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Cosmological solutions with torsion effects in Polynomial Affine Gravity

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Abstract. The Polynomial Affine Gravity it is an alternative gravitational model, where the interactions are mediated solely by the affine connection, instead of the metric tensor. In this paper, we explore the space of solutions to the field equations, when the torsion fields are turned on inf the frame of cosmology. Moreover, we explore how to generate metric descendant structures coming from the space of solutions.

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1 Introduction

Einstein's theory of General Relativity (GR) is currently the most successful theory to describe gravitational interactions, exhibiting excellent agreement between theoretical predictions and observational data in a variety of scenarios [1–3], from laboratory and solar system scales, to the orbital motions of binary pulsars, cosmology and even gravitational waves from colliding compact objects [4, 5] and the shadows of supermassive black holes.

In its traditional formulation, GR is a theory in which the gravitational interaction is solely mediated by the metric tensor [6, 7], from which other quantities such as a covariant derivative or curvature tensors can be derived. The nonlinear character of the equations make it necessary to use certain strategies and impose symmetries to obtain a simplified form that allows us to perform explicit computations. In cosmological scenarios, for instance, homogeneity and isotropy [8–14] turn the original set of coupled, nonlinear differential equations for ten independent variables into a second-order equation for a single function, the scale factor [15]. The evolution of this factor is determined by the matter-energy sources, which may include a cosmological constant [16] or some other form of dark energy, and the curvature of the spatial sections of the foliated space-time.

Despite its success, GR also faces difficulties that suggest that a more fundamental description of the gravitational interactions is necessary. Its combination with quantum theory indicates that an ultraviolet completion is needed [17–20], while the need to include a dark sector, for which no direct evidence exists, that dominates the cosmic evolution and the dynamics of galaxies and clusters [21–25], may point towards new gravitational dynamics in the infrared. As a result, it is becoming generally accepted that GR should be viewed as an effective theory that may require extensions at very short and very large length scales. Obviously, the kind of modifications needed are the million dollar question, and multiple alternatives can be found in the literature. Among those, theories of the $f(R)$ type [26–33], scalar-tensor theories, and various extensions of them [34–39], among others, have become very popular in the last two decades for theoretical and phenomenological reasons, offering a variety of strategies and alternative mechanisms to justify the accelerated cosmic expansion, inflationary scenarios, and the possible existence of exotic compact objects.

A fundamental ingredient in the construction of any gravity theory is the type of fields associated with the gravitational interaction. The traditional approach assumes that the underlying geometry is of Riemannian (or pseudo-Riemannian) type, being described solely by the metric tensor. Alternative approaches in which metric and connection are treated as equally fundamental and independent fields are also gaining attention in the last years. The role that torsion and non-metricity could have at cosmic scales and in strong gravity scenarios offers a window to explore new gravitational phenomena beyond the Riemannian framework. This can be viewed as a complementary approach to the modified theories scenario but considering modified geometry instead.

The freedom contained in the connection, with up to 64 independent components, offers a vast range of options to generate new gravitational phenomena and even to potentially accommodate adaptations needed to build a framework more suitable to incorporate quantum phenomena [40]. In this sense, it is important to note that the most successful description of the fundamental interactions is based on gauge theories, which encode the dynamics of the connections of the symmetry groups of the standard model of elementary particles. It is thus natural to consider if a purely connection-based formulation of gravity is possible, such that it could be represented in a form more closely related to the other interactions. The solution to this question is far from trivial, though some examples of purely affine theories exist in the literature.

Looking back at the literature on purely affine gravity theories, the model proposed by Sir Arthur Eddington [41, 42] represents a simple example that nonetheless illustrates the key challenges faced by this type of theories. Eddington's theory is defined by the square root of the determinant of the symmetric part of the Ricci tensor, which is a diffeomorphism invariant quantity [43]. The variation of this action (with respect to the affine connection) can be manipulated to obtain the well-known Einstein equations in vacuum coupled to an arbitrary cosmological constant (see, for instance, [44], where the role of the antisymmetric part of the Ricci is also analyzed). As a result, the Ricci tensor can be interpreted as an emergent metric tensor (as long as the cosmological constant does not vanish). Despite is formal resemblance with GR, Eddington's theory faces evident difficulties when trying to couple gravity to matter fields, as there is no clear mechanism to build a suitable matter action in the absence of a metric tensor. Some recent attempts in this direction can be found in [45–49], where a metric tensor, that couples only to the matter sector is considered. Other formulations of purely affine theories have been inspired by the canonical approach to quantum gravity [50], where the only dynamical field is an $SU(2)$ connection. Eddington's theory has also inspired metric-affine formulations in recent years in the form of determinantal Born-Infeld like actions [51–56].

An alternative approach to the purely affine formulation of gravity is provided by the Polynomial Affine Gravity (PAG) model. The PAG action is designed following a reasoning that parallels the *dimensional analysis* technique of field theories, considering the irreducible terms that can be constructed out of the affine connection and its first derivatives and that preserve the invariance under diffeomorphisms. Because of the absence of a metric tensor, there is a geometric constraint to satisfy in order to build the most general scalar densities in the affine geometry, which leads to a finite number of terms in the action. This property is usually referred to as the *rigidity* of the model.

