

Clemson Analysis Prelim Solutions

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Winter 2010

1. (a) *Proof.* If E is bounded, E is pre-compact since \mathbb{R} is finite (one) dimensional. If $f(E)$ is unbounded, then there exists $\{x_n\} \subseteq E$ such that $f(x_n) \rightarrow \infty$. Since E is precompact $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ with limit $x \in \mathbb{R}$. Then, since f is continuous,

$$f(x) = \lim_{k \rightarrow \infty} f(x_{n_k}) = \infty$$

However, since f maps \mathbb{R} to \mathbb{R} , $f(x)$ cannot be ∞ . □

- (b) *Proof.* Since f is uniformly continuous, there exists $\delta > 0$ such that whenever $|x - y| < \delta$,

$$|f(x) - f(y)| < 1$$

Since E bounded, it can be covered by finitely many balls of radius δ , say $\{B(x_i, \delta)\}_{i=1}^N$. Then,

$$f(E) = \cup_{i=1}^N f(B(x_i, \delta))$$

Fix i , for any $f(y) \in f(B(x_i, \delta))$,

$$|f(y) - f(x_i)| \leq 1$$

So $f(B(x_i, \delta))$ is bounded. Then, a finite union of bounded sets is also bounded. □

Counterexample: $E = (0, 1)$ and $f(x) = 1/x$. $f(E) = (1, \infty)$.

3. (a) *Proof.* Recall the Bessel inequality for any orthonormal set $\{e_n\}$ in an inner product space, X . For and $f \in X$,

$$\sum |\langle f, e_n \rangle|^2 \leq \|f\|^2$$

In particular, $\langle f, e_n \rangle \rightarrow 0$ as $n \rightarrow \infty$ for any $f \in X$. Now, since

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{2\pi}} \right\}$$

form an orthonormal set in $C[-\pi, \pi]$, we have

$$\int_{-\pi}^{\pi} \sin(2nx) f(x) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any $f \in C[-\pi, \pi]$. Then,

$$\int_{-\pi}^{\pi} \sin^2(nx) f(x) dx = \frac{1}{2} \int_{-\pi}^{\pi} f(x) dx - \frac{1}{2} \int_{-\pi}^{\pi} \sin(2nx) f(x) dx \rightarrow \frac{1}{2} \int_{-\pi}^{\pi} f(x) dx$$

□

(b) *Proof.* For any $f \in C[-\pi, \pi]$, $n \in \mathbb{N}$,

$$\begin{aligned} \left| \int_{-\pi}^{\pi} \frac{x^n}{\pi^n} f(x) dx \right|^2 &\leq \int_{-\pi}^{\pi} \frac{x^{2n}}{\pi^{2n}} dx \int_{-\pi}^{\pi} |f(x)|^2 dx \\ &= \frac{\pi^{2n+1} - (-\pi)^{2n+1}}{(2n+1)\pi^{2n}} \|f\|_{L^2}^2 \\ &= \frac{2\pi}{2n+1} \|f\|_{L^2}^2 \end{aligned}$$

which goes to 0 as $n \rightarrow \infty$. □

9. (a) There exists $\varepsilon_n \searrow 0$ such that

$$\mu\{|f - g| \geq \varepsilon_n\} \leq \varepsilon_n$$

Then,

$$\begin{aligned} \mu\{f \neq g\} &= \mu\{|f - g| > 0\} = \mu\left(\bigcup_n \{|f - g| \geq \varepsilon_n\}\right) \\ &= \lim_{n \rightarrow \infty} \mu\{|f - g| \leq \varepsilon_n\} \leq \lim_{n \rightarrow \infty} \varepsilon_n = 0 \end{aligned}$$

(b) We only need to show the triangle inequality. Let $t, s > 0$, f, g, h measurable functions.

If $|f - h| \leq t$ and $|g - h| \leq s$, then

$$|f - g| \leq |f - h| + |g - h| \leq t + s$$

Thus, $\{|f - h| \leq t\} \cap \{|g - h| \leq s\} \subset \{|f - g| \leq t + s\}$. Then, taking complements, we have

$$\{|f - h| > t\} \cup \{|g - h| > s\} \supset \{|f - g| > t + s\}$$

Therefore, $\mu\{|f - g| > t + s\} \leq \mu\{|f - h| > t\} + \mu\{|g - h| > s\}$. Let $\delta > 0$. There exists $\varepsilon_1, \varepsilon_2$ such that

$$\mu\{|f - h| > \varepsilon_1\} < \varepsilon_1 \text{ and } \varepsilon_1 < \rho(f, h) + \delta/2$$

and similarly for ε_2 and $|g - h|$. Therefore,

$$\mu\{|f - g| > \varepsilon_1 + \varepsilon_2\} \leq \mu\{|f - h| > \varepsilon_1\} + \mu\{|g - h| > \varepsilon_2\} < \varepsilon_1 + \varepsilon_2$$

So,

$$\begin{aligned} \rho(f, g) &= \inf\{\varepsilon : \mu\{|f - g| > \varepsilon\} < \varepsilon\} \\ &\leq \varepsilon_1 + \varepsilon_2 \\ &\leq \rho(f, h) + \rho(g, h) + \delta \end{aligned}$$

for any $\delta > 0$. This proves the Triangle Inequality.

Summer 2010

3. (a)

$$\|Tf\|_\infty = \sup_{x \in [0,1]} |x^2 f(x)| \leq \sup_{x \in [0,1]} |f(x)| = \|f\|_\infty$$

(b) For $f \equiv 1$, $\|Tf\| = 1$ and $\|f\| = 1$.

(c) By triangle inequality, $\|(I+T)f\|_\infty \leq \|f\|_\infty + \|Tf\|_\infty \leq 2\|f\|$. So, we only need to show $\|I+T\| = 2$. Again, this follows from taking $f \equiv 1$.

$$\|(I+T)f\| = \sup_{x \in [0,1]} |1+x^2| = 2$$

4. (a) We will actually show more, namely that

$$\|x\|_q \leq \|x\|_p \quad \text{for } 1 \leq p < q < \infty$$

Proof. Let $x \in \ell^p$. Define

$$y = \frac{x}{\|x\|_p}$$

Then, $|y_i| \leq 1$ for every $i \in \mathbb{N}$. This implies $|y_i|^q \leq |y_i|^p$ for all i . Therefore,

$$\|y\|_q^q = \sum |y_i|^q \leq \sum |y_i|^p = \sum \frac{|x_i|^p}{\|x\|_p^p} = \frac{\sum |x_i|^p}{\sum |x_i|^p} = 1$$

so $\|y\|_q \leq 1$. But this implies

$$\frac{\|x\|_q}{\|x\|_p} = \left\| \frac{x}{\|x\|_p} \right\|_q = \|y\|_q \leq 1$$

□

5. (a) Let $f \in L^q$, $1 \leq p < q < \infty$. we will apply holder inequality with exponent $q/p > 1$.

$$\int |f|^p \leq \left(\int |f|^q \right)^{p/q} \left(\int 1 \right)^{1-p/q} = \left(\int |f|^q \right)^{p/q} (\mu(X))^{1-p/q} < \infty$$

(b) Take

$$f(x) = \begin{cases} 0 & x = 0 \\ x^{-1/2} & 0 < |x| \leq 1 \\ 0 & |x| < 1 \end{cases}$$

$$g(x) = \begin{cases} x^{-1} & |x| \geq 1 \\ 1 & |x| < 1 \end{cases}$$

6. There are many options for f_n :

$$n^2\chi_{[0,1/n]} \quad n\chi_{[n,n+1]} \quad \chi_{[n,2n]}$$

Then, just take

$$g_n = \frac{1}{n}f_n$$

and

$$h_n = (-1)^n g_n$$

7.

Winter 2012

Summer 2012

4. (a) We will show K is totally bounded. Let $\epsilon > 0$. For $x, y \in K$,

$$d_S(x, y) = \sum_i \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)} \leq \sum_i \frac{2}{2^i} < \infty$$

So, there exists N such that

$$\sum_{i=N+1}^{\infty} \frac{1}{2^{i-1}} < \epsilon/2$$

Consider the set $M = \{x \in K : x_i = 0, i > N\} \subseteq K$. M is compact since it is finite dimensional and bounded. So, there is an $\epsilon/2$ net for M which will be an ϵ net for K .

- (b) False. Consider the sequence $\{e_n\} \subseteq K$, which is entirely zero except the n -th entry. For $n \neq m$,

$$d_{\infty}(e_n, e_m) = 1$$

so this sequence cannot have a convergent subsequence.

5. (a) Define $g_n = \sum_{k=1}^n f_k$. Since f_k are non-negative, $\{g_n\}$ is monotone. Moreover,

$$g_n \nearrow \sum_{k=1}^{\infty} f_k$$

Then, by Monotone Convergence Theorem,

$$\sum_{k=1}^{\infty} \int f_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int f_k = \lim_{n \rightarrow \infty} \int g_n = \int \sum_{k=1}^{\infty} f_k$$

- (b) By part (a),

$$\infty > \sum \int |f_k - f| = \int \sum |f_k - f|$$

Since $\sum |f_k - f|$ is integrable, it is finite almost everywhere. Let $E \subseteq \mathbb{R}$ be the set of measure zero where it may not be finite. Let $x \notin E$. Let $\epsilon > 0$. There exists N such that

$$|f_n(x) - f(x)| \leq \sum_{k=N}^{\infty} |f_k(x) - f(x)| < \epsilon$$

for all $n \geq N$.

