Chapter 4

Time-Domain Signal Analysis

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Contents

- Discrete-time system properties
- System representation – difference equation and impulse response
- Convolution
- Discrete cross-correlation

Discrete-Time System Properties

- Discrete–time system
  - Mathematical transform that maps the input signal $x[n]$ into output signal $y[n]$
  - Transformation $T[\cdot]$

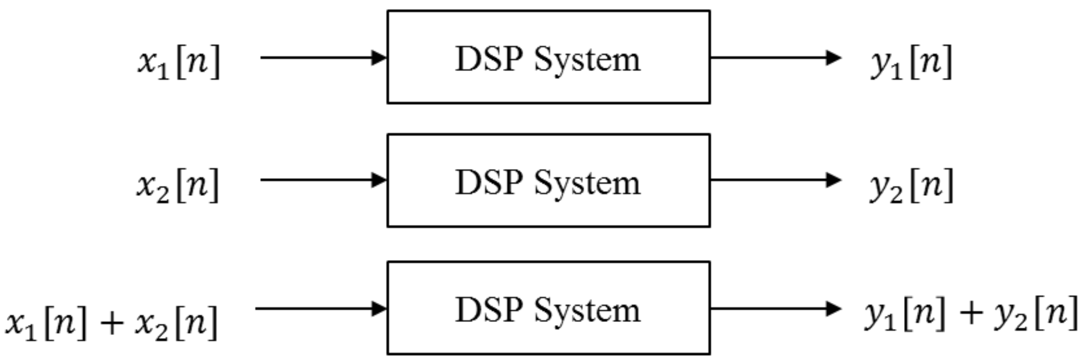
$$y[n] = T[x[n]]$$

$$x[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n] = T[x[n]]$$

Additivity

- For any signal $x_1[n]$ and $x_2[n]$

$$T[x_1[n] + x_2[n]] = T[x_1[n]] + T[x_2[n]]$$



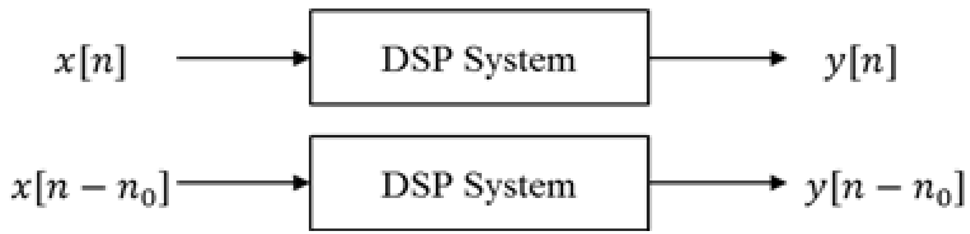
- Homogeneity
  - For any constant c and for any signal $x[n]$

$$T[cx[n]] = cT[x[n]]$$

- Linearity
  - For any constant c_1, c_2 and for any signal $x_1[n]$, $x_2[n]$

$$T[c_1 x_1[n] + c_2 x_2[n]] = c_1 T[x_1[n]] + c_2 T[x_2[n]]$$

Time-invariance



- compare $y[n-n_0]$ to $T\big[x[n-n_0]\big]$ to test for time-invariance. If they are same, then the system is time-invariance

- Ex) Check for the time-invariance
  - $y[n] = x^2[n]$
    - The response of the system to $x'[n] = x[n-n_0]$ is

      $y'[n] = [x'[n]]^2 = x^2[n-n_0]$

      On the other hand $y[n-n_0] = x^2[n-n_0]$

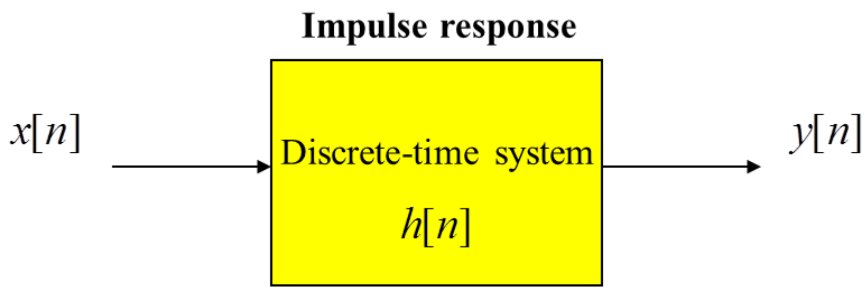
      Because $y'[n] = y[n-n_0]$, the system is time – invariant.
  - $y[n] = x[n] + x[-n]$
    - The response of the system to $x'[n] = x[n-n_0]$ is

      $y'[n] = x[n-n_0] + x[-n-n_0]$

      On the other hand, $y[n-n_0] = x[n-n_0] + x[-(n-n_0)] = x[n-n_0] + x[-n+n_0]$

      Because $y'[n] \neq y[n-n_0]$, the system is time – varying.

