# Multiphase flows
# Lecture 2

## Equation of motion of a single particle, forces on individual particles

Lagrangian framework: evolution of properties associated with the *i-th* particle (*X, V, R (radius)* – Lagrangian coordinates)

$$\frac{\mathrm{d}\boldsymbol{X}_{(i)}}{dt} = \boldsymbol{V}_{(i)} \qquad \frac{\mathrm{d}\boldsymbol{V}_{(i)}}{dt} = \boldsymbol{A}_{(i)} \qquad \frac{\mathrm{d}R_{(i)}}{dt} = \Theta_{(i)}$$

$A_{(i)}$ – acceleration experienced by the particle

$\Theta_{(i)}$ - rate of change of radius due to interphase mass transfer (e.g. vaporization)

# Lagrangian vs. Eulerian reference frames

Eulerian reference frame

$$\boldsymbol{v}(\boldsymbol{x}, t)$$

$$p(\boldsymbol{x}, t)$$

- Velocity $\boldsymbol{v}$ and pressure $p$ location $\boldsymbol{x}$ at time $t$

Lagrangian reference frame

$$\boldsymbol{X}(\boldsymbol{a}, t)$$

- Position $\boldsymbol{X}$ of element $\boldsymbol{a}$ at time $t$

$$\boldsymbol{v}(\boldsymbol{X}(\boldsymbol{a}, t), t) = \frac{\partial \boldsymbol{X}}{\partial t}(\boldsymbol{a}, t)$$

Relation between the Eulerian and the Lagrangian descriptions

## Assumptions: Incompressible flow, translational, unbounded, small sphere, slow variation of characteristics, *Re*$_P$ << 1

Maxey and Riley (1983)

$$m_P \frac{d\boldsymbol{u}_P}{dt} = \sum \boldsymbol{F}_i \quad \Longleftrightarrow \quad m_P \frac{d\boldsymbol{u}_P}{dt} = -\int_{S_P} \boldsymbol{\sigma}_{fij} n_{pj} \mathrm{d}S + m_P \boldsymbol{g}$$

$$m_P \frac{d\boldsymbol{u}_P}{dt} = \sum \boldsymbol{F}_i \quad \Longleftrightarrow \quad m_P \frac{du_{P,i}}{dt} = \oint_S \underbrace{\left[-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]}_{\text{Stress tensor of the fluid phase (to be evaluated on the surface of the sphere)}} n_j dS + m_P g_i$$

# A possible way to go in the derivation - Decomposition of the stress tensor

$$-\int_{S_p} \sigma_{fij} n_{1j} dS = \underbrace{\int_{S_p} \overset{o}{\sigma}_{fij}\, n_{2j} dS}_{1} + \underbrace{\int_{S_p} \delta\sigma_{fij} n_{2j} dS}_{2}$$

1- Stress tensor on a surface of a particle (if the particle were not present)
2 – Perturbation (due to the presence of a particle)

# Contributions – identification of different forces

- Drag force
- Lift force
- History force
- Added mass force
- Pressure gradient force
- Force due to Brownian motion
- …

# Important to have in mind:

- Analytical solution available only for the Stokes regime
- Consideration of heat and mass transfer requires solution of two additional PDE (e.g. droplet diameter and droplet temperature)
- What happens for higher particle Re numbers?

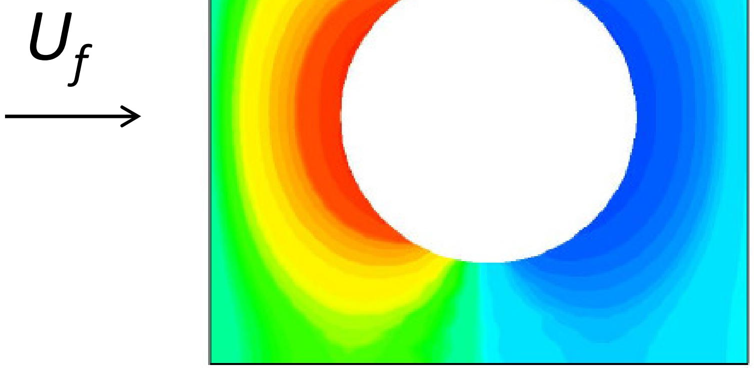
# Fluid – particle interaction force (normal and tangential component)

$$F_{\text{fluid},z} = F_{n,z} + F_{t,z}$$

$$= \int_0^{2\pi} \int_0^{\pi} \left(-p|_{r=R} \cos\theta\right) R^2 \sin\theta d\theta d\varphi$$

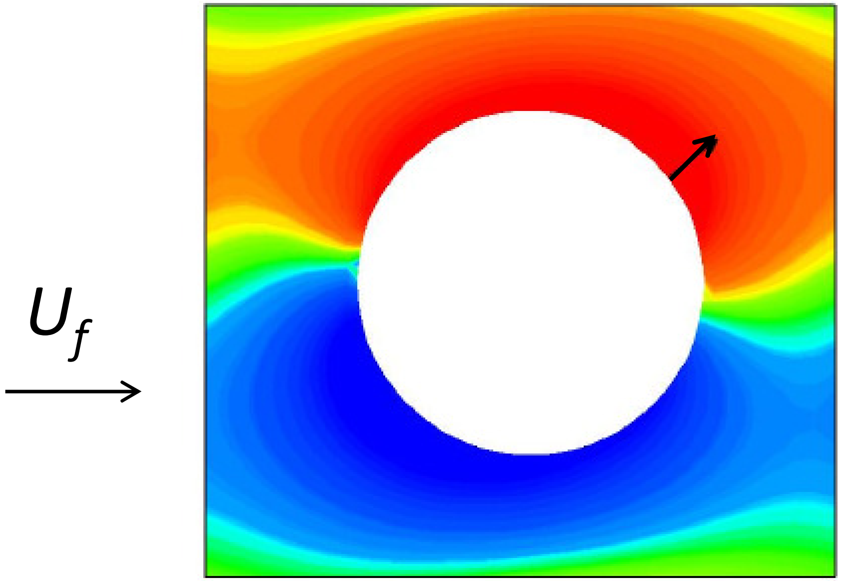
$$+ \int_0^{2\pi} \int_0^{\pi} \left(\tau_{r\theta}|_{r=R} \sin\theta\right) R^2 \sin\theta d\theta d\varphi,$$

Distribution of pressure – flow over a *stationary particle*

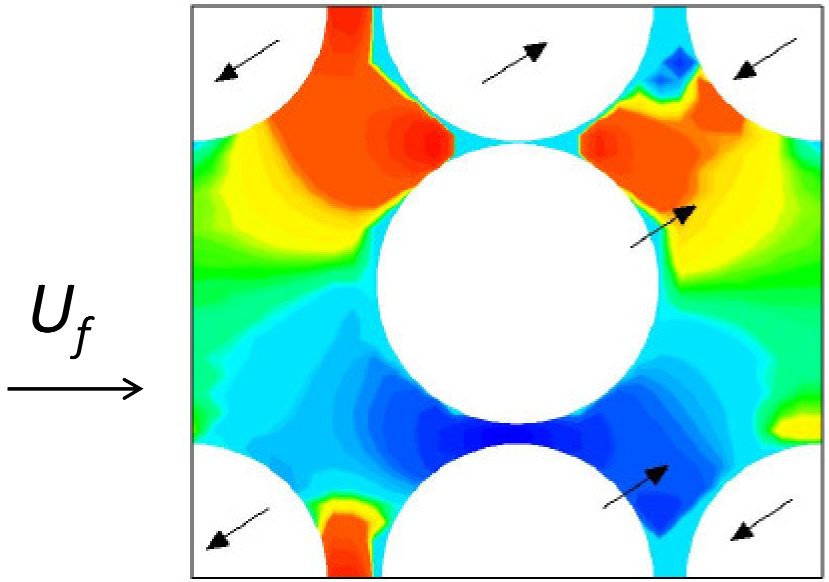


# What happens in other cases (contours of pressure are shown)?

*A particle* moving in a different direction than the mean fluid flow




*Collection of particles* moving in a different direction than the mean fluid flow