Moreover, it is possible to couple a scalar field to the affine model using the same *dimensional analysis* principle and, as a consequence, the *rigidity* of the model is inherited by the coupling mechanism. This approach avoids the use of a metric to build the matter action, bypassing also the difficulties of Eddington's approach, and provides a new landscape to build purely affine gravity theories. The model has been studied in Refs. [57–63] and in this work, we explore the consequences of including torsion effects coming from the antisymmetric part of the affine connection in four dimensions in cosmological settings.

The paper is organized as follows: In Section 2, we present a brief overview on how to built the polynomial affine model of gravity, highlighting its features and the method to build up the Ansatze compatible with the cosmological symmetries. A complete scan of cosmological solutions to the field equations is presented in Section 3. In Section 4, we analyses and discuss the cosmological solutions, and provided a physical interpretation of the solutions by obtaining metric-descendant structures coming from the irreducible fields of the affine connection. Final remarks are presented in Section 5.

2 Polynomial Affine Gravity

As said earlier, the Polynomial Affine Gravity is an alternative gravitational model whose fundamental field is the affine connection, endowing the manifold only with an affine structure $\mathcal{M}(\Gamma)$. In order to build the action, it is convenient to decomposes the affine connection as

$$\begin{aligned} \hat{\Gamma}_{\alpha}{}^{\beta}{}_{\gamma} &= \hat{\Gamma}_{(\alpha}{}^{\beta}{}_{\gamma)} + \hat{\Gamma}_{[\alpha}{}^{\beta}{}_{\gamma]}, \\ &= \Gamma_{\alpha}{}^{\beta}{}_{\gamma} + \mathcal{B}_{\alpha}{}^{\beta}{}_{\gamma} + \delta^{\beta}_{[\gamma}\mathcal{A}_{\alpha]}, \end{aligned} \tag{1}$$

where the first term corresponds to the symmetric part of the connection $\hat{\Gamma}_{(\alpha}{}^{\beta}{}_{\gamma)} = \Gamma_{\alpha}{}^{\beta}{}_{\gamma}$, and the last two terms are related to the torsion tensor. The former represents the purely tensorial (traceless) part of the torsion $\mathcal{B}_{\alpha}{}^{\beta}{}_{\gamma}$, while the latter is a pure vectorial object $\mathcal{A}_{\alpha}$. Additionally, the introduction of the volume element is necessary and, in the absence of a metric tensor, one can use the wedge product to define it as $\mathrm{d}V^{\alpha\beta\gamma\delta} = \mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{\beta} \wedge \mathrm{d}x^{\gamma} \wedge \mathrm{d}x^{\delta}$. It is worth emphasizing that the action must preserve the invariance under diffeomorphisms, which is broken by the symmetric part of the affine connection, and as a consequence of this, the symmetric part must appear in the action only through the covariant derivative $\Gamma_{\alpha}{}^{\beta}{}_{\gamma} \to \nabla^{\Gamma}$. Therefore, the fundamental building blocks of the affine model are

$$\nabla^{\Gamma}_{\alpha}, \mathcal{B}_{\alpha}{}^{\beta}{}_{\gamma}, \mathcal{A}_{\alpha}, \mathrm{d}V^{\alpha\beta\gamma\delta}. \tag{2}$$

In order to build up the action, we use a sort of *dimensional analysis* technique which has been reviewed in [58, 61]. The method allows one to consider every scalar densities composed by powers of Eq. (2) by using the operators $\mathcal{N}$ and $\mathcal{W}$ which count the number of free indices and the weight of the field, respectively. The analysis provides a geometrical constraint equation that limits the number of configurations. For each configuration, there are multiple permutations that need to be analyzed by taking into account the symmetries of the fundamental fields (see [58, 61] for more details on this procedure). A three-dimensional version of this model was developed following this approach in Refs. [62, 63].

The most general action (up to topological invariants and boundary terms) in four dimensions is given by

$$
\begin{aligned}
S = \int \mathrm{d}V^{\alpha\beta\gamma\delta} \Big[& B_1 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\rho} \mathcal{B}_{\alpha}{}^{\nu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} + B_2 \mathcal{R}_{\alpha\beta}{}^{\mu}{}_{\rho} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu} \\
& + B_3 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\nu}{}_{\gamma} \mathcal{A}_{\delta} + B_4 \mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\rho} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} \mathcal{A}_{\sigma} \\
& + B_5 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{A}_{\sigma} + C_1 \mathcal{R}_{\mu\alpha}{}^{\mu}{}_{\nu} \nabla_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \\
& + C_2 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} + D_1 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \nabla_{\beta} \mathcal{B}_{\gamma}{}^{\lambda}{}_{\delta} \\
& + D_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\nu} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} + D_3 \mathcal{B}_{\alpha}{}^{\mu}{}_{\nu} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \nabla_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\delta} \\
& + D_4 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \nabla_{\lambda} \mathcal{A}_{\sigma} + D_5 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\sigma} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \\
& + D_6 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\gamma} \nabla_{\lambda} A_{\delta} + D_7 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\lambda} \nabla_{\gamma} A_{\delta} \\
& + E_1 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} + E_2 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\gamma} \mathcal{A}_{\delta} \\
& + F_1 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\rho} \mathcal{B}_{\sigma}{}^{\rho}{}_{\lambda} + F_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\lambda} \mathcal{B}_{\delta}{}^{\lambda}{}_{\rho} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu} \\
& + F_3 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \mathcal{A}_{\delta} + F_4 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{A}_{\mu} \mathcal{A}_{\nu} \Big].
\end{aligned}
\tag{3}
$$

In the above action, the covariant derivative and the curvature tensor are defined with respect to the symmetric part of the connection, meaning that $\nabla = \nabla^{\Gamma}$ and $\mathcal{R} = \mathcal{R}^{\Gamma}$.

