Summary on Black Hole Entropy

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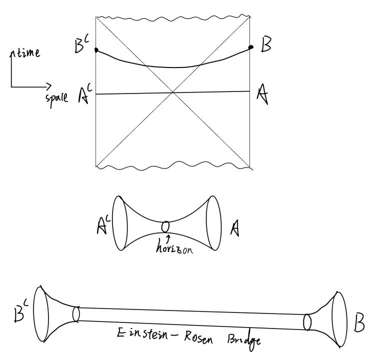
1 Black Hole in AdS

1.1 Black hole geometry

1.1.1 Penrose diagram

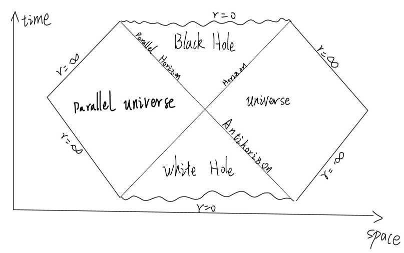
A Penrose diagram is a spacetime diagram arranged to make clear the complete causal structure of any given geometry. Roger Penrose, who invented this kind of diagram in the early 1950s, himself calls them conformal diagrams. Each curve like A, A^cthat cross the Penrose diagram is a bulk Cauchy slice σ,which is a time-refection invariant and thought the bifurcate horizon m_{hor}(point in the middle). Two plots below are the induced geometry for Cauchy slice σ.
1. Lights goes 45 degrees of angle from upward vertical.
2. Points at infinity are included in the diagram. Horizon in Penrose diagram refers to the minimal



surface on time slice that separate A from A^c. As time evolves, the length of horizon in between grow linearly with time. The bridge in between A and A compliment is call Einstein-Rosen Bridge.

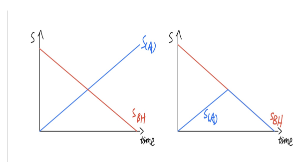
1.1.2 Penrose diagram of the complete, analytically extended Schwarzschild geometry

The Schwarzschild geometry has a simple mathematical form, and that form can be extended analytically. The mathematical extension consists of a second copy of the Schwarzschild geometry, reversed in time, glued along the Antihorizon. The complete analytic extension of the Schwarzschild geometry contains not only a Universe and a Black Hole, but also a Parallel Universe and a White Hole.



1.1.3 Evaporating black hole puzzle

Here comes a little puzzle for the black hole entropy: Hawking radiation is entangled with modes inside black hole. So as time evolves, the entanglement entropy increases linearly. Meanwhile, black hole will shrink as time passed by, so the entropy of black hole will decrease. Question pop out here, will at some time, the entropy of entanglement equals to entropy of black hole, then we have $S(A) < S_{BH}$? The answers is no, using holographic duality, put black hole in a box, couple to bath



B to absorb the radiation.

$$S(A) = S(B) = min_m(\frac{area\ of\ m}{plank\ area} + S(\tilde{A}))$$

At first, no surface m or region$\tilde{B}$, $S(A) = S(\tilde{A})$, entanglement grows with time. As it reaches critical point: $S(A) = B_{BH}$, interior switches to $\tilde{B}$-"Island", after that, $S(A)$ follows S_{BH} decreases.

1.2 Bekenstein-Hawking Entropy

The Bekenstein-Hawking entropy is the amount of entropy that must be assigned to a black hole in order for it to comply with the laws of thermodynamics as they are interpreted by observers external to that black hole.

The event horizon area of a black hole cannot decrease; it increases in most transformations of the black hole. This increasing behavior is reminiscent of thermodynamic entropy of closed systems. Thus it is reasonable that the black hole entropy should be a monotonic function of area, and it turns out to be simplest such function.

Define A as area of the event horizon(surface area of a black hole), then the dimensionless form of black hole entropy could be written as:

$$S_{BH} = \frac{A}{4G_N} = \frac{A}{4{l_p}^2}$$

As G_N stands for Newtonian gravity, l_p is the plank length, and $\hbar$ is the Planck-Dirac constant(Reduced Planck constant), they have relationship as $l_p = \sqrt{\frac{G\hbar}{c^3}}$. So be more precisely, the Bekenstein-Hawking entropy should be written as:

$$S_{BH} = \frac{A}{4{l_p}^2} = \frac{c^3 A}{4G\hbar}$$

For some more symmetric situations, we could precisely calculate the black hole area. Take a spherical symmetric and stationary or we say "Schwarzschild black hole", we have black hole's mass M, horizon's radius $r_h = \frac{2GM}{c^2}$, area count as a circle $4\pi {r_h}^2$, thus we could easily get the area of black hole as

$$A = 16\pi(GM/c^2)^2$$

1.3 Ryu-Takiyanagi formula

The entanglement revolution in AdS/CFT was initiated by the discovery of the Ryu-Takayanagi formula, which shows that the emergent geometry actually encodes entanglement in the boundary field theory in a very direct way.

