \subseteq \mathbb{R}$  is open. Now, notice that  $\bar{U} = \mathbb{R}$  since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  and  $\mathbb{Q} \subseteq U$ . Then,

$$\lambda(\partial U) = \lambda(\bar{U} \setminus U) = \infty$$

9. *Proof.*  $(\Rightarrow)$  let  $\lambda(E) = M > 0$ . Let  $f_n \xrightarrow{\lambda} 0$ . Then, for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\inf \{c > 0 : \lambda\{|f_n| > c\} < \epsilon\} < \epsilon$$

for all  $n \geq N$  which implies

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

Now, we will use the fact that

$$x \mapsto \frac{x}{x+1}$$

is monotone increasing and  $\leq 1$ .

$$\begin{aligned} \int_E \frac{|f_n|}{1+|f_n|} &= \int_{E \cap \{|f_n| > \epsilon\}} \frac{|f_n|}{1+|f_n|} + \int_{E \cap \{|f_n| < \epsilon\}} \frac{|f_n|}{1+|f_n|} \\ &\leq \int_{E \cap \{|f_n| > \epsilon\}} 1 + \int_E \frac{\epsilon}{1+\epsilon} \\ &\leq \lambda\{|f_n| > \epsilon\} + \lambda(E) \left( \frac{\epsilon}{1+\epsilon} \right) \\ &< \epsilon + M \left( \frac{\epsilon}{1+\epsilon} \right) \rightarrow 0 \end{aligned}$$

as  $\epsilon \rightarrow 0$ .

( $\Leftarrow$ ) Let  $\epsilon > 0$ . There exists  $N \in \mathbb{N}$  such that

$$\frac{\epsilon^2}{1+\epsilon} > \int_E \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n| > \epsilon\}} \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n| > \epsilon\}} \frac{\epsilon}{1+\epsilon} = \lambda\{|f_n| > \epsilon\} \frac{\epsilon}{1+\epsilon}$$

so

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

for all  $n \geq N$ . Thus,

$$\|f_n\|_\lambda = \inf\{c > 0 : \lambda\{|f_n| > c\} < c\} < \epsilon$$

□

10. *Proof.* Define

$$F_i := \bigcup_{n=i}^{\infty} E_n$$

for each  $i \in \mathbb{N}$ . Notice that  $F_i$  are reverse nested (i.e.  $F_{i+1} \subseteq F_i$  therefore  $F_i^c \subseteq F_{i+1}^c$ )  
Then,

$$\mu(F_i) = \mu\left(\bigcup_{n=i}^{\infty} E_n\right) \leq \sum_{n=i}^{\infty} \mu(E_n) \rightarrow 0$$

as  $i \rightarrow \infty$ . Now,

$$\begin{aligned} \mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n\right) &= \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \mu\left(F_1 \cap \bigcap_{k=2}^{\infty} F_k\right) = \mu\left(F_1 \setminus \bigcup_{k=2}^{\infty} F_k^c\right) \\ &= \mu(F_1) - \mu\left(\bigcup_{k=2}^{\infty} F_k^c\right) = \mu(F_1) - \lim_{k \rightarrow \infty} \mu(F_k^c) \\ &= \lim_{k \rightarrow \infty} \mu(F_1 \setminus F_k^c) = \lim_{k \rightarrow \infty} \mu(F_1 \cap F_k) \\ &= \lim_{k \rightarrow \infty} \mu(F_k) = 0 \end{aligned}$$

□

## Summer 2015

1. (a) *Proof.* Let  $\epsilon > 0$ , pick  $N \in \mathbb{N}$  such that

$$\sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all  $n \geq N$ . This can be done since  $\sum M_n < \infty$ . Now, for all  $x \in \mathbb{R}$ ,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all  $n \geq N$ . Therefore

$$\sum_{k=1}^{\infty} f_k(x)$$

is uniformly convergent. □

- (b) Define

$$f_n(x) := \begin{cases} \frac{1}{n} & n \leq x < n+1 \\ 0 & \text{otherwise} \end{cases} \quad \forall n \in \mathbb{N}$$

Then, clearly  $\sum f_n(x)$  is convergent pointwise and

$$\sum_{n=1}^{\infty} \|f_n\|_{\infty} \leq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Now we need to show this convergence is actually uniform. Let  $\epsilon > 0$ . Pick  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \epsilon$ . Then, for all  $x \in \mathbb{R}$ ,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \frac{1}{n+1} \leq \frac{1}{N} < \epsilon$$

for all  $n \geq N$ .

2. *Proof.* First we show  $(A^{\perp})^{\perp}$  is a closed subspace containing  $A$ . Clearly  $A \subset (A^{\perp})^{\perp}$ . Let  $x, y \in (A^{\perp})^{\perp}$  and  $a, b \in \mathbb{C}$ . Then,

$$\langle ax + by | z \rangle = a \langle x | z \rangle + b \langle y | z \rangle = 0 + 0 = 0 \quad \forall z \in A^{\perp}$$

Let  $\{x_n\}_{n=1}^{\infty} \subset (A^{\perp})^{\perp}$  such that  $x_n \rightarrow x$ .

$$\langle x | z \rangle = \lim_{n \rightarrow \infty} \langle x_n | z \rangle = \lim_{n \rightarrow \infty} 0 = 0 \quad \forall z \in A^{\perp}$$

So, we have shown  $\overline{\text{span} A} \subset (A^{\perp})^{\perp}$ . Now, let  $x \in (A^{\perp})^{\perp}$ . Then,

$$d(x, \overline{\text{span} A}) = \sup_{\substack{y \in (\overline{\text{span} A})^{\perp} \\ \|y\| \leq 1}} |\langle x | y \rangle| = \sup_{y \in A^{\perp}, \|y\| \leq 1} |\langle x | y \rangle| = 0$$

So  $x \in \overline{\text{span} A}$  since it is closed. □

3. (a) *Proof.* (i) Clearly,  $d_s(A, A) = 0$ . Now, suppose  $d_s(A, B) = 0$ . Then for all  $\epsilon > 0$  and  $x \in A$ ,

$$d(x, B) \leq \epsilon$$

thus  $d(x, B) = 0$  so  $x \in B$  since  $B$  is closed. Thus  $A \subseteq B$ . Likewise  $B \subseteq A$ . so  $A = B$ .

(ii) Clearly  $d_s(A, B) = d_s(B, A)$ .

(iii) Let  $C \subseteq X$  be closed. Let  $\epsilon_1 > 0$  be such that  $A_{\epsilon_1} \subset C$  and  $C_{\epsilon_1} \subseteq A$ . Let  $\epsilon_2 > 0$  such that  $B_{\epsilon_2} \subset C$  and  $C_{\epsilon_2} \subseteq B$ . Then,

$$A_{\epsilon_1 + \epsilon_2} \subset C_{\epsilon_2} \subset B \quad \text{and} \quad B_{\epsilon_1 + \epsilon_2} \subset C_{\epsilon_1} \subset A$$

So,  $d_s(A, B) \leq \epsilon_1 + \epsilon_2$  for all such  $\epsilon_1, \epsilon_2$ . Therefore,

$$d_s(A, B) \leq \inf\{\epsilon_1\} + \inf\{\epsilon_2\} = d_s(A, C) + d_s(C, B)$$

□

- (b) If the sets are not closed, then the first property of the metric fails.  $d_s(A, A) = 0$  but  $d_s(A, B) = 0$  does not necessarily  $A = B$ . Consider  $X = \mathbb{R}$  and  $A = [0, 1]$  and  $B = (0, 1)$ .  $d_s(A, B) = 0$  but  $A \neq B$ .

4. (a) *Proof.* First, we show  $T$  is bounded:

$$\begin{aligned} \|Tf\|_\infty &= \sup_{t \in [0, 1]} \left| \int_0^t s f(s) ds \right| \leq \sup_{t \in [0, 1]} \int_0^t s |f(s)| ds \\ &\leq \|f\|_\infty \sup_{t \in [0, 1]} \int_0^t s ds \leq \|f\|_\infty \int_0^1 s ds = \frac{1}{2} \|f\|_\infty \end{aligned}$$

so  $\|T\| \leq \frac{1}{2}$ . Let  $f, g \in C[0, 1]$  and  $a, b \in \mathbb{R}$ . Then,

$$T(af + bg)(t) = \int_0^t s(af + bg)(s) ds = a \int_0^t s f(s) ds + b \int_0^t s g(s) ds = a(Tf)(t) + b(Tg)(t)$$

so  $T$  is linear. □

- (b) *Proof.* Let  $f(t) = 1$  for all  $t \in [0, 1]$ . Then,  $\|f\|_\infty = 1$  and

$$\|Tf\|_\infty = \sup_{t \in [0, 1]} \left| \int_0^t s ds \right| = \sup_{t \in [0, 1]} \frac{t^2}{2} = \frac{1}{2}$$

so  $\|T\| = \frac{1}{2}$ . □

5. (a) *Proof.* For every  $n \in \mathbb{N}$  there exists  $E_n \subseteq X$  such that  $\mu(E_n) < \frac{1}{n^2}$  and  $f_k \rightarrow f$  uniformly on  $X \setminus E_n$ . Let

$$E := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

Then,  $\mu(E) = 0$  (For proof see Winter 15 #10) since

$$\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Now, for  $x \notin E$ , there exists  $k$  such that  $x \notin \bigcup_{n=k}^{\infty} E_n$  so  $x \notin E_n$  for all  $n \geq k$  (However we only need it to hold for a single set,  $E_k$ . So, since  $x \in E_k^c$ ,

$$f_n(x) \rightarrow f(x)$$

as  $n \rightarrow \infty$ . Therefore  $f_n \rightarrow f$  pointwise a.e. □

- (b) *Proof.* Let  $\epsilon > 0$ . Then, there exists some  $E_\epsilon$  such that  $\mu(E_\epsilon) < \epsilon$  and  $f_n \rightarrow f$  uniformly on  $E_\epsilon$ . Moreover, there exists  $N \in \mathbb{N}$  such that

$$|f_n(x) - f(x)| < \epsilon$$

for all  $n \geq N$ ,  $x \in E_\epsilon^c$ . Then,

$$\mu\{|f_n - f| > \epsilon\} \leq \mu(E_\epsilon) < \epsilon$$

so

$$\|f_n - f\|_\mu = \inf\{c > 0 : \mu\{|f_n - f| > c\} < c\} < \epsilon$$

therefore  $f_n \rightarrow f$  in measure. □

6. *Proof.* Let  $E \subset [a, b]$  be Borel measurable with  $\lambda(E) > 0$ . Let  $\{q_n\}$  be an enumeration of the rational numbers in the interval  $[0, 1]$ . Set

$$F = \bigcup_n (E + q_n)$$

If  $\{E + q_n\}$  are all disjoint, then,  $\lambda(F) = \sum_{n=1}^{\infty} \lambda(E + q_n) = \sum_{n=1}^{\infty} \lambda(E) = \infty$  since  $\lambda(E) > 0$ . But this is a contradiction since  $F \subseteq [a, b + 1]$  which has finite Lebesgue measure. Thus there exists  $x \in (E + q_n) \cap (E + q_m)$  for some  $n$  and  $m$  not equal (so  $q_n \neq q_m$ ). Then, there exists  $y, z \in E$  such that

$$y + q_n = x = z + q_m$$

so  $y - z = q_m - q_n \in \mathbb{Q} \setminus \{0\}$ . □

7. (a) False. Consider the following function with a “spike” at every natural number,  $n \geq 2$ .

$$f(x) := \begin{cases} \nearrow & n \leq x \leq n + \frac{1}{n^3} \\ n & x = n + \frac{1}{n^3} \\ \searrow & n + \frac{1}{n^3} \leq x \leq n + \frac{2}{n^3} \\ 0 & \text{else} \end{cases}$$