One of the most important features of this model is that the lack of a metric tensor implies that the number of terms in the action is finite. This property is known as the *rigidity* of the model. The fact that all the coupling constants are dimensionless suggests that the model is power-counting renormalizable, so that in the hypothetical scenario of its quantization all possible counter-terms should have the form of the ones already written in Eq. (3). This is something desirable from a stand point of Quantum Field Theory view. The reason is that the superficial degree of divergence vanishes. Additionally, the dimensionless nature of the coupling constants also suggests a conformal symmetry, at least at a classical level. An explicit implementation of this idea is likely to require an understanding of the projective invariance of the theory, along the lines of [64], though this aspect will not be explored in this paper.

In the torsion-free sector, the field equations coming from the variation of the action turn out to be a generalization of the Einstein's vacuum field equations []. In that sense, the space of solutions of GR in vacuum is a subspace of solutions of the Polynomial Affine Gravity. Finally, it is possible to couple a scalar field using the *dimensional analysis* technique introduced above without the necessity of having a metric structure on the manifold. The kinetic terms would couple to a combination of tensors and tensor densities, while the potential term should rescale the volume element. ln the torsionless limit, the resulting equations turn out to recover Einsteins's theory coupled to a scalar field, see Ref. [65], giving support to this approach.

Since we are interested in the study of cosmology, we need to impose first the symmetries of the cosmological principle on the irreducible fields associated to the affine connection, which are Γ, $\mathcal{B}$ and $\mathcal{A}$. In order to build up the ansatz, we compute the Lie derivative of each irreducible field along the Killing vectors ξ_i that generate the symmetries of homogeneity (translations) $\mathcal{P}_i$ and isotropy (rotations) $\mathcal{J}_i$. A derivation of the Killing vectors along with an explicit computation of the Lie derivative can be found in Ref. [66]. In what follows we shall briefly summarize the results.

The computation of the Lie derivative of Γ along the Killing vectors determines its coefficients:

$$
\begin{aligned}
\Gamma_{t}{}^{t}{}_{t} &= f(t), & \Gamma_{i}{}^{t}{}_{j} &= g(t) S_{ij}, \\
\Gamma_{t}{}^{i}{}_{j} &= h(t)\delta^{i}_{j}, & \Gamma_{i}{}^{j}{}_{k} &= \gamma_{i}{}^{j}{}_{k},
\end{aligned}
\tag{4}
$$

where S_{ij} is a three-dimensional rank two symmetric tensor defined as

$$
S_{ij} = \begin{pmatrix} \frac{1}{1-\kappa r^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix},
$$

and γ is the three-dimensional symmetric connection compatible with the desired symmetries, which can be written as

$$
\begin{aligned}
\gamma_{r}{}^{r}{}_{r} &= \frac{\kappa r}{1-\kappa r^2}, & \gamma_{\theta}{}^{r}{}_{\theta} &= \kappa r^3 - r, \\
\gamma_{\varphi}{}^{r}{}_{\varphi} &= \left(\kappa r^3 - r\right)\sin^2\theta, & \gamma_{r}{}^{\theta}{}_{\theta} &= \frac{1}{r}, \\
\gamma_{\varphi}{}^{\theta}{}_{\varphi} &= -\cos\theta\sin\theta, & \gamma_{r}{}^{\varphi}{}_{\varphi} &= \frac{1}{r}, \\
\gamma_{\theta}{}^{\varphi}{}_{\varphi} &= \frac{\cos\theta}{\sin\theta}.
\end{aligned}
$$

Additionally, the affine function $f(t)$ can be set to zero by a re-parametrization of the t coordinate [62]. Therefore, there are only two nontrivial functions to define completely the symmetric part of the connection.

Following a similar procedure for the torsion tensor, it is possible to determine the ansatz compatible with the required symmetries. For its traceless part $\mathcal{B}$, the nonvanishing components are

$$
\begin{aligned}
\mathcal{B}_{\theta}{}^{r}{}_{\varphi} &= \psi(t) r^2 \sin\theta \sqrt{1-\kappa r^2}, \quad \mathcal{B}_{r}{}^{\theta}{}_{\varphi} = \frac{\psi(t)\sin\theta}{\sqrt{1-\kappa r^2}}, \\
\mathcal{B}_{r}{}^{\varphi}{}_{\theta} &= \frac{\psi(t)}{\sqrt{1-\kappa r^2}\sin\theta},
\end{aligned}
\tag{5}
$$

while the nontrivial component of its vectorial part $\mathcal{A}$ is

$$
\mathcal{A}_t = \eta(t). \tag{6}
$$

Finally, the field equations are obtained by varying the action with respect to each irreducible field using Kijowkski's formalism, see Ref. [61,67]. The complete set of field equations for each irreducible field can be found in Refs. [61,68].

