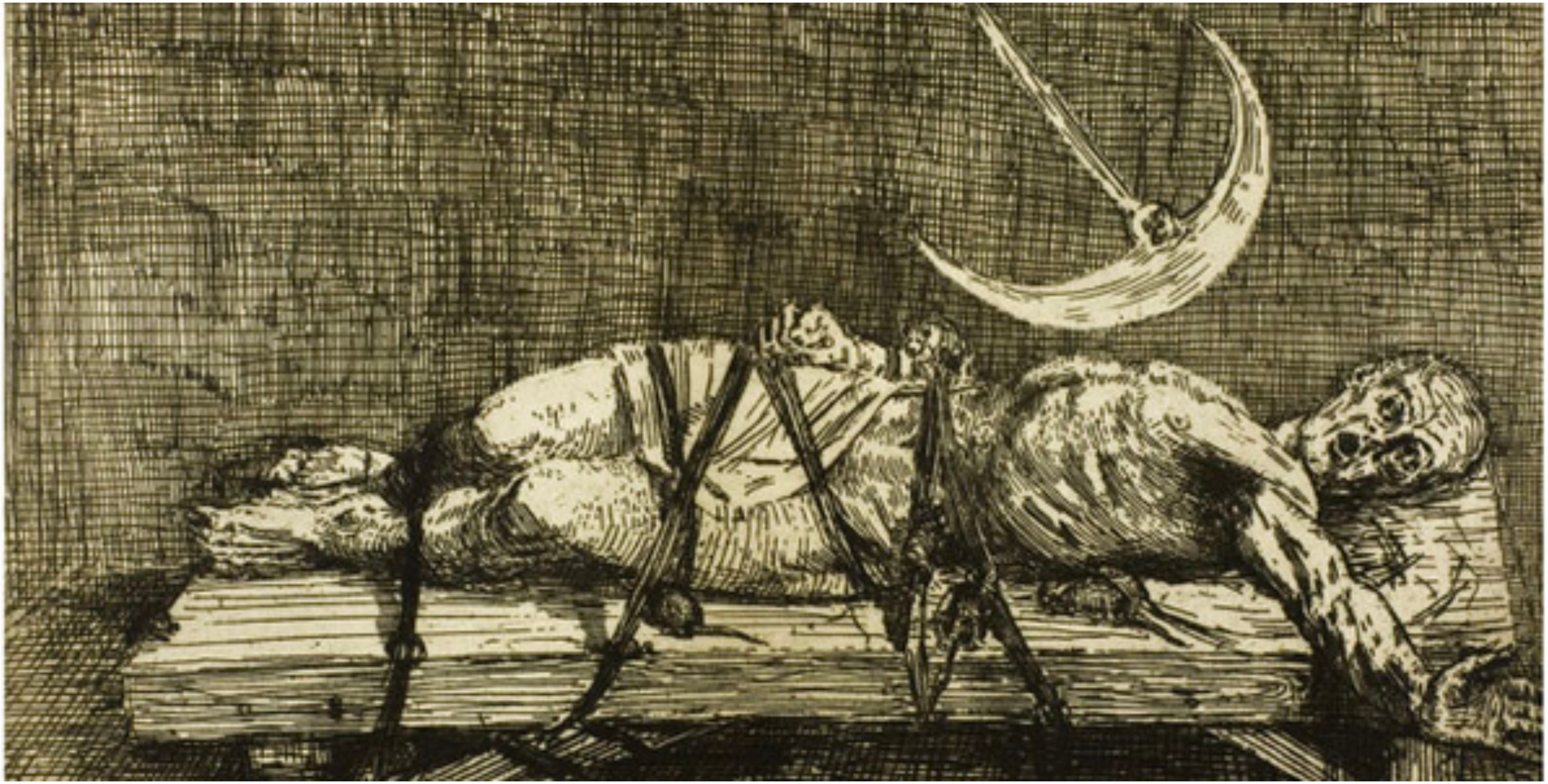
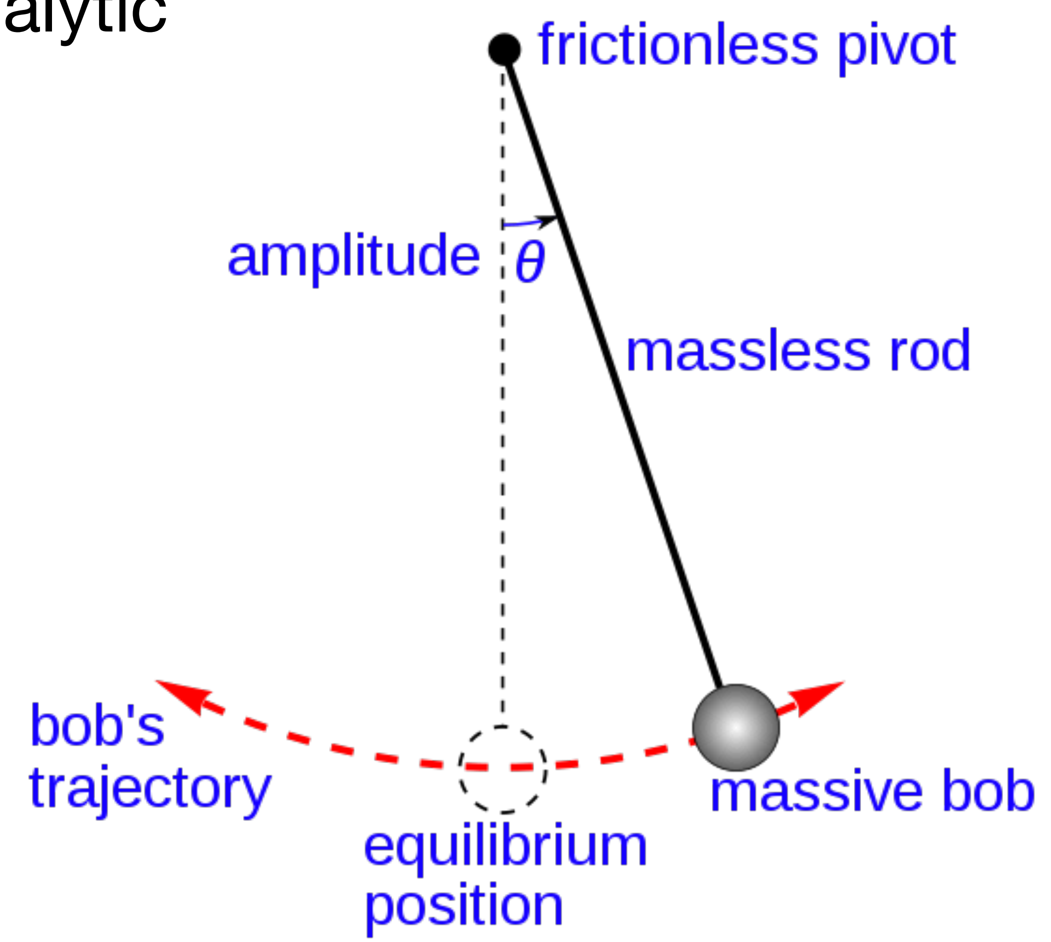


Lecture 15: Numerical ODEs

The pendulum

- While seemingly simple the solution is not analytic
  - $m\ell\ddot{\theta} = -mg\sin(\theta)$
  - $\frac{1}{2}\dot{\theta}^2 = \frac{g}{\ell}\left(\cos\theta - \cos\theta_0\right)$
  - $\int \frac{d\theta}{\sqrt{\left(\cos\theta - \cos\theta_0\right)}} = \int \sqrt{2\frac{g}{\ell}}dt$