Notice that

$$\int_{\mathbb{R}} f(x) dx = \sum_{n=2}^{\infty} \frac{1}{2} \cdot \frac{2}{n^3} \cdot n = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$$

but

$$\limsup_{x \rightarrow \infty} |f(x)| = \infty$$

- (b) *Proof.* Let  $f$  be integrable and differentiable and let  $D > 0$  such that  $|f'(x)| \leq D$  for all  $x \in \mathbb{R}$ . Fix  $x \in \mathbb{R}$ . Using the mean-value theorem, for all  $y \in \mathbb{R}$  such that

$$|x - y| \leq \frac{f(x)}{D},$$

we know that

$$f(y) \geq f(x) - |x - y|D$$

Suppose without loss of generality that

$$\limsup_{x \rightarrow \infty} f(x) = M$$

for some  $M > 0$ . Then for all  $n \in \mathbb{N}$ , there exists  $x_n \geq n$  such that

$$f(x_n) \geq \frac{M}{2}$$

Then,

$$\int_{\mathbb{R}} f(x) dx \geq \sum_{n=1}^{\infty} \frac{1}{2} \cdot \min \left\{ \frac{f(x_n)}{D}, 1 \right\} \cdot f(x_n) \geq \sum_{n=1}^{\infty} \frac{1}{8} \cdot \min \left\{ \frac{M}{D}, 1 \right\} \cdot M = \infty$$

which contradicts the fact that  $f$  is integrable. □

8. *Proof.* (a  $\implies$  b)

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^{\infty} \int_{F_k} 2^k dm \leq \sum_{k=-\infty}^{\infty} \int_{F_k} f dm = \int_{\mathbb{R}} f dm$$

(b  $\implies$  c) First, notice that  $E_k \cup F_{k-1} = E_{k-1}$  and the union is disjoint therefore

$$m(F_k) = m(E_{k+1}) - m(E_k)$$

Mutlply by  $2^k$  and sum from  $-N$  to  $N$  we have

$$\begin{aligned} \sum_{k=-N}^N 2^k m(F_k) &= \sum_{k=-N}^N 2^k m(E_{k+1}) - \sum_{k=-N}^N 2^k m(E_k) = \frac{1}{2} \sum_{k=-N}^N 2^{k+1} m(E_{k+1}) - \sum_{k=-N}^N 2^k m(E_k) \\ &= -\frac{1}{2} \sum_{k=-N+1}^{N-1} 2^k m(E_k) + 2^N m(E_{N+1}) - 2^{-N} m(E_{-N}) \end{aligned}$$

The final two terms can be bounded by  $\int f$ :  $2^N m(E_N) \leq \int_{E_N} f \, dm \leq \int_{\mathbb{R}} f \, dm < \infty$ .  
Therefore, for any  $N$ ,

$$\sum_{k=-(N-1)}^{N-1} 2^k m(E_k) \leq -2 \sum_{k=-\infty}^{\infty} 2^k m(F_k) + 4 \int_{\mathbb{R}} f \, dm < \infty$$

(c  $\implies$  a) Notice that since  $f$  is non-negative,  $\mathbb{R} = \{f = 0\} \cup E_k$ .

$$\int f \, dm = \sum_{k=-\infty}^{\infty} \int_{F_k} f \, dm \leq \sum_{k=-\infty}^{\infty} \int_{F_k} 2^{k+1} \, dm = 2 \sum_{k=-\infty}^{\infty} m(F_k) \leq 2 \sum_{k=-\infty}^{\infty} m(E_k)$$

□



## Winter 2016

1. Recall

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

So for  $x = 1$ ,

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

2. *Proof.* Since  $\ell^2$  is a Hilbert space,  $A$  being dense in  $\ell^2$  is equivalent to

$$A^\perp = \{0\}$$

Let  $x = (x_1, x_2, \dots) \in A^\perp$ . Then,  $\langle x, a \rangle_{\ell^2} = 0$  for all  $a \in A$ . Notice that  $a = e^{(k)} = (0, \dots, 0, \underset{k^{th}}{1}, 0, \dots)$  is in  $A$  for any  $k \in \mathbb{N}$ .

$$0 = \langle x, e^{(k)} \rangle = \sum_{i=1}^{\infty} x_i e_i^{(k)} = x_k$$

for  $k \in \mathbb{N}$ . Therefore  $x = 0$ . Now we show the same thing for  $A^c$ . Let  $y \in (A^c)^\perp$ . Also, define  $f^{(k)} = e^{(k)} - e^{(k+1)} \in A^c$ . Then,

$$0 = \langle y, f^{(k)} \rangle = y_k - y_{k+1}$$

So  $y_k = y_{k+1}$  for all  $k \in \mathbb{N}$ . Thus  $y$  is a constant sequence. The only constant sequence in  $\ell^2$  is the zero sequence therefore  $y = 0$ .  $\square$

3. *Proof.* First we show subspace. Let  $x_1 + y_1, x_2 + y_2 \in X + Y$  and  $a, b \in \mathbb{R}$ . Then,

$$a(x_1 + y_1) + b(x_2 + y_2) = (ax_1 + bx_2) + (ay_1 + by_2) \in X + Y$$

Now we show closure. Let  $\{(x_n + y_n)\}_{n=1}^{\infty}$  be a sequence in  $X + Y$  with limit  $z$ . This sequence is also Cauchy. So, using the fact that  $X \perp Y$ ,

$$\begin{aligned} \|(x_n + y_n) - (x_m + y_m)\|^2 &= \|(x_n - x_m) + (y_n - y_m)\|^2 \\ &= \langle (x_n - x_m) + (y_n - y_m), (x_n - x_m) + (y_n - y_m) \rangle \\ &= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (x_n - x_m), (y_n - y_m) \rangle \\ &\quad + \langle (y_n - y_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\ &= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\ &= \|x_n - x_m\|^2 + \|y_n - y_m\|^2 \end{aligned}$$

and thus  $\{x_n\}$  and  $\{y_n\}$  are both Cauchy. Since  $\mathcal{H}$  is a Hilbert space, they are convergent to some  $x, y$  respectively. Since  $X, Y$  are closed,  $x \in X$  and  $y \in Y$ . Then,

$$z = \lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n = x + y \in X + Y$$

Therefore  $X + Y$  is closed.  $\square$

4. *Proof.* Since  $Y$  is a Banach space,  $\mathcal{B}(X, Y)$  is also a Banach space. In a Banach space, any absolutely convergent series is convergent. Since  $\|T\| < 1$ ,

$$\sum_{n=0}^{\infty} \|T\|^n < \infty$$

So

$$\sum_{n=0}^{\infty} T^n \in \mathcal{B}(X, Y)$$

□

5. (a) *Proof.* First, to show  $T$  is well-defined we need to show  $T\xi$  is continuous for a fixed  $\xi$ . This follows from the fact that for  $n > m$ ,

$$\left\| \sum_{k=0}^n a_k \xi_k x^k - \sum_{k=0}^m a_k \xi_k x^k \right\|_{\infty} = \sup_{x \in [0,1]} \left| \sum_{k=m+1}^n a_k \xi_k x^k \right| \leq \|a\|_{\infty} \sum_{k=m+1}^n |\xi_k| \rightarrow 0$$

as  $n, m \rightarrow \infty$  since  $\xi \in \ell^1$ . Thus, this sequence of partial sums is Cauchy in  $\|\cdot\|_{\infty}$ . Since  $(C[0, 1], \|\cdot\|_{\infty})$  is a Banach space, it's limit,  $T\xi \in C[0, 1]$ . To show linearity, let  $\xi, \zeta \in \ell^1$  and  $\alpha, \beta \in \mathbb{R}$ .

$$\begin{aligned} T(\alpha\xi + \beta\zeta)(x) &= \sum_{k=0}^{\infty} a_k (\alpha\xi_k + \beta\zeta_k) x^k = \alpha \sum_{k=0}^{\infty} a_k \xi_k x^k + \beta \sum_{k=0}^{\infty} a_k \zeta_k x^k \\ &= \alpha T(\xi)(x) + \beta T(\zeta)(x) \end{aligned}$$

□

- (b) *Proof.*

$$\|T(\xi)\|_{\infty} = \sup_{x \in [0,1]} |T(\xi)(x)| = \sup_{x \in [0,1]} \left| \sum_{k=0}^{\infty} a_k \xi_k x^k \right| \leq \|a\|_{\infty} \sum_{k=0}^{\infty} |\xi_k| = \|a\|_{\infty} \cdot \|\xi\|_1$$

So  $\|T\| \leq \|a\|_{\infty}$ . We claim this is actually the norm. For  $\epsilon > 0$  there exists  $a_n \in a$  such that

$$|a_n| > \|a\|_{\infty} - \epsilon$$

Pick  $\xi^{(n)} = (0, \dots, 0, \overset{n^{th}}{1}, 0, \dots) \in \ell^1$ . Then,

$$\|T\xi^{(n)}\|_{\infty} = \sup_{x \in [0,1]} \left| \sum_{k=0}^{\infty} a_k \xi_k^{(n)} x^k \right| = \sup_{x \in [0,1]} |a_n x^n| = |a_n| > \|a\|_{\infty} - \epsilon$$

Since there exists such  $\xi^{(n)}$  for all  $\epsilon > 0$ ,  $\|T\| = \|a\|_{\infty}$ . □

6. (i) LDCT cannot be applied to  $f_n$  since any  $k$  which bounds every  $f_n$  above, must be greater than 1 everywhere thus  $\int_{\mathbb{R}} k = \infty$ .

- (ii) LDCT cannot be applied to  $g_n$  since any  $k$  which bounds every  $g_n$  above, must be greater than  $1/x$  everywhere thus  $\int_{\mathbb{R}} k \geq \int_{\mathbb{R}} 1/x = \infty$ .
- (iii) LDCT can be applied since for  $k = 1/x^2$ ,  $|h_n| \leq k$  and

$$\int_{\mathbb{R}} \frac{1}{x^2} dx < \infty$$

7. *Proof.* Let  $E \subset \mathbb{R}$ ,  $\epsilon \in (0, 1)$ . Set  $\delta = m^*(E)(1/\epsilon - 1) > 0$ . By definition of outer measure, there exists an open set  $G \supset E$  such that  $m^*(E) + \delta > m^*(G) = m(G)$ . Then,

$$\epsilon m(G) < \epsilon(m^*(E) + \delta) = \epsilon m^*(E)(1 + 1/\epsilon - 1) = m^*(E)$$

Moreover, since  $G$  is open, it can be written as a countable, disjoint union of open intervals, say  $I_k$ . Then,

$$\sum_k \epsilon m(I_k) = \epsilon m(G) < m^*(E) = m^*(E \cap G) \leq \sum_k m^*(E \cap I_k)$$

Therefore, at least one term in the left hand sum must be smaller than one term in the right hand sum, i.e. there exists  $k$  such that  $\epsilon m^*(I_k) = \epsilon m(I_k) < m^*(E \cap I_k)$ .  $\square$

8. *Proof.* Define  $A_n := \{x \in [0, 1] : n + 1 > |f(x)| \geq n\}$

$$\sum_{n=1}^{\infty} n \lambda(A_n) = \sum_{n=1}^{\infty} \int_{A_n} n dx \leq \sum_{n=1}^{\infty} \int_{A_n} f(x) dx = \int_0^1 f(x) < \infty$$

So,

$$\begin{aligned} \lim_{n \rightarrow \infty} n \lambda(\{x \in [0, 1] : |f(x)| \geq n\}) &= \lim_{n \rightarrow \infty} n \lambda\left(\bigcup_{k=n}^{\infty} A_k\right) = \lim_{n \rightarrow \infty} n \sum_{k=n}^{\infty} \lambda(A_k) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} k \lambda(A_k) = 0 \end{aligned}$$

$\square$

9. (a) *Proof.* Let  $f(x) > 0$  for  $x \in [0, 1]$  and  $E \subseteq [0, 1]$  such that  $\lambda(E) > 0$ . Suppose  $\int_E f d\lambda = 0$ . Then

$$f(x) = 0$$

for almost every  $x \in E$ . However, since  $\lambda(E) > 0$  there exists  $x \in E$  such that  $f(x) > 0$  which is a contradiction.  $\square$

- (b) First we prove the following fact:  $\mu(\limsup E_n) \geq \limsup \mu(E_n)$ . Indeed, set  $F_k = \bigcup_{n=k}^{\infty} E_n$ .  $F_k$  are decreasing.

$$\mu(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n) = \mu(\bigcap_{k=1}^{\infty} F_k) = \inf_k \mu(F_k) = \inf_k \mu(\bigcup_{n=k}^{\infty} E_n) \geq \inf_k \sup_{n \geq k} \mu(E_n)$$

