# From EFE to Sigularity

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## 1 Einstein Field Equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

- $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, is called Einstein Tensor, represents the curvature of spacetime due to matter.
- $\Lambda$ is called cosmological constant, and acts as a force that can accelerate the expansion of the universe
- $\kappa = \frac{8\pi G_N}{c^4}$, is a coefficient, and $T_{\mu\nu}$ is the Einstein stress tensor

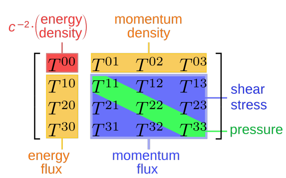


Figure 1: Stress tensor

- And our goal is to solve for $g_{\mu\nu}$, the metric to describe spacetime satisfies assumptions.

## 2 What metric to describe Black Hole

**A black hole, should be black**

Topologically, all light rays orthogonal to the trapped surface converge when traced toward the future. This is contrary to a spherical surface in flat space, where outward-directed light-rays diverge. So we come to a few conditions that Schwarzchild metric need to satisfy:

- It needs to be spherically symmetric
- Non-rotating, for simplicity

- Start from vacuum, $G_{\mu\nu} = 0$

To describe in spherical coordinates, general static spherically symmetric metric in Schwarzschild coordinates can be written as:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Substituting this ansatz into the Einstein field equations and solving for $\alpha(r)$ and $\beta(r)$, we find:

$$e^{2\beta(r)} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}, e^{2\alpha(r)} = 1 - \frac{2GM}{c^2 r}$$

Where $M$ is the mass of the central object, the Schwarzschild metric is thus obtained:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

**Schwarzchild in Eddington–Finkelstein coordinates**
We do a substitution of variable, introduce the advanced spacetime variable $v$

$$dv = dt + \frac{dr}{1 - \frac{2GM}{c^2 r}}$$

And then rewrite the Schwarzchild metric in a new form involving the singularity at $r = \frac{2GM}{c^2}$ (the Schwarzschild radius).

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

This form is the original form that appeared in R.Penrose's paper(1965), and we will introduce how it works in visualization of black hole.

# 3 Visualization of Black Hole

## 3.1 Conformal Structure

Conformal structure is a technique R.Penrose introduced in his 1963 paper, such a transformation keep angles retained but scale changed. The infinity is treated as 3-dimensional boundary $\mathfrak{g}$ to a finite 4-dimensional conformal region $\mathcal{M}$.

Accordingly, we assign a new unphysical metric$g_{\mu\nu}$ to spacetime, which is conformal to original physical metric $\tilde{g_{\mu\nu}}$

$$g_{\mu\nu} = \Omega^2 \tilde{g_{\mu\nu}}$$

In this diagram, three "points" $I^0, I^+, I^-$ represent the spatial infinity, future infinity, and past infinity. The boundary, two null hyper-surface $\mathfrak{g}^+, \mathfrak{g}^-$ representing the future and past null infinities, topologically a 3-dimensional cylinders.

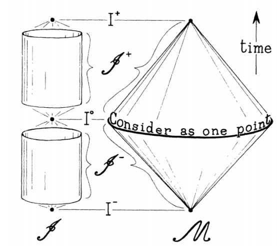


Figure 2: Conformal Structure

## 3.2 Singularity Theorem

### 3.2.1 Raychaudhuri and Komar singularity theorems

The first Singularity theorem has been derived by from eqations:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^{\mu}U^{\nu}$$

In which the expansion $\theta = \nabla_{\mu}U^{\mu}$, Shear $\sigma_{\mu\nu} = \nabla_{(\mu}U_{\nu)} - \frac{1}{3}\theta h_{\mu\nu} - \omega_{\mu\nu}$, rotation $\omega_{\mu\nu} = \nabla_{[\mu}U_{\nu]}$ and timelike convergence condition: $R_{\mu\nu}U^{\mu}U^{\nu} \geq 0$ defined respectively.
Indicates a perfect fluid whose velocity vector field is geodesic and irrotational. If the expansion is negative at an instant of time and convergence holds, then the energy density $\rho$ of the fluid diverges in the finite future along every integral curve of velocity vector field.
Translate: The expansion scalar in the Raychaudhuri equations becomes negative, implying a convergence of geodesics. This convergence suggests the formation of a singularity where densities and curvature can become infinite.

