Introduction to Reinforcement Learning

2025. 1st semester

단국대학교
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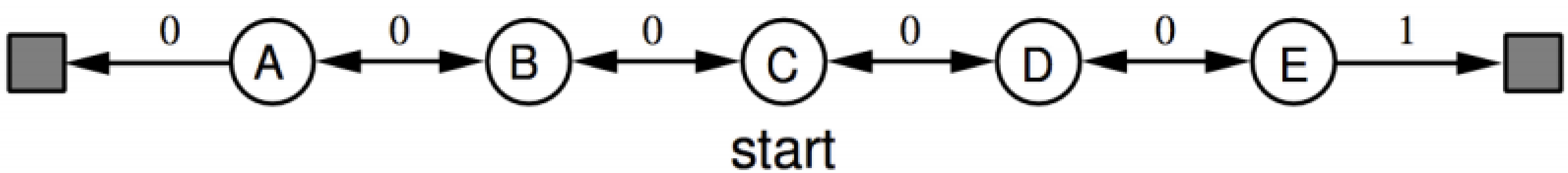
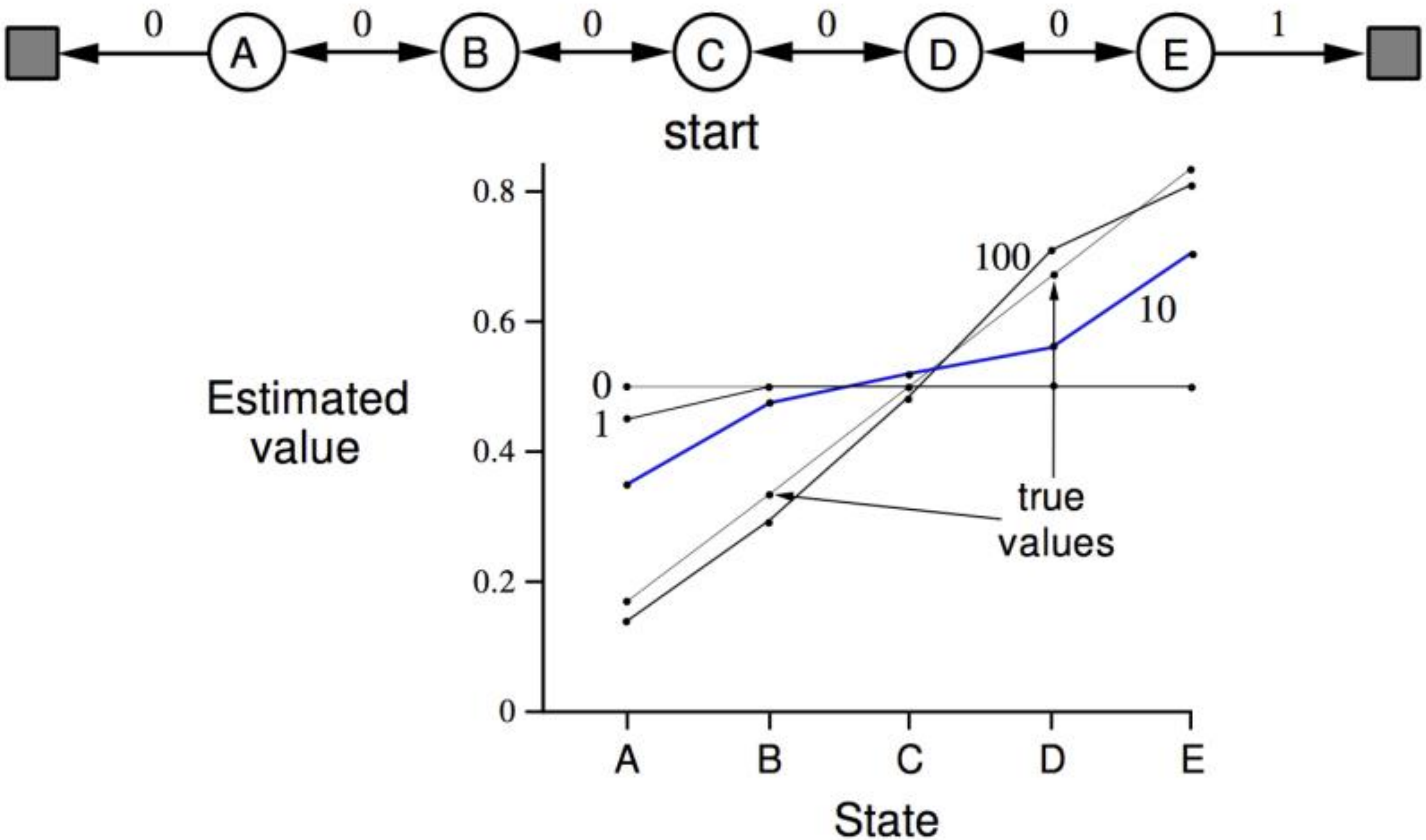
Mobile System Engineering

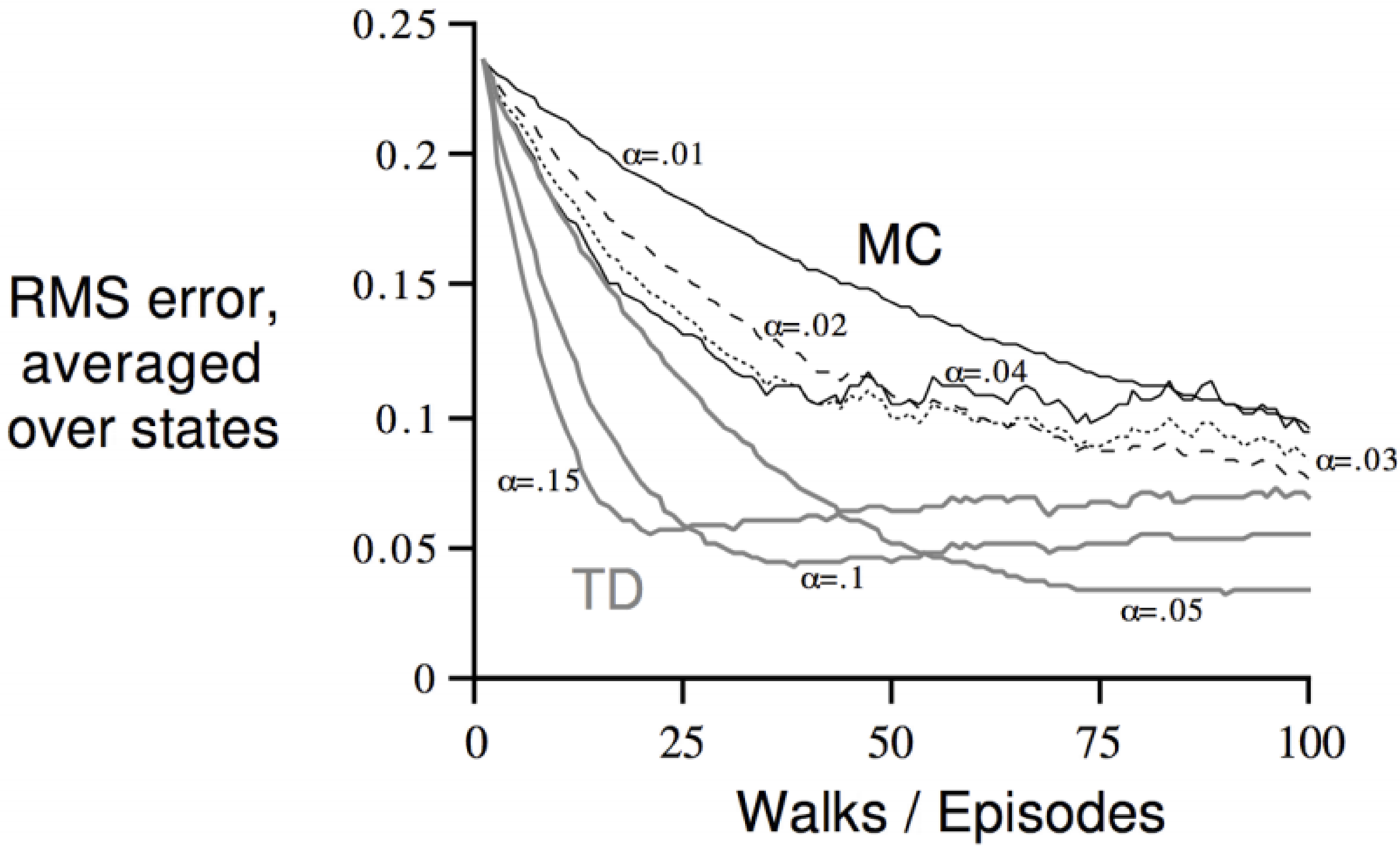
Random Walk Example



Random Walk Example



Random Walk: MC vs. TD



Batch MC and Batch TD(0)

- MC and TD converge: $V(s) \to v_\pi(s)$ as experience $\to \infty$
- But what about batch solution for finite experience?

$$s_1^1, a_1^1, r_2^1, ..., s_{T_1}^1$$
$$\vdots$$
$$s_1^K, a_1^K, r_2^K, ..., s_{T_K}^K$$

  - e.g. Repeatedly sample episode $k \in [1, K]$
  - Apply MC or TD(0) to episode k

AB example

Two states A, B; no discounting; 8 episodes of experience

$A, 0, B, 0$

$B, 1$

$B, 1$

$B, 1$

$B, 1$

$B, 1$

$B, 1$

$B, 0$

What is $V(A), V(B)$?

