DMTH237 Discrete Mathematics II — Assignment 2

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1. As the lines described by each equation have the same slope – that is, they are parallel – they will either intersect at infinitely many points or none at all. In order for the lines to intersect they must be equivalent for all $x, y, k \in \mathbb{R}$. In other words,
$$\lambda(x - y - 2) = 3x - 3y - k,$$
where λ is some constant.

   Now, setting $\lambda = 3$ and $k = 6$, the system has infinitely many solutions. When $k \neq 6$, no solutions exist.

2. (a)
$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right) \tag{1}$$
$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right)$$
$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 17/5 & -8/5 & 7/5 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right) \tag{2}$$

   Thus, the system is inconsistent and no solutions exist.

   (b) The matrix for the given system is
$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & -1 \end{array}\right).$$

   We notice that the only difference between this and the (left) matrix in equation (1) is the last entry, which is now -1 instead of 5. So, we simply subtract $5 - (-1) = 6$ from the corresponding entry in equation (2) to obtain
$$\left(\begin{array}{cccc|c} 1 & 0 & 17/5 & -8/5 & 7/5 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$
   the reduced system. Thus we have
$$\begin{aligned} 5x + 17z - 8w &= 7, \\ 5y + 6z + 6w &= 1, \end{aligned}$$
   as our solution.

   Note that if, during row reduction in part (a), we multiplied the last row by a scalar c, we would have subtracted $c(5 - (-1))$ instead.

3. (a) Augmenting the given matrix with the identity matrix, we have

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right).$$

Therefore the inverse is

$$\begin{pmatrix} 1 & -2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

(b) The inverse of a non-square matrix is undefined.

(c) Augmenting the given matrix with the identity matrix and performing row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & 1 & 5 & 0 & 1 & 0 \\ 0 & -3 & -8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & 8 & 1 & 1 & 0 \\ 0 & -3 & -8 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & 8 & 1 & 1 & 0 \\ 0 & -3 & -8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 8/3 & 1/3 & 1/3 & 0 \\ 0 & -3 & -8 & 0 & 0 & 1 \end{array}\right)$$

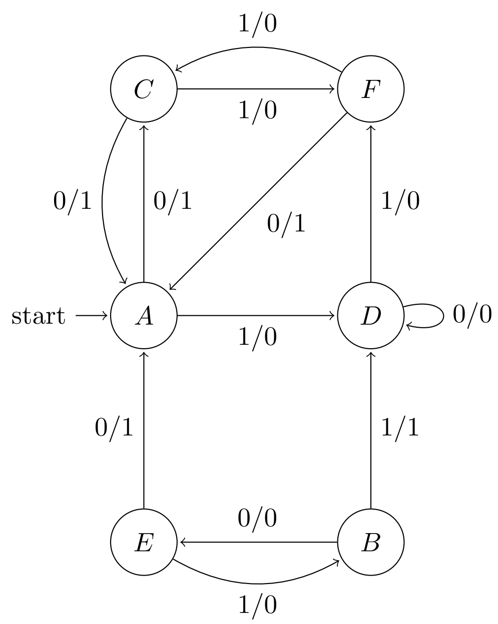
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 8/3 & 1/3 & 1/3 & 0 \\ 0 & -3 & -8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -7/3 & 1/3 & -2/3 & 0 \\ 0 & 1 & 8/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

From this we see the matrix does not have an inverse as it is singular.

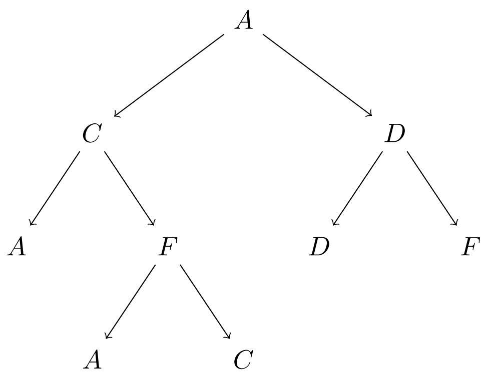
4. (a) Below is a table showing input, output and the corresponding state transitions.

Input	0	1	1	0	1	1	1	0	1	1	0	0	1	
Output	1	0	0	1	0	0	0	1	0	0	1	1	0	
State	A	C	F	C	A	D	F	C	A	D	F	A	C	F

(b) Shown is a state diagram corresponding to the Mealy machine in question 4.



(c) A tree diagram representing (the accessible part of) the Mealy machine is shown in figure 1a below.



(a)

	Transition		Output	
	0	**1**	**0**	**1**
$\to A$	C	D	1	0
C	A	F	1	0
D	D	F	0	0
F	A	C	1	0

(b)

Figure 1

Notice that every "leaf state" was also included in the tree diagram as a "branch state." Thus, only the states shown in figure 1a are accessible. We therefore have the transition table in figure 1b.

(d) The 0-equivalence classes are shown in the table below under the $\equiv_0$ column.