### 3.2.2 Penrose singularity theorems

If the space-time contains a non-compact Cauchy hypersurface $\Sigma$ and a closed future-trapped surface, and if the convergence condition $R_{\mu\nu}U^{\mu}U^{\nu} \geq 0$ holds for null fluid velocity vector field, then there are future incomplete null geodesics.(Spacetime contains singularity)

## 3.3 Singularity Visualization

**What is singularity?**
Consider a spherically symmetrical matter distribution of finite radius in $C^3$ which collapses symmetrically. When sufficient thermal energy has been radiated away, the body contracts and continues to contract until a physical singularity is encountered at r = 0 and forms an infinitely dense point called "singularity"

**Back to Schwarzchild metric**

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

In which the advanced spacetime variable $dv = dt + \frac{dr}{1-\frac{2GM}{c^2 r}}$ Schwarzschild met-

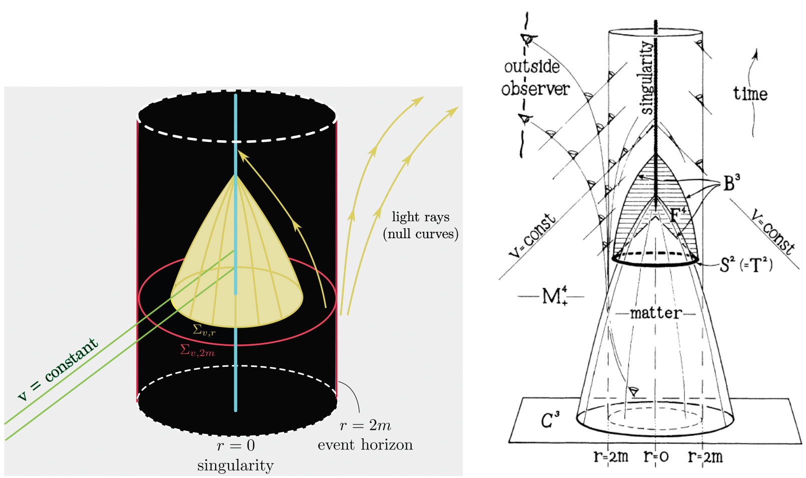


Figure 3: Collapsing Star

ric in the ingoing Eddington-Finkelstein coordinates (one space dimension suppressed)
Recall the Schwarzchild radius $r = \frac{2GM}{c^2}$, define such a 2-sphere with S-radius. The matter inside such a sphere contracts to singularity and is invisible(takes infinity time because light rays are bent and asymptotic to singularity) to the outside observer. This sphere is an example of a trapped surface. A point to

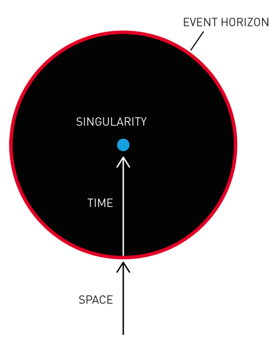


Figure 4: Cross section of event horizon

notice here is inside the horizon, direction of time axis pointing radially inward.

This could be observed from Schwarzchild metric:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

As $r < 2M$, $(1 - \frac{2GM}{c^2 r}) < 0$, we could either think the sign changed to swap the term or add an $i$ to rotate the unit vector by $\pi/2$ **No escape from black hole**

## 3.4 Penrose Diagram

As stated in section 3.1, in conformal structure we can draw infinity in a finite diagram. So the asymptotic problem in star collapsing could also be conclude into a Penrose diagram.

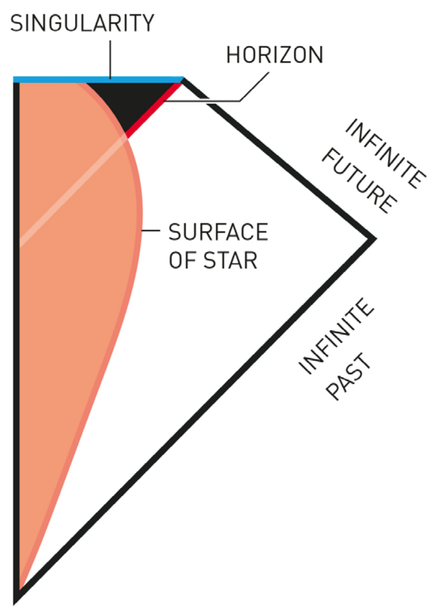


Figure 5: Collapsing Star in Penrose Diagram

## 3.5 Hawking Bekenstein Entropy

Take a spherically symmetric and stationary or we say "Schwarzschild black hole", we have black hole's mass $M$, horizon's radius $r_h = \frac{2GM}{c^2}$, area count as a circle $4\pi {r_h}^2$, thus we could easily get the area of black hole as

$$A = 16\pi(GM/c^2)^2$$

The event horizon area of a black hole cannot decrease; it increases in most transformations of the black hole. This increasing behavior is reminiscent of thermodynamic entropy of closed systems. Thus it is reasonable that the black hole entropy should be a monotonic function of area, and it turns out to be simplest such function.

$$S_{BH} = \frac{A}{4G_N} = \frac{A}{4{l_p}^2}$$

As $G_N$ stands for Newtonian gravity, $l_p$ is the plank length, and $\hbar$ is the Planck-Dirac constant(Reduced Planck constant), they have relationship as $l_p = \sqrt{\frac{G\hbar}{c^3}}$. So be more precisely, the Bekenstein-Hawking entropy should be written as:

$$S_{BH} = \frac{A}{4{l_p}^2} = \frac{c^3 A}{4G\hbar}$$