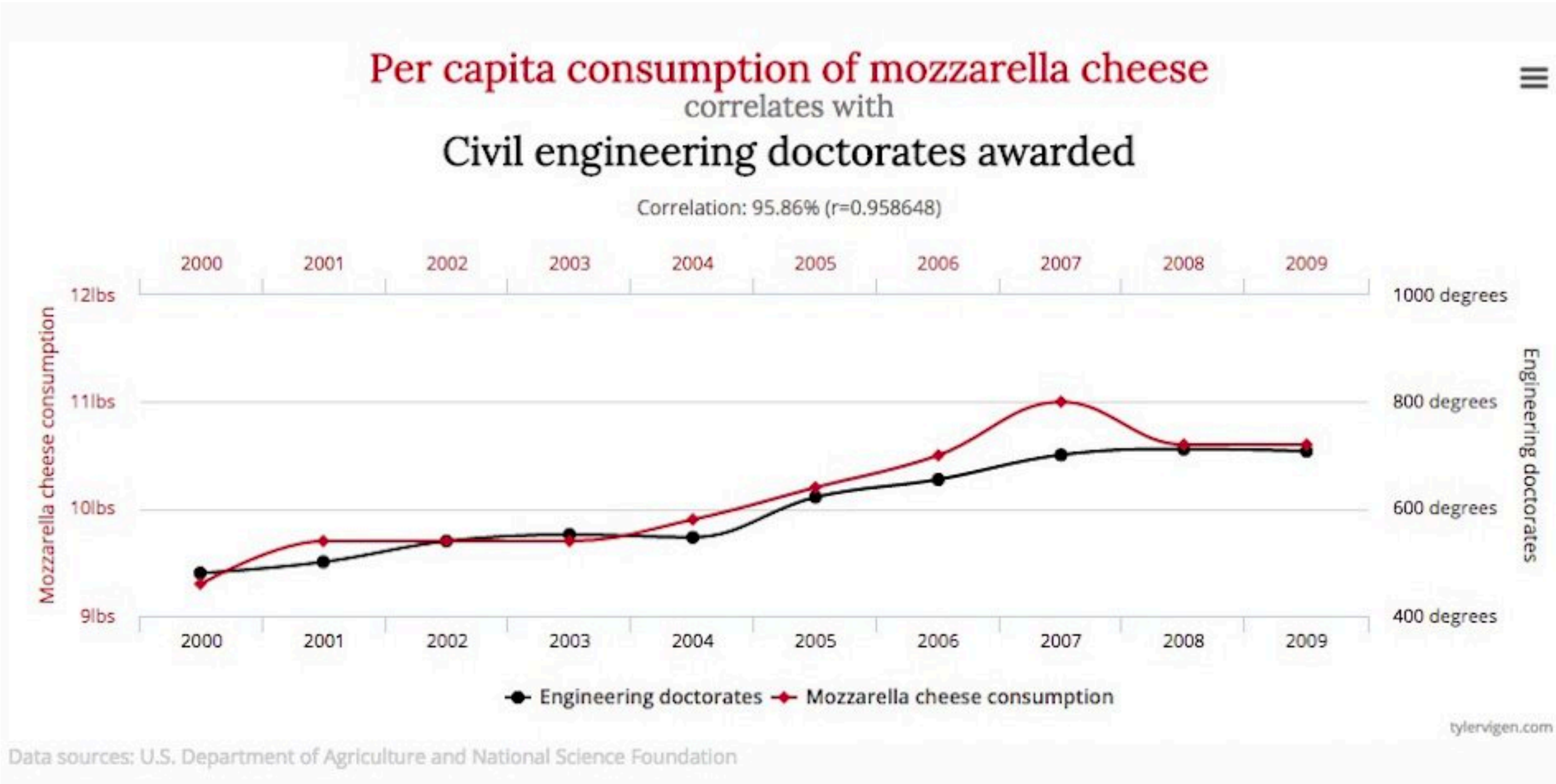
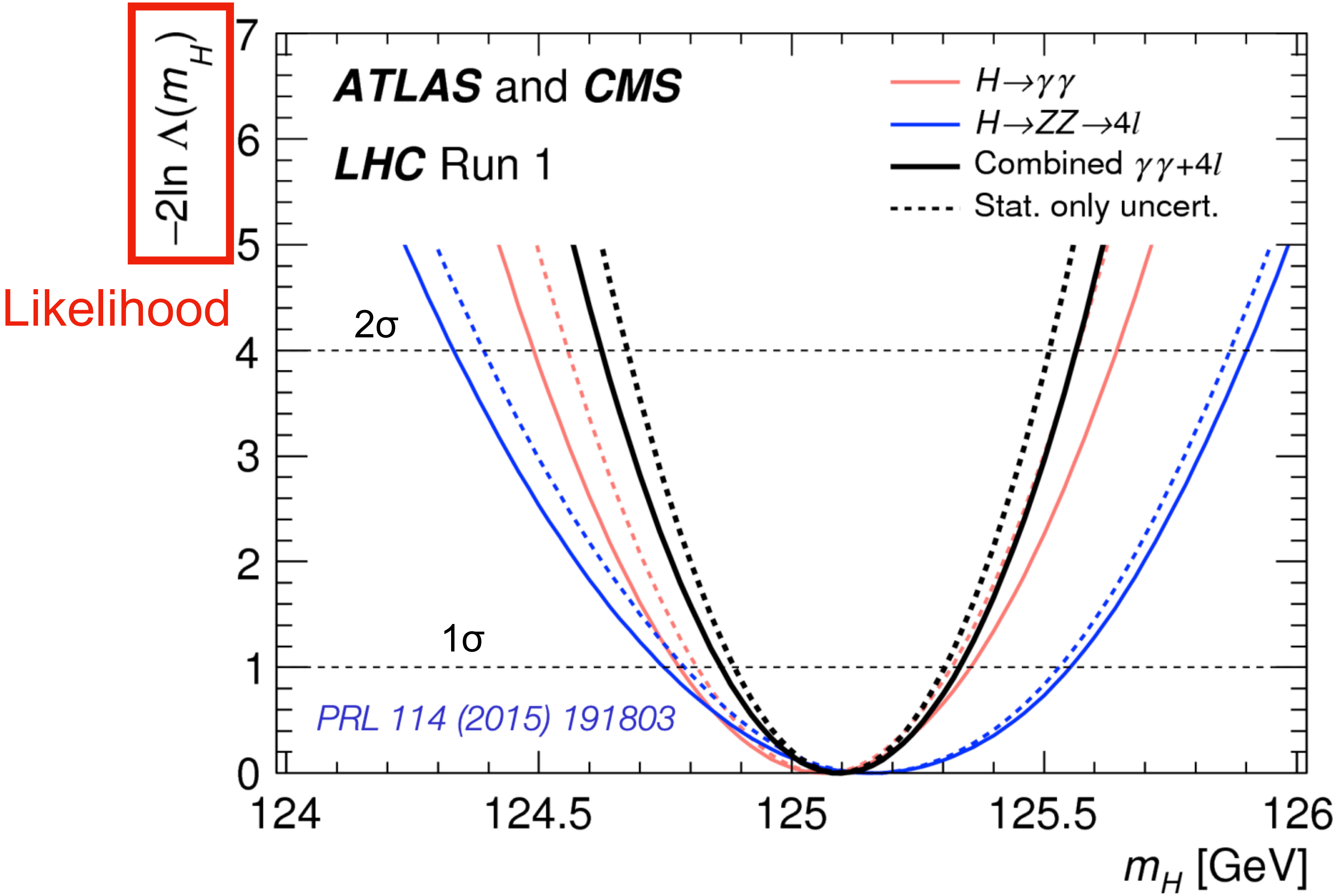
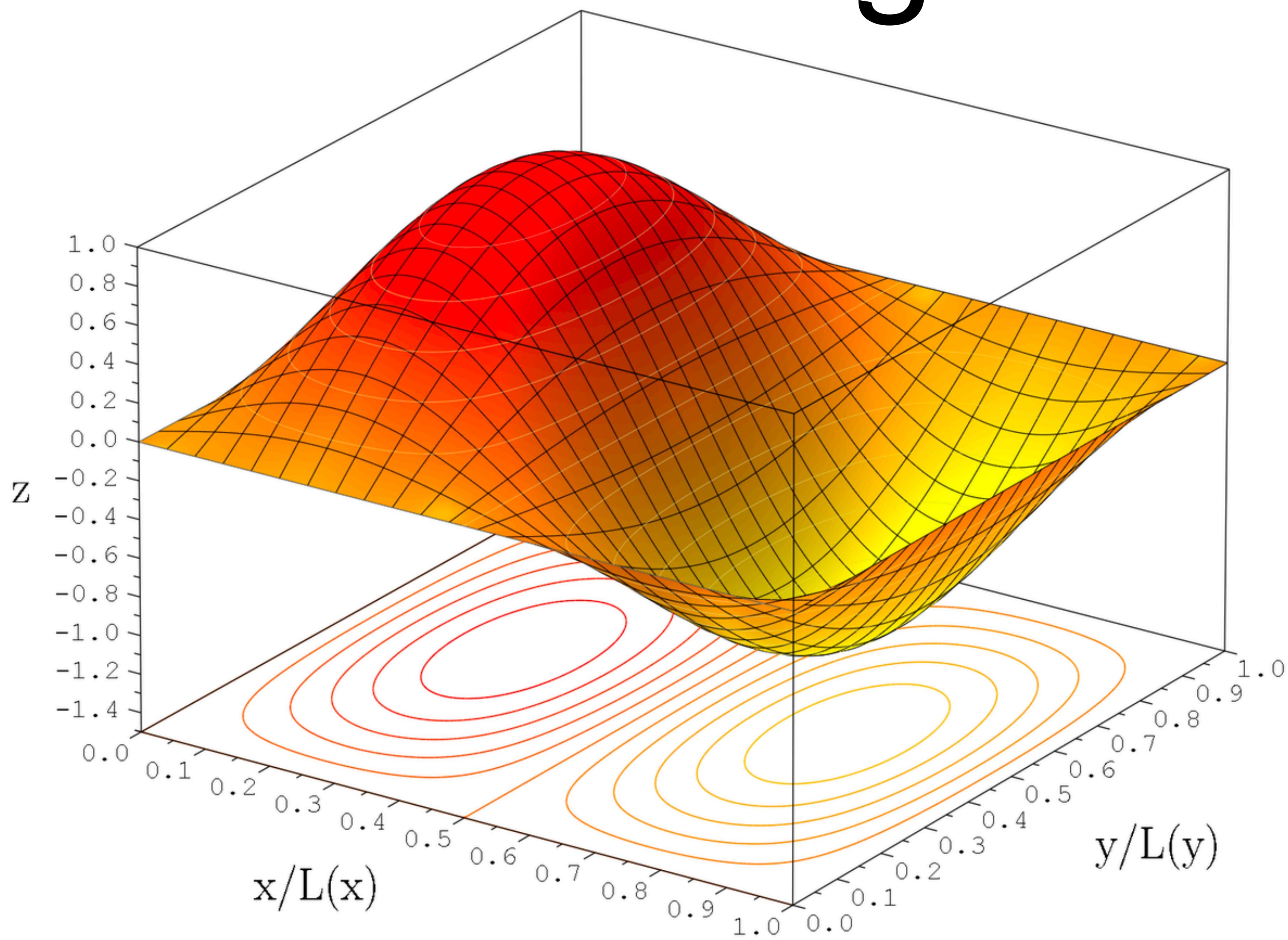