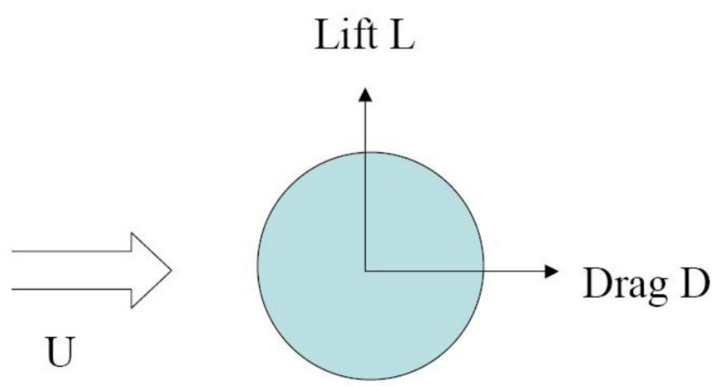
# Drag force

For highly viscous flows: Viscous drag (Stokes law)

$$F_D = 3\pi\,\mu_f\, d_P\left(\mathbf{u}_f - \mathbf{u}_\mathrm{P}\right)$$

Form drag: 1/3 $F_D$

Skin drag: 2/3 $F_D$



# Drag force

Generalized expression

$$F_D = \frac{1}{2}\rho_f \frac{d_p^2\pi}{4} C_D \left|u_f - u_p\right|\left(u_f - u_p\right)$$

Comparison with the Stokes drag

$$C_D = \frac{24}{Re_P}$$

$$Re_P = \frac{\rho_f \left(\boldsymbol{u}_{\mathrm{f}} - \boldsymbol{u}_P\right) d_{\mathrm{p}}}{\mu_f}$$