- Linear time-invariant (LTI) system
  - Satisfy both linear and time-invariant property
  - If LTI system, then convolution output



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- "*" convolution operator
- Impulse response $h[n]$ – mathematical model for system

- Causality
  - If for any n_0, the response for the system at time n_0 depends only on the input up to $n = n_0$
    - Output cannot depend on the future input
  - An LTI system is causal if and only if (iff)

$$h[n] = 0, \qquad for\; n < 0$$

- Ex) Decide the causality of the system

  - $y[n] = x[n] + x[n-1]$ - causal

  - $y[n] = x[n] + x[n+1]$ - non-causal

- Stability
  - BIBO (Bounded Input Bounded Output)

    If for any input that is bounded , $|x[n]| \leq A < \infty$

    the output will be bounded $|y[n]| \leq B < \infty$

  - For LTI system, BIBO stability is guaranteed if

$$|y[n]| < \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| < A \sum_{k=-\infty}^{\infty} |h[k]| \leq B < \infty$$

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

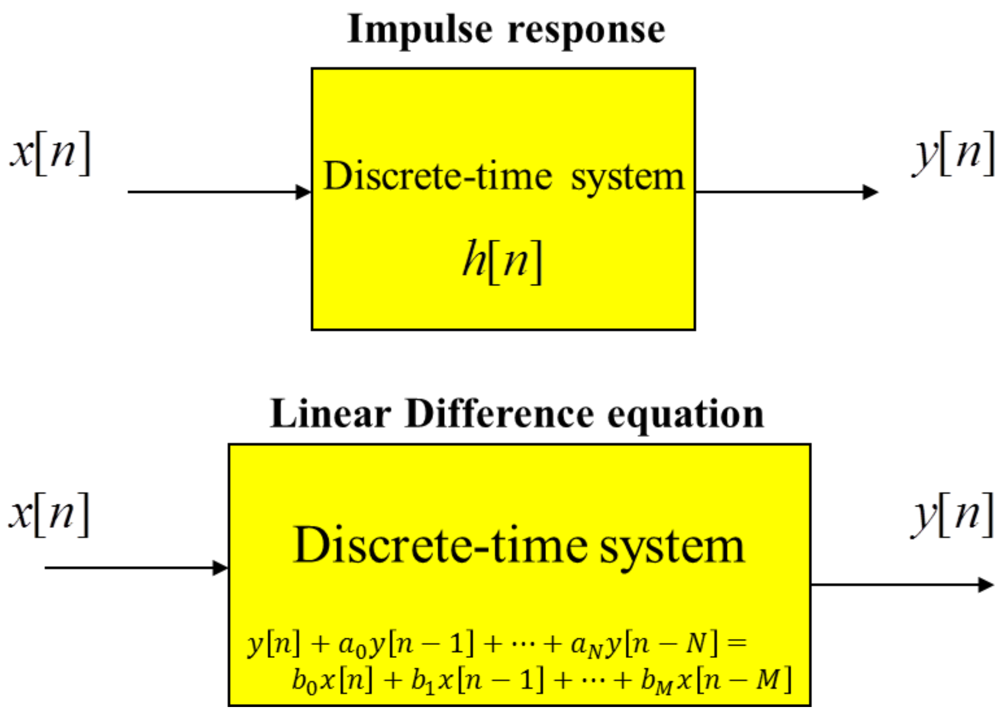
- Ex) Check for the stability $h[n] = a^n u[n]$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |a^n| u[n] = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|}$$

- The system will be stable when $|a| < 1$

System Representation

- Impulse response & Difference equation



- Linear difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] + a_1\ y[n-1] + \dots + a_N y[n-N] = b_0 x[n] + b_1\ x[n-1] + \dots + b_M x[n-M]$$

- Recursive system – IIR (Infinite Impulse Response)
  - Output depends on both input and output

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$= -a_1\ y[n-1] - \dots - a_N y[n-N] + b_0 x[n] + b_1\ x[n-1] + \dots + b_M x[n-M]$$

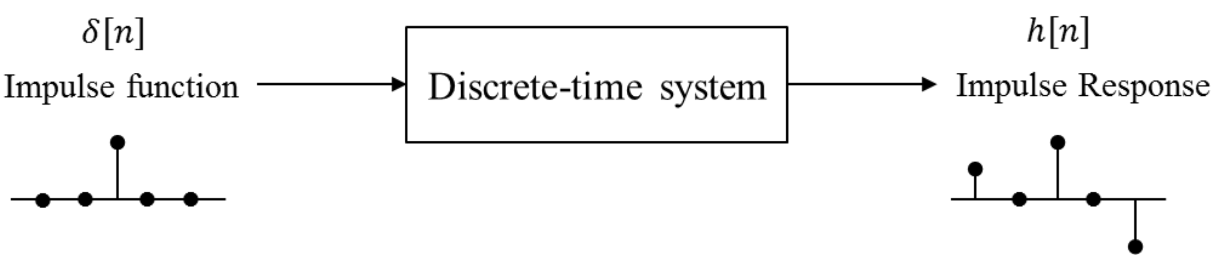
- Non-recursive system – FIR (Finite Impulse Response)
  - Output only depends on input