*Proof.* Fix  $\epsilon \in (0, 1]$ . Suppose  $\inf_{\lambda(E) \geq \epsilon} \int_E f d\lambda = 0$ . Then, for each  $n$  there exists  $E_n$  with  $\lambda(E_n) \geq \epsilon$  and

$$\int_{E_n} f d\lambda < \frac{1}{n^2}$$

Then, consider  $E = \limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$ . By the fact above,  $\mu(E) \geq \epsilon$ . By part (a), this means  $\int_E f d\lambda > 0$ . However,

$$\int_E f d\lambda = \int_{\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n} f d\lambda \leq \int_{\bigcup_{n=k}^{\infty} E_n} f d\lambda \leq \sum_{n=k}^{\infty} \int_{E_n} f d\lambda \leq \sum_{n=k}^{\infty} \frac{1}{n^2}$$

for any  $k$ . Therefore  $\int_E f d\lambda = 0$  which is a contradiction. □

## Summer 2016

1. (a) *Proof.* We show that  $f_n$  does not converge uniformly on the half-open interval  $[0, 1)$ . The pointwise limit is clearly  $f(x) = 0$  for  $x \in [0, 1)$ . If  $\{f_n\}$  converges uniformly, then it must converge to  $f$ , the pointwise limit. Let  $\epsilon > 0$ . For any  $n \in \mathbb{N}$  there exists  $x \in (0, 1]$  such that

$$1 > x > \left(\frac{\epsilon}{1-\epsilon}\right)^{1/n}$$

Then,

$$|f(x)| > \epsilon$$

so  $\{f_n\}$  does not converge uniformly on  $[0, 1)$  therefore it does converge uniformly on  $[0, 1]$ .  $\square$

- (b) *Proof.* Notice that

$$f_n(x) \leq 1$$

for  $x \in [0, 1]$ . Since  $\int_0^1 1 \, dx < \infty$ , by Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \rightarrow \infty} f_n(x) \, dx = \int_0^1 0 \, dx = 0$$

$\square$

2. False. Counterexample:

Consider  $\{x^{(n)}\}_{n=1}^\infty \subset X$  where

$$x^{(n)} := (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots) \in X$$

Then,  $\{x^{(n)}\}_{n=1}^\infty$  is Cauchy: For  $\epsilon > 0$ , pick  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \epsilon$ . So, for all  $n, m \geq N$  ( $n > m$ ),

$$d(x^{(n)}, x^{(m)}) = \sup_{i \in \mathbb{N}} |x_i^{(n)} - x_i^{(m)}| = \frac{1}{m} < \frac{1}{N} < \epsilon$$

However,  $x_n \rightarrow (1, \frac{1}{2}, \frac{1}{3}, \dots)$  which is not in  $X$ .

3. *Proof.* Let  $\{y_n\}_{n=1}^\infty \subset K$ . Let  $\{y_{n_k}\}_k$  denote the set of distinct elements of  $\{y_n\}_{n=1}^\infty$ . If  $\{y_{n_k}\}_k$  is finite, then there exists some  $m \in \mathbb{N}$  such that  $y_m$  occurs infinitely many times in  $\{y_n\}_{n=1}^\infty$  thus the constant sequence  $\{y_m\}$  is a convergent subsequence of  $\{y_n\}_{n=1}^\infty$ . On the other hand if  $\{y_{n_k}\}_k$  is infinite, then

$$\{y_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=0}^\infty$$

is a subsequence of a convergent sequence so it is itself convergent to  $\lim_{n \rightarrow \infty} x_n = x_0 \in K$ .  $\square$

4. *Proof.* First, notice

$$\begin{aligned}
(T - S)^3 &= (T^2 - ST - TS + S^2)(T - S) \\
&= (T - 2ST + S)(T - S) \\
&= (T^2 - 2ST^2 + ST - ST + 2S^2T - S^2) \\
&= (T - 2ST + ST - ST + 2ST - S) \\
&= (T - S)
\end{aligned}$$

Then, by Cauchy-Schwarz for the operator norm,

$$\|T - S\| = \|(T - S)^3\| \leq \|T - S\|^3$$

Therefore

$$1 \leq \|T - S\|^2$$

and

$$\|T - S\| \geq 1$$

□

5. *Proof.* Let  $n, m \in \mathbb{N}$ . Without loss of generality, let  $n > m$ . First,

$$\|x_m\|^2 = \langle x_n, x_m \rangle \leq \|x_n\| \cdot \|x_m\|$$

so  $\{\|x_n\|\}_{n=1}^\infty$  is monotone decreasing. Moreover it is bounded below by 0 so it is convergent to some  $K \in \mathbb{R}$ . Moreover, since

$$\lim_{n \rightarrow \infty} \|x_n\|^2 = \left( \lim_{n \rightarrow \infty} \|x_n\| \right)^2 = K^2$$

$\{\|x_n\|^2\}_{n=1}^\infty$  is convergent and therefore Cauchy. Then, for  $n > m$ ,

$$\begin{aligned}
\|x_n - x_m\|^2 &= \langle x_n - x_m, x_n - x_m \rangle \\
&= \|x_n\|^2 - \langle x_n, x_m \rangle - \langle x_m, x_n \rangle + \|x_m\|^2 \\
&= \|x_n\|^2 - \|x_m\|^2 - \|x_m\|^2 + \|x_m\|^2 \\
&= \|x_n\|^2 - \|x_m\|^2 \\
&= \left| \|x_n\|^2 - \|x_m\|^2 \right| \rightarrow 0
\end{aligned}$$

So  $\{x_n\}_{n=1}^\infty$  is Cauchy and therefore convergent since  $\mathcal{H}$  is a Hilbert space. □

6. *Proof.* Let  $A = (0, 1)$  and  $B = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ . Then,

$$d(A, B) = \lambda(A \Delta B) = \lambda(\{\frac{1}{2}\}) = 0$$

but  $A \neq B$ . Thus the first property of a metric  $d(A, B) = 0 \implies A = B$  fails. □

7. *Proof.* ( $\Leftarrow$ ) Fix  $\epsilon > 0$ . Then, there exists an open set  $\mathcal{O} \supseteq A$  such that

$$\lambda(\mathcal{O} \setminus A) < \epsilon$$

Thus  $\mathcal{O} \setminus A \in \mathcal{L}$ , the  $\sigma$ -algebra of Lebesgue-measurable sets. Moreover, since  $\mathcal{O}$  is open, it is also Lebesgue measurable. Thus,

$$A = \mathcal{O} \setminus (\mathcal{O} \setminus A) \in \mathcal{L}$$

since  $\mathcal{L}$  is closed under set-minus.

( $\Rightarrow$ ) Let  $A$  be Lebesgue measurable. Then,

$$\lambda(A) = \lambda^*(A) = \inf_{A \subseteq \bigcup_n I_n} \sum_{n=1}^{\infty} \lambda(I_n)$$

where  $I_n = [a_n, b_n)$ . Now, let  $\epsilon > 0$ . By definition of inf, there exists  $\{I_n\}_{n=1}^{\infty}$  such that

$$\lambda(A) + \frac{\epsilon}{2} > \sum_{n=1}^{\infty} \lambda(I_n) \quad \text{and} \quad A \subseteq \bigcup_{n=1}^{\infty} I_n$$

Now, define

$$J_n = \left( a_n, b_n + \frac{\epsilon}{2^{n+1}} \right)$$

Then,  $I_n \subset J_n$  for all  $i$  and for  $\mathcal{O} := \bigcup_{n=1}^{\infty} J_n$

$$\lambda(\mathcal{O} \setminus A) = \lambda(\mathcal{O}) - \lambda(A) \leq \sum_{n=1}^{\infty} \lambda(J_n) - \lambda(A) = \sum_{n=1}^{\infty} \left( \lambda(I_n) + \frac{\epsilon}{2^{n+1}} \right) - \lambda(A) < \epsilon$$

□

8. (a) *Proof.* Proof by contraposition. Suppose  $\lambda(E_n) = 0$  for all  $n \in \mathbb{N}$ . Then,

$$\lambda(\{x \in I : f(x) > 0\}) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} \lambda(E_n) = 0$$

since  $\{E_n\}$  are nested. □

(b) *Proof.* Suppose the assumption holds and that  $\lambda(\{x \in I : f(x) > 0\}) > 0$ . Then, by part (a), there exists some  $n \in \mathbb{N}$  such that  $\lambda(E_n) > 0$ . Since the measure of  $E_n$  is positive, it contains infinitely many points. Now, pick  $x_1, \dots, x_{n \cdot M} \in E_n$ , then,

$$f(x_1) + \dots + f(x_{n \cdot M}) > \frac{1}{n} + \dots + \frac{1}{n} = Mn \left( \frac{1}{n} \right) = M$$

which is a contradiction. □

9. INCOMPLETE

*Proof.* ( $\Rightarrow$ ) By the Triangle Inequality,

$$\|f_n\|_1 \leq \|f_n - f\|_1 + \|f\|_1$$

and

$$\|f\|_1 \leq \|f - f_n\|_1 + \|f_n\|_1$$

therefore

$$|\|f_n\|_1 - \|f\|_1| \leq \|f_n - f\|_1 \rightarrow 0$$

as  $n \rightarrow \infty$ .

( $\Leftarrow$ )

□

10. *Proof.* Applying Holder's Inequality,

$$\begin{aligned} \sum_{n=0}^{\infty} \int_n^{n+1} f(x) dx &\leq \sum_{n=0}^{\infty} \left( \int_n^{n+1} f(x)^2 dx \right)^{1/2} \left( \int_n^{n+1} 1^2 dx \right)^{1/2} \\ &\leq \left( \int_{\mathbb{R}} f(x)^2 dx \right)^{1/2} = \|f\|_{L^2(\mathbb{R})} < \infty \end{aligned}$$

therefore

$$\lim_{n \rightarrow \infty} \int_n^{n+1} f(x) dx = 0$$

□



# Winter 2017

1. Notice that

$$\sum \frac{\sin(nx)}{n}$$

is the Fourier series of the function  $x \mapsto \frac{\pi - x}{2}$ . Indeed,

$$\frac{\pi}{2} \int_{-\pi}^{\pi} \sin(nx) dx - \frac{1}{2} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{x \cos(nx)}{2n} \Big|_{x=-\pi}^{\pi} - \frac{1}{2n} \int_{-\pi}^{\pi} \cos(nx) dx =$$

2. (a) *Proof.* (i) First, notice that

$$-M|x - y| \leq f(x) - f(y) \leq M|x - y|$$

therefore  $|f(x)| \leq |f(y)| + M|x - y|$  for all  $x, y \in \mathbb{R}$ . Therefore,

$$\begin{aligned} 0 \leq d(f, g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x)| + |g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(0)| + M|x| + |g(0)| + M|x| \\ &\leq \sum_{n=1}^{\infty} \frac{f(0) + g(0) + 2Mn}{2^n} < \infty \end{aligned}$$

so  $d(f, g)$  is well-defined and non-negative.

(ii) Clearly  $d(f, f) = 0$ . Assume  $d(f, g) = 0$ . Then  $\sup_{x \in [-n, n]} |f(x) - g(x)| = 0$  for all  $n$  thus  $f(x) = g(x)$  on  $\mathbb{R}$ .

(iii) Clearly  $d(f, g) = d(g, f)$ .