Lecture 7: Correlations

Understanding Best Fit

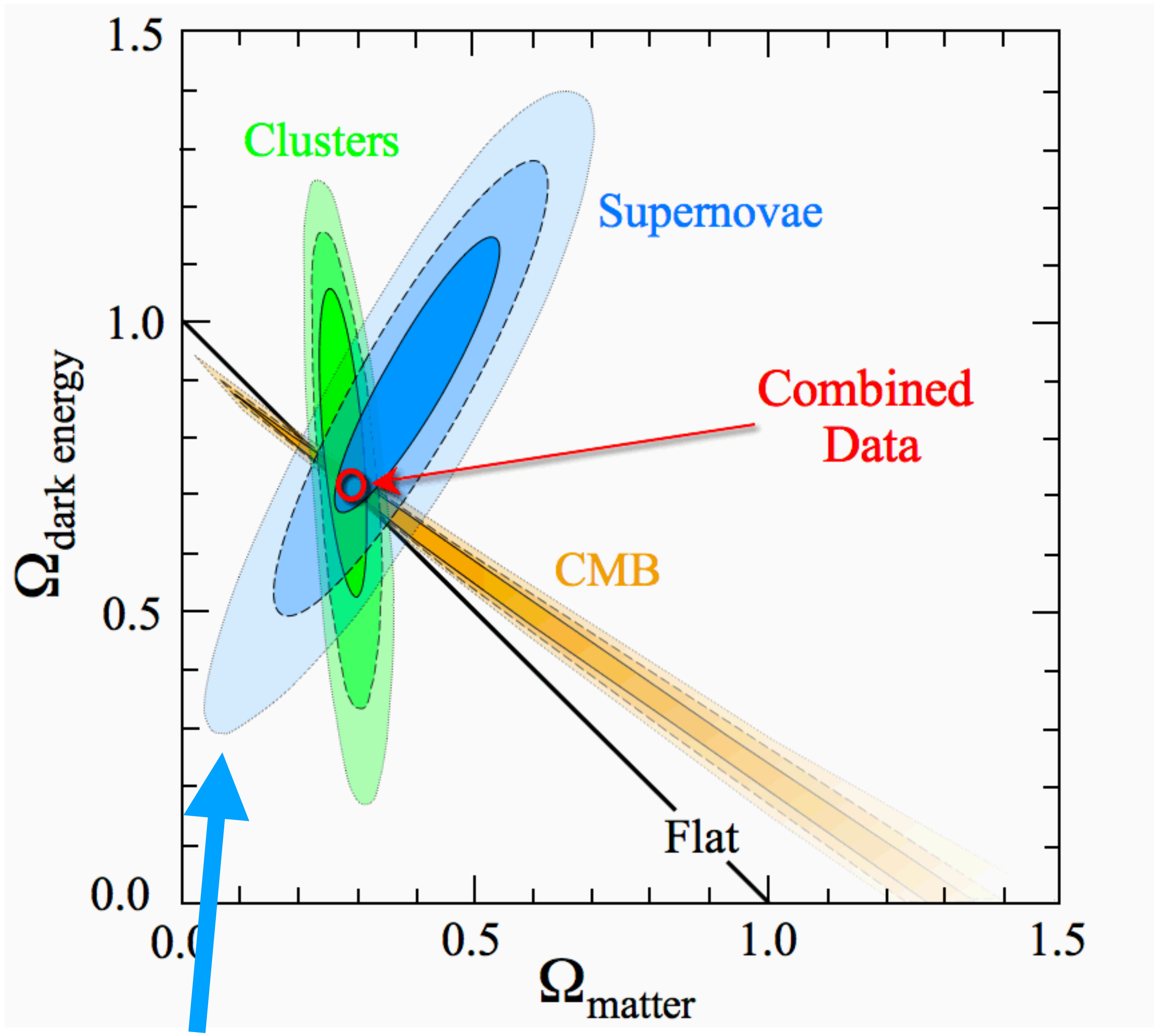


Minimizing A Surface



- How do we get to the minimum of that

Making a 2D Minima



How Can we make the contours on the supernova data?