Elliptic Integral : This is what actually

Numerical Simulation

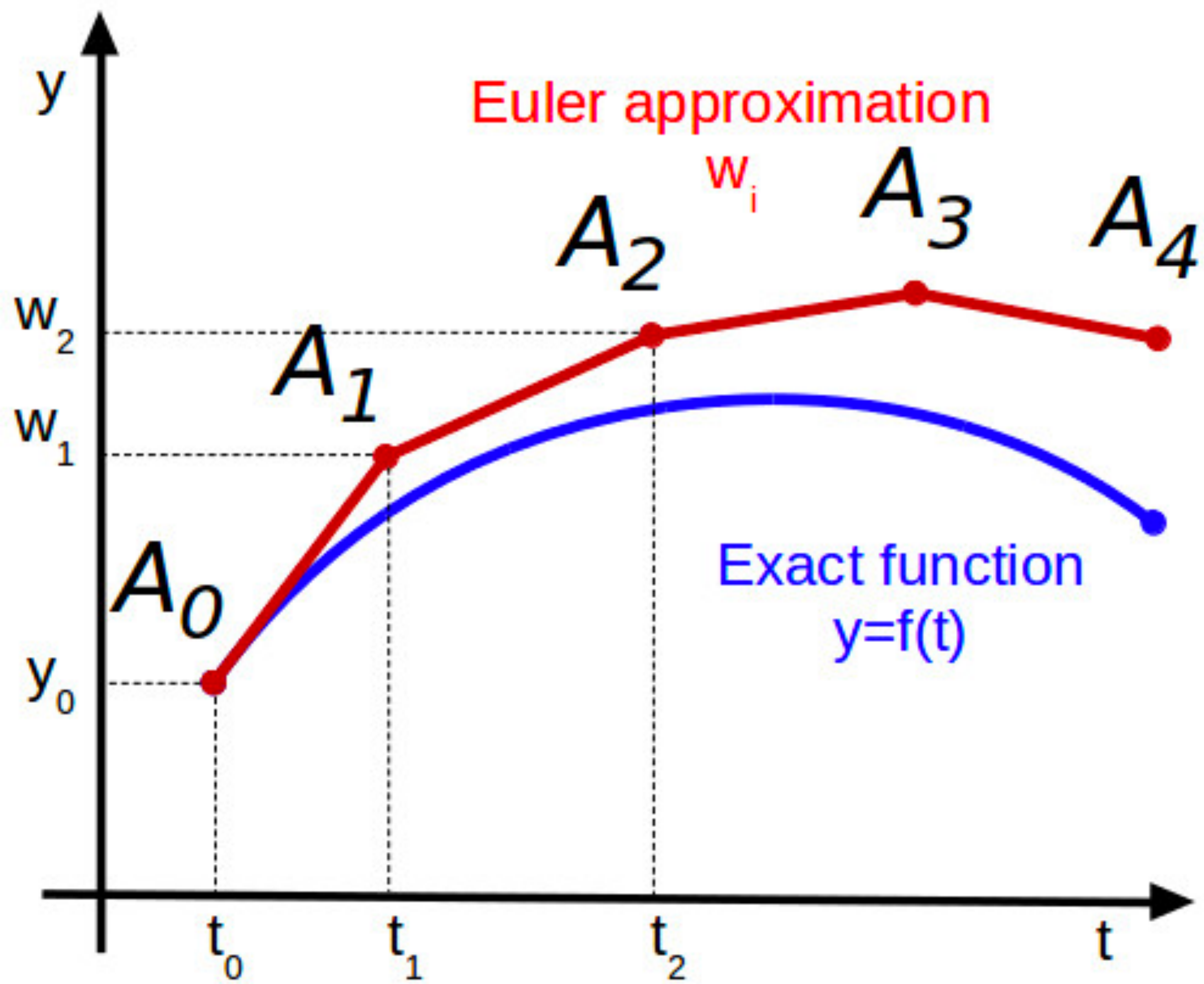
- This part of the class will cover numerical simulation
  - Typically this involves stepping through a simulation
  - Simplest stepping involves computing velocity/acceleration
  - Stepping through the forces :
    - $\dfrac{d\vec{x}}{dt} = \vec{v}(t) \rightarrow \vec{x}(t) = \int d\vec{x} = \int \vec{v}(t)dt$
    - $\dfrac{d\vec{v}}{dt} = \vec{a}(t) \rightarrow \vec{v}(t) = \int d\vec{v} = \int \dfrac{\overrightarrow{F}(t)}{m}dt$

What can we do to step

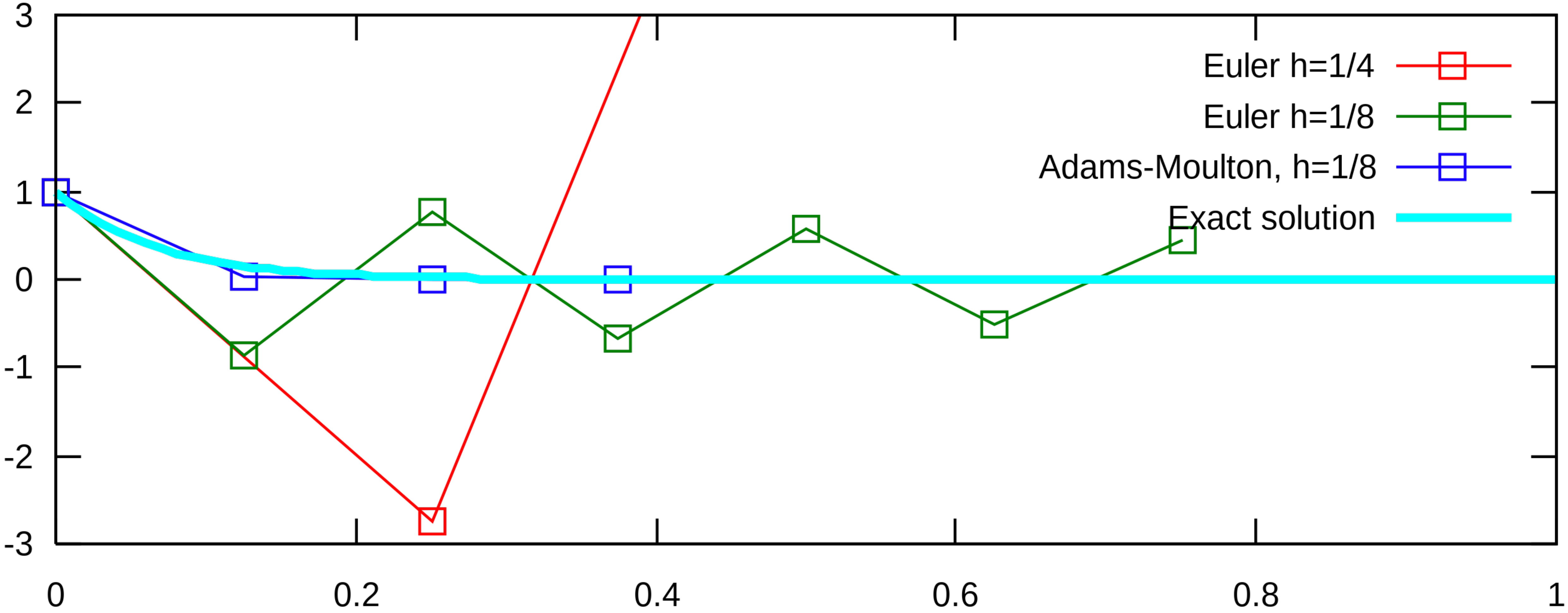
- For some time interval Δt, we can assume that
- $\vec{v}(t) \approx v_t$ (a constant for a short time)
- $\vec{a}(t) \approx a_t$ (a constant for a short time)
- From this base assumption, we can start to approximate
- These lead to a model

Tiers of approximation

- Strategy to linearize
  - Rely on Slope take appropriate timesteps

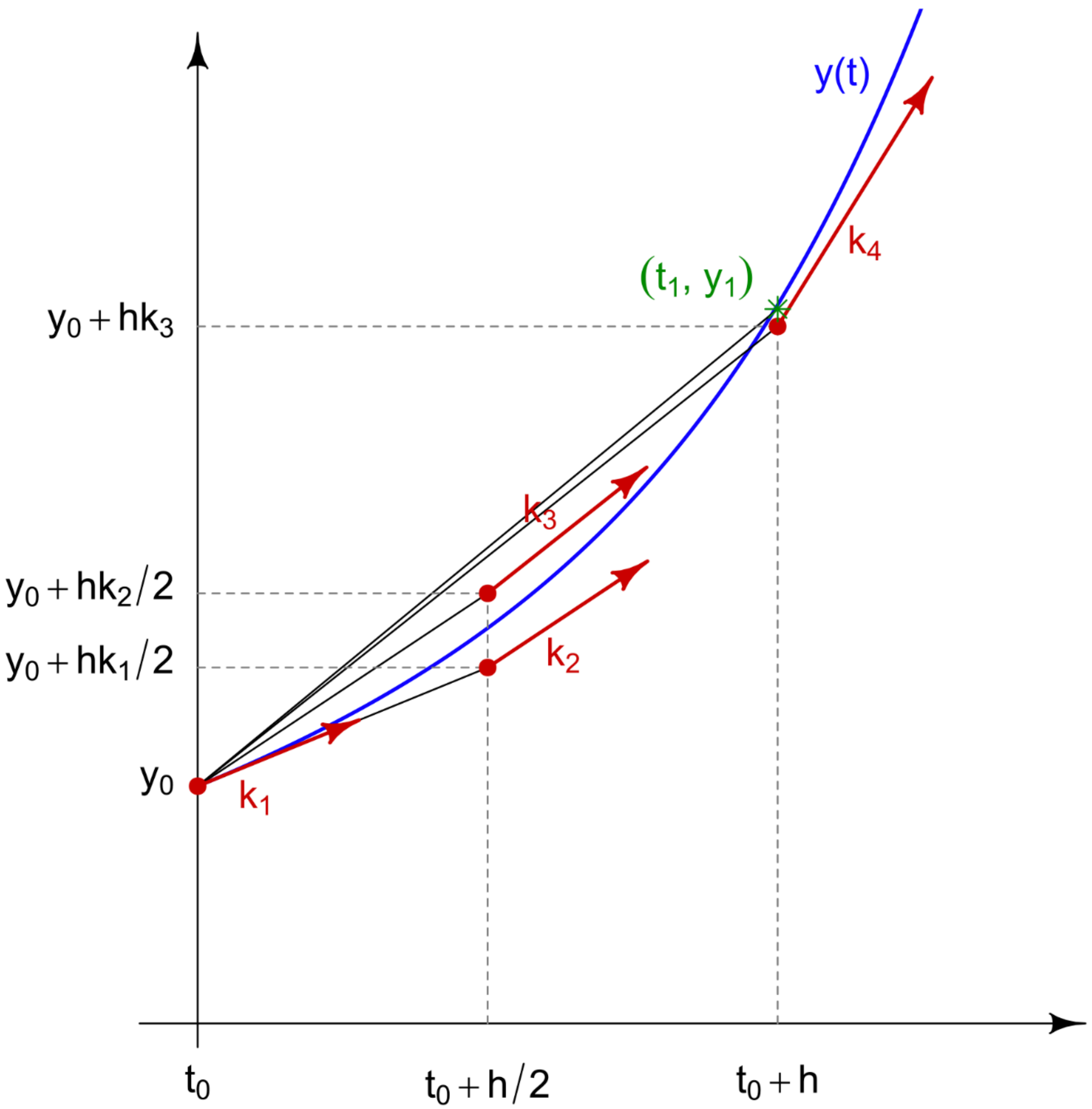


ODE Stiffnesss



- Stiff ODEs breakdown when step size too large
  - Stiffness is a sign of a difficult ODE