7. See Winter 15 #9

Winter 2013

8. See Summer 13 #5

Summer 2013

3. (a) B is complete. Let $\{f_n\} \subseteq C[0, 1]$ be a Cauchy sequence in ρ_∞ . Since $(C[0, 1], \rho_\infty)$ is a complete metric space, there exists $f \in C[0, 1]$ such that $f_n \rightarrow f$ in ρ_∞ . Now, we claim that $f \in B$. For any $\epsilon > 0$ there exists N such that

$$\rho(f, f_n) < \epsilon \quad \forall n \geq N$$

Then,

$$\sup_{0 \leq t \leq 1} |f(t)| = \rho(f, 0) \leq \rho(f, f_n) + \rho(f_n, 0) < \epsilon + 1$$

but $\epsilon > 0$ was arbitrary so

$$\sup_{0 \leq t \leq 1} |f(t)| \leq 1$$

- (b) Consider the spike functions, $\{f_n\}$. For $n \neq m$,

$$\rho(f_n, f_m) = 1$$

so there cannot be a convergent subsequence.

4. (a) Let $f \in L^2(\mu)$. Then, using the Cauchy-Schwarz inequality, we compute

$$\begin{aligned} \|Af\|_{L^2(\mu)}^2 &= \int_X \left(\int_X K(x, y) f(y) d\mu(y) \right)^2 d\mu(x) \\ &\leq \int_X \left(\int_X K(x, y)^2 d\mu(y) \right) \left(\int_X f(y)^2 d\mu(y) \right) d\mu(x) \\ &= \left(\int_X f(y)^2 d\mu(y) \right) \int_X \left(\int_X K(x, y)^2 d\mu(y) \right) d\mu(x) = \|f\|_{L^2(\mu)}^2 \|K\|_{L^2(\mu \times \mu)}^2 \end{aligned}$$

Therefore $\|A\| \leq \|K\|_{L^2(\mu \times \mu)}$.

- (b) First we note that the correspondence $K \mapsto A$ is linear due to the linearity of the integral. So, it suffices to prove the following: Let $K \in L^2(\mu \times \mu)$ such that for any $f \in L^2(\mu)$,

$$\int_X K(x, y) f(y) d\mu(y) = 0$$

for almost every $x \in X$ (w.r.t μ). Then, $K = 0$ a.e. To prove this, suppose there exists $E \subseteq X \times X$ such that E has positive $\mu \times \mu$ measure in the sense that

$$(\mu \times \mu)(E) := \int_X \int_X \mathbf{1}_E(x, y) d\mu(x) d\mu(y) > 0$$

So we define the measure $\mu \times \mu$ on the cylinder $X \times X$ in this way.

5. ($a \Rightarrow b$) Let $x = \hat{m} + e$ where \hat{m} is the closest point to x in M . Let $y \in M$ non-zero. For any $t \in \mathbb{C}$, $ty \in M$ so

$$\|x - \hat{m}\|^2 \leq \|e - ty\|^2 = \|e\|^2 - 2\operatorname{Re}\langle e, ty \rangle + \|ty\|^2$$

which implies

$$\operatorname{Re} \bar{t} \langle e, y \rangle \leq |t|^2 \|y\|^2$$

Take $t = \overline{\langle e, y \rangle} \|y\|^{-2}$. Then we have

$$\frac{|\langle e, y \rangle|^2}{\|y\|^2} \leq \frac{|\langle e, y \rangle|^2}{2\|y\|^2}$$

therefore $\langle e, y \rangle = 0$.

($b \Rightarrow a$) Let $x = \hat{m} + e$ for $\hat{m} \in M$ and $e \in M^\perp$. For any $y \in M$,

$$\|x - y\|^2 = \|e - (y - \hat{m})\|^2 = \|e\|^2 - 2\operatorname{Re}\langle e, y - \hat{m} \rangle + \|y - \hat{m}\|^2 = \|e\|^2 + \|y\|^2 \geq \|e\|^2 = \|x - \hat{m}\|^2$$

Suppose \tilde{m} is another closest point. Set $d = d(x, M)$. By the parallelogram identity,

$$\|\tilde{m} - \hat{m}\|^2 = \|x - \tilde{m} - (x - \hat{m})\|^2 = 2d^2 + 2d^2 - \|2x - 2(\tilde{m} + \hat{m})\|^2 \leq 4d^2 - 4d^2 = 0$$

6.

$$f_n = n \mathbf{1}_{[0, 1/n]}$$

7. (a) We will show that for $h \geq 0$ measurable, $\int h = 0 \implies h = 0$ a.e. Indeed, consider $A_n = \{n^{-1} > h \geq (n+1)^{-1}\}$ for each $n \in \mathbb{N}$ and $A_0 = \{h \geq 1\}$. Then, for any $n \in \mathbb{N} \cup \{0\}$,

$$(n+1)^{-1} \mu(A_n) \leq \int_{A_n} h \, d\mu \leq \int_{\mathbb{R}} h \, d\mu = 0$$

Therefore $\mu(A_n) = 0$. So, $\{h \neq 0\} = A = \cup A_n$ which has measure zero.

Now, we apply this to the problem by taking $h = g - f$. Then, $h \geq 0$ and

$$\int h = \int g - f = \int g - \int f = 0$$

Then by the above lemma, $h = 0$ so $f = g$ a.e.

- (b) If f and g are continuous, then being equal almost everywhere will imply they are equal everywhere. Indeed, suppose there exists $x_0 \in \mathbb{R}$ such that $f(x_0) < g(x_0)$. Then, $f - g$ is continuous around x_0 so there exists $\epsilon > 0$ such that

$$f(x) < g(x) \quad \text{for } |x - x_0| < \epsilon$$

However, $\lambda(\{|x - x_0| < \epsilon\}) = \epsilon$ so $f \neq g$ on a set of measure ϵ which contradicts $f = g$ a.e.

(c) INCOMPLETE

8. (a) First, \mathcal{M} is clearly a linear space since linear combinations of finite signed measures are still finite signed measures. Now we show that total variation is a norm on \mathcal{M} . If μ has total variation 0, this means

$$\mu_+(X) = \mu_-(X) = 0$$

so X is a null set of both μ_+ and μ_- . Thus for every $E \subseteq X$, $\mu(E) = \mu_+(E) - \mu_-(E) = 0 - 0 = 0$. Thus μ is the zero measure. By definition, $|\alpha\mu| = \alpha\mu_+ + \alpha\mu_- = \alpha|\mu|$. Finally, to check the triangle inequality, let $\mu, \lambda \in \mathcal{M}$. Let $A \cup B = X$ be a Hahn decomposition of X with respect to the signed measure $(\mu + \lambda)$. Then,

$$(\mu + \lambda)_+(X) = \mu(A) + \lambda(A) \leq \mu_+(A) + \lambda_+(A) \leq \mu_+(X) + \lambda_+(X)$$

and

$$(\mu + \lambda)_-(X) = -\mu(B) - \lambda(B) \leq \mu_-(B) + \lambda_-(B) \leq \mu_-(X) + \lambda_-(X)$$

Therefore

$$\begin{aligned} |\mu + \lambda|(X) &= (\mu + \lambda)_+(X) + (\mu + \lambda)_-(X) \leq \mu_+(X) + \lambda_+(X) + \mu_-(X) + \lambda_-(X) \\ &= |\mu|(X) + |\lambda|(X) \end{aligned}$$

- (b) Let μ be a σ -finite measure. Clearly $\mathcal{L}_\nu = \{\mu \in \mathcal{M} : \mu \ll \nu\}$ is a linear subspace since if $\lambda, \mu \ll \nu$, and $\nu(E) = 0$, then

$$\alpha\lambda(E) + \beta\mu(E) = 0$$

for any scalars α, β . We note the crucial property of this subspace. If $\mu \ll \nu$, then the null sets of ν are also null sets of μ_+ and μ_- . Indeed, let $E \subset X$ such that $\nu(E) = 0$. Let $A \cup B$ be a Hahn decomposition for μ . Then,

$$\nu(A \cap E) \leq \nu(E) = 0 \quad \nu(B \cap E) \leq \nu(E) = 0$$

so $\nu(A \cap E) = \nu(B \cap E) = 0$. Therefore

$$\mu_+(E) = \mu(A \cap E) = 0 \quad \mu_-(E) = \mu(B \cap E) = 0$$

Now, let $\{\mu_n\} \subseteq \mathcal{L}_\nu$ converge to μ in the total variation norm. Then, let $E \subset X$ such that $\nu(E) = 0$. Then, $|\mu_n|(E) = 0$. By the reverse triangle inequality (a consequence of the triangle inequality for $\|\cdot\|$ shown above)

$$|\mu|(E) = ||\mu|(E) - |\mu_n|(E)|| \leq |\mu - \mu_n|(E) \leq |\mu - \mu_n|(X) = \|\mu - \mu_n\| \rightarrow 0$$

so $\mu(E) = 0$ and $\mu \in \mathcal{L}_\nu$.

- (c) Let $f \in L^1(X, \mathcal{F}, \nu)$. Then,

$$\mu(A) = \int_A f d\nu$$

defines a signed measure for $A \subseteq X$. We only need to check that this pairing is isometric and onto. Surjectivity follows from the Radon-Nikodym theorem which states that if $\rho \ll \lambda$, then there exists λ -measurable g such that

$$\rho = g d\lambda$$

Then, to check the norms are preserved, we first show that the Hahn decomposition of μ corresponds to the positive and negative parts of f . Indeed, let $A = \{f \geq 0\}$. Then, for any $E \subseteq A$,

$$\mu(E) = \int_E f d\nu \geq 0$$

Similarly, for $B = \{f < 0\}$, $F \subseteq B$,

$$\mu(F) = \int_F f d\nu \leq 0$$

So $A \cup B$ is a Hahn decomposition for μ . Therefore,

$$\begin{aligned} \int_X |f| d\nu &= \int_X f^+ + f^- d\nu = \int_X f^+ d\nu + \int_X f^- d\nu = \int_A f^+ d\nu + \int_B f^- d\nu \\ &= \mu_+(A) + \mu_-(B) = \mu_+(X) + \mu_-(X) = |\mu|(X) \end{aligned}$$

9.