In 2008, Shinsei Ryu and Tadashi Takayanagi proposed a holographic derivation for entanglement entropy in CFT from AdS/CFT correspondence. In the paper, it claims that the d +1 dimensional conformal field theories (CFT_{d+1}) are equivalent to the (super)gravity on d+2 dimensional antideSitter space AdS_{d+2}.

from the statement above we could give some conclusion on properties of γ_A: 1. γ_A has the same boundary with A. 2. γ_A is homologous to A. 3. γ_A extremizes the area. If there are multiple extremal surfaces, γ_A has the least area.

Based on CFT/AdS duality, define the entanglement entropy S_A in a CFT on $\mathbb{R}\times S^d$ for a subsystem A that has an arbitrary d-1 dimensional boundary $\partial A \in \mathbb{R}^d$

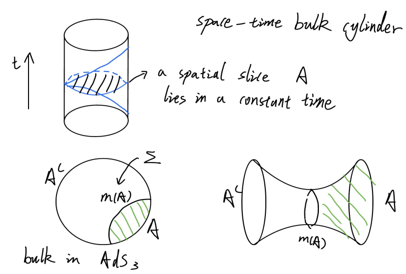
$$S_A = \frac{Area\ of\ \gamma_A}{4G_N^{d+2}}$$

In which γ_A is the d dimensional static minimal surface in AdS_{d+2} spatial slice Σ with boundary as ∂A, G_n^{d+2} is the $(d+2)$ dimensional newton gravitation constant.

For a 3D case, γ is a codimension-2, space-like extremal surface in the dual geometry, anchored to the AdS boundary such that $\partial\gamma_A = \partial A$, γ_A lies on a specific space-like slice, so the following plot for it is suppressed in orthogonal to the time direction.

1.4 HRT formula

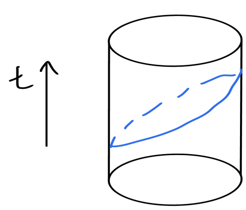
Let's get back to the Penrose diagram. we take a curve $A - A^C$ from the plot and put it into a slice of time-space bulk cylinder. Like we mentioned above, Ryu-Takiyanagi formula gives a solution to the entanglement entropy for this constant-time slice. But it is very clear RT has a restriction to static bulk spacetimes and to boundary regions sitting inside constant-time slices. If we search for a covariant generalization, then we can no longer appeal to the constant-time slice in the bulk, with its Euclidean induced geometry, but rather have to contend with the full Lorentzian spacetime.



As a generalization of RT formula, Hubeny-Rangamani-Takayanagi have proposed a a covariant generalization of the holographic entanglement entropy.

$$S(A) = \frac{1}{4G_N} \underset{extremal\ m \sim A}{min} \ area(m)$$

(As in the RT formula, m has the same dimensionality as A, so it is codimension two.) The choice to replace a minimization procedure by an extremization one when passing from a static situation to a dynamical one is a very natural one.



1.5 Examples in holographic entanglement entropy

1.5.1 In AdS_3/CFT_2

1.5.2 In higher dimensions

2 Holographic Entanglement Entropy

2.1 Holographic dualities

2.2 Holographic Entropy Bound

2.3 Entanglement entropy and Holography

2.4 Geometry&Entanglement

2.5 Properties

2.5.1 Entropy inequalities from HRT

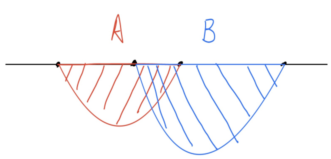
2.5.2 Relation between subsystem and full-system measures

2.5.3 Entropy for pure state

For the state in which density metric ρ is pure, the entropy of state A,

$$S(A) = S(A^c)$$

2.5.4 Subadditivity



For regions A and B lies in a time-symmetric slice Σ, the entropy of subsystem A and B have a relation to entropy of the whole system:

$$S(AB) \leq S(A) + S(B)$$

And it's not hard to prove this, since the local minimal surface $m(A) \cup m(B)$ cannot have area bigger than area $m(A) + m(B)$.

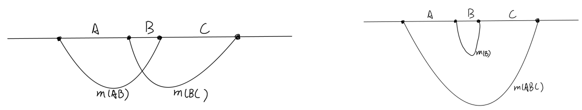
And also a point needs to be mentioned here is the special case of Subadditivity, is the Rocky-Liebe:

$$|S(A) - S(B)| \leq S_{tot}$$

2.5.5 Strong Subadditivity

$$S(AB) + S(BC) \leq S(B) + S(ABC)$$

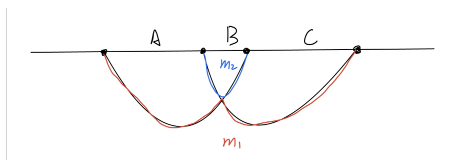
Assume we find the minimal surface $m(AB)$ and $m(BC)$ stands for boundaries AB and BC. And use notation $\tilde{r}$ to represent the region. Then we could have the following relations.