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$= b_0 x[n] + b_1 \; x[n-1] + \ldots + b_M x[n-M]$$

Impulse response

- **Definition:** Impulse response is defined as a response of a system to the unit impulse function



- Depending on the form of $h[n]$ - FIR or IIR

- Ex)) Non-recursive ; FIR system

$$y[n] = 0.25(x[n] + x[n-1] + x[n-2] + x[n-3])$$

- Replace by $x[n] = \delta[n]$ and $y[n] = h[n]$

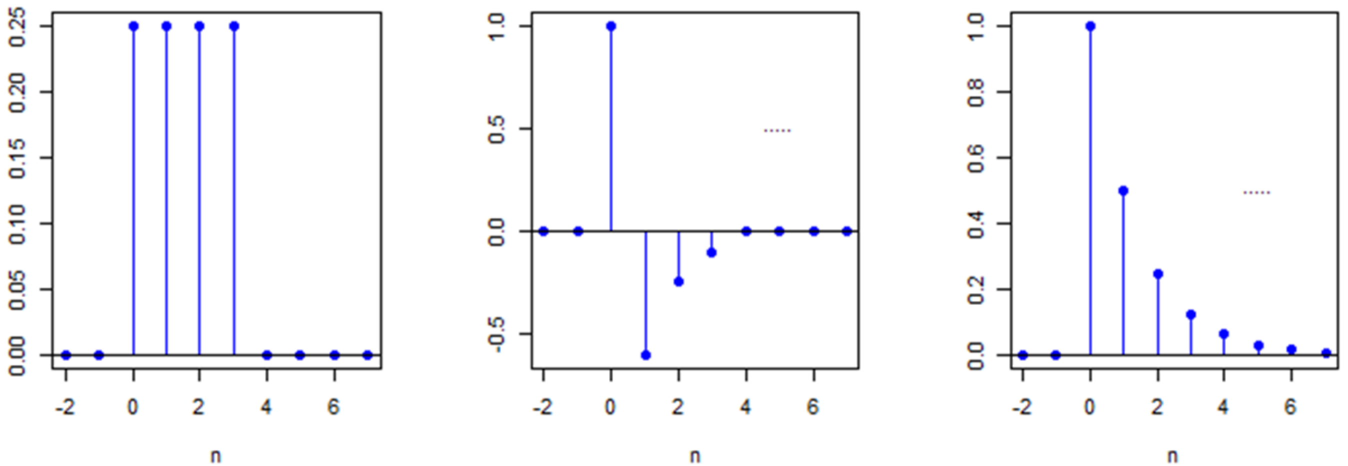
$$h[n] = 0.25(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3])$$

$$\text{Starting with } n = 0$$

$$h[0] = 0.25, h[1] = 0.25, h[2] = 0.25, h[3] = 0.25$$

$$h[4] = 0, h[5] = 0 \ldots$$

$$h[n] = \{\mathbf{0.25}, 0.25, 0.25, 0.25\}$$



(a) (b) (C)

- Called Finite Impulse Response (FIR), since IR has finite length

- Ex) Recursive – IIR system

$$y[n]-0.4y[n-1]=x[n]-x[n-1]$$

- Replace by $x[n]=\delta[n]$ and $y[n]=h[n]$

$$h[n]=0.4h[n-1]+\delta[n]-\delta[n-1]$$

$$\text{Starting with n}=0$$

$$\text{h}[0]=0.4h[-1]+\delta[0]-\delta[-1]=1, h[1]=-0.6$$

$$h[2]=-0.24, h[3], h[4], \ldots\ldots$$

Infinite impulse response

- Ex) Recursive – IIR system

$$y[n]-\alpha y[n-1]=x[n]$$

- Replace by $x[n]=\delta[n]$ and $y[n]=h[n]$

$$h[n]=\alpha h[n-1]+\delta[n],\qquad h[-1]=0$$

$$h[0]=\alpha h[-1]+\delta[0]=1,\quad h[1]=\alpha h[0]=\alpha$$

$$h[2]=\alpha h[1]=\alpha^2, h[3]=\alpha h[2]=\alpha^3,\ \ h[4],\ldots\ldots$$

$$h[n]=\alpha^n,\ n\geq 0 \qquad \text{or} \qquad h[n]=\alpha^n u[n]$$

- Ex) Decide whether the system is FIR or IIR. Determine the stability and causality
  - $h[n] = \{2, \mathbf{1}, 1, 3\}$
    - FIR system, stable, but non- causal
  - $h[n] = (-0.5)^n u[n]$
    - IIR system, stable since $\displaystyle\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} |-0.5|^k = \frac{1}{1-0.5} = 2 < \infty$ causal
    - How about $h[n] = (1.3)^n u[n]$?