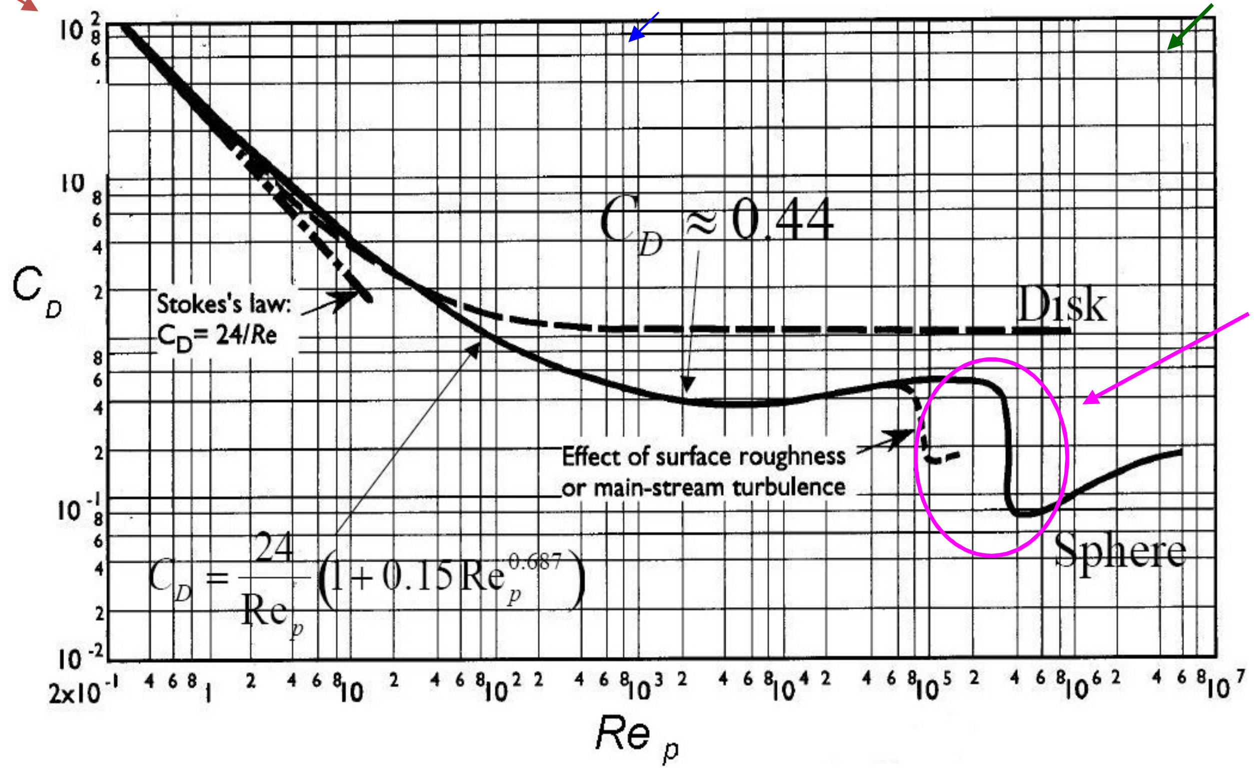
# **Drag force**: friction + form drag

Stokes regime

Transition region

Newton regime

Transition to a turbulent boundary layer

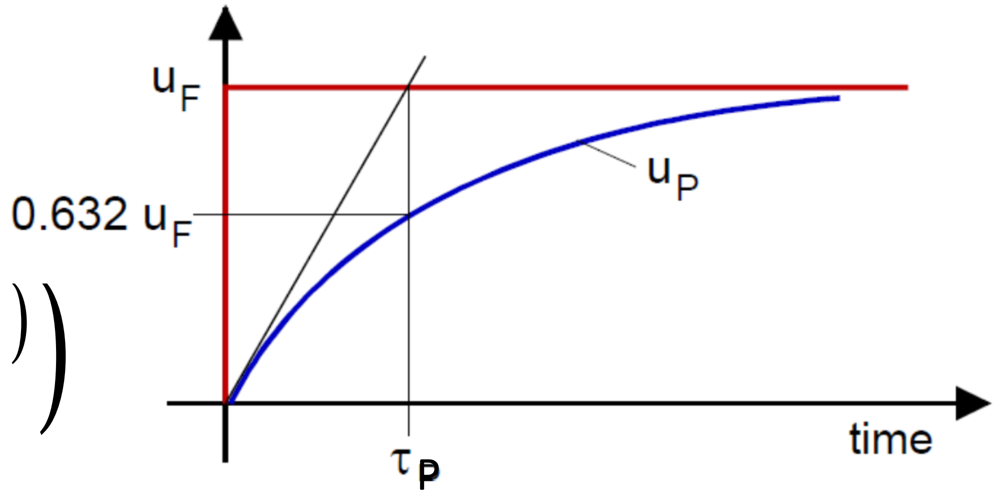


# Particle response time - derivation

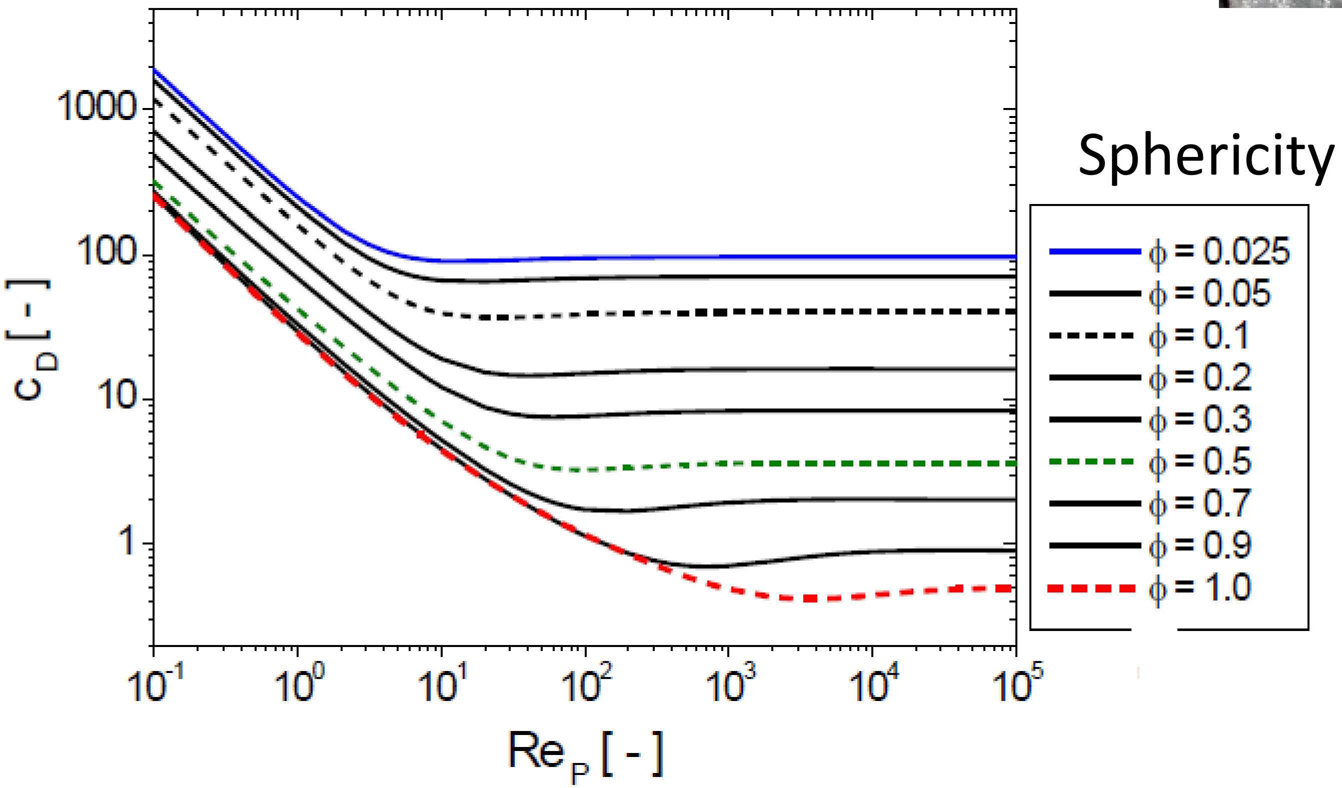
$$m_P \frac{du_P}{dt} = \frac{1}{2}\rho_f \frac{d_p^2 \pi}{4} C_D \left|u_f - u_p\right| \left(u_f - u_p\right)$$

$$\frac{du_P}{dt} = \frac{18\mu_f}{\rho_p d_p^2} \frac{C_D \operatorname{Re}_p}{24} \left(u_f - u_p\right) \Longrightarrow \frac{du_P}{dt} = \frac{\left(u_f - u_p\right)}{\tau_p}$$

$$u_P = u_f \left(1 - e^{-\left(t/\tau_p\right)}\right)$$



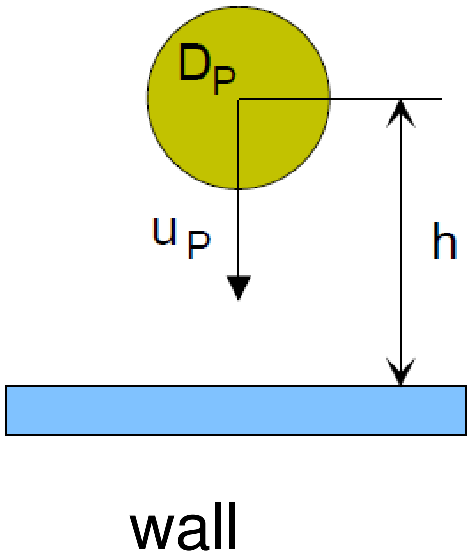
# Drag force - Effect of particle shape?



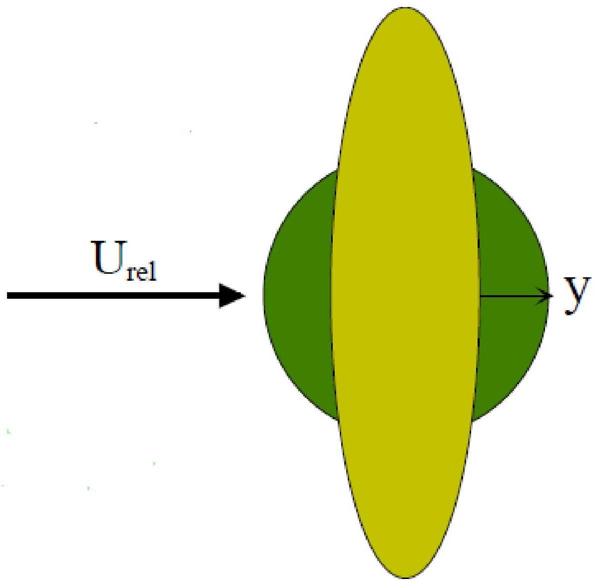
# Drag force - Effect of wall?

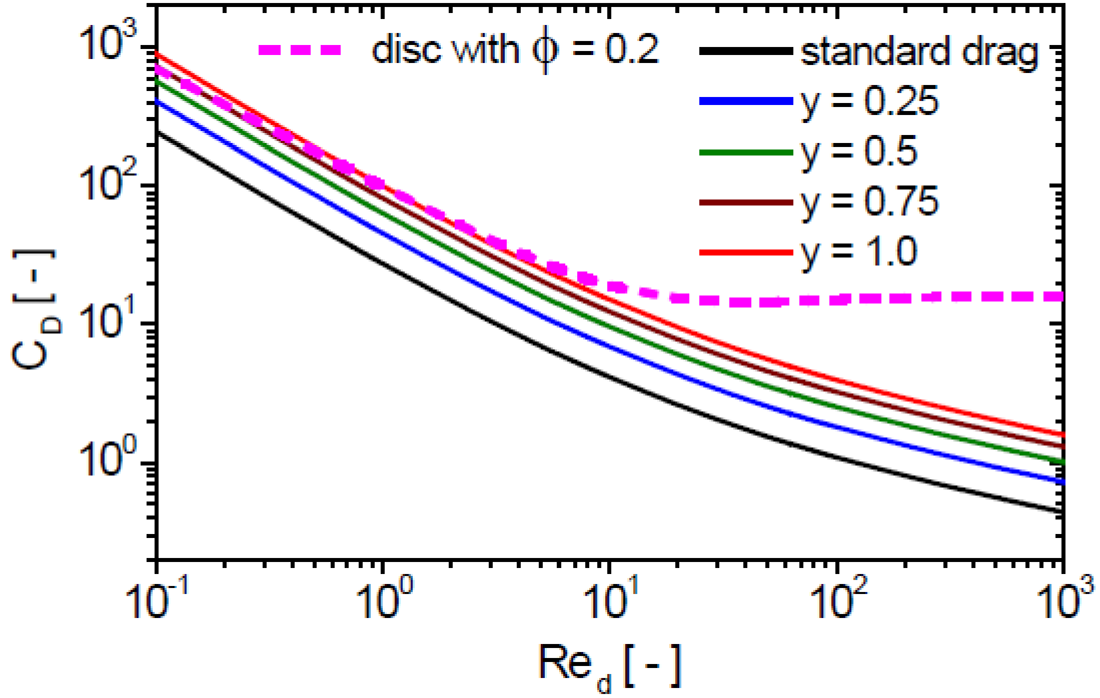
→ Drag force increases – introduction of correction coefficients in modelling

