Einstein Field Equations and BEYOND

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Contents

1 Einstein field equations

1.1 Stress Energy tensor

The stress-energy tensor is associated with the conservation of energy and momentum. The component T^{00} represents $c^{-2}*$ energy density, the column T^{i0} represents energy flux, row T^{0i} represents momentum density. In the matrix T^{ij}, diagonal components are pressure while the upper half is shear stress and lower half is momentum flux. Due to $T^{\mu\nu} = T^{\nu\mu}$, the tensor is symmetric across its main diagonal.

As a rank-2 tensor, we may visualize its components in a matrix-like array:

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{0i} \\ T^{i0} & T^{ij} \end{pmatrix} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}$$

For (3+1)-dimensions pure electrodynamics, the stress tensor is

$$T^{\mu\mu} = \frac{1}{\mu_0}(F^{\mu}_{\rho}F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\lambda}F^{\rho\lambda})$$

. In the equation $F^{\mu\nu}$ is a rank-2 Lorentz tensor that related to Lorentz vector force in electrodynamics. $F^{\mu} = qu_{\nu}F^{\mu\nu}$, while the Lorentz vector force $F^{\mu} = \gamma(|\vec{v}|)q(\vec{E} + v \times \vec{B})$. Under pure electrodynamics, we can show the tensor is conserved: $\partial_{\mu}T^{\mu\nu} = 0$ and the trace of stress tensor is zero:

$$T^{\mu}_{\nu} = \eta_{\mu\nu}T^{\mu\nu} = \frac{1}{\mu_0}(\eta_{\mu\nu}F^{\mu}_{\rho}F^{\nu\rho} - \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F_{\rho\lambda}F^{\rho\lambda}) =$$

$$\frac{1}{\mu_0}(F_{\nu\rho}F^{\nu\rho} - \frac{1}{4}\delta^{\mu}_{\mu}F_{\rho\lambda}F^{\rho\lambda}) = \frac{1}{\mu_0}(1 - \frac{d}{4})F_{\mu\nu}F^{\mu\nu}$$

Where $\delta^{\mu}_{\mu} = d$ in (3+1)-dimensions, d=4. Hence $T^{\mu}_{\mu} = 0$

1.2 Deduction of field equation

Einstein's equations, are known as "field equations" since they determine the gravitational field by formalizing the intuition "curvature=energy". BY intuition, the Einstein equation should be covariant as a tensor equation and has two sides describing curvature and energy. For the curvature side. Knowing Einstein's equations is solving for the metric $g_{\mu\nu}$ itself is a rank-2 tensor, so the tensor equation should also be a rank-2 equation for both sides. The simplest rank-2 curvature covariant expression is $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, since

$$\nabla^{\mu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = \nabla^{\mu}R_{\mu\nu} - \frac{1}{2}\nabla^{\mu}(g_{\mu\nu}R) = \frac{1}{2}\nabla_{\nu}R - \frac{1}{2}g_{\mu\nu}\nabla^{\mu}R = 0$$

That's the left-hand side of Einstein equations, and correspondingly the right side should be a coefficient time the stress-energy tensor, written as $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}$

To determine this coefficient κ^2, we need to matcht he equation with the non-relativistic weak-gravitational limit, and check the condition we derived match Newton's Gravitational euation $\nabla^2\Phi = 4\pi G_N \rho$. By taking the non-relativistic limit for stress-energy tensor, the only non-zero component is $T^{00} \approx c^2\rho$. Also for the LHS, under nonrelativistic, weak gravity condition, the metric $g_{\mu\nu}$ is also differ from $\eta_{\mu\nu}$ a little variation. In particular, the component relavant to Newton's gravitational potential is $g_{00} \approx -1 - \frac{2\Phi}{c^2}$To gain the relavant Ricci tensor, we are looking the dominant $\mu = \nu = 0$ to match up both sides of the equation:

$$R_{00} = \partial_\mu \Gamma^\mu_{00} - \partial_0 \Gamma^\mu_{\mu 0} + \Gamma^\mu_{\mu\nu}\Gamma^\nu_{00} - \Gamma^\mu_{0\nu}\Gamma^\nu_{\mu 0}$$
$$\approx \partial_i \Gamma^i_{00} - \Gamma^0_{0i}\Gamma^i_{00}$$

