1. (a)

$$\begin{aligned}(2-3i)\times(1+5i) &= 2(1+5i)-3i(1+5i)\\ &= 2+10i-3i+15\\ &= 17+7i\end{aligned}$$

(b)

$$\begin{aligned}\left(\tfrac{1}{3}-\tfrac{3}{8}i\right)\times\left(\tfrac{1}{2}+\tfrac{1}{7}i\right) &= \tfrac{1}{3}\left(\tfrac{1}{2}+\tfrac{1}{7}i\right)-\tfrac{3}{8}i\left(\tfrac{1}{2}+\tfrac{1}{7}i\right)\\ &= \tfrac{1}{6}+\tfrac{1}{21}i-\tfrac{3}{16}i+\tfrac{3}{56}\\ &= \tfrac{37}{168}-\tfrac{47}{336}i\end{aligned}$$

2. (a)

$$\begin{aligned}r &= \sqrt{0^2+1^2}=1\\ \theta &= \frac{\pi}{2}\\ z &= 1\angle\tfrac{\pi}{2}\end{aligned}$$

(b)

$$\begin{aligned}r &= \sqrt{(\sqrt{3})^2+3^2}=2\sqrt{3}\\ \theta &= \arctan\left(\tfrac{-3}{\sqrt{3}}\right)=-\tfrac{\pi}{3}\\ z &= 2\sqrt{3}\angle\tfrac{5\pi}{3}\end{aligned}$$

(c)

$$\begin{aligned}r &= \sqrt{3^2+(\sqrt{3})^2}=2\sqrt{3}\\ \theta &= \arctan\left(\tfrac{\sqrt{3}}{3}\right)=\tfrac{\pi}{6}\\ z &= 2\sqrt{3}\angle\tfrac{\pi}{6}\end{aligned}$$

(d)

$$\begin{aligned}r &= \sqrt{3^2+(\sqrt{3})^2}=2\sqrt{3}\\ \theta &= \arctan\left(\tfrac{\sqrt{3}}{3}\right)+\pi=\tfrac{7\pi}{6}\\ z &= 2\sqrt{3}\angle\tfrac{7\pi}{6}\end{aligned}$$

(e)

$$\begin{aligned}r &= \sqrt{3^2+3^2}=3\sqrt{2}\\ \theta &= \arctan\left(\tfrac{3}{3}\right)+\pi=\tfrac{5\pi}{4}\\ z &= 3\sqrt{2}\angle\tfrac{5\pi}{4}\end{aligned}$$

3. Using the results from part question two we write

(a)

$$\begin{aligned}z^2 &= 1^2\angle 2\times\tfrac{\pi}{2}\\ &= 1\angle\pi\end{aligned}$$

$$\begin{aligned}z^{42} &= 1^{42}\angle 42\times\tfrac{\pi}{2}\\ &= 1\angle 21\pi\\ &= 1\angle\pi\end{aligned}$$

(b)

$$z^2 = (2\sqrt{3})^2 \angle 2 \times \tfrac{5\pi}{3} \qquad z^{42} = (2\sqrt{3})^{42} \angle 42 \times \tfrac{5\pi}{3}$$
$$= 12\angle \tfrac{10\pi}{3} \qquad = (2\sqrt{3})^{42}\angle 70\pi$$
$$= 12\angle \tfrac{4\pi}{3} \qquad = (2\sqrt{3})^{42}\angle 0$$

(c)

$$z^2 = (2\sqrt{3})^2 \angle 2 \times \tfrac{\pi}{6} \qquad z^{42} = (2\sqrt{3})^{42} \angle 42 \times \tfrac{\pi}{6}$$
$$= 12\angle \tfrac{\pi}{3} \qquad = (2\sqrt{3})^{42}\angle 7\pi$$
$$= (2\sqrt{3})^{42}\angle \pi$$

(d)

$$z^2 = (2\sqrt{3})^2 \angle 2 \times \tfrac{7\pi}{6} \qquad z^{42} = (2\sqrt{3})^{42} \angle 42 \times \tfrac{7\pi}{6}$$
$$= 12\angle \tfrac{7\pi}{3} \qquad = (2\sqrt{3})^{42}\angle 49\pi$$
$$= 12\angle \tfrac{\pi}{3} \qquad = (2\sqrt{3})^{42}\angle \pi$$