We have shown many times before that if $\sum \lambda(E_n) < \infty$, then $\lambda(\limsup_n E_n) = 0$. Set

$$E_n = [r_n - 2^{-n-1}, r_n + 2^{-n-1}]$$

Then,

$$\sum_n \lambda(E_n) = \sum_n 2^{-n} < \infty$$

Set $E = \limsup_{n \rightarrow \infty} E_n$. Then, $\lambda(E) = 0$. So, for any $x \notin E$, we have that there exists $k \in \mathbb{N}$ such that $x \notin E_n$ for all $n \geq k$. Therefore, $f(x)$ is only nonzero for finitely many indices so the sum must converge at x .

(b) Set

$$X_n = \left(\bigcup_{k \neq n} E_k \right)^c$$

Then, $\mathbb{R} = \cup X_n$ and, $\mu(X_n) \leq 1$ since $f_k = 0$ on X_n for $n \neq k$. Indeed,

$$\mu(X_n) = \int_{X_n} \sum f_k d\lambda = \int_{X_n} f_n \leq \int f_n = 1$$

(c) To show $\mu \ll \lambda$, let $E \subset \mathbb{R}$ such that $\lambda(E) = 0$. Then, integration over a set of measure zero is also zero so $\mu(E) = 0$.

- (d) Without loss of generality, we can just show that each open ball has infinite measure since every open set contains an open ball. Let $B(x, \epsilon) \subseteq \mathbb{R}$. Then, there exists a subsequence of $\{r_n\}$ such that $\{r_{n_k}\} \subseteq B(x, \epsilon/2)$. Moreover, since the radii of E_{n_k} are decreasing, there exists N such that $E_{n_k} \subseteq B(x, \epsilon)$ for all $k \geq N$. Thus,

$$\mu(B(x, \epsilon)) = \int_{B(x, \epsilon)} \sum f_n d\lambda \geq \int_{B(x, \epsilon)} \sum_{k=N}^{\infty} f_{n_k} d\lambda = \sum_{k=N}^{\infty} \int_{B(x, \epsilon)} f_{n_k} \geq \sum_{k=N}^{\infty} \int_{E_{n_k}} f_{n_k} = \infty$$

Winter 2014

8. See Summer 13 #9 (a)
9. See Summer 13 #7
10. See Summer 13 #9

Summer 2014

1. (a) *Proof.* Suppose f is discontinuous at some $x \in (0, 1)$. Then there exists $\epsilon > 0$ such that $\forall \delta > 0$ there exists $y \in B(x, \delta)$ such that

$$|f(x) - f(y)| \geq \epsilon$$

However, consider the compact interval $[x - \gamma, x + \gamma] \subseteq (0, 1)$ for some $\gamma > 0$. Then, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \epsilon/3$$

for all $z \in [x - \gamma, x + \gamma]$, $n \geq N$. □

- (b) False. Let

$$f_n(x) = \frac{1}{x} + \frac{1}{n} \quad f(x) = \frac{1}{x}$$

Then, $f_n \rightarrow f$ uniformly on $(0, 1)$ but f is not uniformly continuous.

- (c) *Proof.* □

9. See Summer 13 #7

Winter 2015

1. *Proof.* For $u = 1 + n^2x^2$, $du = 2n^2x dx$,

$$\|f_n - 0\|_1 = \int_0^1 \frac{nx}{1 + n^2x^2} dx = \int_1^2 \frac{du}{2nu} = \frac{1}{2n} [\ln(2) - \ln(1)] \rightarrow 0$$

as $n \rightarrow \infty$. Therefore $f_n \rightarrow 0$ in $L^1[0, 1]$. Now, since $\|\cdot\|_\infty$ and $\|\cdot\|_{\sup}$ coincide on continuous functions,

$$\|f_n - 0\|_\infty = \sup_{x \in [0, 1]} |f_n(x)| \geq f_n(n^{-3/2}) = \frac{n^{-1/2}}{1 + n^{-1}} \rightarrow \infty$$

as $n \rightarrow \infty$. So $f_n \not\rightarrow 0$ in $L^\infty[0, 1]$. □

2. *Proof.* Let f be convex. Let $x_n \searrow x$. Then, define $t_n \in [0, 1]$ by

$$(1 - t_n)1 + t_nx = x_n$$

Notice that $t_n \rightarrow 1$ as $n \rightarrow \infty$. Then,

$$f(x_n) = f(t_nx + (1 - t_n)1) \leq t_nf(x) + (1 - t_n)f(1)$$

also define $s_n \in [0, 1]$ such that

$$(1 - s_n)(-1) + s_nx_n = x$$

then $s_n \rightarrow 1$ as $n \rightarrow \infty$ so

$$f(x) = f(s_nx_n + (1 - s_n)(-1)) \leq s_nf(x_n) + (1 - s_n)f(-1)$$

Combining this, we get

$$f(x) \leq s_nf(x_n) + (1 - s_n)f(-1) \leq s_nt_nf(x) + s_n(1 - t_n)f(1) + (1 - s_n)f(-1)$$

Notice that the RHS converges to $f(x)$ as $n \rightarrow \infty$ so by the Squeeze theorem

$$\lim_{n \rightarrow \infty} f(x_n) = f(x)$$

So f is right continuous. To show left continuity, we follow the same steps but modify t_n and s_n so they are convex combinations with the opposite endpoints. Therefore f is continuous. □

3. *Proof.* For each $n \in \mathbb{N}$ there exists $x_n \in X$ such that

$$d(x_n, f(x_n)) < \frac{1}{n}$$

Since X is compact, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$ with limit x . Then,

$$d(x, f(x)) \leq d(x, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) + d(f(x_{n_k}), f(x)) \rightarrow 0$$

as $k \rightarrow \infty$ by construction of x_{n_k} and since f is continuous. Thus $f(x) = x$. □

4. (a) *Proof.* Let $\{y_n\} \subseteq Y$ be Cauchy. There exists $\{x_n\} \subseteq X$ such that $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is Cauchy and therefore convergent to some $x \in X$. Then,

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} f(x_n) = f(x) \in Y$$

□

- (b) False. Let $X = (0, 1)$, $Y = \mathbb{R}$. Let $d_Y = d_X = |(\cdot) - (\cdot)|$. Let $f(x) = 1/x$. Then, clearly

$$|x_1 - x_2| \leq \left| \frac{x_1}{x_1 x_2} - \frac{x_2}{x_1 x_2} \right| = |f(x_2) - f(x_1)|$$

so the inequality holds. Additionally, Y is complete but X is not.

5. *Proof.*

$$\|S(a)\|_2 = \sqrt{\sum_{n=1}^{\infty} s_n^2 a_n^2} \leq \|s\|_{\infty} \|a\|_2$$

For each $k \in \mathbb{N}$, there exists $s_{n_k} \in s$ such that

$$|s_{n_k}| > \|s\|_{\infty} - \frac{1}{k}$$

Then, consider $e_{n_k} = (0, \dots, 0 \overset{n_k}{1}, 0, \dots) \in \ell^2$. $\|e_{n_k}\|_2 = 1$ so

$$\|S(e_{n_k})\|_2 = |s_{n_k}| > \|s\|_{\infty} - \frac{1}{k}$$

for all $k \in \mathbb{N}$ thus

$$\|S\| = \|s\|_{\infty}$$

□

6. *Proof.* First, notice that T is bounded below:

$$\|x\|^2 \leq \langle Tx, x \rangle \leq \|Tx\| \cdot \|x\|$$

so, $\|Tx\| \geq \|x\|$ for all $x \in \mathcal{H}$. Now, we show one-to-one. Let $x \in \mathcal{H}$ such that $Tx = 0$. Then,

$$0 = \|Tx\| \geq \|x\|_{\mathcal{H}} \geq 0$$

so $x = 0$. Next, we show T has a closed range. Let $x_n \in \mathcal{H}$ such that $Tx_n \rightarrow y$ for some $y \in \mathcal{H}$. Then,

$$\|Tx_n - Tx_m\| \geq \|x_n - x_m\|$$

for all $n, m \in \mathbb{N}$. So, $\{x_n\}$ is Cauchy. Thus, there exists $x \in \mathcal{H}$ such that $x_n \rightarrow x$. Since T is bounded,

$$y = \lim_{n \rightarrow \infty} Tx_n = Tx$$

so $y \in \text{Ran} T$. Finally, we show T is onto. For $w \in (\text{Ran} T)^{\perp}$

$$\langle Tv, w \rangle = 0$$

for all $v \in \mathcal{H}$. In particular, for $v = w$,

$$0 = \langle Tw, w \rangle \geq \|w\|^2 \geq 0$$

which implies $w = 0$. Thus, $(\text{Ran} T)^\perp = \{0\}$ so $\text{Ran} T = \overline{\text{Ran} T} = \mathcal{H}$. We have T is one-to-one and onto therefore is it invertible so $Tx = y$ has a unique solution for every $y \in \mathcal{H}$. \square

7. Solution by Hao Chen and Walton Green (4/18)

Proof. We will prove the contrapositive of the statement. Suppose $\{E_k\}_{k=1}^n$ are Borel subsets of $[0, 1]$ such that

$$\lambda\left(\bigcap_{k=1}^n E_k\right) = 0$$

Then, we have that

$$1 = \lambda([0, 1]) = \lambda\left[\left(\bigcap_{k=1}^n E_k\right)^c\right] = \lambda\left(\bigcup_{k=1}^n E_k^c\right)$$