(iv)

$$\begin{aligned} d(f, g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} (|f(x) - h(x)| + |h(x) - g(x)|) \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \left( \sup_{x \in [-n, n]} |f(x) - h(x)| + \sup_{x \in [-n, n]} |h(x) - g(x)| \right) \\ &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - h(x)| + \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |h(x) - g(x)| \\ &= d(f, h) + d(h, g) \end{aligned}$$

□

(b) *Proof.* Let  $\{f_k\}_{k=1}^\infty \subseteq \mathcal{L}$  be Cauchy in  $d$ . Fix  $x \in \mathbb{R}$ . Then,  $x \in [-N, N]$  for some  $N \in \mathbb{N}$ . For any  $k, \ell \in \mathbb{N}$ ,

$$|f_k(x) - f_\ell(x)| \leq 2^N 2^{-N} \sup_{x \in [-N, N]} |f_k(x) - f_\ell(x)| \leq 2^N d(f_k, f_\ell) \rightarrow 0$$

as  $k, \ell \rightarrow \infty$ . Therefore  $\{f_k(x)\}_{k=1}^\infty \subseteq \mathbb{R}$  is Cauchy for each  $x$  and therefore convergent since  $\mathbb{R}$  is complete. Then define

$$f(x) := \lim_{k \rightarrow \infty} f_k(x)$$

First, we show  $f \in \mathcal{L}$ . Fix  $x, y \in \mathbb{R}$ . For  $\epsilon > 0$  there exists  $n_1, n_2 \in \mathbb{N}$  such that

$$|f_k(x) - f(x)| < \epsilon \quad |f_\ell(y) - f(y)| < \epsilon \quad \forall k > n_1 \ell > n_2$$

Then, for  $n = \max\{n_1, n_2\}$ ,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 2\epsilon + M|x - y|$$

so  $|f(x) - f(y)| \leq M|x - y|$  and  $f \in \mathcal{L}$ . Now we will show  $f_k \rightarrow f$  in  $d$ . Let  $\epsilon > 0$ . Since  $\{f_k\}$  is Cauchy in  $d$ ,  $\{d(f_k, 0)\}$  is uniformly bounded, i.e. there exist  $C > 0$  such that  $d(f_k, 0) \leq C$  for all  $k$ . Indeed, there exists  $N$  such that  $d(f_k, f_j) < 1$  for  $j, k \geq N$ . Thus,

$$d(f_k, 0) \leq d(f_k, f_N) + d(f_N, 0) \leq 1 + d(f_N, 0)$$

So  $d(f_k, 0) \leq \max_{j=1, \dots, N} \{1 + d(f_j, 0)\}$  for all  $k$ . Thus,

$$d(f_k, f) \leq C + d(f, 0)$$

for all  $k$  so there exists  $N$  such that

$$\sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| < \epsilon/2$$

for all  $k$ . Moreover, since  $f_k(x) \rightarrow f(x)$  for each  $x \in [-N, N]$ ,  $f_k \rightarrow f$  uniformly on  $[-N, N]$  since it is closed and bounded. Therefore we can take  $k$  large enough so that

$$\sum_{n=1}^N 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| < N 2^{-N} \sup_{x \in [-N, N]} |f_k(x) - f(x)| < \epsilon/2$$

Then,

$$d(f_k, f) = \sum_{n=1}^N + \sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| \leq \epsilon$$

for  $k$  large enough. □

3. *Proof.* Let  $f \in C^1[0, 1]$ .

$$|\varphi_0(f)| = |f'(0)| \leq \sup_{x \in [0, 1]} |f'(x)| \leq \|f\|$$

So  $\|\varphi_0\| \leq 1$ . We will show  $\|\varphi_0\| = 1$ . Consider the sequence defined

$$f_n(x) := \frac{\sin(nx)}{n}$$

Notice  $\|f_n\| = 1/n + 1$  and  $|\varphi_0(f_n)| = 1$ . Thus,

$$1 \geq \|\varphi_0\| = \sup_{f \neq 0} \frac{|\varphi_0(f)|}{\|f\|} \geq \sup_{n \in \mathbb{N}} \frac{|\varphi_0(f_n)|}{\|f_n\|} = \sup_{n \in \mathbb{N}} \frac{1}{1 + 1/n} = 1$$

so  $\|\varphi_0\| = 1$ . □

4. (a) *Proof.* Let  $x \in \ell^2$ . Then for  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$\sum_{k=N+1}^{\infty} |x_k|^2 < \epsilon$$

Then, for  $y = (x_1, x_2, \dots, x_N, 0, 0, \dots) \in Y$ ,

$$\|x - y\|_2^2 = \sum_{k=1}^{\infty} |x_k - y_k|^2 = \sum_{k=N+1}^{\infty} |x_k - 0|^2 < \epsilon$$

so  $Y$  is dense in  $\ell^2$ . □

(b) *Proof.* By Cauchy-Schwarz,

$$\left| \sum_{k=1}^n x_k \right| \leq \left( \sum_{k=1}^n |1|^2 \right)^{1/2} \left( \sum_{k=1}^n |x_k|^2 \right)^{1/2} = \sqrt{n} \left( \sum_{k=1}^n |x_k|^2 \right)^{1/2}$$

Moreover, if  $x \in \ell^2$ , then  $\sum_{k=1}^{\infty} |x_k|^2$  converges so we can bound the final term by  $\|x\|_2$ . □

(c) *Proof.* Let  $x \in \ell^2$ ,  $\epsilon > 0$ . By part (a) there exists  $y \in Y$  such that  $\|x - y\|_2 \leq \epsilon/2$ . Then,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left| \sum y_n \right| = 0$$

since the second term is bounded and the first is decreasing to 0. So, there exists  $N$  such that

$$\frac{1}{\sqrt{n}} \left| \sum y_n \right| < \epsilon/2 \quad \text{for } n \geq N$$

By triangle inequality for  $|\cdot|$  and part(b),

$$\frac{1}{\sqrt{n}} \left| \sum x_n \right| \leq \frac{1}{\sqrt{n}} \left| \sum x_n - y_n \right| + \frac{1}{\sqrt{n}} \left| \sum y_n \right| < \|x - y\|_2 + \epsilon/2 < \epsilon$$

for  $n \geq N$  so

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left| \sum x_n \right| = 0$$

□

5. *Proof.* First notice that

$$0 \leq \int_0^\infty \frac{x}{1+x^3} dx = \int_0^1 + \int_1^\infty \frac{x}{1+x^3} dx \leq \int_0^1 1 dx + \int_1^\infty \frac{1}{x^2} dx < \infty$$

Then, notice that  $\frac{x}{1+x^n} \leq \frac{x}{1+x^{n+1}}$  for  $x \in (0, 1)$ . Therefore, by monotone convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{x}{1+x^n} dx = 0$$

since  $x/(1+x^n) \rightarrow 0$  pointwise on  $(0, 1)$ . Moreover,

$$\int_1^\infty \frac{x}{1+x^n} dx$$

is monotone decreasing and bounded below by zero therefore

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{x}{1+x^n} dx = \lim_{n \rightarrow \infty} \int_0^1 + \int_1^\infty \frac{x}{1+x^n} dx$$

exists. Moreover,

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{x}{1+x^n} dx = \int_1^\infty \lim_{n \rightarrow \infty} \frac{x}{1+x^n} dx = 0$$

by the Lebesgue dominated convergence theorem since

$$\frac{x}{1+x^n} \leq \frac{x}{x^n} = x^{1-n}$$

which is integrable on  $(1, \infty)$  for  $n \geq 3$ . □

6. (a) *Proof.* Set  $f(x) = \mathbf{1}_{\liminf_n A_n}$ .

(i)  $f(x) = 1 \iff x \in \cup_k \cap_{n=k}^\infty A_n$ . So, there exists  $k$  such that  $x \in A_n$  ( $\mathbf{1}_{A_n}(x) = 1$ ) for all  $n \geq k$ . So,  $\lim_{n \rightarrow \infty} \mathbf{1}_{A_n}(x) = 1$  (so  $\liminf$  is also 1).

(ii) Suppose  $f(x) = 0$ . For each  $k$  there exists  $n \geq k$  such that  $x \notin A_n$  ( $\mathbf{1}_{A_n}(x) = 0$ ) so  $\liminf_n \mathbf{1}_{A_n} = 0$ . □

(b) By Fatou's Lemma,

$$\mu(\liminf_n A_n) = \int_X f d\mu = \int_X \liminf_n \mathbf{1}_{A_n} d\mu \leq \liminf_n \int_X \mathbf{1}_{A_n} d\mu = \liminf_n \mu(A_n)$$

7. *Proof.* Define  $f = \sup_N \sum_{n=1}^N f_n$ .  $f$  is a measurable function, moreover, since  $f_n$  are non-negative,  $\sum f_n \nearrow f$ . So, by Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f = \sum \int f_n \leq \sum \frac{1}{n^2} < \infty$$

So  $f$  is non-negative and integrable. We claim this implies  $f < \infty$  a.e. If not, then there exists  $E$  with  $\lambda(E) > 0$  and  $f = \infty$  on  $E$ . Then,

$$\int_{\mathbb{R}} f \geq \int_E f = \infty$$

so  $f < \infty$  a.e. □

8. (a) *Proof.* By Hölder's Inequality,

$$\left| \int f_n d\mu - \int f d\mu \right| \leq \int |f - f_n| d\mu \leq \|f - f_n\|_{\infty} \int d\mu = \|f - f_n\|_{\infty} \mu(X) \rightarrow 0$$

as  $n \rightarrow \infty$ . □

## Summer 2017

2. (a) Suppose  $f_n(x) = f(x^n)$  converges uniformly to some function  $g$ . Since each  $f_n$  is continuous,  $g$  must also be continuous. However, for each  $x \in [0, 1)$ ,  $f_n(x) \rightarrow f(0)$ . Moreover,  $f_n(1) = f(1)$ . So,

$$g(x) = \begin{cases} f(0) & x < 1 \\ f(1) & x = 1 \end{cases}$$

which is not continuous.

- (b) Since  $f$  is continuous, there exists  $C > 0$  such that  $f(x) \leq C$  for all  $x \in [0, 1]$ . The constant function  $h(x) = C$  is integrable on  $(0, 1)$  so by Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx = \int_0^1 g(x) dx = f(0).$$

3.

$$\|Ta\|_1 = \sum_{k=1}^{\infty} |a_k b_k| \leq \|a\|_{\infty} \|b\|_1$$

Moreover,  $\|T\| = \|b\|_1$  which can be seen from applying  $T$  to the constant sequence  $\mathbf{1} = (1, 1, 1, \dots)$ .

$$\|T\mathbf{1}\|_1 = \sum_{k=1}^{\infty} |b_k| = \|b\|_1.$$

5. Hao Chen

*Proof.* To show the orthonormal set  $\{f_n\}$  is an orthonormal basis we will show that  $\{f_n\}^{\perp} = \{0\}$ . If not, then there exists  $x \neq 0$  such that  $\langle x, f_n \rangle = 0$  for all  $n$ . However, by Parseval's identity and the Cauchy-Schwarz inequality,

$$\|x\|^2 = \sum |\langle x, e_n \rangle|^2 = \sum |\langle x, e_n - f_n \rangle|^2 \leq \sum \|x\|^2 \|e_n - f_n\|^2 < \|x\|^2$$

but this is a contradiction so  $\{f_n\}^{\perp} = \{0\}$ . □

A more complicated proof by Walton:

*Proof.* Let  $c = \sum \|e_n - f_n\|^2 < 1$ . Define  $T : \mathcal{H} \rightarrow \mathcal{H}$  by sending  $x = \sum \langle x, e_n \rangle e_n \mapsto \sum \langle x, e_n \rangle f_n$ . The second sum converges by the Bessel inequality. Now, by the Cauchy-Schwarz inequality and Parseval's identity,

$$\|(I - T)x\|^2 = \left\| \sum \langle x, e_n \rangle (e_n - f_n) \right\|^2 \leq \sum |\langle x, e_n \rangle|^2 \sum \|e_n - f_n\|^2 = c \|x\|^2$$

So,  $\|T - I\| \leq \sqrt{c} < 1$ . We claim that this means  $T$  is invertible. Indeed, set

$$S = \sum_{n=0}^{\infty} (I - T)^n$$

The sum is absolutely convergent since  $\|I - T\| < 1$  so  $S$  is bounded linear operator since  $\mathcal{L}(\mathcal{H})$  is a Banach space. Moreover,

$$S - TS, S - ST = \sum_{n=1}^{\infty} (I - T)^n = S - (I - T)^0 = S - I$$

so  $S = T^{-1}$ . Now, let  $y \in \mathcal{H}$ . Then, there exists  $x$  ( $T^{-1}y$ ) such that  $Tx = y$ . Therefore,

$$y = \sum \langle x, e_n \rangle f_n \quad (1)$$

and therefore  $\overline{\text{span}}\{f_n\} = \mathcal{H}$ .  $\square$

Remark: This acutally holds if  $\sum \|e_n - f_n\|^2 < \infty$ .