Runge-Kutta



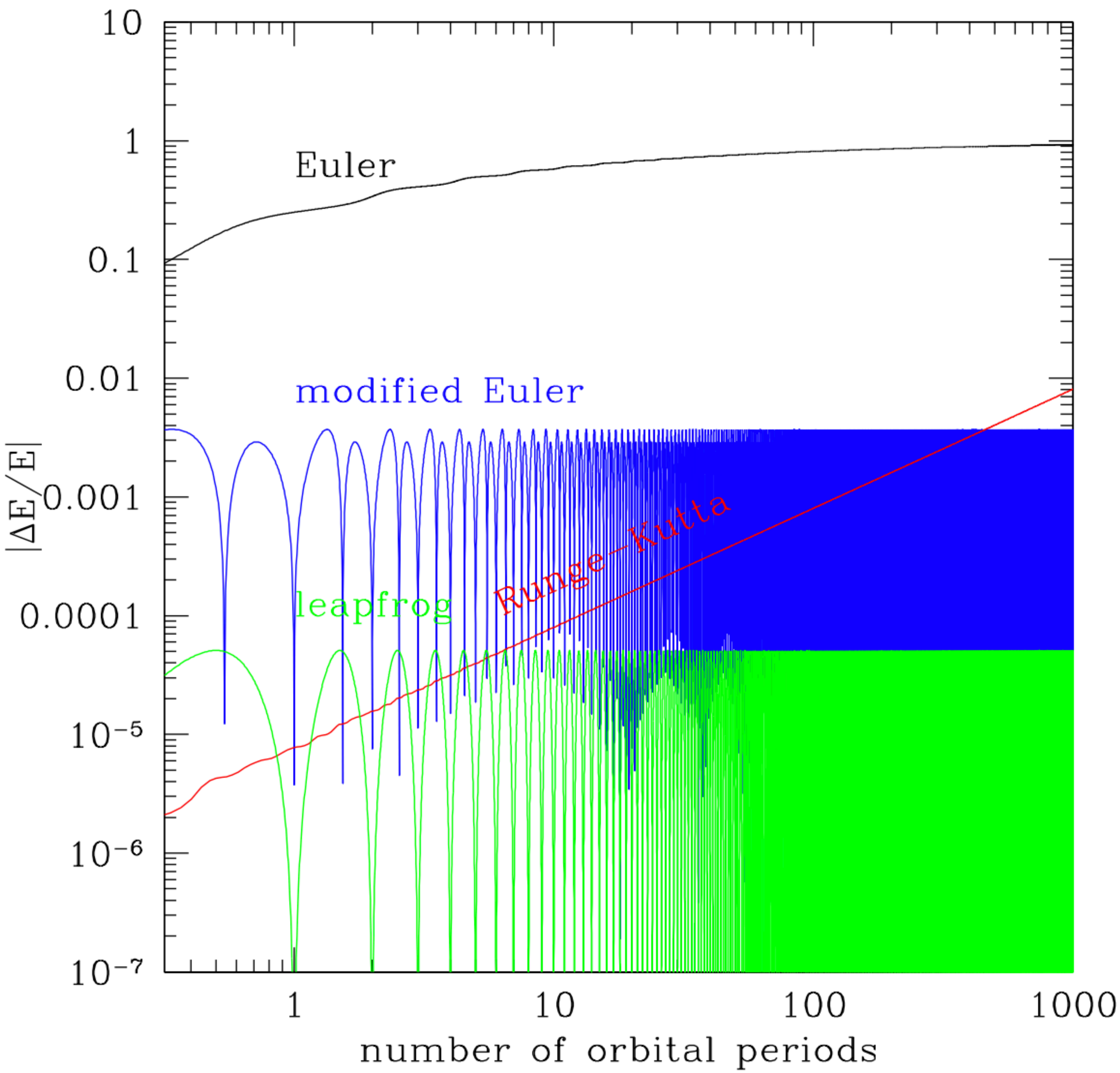
$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h,$$
$$t_{n+1} = t_n + h$$

for $n = 0, 1, 2, 3, \ldots$, using[3]

$$\begin{aligned}
k_1 &= f(t_n, y_n),\\
k_2 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right),\\
k_3 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right),\\
k_4 &= f\left(t_n + h, y_n + hk_3\right).
\end{aligned}$$

- Construct 4 or more steps to get to the next one
  - For Pendulum we have to intertwine velocity and position