Friedmann Equations

$$\left(\frac{h}{h_0}\right)^2 = (\Omega_m + \Omega_{\mathrm{DM}})a^{-3} + \Omega_r a^{-4} + \Omega_\kappa a^{-2} + \Omega_\Lambda$$

$$\boxed{1 = (\Omega_m + \Omega_{\mathrm{DM}}) + \Omega_r + \Omega_\kappa + \Omega_\Lambda}$$

Formula at time now

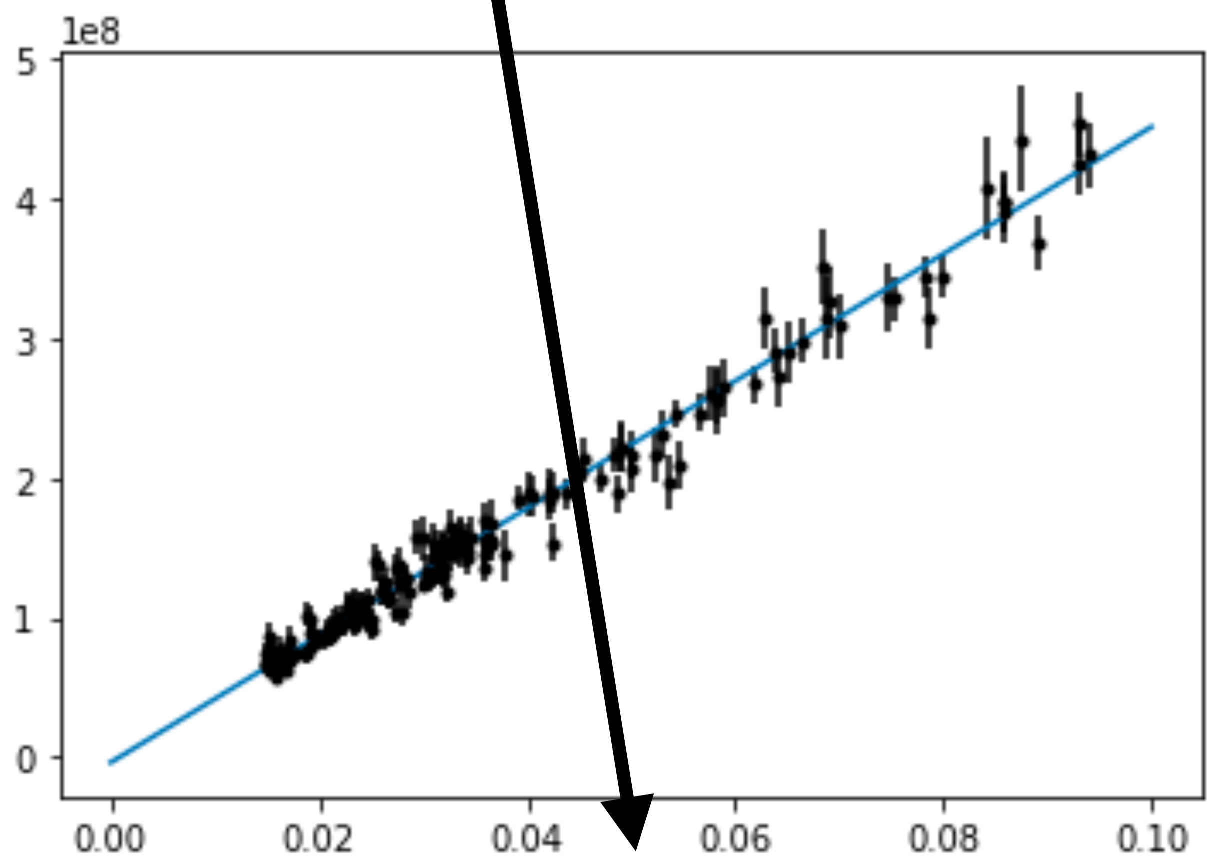
$$\boxed{1 = \Omega_M + \Omega_\Lambda \text{ or } \Omega_\Lambda = 1 - \Omega_M}$$

Removing Terms

$$\boxed{\left(\frac{h}{h_0}\right)^2 = (\Omega_M)a^{-3} + 1 - \Omega_M}$$

Friedmann Equations

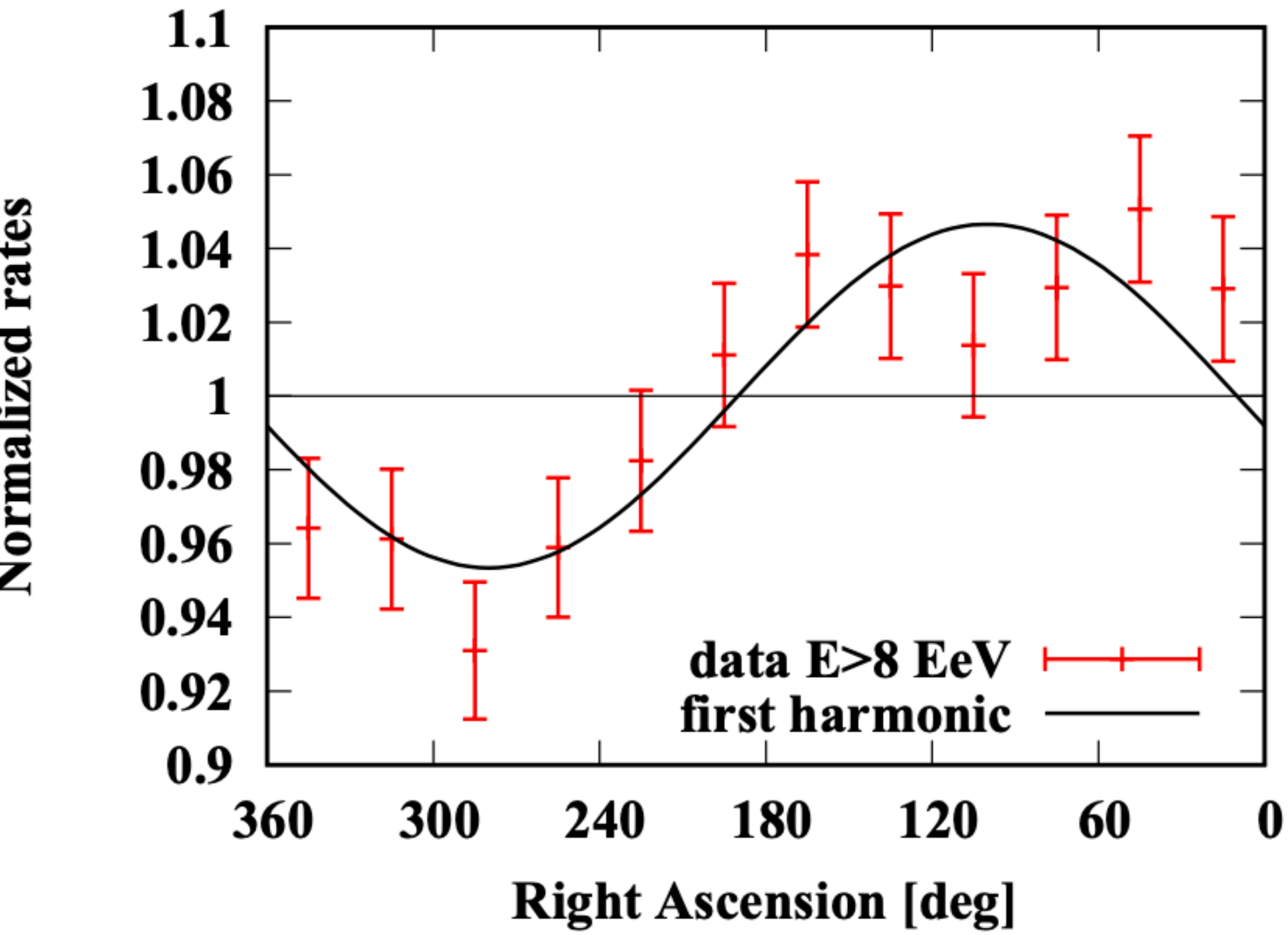
$$d(z) = ct' = (1+z)ct = (1+z)\frac{c}{h_0}\int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + 1 - \Omega_M}}$$