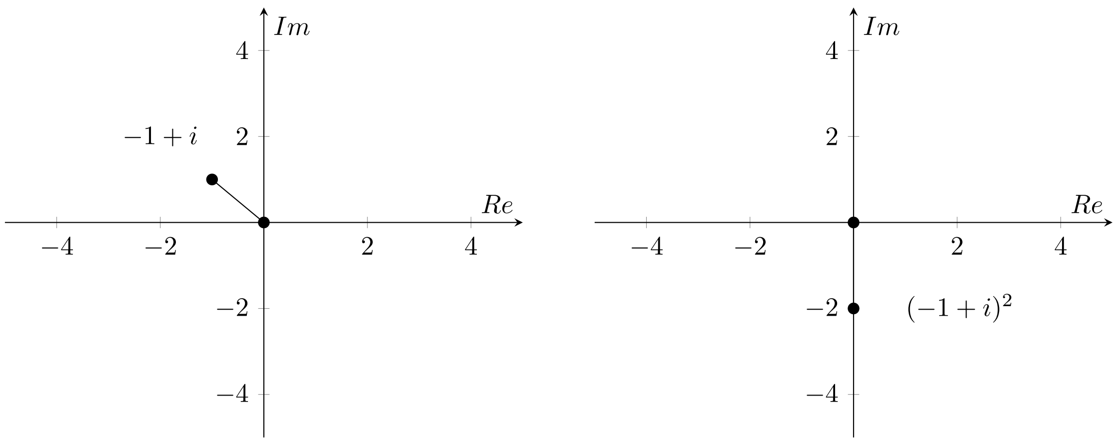
(e)

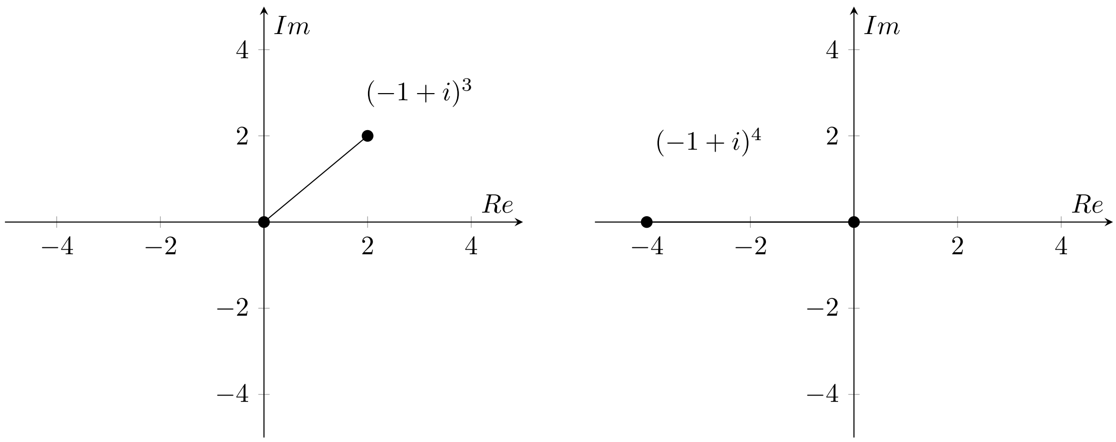
$$z^2 = (3\sqrt{2})^2 \angle 2 \times \tfrac{5\pi}{4} \qquad z^{42} = (3\sqrt{2})^{42} \angle 42 \times \tfrac{5\pi}{4}$$
$$= 18\angle \tfrac{5\pi}{2} \qquad = (3\sqrt{2})^{42}\angle \tfrac{105\pi}{2}$$
$$= 18\angle \tfrac{\pi}{2} \qquad = (3\sqrt{2})^{42}\angle \tfrac{\pi}{2}$$

4. The first four powers of $-1+i$ are given by

$$\begin{aligned}(-1+i) &= \sqrt{2}\angle \tfrac{3\pi}{4} = z,\\ (-1+i)^2 &= 2\angle \tfrac{6\pi}{4},\\ (-1+i)^3 &= 2\sqrt{2}\angle \tfrac{9\pi}{4},\\ (-1+i)^4 &= 4\angle \tfrac{12\pi}{4}.\end{aligned}$$

The plots of these are as follows.