Therefore,

$$n = \sum_{k=1}^n \lambda([0, 1]) = \sum_{k=1}^n \lambda(E_k) + \lambda(E_k^c) \geq \sum_{k=1}^n \lambda(E_k) + \lambda\left(\bigcup_{k=1}^n E_k^c\right) = \sum_{k=1}^n \lambda(E_k) + 1$$

$$\text{so } \sum_{k=1}^n \lambda(E_k) \leq n - 1. \quad \square$$

8. Let $\{q_n\}_{n=1}^\infty \subseteq \mathbb{R}$ be an enumeration of the rational numbers. Then, let

$$U := \bigcup_{n=1}^\infty \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$$

So,

$$\lambda(U) \leq \sum_{n=1}^\infty \frac{2}{n^2} = 2 < \infty$$

and $U \subseteq \mathbb{R}$ is open. Now, notice that $\bar{U} = \mathbb{R}$ since \mathbb{Q} is dense in \mathbb{R} and $\mathbb{Q} \subseteq U$. Then,

$$\lambda(\partial U) = \lambda(\bar{U} \setminus U) = \infty$$

9. *Proof.* (\Rightarrow) let $\lambda(E) = M > 0$. Let $f_n \xrightarrow{\lambda} 0$. Then, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\inf \{c > 0 : \lambda\{|f_n| > c\} < \epsilon\} < \epsilon$$

for all $n \geq N$ which implies

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

Now, we will use the fact that

$$x \mapsto \frac{x}{x+1}$$

is monotone increasing and ≤ 1 .

$$\begin{aligned} \int_E \frac{|f_n|}{1+|f_n|} &= \int_{E \cap \{|f_n| > \epsilon\}} \frac{|f_n|}{1+|f_n|} + \int_{E \cap \{|f_n| < \epsilon\}} \frac{|f_n|}{1+|f_n|} \\ &\leq \int_{E \cap \{|f_n| > \epsilon\}} 1 + \int_E \frac{\epsilon}{1+\epsilon} \\ &\leq \lambda\{|f_n| > \epsilon\} + \lambda(E) \left(\frac{\epsilon}{1+\epsilon} \right) \\ &< \epsilon + M \left(\frac{\epsilon}{1+\epsilon} \right) \rightarrow 0 \end{aligned}$$

as $\epsilon \rightarrow 0$.

(\Leftarrow) Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$\frac{\epsilon^2}{1+\epsilon} > \int_E \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n| > \epsilon\}} \frac{|f_n|}{1+|f_n|} \geq \int_{\{|f_n| > \epsilon\}} \frac{\epsilon}{1+\epsilon} = \lambda\{|f_n| > \epsilon\} \frac{\epsilon}{1+\epsilon}$$

so

$$\lambda\{|f_n| > \epsilon\} < \epsilon$$

for all $n \geq N$. Thus,

$$\|f_n\|_\lambda = \inf\{c > 0 : \lambda\{|f_n| > c\} < c\} < \epsilon$$

□

10. *Proof.* Define

$$F_i := \bigcup_{n=i}^{\infty} E_n$$

for each $i \in \mathbb{N}$. Notice that F_i are reverse nested (i.e. $F_{i+1} \subseteq F_i$ therefore $F_i^c \subseteq F_{i+1}^c$)
Then,

$$\mu(F_i) = \mu\left(\bigcup_{n=i}^{\infty} E_n\right) \leq \sum_{n=i}^{\infty} \mu(E_n) \rightarrow 0$$

as $i \rightarrow \infty$. Now,

$$\begin{aligned} \mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n\right) &= \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \mu\left(F_1 \cap \bigcap_{k=2}^{\infty} F_k\right) = \mu\left(F_1 \setminus \bigcup_{k=2}^{\infty} F_k^c\right) \\ &= \mu(F_1) - \mu\left(\bigcup_{k=2}^{\infty} F_k^c\right) = \mu(F_1) - \lim_{k \rightarrow \infty} \mu(F_k^c) \\ &= \lim_{k \rightarrow \infty} \mu(F_1 \setminus F_k^c) = \lim_{k \rightarrow \infty} \mu(F_1 \cap F_k) \\ &= \lim_{k \rightarrow \infty} \mu(F_k) = 0 \end{aligned}$$

□

Summer 2015

1. (a) *Proof.* Let $\epsilon > 0$, pick $N \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all $n \geq N$. This can be done since $\sum M_n < \infty$. Now, for all $x \in \mathbb{R}$,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_n < \epsilon$$

for all $n \geq N$. Therefore

$$\sum_{k=1}^{\infty} f_k(x)$$

is uniformly convergent. □

- (b) Define

$$f_n(x) := \begin{cases} \frac{1}{n} & n \leq x < n+1 \\ 0 & \text{otherwise} \end{cases} \quad \forall n \in \mathbb{N}$$

Then, clearly $\sum f_n(x)$ is convergent pointwise and

$$\sum_{n=1}^{\infty} \|f_n\|_{\infty} \leq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Now we need to show this convergence is actually uniform. Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then, for all $x \in \mathbb{R}$,

$$\left| \sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \frac{1}{n+1} \leq \frac{1}{N} < \epsilon$$

for all $n \geq N$.

2. *Proof.* First we show $(A^{\perp})^{\perp}$ is a closed subspace containing A . Clearly $A \subset (A^{\perp})^{\perp}$. Let $x, y \in (A^{\perp})^{\perp}$ and $a, b \in \mathbb{C}$. Then,

$$\langle ax + by | z \rangle = a \langle x | z \rangle + b \langle y | z \rangle = 0 + 0 = 0 \quad \forall z \in A^{\perp}$$

Let $\{x_n\}_{n=1}^{\infty} \subset (A^{\perp})^{\perp}$ such that $x_n \rightarrow x$.

$$\langle x | z \rangle = \lim_{n \rightarrow \infty} \langle x_n | z \rangle = \lim_{n \rightarrow \infty} 0 = 0 \quad \forall z \in A^{\perp}$$

So, we have shown $\overline{\text{span} A} \subset (A^{\perp})^{\perp}$. Now, let $x \in (A^{\perp})^{\perp}$. Then,

$$d(x, \overline{\text{span} A}) = \sup_{\substack{y \in (\overline{\text{span} A})^{\perp} \\ \|y\| \leq 1}} |\langle x | y \rangle| = \sup_{y \in A^{\perp}, \|y\| \leq 1} |\langle x | y \rangle| = 0$$

So $x \in \overline{\text{span} A}$ since it is closed. □

3. (a) *Proof.* (i) Clearly, $d_s(A, A) = 0$. Now, suppose $d_s(A, B) = 0$. Then for all $\epsilon > 0$ and $x \in A$,

$$d(x, B) \leq \epsilon$$

thus $d(x, B) = 0$ so $x \in B$ since B is closed. Thus $A \subseteq B$. Likewise $B \subseteq A$. so $A = B$.

(ii) Clearly $d_s(A, B) = d_s(B, A)$.

(iii) Let $C \subseteq X$ be closed. Let $\epsilon_1 > 0$ be such that $A_{\epsilon_1} \subset C$ and $C_{\epsilon_1} \subseteq A$. Let $\epsilon_2 > 0$ such that $B_{\epsilon_2} \subset C$ and $C_{\epsilon_2} \subseteq B$. Then,

$$A_{\epsilon_1 + \epsilon_2} \subset C_{\epsilon_2} \subset B \quad \text{and} \quad B_{\epsilon_1 + \epsilon_2} \subset C_{\epsilon_1} \subset A$$

So, $d_s(A, B) \leq \epsilon_1 + \epsilon_2$ for all such ϵ_1, ϵ_2 . Therefore,

$$d_s(A, B) \leq \inf\{\epsilon_1\} + \inf\{\epsilon_2\} = d_s(A, C) + d_s(C, B)$$

□

(b) If the sets are not closed, then the first property of the metric fails. $d_s(A, A) = 0$ but $d_s(A, B) = 0$ does not necessarily $A = B$. Consider $X = \mathbb{R}$ and $A = [0, 1]$ and $B = (0, 1)$. $d_s(A, B) = 0$ but $A \neq B$.