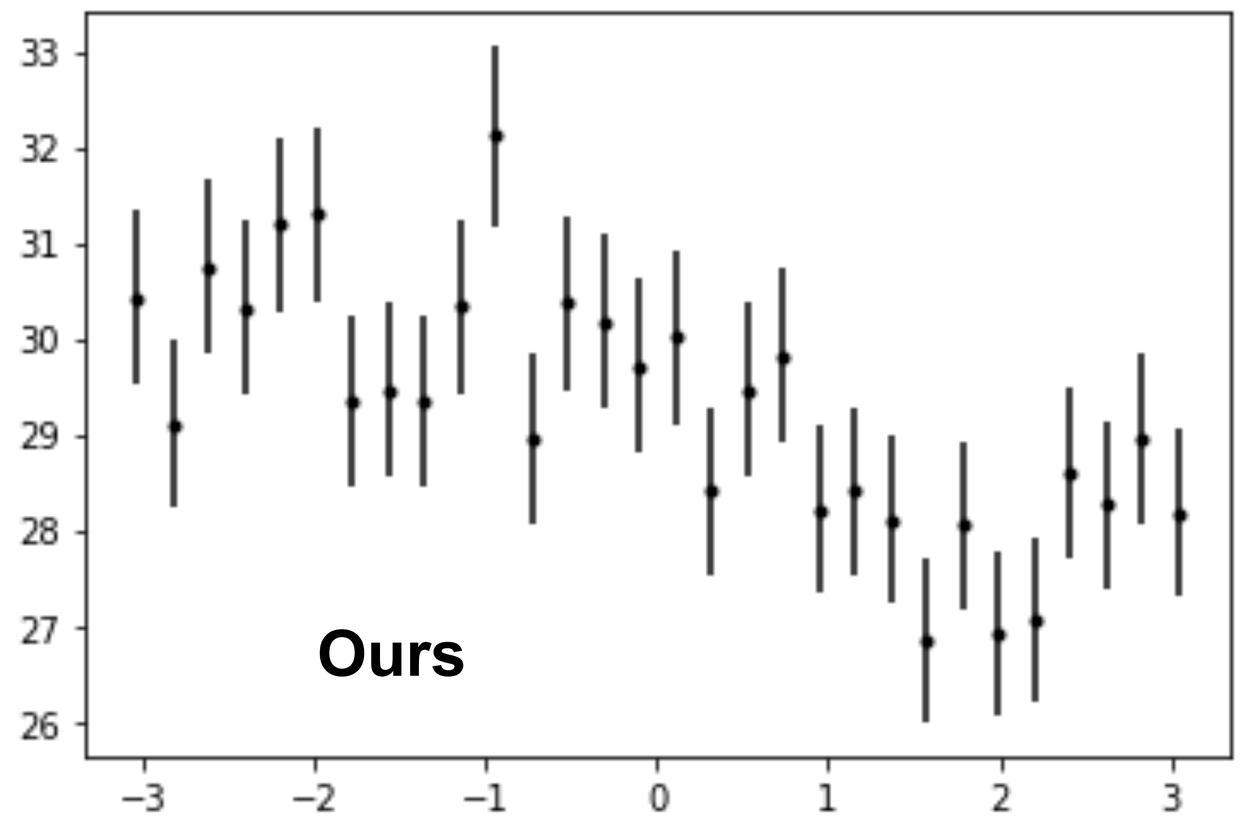


Distance

Redshift

Cosmic Ray Data





Multiple Dimensions

- For N variables the expansion is the same

$$\chi^2(x_i,\vec{\theta}) = \chi^2_{min}(x_i,\vec{\theta}) + \boxed{\frac{1}{2}(\theta_i-\theta_0)^T \boxed{\frac{\partial^2}{\partial\theta_i\theta_j}\chi^2_{min}(x_i,\vec{\theta}_0)}(\theta_j-\theta_0)}$$

χ^2 distribution of 1 degree of freedom

V[$\chi^2(x)$]=1

Hessian of
the χ^2 distribution

$$\Delta\chi^2 = 2\Delta\log\mathrm{L} = 1$$

For one degree of freedom

This is known as Wilk's Theorem

$$\boxed{\sigma^2_{ij} = \left(\frac{\partial^2 \log L}{\partial\theta_i\theta_j}\right)^{-1}}$$

2D examples

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\begin{pmatrix} \theta_a - \theta_{a-min} & \theta_b - \theta_{b-min} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \chi^2}{\partial \theta_a^2} & \frac{\partial^2 \chi^2}{\partial \theta_a \partial \theta_b} \\ \frac{\partial^2 \chi^2}{\partial \theta_a \partial \theta_b} & \frac{\partial^2 \chi^2}{\partial \theta_b^2} \end{pmatrix} \begin{pmatrix} \theta_a - \theta_{a-min} \\ \theta_b - \theta_{b-min} \end{pmatrix}$$

$$\frac{\partial^2 \chi^2}{\partial \theta_a \partial \theta_b} \approx 0$$

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\begin{pmatrix} \Delta\theta_a & \Delta\theta_b \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \chi^2}{\partial \theta_a^2} & 0 \\ 0 & \frac{\partial^2 \chi^2}{\partial \theta_b^2} \end{pmatrix} \begin{pmatrix} \Delta\theta_a \\ \Delta\theta_b \end{pmatrix}$$

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\left(\Delta\theta_a^2 \frac{\partial^2 \chi^2}{\partial \theta_a^2} + \Delta\theta_b^2 \frac{\partial^2 \chi^2}{\partial \theta_b^2} \right)$$

$$= \chi^2_{min}(x,\vec{\theta}) + \left(\frac{\Delta\theta_a^2}{\sigma_{\theta_a}^2} + \frac{\Delta\theta_b^2}{\sigma_{\theta_b}^2} \right)$$

2D examples

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\begin{pmatrix} \theta_a - \theta_{a-min} & \theta_b - \theta_{b-min} \end{pmatrix} \begin{pmatrix} \frac{\partial^2\chi^2}{\partial\theta_a^2} & \frac{\partial^2\chi^2}{\partial\theta_a\partial\theta_b} \\ \frac{\partial^2\chi^2}{\partial\theta_a\partial\theta_b} & \frac{\partial^2\chi^2}{\partial\theta_b^2} \end{pmatrix} \begin{pmatrix} \theta_a - \theta_{a-min} \\ \theta_b - \theta_{b-min} \end{pmatrix}$$

$$\frac{\partial^2\chi^2}{\partial\theta_a\partial\theta_b} \approx 0$$

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\begin{pmatrix} \Delta\theta_a & \Delta\theta_b \end{pmatrix} \begin{pmatrix} \frac{\partial^2\chi^2}{\partial\theta_a^2} & 0 \\ 0 & \frac{\partial^2\chi^2}{\partial\theta_b^2} \end{pmatrix} \begin{pmatrix} \Delta\theta_a \\ \Delta\theta_b \end{pmatrix}$$

$$\chi^2(x,\vec{\theta}) = \chi^2_{min}(x,\vec{\theta}) + \frac{1}{2}\left(\Delta\theta_a^2 \frac{\partial^2\chi^2}{\partial\theta_a^2} + \Delta\theta_b^2 \frac{\partial^2\chi^2}{\partial\theta_b^2} \right)$$

$$= \boxed{\chi^2_{min}(x,\vec{\theta}) + \left(\frac{\Delta\theta_a^2}{\sigma^2_{\theta_a}} + \frac{\Delta\theta_b^2}{\sigma^2_{\theta_b}} \right)}$$

Ellipse

Relating all the 2Ds

$$\frac{2}{\sigma^2} = \frac{\partial^2 \chi^2}{\partial\theta_i \partial\theta_j}$$

$$\sigma^2 = 2\left(\frac{\partial^2 \chi^2}{\partial\theta_i \partial\theta_j}\right)^{-1}$$

Wilk's Theorem

$$\begin{pmatrix} \frac{\partial^2 \chi^2}{\partial\theta_a^2} & 0 \\ 0 & \frac{\partial^2 \chi^2}{\partial\theta_b^2} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{2}{\sigma_a^2} & 0 \\ 0 & \frac{2}{\sigma_b^2} \end{pmatrix}$$

Wilk's For Uncorrelated Parameters

For correlated Paramters Can always Diagonalize

$$A^{-1} 2 \begin{pmatrix} \frac{\partial^2 \chi^2}{\partial\theta_a^2} & \frac{\partial^2 \chi^2}{\partial\theta_a \partial\theta_b} \\ \frac{\partial^2 \chi^2}{\partial\theta_a \partial\theta_b} & \frac{\partial^2 \chi^2}{\partial\theta_b^2} \end{pmatrix}^{-1} \quad A = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

