Linear Programming : Duality and Graphical Solution

Section 1 : Formulate primal & dual
Section 2 : Apply properties of primals & duals
Section 3 : Given primal , solve for optimal solution using R / SAS
Section 4 : Given primal , solve for optimal solution & perform sensitivity analysis graphically

Example 1 (w/o lower bounds)

We are producing products 1 and 2 with unit profits \$3 and \$5 respectively. Their production requires resources R1 , R2 and R3 , available at 230 , 250 , 120 units respectively. To produce a unit of product 1, 2 unit of R1 and 1 unit of R2 is required. To produce a unit of product 2, 1 unit of R1 , 2 units of R2 and 1 unit of R3 is required. What is the optimal production level that maximises profit?

Primal (n variables , m constraints)

max $z = c^T x$
subject to $Ax \leq b$
$x \geq 0$

$$\max \quad z = 3x_1 + 5x_2$$
$$\text{subject to} \quad \begin{aligned} 2x_1 + x_2 &\leq 230 \\ x_1 + 2x_2 &\leq 250 \\ x_2 &\leq 120 \\ x_1 , x_2 &\geq 0 \end{aligned}$$

Dual (m variables , n constraints)

min $w = b^T y$
subject to $A^T y \geq c$
$y \geq 0$

$$\min \quad w = 230y_1 + 250y_2 + 120y_3$$
$$\text{subject to} \quad \begin{aligned} 2y_1 + y_2 &\geq 3 \\ y_1 + y_2 + y_3 &\geq 5 \\ y_1 , y_2 , y_3 &\geq 0 \end{aligned}$$

▷ Economic interpretation of dual problem :

↳ Part I : meaning of dual variables

y_1 = price paid for 1 unit of resource 1
y_2 = price paid for 1 unit of resource 2
y_3 = price paid for 1 unit of resource 3
total price paid = $230y_1 + 250y_2 + 120y_3$

↳ Part II : meaning of dual constraints

A buyer must be willing to pay ≥ \$3 for a basket of 2 units of R1 and 1 unit of R2
A buyer must be willing to pay ≥ \$5 for a basket of 1 unit of R1 , 2 units of R2 & 1 unit of R3

Note : If LP has lower bound constraints (eg $x_1 \geq 100$) , Part I is valid for dual variables which do not correspond to lower bound constraints . Part II is invalid.

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Property 1 : Weak duality

Let x be any feasible solution to the primal problem , and y be any feasible solution to the dual problem.
$\Rightarrow$ z-value of $x \leq$ w-value of y $(c^T x \leq b^T y)$

Property 2 : Strong duality

Let $\bar{x}$ be any feasible solution to the primal problem , and $\bar{y}$ be any feasible solution to the dual problem.
$\Rightarrow$ If $(c^T \bar{x} = b^T \bar{y})$, $\bar{x}$ is the optimal for the primal problem and $\bar{y}$ is optimal for the dual problem.
$\Rightarrow$ Optimal solution is primal feasible & dual feasible : $Ax \leq b$ and $A^T y \geq c$

Note : an LP must fall into one of 4 categories:
① has 1 unique solution
② has multiple solutions $\Rightarrow$ has infinitely many solutions
③ is infeasible (feasible region is an empty set)
④ is unbounded (there are points in feasible region with arbitrarily large / small z for max / min problem)

Property 3 : dual activity & binding constraint & slack

▷ slack . $s = b - Ax$ for primal problem and $s = c - A^T y$ for dual problem

▷ Shadow price / dual activity / dual multiplier of a constraint :
↑ 1 unit in RHS of a constraint → ↑ ? units in obj fn ← Note : only true within dual region

▷ constraint is binding ⇔ RHS = activity ⇔ slack = 0
constraint is not binding ⇔ RHS ≠ activity ⇔ slack ≠ 0 ⇔ dual activity = 0

▷ It is possible for slack $(s_i) = 0$ and dual activity $(y_i) = 0$

▷ $y_i > 0 \Rightarrow s_i = 0$, $y_i = 0 \Rightarrow s_i \geq 0$
$s_i > 0 \Rightarrow y_i = 0$, $s_i = 0 \Rightarrow y_i \geq 0$

▷ When dual problem has no solution , primal problem is unbounded above.