Moving Average (MA) Filter

- Most common and simplest filter
  - Noise reduction, long-term trend prediction

- DE for causal L-point MA filter

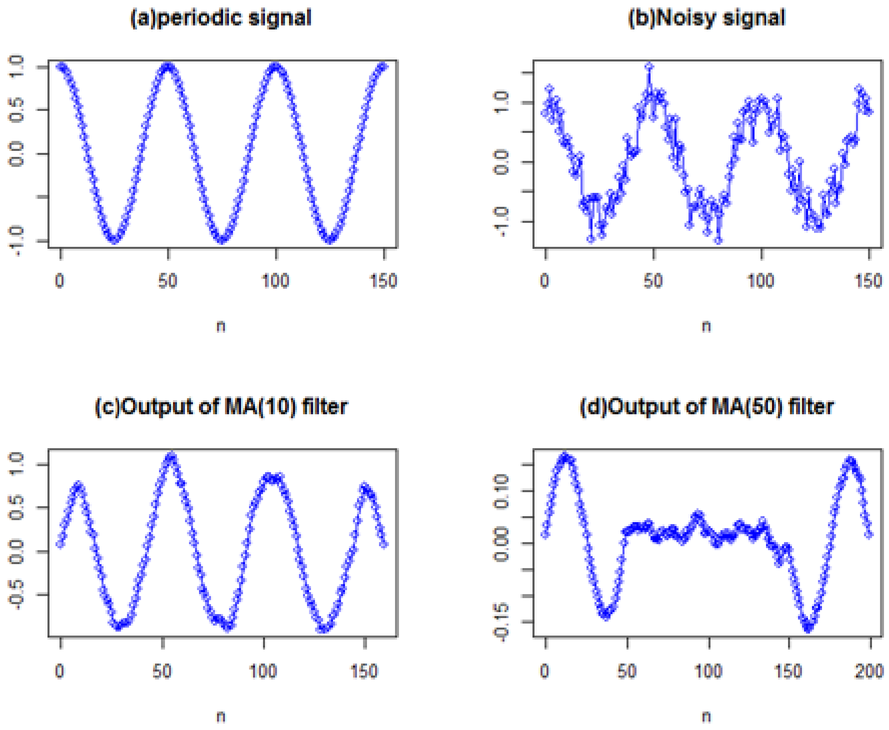
$$y[n] = \frac{1}{L}\{x[n] + x[n-1] + \dots + x[n-(L-1)]\}$$

$$= \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

$$h[n] = \frac{1}{L}\{1,1,1,\dots,1,1\}$$

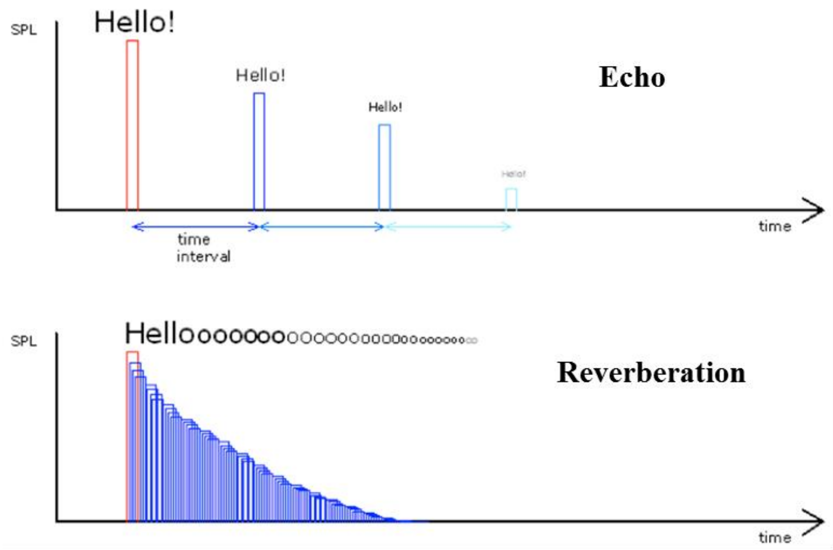
  - For L=5 $\quad y[80] = \dfrac{x[80] + x[79] + x[78] + x[77] + x[76]}{5}$

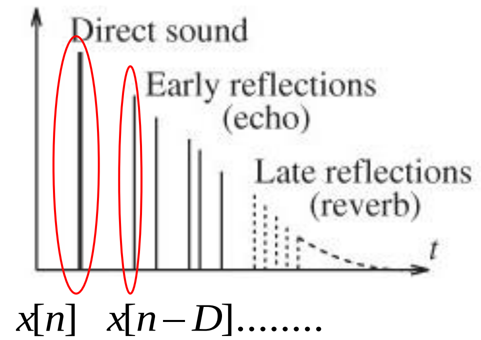
- “periodic signal + random noise”
  - Apply L=10 MA filter, L=50 MA filter