3 Cosmological solutions

Varying the action (3) and using the ansatz in Eqs. (4), (5) and (6), the field equations become

$$
\left(B_3\left(\dot{g} + gh + 2\kappa\right) - 2B_4\left(\dot{g} - gh\right) + 2D_6\eta g - 2F_3\psi^2\right)\psi = 0, \tag{7}
$$

$$(B_3\eta\psi - 2B_4\eta\psi + C_1(\dot{\psi} - 2h\psi))g = 0, \tag{8}$$

$$\begin{aligned}(B_3 + 2B_4)\,\eta g\psi + 2C_1(\kappa\psi + 4gh\psi - g\dot{\psi} - \psi\dot{g}) \\ - 2\psi^3(D_1 - 2D_2 + D_3) = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}B_3(\eta(h\psi - \dot{\psi}) - \psi\dot{\eta}) - 2B_4(\eta(-h\psi - \dot{\psi}) - \psi\dot{\eta}) \\ + C_1(4h^2\psi + 2\psi\dot{h} - \ddot{\psi}) + D_6\eta^2\psi = 0\ ,\end{aligned} \tag{10}$$

$$\begin{aligned}B_3(\dot{g} + gh + 2\kappa)\eta - 2B_4\,(\dot{g} - gh)\,\eta \\ + C_1(2\kappa h + 4gh^2 + 2g\dot{h} - \ddot{g}) \\ - 6h\psi^2\,(D_1 - 2D_2 + D_3) \\ + D_6\eta^2 g - 6F_3\eta\psi^2 = 0\end{aligned} \tag{11}$$

where $\mathcal{A}$ and $\mathcal{B}$ yield only one equation each, namely, Eq. (7) and Eq. (11), respectively, while the remaining three come from Γ. Notice that we have four unknown functions of time ($g(t)$, $h(t)$, $\psi(t)$ and $\eta(t)$) while there are five differential equations, meaning that the system is overdetermined. We will show that, even so, the system can actually be solved analytically without any assumption. For this purpose, we have developed a *logical scheme* that allows us to seek systematically the solutions by branches. First, notice that Eqs. (7) and (8) follow the form

$$\mathcal{F}(g, \dot{g}, h, \psi, \eta)\psi = 0, \tag{12}$$

$$\mathcal{G}(h, \psi, \dot{\psi}, \eta)g = 0, \tag{13}$$

where

$$\begin{aligned}\mathcal{F}(g, \dot{g}, h, \psi, \eta) \equiv B_3\,(\dot{g} + gh + 2\kappa) - 2B_4\,(\dot{g} - gh) \\ + 2D_6\eta g - 2F_3\psi^2.\end{aligned} \tag{14}$$

$$\mathcal{G}(h, \psi, \dot{\psi}, \eta) \equiv B_3\eta\psi - 2B_4\eta\psi + C_1\left(\dot{\psi} - 2h\psi\right). \tag{15}$$

Thus, using Eqs. (12) and (13) it is possible to distinguish four different branches:

- **First branch:** $\mathcal{F}(g, h, \psi, \eta) = 0 \ \wedge\ \mathcal{G}(h, \psi, \eta) = 0$.
- **Second branch:** $\mathcal{F}(g, h, \psi, \eta) = 0 \ \wedge\ g = 0$.
- **Third branch:** $\mathcal{G}(h, \psi, \eta) = 0 \wedge \psi = 0$.
- **Fourth branch:** $\psi = 0 \wedge g = 0$.

The first branch is the most interesting one in terms of the richness of solutions, while branch four is the simplest one.

3.1 First branch

This branch contains the most general case subject to $\mathcal{F}(g, h, \psi, \eta) = 0$ and $\mathcal{G}(h, \psi, \eta) = 0$, with the field equations being

$$B_3\,(\dot{g} + gh + 2\kappa) - 2B_4\,(\dot{g} - gh) + 2D_6\eta g - 2F_3\psi^2 = 0, \tag{16}$$

$$B_3\eta\psi - 2B_4\eta\psi + C_1\left(\dot{\psi} - 2h\psi\right) = 0, \tag{17}$$

$$\begin{aligned}(B_3 + 2B_4)\,\eta g\psi + 2C_1\left(\kappa\psi + 4gh\psi - g\dot{\psi} - \psi\dot{g}\right) \\ + 2\psi^3\,(2D_2 - D_1 - D_3) = 0,\end{aligned} \tag{18}$$

$$\begin{aligned}B_3&\left(\eta\left(h\psi - \dot{\psi}\right) - \psi\dot{\eta}\right) \\ &- 2B_4\left(\eta\left(-h\psi - \dot{\psi}\right) - \psi\dot{\eta}\right) \\ &+ C_1\left(4h^2\psi + 2\psi\dot{h} - \ddot{\psi}\right) + D_6\eta^2\psi = 0,\end{aligned} \tag{19}$$

$$\begin{aligned}B_3\,&(\dot{g} + gh + 2\kappa)\,\eta - 2B_4\,(\dot{g} - gh)\,\eta \\ &+ C_1\left(2\kappa h + 4gh^2 + 2g\dot{h} - \ddot{g}\right) \\ &+ 6h\psi^2\,(2D_2 - D_1 - D_3) \\ &+ D_6\eta^2 g - 6F_3\eta\psi^2 = 0.\end{aligned} \tag{20}$$

To find an expression for $\eta(t)$, one must solve Eq.(17)

$$\eta(t) = \left(\frac{2h\psi - \dot{\psi}}{\psi}\right)\left(\frac{C_1}{B_3 - 2B_4}\right). \tag{21}$$