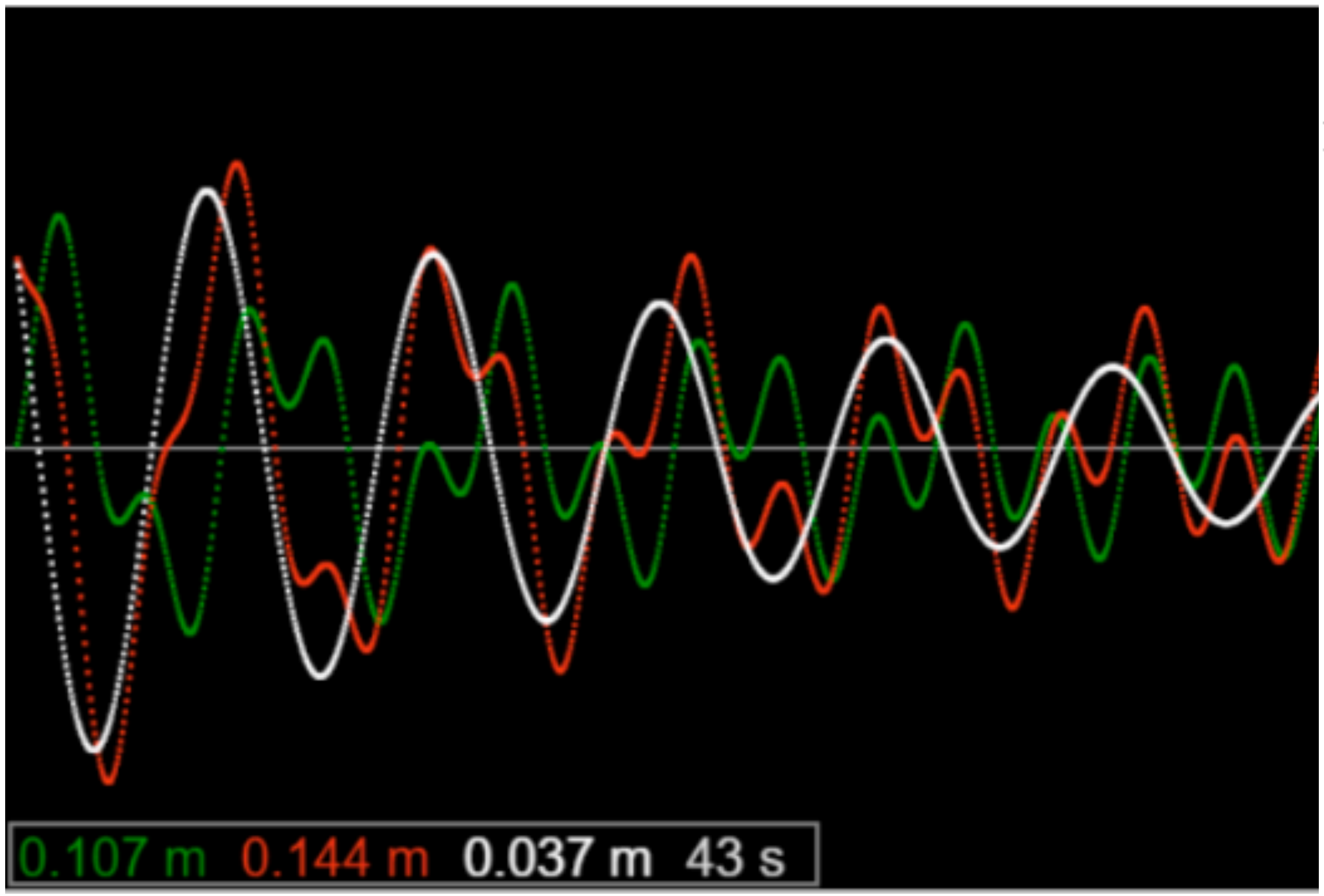
Precision

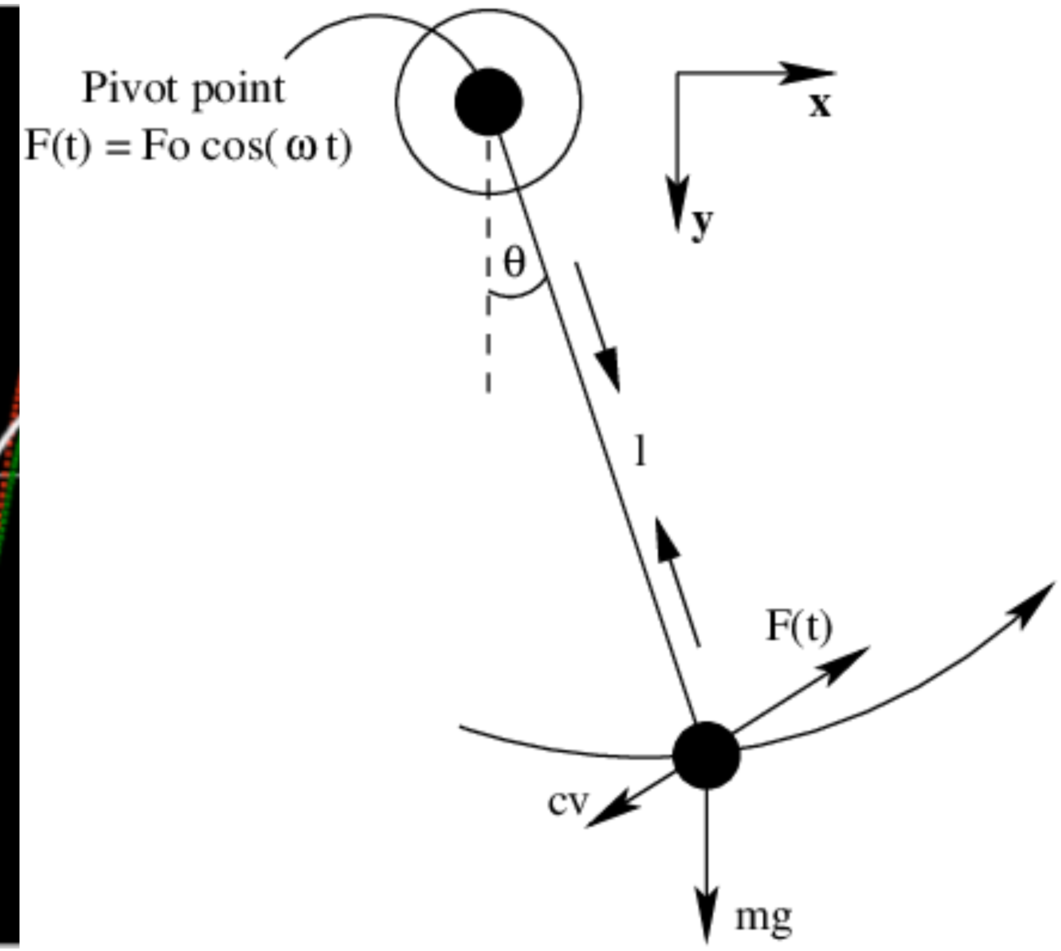


Steps (planetary simulation)

- Each step has its own benefits and limitations
- Can see this from precision over time for the left approximations

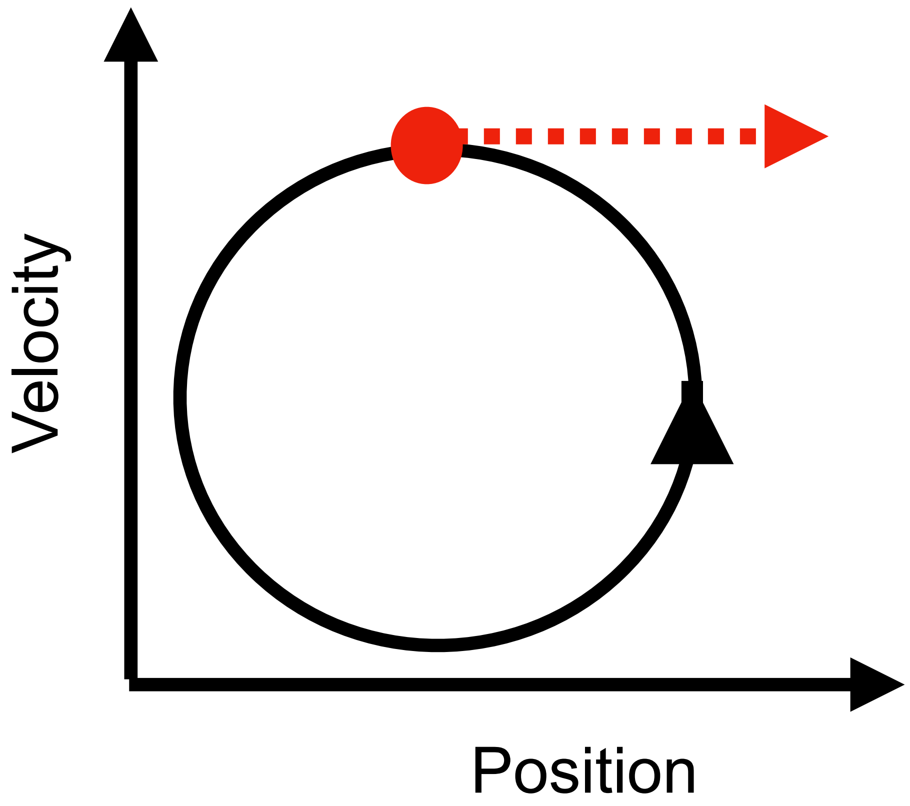
Damped Driven Harmonic Oscillator





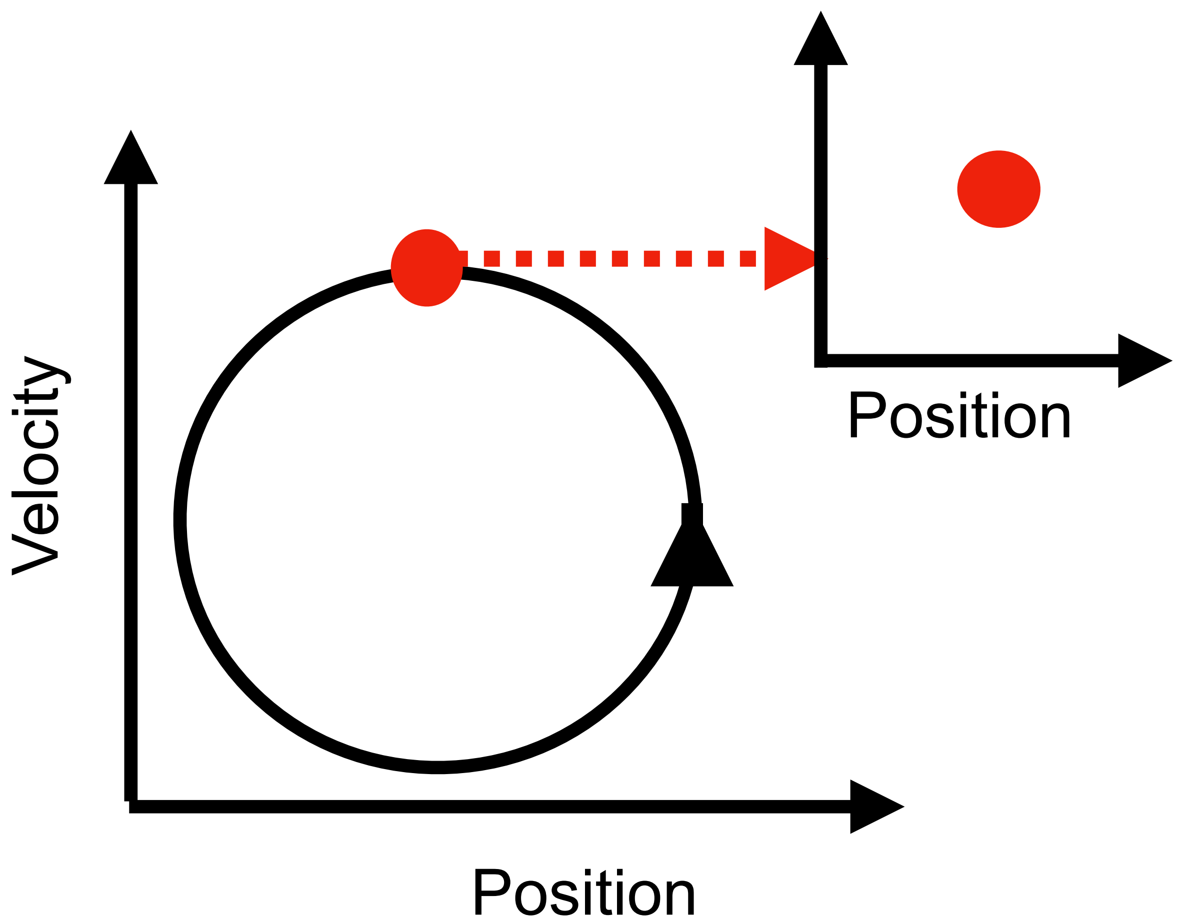
- We can extend our simulation towards damped driven HO
  - Dynamics here are fun and interesting
  - But we need a good integrator to understand it