$$\frac{C_D}{C_{D,Stokes}} = 1 + const\,\frac{d_p}{h}$$

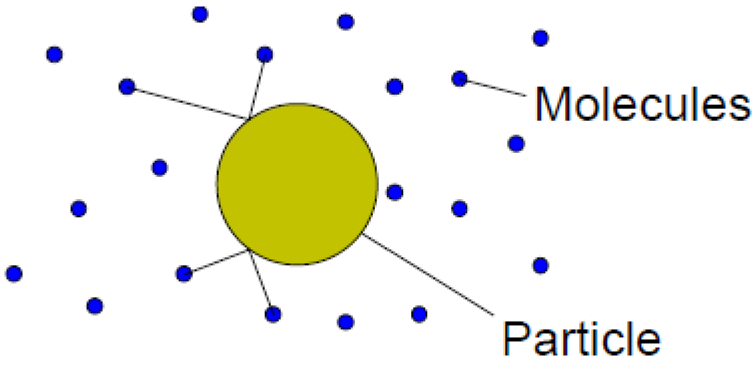


# Drag force – Oscillating and/or distorted particles



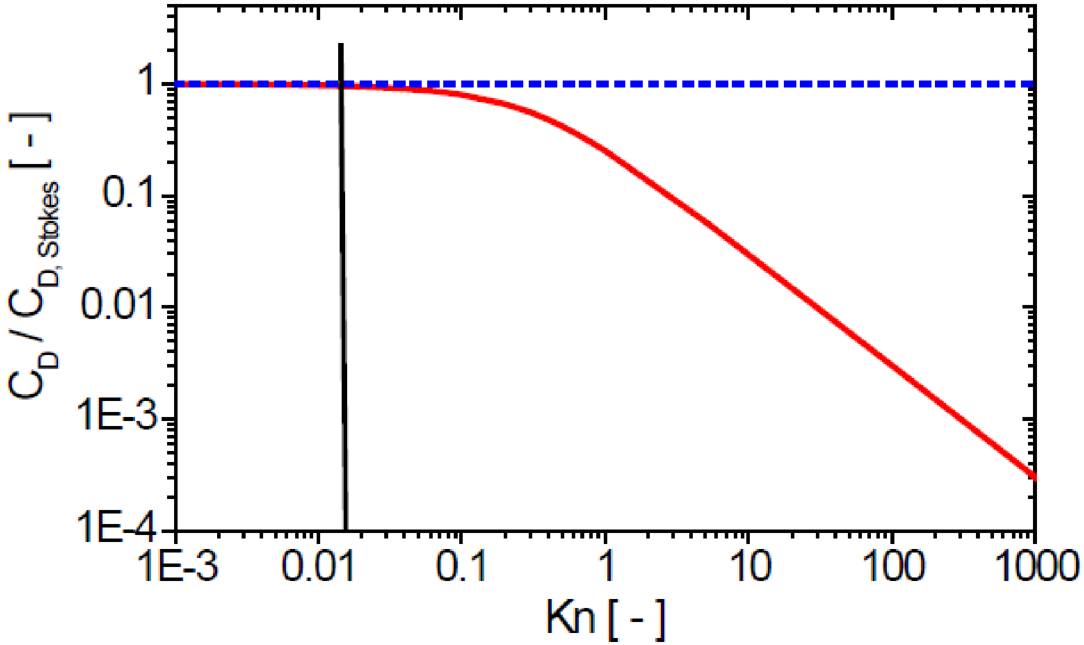



# Drag force – what if the flow is not a continuum?



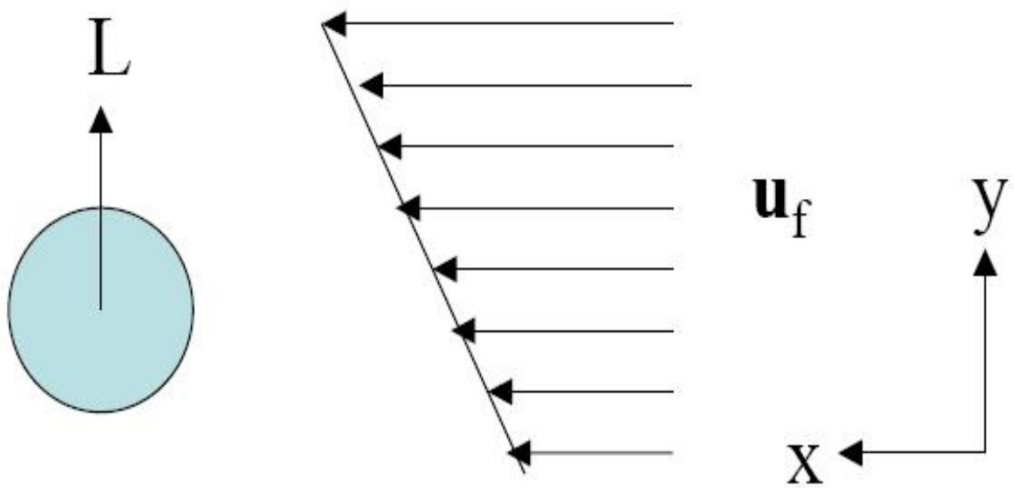

A particle "sees" individual molecules

*Kn* – particle Knudsen number



# Transverse forces

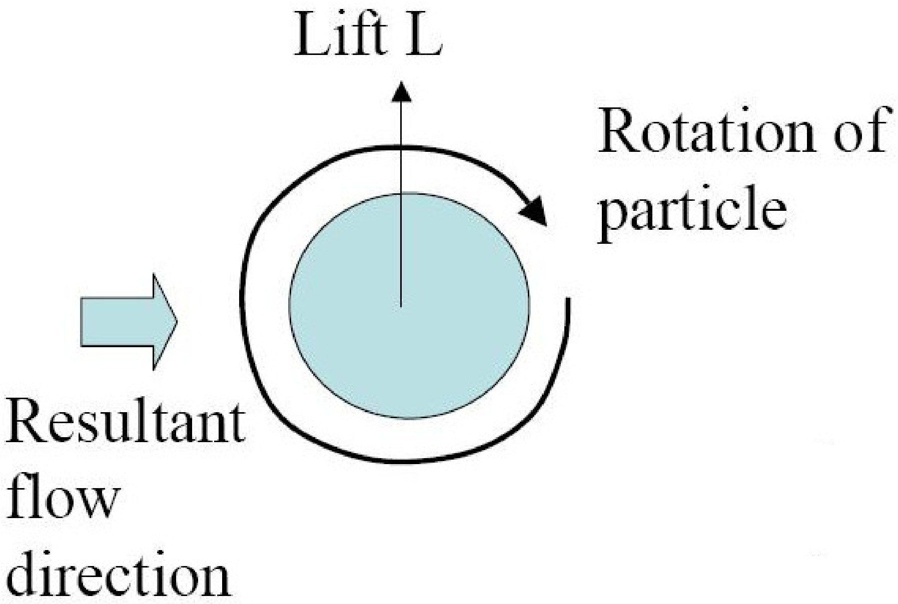
- if the flow is non uniform – presence of a velocity gradient
- if the particle is rotating (velocity gradient, collisions)
- if the particle moves in the vicinity of a wall