- Long-term prediction
  - Smooth out short-term fluctuation and highlight longer-term trends
  - Blue : L=15 MA, Red: L=50 MA

Echo and Reverberation





- Echo – comb filter (FIR)

$$y[n] = x[n] + \alpha x[n-D], \quad \alpha < 1 \ , \ D = \text{delay}$$

- Reverberation – ex) concert hall

$$y[n] = x[n] + \alpha x[n-D] + \alpha^2 x[n-2D] + \alpha^3 x[n-3D] + \dots$$

$$y[n] - \alpha y[n-D] = x[n]$$

- Ex) Find out IR of an echo and reverberation system
  - Echo system - FIR $y[n] = x[n] + \alpha x[n-D], \ \alpha < 1$
    - By substituting $x[n] = \delta[n]$ $y[n] = h[n]$

$h[n] = \delta[n] + \alpha\delta[n-D]$ and causality $h[n] = 0$ for n<0

$$h[0] = 1, h[1] = 0, \dots, h[D] = \alpha, h[D+1] = 0, \dots$$

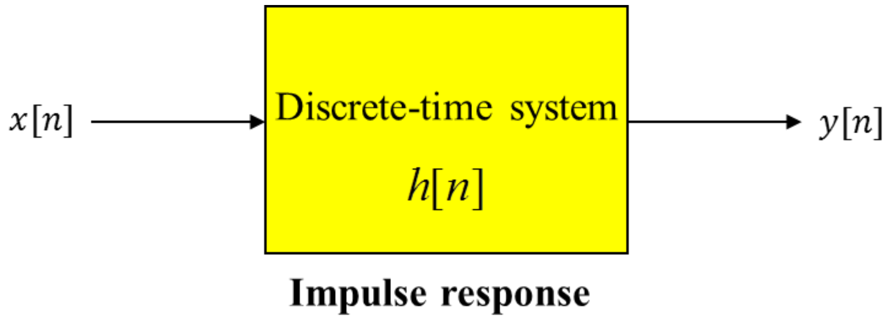
$$h[n] = \{1,0,\dots,0,\alpha\}$$

  - Reverb system – IIR
    - By substituting $x[n] = \delta[n]$ $y[n] = h[n]$

$h[n] = \alpha h[n-D] + \delta[n]$ and causality $h[n] = 0$ for n<0

$$h[0] = \alpha h[-D] + \delta[0] = 1, \dots, \ h[D] = \alpha, \dots h[2D] = \alpha^2, \dots$$

Convolution



- For LTI system

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution property

- Commutative

$$x[n]*h[n]=h[n]*x[n]$$

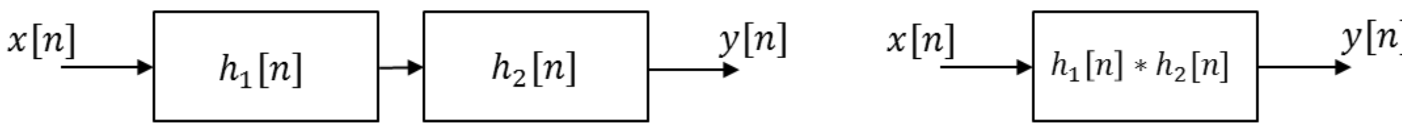
- Distributive

$$x[n]*\{h_1[n]+h_2[n]\}=x[n]*h_1[n]+x[n]*h_2[n]$$

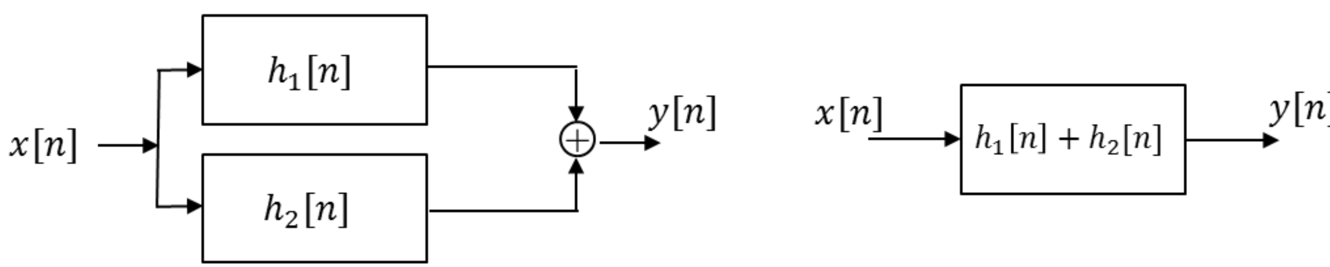
- Associative

$$\{x[n]*h_1[n]\}*h_2[n]=x[n]*\{h_1[n]*h_2[n]\}$$

commutative property



associative property



distributive property

- Mathematical operation of convolution
  - Two different approaches
    - Direct evaluation
    - Graphical approach
  - **Direct evaluation**
    - When the sequences simple or may be described by simple closed-form mathematical expression