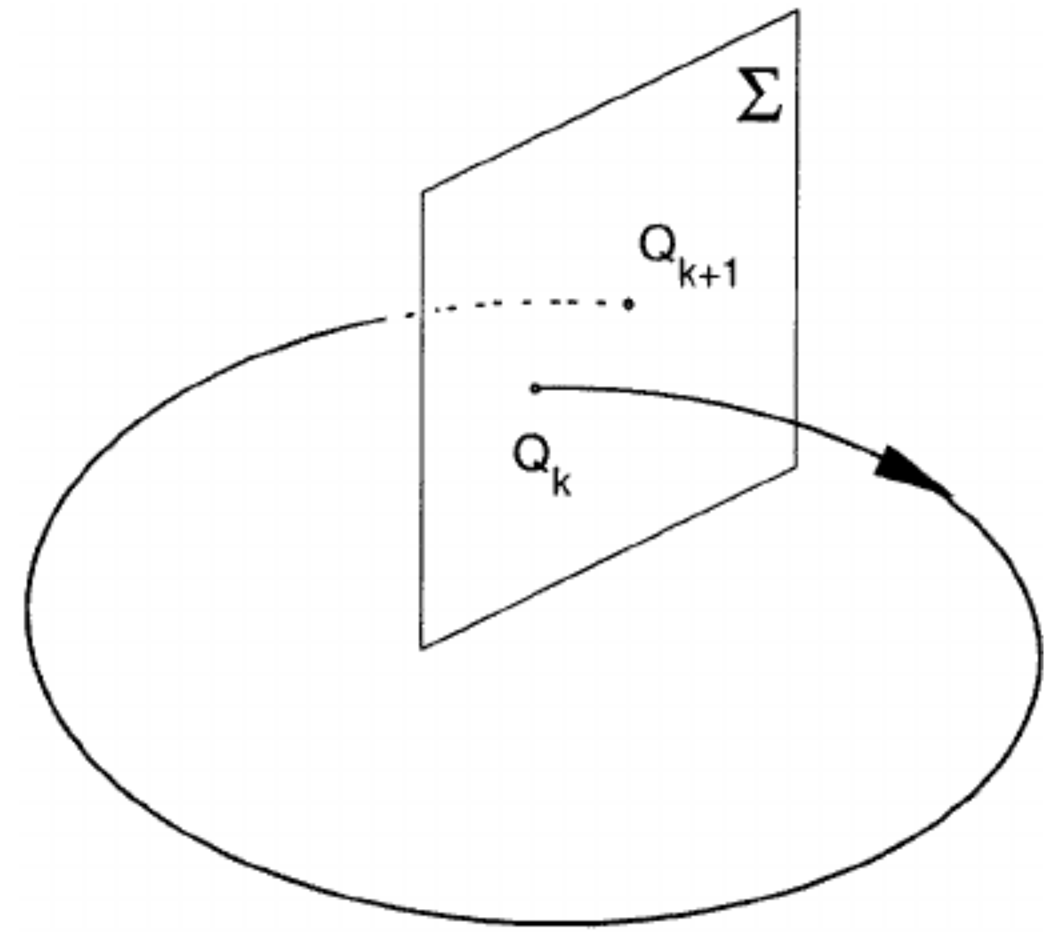
Poincare Map



- Looking at the evolution for a fixed velocity or position point that a trajectory oscillates through defines a poincare map

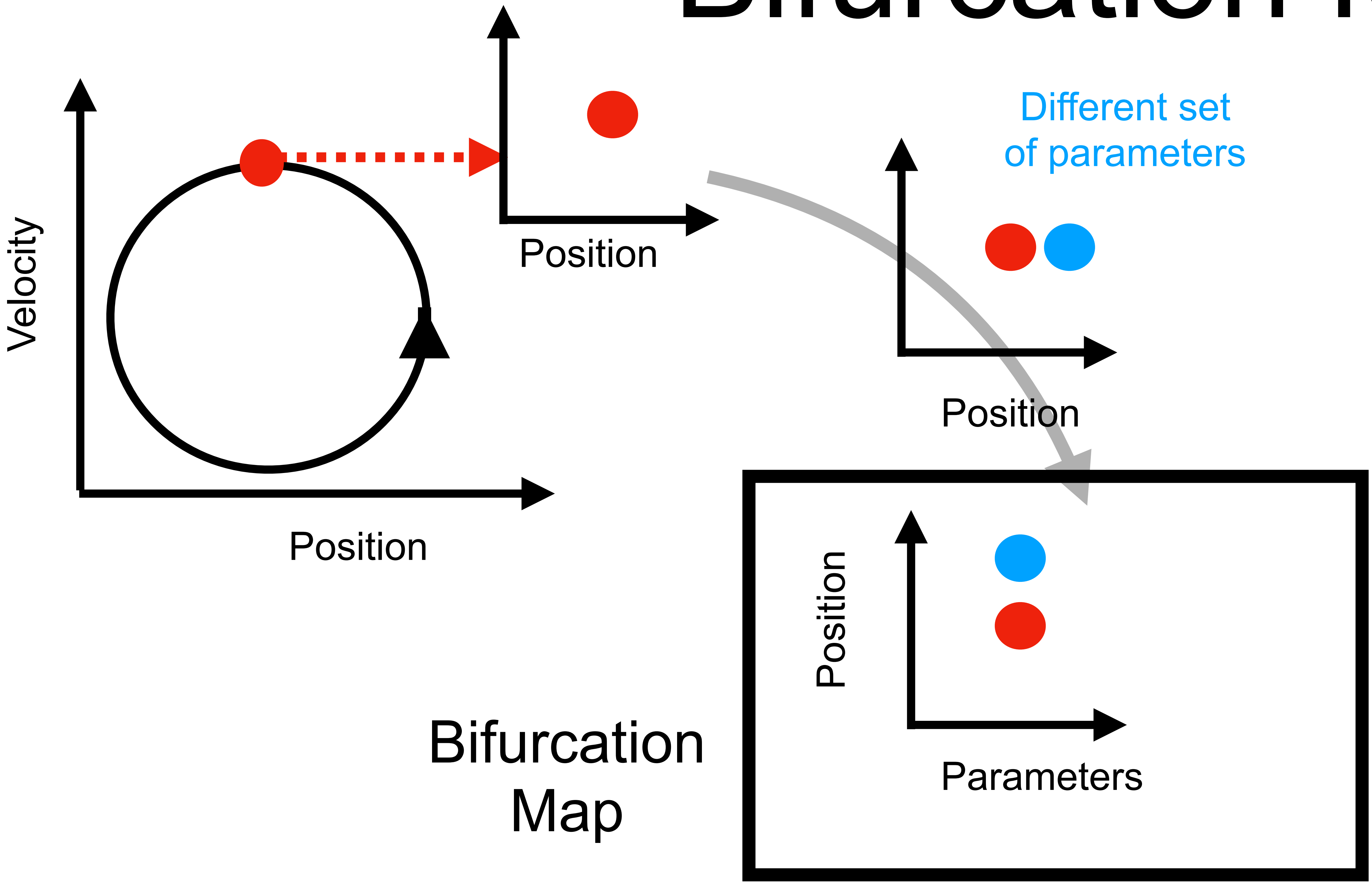
Poincare Map





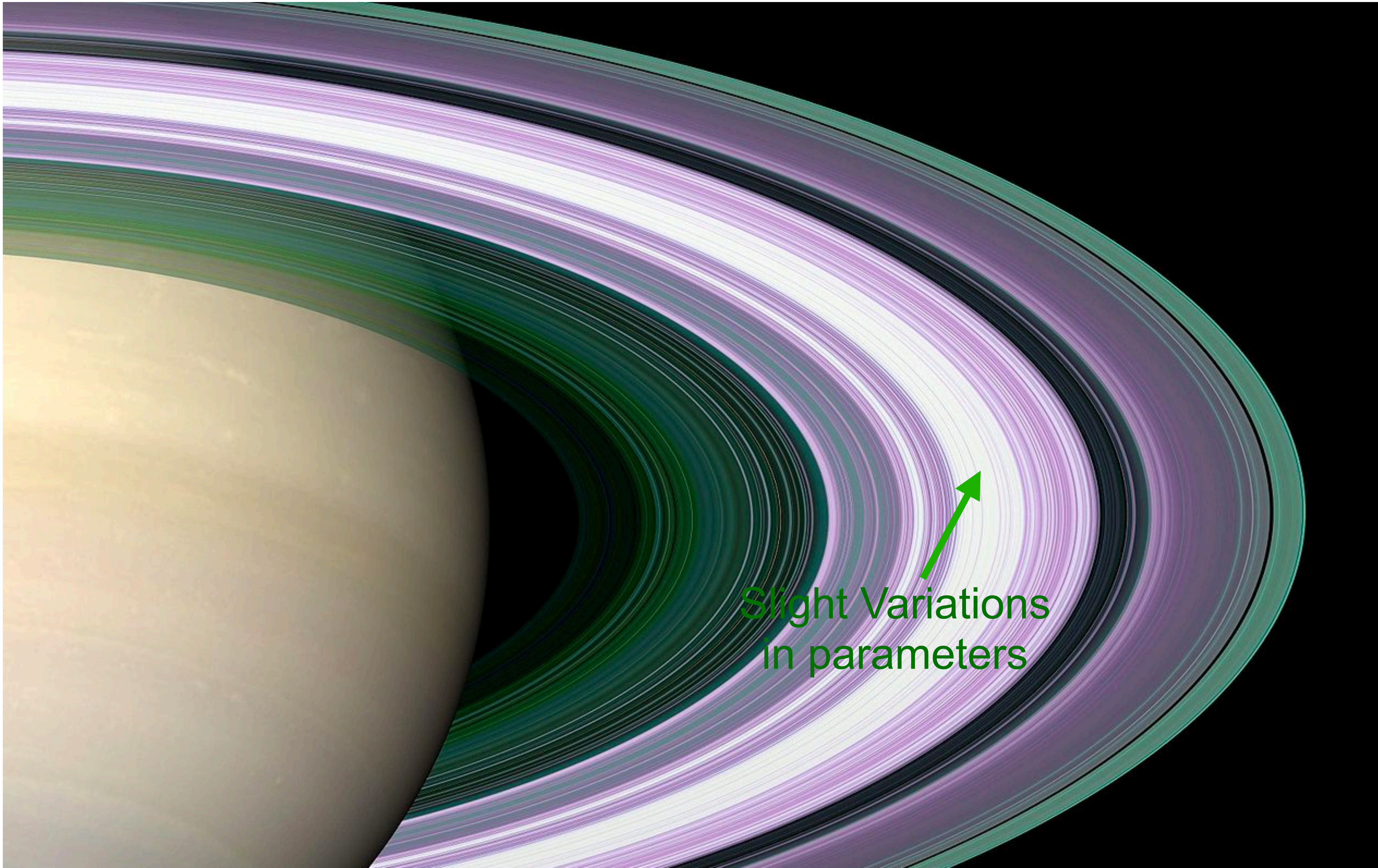
- Looking at the evolution for a fixed velocity or position point that a trajectory oscillates through defines a poincare map