4. (a) *Proof.* First, we show T is bounded:

$$\begin{aligned} \|Tf\|_\infty &= \sup_{t \in [0, 1]} \left| \int_0^t s f(s) ds \right| \leq \sup_{t \in [0, 1]} \int_0^t s |f(s)| ds \\ &\leq \|f\|_\infty \sup_{t \in [0, 1]} \int_0^t s ds \leq \|f\|_\infty \int_0^1 s ds = \frac{1}{2} \|f\|_\infty \end{aligned}$$

so $\|T\| \leq \frac{1}{2}$. Let $f, g \in C[0, 1]$ and $a, b \in \mathbb{R}$. Then,

$$T(af + bg)(t) = \int_0^t s(af + bg)(s) ds = a \int_0^t s f(s) ds + b \int_0^t s g(s) ds = a(Tf)(t) + b(Tg)(t)$$

so T is linear. □

(b) *Proof.* Let $f(t) = 1$ for all $t \in [0, 1]$. Then, $\|f\|_\infty = 1$ and

$$\|Tf\|_\infty = \sup_{t \in [0, 1]} \left| \int_0^t s ds \right| = \sup_{t \in [0, 1]} \frac{t^2}{2} = \frac{1}{2}$$

so $\|T\| = \frac{1}{2}$. □

5. (a) *Proof.* For every $n \in \mathbb{N}$ there exists $E_n \subseteq X$ such that $\mu(E_n) < \frac{1}{n^2}$ and $f_k \rightarrow f$ uniformly on $X \setminus E_n$. Let

$$E := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$$

Then, $\mu(E) = 0$ (For proof see Winter 15 #10) since

$$\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Now, for $x \notin E$, there exists k such that $x \notin \bigcup_{n=k}^{\infty} E_n$ so $x \notin E_n$ for all $n \geq k$ (However we only need it to hold for a single set, E_k . So, since $x \in E_k^c$,

$$f_n(x) \rightarrow f(x)$$

as $n \rightarrow \infty$. Therefore $f_n \rightarrow f$ pointwise a.e. □

- (b) *Proof.* Let $\epsilon > 0$. Then, there exists some E_ϵ such that $\mu(E_\epsilon) < \epsilon$ and $f_n \rightarrow f$ uniformly on E_ϵ . Moreover, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$, $x \in E_\epsilon^c$. Then,

$$\mu\{|f_n - f| > \epsilon\} \leq \mu(E_\epsilon) < \epsilon$$

so

$$\|f_n - f\|_\mu = \inf\{c > 0 : \mu\{|f_n - f| > c\} < c\} < \epsilon$$

therefore $f_n \rightarrow f$ in measure. □

6. *Proof.* Let $E \subset [a, b]$ be Borel measurable with $\lambda(E) > 0$. Let $\{q_n\}$ be an enumeration of the rational numbers in the interval $[0, 1]$. Set

$$F = \bigcup_n (E + q_n)$$

If $\{E + q_n\}$ are all disjoint, then, $\lambda(F) = \sum_{n=1}^{\infty} \lambda(E + q_n) = \sum_{n=1}^{\infty} \lambda(E) = \infty$ since $\lambda(E) > 0$. But this is a contradiction since $F \subseteq [a, b + 1]$ which has finite Lebesgue measure. Thus there exists $x \in (E + q_n) \cap (E + q_m)$ for some n and m not equal (so $q_n \neq q_m$). Then, there exists $y, z \in E$ such that

$$y + q_n = x = z + q_m$$

so $y - z = q_m - q_n \in \mathbb{Q} \setminus \{0\}$. □

7. (a) False. Consider the following function with a “spike” at every natural number, $n \geq 2$.

$$f(x) := \begin{cases} \nearrow & n \leq x \leq n + \frac{1}{n^3} \\ n & x = n + \frac{1}{n^3} \\ \searrow & n + \frac{1}{n^3} \leq x \leq n + \frac{2}{n^3} \\ 0 & \text{else} \end{cases}$$

Notice that

$$\int_{\mathbb{R}} f(x) dx = \sum_{n=2}^{\infty} \frac{1}{2} \cdot \frac{2}{n^3} \cdot n = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty$$

but

$$\limsup_{x \rightarrow \infty} |f(x)| = \infty$$

- (b) *Proof.* Let f be integrable and differentiable and let $D > 0$ such that $|f'(x)| \leq D$ for all $x \in \mathbb{R}$. Fix $x \in \mathbb{R}$. Using the mean-value theorem, for all $y \in \mathbb{R}$ such that

$$|x - y| \leq \frac{f(x)}{D},$$

we know that

$$f(y) \geq f(x) - |x - y|D$$

Suppose without loss of generality that

$$\limsup_{x \rightarrow \infty} f(x) = M$$

for some $M > 0$. Then for all $n \in \mathbb{N}$, there exists $x_n \geq n$ such that

$$f(x_n) \geq \frac{M}{2}$$

Then,

$$\int_{\mathbb{R}} f(x) dx \geq \sum_{n=1}^{\infty} \frac{1}{2} \cdot \min \left\{ \frac{f(x_n)}{D}, 1 \right\} \cdot f(x_n) \geq \sum_{n=1}^{\infty} \frac{1}{8} \cdot \min \left\{ \frac{M}{D}, 1 \right\} \cdot M = \infty$$

which contradicts the fact that f is integrable. □

8. *Proof.* (a \implies b)

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^{\infty} \int_{F_k} 2^k dm \leq \sum_{k=-\infty}^{\infty} \int_{F_k} f dm = \int_{\mathbb{R}} f dm$$

(b \implies c) First, notice that $E_k \cup F_{k-1} = E_{k-1}$ and the union is disjoint therefore

$$m(F_k) = m(E_{k+1}) - m(E_k)$$

Mutlply by 2^k and sum from $-N$ to N we have

$$\begin{aligned} \sum_{k=-N}^N 2^k m(F_k) &= \sum_{k=-N}^N 2^k m(E_{k+1}) - \sum_{k=-N}^N 2^k m(E_k) = \frac{1}{2} \sum_{k=-N}^N 2^{k+1} m(E_{k+1}) - \sum_{k=-N}^N 2^k m(E_k) \\ &= -\frac{1}{2} \sum_{k=-N+1}^{N-1} 2^k m(E_k) + 2^N m(E_{N+1}) - 2^{-N} m(E_{-N}) \end{aligned}$$

The final two terms can be bounded by $\int f$: $2^N m(E_N) \leq \int_{E_N} f \, dm \leq \int_{\mathbb{R}} f \, dm < \infty$.
Therefore, for any N ,

$$\sum_{k=-(N-1)}^{N-1} 2^k m(E_k) \leq -2 \sum_{k=-\infty}^{\infty} 2^k m(F_k) + 4 \int_{\mathbb{R}} f \, dm < \infty$$

(c \implies a) Notice that since f is non-negative, $\mathbb{R} = \{f = 0\} \cup E_k$.

$$\int f \, dm = \sum_{k=-\infty}^{\infty} \int_{F_k} f \, dm \leq \sum_{k=-\infty}^{\infty} \int_{F_k} 2^{k+1} \, dm = 2 \sum_{k=-\infty}^{\infty} m(F_k) \leq 2 \sum_{k=-\infty}^{\infty} m(E_k)$$

□

Winter 2016

1. Recall

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

So for $x = 1$,

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

2. *Proof.* Since ℓ^2 is a Hilbert space, A being dense in ℓ^2 is equivalent to

$$A^\perp = \{0\}$$

Let $x = (x_1, x_2, \dots) \in A^\perp$. Then, $\langle x, a \rangle_{\ell^2} = 0$ for all $a \in A$. Notice that $a = e^{(k)} = (0, \dots, 0, \underset{k^{th}}{1}, 0, \dots)$ is in A for any $k \in \mathbb{N}$.

$$0 = \langle x, e^{(k)} \rangle = \sum_{i=1}^{\infty} x_i e_i^{(k)} = x_k$$

for $k \in \mathbb{N}$. Therefore $x = 0$. Now we show the same thing for A^c . Let $y \in (A^c)^\perp$. Also, define $f^{(k)} = e^{(k)} - e^{(k+1)} \in A^c$. Then,

$$0 = \langle y, f^{(k)} \rangle = y_k - y_{k+1}$$

So $y_k = y_{k+1}$ for all $k \in \mathbb{N}$. Thus y is a constant sequence. The only constant sequence in ℓ^2 is the zero sequence therefore $y = 0$. \square

3. *Proof.* First we show subspace. Let $x_1 + y_1, x_2 + y_2 \in X + Y$ and $a, b \in \mathbb{R}$. Then,

$$a(x_1 + y_1) + b(x_2 + y_2) = (ax_1 + bx_2) + (ay_1 + by_2) \in X + Y$$

Now we show closure. Let $\{(x_n + y_n)\}_{n=1}^{\infty}$ be a sequence in $X + Y$ with limit z . This sequence is also Cauchy. So, using the fact that $X \perp Y$,

$$\begin{aligned} \|(x_n + y_n) - (x_m + y_m)\|^2 &= \|(x_n - x_m) + (y_n - y_m)\|^2 \\ &= \langle (x_n - x_m) + (y_n - y_m), (x_n - x_m) + (y_n - y_m) \rangle \\ &= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (x_n - x_m), (y_n - y_m) \rangle \\ &\quad + \langle (y_n - y_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\ &= \langle (x_n - x_m), (x_n - x_m) \rangle + \langle (y_n - y_m), (y_n - y_m) \rangle \\ &= \|x_n - x_m\|^2 + \|y_n - y_m\|^2 \end{aligned}$$

and thus $\{x_n\}$ and $\{y_n\}$ are both Cauchy. Since \mathcal{H} is a Hilbert space, they are convergent to some x, y respectively. Since X, Y are closed, $x \in X$ and $y \in Y$. Then,

$$z = \lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n = x + y \in X + Y$$

Therefore $X + Y$ is closed. \square

4. *Proof.* Since Y is a Banach space, $\mathcal{B}(X, Y)$ is also a Banach space. In a Banach space, any absolutely convergent series is convergent. Since $\|T\| < 1$,

$$\sum_{n=0}^{\infty} \|T\|^n < \infty$$

So

$$\sum_{n=0}^{\infty} T^n \in \mathcal{B}(X, Y)$$

□

5. (a) *Proof.* First, to show T is well-defined we need to show $T\xi$ is continuous for a fixed ξ . This follows from the fact that for $n > m$,

$$\left\| \sum_{k=0}^n a_k \xi_k x^k - \sum_{k=0}^m a_k \xi_k x^k \right\|_{\infty} = \sup_{x \in [0,1]} \left| \sum_{k=m+1}^n a_k \xi_k x^k \right| \leq \|a\|_{\infty} \sum_{k=m+1}^n |\xi_k| \rightarrow 0$$

as $n, m \rightarrow \infty$ since $\xi \in \ell^1$. Thus, this sequence of partial sums is Cauchy in $\|\cdot\|_{\infty}$. Since $(C[0, 1], \|\cdot\|_{\infty})$ is a Banach space, it's limit, $T\xi \in C[0, 1]$. To show linearity, let $\xi, \zeta \in \ell^1$ and $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} T(\alpha\xi + \beta\zeta)(x) &= \sum_{k=0}^{\infty} a_k (\alpha\xi_k + \beta\zeta_k) x^k = \alpha \sum_{k=0}^{\infty} a_k \xi_k x^k + \beta \sum_{k=0}^{\infty} a_k \zeta_k x^k \\ &= \alpha T(\xi)(x) + \beta T(\zeta)(x) \end{aligned}$$