Certainty Equivalence

- MC converges to solution with minimum mean-squared error
  - Best fit to the observed returns

$$\sum_{k=1}^{K} \sum_{t=1}^{T_k} \left(G_t^k - V(s_t^k)\right)^2$$

- TD(0) converges to solution of max likelihood Markov model
  - Solution to the MDP $\langle \mathcal{S}, \mathcal{A}, \hat{\mathcal{P}}, \hat{\mathcal{R}}, \gamma \rangle$ that best fits the data

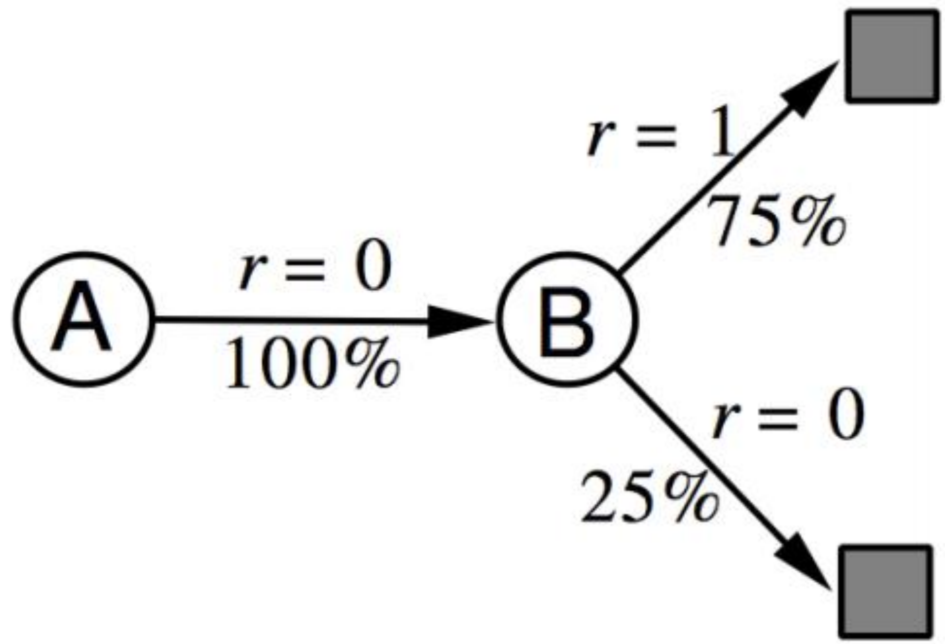
$$\hat{\mathcal{P}}_{s,s'}^{a} = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k, s_{t+1}^k = s, a, s')$$

$$\hat{\mathcal{R}}_{s}^{a} = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k = s, a) r_t^k$$

AB example

Two states A, B; no discounting; 8 episodes of experience

A, 0, B, 0
B, 1
B, 1
B, 1
B, 1
B, 1
B, 1
B, 0



What is $V(A), V(B)$?

Certainty Equivalence

- *In Batch MC*
  - $V(A) = 0$, $V(B) = 3/4$

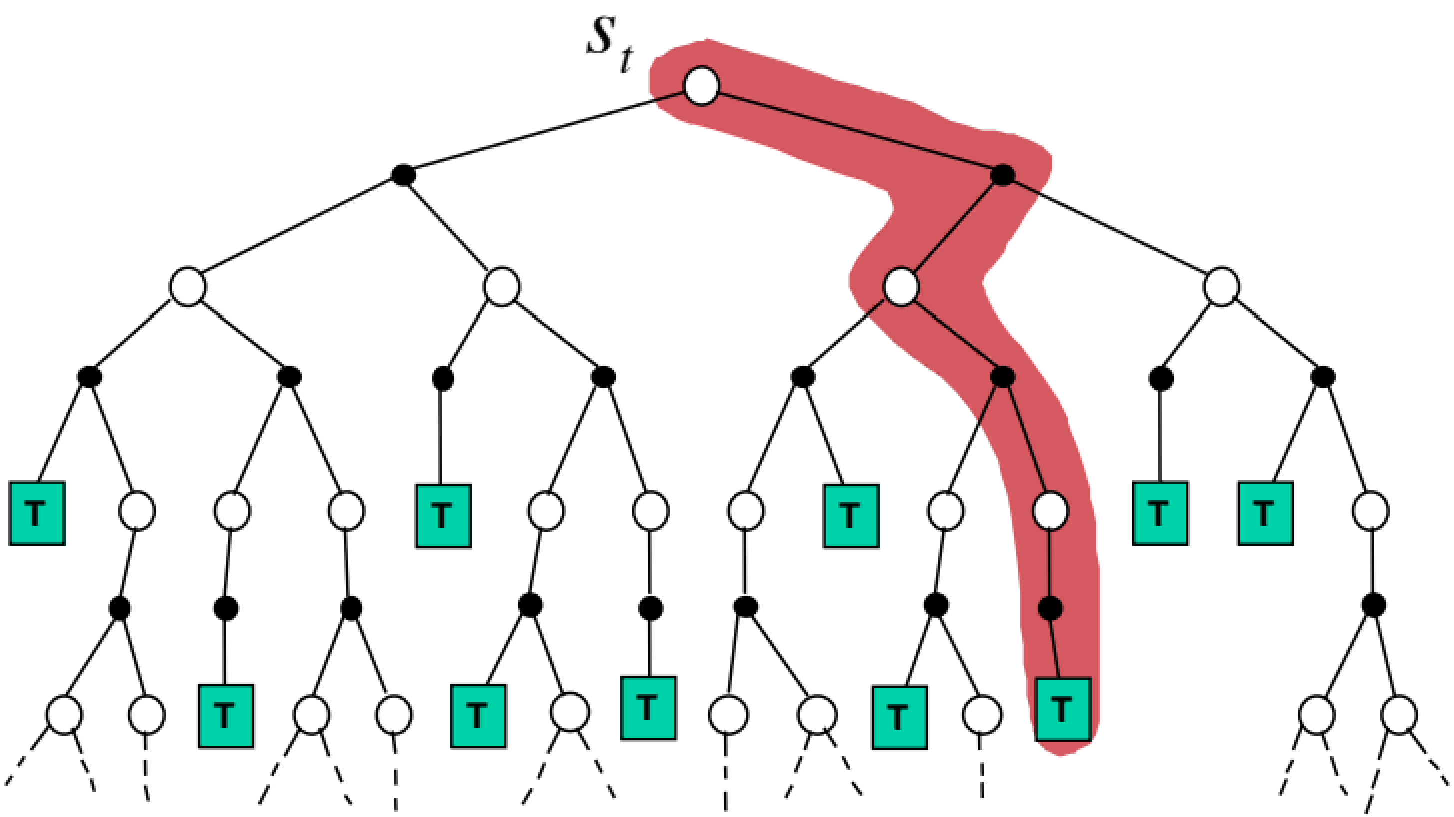
- *In Batch TD(0)*
  - $V(A) = 3/4$, $V(B) = 3/4$

Advantages and Disadvantages of MC vs. TD (3)

- *TD exploits Markov property*
  - *Usually more efficient in Markov environments*
- *MC does not exploit Markov property*
  - *Usually more effective in non-Markov environments*