▷ $\leq$ constraint $\Rightarrow u \geq 0$
$\geq$ constraint $\Rightarrow u \leq 0$
$=$ constraint $\Rightarrow u \leq 0$ or $u \geq 0$
} maximisation problem

Miscellaneous properties on transformations

Suppose original problem is $z = c^T x$ $\Rightarrow$ optimal solution. $\bar{z}, \bar{x}, \bar{u}$

s.t. $Ax + b \leq 0$

$x \geq 0$

Optimal solution

Case 1

max $z = \alpha c^T x$, $\alpha > 0$ $\Rightarrow$ $\alpha \bar{z}$, $\bar{x}$, $\alpha \bar{u}$

subject to $Ax \leq b$

$x \geq 0$

Case 2

min $z = \alpha c^T x$, $\alpha > 0$ $\Rightarrow$ $z = 0$, $\bar{x} = \underset{\sim}{0}$, $\bar{u} = \underset{\sim}{0}$

subject to $Ax \leq b$

$x \geq 0$

Case 3

max $z = \alpha c^T x$, $\alpha < 0$ $\Rightarrow$ new solution

subject to $Ax \leq b$

$x \geq 0$

Case 4

min $z = \alpha c^T x$, $\alpha < 0$ $\Rightarrow$ $\alpha \bar{z}$, $\bar{x}$, $\alpha \bar{u}$

subject to $Ax \leq b$

$x \geq 0$

Case 5

max $z = c^T x$

subject to $\alpha A x \leq \alpha b$, $\alpha > 0$ $\Rightarrow$ $\bar{z}$, $\bar{x}$, $\frac{\bar{u}}{\alpha}$

$x \geq 0$

Case 5

max $z = c^T x$

subject to $\alpha A x \leq \alpha b$, $\alpha < 0$ $\Rightarrow$ new solution

$x \geq 0$

* Note: The transformations can be applied in combinations.

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Example 2 (w̄ lower bound constraints)

maximise profit, $z = 8x_1 + 12x_2$
subject to

labour constraint	$3x_1 + 5x_2 \leq 200$
budget constraint	$10x_1 + 15x_2 \leq 8000$
lower bound 1	$x_1 \geq 100$
lower bound 2	$x_2 \geq 200$

SAS input

```
data example;
input _id_ $ x1 x2 _type_ $ _rhs_;
datalines;
object 8 12 max .
labor 0.3 0.5 LE 200
budget 10 15 LE 8000
x1lower 1 0 GE 100
x2lower 0 1 GE 200
;
Proc lp rangeprice rangerhs;
run;p
```

SAS output

"How much you have"

"How much you use"

The LP Procedure

Constraint Summary

Row	Constraint Name	Type	S/S Col	Rhs	Activity	Dual Activity
1	object	OBJECTVE	.	0	5066.6667	.
2	labor	LE	3	$\boxed{200}$	200	26.666667
3	budget	LE	4	8000	6333.3333	0
4	x1lower	GE	5	100	333.33333	0
5	x2lower	GE	6	200	200	$\boxed{-1.333333}$

If constraint of $x_2 \geq 200$ is changed to $x_2 \geq 201$,
total profit ↓ by $1.33
↳ indeed, new production level occurs at intersection b/w labour & x_2 lower constraints

$x_2 = 201 \quad x_1 = \frac{200 - 0.5(201)}{0.3} = 333\frac{2}{3}$

Δ profit $= \left[8(331\frac{2}{3}) + 12(201)\right] - 5066\frac{2}{3} = -1\frac{1}{3}$

ΔRHS of constraint

RHS Range Analysis

Row	Minimum Phi: Rhs	Minimum Phi: Leaving	Minimum Phi: Objective	Maximum Phi: Rhs	Maximum Phi: Leaving	Maximum Phi: Objective
labor	$\boxed{130}$	x1lower	$\boxed{3200}$	$\boxed{250}$	budget	$\boxed{6400}$
budget	6333.3333	budget	5066.6667	INFINITY	.	.
x1lower	-INFINITY	.	.	333.33333	x1lower	5066.6667
x2lower	0	x2	5333.3333	340	x1lower	4880

As long as $\boxed{130} \leq$ labour $\leq \boxed{250}$, the optimal solution is the intersection of labour and x_2 lower constraints