Bifurcation Map



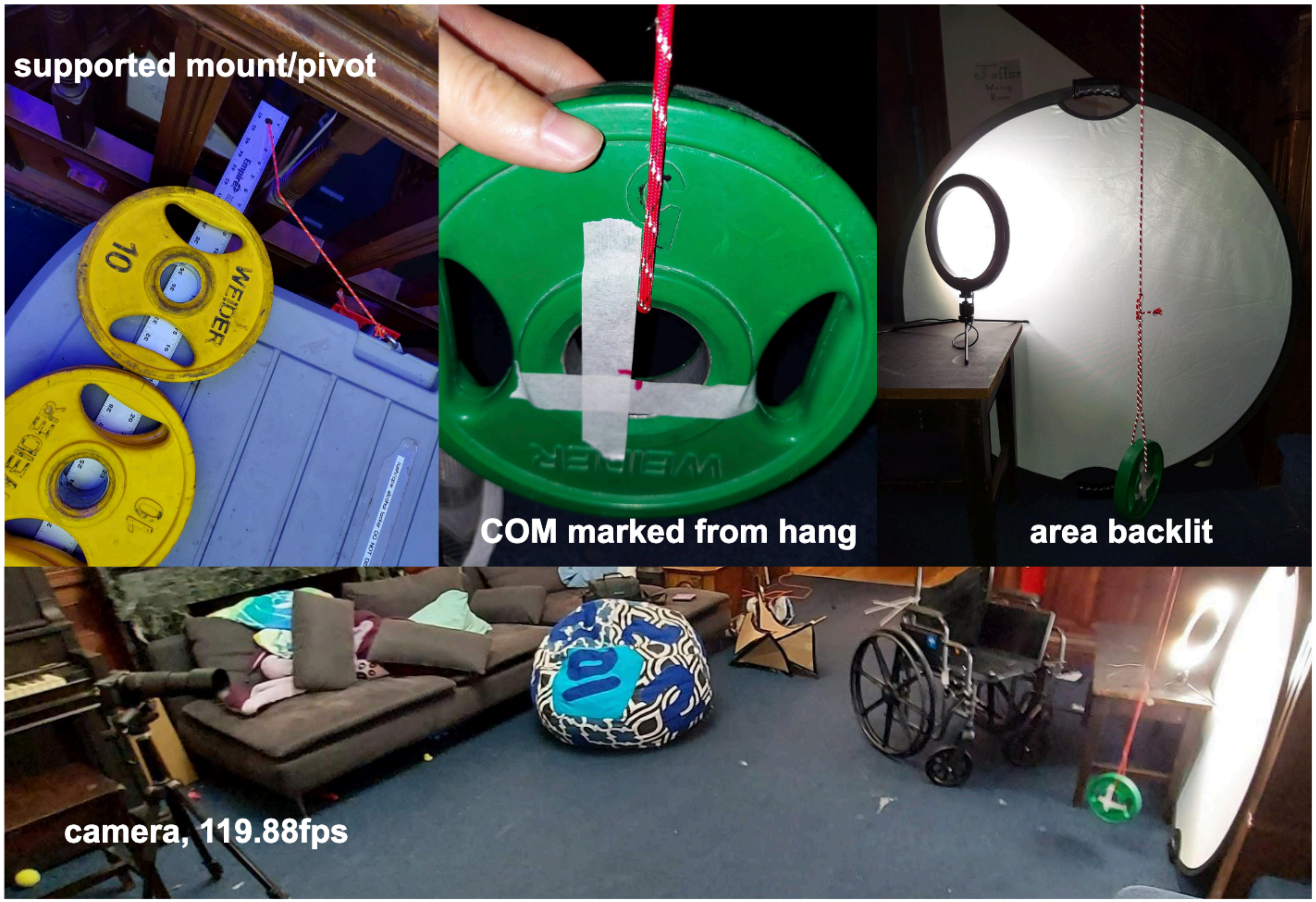
- We can look at behavior over parameters

Saturn's Rings



High quality Pendulum data

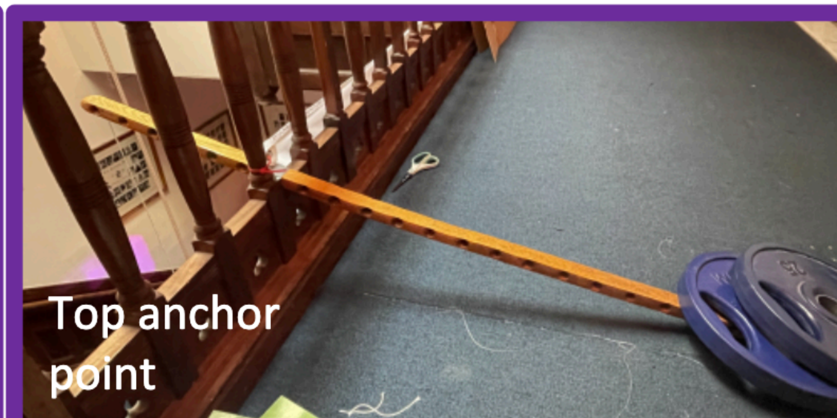
Apparatus



Pendulum designed to make length measurement more repeatable, improve small angle approx. and facilitate timing via computer vision, with low damping.

length approx. 4 m
displacement < 1.5°
mass approx. 5 lbs

High quality Pendulum data



Length:
$10.7886 \pm 0.0032\ m$

Period measurement:
phone camera + Jade’s
computer vision program
30fps $\rightarrow \sigma = 0.0096\ s$