	Transition		Output		
	0	**1**	**0**	**1**	$\equiv_0$
$\to A$	C	D	1	0	0
C	A	F	1	0	0
D	D	F	0	0	1
F	A	C	1	0	0

Although though this machine has only two 0-equivalence classes, there can be up to four. This is because the output set, whose cardinality is 2, depends on the input set, whose cardinality is also 2. (And $2 \times 2 = 4$.)

(e) The table showing equivalence classes is as follows.

	Transition		Output								
	0	**1**	**0**	**1**	$\equiv_0$	**0**	**1**	$\equiv_1$	**0**	**1**	$\equiv_2$
$\to A$	C	D	1	0	0	0	1	0	1	2	0
C	A	F	1	0	0	0	0	1	0	1	1
D	D	F	0	0	1	1	0	2	2	1	2
F	A	C	1	0	0	0	0	1	0	1	1

Identifying states C and F yields the Mealy machine shown below.

	Transition		Output	
	0	**1**	**0**	**1**
$\to A$	C	D	1	0
C	A	C	1	0
D	D	C	0	0

(f) The reduced Mealy machine in standard form is as follows.

	Transition		Output	
	0	**1**	**0**	**1**
$\to 0$	1	2	1	0
1	0	1	1	0
2	2	1	0	0

(g) Performing steps (d) and (e) on the given Mealy machine yields the following table of equivalence classes.

	Transition		Output								
	0	**1**	**0**	**1**	$\equiv_0$	**0**	**1**	$\equiv_1$	**0**	**1**	$\equiv_2$
$\to A$	C	D	1	0	0	0	2	0	2	3	0
B	E	D	0	1	1	0	2	1	4	3	1
C	A	F	1	0	0	0	0	2	0	2	2
D	D	F	0	0	2	2	0	3	3	2	3
E	A	B	1	0	0	0	1	4	0	1	4
F	A	C	1	0	0	0	0	2	0	2	2

Just as in part (e) of this question, states C and F can be identified. The inaccessible states were not however, as none of them were equivalent.

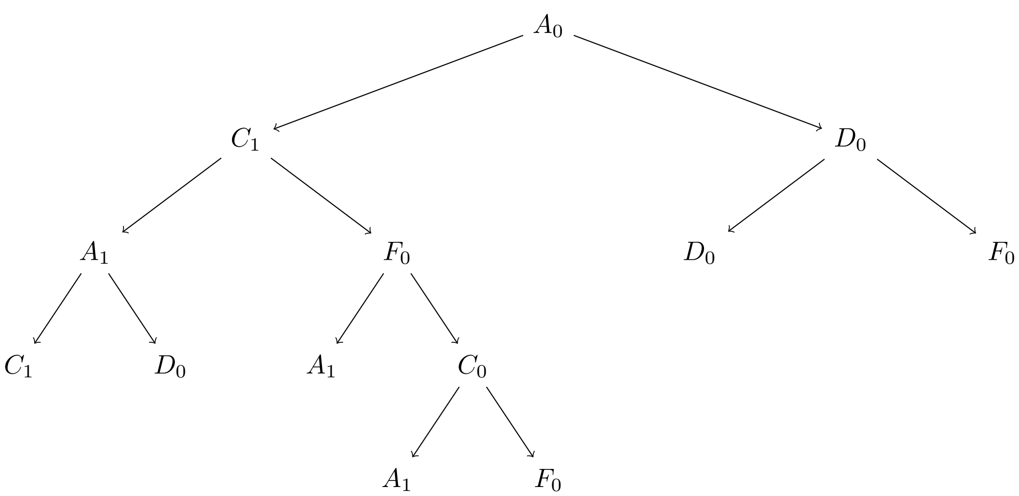
In general, performing step (c) last will yield the same machine, however steps (d) and (e) are made more difficult.

(h) The Mealy machine can be converted to a Moore machine by having a version of each state for each possible output.

For example, in the original machine, either a 0 or a 1 can be outputted in state A. In the corresponding Moore machine there should be one state A_0 which outputs a 0 and another A_1 which outputs 1. The resulting machine is shown below.

	Transition		Output
	0	**1**	
$\to A_0$	C_1	D_0	0
A_1	C_1	D_0	1
B_0	E_0	D_1	0
B_1	E_0	D_1	1
C_0	A_1	F_0	0
C_1	A_1	F_0	1
D_0	D_0	F_0	0
D_1	D_0	F_0	1
E_0	A_1	B_0	0
E_1	A_1	B_0	1
F_0	A_1	C_0	0
F_1	A_1	C_0	1

Now we remove inaccessible states by creating a tree diagram of accesses.



Preserving only the states shown in the above tree diagram (excluding the root) yields the following transition table.

	Transition 0	Transition 1	Output
$\to A_1$	C_1	D_0	1
C_0	A_1	F_0	0
C_1	A_1	F_0	1
D_0	D_0	F_0	0
F_0	A_1	C_0	0

Creating a table to find equivalent states, we have the following.