6. Define  $A_n = \{f \geq 1/n\}$ . Since  $A_n \subseteq A_{n+1}$ ,

$$0 < \mu(\{f > 0\}) = \mu\left(\bigcup_n A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n)$$

Therefore there exists  $n$  such that  $\mu(A_n) > 0$ . Then,

$$\int f \geq \int_{A_n} f \geq \frac{1}{n} \mu(A_n) > 0$$

7. (a) We first show that if  $f$  is integrable, the  $\mu(E_n) \rightarrow 0$  implies  $\int_{E_n} f \rightarrow 0$ . Since  $f$  is integrable, for  $A_n = \{n - 1 \leq |f| \leq n\}$ ,

$$\infty > \int |f| \geq \sum (n - 1) \mu(A_n)$$

Given  $\epsilon > 0$  there exists  $N$  such that  $\sum_{n=N}^{\infty} (n - 1) \mu(A_n) < \epsilon/2$ . Also, we can find  $M$  such that

$$\mu(E_n) < \epsilon/(2N) \quad \forall n \geq M$$

Then, for  $n \geq M$ ,

$$\begin{aligned} \left| \int_{E_n} f \right| &\leq \int_{E_n} |f| = \int_{E_n \cap \{|f| \leq N\}} |f| + \int_{E_n \cap \{|f| > N\}} |f| \\ &\leq N \mu(E_n) + \sum_{k=N}^{\infty} (k - 1) \mu(A_k) \leq N \epsilon/2N + \epsilon/2 = \epsilon \end{aligned}$$

Now we can prove the statement. Let  $a, b \in \mathbb{R}$ . Then there exists  $\{a_n\}, \{b_n\} \subseteq \mathbb{Q}$  such that

$$a_n \rightarrow a \quad b_n \rightarrow b$$

Then,

$$\int_a^b f = \int_a^{a_n} f + \int_{a_n}^{b_n} f + \int_{b_n}^b f$$

The middle term is zero by assumption and applying the above lemma, the first and third terms go to 0.

(b) INCOMPLETE

8. This is a special case of Winter 15 #10.



# Winter 2018 - Yiran Zhu

Yiran Zhu — Clemson - Math

1. Let  $\mathcal{C}(0, 1)$  be the collection of all continuous functions on  $(0, 1)$  which is a unit *open* interval in  $\mathbb{R}$ . Suppose  $\{f_n\}_{n=1}^\infty \subset \mathcal{C}(0, 1)$  converges uniformly to  $f$  on  $(0, 1)$ , i.e.,

$$\|f_n - f\|_\infty = \sup_{t \in (0, 1)} |f_n(t) - f(t)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Can we say  $f \in \mathcal{C}(0, 1)$ ? Prove or disprove.

*Proof.* Fix  $x \in (0, 1)$ . Observe that  $\forall y \in (0, 1)$ ,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq 2\|f_n - f\| + |f_n(x) - f_n(y)| \end{aligned}$$

For  $\epsilon > 0$ , there exists  $N \geq 1$  such that  $\|f_N - f\| < \epsilon/3$ . Furthermore, since  $f_N$  is continuous, there exists  $\delta > 0$  such that  $\forall |x - y| < \delta$ ,  $|f_N(x) - f_N(y)| < \epsilon/3$ .

$$\forall |x - y| < \delta, \quad |f(x) - f(y)| \leq 2\|f_N - f\| + |f_N(x) - f_N(y)| < \epsilon$$

Therefore  $f$  is continuous at  $x$ . Since  $x$  can be an arbitrary number in  $(0, 1)$ ,  $f \in \mathcal{C}(0, 1)$ .  $\square$

2. (Walton) Let  $\{f_n\}$  be a Cauchy sequence in  $B(0, 1)$ . Then, for each  $t \in (0, 1)$ ,  $\{f_n(t)\}$  is a Cauchy sequence in  $\mathbb{R}$ . Thus, since  $\mathbb{R}$  is complete, this sequence converges for each  $t$ . Define  $f(t) = \lim_{n \rightarrow \infty} f_n(t)$ . We claim that  $\|f_n - f\|_\infty \rightarrow 0$ . Indeed, for  $\epsilon > 0$ , there exists  $N$  such that  $\|f_n - f_k\|_\infty < \epsilon/2$  for all  $n, k \geq N$ . Moreover, for each  $t \in (0, 1)$ , there exists  $M_t \geq N$  such that  $|f(t) - f_{M_t}(t)| < \epsilon/2$ . Thus, for each  $t \in (0, 1)$ ,

$$|f_n(t) - f(t)| \leq \|f_n - f_{M_t}\|_\infty + |f_{M_t}(t) - f(t)| < \epsilon$$

for all  $n \geq N$ . Thus,

$$\|f_n - f\|_\infty = \sup_{t \in (0, 1)} |f_n(t) - f(t)| < \epsilon.$$

Therefore  $\|f_n - f\|_\infty \rightarrow 0$ . It only remains to show that  $f \in B(0, 1)$ . This follows from the triangle inequality: There exists  $K > 0$  such that  $\|f_K - f\|_\infty < 1$ . So,

$$\|f\|_\infty \leq \|f - f_K\|_\infty + \|f_K\|_\infty < \infty$$

which shows  $f$  is bounded.

3. If  $x \in Y^\perp$ , then  $\|x - y\|^2 = \|x\|^2 + \|y\|^2 \geq \|x\|^2$ . Conversely, since  $Y$  is a closed subspace,  $H = Y \oplus Y^\perp$ . There exists  $x' \in Y$  and  $x^\perp \in Y^\perp$  such that  $x = x' + x^\perp$ . Then

$$\|x^\perp\|^2 = \|x - x'\|^2 = \|x\|^2 = \|x' + x^\perp\|^2 = \|x'\|^2 + \|x^\perp\|^2$$

Therefore,  $\|x'\|^2 = 0 \Rightarrow x' = 0 \Rightarrow x = x^\perp \in Y^\perp$ .

4. Given a Cauchy sequence  $\{T_n\}_n \subseteq \mathcal{B}(X, Y)$  with respect to operator norm, we need to construct an operator  $T$  such that  $T_n \rightarrow T$ . Note that, for all  $x \in X$ ,  $\{T_n(x)\}_n$  is a Cauchy sequence in  $Y$  and thus convergent to some point in  $Y$ . We denote this extreme point by  $T_x$ . Our mapping is  $T : x \rightarrow T_x$ . Then one can easily check  $T$  is linear and also bounded so  $T \in \mathcal{B}(X, Y)$ . Finally,  $T_n \rightarrow T$  in operator norm.
5. (a)  $\|T\| = 1$ . This norm cannot be attained but can be approached by  $e_n$  as  $n \rightarrow \infty$ .

$$\|T(x)\| = \left\| \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x_n \right\| \leq \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) |x_n| \leq \|x\|_1$$

- (b) Suppose there exists  $x$  with  $\|x\| \leq 1$  such that  $|T(x)| = \|T\| = 1 \geq \|x\|$ . From above inequality, we essentially have

$$\|x_1\| = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) |x_n| = \|x\|_1 - \sum_{n=1}^{\infty} \frac{|x_n|}{n}$$

Hence

$$\sum_{n=1}^{\infty} \frac{|x_n|}{n} = 0 \Rightarrow \forall n \geq 1, x_n = 0 \Rightarrow |T(x)| = 0 \neq 1$$

6. Prove by contradiction. Suppose there is a such measure  $(H, \mathcal{M}, \lambda)$  for Hilbert Space  $H$ . Let  $r = 2$  and  $x = 0$ ,

$$0 < \lambda(B_2(0)) < \infty$$

Since  $\dim H = \infty$ , there is an orthonormal sequence  $\{x_n\}_n \subseteq B_2(0)$ . For all  $x_n$ , we claim that  $B_{1/2}(x_n) \subseteq B_2(0)$ . Indeed,

$$\forall y \in B_{1/2}(x_n), \|y - 0\| \leq \|y - x_n\| + \|x_n\| \leq 1/2 + 1 = 3/2 < 2 \Rightarrow y \in B_2(0)$$

Observe that  $\|x_n - x_m\|^2 = \langle x_n - x_m, x_n - x_m \rangle = \|x_n\|^2 + \|x_m\|^2 = 2$  for all  $n \neq m$ . Moreover, if  $n \neq m$ , then we can check that  $B_{1/2}(x_n) \cap B_{1/2}(x_m) = \emptyset$  as follows:

$$\forall y \in B_{1/2}(x_n), \|y_n - x_m\| \geq \|x_n - x_m\| - \|y - x_n\| = \sqrt{2} - \frac{1}{2} > \frac{1}{2} \Rightarrow y \notin B_{1/2}(x_m)$$

By assumption that measure of balls is invariant under translation, we have

$$\forall n \geq 1, \lambda(B_{1/2}(x_n)) = \lambda(B_{1/2}(x_1))$$

Note that  $\cup_{n=1}^{\infty} B_{1/2}(x_n)$ , the union of disjoint balls, is a subset of  $B_2(0)$ .

$$\lambda(B_2(0)) \geq \lambda\left(\cup_{n=1}^{\infty} B_{1/2}(x_n)\right) = \sum_{n=1}^{\infty} \lambda(B_{1/2}(x_n)) = \sum_{n=1}^{\infty} \lambda(B_{1/2}(x_1))$$

Therefore,  $\lambda(B_{1/2}(x_1)) = 0$ . However, we assume that measure of a ball is greater than 0.

7. Let  $(X, \mathcal{M}, \mu)$  be a measure space and  $f \in L^1(X, \mathcal{M}, \mu)$ . Then  $\{x : f(x) \neq 0\}$  is  $\sigma$ -finite with respect to  $\mu$ .

*Proof.* Let  $E_n = \{x : |f(x)| \geq 1/n\}$  and then  $\{x : f(x) \neq 0\} = \cup_{n=1}^{\infty} E_n$ . From Chebyshev's Inequality,

$$\frac{\mu(E_n)}{n} \leq \int_X |f| d\mu < \infty \Rightarrow \mu(E_n) < \infty$$

Therefore,  $\{x : f(x) \neq 0\}$  is  $\sigma$ -finite. □

8. (a) Observe that

$$\bigcup_{n=1}^{\infty} E_n = \{x \in I : f(x) > 0\} \Rightarrow \lambda(\{x \in I : f(x) > 0\}) \leq \sum_{n=1}^{\infty} \lambda(E_n)$$

So  $\lambda(\{x \in I : f(x) > 0\}) > 0$  implies that there  $\lambda(E_n) > 0$  for some  $n$ .

- (b) This is also obvious. We show that  $\lambda(E_n) = 0$  for all  $n \geq 1$ . Then the inequality derived in part (a) asserts that  $\lambda(\{x \in I : f(x) > 0\}) = 0$ . Suppose  $\lambda(E_n) > 0$  for some  $n$ , then we pick a finite set  $\{x_1, \dots, x_m\} \subseteq E_n$  where  $m = 2Mn$ .

$$\sum_{n=1}^m f(x_n) \geq \sum_{n=1}^m \frac{1}{n} = 2M > M$$

However, by assumption,  $\sum_{n=1}^m f(x_n) \leq M$ . Therefore,  $\lambda(E_n) = 0$  holds for all  $n \geq 1$ .

9. This is a direct application of *Monotone Convergence Theorem*. Let  $h_m(x) = \sum_{n=1}^m f(x+n)$ .

$$0 \leq h_1(x) \leq h_2(x) \leq \dots \leq h_m(x) \leq h_{m+1}(x) \leq \dots; \quad \lim_{m \rightarrow \infty} h_m(x) = \sum_{n=1}^{\infty} f(x+n) = g(x)$$

*Monotone Convergence Theorem* says

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}} h_m(x) d\mu = \int_{\mathbb{R}} \lim_{m \rightarrow \infty} h_m(x) d\mu = \int_{\mathbb{R}} g(x) d\mu$$

Let's compute the left hand side,

$$\int_{\mathbb{R}} h_m(x) d\mu = \int_{\mathbb{R}} \sum_{n=1}^m f(x+n) d\mu = \sum_{n=1}^m \int_{\mathbb{R}} f(x+n) d\mu = m \int_{\mathbb{R}} f(x) d\mu$$

Recall that Lebesgue measure is invariant under translation. Combine above two equations,

$$\lim_{m \rightarrow \infty} m \int_{\mathbb{R}} f(x) d\mu = \int_{\mathbb{R}} g(x) d\mu < \infty \Rightarrow \int_{\mathbb{R}} f(x) d\mu = 0$$

Since  $f(x)$  is nonnegative,  $\int_{\mathbb{R}} f d\mu = 0$  is equivalent to  $f = 0$  a.e.