**Saffmann force**

# Transverse forces

**Magnus force**



***Saffmann lift force*** – presence of a velocity gradient in the direction normal to the main flow

$$\mathbf{F}_{saff} = 1.61\, d_P^2 \sqrt{\rho_f \mu_f}\, \frac{\left(\mathbf{u}_f - \mathbf{u}_\mathrm{P}\right) \times \boldsymbol{\omega}_f}{\sqrt{\left|\omega_f\right|}}; \qquad \boldsymbol{\omega}_f = rot\ \mathbf{u}_f$$

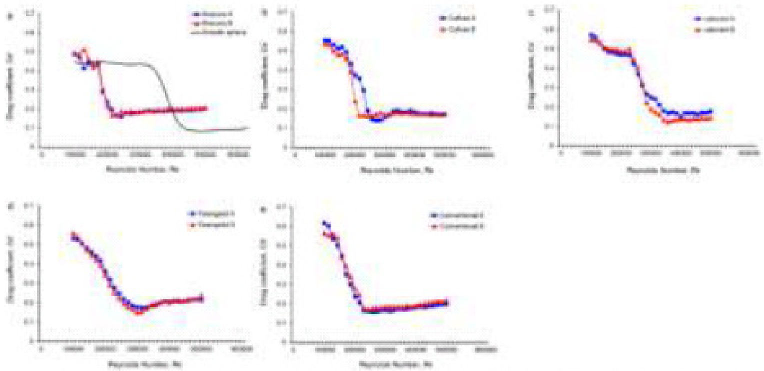
***Magnus lift force*** – particle rotating in a fluid flow (rotation introduced due to, e.g., collisions)

$$\mathbf{F}_{mag} = \frac{\pi}{8} d_P^3\, \rho_f \left( \frac{1}{2} \nabla \times \mathbf{u}_\mathrm{f} - \boldsymbol{\omega}_P \right) \times \left(\mathbf{u}_f - \mathbf{u}_\mathrm{P}\right)$$

# Magnus force example: Freekicks!

Figure 1 | Photograph of the wind tunnel test setup.


Figure 2 | Soccer balls used for the test and their panel orientations. (a, b) Adidas Brazuca: small dimple and six panels, (c, d) Adidas Cafusa: small grip texture and 32 modified panels, (e, f) Adidas Jabulani: small ridges or protrusions and eight panels, (g, h) Adidas Teamgeist 2: small protuberances and 14 panels; (i, j) Molten Vantaggio (conventional soccer ball): smooth surface and 32 pentagonal and hexagonal panels. (Photo by



igure 3 | Variation of the drag coefficient with the type of ball and panel orientation. (a) Brazuca, (b) Cafusa, (c) Jabulani, (d) Teamgeist 2, (e) conventional ball.

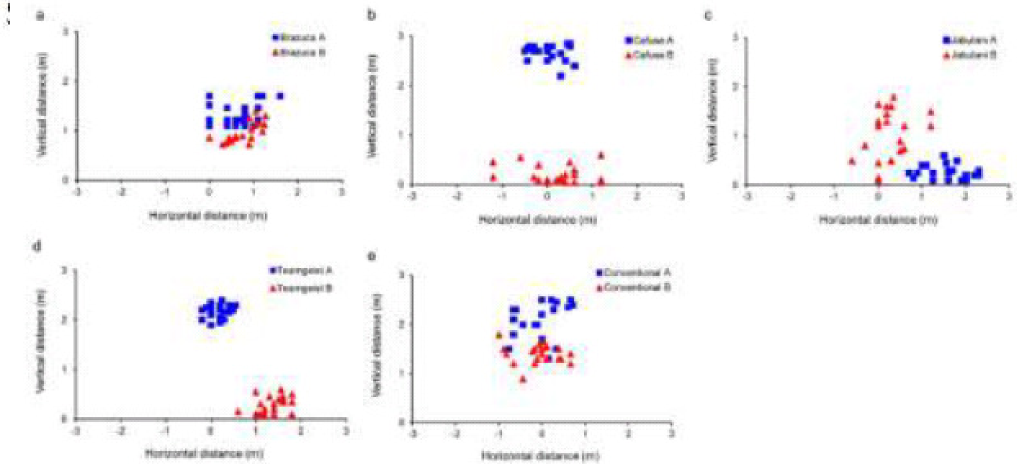


Figure 7 | Comparison of the flight characteristics (points of impact) of the different balls for different panel orientations (initial launch velocity of

Two identical freekicks made at the 2013 FIFA Confederations 25m from the goal with velocity 30 m/s would reach the goal three meters from each other in the vertical direction. Why? Because the ball was rotated 45 degrees before the second freekick

**Hong & Asai, *Scientific Reports* 4 (2014)**

**Added mass force** – acceleration of a certain fraction of the surrounding fluid

$$F_A = \frac{1}{2} m_P \frac{\rho_f}{\rho_P} \frac{d}{dt}\left(u_f - u_p\right)$$

Inertia added to the system

When can this force be neglected?

**History (Basset) force** – delay in the boundary layer development with changing relative velocity

$$F_H = \sqrt{\pi \rho_f \mu_f}\, \frac{m_P}{\rho_P d_P} \int_0^t \frac{1}{\sqrt{1-\tau}} \frac{d}{dt}\left(\mathrm{u}_f - \mathrm{u}_\mathrm{P}\right) d\tau$$

Integration along the entire trajectory for each time step of calculation

# History (Basset) force

Inclusion of the history force transforms Newton's second law for the particle from an ODE to an integro-differential equation (not explicit in $u_p$ or $du_p/dt$)

Effects most pronounced for high frequency unsteady flows when the fluid-to-particle density ratio is high, and for acceleration from rest in a quiescent fluid

**Pressure gradient force** (presence of a local pressure gradient) and the force due to shear

$$F_P = \frac{m_P}{\rho_P}\left(-\nabla p + \mu_f \nabla \tau_{shear}\right)$$

$\Downarrow$ from the Navier-Stokes equations

$$-\nabla p + \nabla \tau_{shear} = \rho_f\left(\frac{D u_f}{D t} - g\right)$$

The total pressure force is:

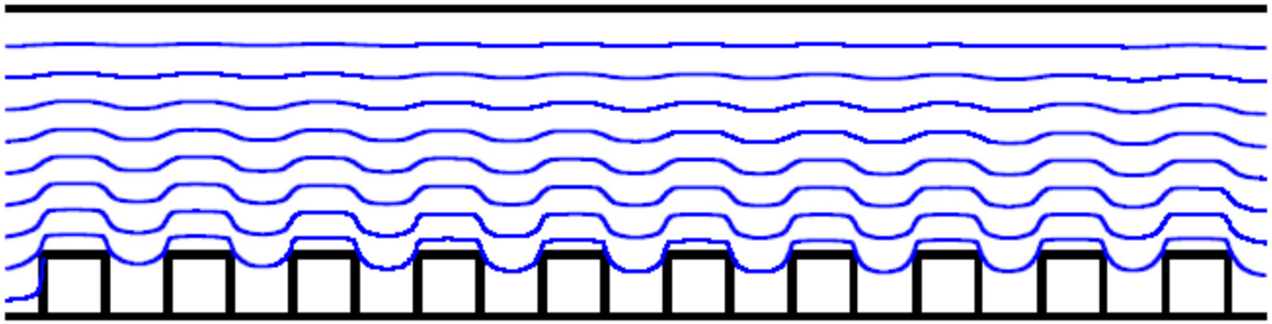
$$F_P = m_P \frac{\rho_f}{\rho_P}\left(\frac{D u_f}{D t} - g\right)$$

When is this force important?

