Anti-de Sitter space and beyonds

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May 2023

1 Maximally symmetric

1.1 Killing vector

First, we consider a metric-preserving transformation $g'_{\mu\nu}(x') = g_{\mu\nu}(x')$, so that the form of metric is invariant under transformation, the function itself remains the same, then the transformation of metric written as

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x')$$

To make the metric invariant, we want the coordinates transformation$x \mapsto x'$ to be isometric. Restricted to an infinitesimal isometric transformation,

$$x^{\mu} \mapsto x'^{\mu} = x^{\mu} + \epsilon \xi^{\mu}$$

Under this infinitesimal coordinate transformation, we could expand metric$g_{\mu}\nu$in Taylor series:

$$g_{\rho\sigma} = g_{\rho\sigma} + \epsilon \xi^{\alpha} \frac{\partial g_{\rho\sigma(x)}}{\partial x^{\alpha}} + \mathcal{O}(\epsilon^2)$$

And the product of partial derivatives becomes

$$\frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x') = \left(\frac{\partial x^{\rho}}{\partial x^{\mu}} + \epsilon \frac{\partial \xi^{\rho}}{\partial x^{\mu}} \right) \left(\frac{\partial x^{\sigma}}{\partial x^{\nu}} + \epsilon \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} \right) = \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} + \epsilon \delta^{\rho}_{\mu} \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} + \epsilon \delta^{\sigma}_{\nu} \frac{\partial \xi^{\rho}}{\partial x^{\mu}} + \mathcal{O}\left(\epsilon^2\right)$$

Plug these two parts into the equation for metric transformation, we get

$$\begin{aligned} g_{\mu\nu}(x) &= \left(\delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} + \epsilon \delta^{\rho}_{\mu} \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} + \epsilon \delta^{\sigma}_{\nu} \frac{\partial \xi^{\rho}}{\partial x^{\mu}} \right) \left(g_{\rho\sigma}(x) + \epsilon \xi^{\alpha} \frac{\partial g_{\rho\sigma}(x)}{\partial x^{\alpha}} \right) \\ &= \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} g_{\rho\sigma}(x) + \epsilon \delta^{\rho}_{\mu} \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x) + \epsilon \delta^{\sigma}_{\nu} \frac{\partial \xi^{\rho}}{\partial x^{\mu}} g_{\rho\sigma}(x) + \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} \epsilon \xi^{\alpha} \frac{\partial g_{\rho\sigma}(x)}{\partial x^{\alpha}} \\ &= g_{\mu\nu}(x) + \epsilon \frac{\partial \xi^{\sigma}}{\partial x^{\nu}} g_{\mu\sigma}(x) + \epsilon \frac{\partial \xi^{\rho}}{\partial x^{\mu}} g_{\rho\nu}(x) + \epsilon \xi^{\alpha} \frac{\partial g_{\mu\nu}(x)}{\partial x^{\alpha}} \end{aligned}$$

Thus, by expanding transformation of metric under an infinitesimal isometry, is equivalent to following condition about the first order of ϵ:

$$\frac{\partial \xi^{\sigma}}{\partial x^{\nu}} g_{\mu\sigma}(x) + \frac{\partial \xi^{\rho}}{\partial x^{\mu}} g_{\rho\nu}(x) + \xi^{\alpha} \frac{\partial g_{\mu\nu}(x)}{\partial x^{\alpha}} = 0$$

Notice that: $\frac{\partial}{\partial x^\nu}\left(\xi^\sigma g_{\mu\sigma}\right) = \frac{\partial \xi^\sigma}{\partial x^\nu} g_{\mu\sigma} + \xi^\sigma \frac{\partial g_{\mu\sigma}}{\partial x^\nu}$ And then we rewrite first order ϵ equivalence relationship as

$$\frac{\partial}{\partial x^\nu}\left(\xi^\sigma g_{\mu\sigma}\right) - \xi^\sigma \frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial}{\partial x^\mu}\left(\xi^\rho g_{\rho\nu}\right) - \xi^\rho \frac{\partial g_{\rho\nu}}{\partial x^\mu} + \xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = 0$$

Substitute $\xi^\rho g_{\rho\nu} = \xi^\nu$, equation above becomes:

$$\partial_\nu \xi_\mu + \partial_\mu \xi_\nu$$

1.2 AdS_n & dS_n

Mathematically, de-Sitter and Anti-de Sitter spaces are defined as **maximally symmetric Lorentzian manifolds with constant scalar curvature.** Formally AdS_n spacetime is denoted as an empty space solution to the Einstein field equations with a negative cosmological constant. While the dS spacetime is with a positive cosmological constant. For detailed derivation, please refer to the section **3.2 Simple solution to Einstein equation**

2 Conformal field theory

2.1 Conformal Transformation

2.2 Conformal Coordinates

2.3 Infinitesimal transformation

2.4 Finite conformal transformations

2.4.1 Line segment and metric

3 Anti-de Sitter Spacetime

3.1 Einstein Field Equation

3.2 Simple solutions to Einstein equation

3.2.1 de-Sitter Space

3.2.2 Anti de-Sitter Space

3.2.3 Minkowski Space

3.3 Essential concepts

1.The Riemann curvature tensor is defined in terms of the metric tensor to express curvature of Riemann manifold, which encodes the geometry of the manifold. Consider a smooth manifold with a metric tensor g that assigns a scalar product to each tangent space at every point. The Riemann curvature tensor R is a four-index tensor that characterizes the curvature of the manifold. Its components are given by:

$$\boxed{R^{\mu}_{\nu\rho\lambda} = \partial_{\rho}\Gamma^{\mu}_{\nu\lambda} - \partial_{\lambda}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\rho\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\rho}}$$

2. The Ricci tensor is a symmetric second-order tensor that arises in the study of curved manifolds in differential geometry. A very straightforward way of defining Ricci curvature is to measure the metrix at a very close point$x^{\mu} + v^{\mu}$ near the point x^{μ}

$$\boxed{R_{\mu\nu} = \partial_{\rho}\Gamma^{\rho}{}_{\mu\nu} - \partial_{\mu}\Gamma^{\rho}{}_{\rho\nu} + \Gamma^{\rho}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\rho\nu}}$$

3. Scalar curvature is defined to compare the volume of ball $B_{\mathbb{R}^d}(\varepsilon)$ with ball Of same radius ϵ in the Euclidean space. The difference between two volumes defines the scalar curvature:

$$\frac{\mathrm{Vol}\left[B_{\mathcal{M}}(x,\varepsilon)\right]}{\mathrm{Vol}\left[B_{\mathbb{R}^d}(\varepsilon)\right]} = 1 - \frac{R(x)}{6(d+2)}\varepsilon^2 + \mathcal{O}\left(\varepsilon^4\right)$$

$$\boxed{R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}}$$

4. Einstein tensor

Appendix A Computation with machine

A.1 Deep learning in AdS/CFT correspondence

A.2 Holographic error correcting codes

Appendix B Manifold

From GTM 51.

Appendix C Gauge/gravity duality

The duality states that there is an equivalence between a gravitational theory in a higher-dimensional spacetime and a quantum field theory living on the lower-dimensional boundary of that spacetime. More specifically, the duality relates a theory of gravity in Anti-de Sitter (AdS) spacetime, which has negative curvature, to a conformal field theory (CFT) living on the boundary of that spacetime.