We notice that the argument of z increases by $\frac{3\pi}{4}$ each time the power increases, so we expect the argument of $(-1+i)^{43}$ to equal $43 \times \frac{3\pi}{4}$.

5. Let $f(x) : \mathcal{B} \to \mathbb{N}$ be a function mapping from a binary string to a natural number. Then f is one-to-one as, for every $b \in \mathcal{B}$, $f(b) = f(n)$ if and only if $a = b$. Also f is onto as we have

$$\begin{aligned} f(0) &= 0, \\ f(1) &= 1, \\ f(10) &= 2, \\ f(100) &= 4, \\ f(101) &= 5 \\ &\vdots \\ f(b_i) &= n_i \end{aligned}$$

so that f clearly maps to every element in the set of natural numbers.

It follows that $\mathcal{B}$ is equinumerous with $\mathbb{N}$.

6. Suppose there exists an onto function $f : A \to \mathcal{P}(A)$ and let $X = \{a \in A \mid a \notin f(a)\}$ be a subset of A not in the range of f.

Now choose some $a \in A$ and assume $a \in f(a)$. By the construction of X, $a \notin X$.

Alternatively, we choose some $a \in A$ and assume $a \notin f(a)$. This leads to a contradiction because, by the definition of the power set, $f(a) \subseteq A$.

Thus there is at least one subset of A that is not an element of $f(A)$, so f cannot be onto. What we have effectively proved is Cantor's theorem.

7. (a)

$$\begin{aligned} M^2 + L &= \{\lambda, 2, 4, 22, 24, 42, 44\} + \{2, 23, 24, 42\} \\ &= \{\lambda, 2, 4, 22, 24, 42, 44, 23\} \end{aligned}$$

Therefore the number of elements in $M^2 + L$ is 8.
Note that every string in $M^2 + L$ except for 22, 23 and 44 occur multiple times, however the resulting set only contains one of each repeated element.

(b)

$$\begin{aligned} LM &= \{2, 23, 24, 42\}\{\lambda, 2, 4\} \\ &= \{2, 23, 24, 42, 22, 232, 242, 422, 234, 244, 424\} \end{aligned}$$

The string 24 is repeated once. Therefore the number of elements in LM is 12.

(c)

$$\begin{aligned} LM^2 &= \{2, 23, 24, 42\}\{\lambda, 2, 4, 22, 24, 42, 44\} \\ &= \{2, 23, 24, 42, \\ &\quad 22, 232, 242, 422, \\ &\quad 234, 244, 424, \\ &\quad 222, 2322, 2422, 4222, \\ &\quad 224, 2324, 2424, 4224, \\ &\quad 2342, 2442, 4242, \\ &\quad 2344, 2444, 4244\} \end{aligned}$$

Considering 24, 242 and 244 occur repeatedly, LM^2 has 25 elements.

8. $((00)^* + 1(11)^*)^*$

9. Using Perl-style regular expression syntax in Notepad++:

   (a) $(0+1)^*(0000000000 + 1111111111)(0+1)^*$ was rewritten as `^(0|1)*(0000000000|1111111111)(0|1)*$`, which matched lines 10, 45 and 87;

   (b) $(0+1)^*(01)^5(0+1)^*$ was rewritten as `^(0|1)*(01){5}(0|1)*$`, which matched lines 4, 10, 15, 20, 35, 51, 66 and 77, and;

   (c) $(0+1)^*0000000(0+1)^{12}1111111(0+1)^*$ was rewritten as `^(0|1)*0000000(0|1){12}1111111(0|1)*$`, which matched line 39.

Below is a screenshot of Notepad++ matching a string using the regular expression from part (a).