□

- (b) *Proof.*

$$\|T(\xi)\|_{\infty} = \sup_{x \in [0,1]} |T(\xi)(x)| = \sup_{x \in [0,1]} \left| \sum_{k=0}^{\infty} a_k \xi_k x^k \right| \leq \|a\|_{\infty} \sum_{k=0}^{\infty} |\xi_k| = \|a\|_{\infty} \cdot \|\xi\|_1$$

So $\|T\| \leq \|a\|_{\infty}$. We claim this is actually the norm. For $\epsilon > 0$ there exists $a_n \in a$ such that

$$|a_n| > \|a\|_{\infty} - \epsilon$$

Pick $\xi^{(n)} = (0, \dots, 0, \overset{n^{th}}{1}, 0, \dots) \in \ell^1$. Then,

$$\|T\xi^{(n)}\|_{\infty} = \sup_{x \in [0,1]} \left| \sum_{k=0}^{\infty} a_k \xi_k^{(n)} x^k \right| = \sup_{x \in [0,1]} |a_n x^n| = |a_n| > \|a\|_{\infty} - \epsilon$$

Since there exists such $\xi^{(n)}$ for all $\epsilon > 0$, $\|T\| = \|a\|_{\infty}$. □

6. (i) LDCT cannot be applied to f_n since any k which bounds every f_n above, must be greater than 1 everywhere thus $\int_{\mathbb{R}} k = \infty$.

- (ii) LDCT cannot be applied to g_n since any k which bounds every g_n above, must be greater than $1/x$ everywhere thus $\int_{\mathbb{R}} k \geq \int_{\mathbb{R}} 1/x = \infty$.
- (iii) LDCT can be applied since for $k = 1/x^2$, $|h_n| \leq k$ and

$$\int_{\mathbb{R}} \frac{1}{x^2} dx < \infty$$

7. *Proof.* Let $E \subset \mathbb{R}$, $\epsilon \in (0, 1)$. Set $\delta = m^*(E)(1/\epsilon - 1) > 0$. By definition of outer measure, there exists an open set $G \supset E$ such that $m^*(E) + \delta > m^*(G) = m(G)$. Then,

$$\epsilon m(G) < \epsilon(m^*(E) + \delta) = \epsilon m^*(E)(1 + 1/\epsilon - 1) = m^*(E)$$

Moreover, since G is open, it can be written as a countable, disjoint union of open intervals, say I_k . Then,

$$\sum_k \epsilon m(I_k) = \epsilon m(G) < m^*(E) = m^*(E \cap G) \leq \sum_k m^*(E \cap I_k)$$

Therefore, at least one term in the left hand sum must be smaller than one term in the right hand sum, i.e. there exists k such that $\epsilon m^*(I_k) = \epsilon m(I_k) < m^*(E \cap I_k)$. \square

8. *Proof.* Define $A_n := \{x \in [0, 1] : n + 1 > |f(x)| \geq n\}$

$$\sum_{n=1}^{\infty} n \lambda(A_n) = \sum_{n=1}^{\infty} \int_{A_n} n dx \leq \sum_{n=1}^{\infty} \int_{A_n} f(x) dx = \int_0^1 f(x) < \infty$$

So,

$$\begin{aligned} \lim_{n \rightarrow \infty} n \lambda(\{x \in [0, 1] : |f(x)| \geq n\}) &= \lim_{n \rightarrow \infty} n \lambda\left(\bigcup_{k=n}^{\infty} A_k\right) = \lim_{n \rightarrow \infty} n \sum_{k=n}^{\infty} \lambda(A_k) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} k \lambda(A_k) = 0 \end{aligned}$$

\square

9. (a) *Proof.* Let $f(x) > 0$ for $x \in [0, 1]$ and $E \subseteq [0, 1]$ such that $\lambda(E) > 0$. Suppose $\int_E f d\lambda = 0$. Then

$$f(x) = 0$$

for almost every $x \in E$. However, since $\lambda(E) > 0$ there exists $x \in E$ such that $f(x) > 0$ which is a contradiction. \square

- (b) First we prove the following fact: $\mu(\limsup E_n) \geq \limsup \mu(E_n)$. Indeed, set $F_k = \bigcup_{n=k}^{\infty} E_n$. F_k are decreasing.

$$\mu(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n) = \mu(\bigcap_{k=1}^{\infty} F_k) = \inf_k \mu(F_k) = \inf_k \mu(\bigcup_{n=k}^{\infty} E_n) \geq \inf_k \sup_{n \geq k} \mu(E_n)$$

Proof. Fix $\epsilon \in (0, 1]$. Suppose $\inf_{\lambda(E) \geq \epsilon} \int_E f d\lambda = 0$. Then, for each n there exists E_n with $\lambda(E_n) \geq \epsilon$ and

$$\int_{E_n} f d\lambda < \frac{1}{n^2}$$

Then, consider $E = \limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$. By the fact above, $\mu(E) \geq \epsilon$. By part (a), this means $\int_E f d\lambda > 0$. However,

$$\int_E f d\lambda = \int_{\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n} f d\lambda \leq \int_{\bigcup_{n=k}^{\infty} E_n} f d\lambda \leq \sum_{n=k}^{\infty} \int_{E_n} f d\lambda \leq \sum_{n=k}^{\infty} \frac{1}{n^2}$$

for any k . Therefore $\int_E f d\lambda = 0$ which is a contradiction. □

Summer 2016

1. (a) *Proof.* We show that f_n does not converge uniformly on the half-open interval $[0, 1)$. The pointwise limit is clearly $f(x) = 0$ for $x \in [0, 1)$. If $\{f_n\}$ converges uniformly, then it must converge to f , the pointwise limit. Let $\epsilon > 0$. For any $n \in \mathbb{N}$ there exists $x \in (0, 1]$ such that

$$1 > x > \left(\frac{\epsilon}{1-\epsilon}\right)^{1/n}$$

Then,

$$|f(x)| > \epsilon$$

so $\{f_n\}$ does not converge uniformly on $[0, 1)$ therefore it does converge uniformly on $[0, 1]$. \square

- (b) *Proof.* Notice that

$$f_n(x) \leq 1$$

for $x \in [0, 1]$. Since $\int_0^1 1 \, dx < \infty$, by Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \lim_{n \rightarrow \infty} f_n(x) \, dx = \int_0^1 0 \, dx = 0$$

\square

2. False. Counterexample:

Consider $\{x^{(n)}\}_{n=1}^\infty \subset X$ where

$$x^{(n)} := (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots) \in X$$

Then, $\{x^{(n)}\}_{n=1}^\infty$ is Cauchy: For $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. So, for all $n, m \geq N$ ($n > m$),

$$d(x^{(n)}, x^{(m)}) = \sup_{i \in \mathbb{N}} |x_i^{(n)} - x_i^{(m)}| = \frac{1}{m} < \frac{1}{N} < \epsilon$$

However, $x_n \rightarrow (1, \frac{1}{2}, \frac{1}{3}, \dots)$ which is not in X .

3. *Proof.* Let $\{y_n\}_{n=1}^\infty \subset K$. Let $\{y_{n_k}\}_k$ denote the set of distinct elements of $\{y_n\}_{n=1}^\infty$. If $\{y_{n_k}\}_k$ is finite, then there exists some $m \in \mathbb{N}$ such that y_m occurs infinitely many times in $\{y_n\}_{n=1}^\infty$ thus the constant sequence $\{y_m\}$ is a convergent subsequence of $\{y_n\}_{n=1}^\infty$. On the other hand if $\{y_{n_k}\}_k$ is infinite, then

$$\{y_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=0}^\infty$$

is a subsequence of a convergent sequence so it is itself convergent to $\lim_{n \rightarrow \infty} x_n = x_0 \in K$. \square

4. *Proof.* First, notice

$$\begin{aligned}
(T - S)^3 &= (T^2 - ST - TS + S^2)(T - S) \\
&= (T - 2ST + S)(T - S) \\
&= (T^2 - 2ST^2 + ST - ST + 2S^2T - S^2) \\
&= (T - 2ST + ST - ST + 2ST - S) \\
&= (T - S)
\end{aligned}$$

Then, by Cauchy-Schwarz for the operator norm,

$$\|T - S\| = \|(T - S)^3\| \leq \|T - S\|^3$$

Therefore

$$1 \leq \|T - S\|^2$$

and

$$\|T - S\| \geq 1$$

□

5. *Proof.* Let $n, m \in \mathbb{N}$. Without loss of generality, let $n > m$. First,

$$\|x_m\|^2 = \langle x_n, x_m \rangle \leq \|x_n\| \cdot \|x_m\|$$

so $\{\|x_n\|\}_{n=1}^\infty$ is monotone decreasing. Moreover it is bounded below by 0 so it is convergent to some $K \in \mathbb{R}$. Moreover, since

$$\lim_{n \rightarrow \infty} \|x_n\|^2 = \left(\lim_{n \rightarrow \infty} \|x_n\| \right)^2 = K^2$$