With the limitation of $\mu = \nu = 0$, the only components of Levi-Civita connection has two temporal indices and one spacial index, those are $\Gamma^i_{00} \approx \frac{1}{c^2}\nabla\Phi$ and $\Gamma^0_{0i} \approx \frac{1}{c^2}\nabla\Phi$, it directly indicates production of two Levi-Civita connection are negligible compare to its partial derivative due to square of the coefficient $\frac{1}{c^2}$ included in the connection. By further inspection:

$$R_{00} \approx \partial_i \Gamma^i_{00} - \Gamma^0_{0i}\Gamma^i_{00}$$

$$\approx \partial_i \Gamma^i_{00} \approx \frac{1}{c^2}\nabla^2\Phi$$

Then we get hands on the spatial components of LHS, the strategy is to solve for spatial components of Ricci tensor by equating both sides of the equation and utilizing spatial components of stress tensor are zero to get all diagonal components of Ricci tensor:

$$R_{ij} - \frac{1}{2}g_{ij}R \approx R_{ij} - \frac{1}{2}\delta_{ij}R \approx 0$$

Which indicates$R_{11} \approx R_{22} \approx R_{33} \approx \frac{1}{2}R$,

$$R = g^{\mu\nu}R_{\mu\nu} \approx -R_{00} + R_{11} + R_{22} + R_{33} = \frac{3}{2}R - R_{00}$$

And solve for $R \approx 2R_{00} \approx \frac{2}{c^2}\nabla^2\Phi$. With all components solved, we put them back to the simple form of field equation: $R_{00} - \frac{1}{2}g_{00}R = \kappa^2 T_{00}$

$$\frac{2}{c^2}\nabla^2\Phi \approx \kappa^2 c^2 \rho$$

We also have the condition from Newtonian equation $\nabla^2\Phi = 4\pi G_N \rho$, write them altogether, we get $\kappa^2 = \frac{8\pi G_N}{c^4}$. Finally, full Einstein Equations are written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G_N}{c^4}T_{\mu\nu}$$

1.3 More about EFE

Start from the tensor equation: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G_N}{c^4}T_{\mu\nu}$, the tensor equation actually has 10 very complicated equations since the tensors are symemtric, encoded all the effects of gravity of our universe in general relativity. These equations are coupled PDEs solving for the metric $g_{\mu\nu}$, which are highly non-linear.

Another form of the equations might be more famous and widely used as: $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$. It is since by covariantising local form of Newton's gravitation law, Einstein realized the metric $g_{\mu\nu}$ is symmetric and divergence-free($g_{\mu\nu,\lambda} = 0$), so the equation itself could even add a term linear to metric $g_{\mu\nu}$, so he added the term metric times cosmological constant to allow for a universe that is not expanding or contracting. In EFE with cosmological constant: $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is called Einstein tensor. Λ is then called cosmological constant and coefficient κ is still equals to $\frac{8\pi G_N}{c^4}$.

1.4 Solutions to EFE and their applications

1.4.1 Minkowski Space

the simplest solution and describes a flat, empty space with no gravitational fields (i.e., the vacuum solution for regions of spacetime without any mass-energy).
The Minkowski solution corresponds to the special case where the stress-energy tensor $T_{\mu\nu}$ is zero everywhere, which implies that there is no matter or non-gravitational energy present to curve spacetime. The cosmological constant Λ is also assumed to be zero in this case. Under these conditions, the Einstein field equations reduce to:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$$

When $T_{\mu\nu} = 0$ and $\Lambda = 0$, this simplifies to the vacuum Einstein equations:

$$R_{\mu\nu} = 0$$

The Minkowski metric, which is used in this spacetime, is given by:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

This metric describes the geometry of spacetime in special relativity and serves as the backdrop for all other solutions. When general relativistic effects due to gravity are negligible, spacetime can be approximated by Minkowski spacetime.