Small angle approximation:
1.06°: $T_{corr} = 0.9999 T_{meas}$

Procedure:
2 minutes damping time
60s recording
Video analysis

Machine Learning Diffeq

- Recently within ML community :
  - The concept of Physics informed ML emerged

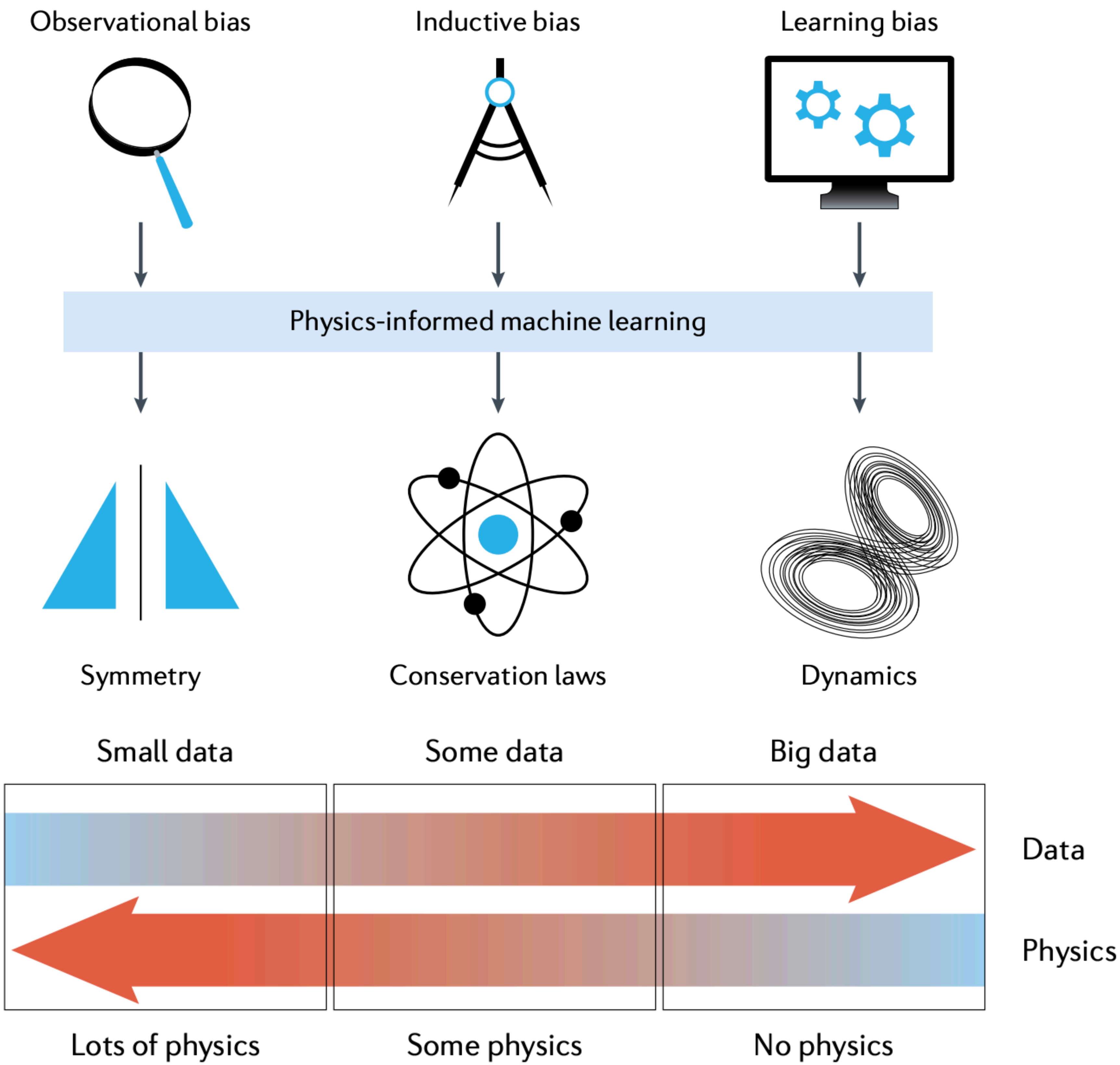
Strategy: $\mathscr{L}_{total} = \mathscr{L}_{NN} + \mathscr{L}_{\text{Diffeq}}$

$$\ddot{\theta} + \mu\dot{\theta} + k\theta = 0$$

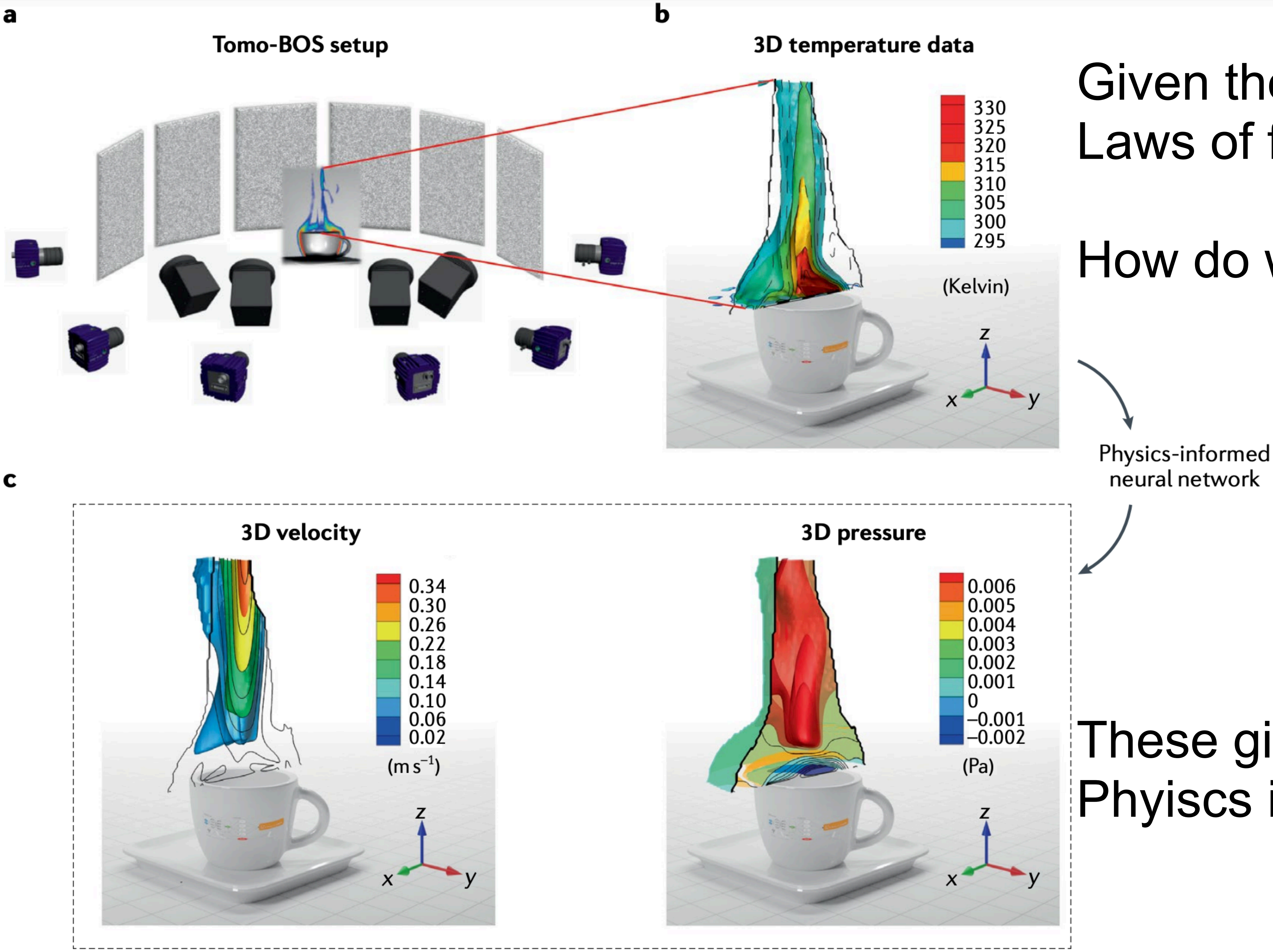
$$\boxed{\mathscr{L}_{\text{Diffeq}} = \left(\ddot{\theta} + \mu\dot{\theta} + k\theta\right)^2}$$