- Ex) Given $x[n] = \{\mathbf{3}, 4, 5\}, \quad h[n] = \{\mathbf{2}, 1\}$
  - By convolution equation

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

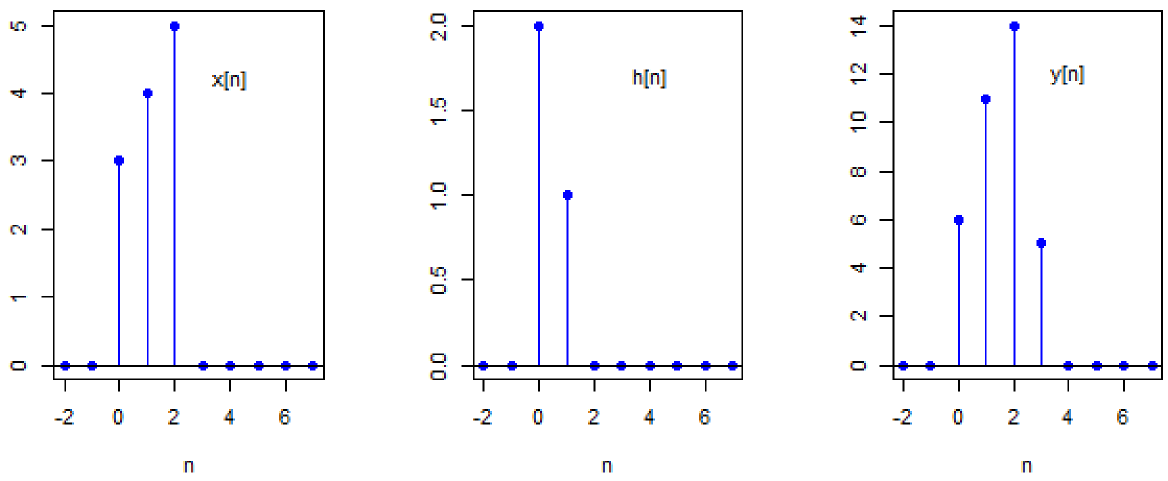
$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = x[0]h[0] = 3 \cdot 2 = 6$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = x[0]h[1] + x[1]h[0] = 3 \cdot 1 + 4 \cdot 2 = 11$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = x[0]h[2] + x[1]h[1] + x[2]h[0] = 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 2 = 14$$

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] = 0 + 0 + 5 \cdot 1 + 0 = 5$$

$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k] = x[0]h[4] + x[1]h[3] + \ldots = 0$$



- Length of output $L_y = L_x + L_h - 1 = 3 + 2 - 1 = 4$
- Note two end-points of output

❖ Ex)

$$x[n] = (0.5)^n u[n] = \begin{cases} (0.5)^n & n \geq 0 \\ 0 & n < 0 \end{cases}, \qquad h[n] = u[n]$$

- From the convolution equation

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} (0.5)^k u[k]u[n-k]$$

$$y[n] = \sum_{k=0}^{n} (0.5)^k = \frac{1-(0.5)^{n+1}}{1-(0.5)}, \quad y[n] = \frac{1-(0.5)^{n+1}}{1-(0.5)} u[n]$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{for } |a| < 1; \quad \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \; ; \quad \sum_{n=0}^{N-1} n = \frac{N(N-1)}{2}$$