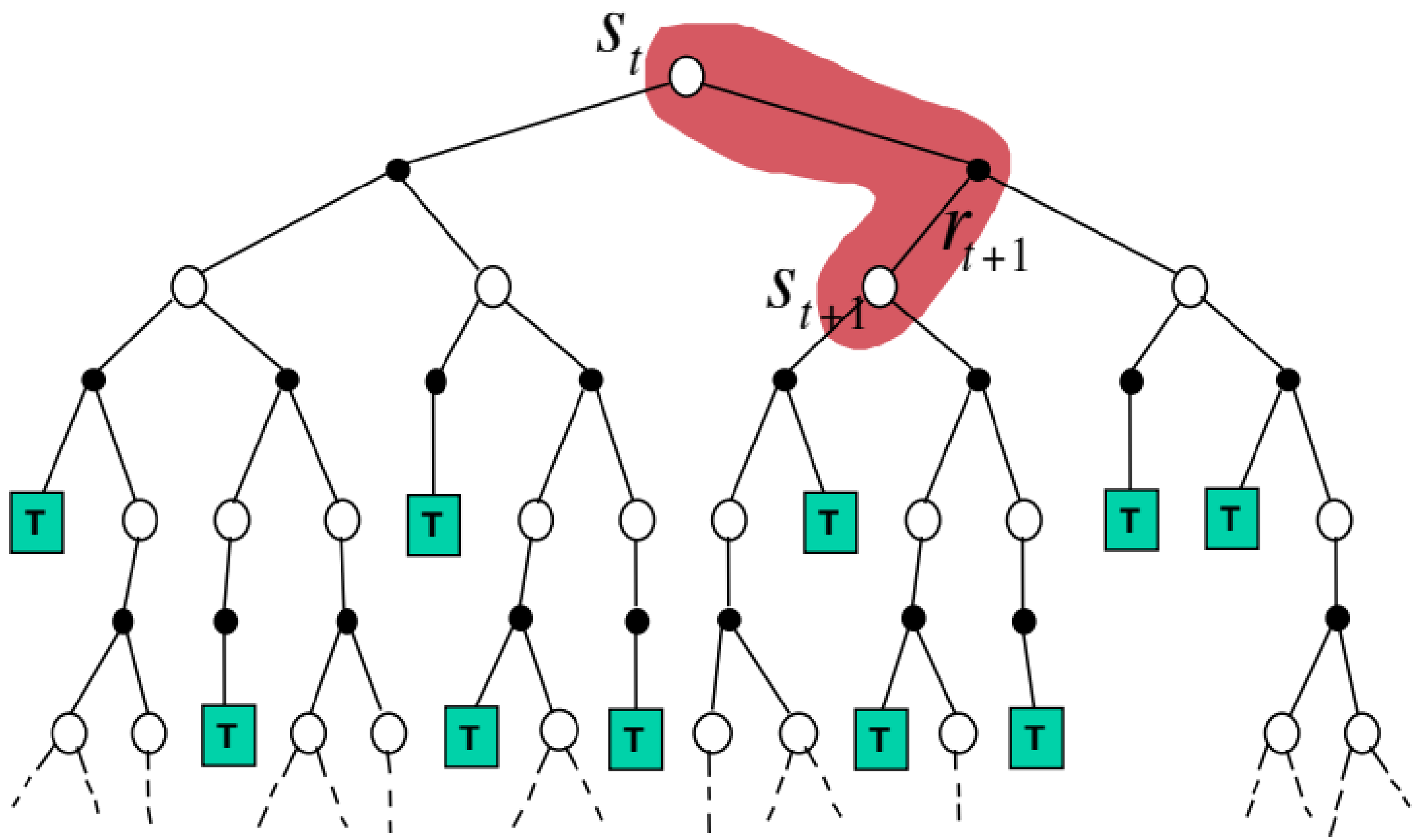
Monte-Carlo Backup

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t - V(S_t) \right)$$



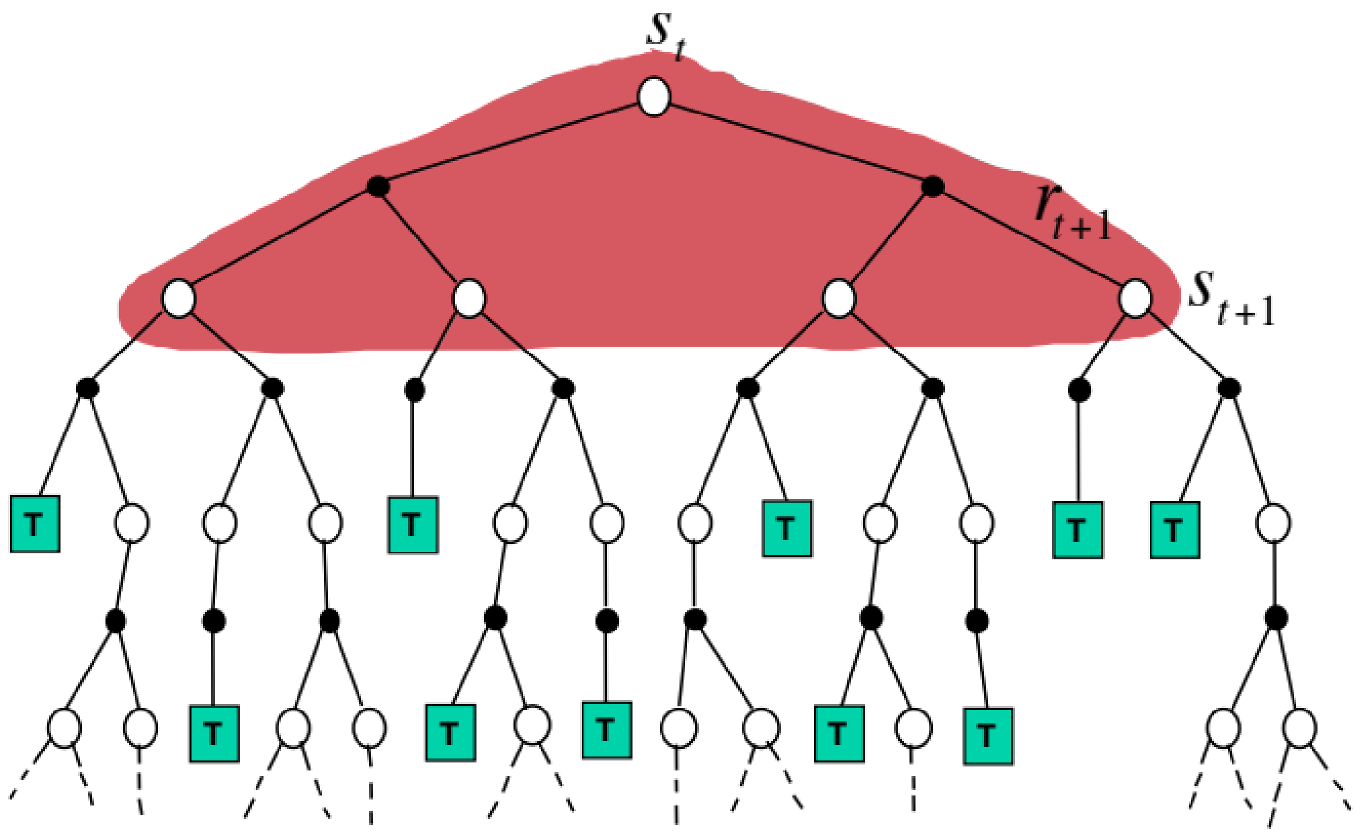
Temporal-Difference Backup

$$V(S_t) \leftarrow V(S_t) + \alpha \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \right)$$



Dynamic Programming Backup

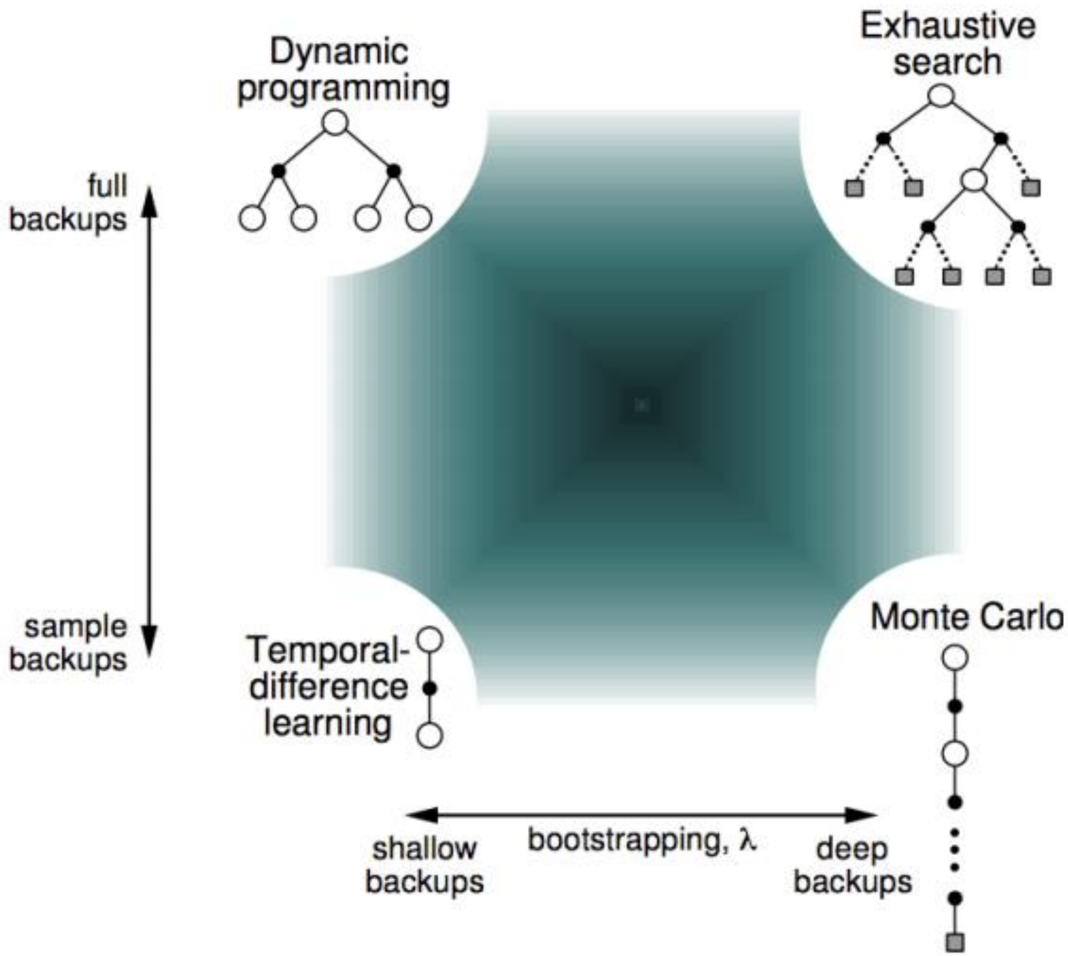
$$V(S_t) \leftarrow \mathbb{E}_\pi \left[R_{t+1} + \gamma V(S_{t+1})\right]$$



