DMTH237 Discrete Mathematics II — Assignment 3

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1. (a) We have $26 \times 2 + 10 + 6 = 68$ available characters. Hence, there are $68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12} = 9,920,671,339,261,325,541,376$ different passwords.

   (b) With only letters and numbers we have $26 \times 2 + 10 = 62$ characters so that there are $62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12}$ passwords which *do not* contain at least one special character. Subtracting this from the number of passwords found in part 1a we have

$$(68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12}) - (62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12})$$
$$= 6,641,514,961,387,068,437,760$$

   which is the number of passwords not containing at least one special character.

   (c) Checking each password would take 314.6 billion years (short scale)—about 23 times the age of the universe!

2. To solve this problem we note the following.

   - Without restricting the value of each digit we have 10^9 possible strings.
   - There are 9^9 strings which do not contain the digit 1 but may contain 3 or 7.
   - There are $9^9 - 8^9$ strings which do not contain the digit 3, contain at least a single 1 and may contain a 7. This is a problem of the same kind as in part 1b.
   - There are $9^9 - 2(8^9 - 7^9) - 7^9$ strings which do not contain the digit 7 but contain at least a single 1 and at least a single 3. This is because there are $8^9 - 7^9$ strings which do not contain the digits 1 and 7 but contain at least a single 3; there are $8^9 - 7^9$ strings which do not contain the digits 3 and 7 but contain at least a single 1 and; there are 7^9 strings which do not contain the digits 1, 3 or 7.

   From this we see that the number of strings which have 1, 3 and 7 appearing at least once is given by

$$\left(10^9\right) - \left(9^9\right) - \left(9^9 - 8^9\right) - \left(9^9 - 2(8^9 - 7^9) - 7^9\right)$$
$$= 200,038,110.$$

3. (a) A teacher has n students and asks them all m questions. Any student who gets a question correct gets a piece of candy. (It is possible that no student is correct, so that no one gets candy.)

   Now, if r questions are answered correctly there are

$$\binom{n+r-1}{r}$$

ways the candy might be distributed among the students. By the rule of sum,

$$\sum_{r=0}^{m} \binom{n+r-1}{r}$$

is the number of different distributions of candy for a varying number of correct answers. The RHS,

$$\binom{n+m}{m},$$

is then the number of ways to distribute the students among the candy, including the case where no students answer correctly.

(b) First we show that the base case (where $m = 0$) holds.

$$\begin{aligned}
\sum_{r=0}^{m} \binom{n+r-1}{r} &= \binom{n+m}{m} \qquad (1)\\
\sum_{r=0}^{0} \binom{n+r-1}{r} &= \binom{n+0}{0}\\
\binom{n+0-1}{0} &= \binom{n+0}{0}\\
1 &= 1
\end{aligned}$$

Now, in order to undertake the inductive step we will use

$$\sum_{j=0}^{n} \binom{j}{k} = \binom{n+1}{k+1}, \qquad \text{for every } n, k \in \mathbb{N} \qquad (2)$$

which, for completeness, we prove as follows:

$$\begin{aligned}
\sum_{j=0}^{n} \binom{j}{k} &= \sum_{j=0}^{n} \left[\binom{j+1}{k+1} - \binom{j}{k+1} \right]\\
&= \left[\binom{1}{k+1} - \binom{0}{k+1} \right] + \left[\binom{2}{k+1} - \binom{1}{k+1} \right] +\\
&\quad \left[\binom{3}{k+1} - \binom{2}{k+1} \right] + \cdots + \left[\binom{n+1}{k+1} - \binom{n}{k+1} \right]\\
&= -\binom{0}{k+1} + \left[\binom{1}{k+1} - \binom{1}{k+1} \right] + \left[\binom{2}{k+1} - \binom{2}{k+1} \right]\\
&\quad + \cdots + \left[\binom{n}{k+1} - \binom{n}{k+1} \right] + \binom{n+1}{k+1}\\
&= \binom{n+1}{k+1}.
\end{aligned}$$