Replacing the above expression for $\eta(t)$ into Eq. (19), leads to two sub-branches for the $h(t)$ function, namely

$$h_I(t) = \frac{\dot{\psi}}{2\psi} \quad \wedge \quad h_{II}(t) = \frac{\dot{\psi}}{\psi}\left(\frac{C_1 D_6}{3B_3^2 - 8B_3B_4 + B_4^2 + 2C_1D_6}\right). \tag{22}$$

Using $h_I(t)$, then Eq. (18) turns into

$$-\,(D_1 - 2D_2 + D_3)\,\psi^3 + C_1\left(\psi\,(\kappa - \dot{g}) + g\dot{\psi}\right) = 0, \tag{23}$$

which allows us to solve $g(t)$ in terms of the $\psi(t)$

$$g(t) = \psi(t)\left(g_0 + \int_1^t \left(\frac{\kappa}{\psi(\tau)} - \psi(\tau)\left(\frac{D_1 - 2D_2 + D_3}{C_1}\right)\right)\mathrm{d}\tau\right), \tag{24}$$

where g_0 is an integration constant. The above expression also solves Eq. (20). Then, Eq. (16) becomes a first-order integro-differential equation

$$\dot{\psi}\left(g_0 + \int_1^t \left(\frac{\kappa}{\psi(\tau)} - \psi(\tau)\alpha\right)\mathrm{d}\tau\right)\beta - \psi^2\gamma + 2\kappa\beta = 0. \tag{25}$$

where α, β and γ are related to the coupling constants by the following relation

$$\alpha = \left(\frac{D_1 - 2D_2 + D_3}{C_1} \right), \tag{26}$$

$$\beta = \left(\frac{3B_3 - 2B_4}{2} \right), \tag{27}$$

$$\gamma = (\beta - 2B_3)\,\alpha + 2F_3, \tag{28}$$

For the special case $\kappa = 0$ and defining $\psi(t) \equiv \dot{\phi}(t)$, Eq.(25) can be turned into a second-order differential equation for $\phi(t)$ of the form

$$\ddot{\phi}\,(g_0 - \phi\alpha)\,\beta - \dot{\phi}^2\gamma = 0, \tag{29}$$

whose solution is

$$\phi(t) = \frac{g_0}{\alpha} + \lambda\,(t - t_0)^{\frac{\alpha\beta}{\alpha\beta+\gamma}}, \tag{30}$$

where λ and t_0 are integration constants. Using the above solution, it is straightforward to recover the original function

$$\psi(t) = \frac{\lambda\alpha\beta}{\alpha\beta + \gamma}\,(t - t_0)^{\frac{-\gamma}{\alpha\beta+\gamma}}. \tag{31}$$

Using the above expression and from the relations in Eqs. (21), (22) and (24) it is direct to find the rest of the affine functions

$$\eta(t) = 0 \tag{32}$$

$$h(t) = -\frac{\gamma}{2\,(\alpha\beta+\gamma)\,(t - t_0)} \tag{33}$$

$$g(t) = \left(\frac{(\alpha\beta+\gamma)g_1 - \alpha^2\beta\lambda^2\,(t - t_0)^{\frac{\alpha\beta}{\alpha\beta+\gamma}}}{\alpha\beta+\gamma} \right)(t - t_0)^{-\frac{\gamma}{\alpha\beta+\gamma}} \tag{34}$$

where g_1 is an integration constant.

Finally, from Eq. (22) the second branch of $h(t)$ leads to a similar first order integro-differential equation which can not be solved analytically, which is why, for the moment we will exclude in this paper.

3.2 Second branch

The second branch imposes $\mathcal{F}(g, h, \psi, \eta) = 0$ and $g(t) = 0$ restrictions which leads to the following field equations:

$$\kappa B_3\psi - F_3\psi^3 = 0, \tag{35}$$

$$\kappa C_1\psi - \psi^3 D = 0, \tag{36}$$

$$\begin{aligned} B_3(\eta(h\psi - \dot{\psi}) - \psi\dot{\eta}) - 2B_4(\eta(-h\psi - \dot{\psi}) - \psi\dot{\eta}) \\ + C_1(4h^2\psi + 2\psi\dot{h} - \ddot{\psi}) + D_6\eta^2\psi = 0, \end{aligned} \tag{37}$$

$$B_3\kappa\eta + C_1\kappa h - 3h\psi^2 D - 3F_3\eta\psi^2 = 0, \tag{38}$$

where $D = D_1 - 2D_2 + D_3$.

From Eq. (35) it is possible to find an expression for $\psi(t)$ in the form

$$\psi(t) = \pm\sqrt{\frac{\kappa B_3}{F_3}}. \tag{39}$$

Using the compatibility condition from Eq. (36), leads to a relation between the coupling constant

$$C_1 F_3 = D B_3. \tag{40}$$

Taking the algebraic expression for C_1 and replacing Eq. (39) in Eq.(38) leads to a relation between $h(t)$ and $\eta(t)$ of the form

$$h(t) = -\eta(t)\frac{F_3}{D} \tag{41}$$

Combining the above result along with Eq. (39) turns Eq. (37) into a first order differential equation of the form

$$\dot{\eta} - \eta^2\left(\frac{D_6}{3B_3 - 2B_4} + \frac{F_3}{D} \right) = 0 \tag{42}$$

whose solution is

$$\eta(t) = \frac{D\,(2B_4 - 3B_3)}{(D\eta_0 + tF_3)\,(3B_3 - 2B_4) + DD_6 t}. \tag{43}$$

Then, it is straightforward to obtain $h(t)$

$$h(t) = \frac{F_3\,(2B_4 - 3B_3)}{(D\eta_0 + tF_3)\,(3B_3 - 2B_4) + DD_6 t} \tag{44}$$

It is worth to remark that the above solutions is only valid for the special case where $\kappa \neq 0$, this can be seen directly from Eqs. (35) and (36).