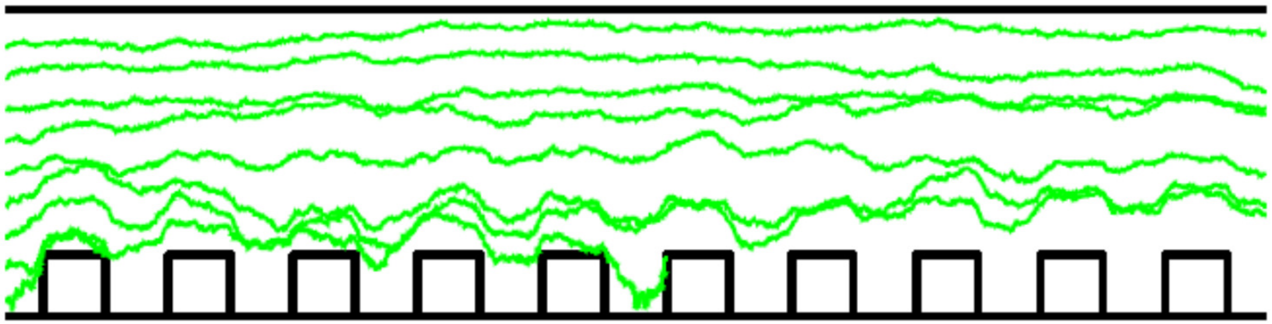
# Pressure gradient force

*”For a small sphere, small compared to the scale of the spatial variations of the undisturbed flow, the effect of the undisturbed fluid stresses both from pressure and viscosity is to produce the same net force as would act on a fluid sphere of the same size. This force must equal the product of the fluid mass and local fluid acceleration.”*

**Force due to Brownian motion** (sub-micron particles)



Drag only



Drag + Brownian

# Force due to Brownian motion

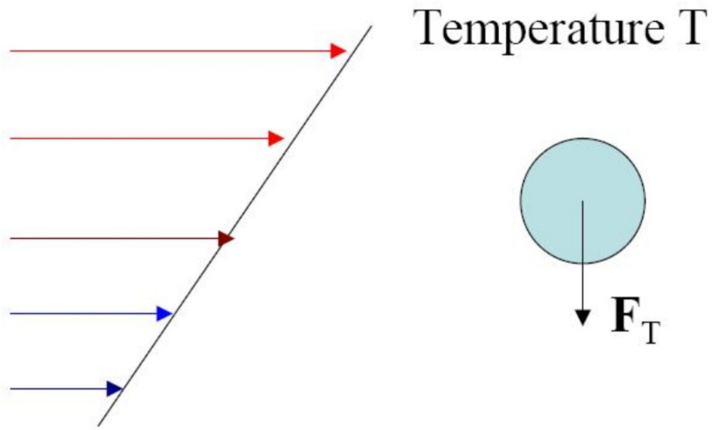
$$F_{Brownian} = \xi m_p \sqrt{\frac{216 \mu k_B T}{\pi d_p{}^5 \rho_p{}^2 C_c \Delta t}}$$

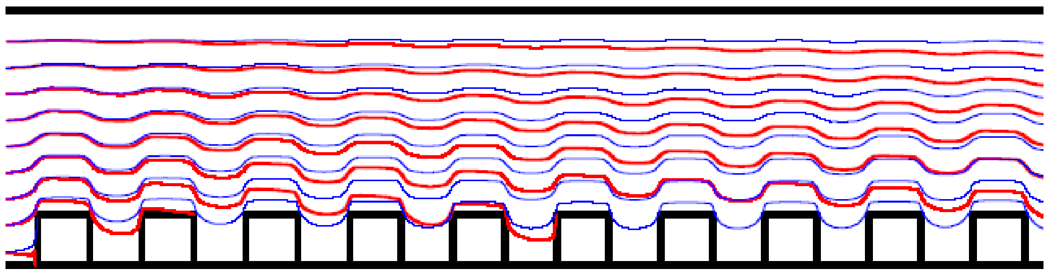
- Can be modelled as a white-noise processes using random numbers
- Transforms Newton's second law for the particle from an ODE to a stochastic differential equation
- Requires very short time steps and many particles

# Thermophoretic force

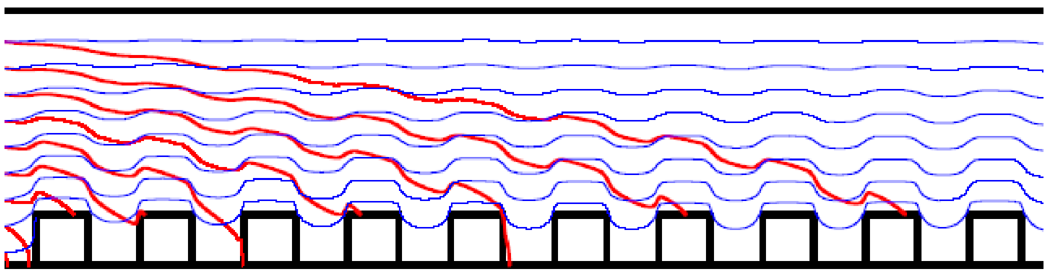
$$F_T = -\frac{6\pi d_p \mu^2 C_s \left(K + C_t \mathrm{Kn}\right)}{\rho\left(1+3C_m \mathrm{Kn}\right)\left(1+2K+2C_t \mathrm{Kn}\right)} \frac{1}{T} \frac{\partial T}{\partial x}$$

Talbot et al., 1980



$$\frac{dT}{dh} = 1 \;\; K/mm$$



$$\frac{dT}{dh} = 10 \;\; K/mm$$

✓ For $Re_p > 1$ and $Re_p >> 1$

1. The same forces as recognized before, but with coefficients introduced
2. Drag coefficient, Lift coefficients, Added mass coefficients, History coefficients
3. For the drag force – successful
4. Other forces with varying success

# Specific features related to bubbles

# Important to have in mind

- Absence of a rigid interface between a bubble and fluid

⬇

✓Consequence: Reduction of the drag force (compared to the corresponding solid particle)

- Contamination of the surface of a bubble

⬇

✓Consequence: Rigid interface again created?