2D Terminology

Covariance Matrix

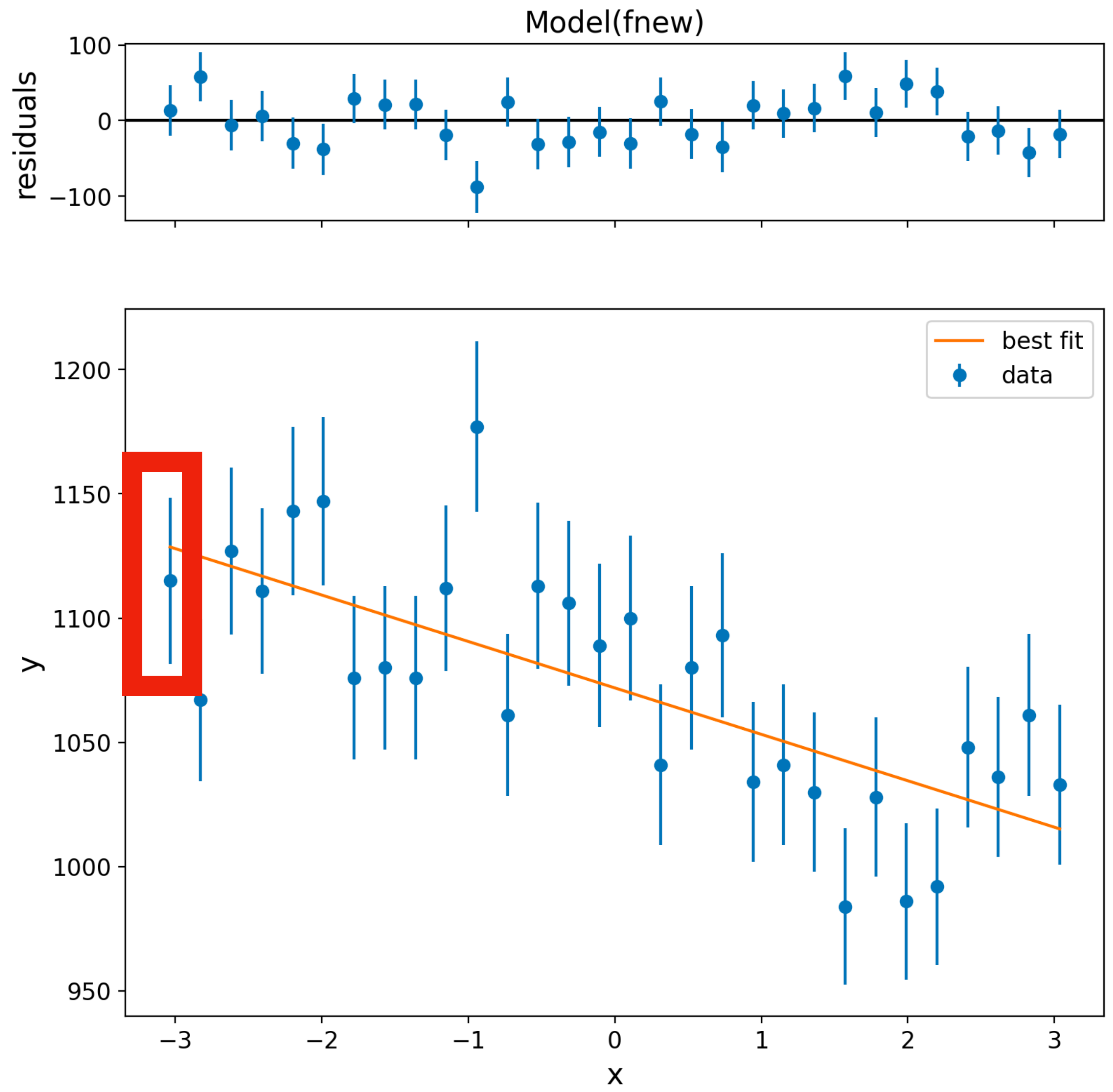
$$\begin{pmatrix} \sigma_a^2 & \mathrm{COV}(a,b) \\ \mathrm{COV}(a,b) & \sigma_b^2 \end{pmatrix} = \sum_{i=1}^{N} \begin{pmatrix} (a_i - \bar{a})^2 & (a_i - \bar{a})(b_i - \bar{b}) \\ (a_i - \bar{a})(b_i - \bar{b}) & (b_i - \bar{b})^2 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1 & \frac{\mathrm{COV}(a,b)}{\sigma_a \sigma_b} \\ \boxed{\frac{\mathrm{COV}(a,b)}{\sigma_a \sigma_b}} & 1 \end{pmatrix}$$