10. (a) This is an immediate result of Cauchy-Schwartz Inequality.

$$\left(\int_B f d\mu\right)^2 = \left(\int_X f \chi_B d\mu\right)^2 \leq \left(\int_X f^2 d\mu\right) \left(\int_X \chi_B^2 d\mu\right) = \mu_B \int_X f^2 d\mu$$

- (b) Let  $f = \sum_{k=1}^n \chi_{A_k}$  where  $\chi_{A_k}$  is a characteristic function of measurable set  $A_k$ . Furthermore, let  $B = \cup_{k=1}^n A_k$ . This inequality holds directly from part (a).

# Summer 2018 - Yiran Zhu

Yiran Zhu — Clemson - Math

1. Let  $\{a_n\}_{n=1}^{\infty}$  be a real sequence with  $a_n \rightarrow 0, n \rightarrow \infty$ . Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a_n = 0$$

*Proof.* For  $\epsilon > 0$ , there exists a  $M \geq 1$  such that  $\forall n \geq M, |a_n| \leq \epsilon/2$ . For  $N \geq M+1$ , we have

$$\begin{aligned} \left| \frac{1}{N} \sum_{n=1}^N a_n \right| &= \left| \frac{1}{N} \sum_{n=1}^M a_n + \frac{1}{N} \sum_{n=M+1}^N a_n \right| \leq \frac{1}{N} \left| \sum_{n=1}^M a_n \right| + \frac{1}{N} \sum_{n=M+1}^N |a_n| \\ &\leq \frac{1}{N} \left| \sum_{n=1}^M a_n \right| + \left( \frac{N-M}{N} \right) \frac{\epsilon}{2} \leq \frac{1}{N} \left| \sum_{n=1}^M a_n \right| + \frac{\epsilon}{2} \end{aligned}$$

Let  $\widehat{N}$  be an integer greater than  $2 \left| \sum_{n=1}^M a_n \right| / \epsilon$  and  $\widehat{M} = \max\{\widehat{N}, M+1\}$

$$\forall N \geq \widehat{M}, \quad \left| \frac{1}{N} \sum_{n=1}^N a_n \right| < \epsilon$$

Equivalently,  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a_n = 0$ . □

2. Let  $X$  be a normed linear space and  $\emptyset \neq Y \subset X$  be a subset with the property that  $X \setminus Y$  is a linear subspace. Show that  $Y$  is dense in  $X$ .

*Proof.* Suppose  $Y$  is not dense in  $X$ . Then there exists a point  $z \in X \setminus Y$  and a number  $r > 0$  such that  $B(z, r) \cap Y = \emptyset$ . Equivalently,  $B(z, r) \subseteq X \setminus Y$ . Then we will show this implies  $Y = \emptyset$ . Pick  $x \in X$  and let  $d = \|x - z\|$ . Then

$$r > \left\| \frac{r(x - z)}{2d} \right\| = \left\| \frac{rx - (r - 2d)z}{2d} - z \right\| \Rightarrow a := \frac{rx - (r - 2d)z}{2d} \in B(z, r) \subseteq X \setminus Y$$

Since  $X \setminus Y$  is a subspace, we have  $x = (2da + (r - 2d)z)/r \in X \setminus Y$ . Note that  $x$  is arbitrarily picked from  $X$ . Therefore,  $X \subseteq X \setminus Y \subseteq X \Rightarrow Y = \emptyset$ . By contradiction,  $Y$  is dense in  $X$ . □

3. Define  $d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$  by  $d(x, y) = |f(x) - f(y)|$  where  $f$  is defined as

$$f(x) = \frac{x}{1 + |x|}, \forall x \in \mathbb{R}$$

Show that  $d$  is a metric on  $\mathbb{R}$  and determine if  $(\mathbb{R}, d)$  is complete.

*Proof.* (i) Positive-definite:  $d(x, y) = |f(x) - f(y)| \geq 0$  and  $d(x, y) = 0 \Leftrightarrow f(x) = f(y)$ . Note that

$$f(x) = f(y) \Leftrightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|} \Leftrightarrow x(1+|y|) = y(1+|x|)$$

Suppose  $x < 0$ , then  $y < 0$  and  $x(1+|y|) = y(1+|x|) \Rightarrow x = y$ . Similarly, if  $x \geq 0$ , then  $y \geq 0$  and  $x(1+|y|) = y(1+|x|) \Rightarrow x = y$ . In a word,  $d(x, y) = 0 \Leftrightarrow x = y$ .

(ii) Symmetric:  $d(x, y) = |f(x) - f(y)| = |f(y) - f(x)| = d(y, x)$

(iii) Triangle inequality:  $d(x, z) \leq d(x, y) + d(y, z)$  follows from

$$|f(x) - f(z)| = |f(x) - f(y) + f(y) - f(z)| \leq |f(x) - f(y)| + |f(y) - f(z)|$$

So  $d$  is a metric. However,  $(\mathbb{R}, d)$  is not complete. Consider sequence  $\{x_n\}_n$  with  $x_n = n$ .

$$d(x_n, x_{n+m}) = \left| \frac{m}{(1+n)(1+n+m)} \right| \leq \frac{1}{n+1}, \quad \forall n \geq 1, \forall m \geq 0$$

Therefore,  $\{x_n\}_n$  is Cauchy. It's obvious that  $\{x_n\}_n$  does not converge in  $\mathbb{R}$ .  $\square$

4. Let  $H$  be a Hilbert space and  $Y_1, Y_2$  be two closed linear subspaces in  $H$ . Denote  $P_1$  and  $P_2$  as the orthogonal projections onto  $Y_1$  and  $Y_2$ , respectively. Show that  $\|P_1 - P_2\| \leq 1$ .

*Proof.* Observe that  $(2P - I)^2 = 4P^2 + I - 4P = I$  holds for all projection  $P$ . In particular, if  $P$  is orthogonal, then, for all  $h \in H$ ,

$$\|(2P - I)h\|^2 = \langle (2P - I)h, (2P - I)h \rangle = \langle h, (2P - I)^2 h \rangle = \langle h, h \rangle = \|h\|^2$$

Therefore,  $\|2P - I\| = 1 \Rightarrow \|P - \frac{1}{2}I\| = \frac{1}{2}$ .

$$\|P_1 - P_2\| \leq \left\| P_1 - \frac{1}{2}I \right\| + \left\| P_2 - \frac{1}{2}I \right\| \leq 1$$

$\square$

5. Assume  $C[0, 1]$  is equipped with the supremum norm and let  $T_n : C[0, 1] \rightarrow C[0, 1]$  be defined by

$$T_n(f) = f\left(x^{1+\frac{1}{n}}\right), \quad \forall n \in \mathbb{N}$$

(a) Show that  $T_n(f) \rightarrow f, n \rightarrow \infty, \forall f \in C[0, 1]$

*Proof.* Fix  $f \in C[0, 1]$ . Since  $[0, 1]$  is compact,  $f$  is also uniformly continuous on  $[0, 1]$ , i.e. for  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\forall x, y \in [0, 1] \text{ s.t. } |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Let's give an estimation for  $g_n(x) := \left| x^{1+\frac{1}{n}} - x \right| = x \left( 1 - x^{\frac{1}{n}} \right)$ . Obviously,  $g_n(x)$  is continuous on  $[0, 1]$  and  $g_n(0) = g_n(1) = 0$ . To find the maximum value of  $g_n(x)$ , we let

$$g'_n(x) = \left( 1 + \frac{1}{n} \right) x^{\frac{1}{n}} - 1 = 0 \Rightarrow \sup_{x \in [0, 1]} g_n(x) = \left( \frac{n}{n+1} \right)^n \frac{1}{n+1} \leq \frac{1}{n+1}$$

Pick  $N \in \mathbb{Z}^+$  such that  $\frac{1}{N+1} < \delta$ , then

$$\forall n \geq N, \forall x \in [0, 1] \quad \left| x^{1+\frac{1}{n}} - x \right| < \frac{1}{N+1} < \delta \Rightarrow \|T_n(f) - f\|_\infty < \epsilon$$

Therefore,  $T_n(f) \rightarrow f$  as  $n \rightarrow \infty$ . □

(b) For each  $n \in N$ , find  $\|T_n - I\|$ .

*Proof.* For  $f \in C[0, 1]$ ,

$$\|(T_n - I)f\| = \|T_n(f) - f\|_\infty = \sup_{x \in [0, 1]} |f(x^{1+\frac{1}{n}}) - f(x)| \leq 2\|f\|_\infty$$

So  $\|T_n - I\| \leq 2$ . Let  $x_0 = \frac{1}{2}$  and  $x_1 = \left(\frac{1}{2}\right)^{1+\frac{1}{n}} \in (0, x_0)$ . Construct a function  $f$  as follows

$$f(x) = \begin{cases} -1 & x \in [0, x_1) \\ -1 + \frac{2(x - x_1)}{x_0 - x_1} & x \in [x_1, x_0] \\ 1 & x \in (x_0, 1] \end{cases}$$

Note that  $f(x_0) = 1$  and  $f(x_1) = -1$ . So  $f$  is continuous and  $\|f\|_\infty = 1$ .

$$|(T_n(f) - f)(x_0)| = |f(x_1) - f(x_0)| = 2 = 2\|f\|_\infty$$

As shown before,  $\|T_n(f) - f\|_\infty \leq 2\|f\|_\infty$ . Thus

$$\|T_n(f) - f\|_\infty = 2\|f\|_\infty \Rightarrow \|T_n - I\| = 2$$

□

6. Assume that  $\lambda$  is the Lebesgue measure on the real line and  $f$  a Lebesgue integrable function on the real line. Show that

$$F(x) := \int_{-\infty}^x f d\lambda$$

is uniformly continuous.

*Proof.* Let  $A_n = \{x \in X \mid |f(x)| \geq n\}$ . Then, Dominated Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \int_{A_n} |f| d\lambda = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f| \chi_{A_n} d\lambda = 0$$

For  $\epsilon > 0$ , there exists  $N \geq 1$  such that

$$\int_{A_N} |f| d\lambda < \frac{\epsilon}{2}$$

Then

$$\forall x, y \in \mathbb{R}, \quad |F(x) - F(y)| = \left| \int_{-\infty}^x f d\lambda - \int_{-\infty}^y f d\lambda \right| = \left| \int_y^x f d\lambda \right| \leq \int_y^x |f| d\lambda$$

Observe that, if  $|x - y| < \frac{\epsilon}{2N}$ , then

$$\int_y^x |f| d\lambda = \int_{[x,y] \cap A_N} |f| d\lambda + \int_{[x,y] \setminus A_N} |f| d\lambda \leq \int_{A_N} |f| d\lambda + N|x - y| < \epsilon$$

□

7. Let  $(X, \mathcal{M}, \mu)$  be a measure space and  $\{A_n\}_n$  be a sequence of sets in  $\mathcal{M}$ . Recall that  $\limsup_{n \rightarrow \infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ .

- (a) Prove that if  $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ , then  $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$

*Proof.* Observe that

$$\mu \left( \limsup_{n \rightarrow \infty} A_n \right) \leq \mu \left( \bigcup_{k=n}^{\infty} A_k \right) \leq \sum_{k=n}^{\infty} \mu(A_k), \quad \forall n \geq 1$$

Since  $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ , for  $\epsilon > 0$ , there exists  $N \geq 1$  such that

$$\sum_{k=N}^{\infty} \mu(A_k) < \epsilon \Rightarrow \mu \left( \limsup_{n \rightarrow \infty} A_n \right) \leq \epsilon$$

Let  $\epsilon \rightarrow 0$ , we derive  $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$ .