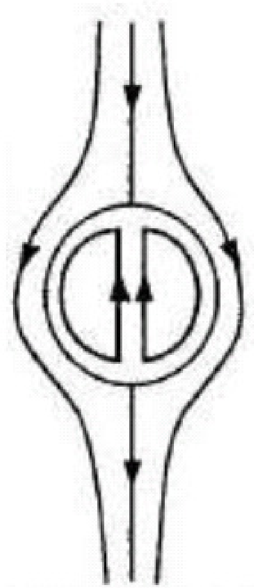
# Internal circulation as a fundamental phenomenon

Tendency to internal circulation given by:

$$\frac{3\kappa + 2}{3\kappa + 3}$$

1 = particle behaves as a rigid body
2/3 = "full" internal circulation

$$\kappa = \frac{viscosity\ of\ particle\ fluid}{viscosity\ of\ carrier\ fluid}$$



# Drag force – what happens for bubbles and drops?

Hadamard-Rybczynski drag law – stresses on the surface induce internal motion - drag coefficient decreased

$$C_{D,HR} = \frac{24}{Re_p}\left(\frac{2/3 + \mu_d/\mu_c}{1 + \mu_d/\mu_c}\right)$$

Droplet in air : the Stokes law is recovered

Bubble in liquid: drag is reduced by 1/3

# Bubble deformation and oscillation

*Weber* number:
(inertia/surface tension)

$$We_b = \frac{\rho_f \, u_{rel}^2 \, d_{eq}}{\sigma}$$

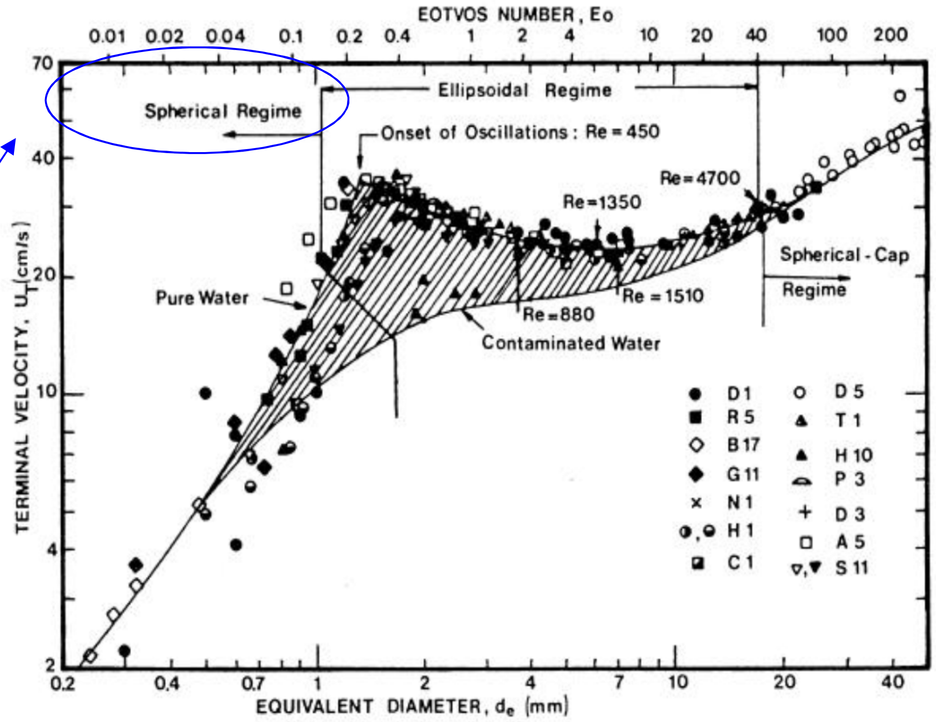
Equivalent hydraulic diameter

Can bubbles maintain spherical shape?

$$E_O = \frac{We_b}{Fr_b} = g \frac{|\rho_f - \rho_b| \, d_{eq}^{\;2}}{\sigma}$$

*Eötvös* number
(buoyancy/surface tension)

Clift (1986)



# Bubble-shape diagram

*Capillary* number

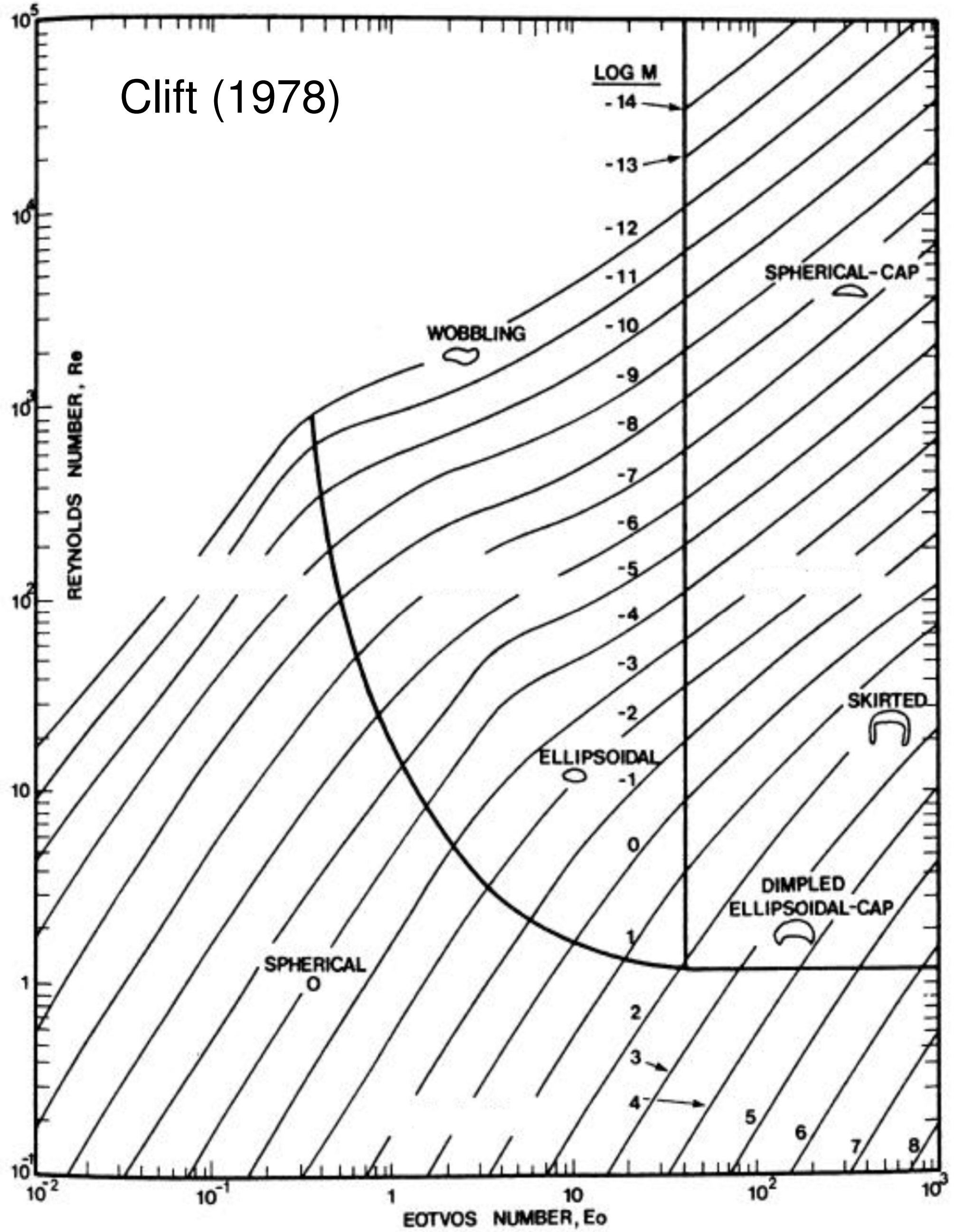
(viscous/surface tension)

$$Ca = \frac{\mu_f U}{\sigma} = \frac{We}{\mathrm{Re}}$$

*Morton* number

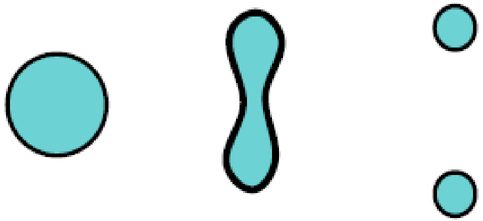
$$Mo = \frac{g\mu_f^{\,4}\left|\rho_f - \rho_b\right|}{\rho_f^2 \sigma^3}$$