∴ current value $= \boxed{200}$ must lie within this range

- If labour $< \boxed{130}$, problem is infeasible (there are no X_1 and X_2 that simultaneously satisfy all constraints)
  (* true in this specific case)
- At labour $= \boxed{130}$, ie when $(x_1, x_2) = (100, 200)$, $Z = 5066\frac{2}{3} + (130 - 200)\, 26\frac{2}{3} = \boxed{3200}$

- If labour > 250, labour constraint becomes redundant, budget constraint becomes active constraint
- At labour $= 250$, $(x_1, x_2) = (500, 200)$, $Z = 8x_1 + 12x_2 = 5066\frac{2}{3} + (250 - 200)\, 26\frac{2}{3} = \boxed{6400}$

Δ objective function coefficient

```
                      Price Range Analysis

   Variable ---------Minimum Phi-------- ---------Maximum Phi--------
Col Name          Price Entering Objective    Price Entering Objective

  1 x1              7.2 x2lower        4800 INFINITY .          INFINITY
  2 x2        -INFINITY .         -INFINITY 13.333333 x2lower  5333.3333
```

- Range of price coefficient of … (in objective function such that $(333\frac{1}{3}, 200)$ is optimal)
  - $x_1 = [7.2, \infty)$, $x_2 = 12$
  - $x_1 = 8$, $x_2 = (-\infty, 13\frac{1}{3}]$

- If unit profit for $x_1 < \boxed{7.2}$, ↓ production of x_1 and ↑ production of x_2 (if possible)
  $\therefore$ ↓ x_1 to 100, $\because$ labour constraint , $x_2 = \frac{200 - 30}{0.5} = 340$, $Z = \boxed{4800}$

- When unit profit of Product 1 increases, we want to ↑ production of product 1 and ↓ production of product 2. But we are already producing min Product 2 subject to x2lower. $\therefore$ Unit profit of product 1 can ↑ $\boxed{\text{indefinitely}}$ and solution remains at $(333\frac{1}{3}, 200)$

R input (lpSolve)

```
library(lpSolve)
library(linprog)

cvec <- c(8,12)
names(cvec) <- c("X1", "X2")

bvec <- c(200, 8000, 100, 200)
names(bvec) <- c("Labor", "Budget", "X1lower", "X2lower")

Amat <- rbind( c(0.3, 0.5),
               c(10 , 15 ),
               c(1  , 0  ),
               c(0  , 1  ) )

res <- solveLP(cvec, bvec, Amat, const.dir = c( "<=", "<=", ">=", ">="), TRUE)
res
```

objective coefficients

RHS of constraints

constraint matrix

R output (lpSolve)

R Console | data.frame 4 x 6 | data.frame 6 x 6

```
Results of Linear Programming / Linear Optimization

Objective function (Maximum): 5066.67

Iterations in phase 1: 2
Iterations in phase 2: 1
Solution
       opt
X1 333.333
X2 200.000

Basic Variables
              opt
X1        333.333
X2        200.000
S Budget  1666.667
S X1lower  233.333

Constraints

All Variables (including slack variables)
```

R Console | data.frame 4 x 6 | data.frame 6 x 6

slack

↑ 1 unit of labour ⇒ ↑ π by \$26.67

	actual <dbl>	dir <chr>	bvec <dbl>	free <dbl>	dual <dbl>	dual.reg <dbl>
Labor	200.000	<=	200	0.000	26.66667	70.000
Budget	6333.333	<=	8000	1666.667	0.00000	1666.667
X1lower	333.333	>=	100	233.333	0.00000	233.333
X2lower	200.000	>=	200	0.000	1.33333	140.000

4 rows

labour can be ↓ from curr 200 to 130 w/o Δ optimal solution

If profit from product 1 $\in [7.2, \infty)$, optimal solution remains as such

R Console | data.frame 4 x 6 | data.frame 6 x 6

	opt <dbl>	cvec <dbl>	min.c <dbl>	max.c <dbl>	marg <dbl>	marg.reg <dbl>
X1	333.333	8	7.2	Inf	NA	NA
X2	200.000	12	-Inf	13.33333	NA	NA
S Labor	0.000	0	-Inf	26.66667	-26.66667	70
S Budget	1666.667	0	NA	0.80000	0.00000	NA
S X1lower	233.333	0	-0.8	Inf	0.00000	NA
S X2lower	0.000	0	-Inf	1.33333	-1.33333	140