□

- (b) Is the converse true? If yes, prove it. If no, give a counter-example.

*Proof.* Converse is not true. Consider  $A_n = [0, 1/n]$ . Then  $\bigcup_{k=n}^{\infty} A_k = A_n = [0, 1/n]$ .

$$\limsup_{n \rightarrow \infty} A_n = \lim_{N \rightarrow \infty} \bigcap_{n=1}^N \bigcup_{k=n}^{\infty} A_k = \lim_{N \rightarrow \infty} \left[ 0, \frac{1}{N} \right] = \{0\}$$

Therefore,

$$\mu \left( \limsup_{n \rightarrow \infty} A_n \right) = 0$$



However,  $\mu(A_n) = \frac{1}{n}$  for all  $n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

□

8. Let  $(X, \mathcal{M}, \mu)$  be a finite measure space. Prove that a monotone increasing sequence of measurable functions  $f_n : X \rightarrow \mathbb{R}$  converges in measure if and only if it converges pointwise a.e..

*Proof.* (i) Suppose  $f_n$  converges to  $f$  pointwise a.e.: By Egorov theorem, for each  $\epsilon > 0$ , there exists a measurable set  $E$  with measure  $\mu(E) < \epsilon$  such that  $f_n$  converges uniformly to  $f$  on  $X \setminus E$ . In other words, for each  $\delta > 0$ , there exists  $N \geq 1$  such that

$$\forall n \geq N, \quad |f_n(x) - f(x)| < \delta, \quad \forall x \in X \setminus E$$

Consequently,

$$\forall n \geq N, \quad A_n := \{x \in X \mid |f_n(x) - f(x)| \geq \delta\} \subseteq E \Rightarrow \mu(A_n) < \epsilon$$

Therefore,  $f_n$  converges to  $f$  in measure.

(Alternate Solution: Walton) Let  $\varepsilon > 0$ . Set  $E_n = \{|f_n - f| < \varepsilon\}$ . Since  $f_n \rightarrow f$  pointwise a.e., letting  $F$  be the set where  $f_n \rightarrow f$ , we have

$$F \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n = \liminf E_n.$$

Therefore,

$$F^c \supset \limsup E_n^c$$

and

$$0 = \mu(F^c) \geq \mu(\limsup E_n^c) \geq \limsup \mu(E_n^c) \geq 0 \quad (2)$$

which implies  $\lim_{n \rightarrow \infty} \{|f_n - f| > \varepsilon\} = 0$  for any  $\varepsilon > 0$  so  $f_n \rightarrow f$  in measure. In the second inequality in (2) we have used the fact that the measure space is finite.

- (ii) Suppose  $f_n$  converges to  $f$  in measure: There exists a subsequence  $\{f_{n_k}\}_k$  converges to  $f$  pointwise a.e.. Let  $E$  be the zero-measure set that  $\{f_{n_k}\}_k$  does not converge to  $f$ . Then fix  $x \in X \setminus E$ , for each  $\epsilon > 0$ , there exists  $N \geq 1$  such that

$$\forall k \geq N, \quad |f_{n_k}(x) - f(x)| < \epsilon$$

Since  $\{f_n\}_n$  is monotone increasing, we have

$$\forall n \geq n_N, \quad |f_n(x) - f(x)| = f(x) - f_n(x) \leq f(x) - f_{n_N}(x) = |f_{n_k}(x) - f(x)| < \epsilon$$

Note that  $\{f_{n_k}\}_k$  is also monotone increasing.

□

9. Suppose that  $g$  is a non-negative Borel measurable function on  $\mathbb{R}$  with  $\int_{\mathbb{R}} g d\lambda = 1$  where  $\lambda$  denotes Lebesgue measure on  $\mathbb{R}$ . For  $k \in \mathbb{N}$  set  $g_k(x) = kg(kx)$ . Let  $f$  be a bounded continuous function. Prove that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} g_k f d\lambda = f(0)$$

*Proof.* Suppose  $\sup_{x \in \mathbb{R}} |f(x)| = M$  and define  $h_k(x) = g(x)f(x/k)$ . Observe that

$$\int_{\mathbb{R}} g_k f d\lambda = \int_{\mathbb{R}} kg(kx)f(x)d\lambda = \int_{\mathbb{R}} g(x)f(x/k)d\lambda = \int_{\mathbb{R}} h_k d\lambda$$

In order to apply Dominated Convergence Theorem, we need to show  $h_k$  is uniformly bounded by a integrable function. Indeed,

$$\forall k \geq 1, \quad |h_k(x)| = |g(x)f(x/k)| \leq Mg(x) \quad \text{and} \quad \int_{\mathbb{R}} Mg d\lambda = M < \infty$$

By DCT,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} h_k d\lambda = \int_{\mathbb{R}} \lim_{k \rightarrow \infty} h_k d\lambda = \int_{\mathbb{R}} f(0)g d\lambda = f(0)$$

□

10. Let  $\lambda$  be the Lebesgue measure on  $(0, 1)$ . Suppose the  $f_n : (0, 1) \rightarrow [0, \infty)$  is a sequence of Borel measurable functions such that  $\int_{(0,1)} f_n d\lambda = 1$  for all  $n \geq 1$  and  $\lim_{n \rightarrow \infty} f_n(x) = x$  for all  $x \in (0, 1)$ .

- (a) Give an example of such a sequence.

*Proof.*

$$f_n(x) = \begin{cases} (n+1)(1-nx) & x \in \left(0, \frac{1}{n}\right) \\ \frac{n}{n-1} \left(x - \frac{1}{n}\right) & x \in \left[\frac{1}{n}, 1\right) \end{cases}$$

Then

$$\int_{(0,1)} f_n d\lambda = \int_{(0, \frac{1}{n})} (n+1)(1-nx) d\lambda + \int_{[\frac{1}{n}, 1)} \frac{n}{n-1} \left(x - \frac{1}{n}\right) d\lambda = \frac{n+1}{2n} + \frac{n-1}{2n} = 1$$

Fix  $x \in (0, 1)$ , there exists  $N \geq 1$  such that  $x > \frac{1}{N}$ , then

$$\forall n \geq N, \quad f_n(x) - x = \frac{n}{n-1} \left(x - \frac{1}{n}\right) - x = \frac{x-1}{n-1}$$

So  $f_n(x) \rightarrow x$  as  $n \rightarrow \infty$ .

□

- (b) Show that one can find  $n \geq 1$  and  $x \in (0, 1)$  such that  $f_n(x)\sqrt{x} \geq 2018$

*Proof.* Prove by contradiction. Suppose not, then

$$\forall n \geq 1, \quad f_n(x) \leq \frac{2018}{\sqrt{x}}$$

Note that

$$\int_{(0,1)} \frac{2018}{\sqrt{x}} d\lambda = 4036 < \infty$$

By Dominated Convergence Theorem,

$$1 = \lim_{n \rightarrow \infty} \int_{(0,1)} f_n d\lambda = \int_{(0,1)} \lim_{n \rightarrow \infty} f_n d\lambda = \int_{(0,1)} x d\lambda = \frac{1}{2}$$

Above equality cannot be true. □

## Winter 2019 - Yiran Zhu

1. Prove by contradiction. Suppose not. There exists  $\epsilon > 0$  such that, for all  $N > 0$ ,  $\exists n > N$  s.t.  $\|x_n - x_0\| \geq \epsilon$ . Therefore, we can pick a subsequence  $\{x'_i\}_i$  such that

$$\forall i \in \mathbb{N}, \|x'_i - x_0\| \geq \epsilon$$

By assumption,  $\{x'_i\}_i$  contains a subsequence  $\{x''_j\}_j$  converging to  $x_0$ . However, by construction, there is no element in  $\{x''_j\}_j$  such that  $\|x''_j - x_0\| < \epsilon$ .

2.  $P([0, 1], d)$  is not a complete metric space. Firstly,  $d$  is essentially a metric which can be proved easily. However,  $P([0, 1], d)$  is not complete. By Taylor's expansion,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Note that  $e^x$  is not a polynomial. Indeed, suppose  $e^x = \sum_{n=0}^k a_n x^n$ . Take  $(k+1)$ -th derivative of both sides, we have  $e^x = 0$ , which is impossible.

Let  $f_m = \sum_{n=0}^m \frac{x^n}{n!}$ . We will prove  $f_m \rightarrow e^x$  w.r.t.  $d$  as  $m \rightarrow \infty$ .

$$d(e^x, f_m) = \int_0^1 |e^x - f_m| dx = \int_0^1 \left| \sum_{n=m+1}^{\infty} \frac{x^n}{n!} \right| dx = \sum_{n=m+1}^{\infty} \frac{1}{(n+1)!}$$

Observe that

$$\sum_{n=m+1}^{\infty} \frac{1}{(n+1)!} < \frac{1}{(m+1)!} + \frac{1}{m!} \sum_{n=m+1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{(m+1)!} + \frac{1}{m!} \sum_{n=m+1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$$

Note that

$$\sum_{n=m+1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{m+1} \Rightarrow \sum_{n=m+1}^{\infty} \frac{1}{(n+1)!} < \frac{2}{(m+1)!}$$

Hence, with  $m \rightarrow \infty$ ,  $d(e^x, f_m) \rightarrow 0$ . To show  $P([0, 1], d)$  is not complete, we need to show sequence  $\{f_m\}_m$  is Cauchy. For  $\epsilon > 0$ , there exist  $N \geq 0$  such that  $\forall m \geq N$ ,  $d(e^x, f_m) < \epsilon/2$ . By triangle inequality,

$$\forall n, m \geq N, d(f_m, f_n) \leq d(f_m, e^x) + d(e^x, f_n) < \epsilon$$

3. Prove by contradiction. Suppose  $X$  is separable and let  $Y$  be a dense countable subset of  $X$ .

$$A \subseteq X \subseteq \bigcup_{y \in Y} B\left(y, \frac{\epsilon}{3}\right)$$

We show that, for each  $y \in Y$ ,  $A \cap B(y, \epsilon/3)$  contains at most one element from  $A$ . Suppose  $A \cap B(y, \epsilon/3)$  contains two distinct elements  $a, b \in A$ . Then

$$d(a, b) \leq d(a, y) + d(y, b) \leq 2\epsilon/3 < \epsilon$$

This is impossible. Therefore,  $A$  contains at most as many elements as  $Y$ . In other words,  $A$  is countable. (Contradiction!)

4. First, there exists a sequence  $\{(a_n, b_n)\}_n$  such that  $d(A, B) = \lim_{n \rightarrow \infty} d(a_n, b_n)$ . Since  $A$  is compact, there exists a convergent subsequence of  $\{a_n\}_n$ . Denote this subsequence as  $\{a_{n,1}\}_n$  and let  $a^* \in A$  be its limit point. Note that  $\{b_{n,1}\}_n$  is a sequence in  $B$  and  $B$  is also compact. So there exists a convergent subsequence  $\{b_{n,2}\}_n$  and we denote its limit point as  $b^* \in B$ . Then we will prove  $d(A, B) = d(a^*, b^*)$ .

$$d(A, B) = \lim_{n \rightarrow \infty} d(a_n, b_n) = \lim_{n \rightarrow \infty} d(a_{n,2}, b_{n,2})$$

For  $\epsilon > 0$ , there exist  $N \geq 0$  such that

$$\forall n \geq N, \quad d(a_{n,2}, b_{n,2}) - d(A, B) < \epsilon$$

Note that

$$d(a^*, b^*) \leq d(a^*, a_{n,2}) + d(a_{n,2}, b_{n,2}) + d(b_{n,2}, b^*)$$

Combine above two inequalities to claim

$$\forall n \geq N, \quad d(a^*, b^*) - d(A, B) < d(a^*, a_{n,2}) + d(b_{n,2}, b^*) + \epsilon$$

Let  $n \rightarrow \infty$ , we conclude  $d(a^*, b^*) - d(A, B) < \epsilon$ . Since  $\epsilon$  can be arbitrarily small,  $d(a^*, b^*) \leq d(A, B)$ . Recall that, by definition,  $d(A, B) \leq d(a^*, b^*) \Rightarrow d(A, B) = d(a^*, b^*)$ .