Bootstrapping and Sampling

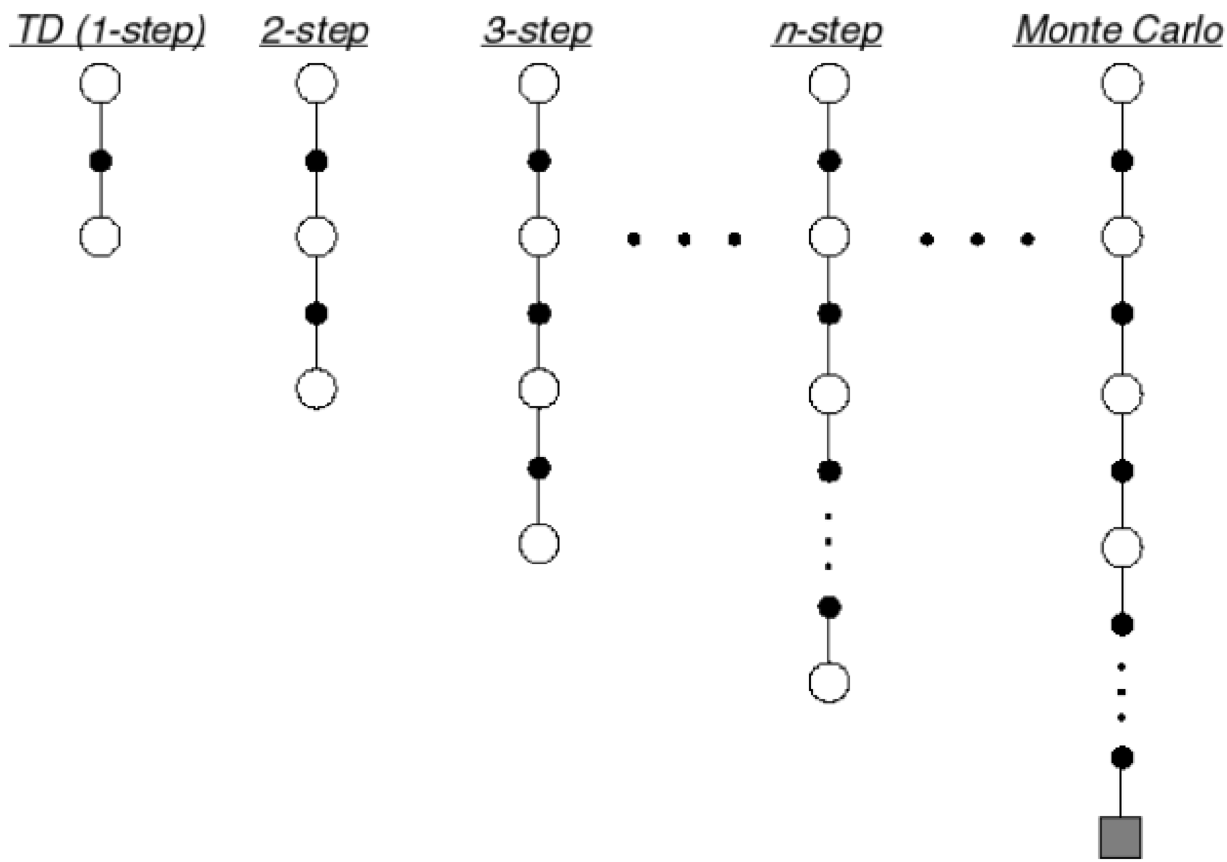
- Bootstrapping: update involves an estimate
  - MC does not bootstrap
  - DP bootstraps
  - TD bootstraps
- Sampling: update samples an expectation
  - MC samples
  - DP does not sample
  - TD samples

Unified View of Reinforcement Learning



n-Step Prediction

- Let TD target look n steps into the future



n-Step Return

- Consider the following n-step returns for $n = 1, 2, \infty$:

$$
\begin{array}{lll}
n = 1 & (TD) & G_t^{(1)} = R_{t+1} + \gamma V(S_{t+1}) \\
n = 2 & & G_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2}) \\
\vdots & & \vdots \\
n = \infty & (MC) & G_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{T-1} R_T
\end{array}
$$

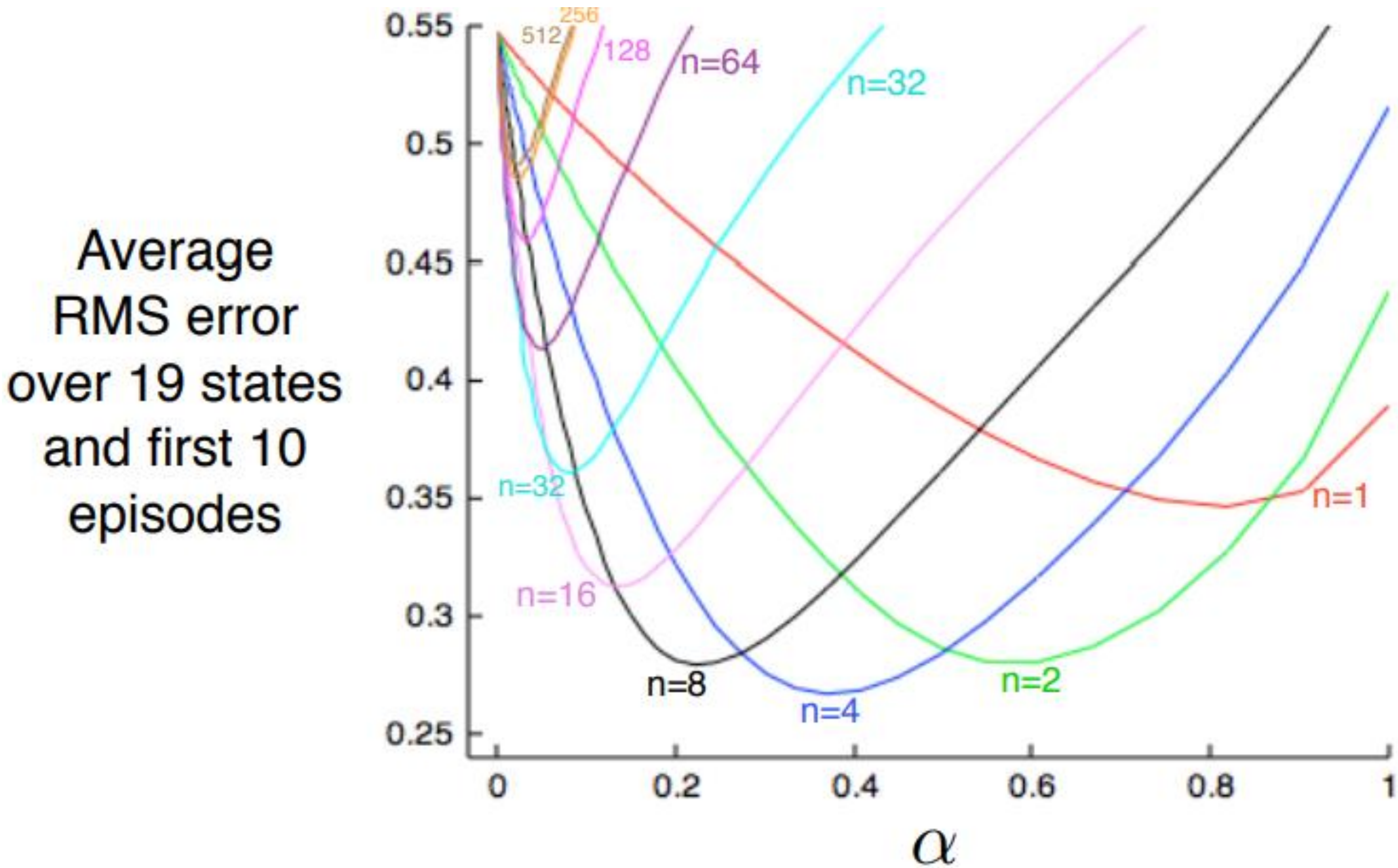
- Define the n-step return

$$G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- n-step temporal-difference learning