Correlation Matrix

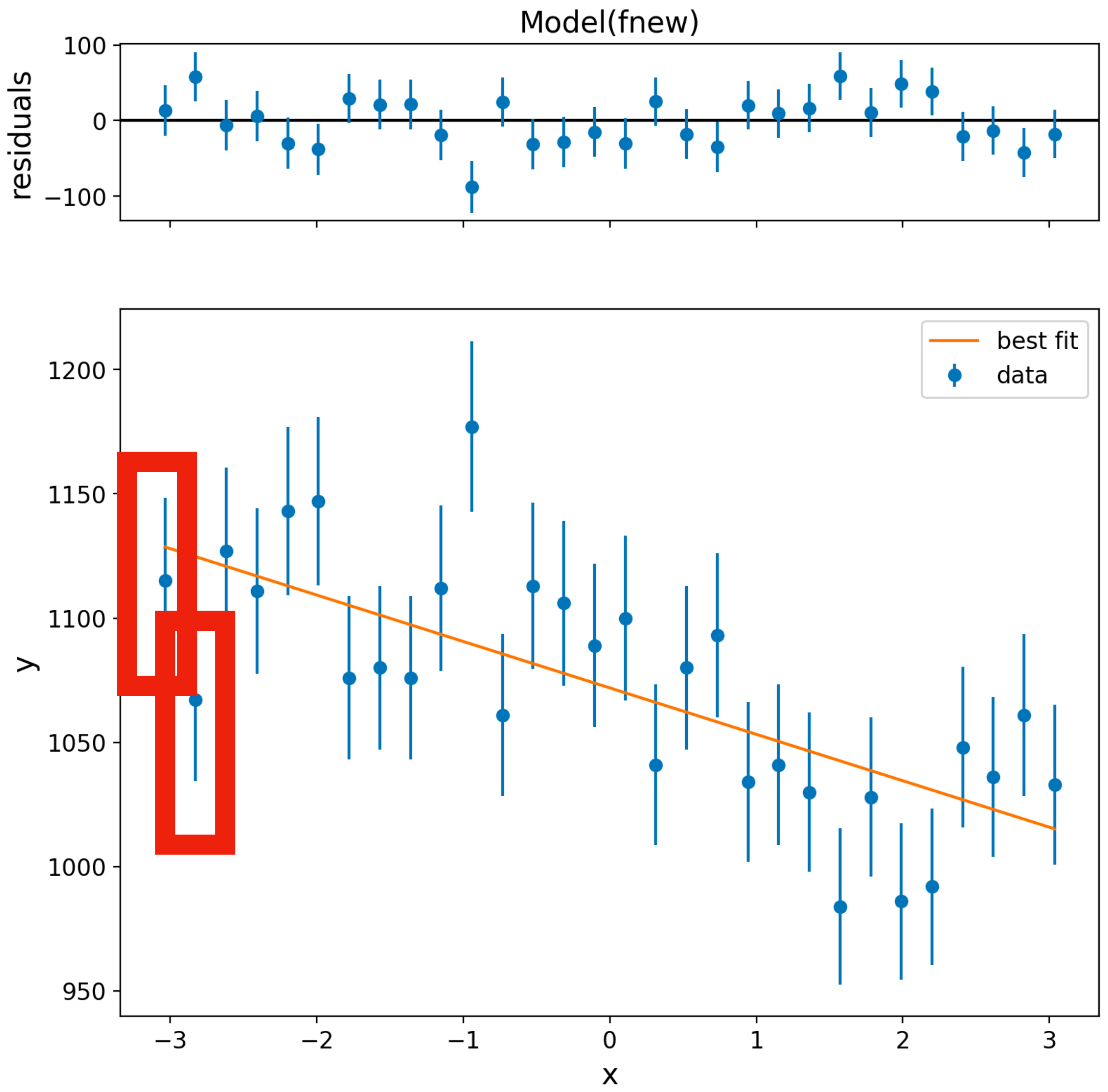
Correlation Coefficient

Toys



Poisson Fluctuate this bin

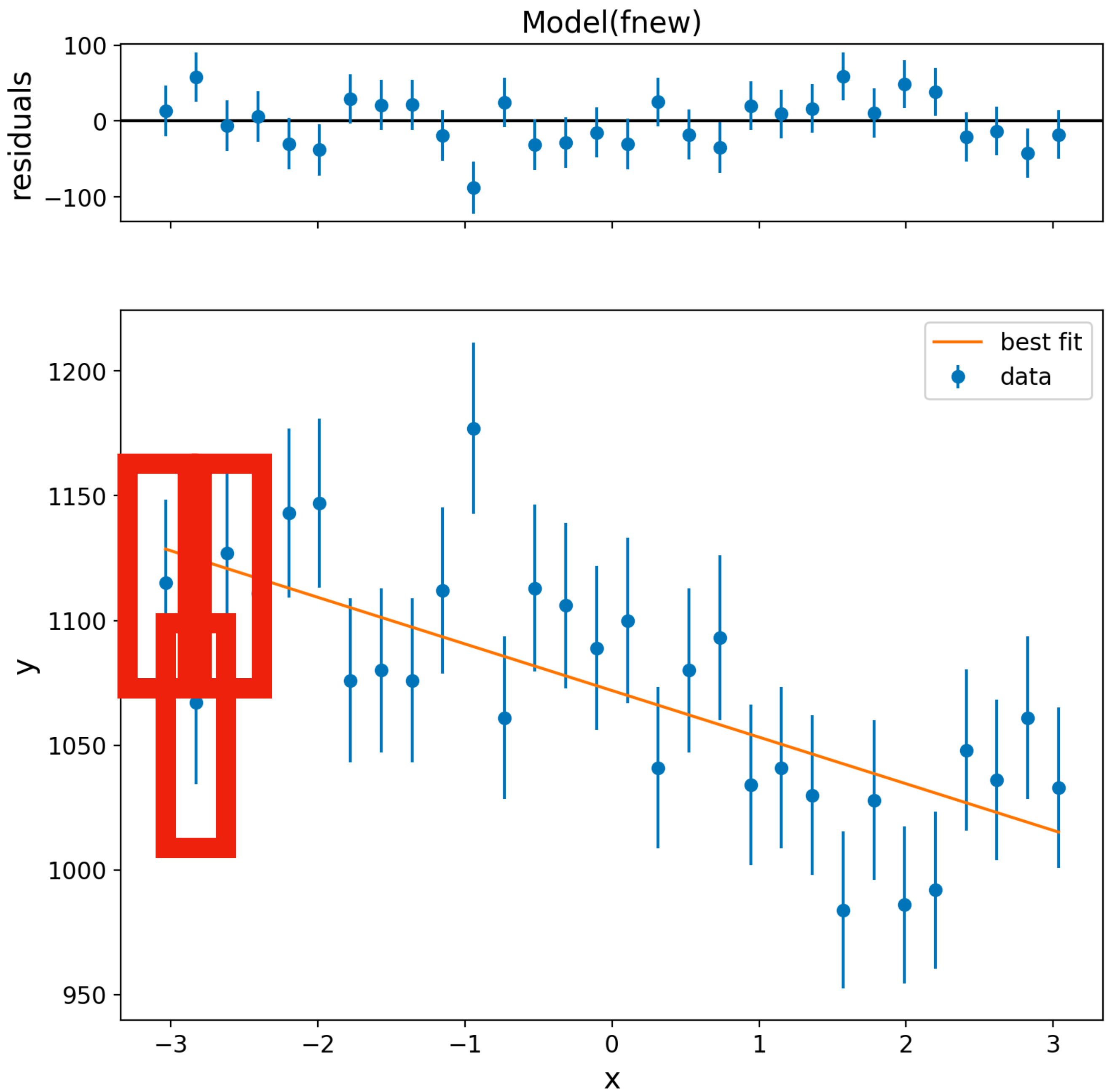
Toys



Poisson Fluctuate this bin

Poisson Fluctuate this bin

Toys

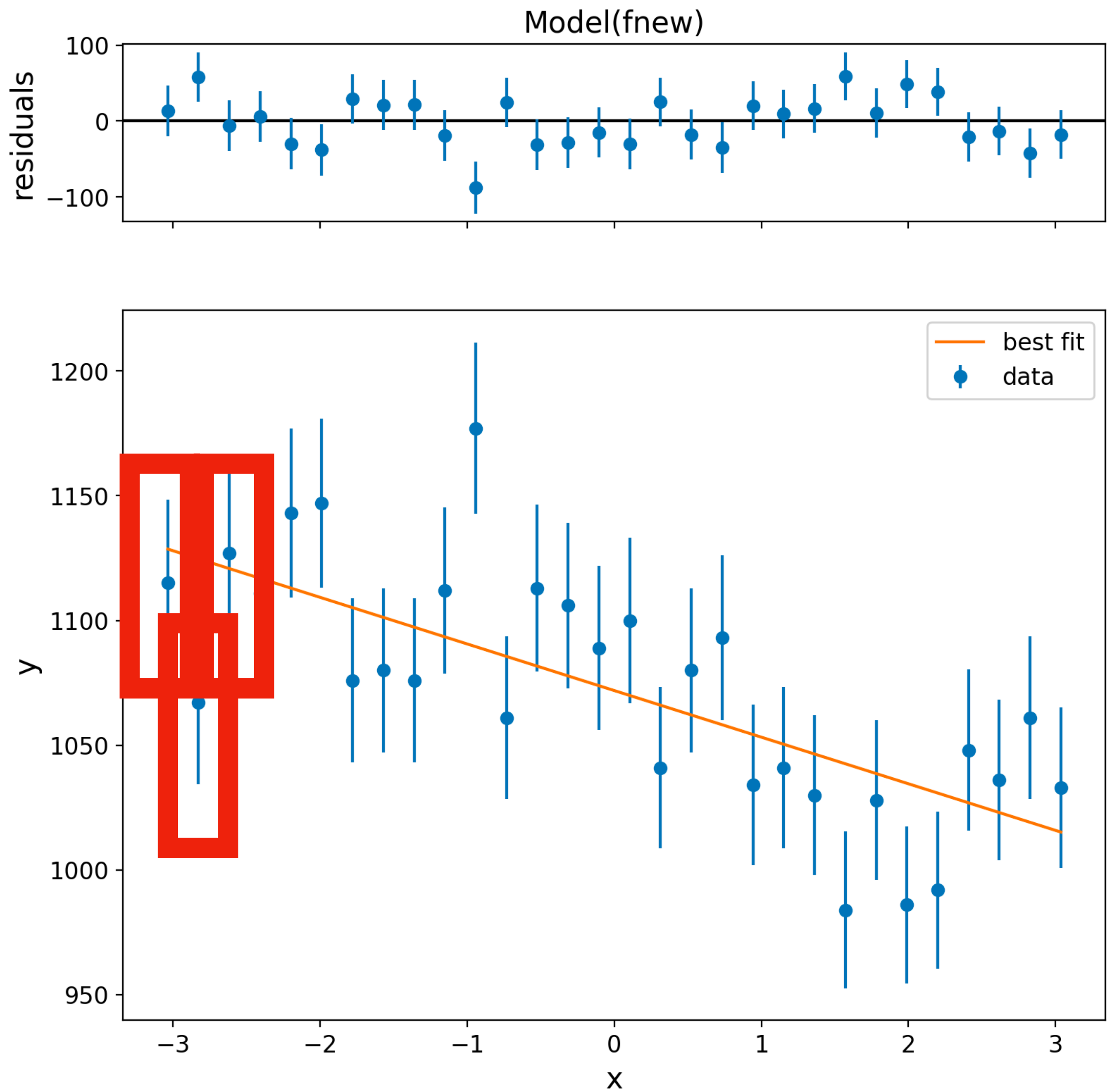


Poisson Fluctuate this bin

Poisson Fluctuate this bin

Poisson Fluctuate this bin