5. By definition,  $T$  is orthogonal  $\Leftrightarrow \langle x, Ty \rangle = \langle Tx, y \rangle$  for all  $x, y \in H$ .

- (a)  $\rightarrow$  (b): for all  $x \in H$ ,  $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^2x \rangle = \langle x, Tx \rangle \leq \|x\|\|Tx\|$ . Therefore,  $\|Tx\| \leq \|x\| \Rightarrow \|T\| \leq 1$ . Since  $T$  is nonzero, there exists  $x \in H$  such that  $Tx \neq 0$ .

$$\|T(Tx)\| = \|Tx\| \Rightarrow \|T\| \geq 1$$

Combine all results, we conclude  $\|T\| = 1$ .

- (b)  $\rightarrow$  (c): We first show that  $\ker(T) \subseteq T(H)^\perp$  by contradiction. Suppose not, there exists  $x, y \in H$  such that  $T(x) = 0$  but  $\langle x, Ty \rangle \neq 0$ . Without loss of generality, we may assume  $\operatorname{Re}\langle x, T(y) \rangle > 0$ . Otherwise, we replace  $y$  by one of  $\{-y, iy, -iy\}$ . Then, for  $\alpha > 0$ ,

$$\|\alpha x - Ty\|^2 = \alpha^2\|x\|^2 + \|Ty\|^2 - 2\alpha \operatorname{Re}\langle x, T(y) \rangle := f(\alpha)$$

Let  $\alpha^* := \operatorname{Re}\langle x, Ty \rangle / (\|x\|^2) > 0$ . We have  $f(\alpha^*) = \|Ty\|^2 - \alpha^* \operatorname{Re}\langle x, Ty \rangle < \|Ty\|^2$ . However,

$$\|Ty\|^2 = \|T(\alpha^*x - Ty)\|^2 \leq \|T\|^2\|\alpha^*x - Ty\|^2 = \|\alpha^*x - Ty\|^2 = f(\alpha^*)$$

This contradiction leads to that  $\ker(T) \subseteq T(H)^\perp$ . Conversely, we need to show  $T(H)^\perp \subseteq \ker(T)$ . For  $y \in T(H)^\perp$ , we observe that  $y - Ty \in \ker(T) \subseteq T(H)^\perp$ , so

$$\langle y - Ty, Ty \rangle = 0 \Rightarrow \langle y, Ty \rangle - \langle Ty, Ty \rangle = 0$$

By choice of  $y$ , we have  $\langle y, Ty \rangle = 0$ . Hence, the right equation above essentially shows that  $\langle Ty, Ty \rangle = 0 \Rightarrow Ty = 0 \Rightarrow y \in \ker(T)$ .

- (c)  $\rightarrow$  (a): For all  $x, y \in H$ ,  $x - Tx, y - Ty \in \ker(T) = (T(H))^\perp$ . Then

$$\langle x - Tx, y - Ty \rangle = \langle x, y - Ty \rangle + \langle Tx, y - Ty \rangle = \langle x, y - Ty \rangle = \langle x, y \rangle - \langle x, Ty \rangle$$

On the other side,

$$\langle x - Tx, y - Ty \rangle = \langle x - Tx, y \rangle - \langle x - Tx, Ty \rangle = \langle x - Tx, y \rangle = \langle x, y \rangle - \langle Tx, y \rangle$$

After canceling  $\langle x, y \rangle$ , we derive  $\langle x, Ty \rangle = \langle Tx, y \rangle$ .

*Alternative proof.* (c)  $\implies$  (a). First we show the range of  $T$  is closed. Let  $Tx_n \rightarrow y$ . We need to show  $y$  is in the range. Since  $T$  is bounded,  $T(Tx_n) \rightarrow Ty$ . But since  $T^2 = T$ ,  $Tx_n \rightarrow Ty$  and by uniqueness of limits  $Ty = y$  so  $y$  is in the range. Therefore

$$H = \text{Ran}(T) + \text{Ran}(T)^\perp = \text{Ran}(T) + \ker(T)$$

so every vector has a unique decomposition in to  $\text{Ran}(T)$  and  $\ker(T)$  and the subspaces are orthogonal.

(a)  $\implies$  (b). Since  $T^2 = T$ ,  $\|T\| = \|T^2\| \leq \|T\|^2$  so  $\|T\| \geq 1$ . However,

$$\|Tx\|^2 + \|(I - T)x\|^2 = \|x\|^2$$

by (i) so  $\|Tx\|^2 \leq \|x\|^2$  and  $\|T\| \leq 1$ .

(b)  $\implies$  (c): We first show that  $\ker(T) \subseteq T(H)^\perp$  by contradiction. Suppose not, there exists  $x, y \in H$  such that  $T(x) = 0$  but  $\langle x, Ty \rangle \neq 0$ . Without loss of generality, we may assume  $\text{Re}\langle x, T(y) \rangle > 0$ . Otherwise, we replace  $y$  by one of  $\{-y, iy, -iy\}$ . Then, for  $\alpha > 0$ ,

$$\|\alpha x - Ty\|^2 = \alpha^2 \|x\|^2 + \|Ty\|^2 - 2\alpha \text{Re}\langle x, T(y) \rangle := f(\alpha)$$

Let  $\alpha^* := \text{Re}\langle x, Ty \rangle / (\|x\|^2) > 0$ . We have  $f(\alpha^*) = \|Ty\|^2 - \alpha^* \text{Re}\langle x, Ty \rangle < \|Ty\|^2$ . However,

$$\|Ty\|^2 = \|T(\alpha^* x - Ty)\|^2 \leq \|T\|^2 \|\alpha^* x - Ty\|^2 = \|\alpha^* x - Ty\|^2 = f(\alpha^*)$$

This contradiction leads to that  $\ker(T) \subseteq T(H)^\perp$ . Conversely, we need to show  $T(H)^\perp \subseteq \ker(T)$ . For  $y \in T(H)^\perp$ , we observe that  $y - Ty \in \ker(T) \subseteq T(H)^\perp$ , so

$$\langle y - Ty, Ty \rangle = 0 \implies \langle y, Ty \rangle - \langle Ty, Ty \rangle = 0$$

By choice of  $y$ , we have  $\langle y, Ty \rangle = 0$ . Hence, the right equation above essentially shows that  $\langle Ty, Ty \rangle = 0 \implies Ty = 0 \implies y \in \ker(T)$ .

□

6. (a) Omitted.

(b) Observe that

$$\forall n \in \mathbb{N}, \quad \frac{e^{-x}}{1 + (x/n)} \leq e^{-x} \text{ on } (0, \infty)$$

And  $\int_0^\infty e^{-x} = 1$ . By dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{e^{-x}}{1 + (x/n)} dx = \int_0^\infty \lim_{n \rightarrow \infty} \frac{e^{-x}}{1 + (x/n)} dx = \int_0^\infty e^{-x} dx = 1$$

7. Define  $A_k = \{x \in E : f(x) \geq 1/k\}$  for each positive integer  $k$ . Since  $f$  is measurable, so is  $A_k$ . Let  $Z = \{x \in E : f(x) = 0\}$ . Follows from problem assumption,  $m(Z) = 0$ . By sub-additivity of measure, together with fact that  $E_n \setminus Z = \cup_{k=1}^\infty (E_n \cap A_k)$ .

$$\forall n \in \mathbb{N}, \quad m(E_n) = m(E_n \setminus Z) = m\left(\bigcup_{k=1}^\infty (E_n \cap A_k)\right) \leq m(E) < \infty$$

For  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$m(E_n) = m\left(\bigcup_{k=1}^\infty (E_n \cap A_k)\right) \leq m\left(\bigcup_{k=1}^N (E_n \cap A_k)\right) + \epsilon \leq \sum_{k=1}^N m(E_n \cap A_k) + \epsilon$$

We will prove that, for each fixed  $k$ ,  $\lim_{n \rightarrow \infty} m(E_n \cap A_k) = 0$ .

$$\int_{E_n} f(x) dx = \int_{E_n \setminus Z} f(x) dx \geq \int_{E_n \cap A_k} f(x) dx \geq \frac{m(E_n \cap A_k)}{k}$$

Let  $n \rightarrow \infty$ , the left hand side goes to 0 and hence  $\lim_{n \rightarrow \infty} m(E_n \cap A_k) = 0$ . As a consequence,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^N m(E_n \cap A_k) = \sum_{k=1}^N \lim_{n \rightarrow \infty} m(E_n \cap A_k) = 0 \Rightarrow \limsup_{n \rightarrow \infty} m(E_n) \leq \epsilon$$

As  $\epsilon$  can be arbitrarily small,  $\limsup_{n \rightarrow \infty} m(E_n) = 0$ . Observe that

$$0 \leq \liminf_{n \rightarrow \infty} m(E_n) \leq \limsup_{n \rightarrow \infty} m(E_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} m(E_n) \text{ exists and } \lim_{n \rightarrow \infty} m(E_n) = 0$$

8. By Cauchy inequality, we have

$$|L| := \left| \int_0^1 (f(x) - \cos(2\pi x))(f(x) - \sin(2\pi x)) dx \right| \leq \sqrt{L_1 L_2} = \frac{1}{9}$$

where  $L_1 = \int_0^1 |f(x) - \cos(2\pi x)|^2 dx$  and  $L_2 = \int_0^1 |f(x) - \sin(2\pi x)|^2 dx$ . Since  $L_1 = L_2 = 1/9$ , we have

$$\int_0^1 f(x) \cos(2\pi x) dx = \int_0^1 f(x) \sin(2\pi x) dx$$

This equality allows us to compute  $L$  directly.

$$\begin{aligned} L &= \int_0^1 (f^2(x) + f(x) \cos(2\pi x) + f(x) \sin(2\pi x) + \cos(2\pi x) \sin(2\pi x)) dx \\ &= \int_0^1 (f^2(x) + 2f(x) \cos(2\pi x)) dx = L_1 - \int_0^1 \cos^2(2\pi x) dx = L_1 - \frac{1}{2} \\ &= -\frac{7}{18} \end{aligned}$$

However, by the inequality given at the beginning,  $|L| \leq 1/9$ . This contradiction leads to that there is no such function  $f(x)$  satisfying  $L_1 = L_2 = 1/9$ .

9. By substitution  $y = \sqrt{x}$ , we convert integral to following form

$$\int_0^1 n\sqrt{x}f(nx)dx = 2 \int_0^1 ny^2f(ny^2)dy$$

Since  $|xf(x)| \rightarrow 0$  as  $x \rightarrow \infty$ , there exists  $\alpha > 0$  such that  $\forall x \geq \alpha$ ,  $|xf(x)| < 1$ . Suppose  $M$  is an upper bound for  $f(x)$  and then we have

$$\forall x \in (0, \infty), \quad |xf(x)| < \alpha M + 1$$

It follows that

$$\forall n \in \mathbb{Z}_+, \quad \forall y \in (0, 1), \quad |ny^2f(ny^2)| < \alpha M + 1$$

Observe that

$$\int_0^1 (\alpha M + 1)dy = \alpha M + 1 < \infty \text{ and } \forall y \in (0, 1), \quad \lim_{n \rightarrow \infty} ny^2f(ny^2) = 0$$

By Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 ny^2f(ny^2)dy = \int_0^1 \lim_{n \rightarrow \infty} ny^2f(ny^2)dy = 0$$

Consequently,  $\int_0^1 n\sqrt{x}f(nx)dx = 0$ .

10. Observe that  $\limsup_{n \rightarrow \infty} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \subseteq \bigcup_{m=k}^{\infty} A_m$  for each fixed  $k$ . Hence

$$\forall k \geq 1, \quad \mu \left( \limsup_{n \rightarrow \infty} A_n \right) \leq \mu \left( \bigcup_{m=k}^{\infty} A_m \right) \leq \sum_{m=k}^{\infty} \mu(A_m)$$

Since  $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ , for  $\epsilon > 0$ , there exists  $N \geq 0$  such that

$$\forall m \geq N, \quad \sum_{n=m}^{\infty} \mu(A_n) < \epsilon \Rightarrow \mu \left( \limsup_{n \rightarrow \infty} A_n \right) < \epsilon$$

As  $\epsilon$  can be arbitrarily small,  $\mu(\limsup_{n \rightarrow \infty} A_n) = 0$ .

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