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t^{(n)} - V(S_t) \right)$$

Large Random Walk Example

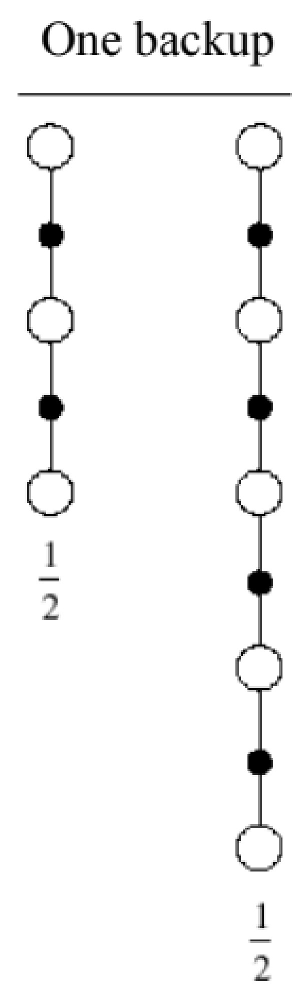


Averaging n-Step Returns

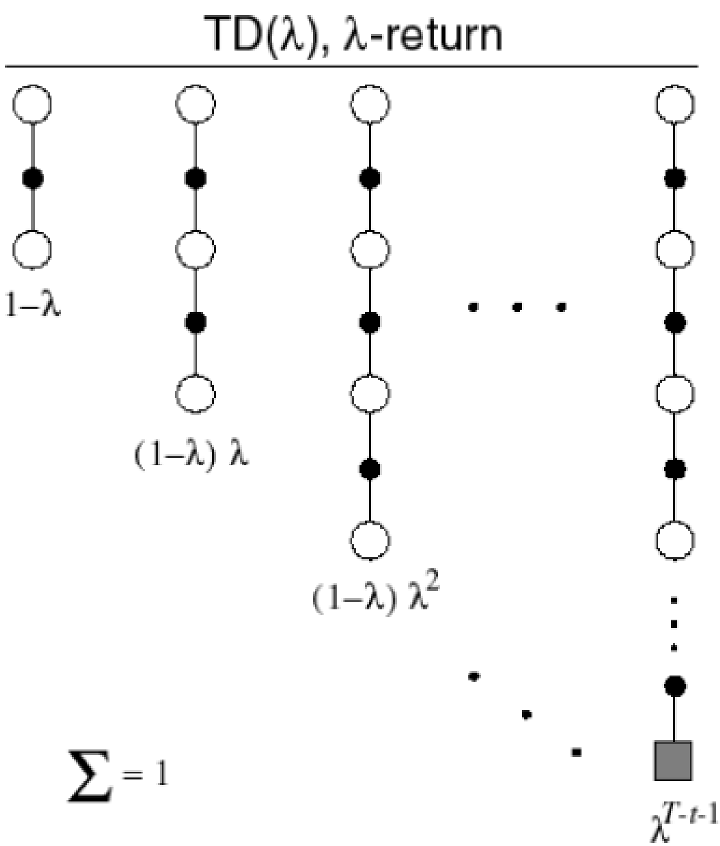
- We can average n-step returns over different n
- e.g. average the 2-step and 4-step returns

$$\frac{1}{2}G^{(2)} + \frac{1}{2}G^{(4)}$$

- Combines information from two different time-steps
- Can we efficiently combine information from all time-steps?



λ-return



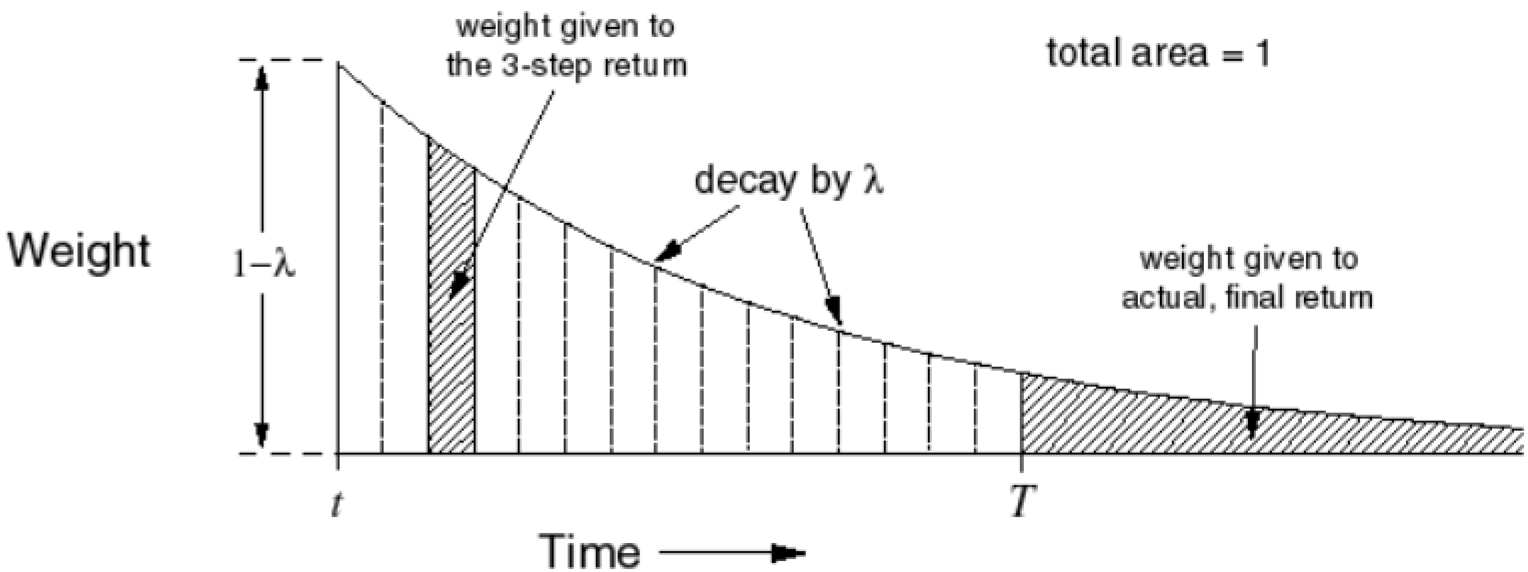
- The λ-*return* G_t^λ combines all n-step returns $G_t^{(n)}$
- Using weight $(1-\lambda)\lambda^{n-1}$

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{(n)}$$

- Forward-view TD(λ)

$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^\lambda - V(S_t)\right)$$

TD(λ) Weighting Function



$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

TD(λ) Weighting Function

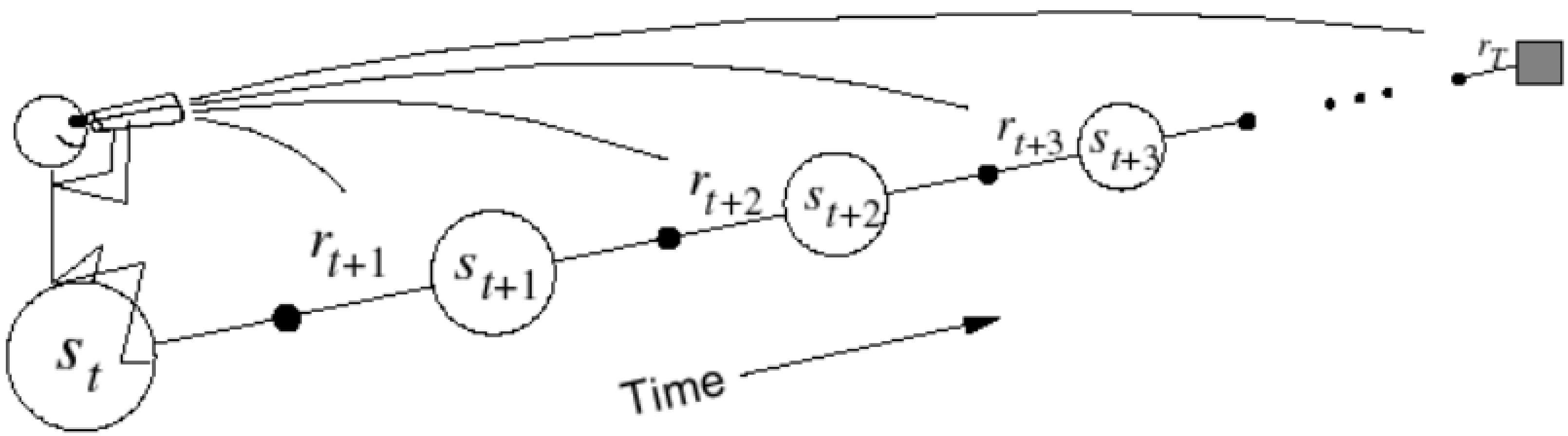
- *More precisely*

$$L_t = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_t^{t+n}(V_t(S_{t+n})).$$