	0	**1**	$\equiv_0$	**0**	**1**	$\equiv_1$	**0**	**1**	$\equiv_2$	**0**	**1**	$\equiv_3$
$\to A_1$	C_1	D_0	1	1	0	0	0	2	0	2	3	0
C_0	A_1	F_0	0	1	0	1	0	1	1	0	1	1
C_1	A_1	F_0	1	1	0	0	0	1	2	0	1	2
D_0	D_0	F_0	0	0	0	2	2	1	3	3	1	3
F_0	A_1	C_0	0	1	0	1	0	1	1	0	1	1

Identifying states C_0 and F_0 leaves us with the table shown below.

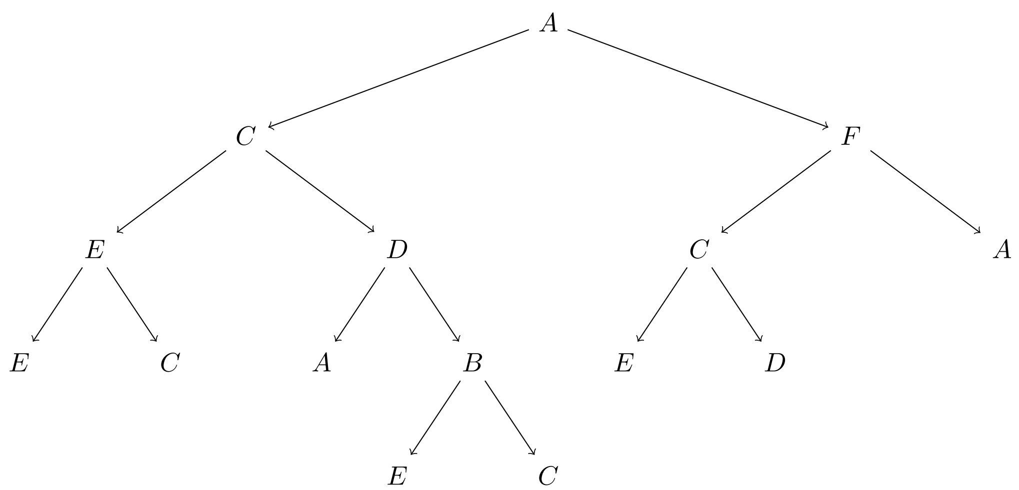
	Transition 0	Transition 1	Output
$\to A_1$	C_1	D_0	1
C_0	A_1	C_0	0
C_1	A_1	C_0	1
D_0	D_0	C_0	0

This Moore machine could have also been obtained by immediately converting from the reduced Mealy Machine in part (e).

(i) The reduced Moore machine is shown below in standard form.

	Transition 0	Transition 1	Output
$\to 0$	1	2	1
1	0	3	1
2	2	3	0
3	0	3	0

5. (a) From the tree diagram below, all states in the given FSA are accessible.

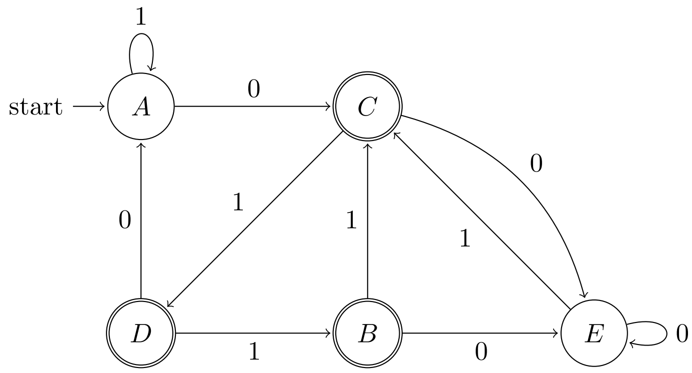


(b) To reduce a finite state machine, inaccessible states (if they exist) must be removed.

(c) The table showing equivalence classes for the given FSA is included below.

	0	**1**	$\equiv_0$	**0**	**1**	$\equiv_1$	**0**	**1**	$\equiv_2$	**0**	**1**	$\equiv_3$	**0**	**1**	$\equiv_4$
$\to A$	C	F	0	1	0	0	1	0	0	1	0	0	2	0	0
B	E	C	1	0	1	1	2	1	1	3	1	1	4	2	1
C	E	D	1	0	1	1	2	1	1	3	2	2	4	3	2
D	A	B	1	0	1	1	0	1	2	0	1	3	0	1	3
E	E	C	0	0	1	2	2	1	3	3	1	4	4	2	4
F	C	A	0	1	0	0	1	0	0	1	0	0	2	0	0

From the table, states A and F are equivalent. This is shown in the state diagram which follows.



(d) The reduced FSA can be also be written as a transition table.

	0	**1**	
$\to A$	C	A	
B	E	C	$*$
C	E	D	$*$
D	A	B	$*$
E	E	C	

(e) In standard form, the machine is as follows.

	0	**1**	
$\to 0$	1	0	
1	2	3	$*$
2	2	1	
3	0	4	$*$
4	2	1	$*$