1.4.2 Schwarzchild space

We begin with the Einstein field equations in the absence of matter (the vacuum equations):

$$G_{\mu\nu} = 0$$

where $G_{\mu\nu}$ is the Einstein tensor. For a spherically symmetric, non-rotating mass, the Schwarzschild metric is a solution to these equations. The spacetime is assumed to be static and spherically symmetric.

A general static, spherically symmetric metric in Schwarzschild coordinates (t, r, θ, ϕ) can be written as:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

where $e^{2\alpha(r)}$ and $e^{2\beta(r)}$ are unknown functions of the radial coordinate r.

Substituting this ansatz into the Einstein field equations and solving for $\alpha(r)$ and $\beta(r)$, we find:

$$e^{2\beta(r)} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$
$$e^{2\alpha(r)} = 1 - \frac{2GM}{c^2 r}$$

Where M is the mass of the central object, the Schwarzschild metric is thus obtained:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Start from the standard Schwarzschild metric in Schwarzschild coordinates:
You then introduce the advanced time coordinate v which is defined by the relation:

$$dv = dt + \frac{dr}{1 - \frac{2GM}{c^2 r}} \tag{1}$$

This effectively "absorbs" the singularity at $r = \frac{2GM}{c^2}$ (the Schwarzschild radius). When you transform the metric using this new coordinate, you get:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

1.4.3 Kerr Newman space

The Kerr-Newman metric is a solution to the Einstein field equations in general relativity that describes the spacetime in the region surrounding a charged, rotating mass. It generalizes the Kerr metric by including charge and the Reissner-Nordström metric by including angular momentum.
Due to its complexity, a full derivation of the Kerr-Newman solution is quite lengthy and technical, and involves advanced methods of solving partial differential equations. However, an outline of the steps typically involved in deriving this solution is given below:

Assumptions and Ansatz: One begins with assumptions about the symmetries of the spacetime in question, such as stationarity and axisymmetry.

With these symmetries in mind, a general form of the metric is proposed (the ansatz), which contains several undetermined functions of the coordinates.

Complexification and Newman-Penrose Formalism: The Newman-Penrose formalism, which uses a set of null vectors to simplify the equations, is often employed. Furthermore, the complexification of spacetime coordinates can simplify calculations significantly.

Solving the Field Equations: The Einstein field equations in vacuum are given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

where $T_{\mu\nu}$ is given by the electromagnetic stress-energy tensor of a point charge and Λ is the cosmological constant, typically set to zero.

Electromagnetic Field: Maxwell's equations in curved spacetime are solved in conjunction with the Einstein field equations to obtain the electromagnetic field associated with the charge of the rotating body.

Integrating the Equations: The resulting partial differential equations are integrated to find the functions in the metric ansatz, depending on the mass M, charge Q, and angular momentum per unit mass a of the rotating body.

Kerr-Newman Metric: The final form of the Kerr-Newman metric in Boyer-Lindquist coordinates (t, r, θ, ϕ) is:

$$ds^2 = -\left(1 - \frac{2Mr - Q^2}{\rho^2}\right) dt^2 - \frac{2a\sin^2\theta(2Mr - Q^2)}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 +$$

$$\left(r^2 + a^2 + \frac{2Mr - Q^2}{\rho^2} a^2 \sin^2\theta\right) \sin^2\theta d\phi^2$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta$$

and

$$\Delta = r^2 - 2Mr + a^2 + Q^2$$

This metric is a central solution in general relativity and is used to model rotating black holes, including their event horizons, ergospheres, and the singularity structure.