If we impose the constraint $\kappa = 0$, then Eq. (35) tells us that $\psi(t) = 0$ solves entirely the other equations, and the remaining functions $h(t)$ and $\eta(t)$ are unknown.

3.3 Third branch

The constraint $\mathcal{G}(h, \psi, \eta) = 0$ and $\psi(t) = 0$ imposes a strong restriction to the system of differential equations, which boils down to

$$\begin{aligned} g\eta^2 D_6 + 2B_4\eta\,(gh - \dot{g}) + B_3\eta\,(2\kappa + gh + \dot{g}) \\ + C_1\left(2h\,(\kappa + 2gh) + 2g\dot{h} - \ddot{g}\right) = 0. \end{aligned} \tag{45}$$

The above differential equation has three unknown functions of time $h(t)$, $g(t)$ and $\eta(t)$ which cannot be solved without further restriction, or by providing an ansatz for two functions.

3.4 Fourth branch

In the fourth branch defined by $g(t) = 0$ and $\psi(t) = 0$, the field equations are reduced to a single algebraic equation

$$\kappa\left(hC_1 + B_3\eta\right) = 0, \tag{46}$$

the above equation can be solved by setting $\kappa = 0$, then, the functions $h(t)$ and $\eta(t)$ are undetermined, or by the relation $\eta(h) = -h(t)\left(\frac{C_1}{B_3}\right)$, where $h(t)$ is an arbitrary function.

4 Analysis of the solutions

There are only two nontrivial and analytic solutions to the field equations, which were found in Sec. 3.1 and 3.2. Nonetheless, we are going to strict ourselves to analyses only the former branch. The reason for this, will be clear in the next subsection.

In order to simplify our calculations, we shall set up the integration constants to $t_0 \to 0$, $\lambda \to 1$ and $g_1 \to 0$. Hence, the affine functions are simplify up to

$$h(t) = -\frac{\gamma}{2t\left(\alpha\beta+\gamma\right)}, \tag{47}$$

$$g(t) = -\left(\frac{\alpha^2\beta}{\alpha\beta+\gamma}\right)t^{\frac{\alpha\beta-\gamma}{\alpha\beta+\gamma}}, \tag{48}$$

$$\psi(t) = \left(\frac{\alpha\beta}{\alpha\beta+\gamma}\right)t^{\frac{-\gamma}{\alpha\beta+\gamma}}, \tag{49}$$

$$\eta(t) = 0, \tag{50}$$

and therefore, the behavior of the functions is reduced to the numerical values of the constants α, β and γ.

4.1 Emergent metrics

The physical implications of the solutions can be found through the descendant metric structures coming from the irreducible fields of the affine connection in the space of solutions to the field equations. For starts, lets recall the definition of a metric tensor

Definition. Let $\mathcal{M}$ be a smooth manifold of dimension n. At each point $p \in \mathcal{M}$ there is a vector space $T_p\mathcal{M}$, called the tangent space. A metric tensor at the point p is a function $g_p(X_p, Y_p)$ which takes as inputs a pair of tangent vectors X_p and Y_p at p and produces a real number, so that the following conditions are satisfied: i) g is bilinear, meaning that it is linear sperately in each argument, ii) g is symmetric provided that for all vector X_p and Y_p we have $g_p(X_p, Y_p) = g_p(Y_p, X_p)$, iii) g is nondegenerate, and therefore the tensor can be inverted.

In the space of solution, the first emergent metric comes from the Ricci tensor $\mathcal{R}_{\mu\nu}$, which is defined by the contraction of the Riemann tensor.[1]

$$\hat{\Gamma} \to \nabla^{\Gamma} \to \mathcal{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} \to \mathcal{R}_{\beta\delta}. \tag{51}$$

A second metric structure, comes from the contraction of the product of two torsion tensor.[2] This idea was first introduce by Poplwaski in Ref. [69], and the metric structure is defined as follow

$$\mathcal{P}_{\alpha\delta} = \left(\mathcal{B}_{\alpha}{}^{\beta}{}_{\gamma} + \delta^{\beta}_{[\gamma}\mathcal{A}_{\alpha]}\right)\left(\mathcal{B}_{\beta}{}^{\gamma}{}_{\delta} + \delta^{\gamma}_{[\delta}\mathcal{A}_{\beta]}\right). \tag{52}$$

Using the cosmological ansatz presented in Eq. (4), the Ricci tensor is defined as follow

$$\mathcal{R}_{tt} = \dot{h} + h^2, \qquad \mathcal{R}_{rr} = \frac{\dot{g} + gh + 2\kappa}{1-\kappa r^2}, \tag{53}$$

and the Poplawski's metric is computed using Eqs. (5) and (6)

$$\mathcal{P}_{tt} = \eta^2, \qquad \mathcal{P}_{rr} = -\frac{2\psi^2}{1-\kappa r^2}. \tag{54}$$

In order to use the Ricci tensor $\mathcal{R}_{\mu\nu}$ or the Poplawski $\mathcal{P}_{\mu\nu}$ as an emergent metric tensor, is necessary that the tensor is well defined, meaning, it can be invertible. If the tensors are well behaved, then the factors $(\dot{g} + gh + 2\kappa)$ or $\left(-2\psi^2\right)$ could play an analogue role to the scale factor $a^2(t)$ from FLRW, moreover, the factor $\dot{h} + h^2$ or η^2 could be associated to the lapse function. Therefore, it is crucial to have non vanishing affine functions, and also, that the affine functions must be time dependent, this is the reason on why we exlcuded the second branch on this analysis, that is because it has a vanishing $g(t)$ and a constant $\psi(t)$ function, providing a degenerate Ricci and Poplawski effective tensor.