6 rows

R input (lp)

```
library(lpSolve)

lp.obj <- c(8,12)

lp.con <- rbind( c(0.3, 0.5),
               c(10 , 15 ),
               c(1  , 0  ),
               c(0  , 1  ) )

lp.dir <- c("<=", "<=", "<=", "<=")

lp.rhs <- c(200, 8000, 100, 200)

sol <- lp("max", lp.obj, lp.con, lp.dir, lp.rhs, compute.sens = TRUE)
```

R output (lp)

```
> sol
Success: the objective function is 3200
> sol$solution
[1] 100 200
> sol$duals
[1]  0  0  8 12  0  0
> sol$sens.coef.from
[1] 8.881784e-16 0.000000e+00
> sol$sens.coef.to
[1] 1e+30 1e+30
> sol$duals.from
[1] -1.000000e+30 -1.000000e+30 -1.421085e-14  0.000000e+00 -1.000000e+30 -1.000000e+30
> sol$duals.to
[1] 1.000000e+30 1.000000e+30 3.333333e+02 3.400000e+02 1.000000e+30 1.000000e+30
```

primal

Δ obj fn value w/o Δ optimal soln

dual

Δ obj fn value w/o Δ optimal soln

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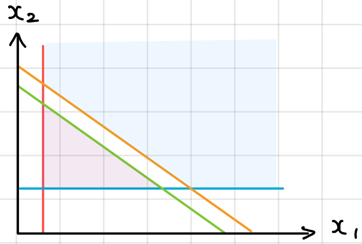
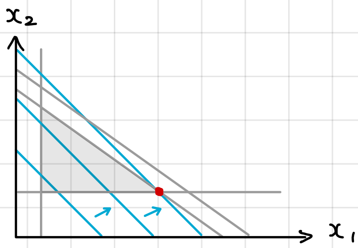
* The examples below are not exhaustive of all possibilities arising from different directions and magnitudes of change

Constraints & feasible region



: constraints
: feasible region for maximisation problem
: feasible region for minimisation problem

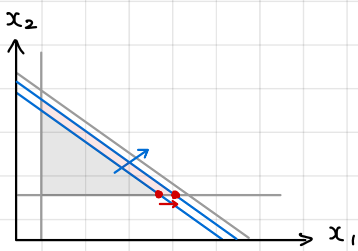
Optimal solution



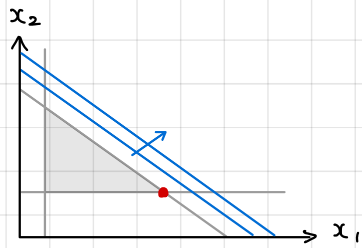
: isoprofit lines
: optimal solution

Sensitivity analysis

Δ RHS of constraint



: expansion of feasible region

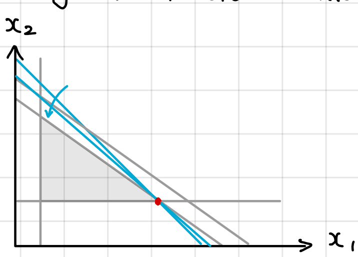


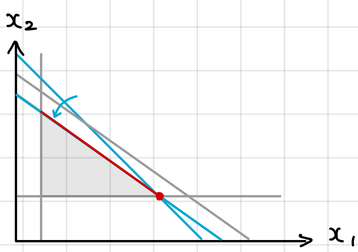
$a_1x_1 + a_2x_2 \leq b \Rightarrow a_1x_1 + a_2x_2 \leq (b + \delta)$

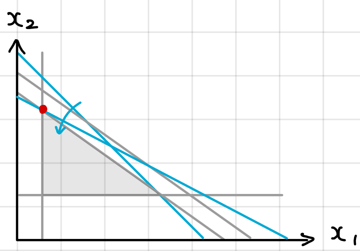
If $\delta > 0$: Feasible region enlarges or remains unchanged.

∴ max / min problem : obj fn value may remain unchanged or ↑ / ↓ respectively

Δ objective function coefficients







max $z = c_1x_1 + c_2x_2 \Rightarrow$ max $z = (c_1 + \delta)x_1 + c_2x_2$

If $\delta < 0$, objective function becomes flatter. If $\delta > 0$, objective function becomes steeper.