Graphical approach

- Induced from the calculation procedure of the convolution equation

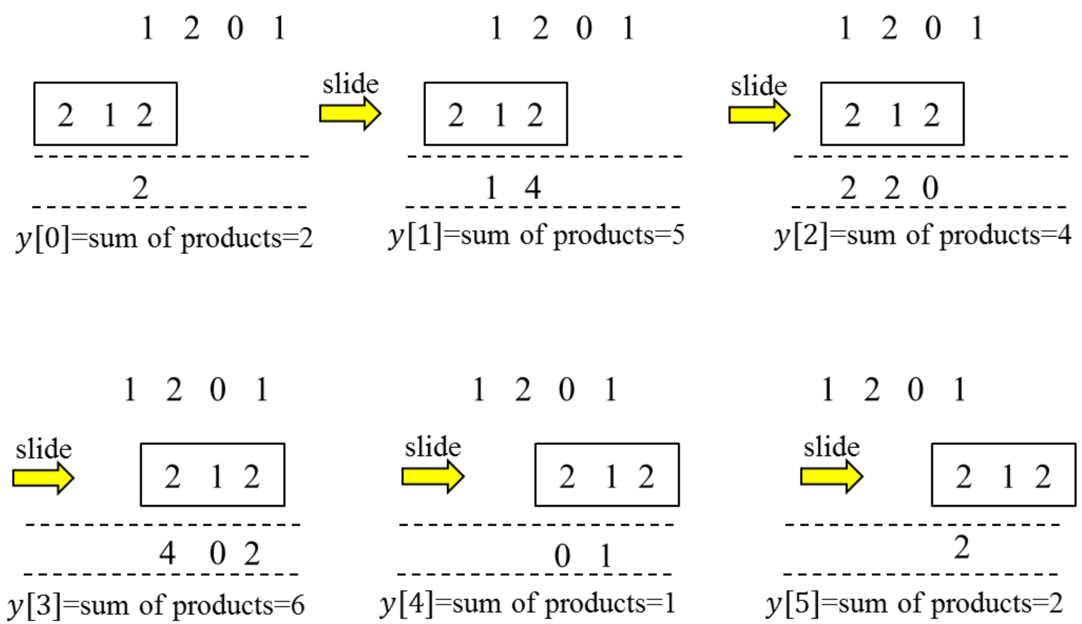
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k], \quad y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k], \quad y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k], \dots$$

- To obtain output
  - First time-reverse $h[k]$
  - Then slide $h[-k]$ by the amount of index n
  - Do the multiplication and sum with input $x[k]$, and repeat

- Ex) Do convolution using graphical method

$$x[n] = \{\mathbf{1},2,0,1\}, \qquad h[n] = \{\mathbf{2},1,2\}$$



Discrete Cross-Correlation

Cross-correlation

- Measure of similarity of two series as a function of the lag of one relative to the other
- Commonly used for searching a long signal for a shorter, known feature and time-delay analysis

$$r_{xh}[n] = x[n] ** h[n] = \sum_{k=-\infty}^{\infty} x[k]h[k-n]$$

  - “**” – cross correlation notation

- Correlation $r_{xh}[n] = x[n]**h[n] \neq 0$
- Un-correlation – no similarity $r_{xh}[n] = x[n]**h[n] = 0$

- Cross-correlation in terms of convolution

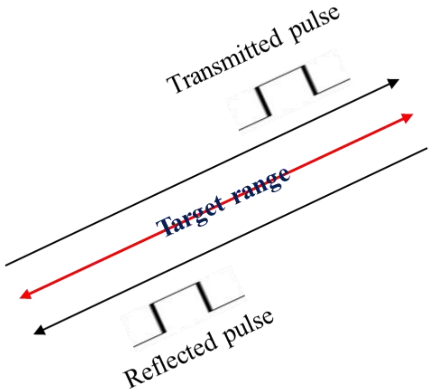
$$r_{xh}[n] = x[n]**h[n] = x[n]*h[-n]$$

- **Autocorrelation**

$$r_{xx}[n] = x[n]**x[n] = x[n]*x[-n]$$

$$|r_{xx}[n]| \leq r_{xx}[0]$$

- Radar target ranging
  - Estimate target distance with radar signal $x[n]$ and target reflected signal $s[n]$



$$s[n] = \alpha x[n-D] + p[n], \quad \alpha < 1$$

$$\boldsymbol{target\ range} = \boldsymbol{d} = \frac{\mathbf{1}}{\mathbf{2}}\frac{\boldsymbol{c}\cdot\boldsymbol{D}}{\boldsymbol{f_s}}$$

c =speed of light, D =round trip time
f_s =sampling rate

- Correlation receiver
  - A device that performs the cross-correlation

$$r_{sx}[n] = s[n]**x[n] = s[n]*x[-n]$$

  - In general, one can assume that the noise $p[n]$ is un-correlate with a radar signal x$[n]$