1.4.4 de-Sitter Space

1.4.5 Anti de-Sitter Space

1.5 AdS_n & dS_n

Mathematically, de-Sitter and Anti-de-Sitter spaces are defined as **maximally symmetric Lorentzian manifolds with constant scalar curvature.** Formally AdS_n spacetime is denoted as an empty space solution to the Einstein field equations with a negative cosmological constant. While the dS spacetime is with a positive cosmological constant. For detailed derivation, please refer to the section **3.2 Simple solution to Einstein equation**

2 Conformal field theory

2.1 Conformal Transformation

2.2 Conformal Coordinates

2.3 Infinitesimal transformation

2.4 Finite conformal transformations

2.4.1 Line segment and metric

3 Gravitational Collapse and Spacetime Singularity

4 Anti-de Sitter Spacetime

4.1 Geometry of Anti de Sitter

4.2 Essential concepts

1.The Riemann curvature tensor is defined in terms of the metric tensor to express curvature of Riemann manifold, which encodes the geometry of the manifold. Consider a smooth manifold with a metric tensor g that assigns a scalar product to each tangent space at every point. The Riemann curvature tensor R is a four-index tensor that characterizes the curvature of the manifold. Its components are given by:

$$\boxed{R^{\mu}_{\nu\rho\lambda} = \partial_\rho \Gamma^{\mu}_{\nu\lambda} - \partial_\lambda \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\rho\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\rho}}$$

2. The Ricci tensor is a symmetric second-order tensor that arises in the study of curved manifolds in differential geometry. A very straightforward way of defining Ricci curvature is to measure the metrix at a very close point$x^\mu + v^\mu$ near the point x^μ

$$\boxed{R_{\mu\nu} = \partial_\rho \Gamma^{\rho}{}_{\mu\nu} - \partial_\mu \Gamma^{\rho}{}_{\rho\nu} + \Gamma^{\rho}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\rho\nu}}$$

3. Scalar curvature is defined to compare the volume of ball $B_{\mathbb{R}^d}(\varepsilon)$ with ball Of same radius ϵ in the Euclidean space. The difference between two volumes defines the scalar curvature:

$$\frac{\operatorname{Vol}\left[B_{\mathcal{M}}(x,\varepsilon)\right]}{\operatorname{Vol}\left[B_{\mathbb{R}^d}(\varepsilon)\right]} = 1 - \frac{R(x)}{6(d+2)}\varepsilon^2 + \mathcal{O}\left(\varepsilon^4\right)$$

$$\boxed{R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}}$$

4. Einstein tensor

4.3 Global Coordinates

4.3.1 Line segment and metric

4.3.2 Riemann curvature tensor

4.3.3 Ricci tensor and scalar curvature

4.3.4 Einstein tensor

4.3.5 Geodesic

4.4 Poincare Coordinates

4.4.1 Line segment and metric

4.4.2 Riemann curvature tensor

4.4.3 Ricci tensor and scalar curvature

4.4.4 Einstein tensor

4.4.5 Geodesic

4.5 Static Coordinates

4.5.1 Line segment and metric

4.5.2 Riemann curvature tensor

4.5.3 Ricci tensor and scalar curvature

4.5.4 Einstein tensor

4.5.5 Geodesic

4.5.6 Riemann curvature tensor

4.5.7 Ricci tensor and scalar curvature

4.5.8 Einstein tensor

4.5.9 Geodesic

4.6 N-dimensional Anti-de Sitter

4.7 From Geodesic to bulk-boundary correspondence

5 Maximally Symmetry

5.1 Isometries of the Metric

Under coordinate transformation $x \mapsto x'$, the metric is form-invariant, which means $g'_{\mu\nu}(x') = g_{\mu\nu}(x')$, the function itself remains the same, then the transformation of metric written as