Using Eqs. (47), (48) and (53) we compute the components of the Ricci tensor

$$\mathcal{R}_{tt} = -\frac{3\gamma\left(2\alpha\beta+3\gamma\right)}{4t^2\left(\alpha\beta+\gamma\right)^2}, \tag{55}$$

$$\mathcal{R}_{rr} = \Sigma\left(\alpha,\beta,\gamma\right)t^{\frac{-2\gamma}{\alpha\beta+\gamma}}, \tag{56}$$

where the constant $\Sigma\left(\alpha,\beta,\gamma\right)$ is defined as

$$\Sigma\left(\alpha,\beta,\gamma\right) = \frac{\alpha^2\beta\left(3\gamma-2\alpha\beta\right)}{2\left(\alpha\beta+\gamma\right)^2}. \tag{57}$$

Notice that if $\alpha\beta+\gamma \neq 0$ and neither constant α, β and γ are trivial, then Ricci tensor can acts as a metric tensor. In that particular case, the spatial part of the ricci tensor plays an analogue role to the scale factor in the FLRW universe, thus

$$\Sigma\left(\alpha,\beta,\gamma\right)t^{\frac{-2\gamma}{\alpha\beta+\gamma}} \longleftrightarrow a^2(t). \tag{58}$$

Therefore, we can use the Ricci tensor to define the notion of distance, as long as is well behaved (nondegenerate). For this metric, unlike the one in FLRW universe, we have a singularity in the temporal part of the Ricci tensor at $t = 0$, meaning that the associated lapse function in

[1] The Riemann tensor is defined by the commutator of covariant derivatives acting on a vector, therefore, it does not required the existence of a metric structure.

[2] This comes from the antisymmetric part of the affine connection

this affine geometry is not well defined at that particular time. Additionally, the metriciy condition is not satisfied, meaning $\nabla_\lambda \mathcal{R}_{\mu\nu}$ does not vanishes, and as a consequence of this, there are non-metricity effects $\mathcal{Q}^{\mathcal{R}}_{\mu\nu\lambda}$.

We can also compute Poplawski's metric like tensor which was defined in Eq. (54) using Eqs. (49) and (50), leading to

$$\mathcal{P}_{tt} = 0 \qquad \mathcal{P}_{rr} = \alpha^2\beta^2 \left(t\left(\alpha\beta+\gamma\right)\right)^{-\frac{2\gamma}{\alpha\beta+\gamma}}, \tag{59}$$

however, unlike the Ricci tensor, the Poplawski metric like tensor, is a degenerate tensor which can not be inverted, and therefore, it can not act as a metric tensor.

4.2 Matter interpretation from torsion fields

The cosmological solutions allow us to have a well defined Ricci tensor, whose spatial part was defined in Eq. (56), this is a power-law solution, where the coupling constant α, β and γ are due to the presence of the torsion field. This type of solution can not be found in the torsion-free sector.

In order to provide a physical meaning we inquire the value of the parameters α, β and γ, that possess a well-known behavior in General Relativity, specifically in the FLRW universe. In the standard model of cosmology, there are three well-known epochs: the radiation era, the non-relativistic matter era and the accelerated expansion, each with its respective scale factor.

We can find relations between the constants α, β and γ in order to have an affine scale factor $a_f(t)$ that can have the same behavior as the one from FLRW Universe $a(t)$ with the big contrast that: in the standard model of cosmology, the source to get a dynamical factor is the energy-momentum tensor, whereas in the affine geometry, the *source* is the torsion field, specifically its trace-less part.

With the relation $\gamma = -\frac{\alpha\beta}{3}$, the affine scale factor turns to

$$a_f(t) = a_0 t^{1/2}, \tag{60}$$

where a_0 is defined as

$$a_0 = \sqrt{-\frac{27\alpha}{8}}, \tag{61}$$

where α must be a negative number in order to have a positive and real affine scale factor, which translate into an inequality for the coupling constants

$$\frac{D_1 - 2D_2 + D_3}{C_1} < 0 \tag{62}$$

This type of solution allow us to emulate the radiation era.

In the same spirit, considering the case $\gamma = -\frac{2\alpha\beta}{5}$, leads to

$$a_f(t) = a_0 t^{2/3}, \tag{63}$$

where we defiend a_0 as

$$a_0 = \sqrt{-\frac{40\alpha}{9}}, \tag{64}$$

which, requires that $\alpha < 0$, which is consistent with constraint written in Eq. (62). This type of solution allow us to emulate a non-relativistic matter era.