$\{\|x_n\|^2\}_{n=1}^\infty$ is convergent and therefore Cauchy. Then, for $n > m$,

$$\begin{aligned}
\|x_n - x_m\|^2 &= \langle x_n - x_m, x_n - x_m \rangle \\
&= \|x_n\|^2 - \langle x_n, x_m \rangle - \langle x_m, x_n \rangle + \|x_m\|^2 \\
&= \|x_n\|^2 - \|x_m\|^2 - \|x_m\|^2 + \|x_m\|^2 \\
&= \|x_n\|^2 - \|x_m\|^2 \\
&= \left| \|x_n\|^2 - \|x_m\|^2 \right| \rightarrow 0
\end{aligned}$$

So $\{x_n\}_{n=1}^\infty$ is Cauchy and therefore convergent since \mathcal{H} is a Hilbert space. □

6. *Proof.* Let $A = (0, 1)$ and $B = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then,

$$d(A, B) = \lambda(A \Delta B) = \lambda(\{\frac{1}{2}\}) = 0$$

but $A \neq B$. Thus the first property of a metric $d(A, B) = 0 \implies A = B$ fails. □

7. *Proof.* (\Leftarrow) Fix $\epsilon > 0$. Then, there exists an open set $\mathcal{O} \supseteq A$ such that

$$\lambda(\mathcal{O} \setminus A) < \epsilon$$

Thus $\mathcal{O} \setminus A \in \mathcal{L}$, the σ -algebra of Lebesgue-measurable sets. Moreover, since \mathcal{O} is open, it is also Lebesgue measurable. Thus,

$$A = \mathcal{O} \setminus (\mathcal{O} \setminus A) \in \mathcal{L}$$

since \mathcal{L} is closed under set-minus.

(\Rightarrow) Let A be Lebesgue measurable. Then,

$$\lambda(A) = \lambda^*(A) = \inf_{A \subseteq \bigcup_n I_n} \sum_{n=1}^{\infty} \lambda(I_n)$$

where $I_n = [a_n, b_n)$. Now, let $\epsilon > 0$. By definition of inf, there exists $\{I_n\}_{n=1}^{\infty}$ such that

$$\lambda(A) + \frac{\epsilon}{2} > \sum_{n=1}^{\infty} \lambda(I_n) \quad \text{and} \quad A \subseteq \bigcup_{n=1}^{\infty} I_n$$

Now, define

$$J_n = \left(a_n, b_n + \frac{\epsilon}{2^{n+1}} \right)$$

Then, $I_n \subset J_n$ for all i and for $\mathcal{O} := \bigcup_{n=1}^{\infty} J_n$

$$\lambda(\mathcal{O} \setminus A) = \lambda(\mathcal{O}) - \lambda(A) \leq \sum_{n=1}^{\infty} \lambda(J_n) - \lambda(A) = \sum_{n=1}^{\infty} \left(\lambda(I_n) + \frac{\epsilon}{2^{n+1}} \right) - \lambda(A) < \epsilon$$

□

8. (a) *Proof.* Proof by contraposition. Suppose $\lambda(E_n) = 0$ for all $n \in \mathbb{N}$. Then,

$$\lambda(\{x \in I : f(x) > 0\}) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} \lambda(E_n) = 0$$

since $\{E_n\}$ are nested. □

(b) *Proof.* Suppose the assumption holds and that $\lambda(\{x \in I : f(x) > 0\}) > 0$. Then, by part (a), there exists some $n \in \mathbb{N}$ such that $\lambda(E_n) > 0$. Since the measure of E_n is positive, it contains infinitely many points. Now, pick $x_1, \dots, x_{n \cdot M} \in E_n$, then,

$$f(x_1) + \dots + f(x_{n \cdot M}) > \frac{1}{n} + \dots + \frac{1}{n} = Mn \left(\frac{1}{n} \right) = M$$

which is a contradiction. □

9. INCOMPLETE

Proof. (\Rightarrow) By the Triangle Inequality,

$$\|f_n\|_1 \leq \|f_n - f\|_1 + \|f\|_1$$

and

$$\|f\|_1 \leq \|f - f_n\|_1 + \|f_n\|_1$$

therefore

$$|\|f_n\|_1 - \|f\|_1| \leq \|f_n - f\|_1 \rightarrow 0$$

as $n \rightarrow \infty$.

(\Leftarrow)

□

10. *Proof.* Applying Holder's Inequality,

$$\begin{aligned} \sum_{n=0}^{\infty} \int_n^{n+1} f(x) dx &\leq \sum_{n=0}^{\infty} \left(\int_n^{n+1} f(x)^2 dx \right)^{1/2} \left(\int_n^{n+1} 1^2 dx \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} f(x)^2 dx \right)^{1/2} = \|f\|_{L^2(\mathbb{R})} < \infty \end{aligned}$$

therefore

$$\lim_{n \rightarrow \infty} \int_n^{n+1} f(x) dx = 0$$

□

Winter 2017

1. Notice that

$$\sum \frac{\sin(nx)}{n}$$

is the Fourier series of the function $x \mapsto \frac{\pi - x}{2}$. Indeed,

$$\frac{\pi}{2} \int_{-\pi}^{\pi} \sin(nx) dx - \frac{1}{2} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{x \cos(nx)}{2n} \Big|_{x=-\pi}^{\pi} - \frac{1}{2n} \int_{-\pi}^{\pi} \cos(nx) dx =$$

2. (a) *Proof.* (i) First, notice that

$$-M|x - y| \leq f(x) - f(y) \leq M|x - y|$$

therefore $|f(x)| \leq |f(y)| + M|x - y|$ for all $x, y \in \mathbb{R}$. Therefore,

$$\begin{aligned} 0 \leq d(f, g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x)| + |g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(0)| + M|x| + |g(0)| + M|x| \\ &\leq \sum_{n=1}^{\infty} \frac{f(0) + g(0) + 2Mn}{2^n} < \infty \end{aligned}$$

so $d(f, g)$ is well-defined and non-negative.

(ii) Clearly $d(f, f) = 0$. Assume $d(f, g) = 0$. Then $\sup_{x \in [-n, n]} |f(x) - g(x)| = 0$ for all n thus $f(x) = g(x)$ on \mathbb{R} .

(iii) Clearly $d(f, g) = d(g, f)$.

(iv)

$$\begin{aligned} d(f, g) &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - g(x)| \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} (|f(x) - h(x)| + |h(x) - g(x)|) \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \left(\sup_{x \in [-n, n]} |f(x) - h(x)| + \sup_{x \in [-n, n]} |h(x) - g(x)| \right) \\ &= \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f(x) - h(x)| + \sum_{n=1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |h(x) - g(x)| \\ &= d(f, h) + d(h, g) \end{aligned}$$

□

(b) *Proof.* Let $\{f_k\}_{k=1}^\infty \subseteq \mathcal{L}$ be Cauchy in d . Fix $x \in \mathbb{R}$. Then, $x \in [-N, N]$ for some $N \in \mathbb{N}$. For any $k, \ell \in \mathbb{N}$,

$$|f_k(x) - f_\ell(x)| \leq 2^N 2^{-N} \sup_{x \in [-N, N]} |f_k(x) - f_\ell(x)| \leq 2^N d(f_k, f_\ell) \rightarrow 0$$

as $k, \ell \rightarrow \infty$. Therefore $\{f_k(x)\}_{k=1}^\infty \subseteq \mathbb{R}$ is Cauchy for each x and therefore convergent since \mathbb{R} is complete. Then define

$$f(x) := \lim_{k \rightarrow \infty} f_k(x)$$

First, we show $f \in \mathcal{L}$. Fix $x, y \in \mathbb{R}$. For $\epsilon > 0$ there exists $n_1, n_2 \in \mathbb{N}$ such that

$$|f_k(x) - f(x)| < \epsilon \quad |f_\ell(y) - f(y)| < \epsilon \quad \forall k > n_1 \ell > n_2$$

Then, for $n = \max\{n_1, n_2\}$,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 2\epsilon + M|x - y|$$

so $|f(x) - f(y)| \leq M|x - y|$ and $f \in \mathcal{L}$. Now we will show $f_k \rightarrow f$ in d . Let $\epsilon > 0$. Since $\{f_k\}$ is Cauchy in d , $\{d(f_k, 0)\}$ is uniformly bounded, i.e. there exist $C > 0$ such that $d(f_k, 0) \leq C$ for all k . Indeed, there exists N such that $d(f_k, f_j) < 1$ for $j, k \geq N$. Thus,

$$d(f_k, 0) \leq d(f_k, f_N) + d(f_N, 0) \leq 1 + d(f_N, 0)$$

So $d(f_k, 0) \leq \max_{j=1, \dots, N} \{1 + d(f_j, 0)\}$ for all k . Thus,

$$d(f_k, f) \leq C + d(f, 0)$$

for all k so there exists N such that

$$\sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| < \epsilon/2$$

for all k . Moreover, since $f_k(x) \rightarrow f(x)$ for each $x \in [-N, N]$, $f_k \rightarrow f$ uniformly on $[-N, N]$ since it is closed and bounded. Therefore we can take k large enough so that

$$\sum_{n=1}^N 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| < N 2^{-N} \sup_{x \in [-N, N]} |f_k(x) - f(x)| < \epsilon/2$$

Then,

$$d(f_k, f) = \sum_{n=1}^N + \sum_{n=N+1}^{\infty} 2^{-n} \sup_{x \in [-n, n]} |f_k(x) - f(x)| \leq \epsilon$$

for k large enough. □

3. *Proof.* Let $f \in C^1[0, 1]$.