$$\downarrow$$

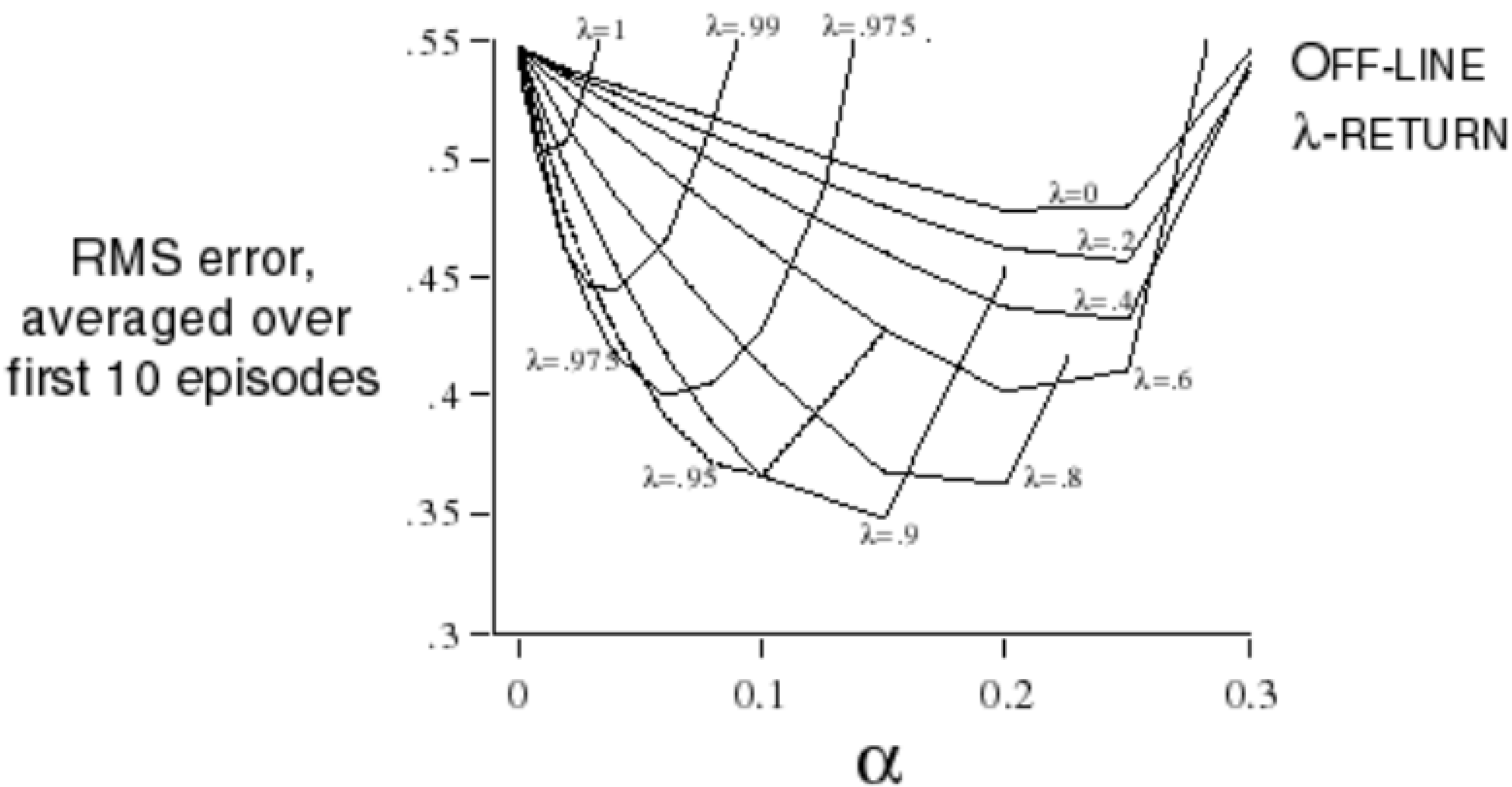
$$L_t \quad = \quad (1-\lambda)\sum_{n=1}^{T-t-1}\lambda^{n-1}G_t^{t+n}(V_t(S_{t+n})) \quad + \quad \lambda^{T-t-1}G_t,$$

Forward-view TD(λ)



- Update value function towards the λ-return
- Forward-view looks into the future to compute G_t^λ
- Like MC, can only be computed from complete episodes

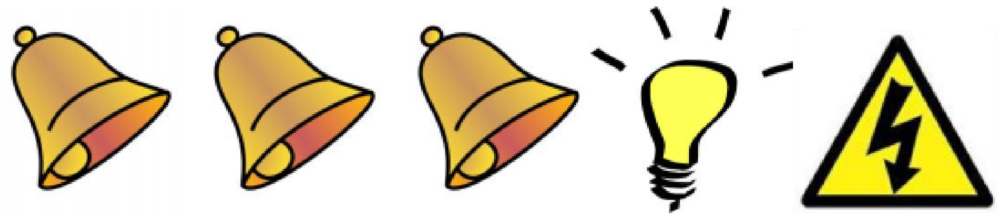
Forward-View TD(λ) on Large Random Walk



Backward View TD(λ)

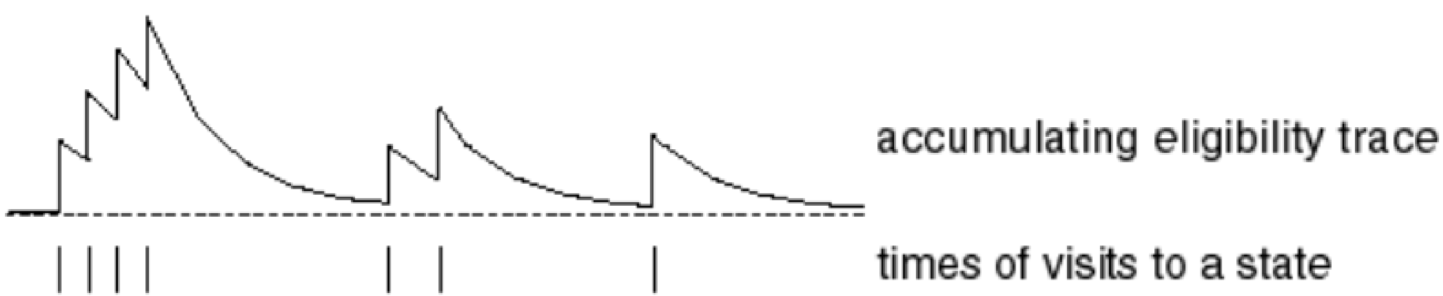
- Forward view provides theory
- Backward view provides mechanism
- Update online, every step, from incomplete sequences

Eligibility Traces



- Credit assignment problem: did bell or light cause shock?
- Frequency heuristic: assign credit to most frequent states
- Recency heuristic: assign credit to most recent states
- *Eligibility traces* combine both heuristics

$$E_0(s) = 0$$
$$E_t(s) = \gamma \lambda E_{t-1}(s) + \mathbf{1}(S_t = s)$$

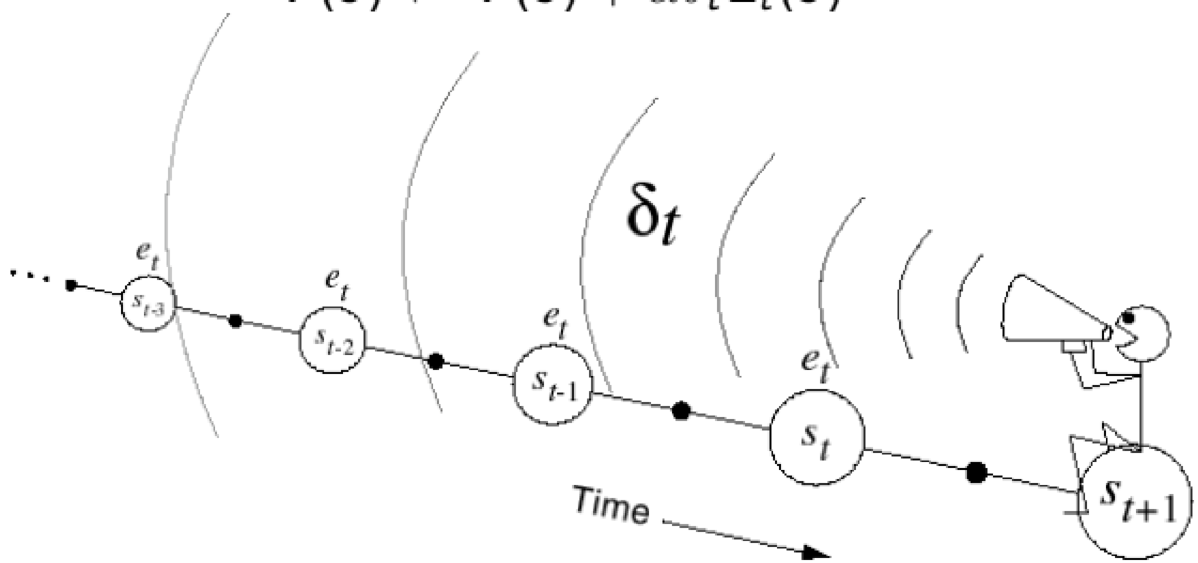


Backward View TD(λ)

- Keep an eligibility trace for every state s
- Update value $V(s)$ for every state s
- In proportion to TD-error δ_t and eligibility trace $E_t(s)$