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x')$$

To make the metric invariant, we want the coordinates transformation$x \mapsto x'$ to be isometric. Restricted to an infinitesimal isometric transformation,

$$x^{\mu} \mapsto x'^{\mu} = x^{\mu} + \epsilon \xi^{\mu}$$

Under this infinitesimal coordinate transformation, we could expand metric$g_{\mu}\nu$in Taylor series:

$$g_{\rho\sigma} = g_{\rho\sigma} + \epsilon \xi^{\alpha} \frac{\partial g_{\rho\sigma(x)}}{\partial x^{\alpha}} + \mathcal{O}(\epsilon^2)$$

And the product of partial derivatives becomes

$$\frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x') = \left(\frac{\partial x^{\rho}}{\partial x^{\mu}} + \epsilon \frac{\partial \xi^{\rho}}{\partial x^{\mu}} \right) \left(\frac{\partial x^{\sigma}}{\partial x^{\nu}} + \epsilon \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} \right) = \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} + \epsilon \delta^{\rho}_{\mu} \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} + \epsilon \delta^{\sigma}_{\nu} \frac{\partial \xi^{\rho}}{\partial x^{\mu}} + \mathcal{O}\left(\epsilon^2\right)$$

Plug these two parts into the equation for metric transformation, we get

$$\begin{aligned} g_{\mu\nu}(x) &= \left(\delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} + \epsilon \delta^{\rho}_{\mu} \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} + \epsilon \delta^{\sigma}_{\nu} \frac{\partial \xi^{\rho}}{\partial x^{\mu}} \right) \left(g_{\rho\sigma}(x) + \epsilon \xi^{\alpha} \frac{\partial g_{\rho\sigma}(x)}{\partial x^{\alpha}} \right) \\ &= \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} g_{\rho\sigma}(x) + \epsilon \delta^{\rho}_{\mu} \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x) + \epsilon \delta^{\sigma}_{\nu} \frac{\partial \xi^{\rho}}{\partial x^{\mu}} g_{\rho\sigma}(x) + \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} \epsilon \xi^{\alpha} \frac{\partial g_{\rho\sigma}(x)}{\partial x^{\alpha}} \\ &= g_{\mu\nu}(x) + \epsilon \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} g_{\mu\sigma}(x) + \epsilon \frac{\partial \xi^{\rho}}{\partial x^{\mu}} g_{\rho\nu}(x) + \epsilon \xi^{\alpha} \frac{\partial g_{\mu\nu}(x)}{\partial x^{\alpha}} \end{aligned}$$

Thus, by expanding the transformation of metric under an infinitesimal isometry, is equivalent to following condition about the first order of ϵ:

$$\frac{\partial \xi^{\sigma}}{\partial x^{\nu}} g_{\mu\sigma}(x) + \frac{\partial \xi^{\rho}}{\partial x^{\mu}} g_{\rho\nu}(x) + \xi^{\alpha} \frac{\partial g_{\mu\nu}(x)}{\partial x^{\alpha}} = 0$$

Notice that: $\frac{\partial}{\partial x^{\nu}} \left(\xi^{\sigma} g_{\mu\sigma} \right) = \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} g_{\mu\sigma} + \xi^{\sigma} \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}}$ And then we rewrite first order ϵ equivalence relationship as

$$\frac{\partial}{\partial x^{\nu}} \left(\xi^{\sigma} g_{\mu\sigma} \right) - \xi^{\sigma} \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial}{\partial x^{\mu}} \left(\xi^{\rho} g_{\rho\nu} \right) - \xi^{\rho} \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} + \xi^{\alpha} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} = 0$$

Substitute $\xi^{\rho} g_{\rho\nu} = \xi^{\nu}$, equation above becomes:

$$\partial_{\nu} \xi_{\mu} + \partial_{\mu} \xi_{\nu} + \xi^{\alpha} (\partial_{\alpha} g_{\mu\nu} - \partial_{\nu} g_{\mu\alpha} - \partial_{\mu} g_{\alpha\nu})$$