$$\tilde{r}(B) = \tilde{r}(AB) \cap \tilde{r}(BC)$$

$$\tilde{r}(ABC) = \tilde{r}(AB) \cup \tilde{r}(BC)$$

We have the homologous relations $m_1 \sim ABC$, $m_2 \sim B$ Surfaces m_1 and m_2 are homologous



to ABC and B via regions $\tilde{r}(AB) \cap \tilde{r}(BC)$ and $\tilde{r}(AB) \cup \tilde{r}(BC)$, so their areas bound $S(B)$ and $S(ABC)$ respectively as:

$$S(ABC) \leq \frac{1}{4G_N} area(m_1),\ S(B) \leq \frac{1}{4G_N} area(m2)$$

On the other hand, m1 and m2 are obtained by swapping parts of m(AB) and m(BC), so they have the same total area:

$$area(m_1) + area(m_2) = area(m(AB)) + area(m(BC))$$

Then the strong subaddativity $S(AB) + S(BC) \leq S(B) + S(ABC)$ can be derived by looking at the edge curves: $m(AB) + m(BC) \leq m(B) + m(ABC)$, so that:

$$S(AB) + S(BC) \leq S(B) + S(ABC)$$

3 Complexity

3.1 Complexity and in quantum circuits

Next, let's consider the concept of Complexity. The definition of complexity initially depends on the computational system, used to evaluate how much time and effort is needed to implement an algorithm or task.

To better define complexity and better apply it to entropy, here we introduce a new concept- qubits. Like a normal bit, $\{0, 1\}$, we have our qubits defined as the quantum states, have the computational basis $\{|0\rangle\,,\ |1\rangle\}$. We could apply operations on one or more quantum states and manipulate them to become the final state that we want.

$$|ini\rangle - \circ - > \ |final\rangle$$

By another way, we could also considered those operations we applied as a series of different operators like $\{Pauli\ matrices,\ Hadmard\ gates,\ phase\ transition,\ or\ control\ not\}$

Given this, it is sensible to define the exact complexity of a target state as the minimum number of gates needed to exactly prepare the state.

$$C_o(|\psi\rangle) = minimal\ \#\ of\ gates\ needed\ to\ produce\ |\psi\rangle\ \ from\ |0\rangle^{\otimes N}$$

As one thing worth notice here is the complexity has upper bound: $C_o(|\psi\rangle) \leq a^N$. In which a is a constant, and the complexity is cannot bigger than exponential.

3.2 Smoothed complexity

Because the definition of complexity is kinda fragile, like we can have a state that is very close in a physical sense to the target, but have mixed in a very small Haar random state, and then the complexity could be very high. This tiny tiny fraction of somthing very complicated. Instead, we can define:

$$C_\epsilon(|\psi\rangle) : min_{||\psi\rangle - |\phi\rangle| \leq \epsilon}(C_o)$$

3.3 Complexity and Entanglement entropy

Look back on the geometry for black hole, if we evolve time forward for both, consider a uniform evolution, t_L=t_R=t. Recall the Bekenstein-Hawking formula

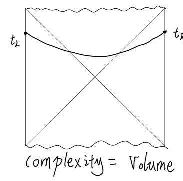
$$S(A) = \frac{m_{hor}}{4G_N}$$

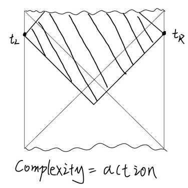
has implied that entropy is proportional to the minimal surface on time slice that separates A and A^c. The analog of the through-the-wormhole HRT surface now depends on the time t, linearly growing with t. Because its area includes a factor that measures the length of the wormhole and this length is growing linearly with time t.

Initially, the entropy of black hole will grow linearly with time as we predicted, until the point where it saturates at its thermal equilibrium value. The saturation of entanglement arises in essence because of the unbounded growth of the interior. The state has finite entropy while the interior keeps growing long after the saturation. Here comes the puzzle: " What is dual to this seemingly unbounded growth of interior?

Leonard has gave an answer to this problem by building a tensor network of Hartman and Maldacena, is that the growth of the interior is dual to the growth in computational complexity of the microscopic state.

3.4 Two points of views, CV and CA





3.4.1 CV

The view complexity=volume evaluates proper volume of extremal Co-dim 1 surface connecting Cauchy surface in boundary theory.(Point hold by Susskind, Ryu, Takiyanagi, Hubeny, Headrick?)

$$C_V(\Sigma) = max_{(\Sigma=\partial B)}[\frac{V(B)}{G_N l}]$$

3.4.2 CA

The point of view complexity=action evaluates gravitational action for Wheeler-Dewitt patch=domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT. (Point hold by Brown, Robert, Swingle and Susskind)

$$C_A(\Sigma) = \frac{I_{WDW}}{\pi\hbar}$$

3.4.3 CV 2.0

Evaluate spacetime volume of WDW patch.(Point hold by Couch, Fischler and Nguyen)

$$C'_V = \frac{V_{WDW}}{G_N l^2}$$

4 Holographic calculation

4.1 Calculation for neutral black holes

4.2 Calculation for charged black holes