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

$$V(s) \leftarrow V(s) + \alpha \delta_t E_t(s)$$



$TD(\lambda)$ and $TD(0)$

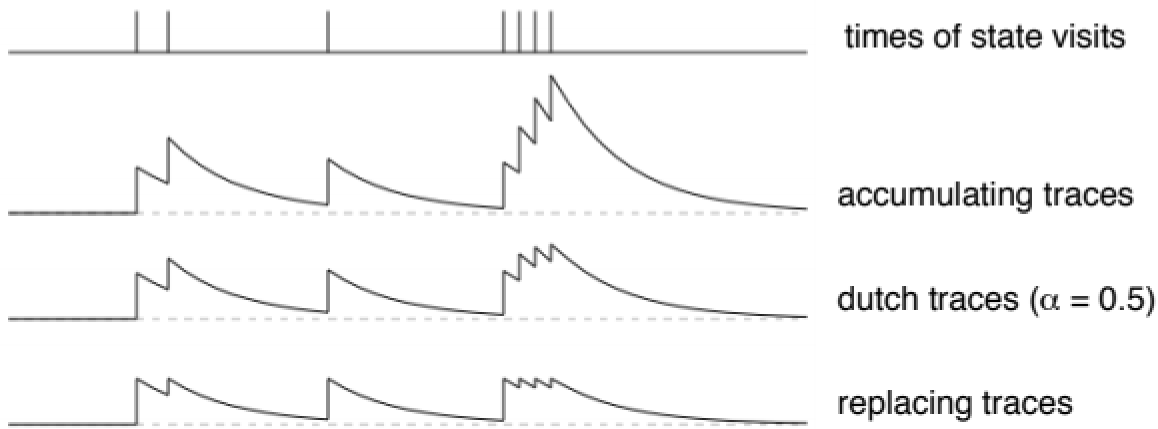
- When $\lambda = 0$, only current state is updated

$$E_t(s) = \mathbf{1}(S_t = s)$$
$$V(s) \leftarrow V(s) + \alpha\delta_t E_t(s)$$

- This is exactly equivalent to TD(0) update

$$V(S_t) \leftarrow V(S_t) + \alpha\delta_t$$

Other TD(λ)



times of state visits
accumulating traces
dutch traces (α = 0.5)
replacing traces

Other TD(λ)

- *Replacing Trace*
  - *Similar with first visit MC*
  - *Set eligibility trace as* $E_t(S_t) = 1.$

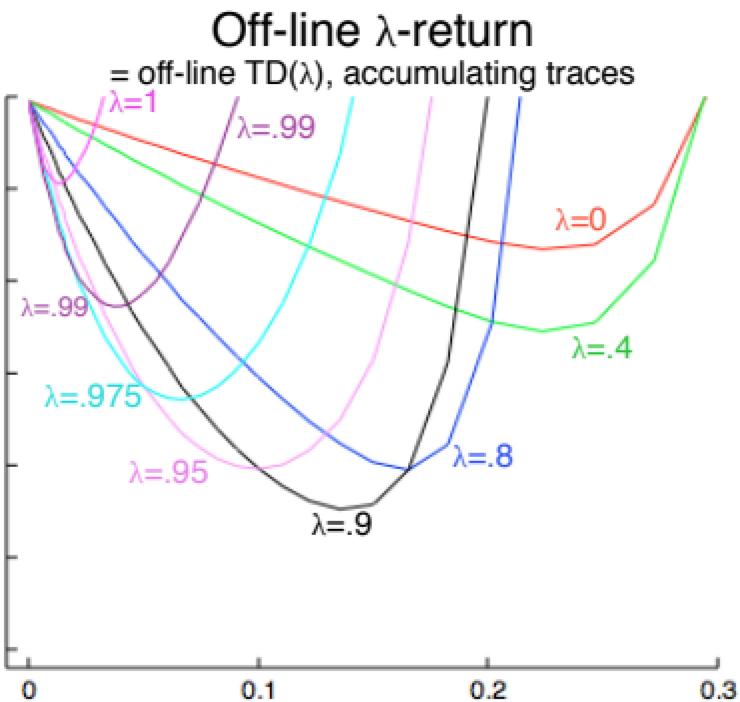
- *Dutch Trace*
  - *A kind of intermediate between accumulating and replacing traces*

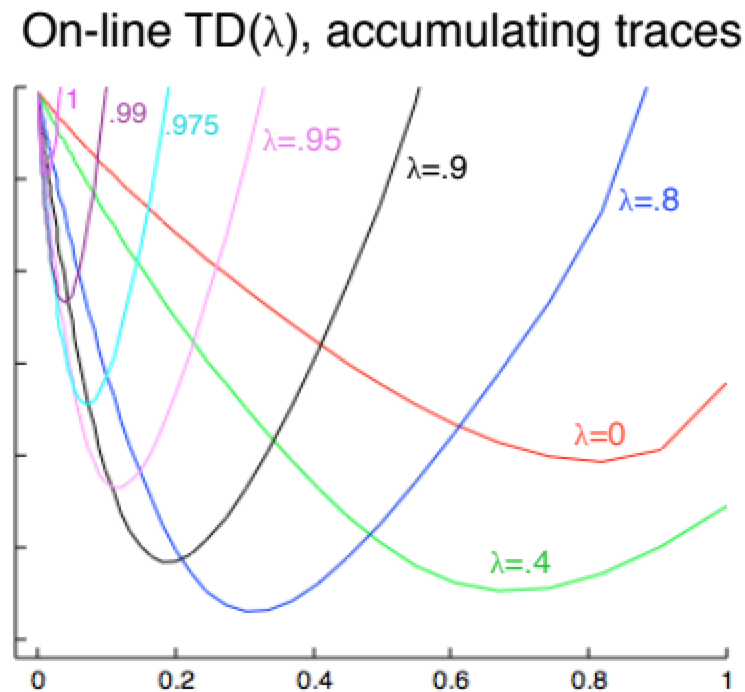
$$E_t(S_t) = (1-\alpha)\gamma\lambda E_{t-1}(S_t) + 1.$$

TD(λ)

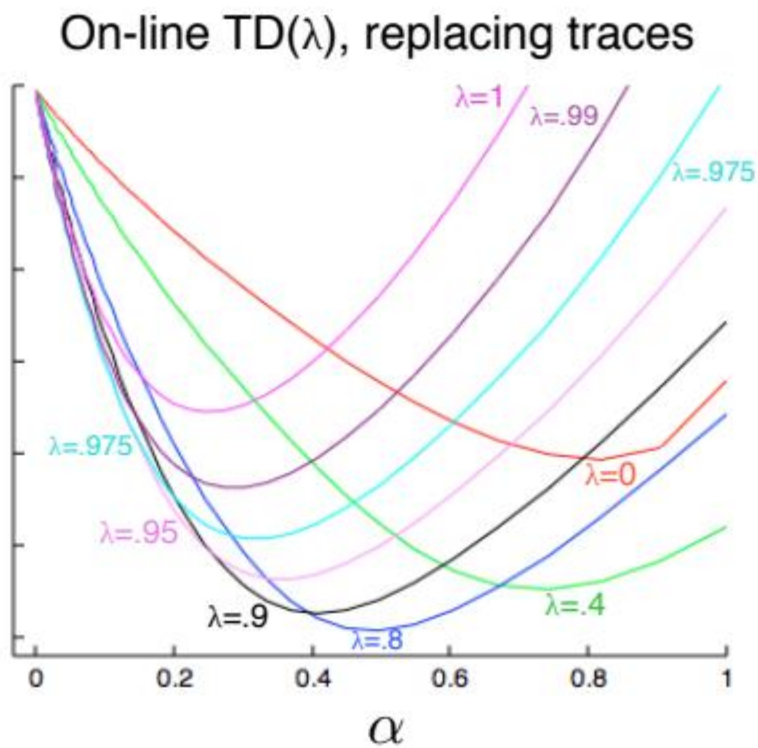
Initialize $V(s)$ arbitrarily (but set to 0 if s is terminal)
Repeat (for each episode):
  Initialize $E(s) = 0$, for all $s \in \mathcal{S}$
  Initialize S
  Repeat (for each step of episode):
    $A \leftarrow$ action given by π for S
    Take action A, observe reward, R, and next state, S'
    $\delta \leftarrow R + \gamma V(S') - V(S)$
    $E(S) \leftarrow E(S) + 1$ (accumulating traces)
    or $E(S) \leftarrow (1 - \alpha)E(S) + 1$ (dutch traces)
    or $E(S) \leftarrow 1$ (replacing traces)
    For all $s \in \mathcal{S}$:
      $V(s) \leftarrow V(s) + \alpha \delta E(s)$
      $E(s) \leftarrow \gamma \lambda E(s)$
    $S \leftarrow S'$
  until S is terminal