Then, Christoffel symbol $\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} (\partial_{\nu} g_{\rho\lambda} + \partial_{\rho} g \lambda \nu - \partial \lambda g_{\nu\rho})$:

$$\partial_{\nu} \xi_{\mu} + \partial_{\mu} \xi_{\nu} - 2\Gamma^{\lambda}_{\mu\nu} = (\partial_{\nu} \xi_{\mu} - \Gamma^{\lambda}_{\mu\nu}) + (\partial_{\mu} \xi_{\nu} - \Gamma^{\lambda}_{\mu\nu})$$

write in term of covariant derivative$\nabla_{\mu} \omega_{\nu} = \partial_{\mu} \omega_{\nu} - \Gamma^{\rho}_{\mu\nu} \omega_{\rho}$, and lower index of Christoffel symbol(g_{ij} is a symmetric matrix) is symmetric, we get: $\nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu} = 0$

5.2 The Killing vector

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

This equation that we have reached is named Killing equation, and the corresponding solution vector is called as the Killing vector fields. Which is usually used to reduce the problem of finding all symmetries of a particular metric down to simply finding the corresponding Killing vectors.

$$\xi_{\mu;\nu;\rho} - \xi_{\mu;\rho;\nu} = -R^\lambda_{\mu\nu\rho}\xi_\lambda$$

Combining this with the first Bianchi identity of the Riemann curvature tensor:

$$\begin{aligned}
&R^\lambda_{\mu\nu\rho} + R^\lambda_{\nu\rho\mu} + R^\lambda_{\rho\mu\nu} = 0 \\
\Rightarrow &R^\lambda_{\mu\nu\rho}\xi_\lambda + R^\lambda_{\nu\rho\mu}\xi_\lambda + R^\lambda_{\rho\mu\nu}\xi_\lambda = 0 \\
\Rightarrow &\xi_{\mu;\nu;\rho} - \xi_{\mu;\rho;\nu} + \xi_{\nu;\rho;\mu} - \xi_{\nu;\mu;\rho} + \xi_{\rho;\mu;\nu} - \xi_{\rho;\nu;\mu} = 0
\end{aligned}$$

Now we want to group the above Killing vector derivatives by the second derivative:

$$(\xi_{\mu;\nu;\rho} - \xi_{\nu;\mu;\rho}) + (\xi_{\nu;\rho;\mu} - \xi_{\rho;\nu;\mu}) + (\xi_{\rho;\mu;\nu} - \xi_{\mu;\rho;\nu}) = 0$$

If we take the Killing equation and take a second covariant derivative:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \Rightarrow \xi_{\mu;\nu;\rho} + \xi_{\nu;\mu;\rho} = 0$$

we can substitute these modified Killing equations into equation above, yields:

$$\xi_{\mu;\nu;\rho} - \xi_{\mu;\rho;\nu} = \xi_{\rho;\nu;\mu}$$

Use this identity for the equation above that includes Riemann tensor:

$$\xi_{\rho;\nu;\mu} = -R^\lambda_{\mu\nu\rho}\xi_\lambda$$

Thus proved the second-order derivative of killing vector depends on itself, with this property, we can construct higher-order covariant derivatives in terms of the Killing vector itself and its first-order covariant derivative, at that same point X. So, any Killing vector can be written as the Taylor expansion:

$$\xi_\mu(x) = A^\lambda_\mu(x;X)\xi_\lambda(X) + B^{\lambda\nu}_\mu(x;X)\xi_{\lambda;\nu}(X)$$

6 Black Hole entropy

6.1 Violation of the Second Law

6.2 The Bekenstein and 't Hooft Propositions

7 Black Hole Entropy in Anti-deSitter

7.1 Entanglement Entropy in Conformal Field Theory

8 Bulk-Boundary correspondence proposal

8.1 Conformal Transformations

8.2 RT Proposal and the AdS3/CFT2 Correspondence

Appendix A Computation with machine

A.1 Deep learning in AdS/CFT correspondence

A.2 Holographic error correcting codes

Appendix B Penrose Diagrams