Constraint on Differential Equation
Aim to approximate learning

Physics Informed ML



Physics Informed ML

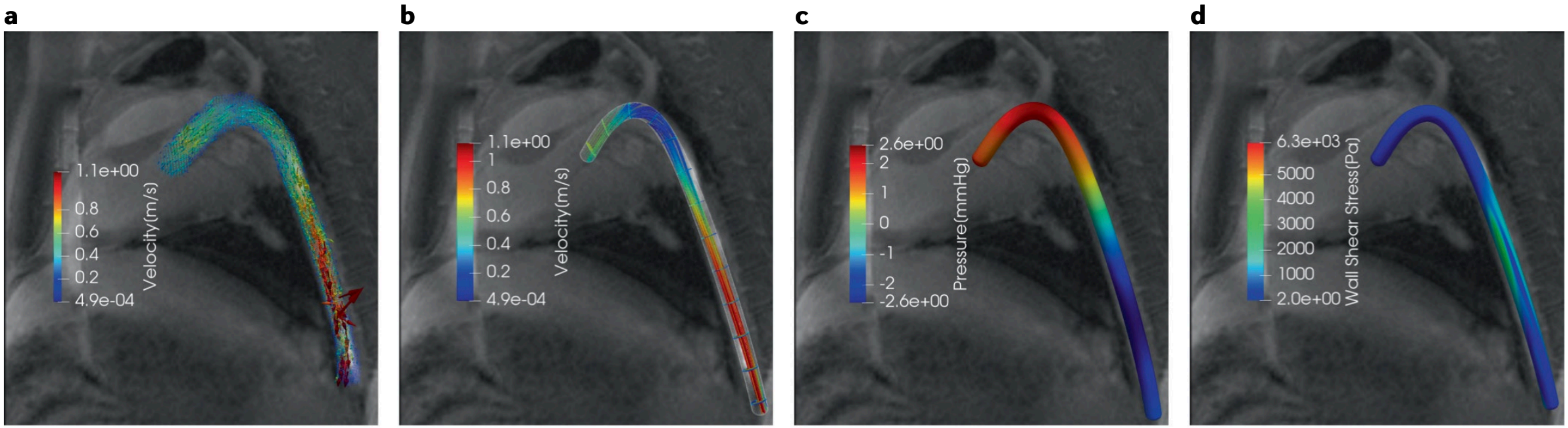


Given the
Laws of fluid flow

How do we model flow

These give us
Phyiscs informed ML

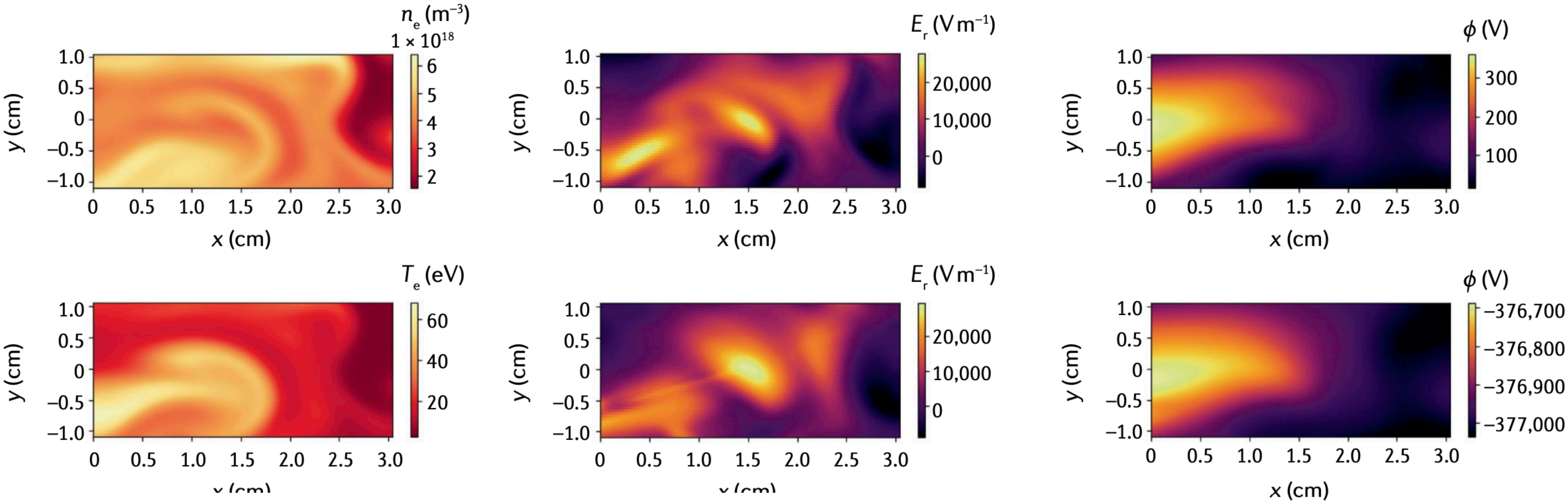
Physics Informed ML

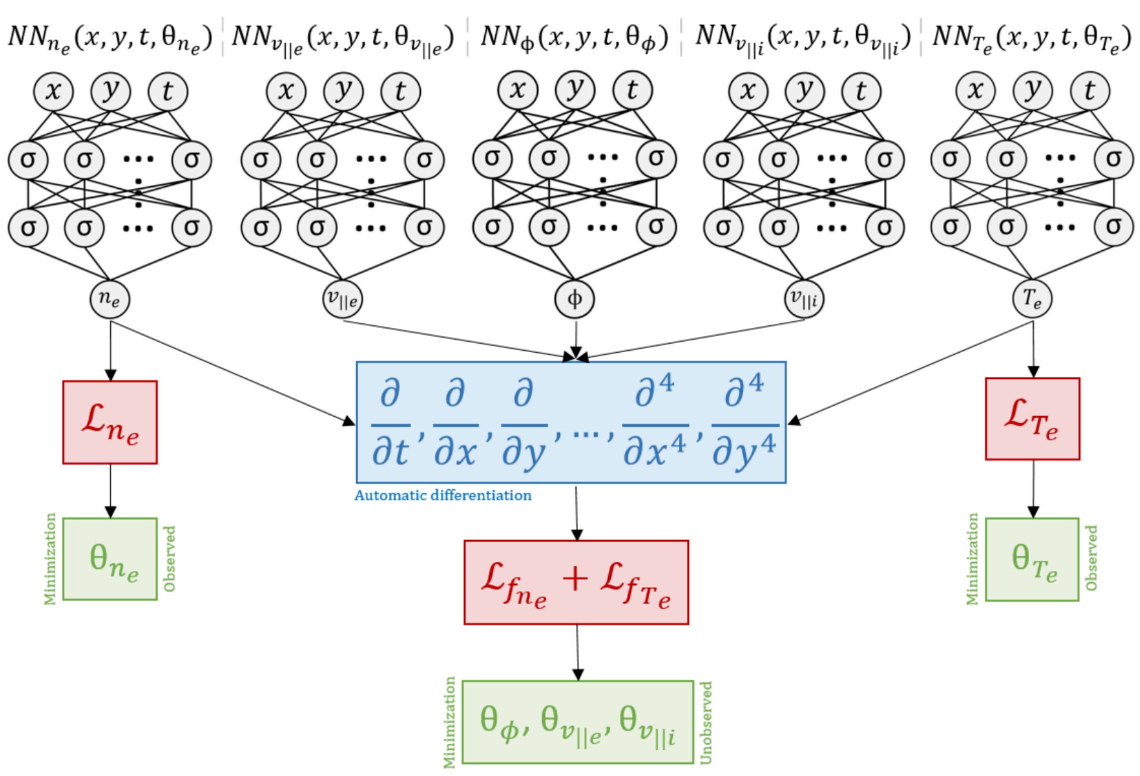


Navier Stokes equation to extrapolate blood flow in system

Navier Stokes equation to extrapolate blood flow in system

Physics Informed ML

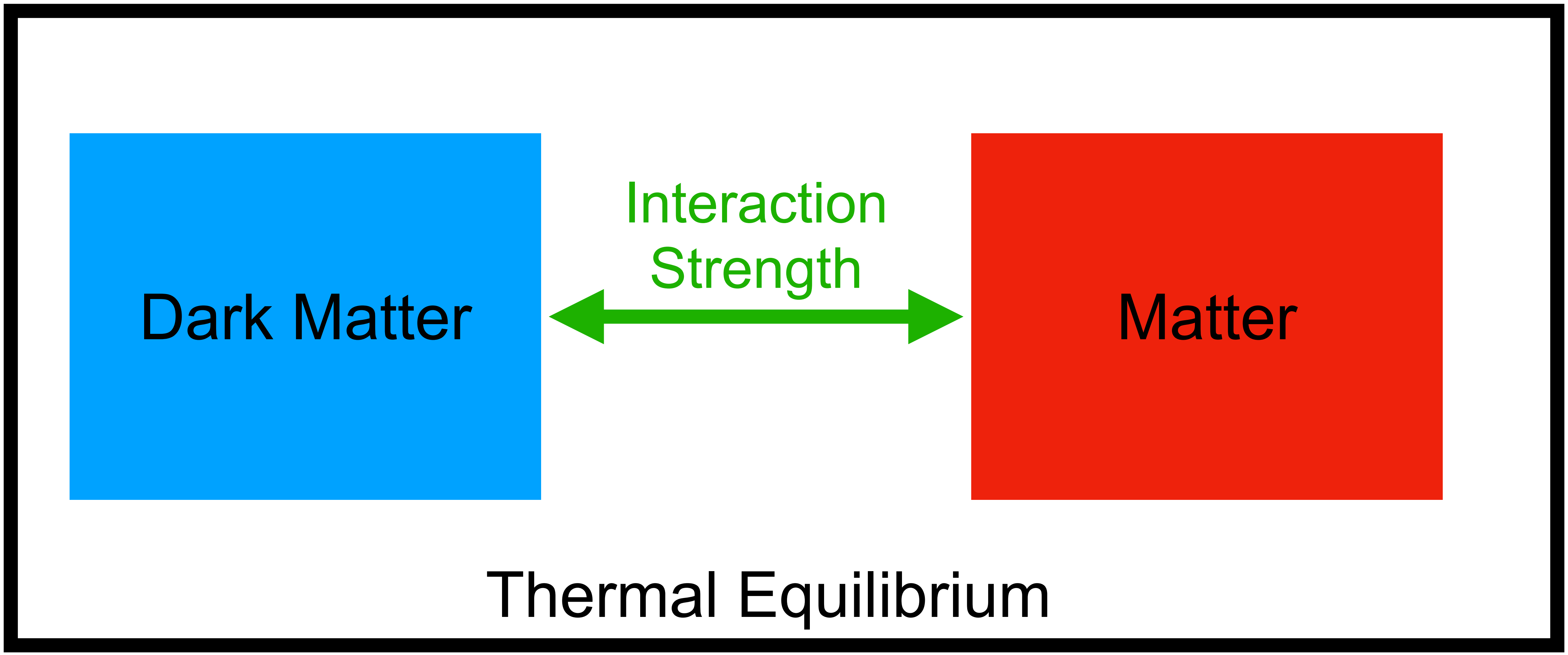




$$\mathcal{L}_{n_e} = \frac{1}{N_0}\sum_{i=1}^{N_0}|n_e^*(x_0^i, y_0^i, z_0^i, t_0^i) - n_{e,0}^i|^2,$$

$$\mathcal{L}_{T_e} = \frac{1}{N_0}\sum_{i=1}^{N_0}|T_e^*(x_0^i, y_0^i, z_0^i, t_0^i) - T_{e,0}^i|^2,$$

Dark Matter


Dark Matter
Interaction
Strength
Matter
Thermal Equilibrium

Dark Matter

Dark

$$\frac{dn}{dt} = -(\text{Interactionrate})(n - n_{\text{eq}})^2$$

er

Thermal Equilibrium

University is Expanding