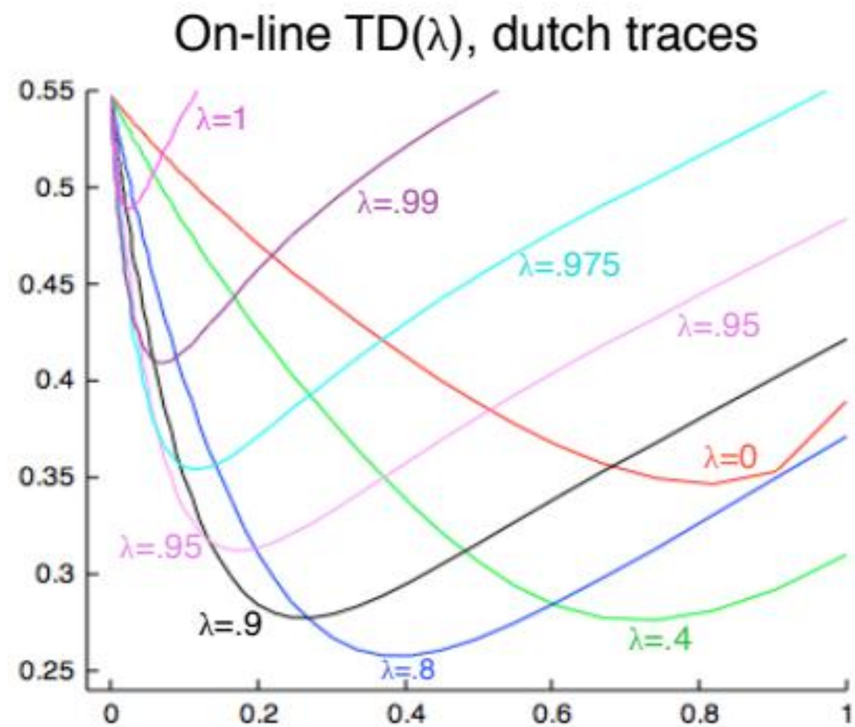
Performance Comparison





Performance Comparison





Question

- *Can you mathematically compare MC and TD?*

Question

- ***Can you mathematically compare MC and TD?***

$$
\begin{aligned}
G_t - V(S_t) &= R_{t+1} + \gamma G_{t+1} - V(S_t) \\
&= R_{t+1} + \gamma G_{t+1} - V(S_t) + \gamma V(S_{t+1}) - \gamma V(S_{t+1}) \\
&= [R_{t+1} + \gamma V(S_{t+1}) - V(S_t)] + [\gamma G_{t+1} - \gamma V(S_{t+1})] \\
&= \delta_t + \gamma [G_{t+1} - V(S_{t+1})] \\
&= \delta_t + \gamma \delta_{t+1} + \gamma^2 [G_{t+2} - V(S_{t+2})] \\
&= \vdots \\
&= \delta_t + \gamma \delta_{t+1} + \gamma^2 \delta_{t+2} + \cdots + \gamma^{T-t-1} \delta_{T-1} + \gamma^{T-t} [G_T - V(S_T)] \\
&= \delta_t + \gamma \delta_{t+1} + \gamma^2 \delta_{t+2} + \cdots + \gamma^{T-t-1} \delta_{T-1} + \gamma^{T-t} [0 - 0] \\
&= \sum_{k=t}^{T-1} \gamma^{k-t} \delta_k
\end{aligned}
$$

TD(λ) and MC

- When $\lambda = 1$, credit is deferred until end of episode
- Consider episodic environments with offline updates
- Over the course of an episode, total update for TD(1) is the same as total update for MC

Theorem

The sum of offline updates is identical for forward-view and backward-view TD(λ)

$$\sum_{t=1}^{T} \alpha \delta_t E_t(s) = \sum_{t=1}^{T} \alpha \left(G_t^{\lambda} - V(S_t) \right) \mathbf{1}(S_t = s)$$

MC and TD(1)

- Consider an episode where s is visited once at time-step k,
- TD(1) eligibility trace discounts time since visit,

$$E_t(s) = \gamma E_{t-1}(s) + \mathbf{1}(S_t = s)$$
$$= \begin{cases} 0 & \text{if } t < k \\ \gamma^{t-k} & \text{if } t \geq k \end{cases}$$

- TD(1) updates accumulate error *online*

$$\sum_{t=1}^{T-1} \alpha \delta_t E_t(s) = \alpha \sum_{t=k}^{T-1} \gamma^{t-k} \delta_t = \alpha \left(G_k - V(S_k) \right)$$

- By end of episode it accumulates total error

$$\delta_k + \gamma \delta_{k+1} + \gamma^2 \delta_{k+2} + \ldots + \gamma^{T-1-k} \delta_{T-1}$$

Telescoping in TD(1)

When $\lambda = 1$, sum of TD errors telescopes into MC error,

$$
\begin{aligned}
&\delta_t + \gamma\delta_{t+1} + \gamma^2\delta_{t+2} + ... + \gamma^{T-1-t}\delta_{T-1} \\
= \; & R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \\
+ \; & \gamma R_{t+2} + \gamma^2 V(S_{t+2}) - \gamma V(S_{t+1}) \\
+ \; & \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3}) - \gamma^2 V(S_{t+2}) \\
& \quad \vdots \\
+ \; & \gamma^{T-1-t} R_T + \gamma^{T-t} V(S_T) - \gamma^{T-1-t} V(S_{T-1}) \\
= \; & R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} ... + \gamma^{T-1-t} R_T - V(S_t) \\
= \; & G_t - V(S_t)
\end{aligned}
$$

TD(λ) and TD(1)

- TD(1) is roughly equivalent to every-visit Monte-Carlo
- Error is accumulated online, step-by-step
- If value function is only updated offline at end of episode
- Then total update is exactly the same as MC

Telescoping in TD(λ)

For general λ, TD errors also telescope to λ-error, $G_t^\lambda - V(S_t)$

$$
\begin{aligned}
G_t^\lambda - V(S_t) = -V(S_t) \quad & + \quad (1-\lambda)\lambda^0 \left(R_{t+1} + \gamma V(S_{t+1})\right) \\
& + \quad (1-\lambda)\lambda^1 \left(R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2})\right) \\
& + \quad (1-\lambda)\lambda^2 \left(R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3})\right) \\
& + \quad \ldots \\
= -V(S_t) \quad & + \quad (\gamma\lambda)^0 \left(R_{t+1} + \gamma V(S_{t+1}) - \gamma\lambda V(S_{t+1})\right) \\
& + \quad (\gamma\lambda)^1 \left(R_{t+2} + \gamma V(S_{t+2}) - \gamma\lambda V(S_{t+2})\right) \\
& + \quad (\gamma\lambda)^2 \left(R_{t+3} + \gamma V(S_{t+3}) - \gamma\lambda V(S_{t+3})\right) \\
& + \quad \ldots \\
= \quad & \quad\quad (\gamma\lambda)^0 \left(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\right) \\
& + \quad (\gamma\lambda)^1 \left(R_{t+2} + \gamma V(S_{t+2}) - V(S_{t+1})\right) \\
& + \quad (\gamma\lambda)^2 \left(R_{t+3} + \gamma V(S_{t+3}) - V(S_{t+2})\right) \\
& + \quad \ldots \\
= \delta_t + \gamma\lambda\delta_{t+1} & + (\gamma\lambda)^2\delta_{t+2} + \ldots
\end{aligned}
$$

Forwards and Backwards TD(λ)

- Consider an episode where s is visited once at time-step k,
- TD(λ) eligibility trace discounts time since visit,

$$
\begin{aligned}
E_t(s) &= \gamma\lambda E_{t-1}(s) + \mathbf{1}(S_t = s) \\
&= \begin{cases} 0 & \text{if } t < k \\ (\gamma\lambda)^{t-k} & \text{if } t \geq k \end{cases}
\end{aligned}
$$

- Backward TD(λ) updates accumulate error *online*

$$
\sum_{t=1}^{T} \alpha\delta_t E_t(s) = \alpha \sum_{t=k}^{T} (\gamma\lambda)^{t-k}\delta_t = \alpha\left(G_k^\lambda - V(S_k)\right)
$$

- By end of episode it accumulates total error for λ-return
- For multiple visits to s, $E_t(s)$ accumulates many errors

Offline Equivalence of Forward and Backward TD

Offline updates

- Updates are accumulated within episode
- but applied in batch at the end of episode

Online Equivalence of Forward and Backward TD

Online updates

- TD(λ) updates are applied online at each step within episode
- Forward and backward-view TD(λ) are slightly different
- NEW: Exact online TD(λ) achieves perfect equivalence
- By using a slightly different form of eligibility trace
- Sutton and von Seijen, ICML 2014

Summary of Forward and Backward TD(λ)

Offline updates	$\lambda = 0$	$\lambda \in (0, 1)$	$\lambda = 1$
Backward view	TD(0)	TD(λ)	TD(1)
	$\|$	$\|$	$\|$
Forward view	TD(0)	Forward TD(λ)	MC
Online updates	$\lambda = 0$	$\lambda \in (0, 1)$	$\lambda = 1$
Backward view	TD(0)	TD(λ)	TD(1)
	$\|$	$\nparallel$	$\nparallel$
Forward view	TD(0)	Forward TD(λ)	MC
	$\|$	$\|$	$\|$
Exact Online	TD(0)	Exact Online TD(λ)	Exact Online TD(1)

$=$ here indicates equivalence in total update at end of episode.