$$Mo = \frac{EoWe^2}{\mathrm{Re}^4}$$

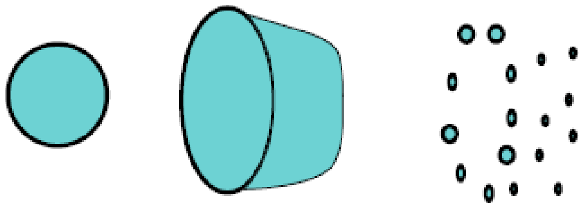


# Example: (secondary) breakup of droplets as a function of *We*

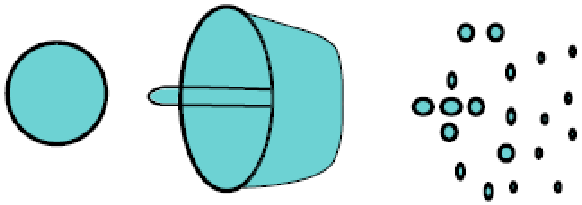
Wierzba, 1990



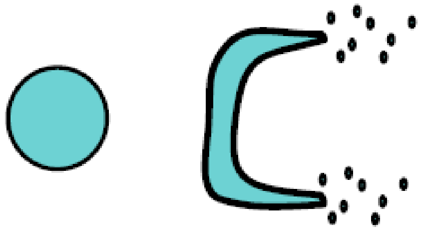
(a) Vibrational breakup, $We_d \approx 12$



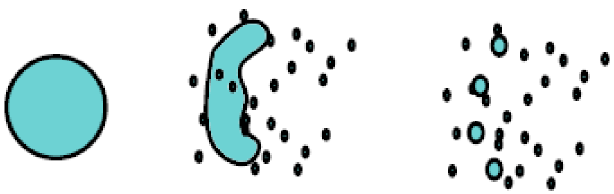
(b) Bag breakup, $We_d < 20$



(c) Bag / streamer breakup, $We_d < 50$



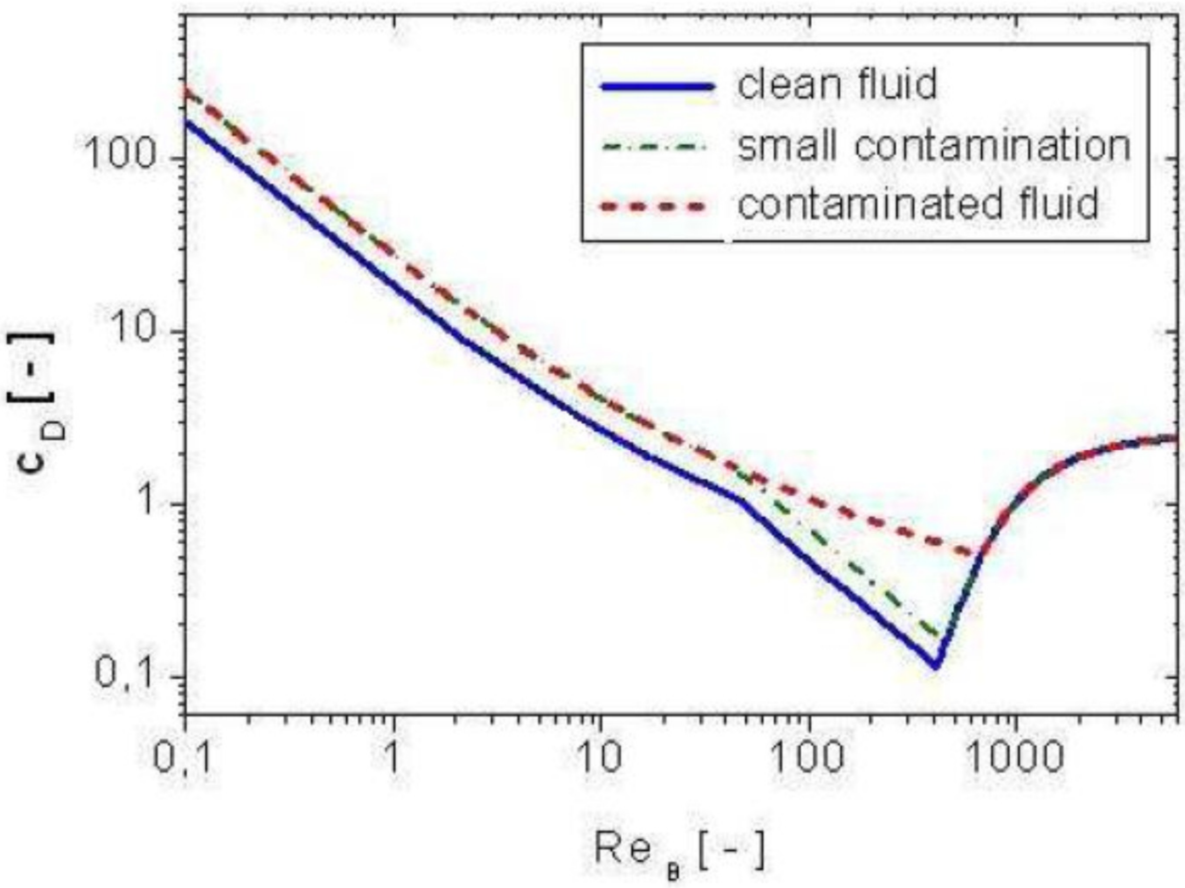
(d) Stripping breakup, $We_d < 100$



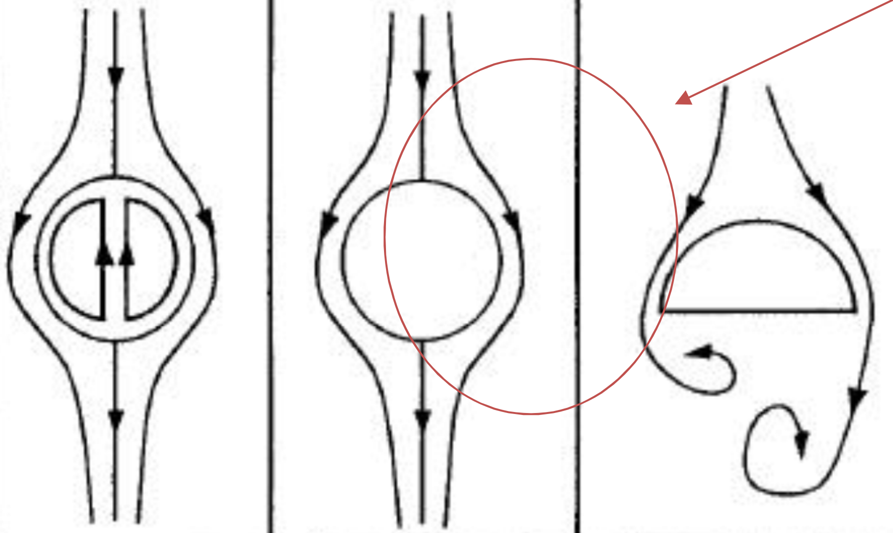
(e) Catastrophic breakup, $We_d > 100$

# Drag force as a function of shape and level of contamination

Tomyama (1998)



Tomyama (1998)

| Shape | spherical | | non-spherical |
|---|---|---|---|
| Motion | rectilinear | | fluctuating |
| Purity | pure | contaminated | both |
| Flow pattern |  | | |
| Governing effects | viscosity | viscosity | surface tension and gravity |
| Relevant dimension-less number | Re | Re | Eo |

Rigid interface re-created?