$$r_{px}[n] = p[n]**x[n] = p[n]*x[-n] = 0$$

$$r_{sx}[n] = s[n]*x[-n] = (\alpha x[n-D] + p[n])*x[-n]$$

$$= \alpha x[n-D]*x[-n] + p[n]*x[-n] = \alpha x[n-D]*x[-n]$$

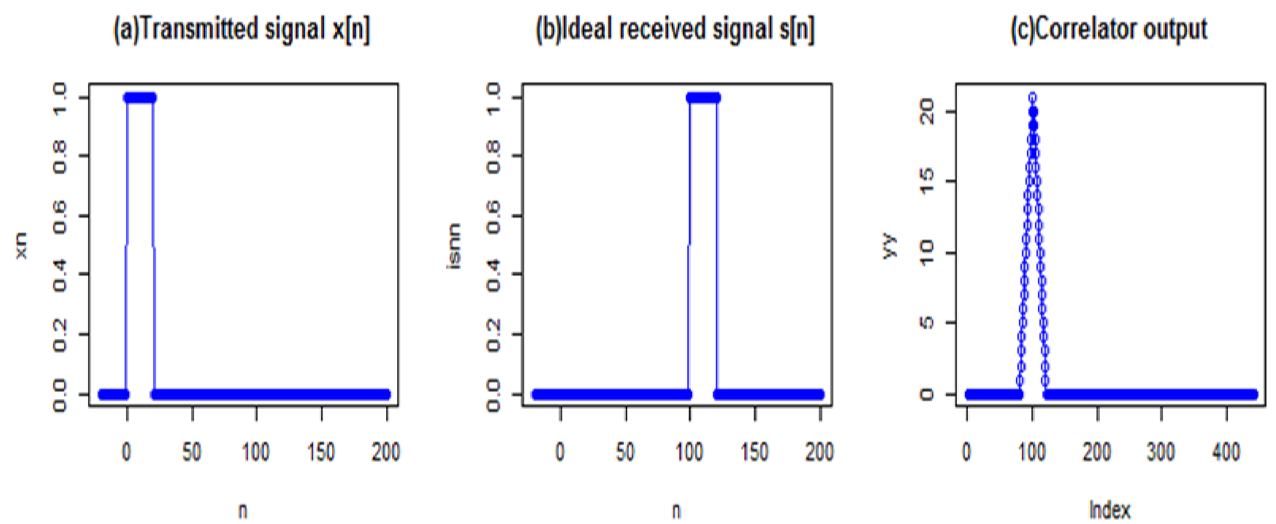
  - Delayed version of autocorrelation $r_{xx}[n]$, peak at D

- Target distance

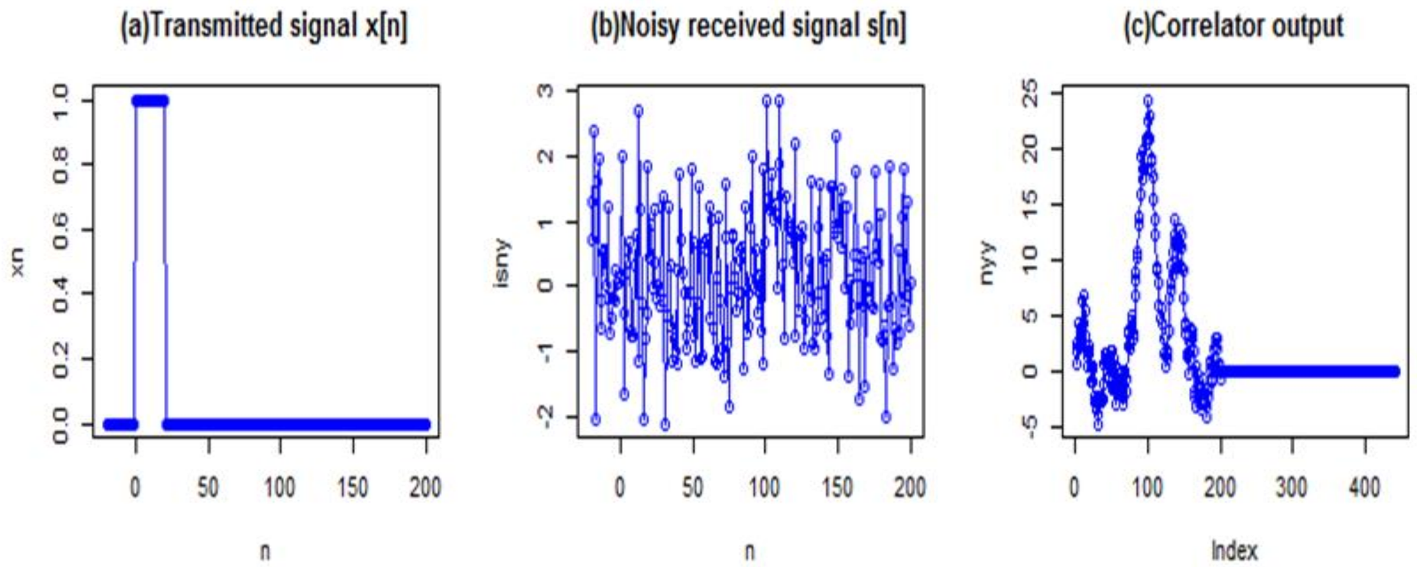
$$d = \frac{0.5 \cdot v \cdot D}{f_s}$$

  - sound velocity $v = 3X10^8 m/\sec$, D= round trip delay time
    f_s sampling rate of the signal $x[n]$

- Ex) Radar targeting
  - Radar signal $x[n]$ with length 20 rectangular pulse
  - Reflected signal $s[n]$ with an delay of 100 sample
  - With no noise condition, $p[n]$=0



- With $p[n]$ random Gaussian noise



Homework

- Exercise Problems
  - 4.1, 4.2, 4.3, 4.4, 4.5 (a)
  - 4.8
  - 4.13