Completing the inductive step, we must have

$$
\begin{aligned}
\sum_{r=0}^{m+1} \binom{n+r-1}{r} &= \binom{n+m+1}{m+1} && \text{(by the induction hypothesis)} \\
\sum_{r=0}^{m+1} \binom{n+r-1}{n-1} &= \binom{n+m+1}{m+1} && \text{(n choose k equals n choose $n-k$)} \\
\sum_{s=n-1}^{n+m} \binom{s}{n-1} &= \binom{n+m+1}{m+1} && \text{(letting $s = n+r-1$)} \\
\binom{n+m+1}{n-1+1} &= \binom{n+m+1}{m+1} && \text{(by equation (2))} \\
\binom{n+m+1}{m+1} &= \binom{n+m+1}{m+1}, && \text{(n choose k equals n choose $n-k$)}
\end{aligned}
$$

which was to be proved.

4. Two integers a_i, a_{i+1} are consecutive if $a_{i+1} > a_i + 1$. Hence we have

$$1 \leq a_1 < a_2 - 1 < a_3 - 2 < a_4 - 3 < a_5 - 4 < a_6 - 5 < a_7 - 6 \leq 50 - 6.$$

Writing $b_i = a_i - (i-1)$, we have

$$1 \leq b_1 < b_2 < \cdots < b_7 \leq 44.$$

Now there clearly exists a bijection between every a_i, b_i. Thus, the number of ways to choose $a_i, 1 \leq i \leq 7$ is equal to the number of ways to choose $b_i, 1 \leq i \leq 7$.

The number of ways to choose the seven different numbers is therefore

$$\binom{44+1}{7} = 45,379,620.$$

5. The left and right machines accept the languages $\mathcal{L}$ and $\mathcal{M}$. Their respective state tables are show below.

	0	**1**	
$\rightarrow A$	B	C	
B	D	E	
C	F	Z	
D	Z	Z	$*$
E	G	G	$*$
F	Z	Z	$*$
G	Z	A	
Z	Z	Z	

	0	**1**	
$\rightarrow R$	S	R	$*$
S	T	S	
T	R	T	

The state tables for the machines which accept the languages $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ are shown below.

	0	**1**	
$\rightarrow A$	B	C	$*$
B	D	E	$*$
C	F	Z	$*$
D	Z	Z	
E	G	G	
F	Z	Z	
G	Z	A	$*$
Z	Z	Z	$*$

	0	**1**	
$\rightarrow R$	S	R	
S	T	S	$*$
T	R	T	$*$

Combining the above tables to find a finite state acceptor for the language $\overline{\mathcal{L}} + \overline{\mathcal{M}}$ yields the mutually equivalent machines shown below.

	0	**1**	λ	
$\to A_0$			AR	
A	B	C		$*$
B	D	E		$*$
C	F	Z		$*$
D	Z	Z		
E	G	G		
F	Z	Z		
G	Z	A		$*$
Z	Z	Z		$*$
R	S	R		
S	T	S		$*$
T	R	T		$*$

	0	**1**	
$\to AR$	BS	CR	$*$
BS	CR	ES	$*$
CR	CR	CR	$*$
ES	GT	GS	$*$
GT	CR	AT	$*$
GS	CR	AS	$*$
AT	BR	CT	$*$
AS	BT	CR	$*$
BR	CR	ER	$*$
CT	FR	CR	$*$
BT	FR	ET	$*$
ER	GS	GR	
FR	CR	CR	
ET	GR	GT	$*$
GR	CR	AR	$*$

	0	**1**	
$\to 0$	1	2	$*$
1	2	3	$*$
2	2	2	$*$
3	4	5	$*$
4	2	6	$*$
5	2	7	$*$
6	8	9	$*$
7	10	2	$*$
8	2	11	$*$
9	12	2	$*$
10	12	13	$*$
11	5	14	
12	2	2	
13	14	4	$*$
14	2	0	$*$

Hence, we have the machine accepting strings in the language $\mathcal{L} \cap \mathcal{M}$, shown below.

	0	**1**	
$\to 0$	1	2	
1	2	3	
2	2	2	
3	4	5	
4	2	6	
5	2	7	
6	8	9	
7	10	2	
8	2	11	
9	12	2	
10	12	13	
11	5	14	$*$
12	2	2	$*$
13	14	4	
14	2	0	