$$|\varphi_0(f)| = |f'(0)| \leq \sup_{x \in [0, 1]} |f'(x)| \leq \|f\|$$

So $\|\varphi_0\| \leq 1$. We will show $\|\varphi_0\| = 1$. Consider the sequence defined

$$f_n(x) := \frac{\sin(nx)}{n}$$

Notice $\|f_n\| = 1/n + 1$ and $|\varphi_0(f_n)| = 1$. Thus,

$$1 \geq \|\varphi_0\| = \sup_{f \neq 0} \frac{|\varphi_0(f)|}{\|f\|} \geq \sup_{n \in \mathbb{N}} \frac{|\varphi_0(f_n)|}{\|f_n\|} = \sup_{n \in \mathbb{N}} \frac{1}{1 + 1/n} = 1$$

so $\|\varphi_0\| = 1$. □

4. (a) *Proof.* Let $x \in \ell^2$. Then for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |x_k|^2 < \epsilon$$

Then, for $y = (x_1, x_2, \dots, x_N, 0, 0, \dots) \in Y$,

$$\|x - y\|_2^2 = \sum_{k=1}^{\infty} |x_k - y_k|^2 = \sum_{k=N+1}^{\infty} |x_k|^2 < \epsilon$$

so Y is dense in ℓ^2 . □

(b) *Proof.* By Cauchy-Schwarz,

$$\left| \sum_{k=1}^n x_k \right| \leq \left(\sum_{k=1}^n |1|^2 \right)^{1/2} \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} = \sqrt{n} \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2}$$

Moreover, if $x \in \ell^2$, then $\sum_{k=1}^{\infty} |x_k|^2$ converges so we can bound the final term by $\|x\|_2$. □

(c) *Proof.* Let $x \in \ell^2$, $\epsilon > 0$. By part (a) there exists $y \in Y$ such that $\|x - y\|_2 \leq \epsilon/2$. Then,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left| \sum y_n \right| = 0$$

since the second term is bounded and the first is decreasing to 0. So, there exists N such that

$$\frac{1}{\sqrt{n}} \left| \sum y_n \right| < \epsilon/2 \quad \text{for } n \geq N$$

By triangle inequality for $|\cdot|$ and part(b),

$$\frac{1}{\sqrt{n}} \left| \sum x_n \right| \leq \frac{1}{\sqrt{n}} \left| \sum x_n - y_n \right| + \frac{1}{\sqrt{n}} \left| \sum y_n \right| < \|x - y\|_2 + \epsilon/2 < \epsilon$$

for $n \geq N$ so

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left| \sum x_n \right| = 0$$

□

5. *Proof.* First notice that

$$0 \leq \int_0^\infty \frac{x}{1+x^3} dx = \int_0^1 + \int_1^\infty \frac{x}{1+x^3} dx \leq \int_0^1 1 dx + \int_1^\infty \frac{1}{x^2} dx < \infty$$

Then, notice that $\frac{x}{1+x^n} \leq \frac{x}{1+x^{n+1}}$ for $x \in (0, 1)$. Therefore, by monotone convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{x}{1+x^n} dx = 0$$

since $x/(1+x^n) \rightarrow 0$ pointwise on $(0, 1)$. Moreover,

$$\int_1^\infty \frac{x}{1+x^n} dx$$

is monotone decreasing and bounded below by zero therefore

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{x}{1+x^n} dx = \lim_{n \rightarrow \infty} \int_0^1 + \int_1^\infty \frac{x}{1+x^n} dx$$

exists. Moreover,

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{x}{1+x^n} dx = \int_1^\infty \lim_{n \rightarrow \infty} \frac{x}{1+x^n} dx = 0$$

by the Lebesgue dominated convergence theorem since

$$\frac{x}{1+x^n} \leq \frac{x}{x^n} = x^{1-n}$$

which is integrable on $(1, \infty)$ for $n \geq 3$. □

6. (a) *Proof.* Set $f(x) = \mathbf{1}_{\liminf_n A_n}$.

(i) $f(x) = 1 \iff x \in \cup_k \cap_{n=k}^\infty A_n$. So, there exists k such that $x \in A_n$ ($\mathbf{1}_{A_n}(x) = 1$) for all $n \geq k$. So, $\lim_{n \rightarrow \infty} \mathbf{1}_{A_n}(x) = 1$ (so \liminf is also 1).

(ii) Suppose $f(x) = 0$. For each k there exists $n \geq k$ such that $x \notin A_n$ ($\mathbf{1}_{A_n}(x) = 0$) so $\liminf_n \mathbf{1}_{A_n} = 0$. □

(b) By Fatou's Lemma,

$$\mu(\liminf_n A_n) = \int_X f d\mu = \int_X \liminf_n \mathbf{1}_{A_n} d\mu \leq \liminf_n \int_X \mathbf{1}_{A_n} d\mu = \liminf_n \mu(A_n)$$

7. *Proof.* Define $f = \sup_N \sum_{n=1}^N f_n$. f is a measurable function, moreover, since f_n are non-negative, $\sum f_n \nearrow f$. So, by Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f = \sum \int f_n \leq \sum \frac{1}{n^2} < \infty$$

So f is non-negative and integrable. We claim this implies $f < \infty$ a.e. If not, then there exists E with $\lambda(E) > 0$ and $f = \infty$ on E . Then,

$$\int_{\mathbb{R}} f \geq \int_E f = \infty$$

so $f < \infty$ a.e. □

8. (a) *Proof.* By Hölder's Inequality,

$$\left| \int f_n d\mu - \int f d\mu \right| \leq \int |f - f_n| d\mu \leq \|f - f_n\|_{\infty} \int d\mu = \|f - f_n\|_{\infty} \mu(X) \rightarrow 0$$

as $n \rightarrow \infty$. □

Summer 2017

5. Hao Chen

Proof. To show the orthonormal set $\{f_n\}$ is an orthonormal basis we will show that $\{f_n\}^\perp = \{0\}$. If not, then there exists $x \neq 0$ such that $\langle x, f_n \rangle = 0$ for all n . However, by Parseval's identity and the Cauchy-Schwarz inequality,

$$\|x\|^2 = \sum |\langle x, e_n \rangle|^2 = \sum |\langle x, e_n - f_n \rangle|^2 \leq \sum \|x\|^2 \|e_n - f_n\|^2 < \|x\|^2$$

but this is a contradiction so $\{f_n\}^\perp = \{0\}$. \square

A more complicated proof by Walton:

Proof. Let $c = \sum \|e_n - f_n\|^2 < 1$. Define $T : \mathcal{H} \rightarrow \mathcal{H}$ by sending $x = \sum \langle x, e_n \rangle e_n \mapsto \sum \langle x, e_n \rangle f_n$. The second sum converges by the Bessel inequality. Now, by the Cauchy-Schwarz inequality and Parseval's identity,

$$\|(I - T)x\|^2 = \left\| \sum \langle x, e_n \rangle (e_n - f_n) \right\|^2 \leq \sum |\langle x, e_n \rangle|^2 \sum \|e_n - f_n\|^2 = c \|x\|^2$$

So, $\|T - I\| \leq \sqrt{c} < 1$. We claim that this means T is invertible. Indeed, set

$$S = \sum_{n=0}^{\infty} (I - T)^n$$

The sum is absolutely convergent since $\|I - T\| < 1$ so S is bounded linear operator since $\mathcal{L}(\mathcal{H})$ is a Banach space. Moreover,

$$S - TS, S - ST = \sum_{n=1}^{\infty} (I - T)^n = S - (I - T)^0 = S - I$$

so $S = T^{-1}$. Now, let $y \in \mathcal{H}$. Then, there exists $x (T^{-1}y)$ such that $Tx = y$. Therefore,

$$y = \sum \langle x, e_n \rangle f_n \tag{1}$$

and therefore $\overline{\text{span}}\{f_n\} = \mathcal{H}$. \square

Remark: This actually holds if $\sum \|e_n - f_n\|^2 < \infty$.

6. Define $A_n = \{f \geq 1/n\}$. Since $A_n \subseteq A_{n+1}$,

$$0 < \mu(\{f > 0\}) = \mu\left(\bigcup_n A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n)$$

Therefore there exists n such that $\mu(A_n) > 0$. Then,

$$\int f \geq \int_{A_n} f \geq \frac{1}{n} \mu(A_n) > 0$$

7. (a) We first show that if f is integrable, the $\mu(E_n) \rightarrow 0$ implies $\int_{E_n} f \rightarrow 0$. Since f is integrable, for $A_n = \{n-1 \leq |f| \leq n\}$,

$$\infty > \int |f| \geq \sum (n-1)\mu(A_n)$$

Given $\epsilon > 0$ there exists N such that $\sum_{n=N}^{\infty} (n-1)\mu(A_n) < \epsilon/2$. Also, we can find M such that

$$\mu(E_n) < \epsilon/(2N) \quad \forall n \geq M$$

Then, for $n \geq M$,

$$\begin{aligned} \left| \int_{E_n} f \right| &\leq \int_{E_n} |f| = \int_{E_n \cap \{f \leq N\}} |f| + \int_{E_n \cap \{f > N\}} |f| \\ &\leq N\mu(E_n) + \sum_{k=N}^{\infty} (k-1)\mu(A_k) \leq N\epsilon/2N + \epsilon/2 = \epsilon \end{aligned}$$

Now we can prove the statement. Let $a, b \in \mathbb{R}$. Then there exists $\{a_n\}, \{b_n\} \subseteq \mathbb{Q}$ such that

$$a_n \rightarrow a \quad b_n \rightarrow b$$

Then,

$$\int_a^b f = \int_a^{a_n} f + \int_{a_n}^{b_n} f + \int_{b_n}^b f$$

The middle term is zero by assumption and applying the above lemma, the first and third terms go to 0.

(b) INCOMPLETE

8. This is a special case of Winter 15 #10.