If we want to have an accelerated expansion due to torsion effects, we need to consider

$$\ddot{a}_f > 0 \to \frac{\alpha^2\beta\gamma(\alpha\beta+\gamma)(3\gamma-2\alpha\beta)}{2(\alpha\beta+\gamma)^4} t^{\frac{\alpha\beta}{\alpha\beta+\gamma}-3} > 0. \tag{65}$$

the above condition, ensures a positive acceleration. In order to have a steady growth acceleration, it is required that

$$\alpha\beta < -\frac{3}{2}\gamma, \tag{66}$$

this conditions ensures that the time evolution will be defined positive. Additionally, we need to ensure that the numerator in Eq. (65) must be positive, from which we distinguish two types of conditions: $\gamma > 0$ and $\beta > 0$, and therefore $\alpha < 0$, which is consistent with the above two cosmological epochs, or $\gamma < 0$ and $\beta < 0$, leading to $\alpha > 0$, which is inconsistent with the above description. .

5 Final remarks

In this work, we have analyzed some cosmological scenarios appearing in the context of the Polynomial Affine Gravity model with torsion effects, generalizing those considered in Refs. [59, 61]. Considering a non vanishing torsion tensor, yields an overdetermined system of differential equations, see Eqs. (7) to (11), unlike the case without torsion effects, whose field equations are reduced to the *harmonic curvature* $\nabla_\alpha \mathcal{R}_{\beta\gamma}{}^{\alpha}{}_{\delta} = 0$, which through Bianchi's identity can be written as an anti symmetrized covariant derivative of the Ricci tensor $\nabla_{[\mu}\mathcal{R}_{\alpha]\beta} = 0$. For this particular case, the Ricci tensor is a Codazzi tensor.

Without torsion effects, the field equation leads to one differential equation for two unknown functions (see eq.(11) with $\eta(t) = 0$), thus, the system can be solved parametrically, for example introducing an ansatz for $h(t)$ or $g(t)$. A priori, there is no physical reason to fix one of the functions, however, one could explore the subspace of this torsion-free field equation, by studying integrability conditions such as $R_{\beta\gamma} = 0$ and $\nabla_\alpha \mathcal{R}_{\beta\gamma} = 0$. The former presents a system of two differential equations for two unknown functions, which can be solved analytically. However, because the field equations requires a vanishing Ricci tensor, in this subspace of solutions it is not possible to find emergent metric tensor.

The latter, yields a three differential equations, which can be solved analytically and provides with an exact solution for $h(t)$ and $g(t)$ functions. Even more, the Ricci tensor is well behaved and it can be used as a metric tensor. This case yields well-known de-Sitter/Anti de-Sitter space of solutions, where the cosmological constant is an integration constant, see Ref. [61].

The introduction of torsion fields with the tensors $\mathcal{B}$ and $\mathcal{A}$, induce nontrivial effects in the dynamics of the system. Due to the presence of the torsion tensor, there are two (possible) emergent descendant metric structures coming from the irreducible fields of the affine connection in the space of solution. Specifically, one comes from the symmetric part of the connection, which allow us to define the covariant derivative ∇^{Γ}, and, the other one comes from the anti-symmetric part of the affine connection, defined by the contraction of the product of two torsion tensors.

Keeping in mind in the standard FLRW model of the universe, there are two fundamental objects, the metric tensor $g_{\mu\nu}$, and the energy-momentum tensor $T_{\mu\nu}$, we could argue that in Polynomial Affine Gravity, the Ricci tensor $\mathcal{R}_{\mu\nu}$ can act as $g_{\mu\nu}$, whereas $\mathcal{P}_{\mu\nu}$ emulate a $T_{\mu\nu}$. In that sense, the Ricci tensor provide us an affine descendent scale factor, while the Poplawski like metric tensor emulate different types of matter effects.

Interestingly, the system of field equations eqs. (7) to (11) present four different branches of solution, of which only two branches can be solved exactly (non parametrically). The first branch can be solved analytically defining completely the symmetric and skew symmetric part of the affine connection. The solutions allow us, to define the Ricci tensor as an emergent metric tensor in the space of solutions, however, the Poplawski tensor is a degenerate tensor, because its temporal component its trivial, and therefore, it can not be interpreted as a metric tensor.

From the solution, it is possible to find relations for the constants α, β and γ that allow us to define an affine scale factor that can emulate the standard scale factor coming from the ΛCDM model, providing us with the three classical eras: radiation era, non-relativistic matter era and the accelerated expansion. However, unlike FRLW Universe where the source of the scale factor dynamics comes from the energy-momentum tensor, in the affine geometry, the *source* comes from having non trivial torsion fields.

Moreover, the second branch also can be solved analytically for the special case where $\kappa \neq 0$. For this particular case, $g(t)$ is trivial and the symmetric part of the affine connection is determined completely by the $h(t)$ function. As a consequence of this solution, the Ricci tensor can be interpreted as a metric tensor, however, the affine scale factor coming from its spatial component is constant and provides no dynamics. Following a similar analysis, the $psi(t)$ it is a constant defined by the geometrical factor κ and the coupling constant, and $\eta(t)$ has a non trivial inverse time dependence, defining completely the skew-symmetric part of the affine connection. The Poplawski metric like tensor $\mathcal{P}_{\mu\nu}$ it is well defined, however, the affine scale factor coming from its spatial it is also a constant (just like the Ricci tensor), and therefore, it has no dynamics.

It has been shown that, the introduction of the torsion field in this affine model of gravity allow us to obtain an affine scale factor coming from the symmetric part of the affine connection in the space of solutions. Moreover, due to torsion effects, it is possible to recover the three classical epochs of the standard model of cosmology, whose *source* in this affine gravity is the torsion tensor instead of an energy-momentum tensor.

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