Toys



Poisson Fluctuate this bin

Poisson Fluctuate this bin

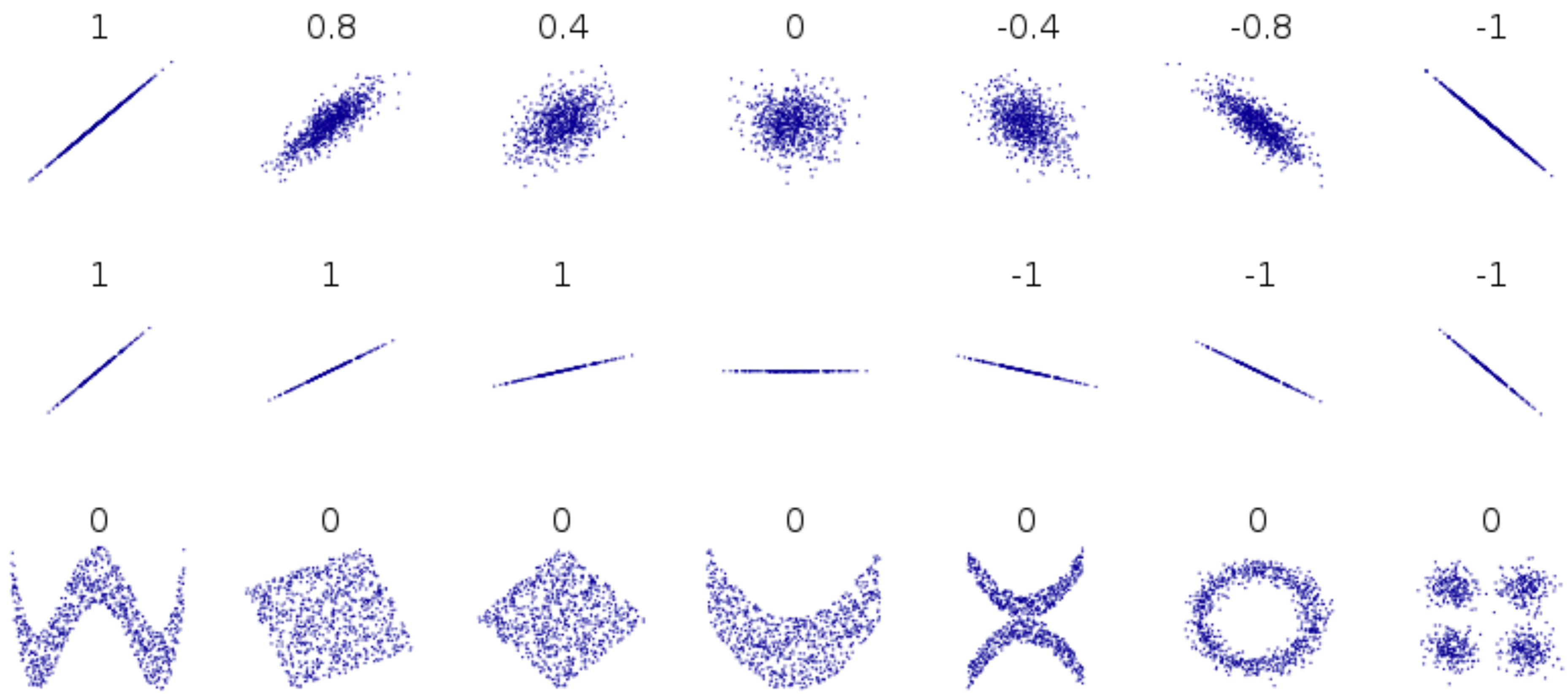
….

Poisson Fluctuate this bin

Correlation Vernacular

$$r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}$$

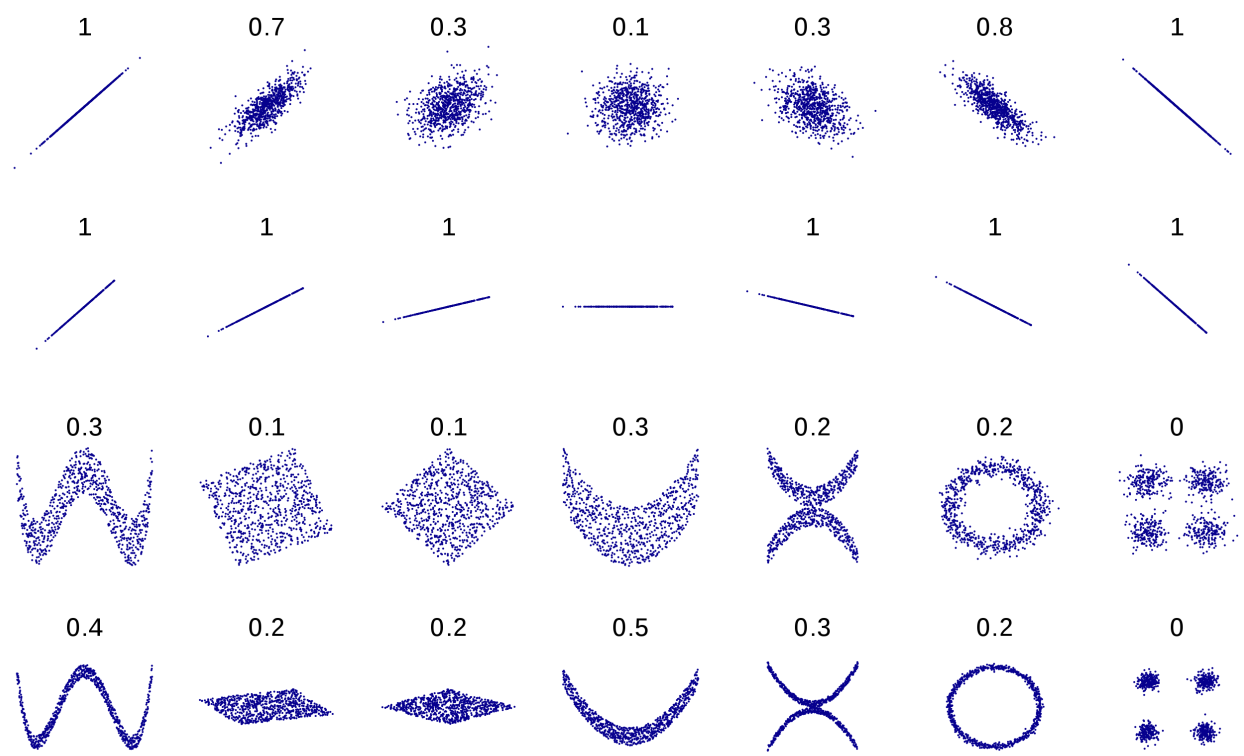
This is what we discussed Last Class
Commonly referred to as
"Pearson Correlation"



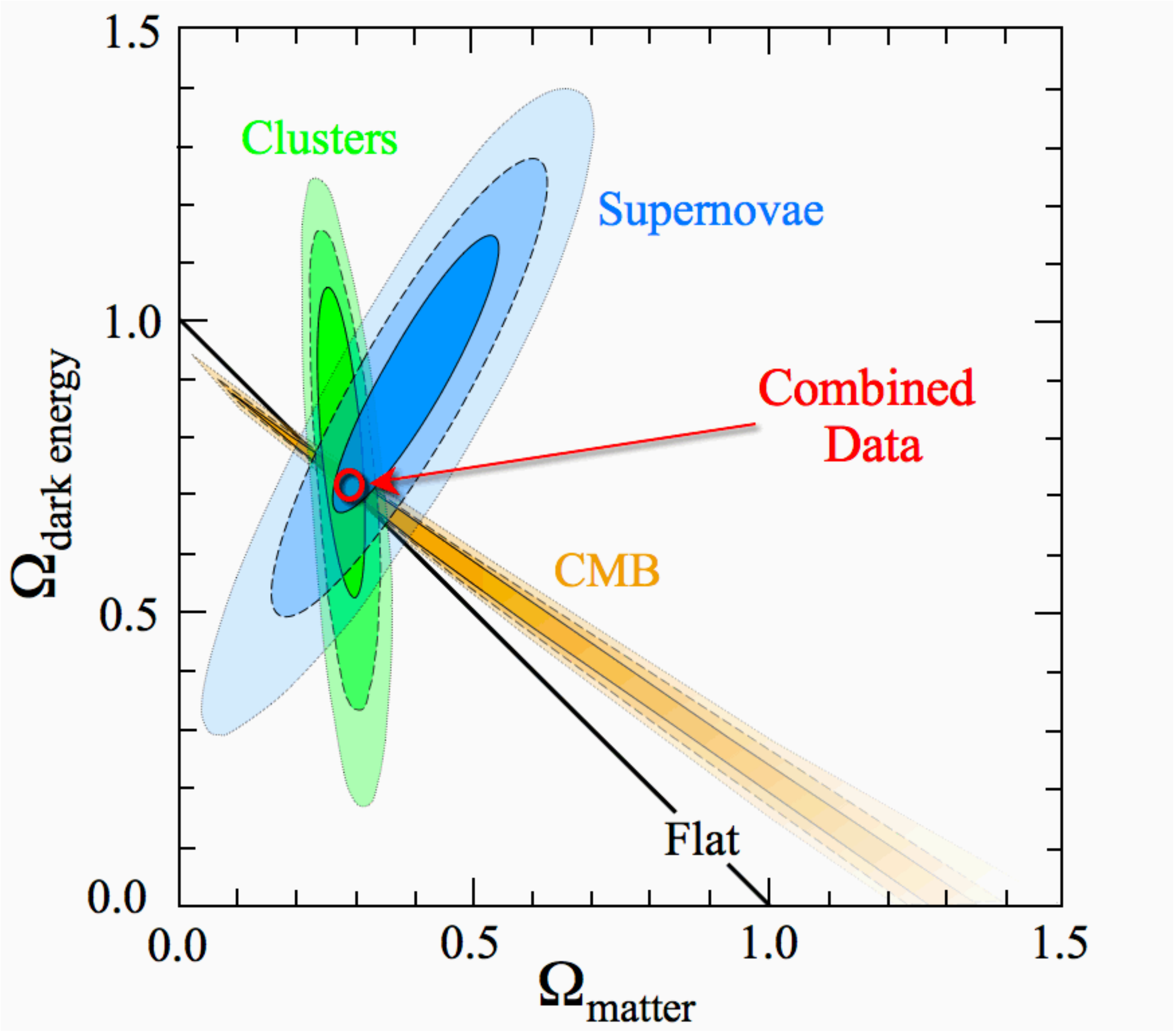
Correlation Vernacular

$$\mathrm{dCor}^2(X,Y) = \frac{\mathrm{dCov}^2(X,Y)}{\sqrt{\mathrm{dVar}^2(X)\;\mathrm{dVar}^2(Y)}},$$

You can construct other correlations
Such as the correlation of distances between points
"Distance Correlation" (Disco)



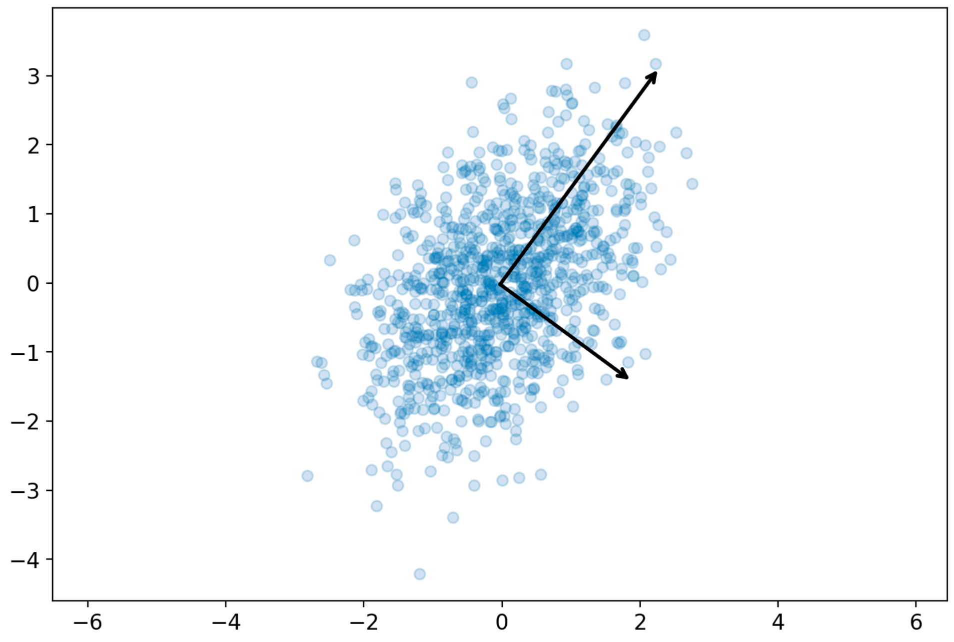
Full 2D Plot



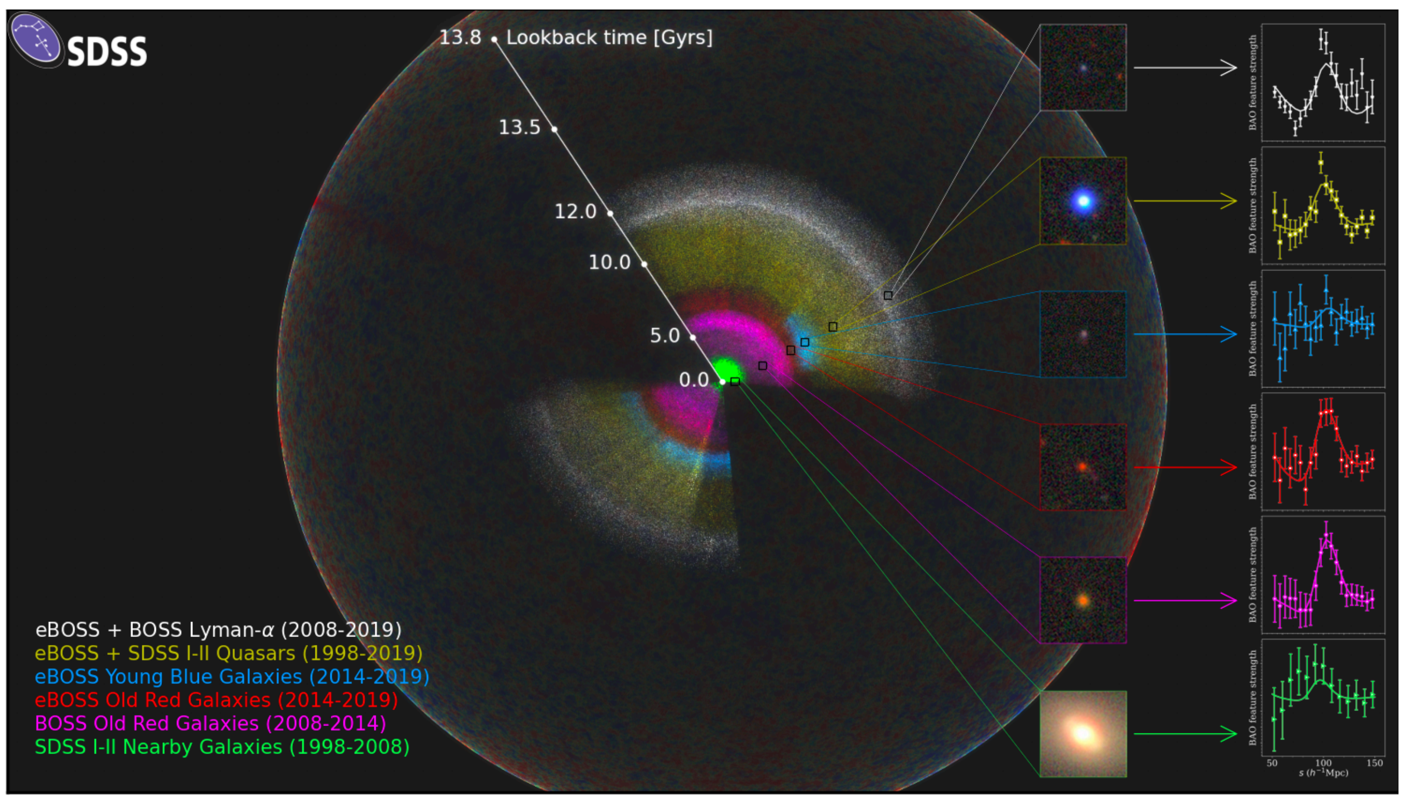
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
$\Omega_{\text{dark energy}}$
Ω_{matter}
Clusters
Supernovae
Combined Data
CMB
Flat

Principal Component Analysis

- Just taking the Eigen-vectors/values of an n-d space
  - Sorted by Eigenvalues, which give rank
  - Number of Eigenvalues above threshold is dimension



SDSS


SDSS
13.8 Lookback time [Gyrs]
13.5
12.0
10.0
5.0
0.0
eBOSS + BOSS Lyman-α (2008-2019)
eBOSS + SDSS I-II Quasars (1998-2019)
eBOSS Young Blue Galaxies (2014-2019)
eBOSS Old Red Galaxies (2014-2019)
BOSS Old Red Galaxies (2008-2014)
SDSS I-II Nearby Galaxies (1998-2008)

Fitting Checklist

- When I fit a function:
  - I minimize Log likelihood (usually approxed by chi2)
  - When I fit does the fit converge
  - Check the final chi2 value
    - Chi2/NDOF should be close to 1
    - Chi2 p-value should be good (5%-95%)
  - Look at the parameters
    - Do my uncertainties make sense? (Too small/large)
      - Are parameters moving far away from expected?
    - Should I do a 2D scan of the parameters?
  - How do I want to express my results?