Maximum Cut

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Abstract

TODO

1 Introduction

2 Mathematical Formulations

3 Relaxations for Max Cut

3.1 Eigenvalue Relaxation

Proposition 1. *The optimal value of z^* in the maxcut problem defined on graph $G = (V, E)$ satisifes*

$$z^* \leq z^{ev} := \frac{n}{4}\lambda_{max}(L(G))$$

Proof. From the previous section we saw that

$$z^* = \frac{1}{4} \max_{\mathbf{z} \in \{-1,1\}} \sum_{i \in V} \sum_{j \in V} w_{ij}(1 - z_i z_j) \tag{1}$$

Since $z_i \in \{-1, 1\}$ it is clear that $z_i z_i = 1$. Then, we can express (1) as

$$\frac{1}{4} \max_{\mathbf{z} \in \{-1,1\}} \sum_{i \in V} \sum_{j \in V} w_{ij}(z_i z_i - z_i z_j) \tag{2}$$

Rearranging these terms we see that

$$\frac{1}{4} \max_{\mathbf{z} \in \{-1,1\}} \sum_{i \in V} z_i \left(\sum_{j \in V} w_{ij} \right) z_i - \sum_{i \in V} z_i \sum_{j \in V} w_{ij} z_j \tag{3}$$

Recall that $D(G)_{ii} = \sum_{j \in V} w_{ij}$ and $D(G)_{ij} = 0$ for $i, j \in V$. The first term in (3) can be expressed as $\mathbf{z}^\top D(G)\mathbf{z}$. Likewise, the weighted adjacency matrix $A(G)_{ii} = 0$ and $A(G)_{ij} = w_{ij}$ for $i, j \in V$. The second term can be expressed as $\mathbf{z}^\top A(G)\mathbf{z}$ and now

$$\frac{1}{4} \max_{\mathbf{z} \in \{-1,1\}} \mathbf{z}^\top D(G)\mathbf{z} - \mathbf{z}^\top A(G)\mathbf{z} \tag{4}$$

Since the matrix product is distributive we have

$$\frac{1}{4} \max_{\mathbf{z} \in \{-1,1\}} \mathbf{z}^\top \left(D(G) - A(G) \right) \mathbf{z} \tag{5}$$

By definition of the Laplacian of G we have that the optimal solution of maxcut can be computed as

$$\frac{1}{4} \max\{\mathbf{z}^\top L(G)\mathbf{z} \,|\, \mathbf{z} \in \{-1, 1\}\} \tag{6}$$

Taking the continuous relaxation of (6), namely $z \in [-1, 1]$, is equivalent to maximizing over the $\|\mathbf{z}\|_\infty = 1$. We can relax this further by maximizing over a ball of radius $\sqrt{n}$. In other words, our region is now defined where $\|\mathbf{z}\| \leq \sqrt{n}$.

$$z^* \leq \frac{1}{4} \max\{\mathbf{z}^\top L(G)\mathbf{z} \,|\, \mathbf{z} \leq \sqrt{n}\} \tag{7}$$

We can define our problem over the unit ball with a simple transformation. Let $\|\mathbf{z}\| = \sqrt{n}\mathbf{y}$ and we now have

$$z^* \leq \frac{n}{4} \max\{\mathbf{y}^\top L(G)\mathbf{y} \mid \|\mathbf{y}\| \leq 1\} \tag{8}$$

Since for any symmetric matrix $\mathbf{A}$, $\max\{\mathbf{x}^\top \mathbf{A}\mathbf{x} \mid \|\mathbf{x}\| \leq 1\} = \lambda_{\max(\mathbf{A})}$ and $L(G)$ is symmetric, then we have

$$z^* \leq z^{ev} := \frac{n}{4}\lambda_{max}(L(G)) \tag{9}$$

$\square$

Proposition 2. *The optimal value of z^* in the maxcut problem defined on graph $G = (V, E)$ satisifes*

$$z^* \leq z^{ev} := -\frac{1}{4}\sum_{i=1}^{n} u_i + \frac{n}{4}\lambda_{max}(L(G) + diag(\mathbf{u}))$$

for all $\mathbf{u} \in \mathbb{R}^n$

Proof. Consider the maximization problem

$$\max\{\mathbf{z}^\top (L(G) + diag(\mathbf{u}))\mathbf{z} | \mathbf{z} \in \{-1, 1\}\}$$

Note that no matter the value of $\mathbf{u} \in \mathbb{R}^n$, $L(G) + diag(\mathbf{u})$ remains to be a symmetric square matrix. As a result,

$$\lambda_{max}(L(G) + diag(\mathbf{u})) = \max\{\mathbf{z}^\top (L(G) + diag(\mathbf{u}))\mathbf{z} | \mathbf{z} \in \{-1, 1\}\}$$

Using the distributive property of the matrix product we can separate this into two separate maximization problems

$$\max\{\mathbf{z}^\top L(G)\mathbf{z} | \mathbf{z} \in \{-1, 1\}\} + \max\{\mathbf{z}^\top diag(\mathbf{u})\mathbf{z} \mid \mathbf{z} \in \{-1, 1\}\}$$

Expanding this further yields the optmization problem

$$\max_{\mathbf{z} \in \{-1,1\}} \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(1 - z_i z_j) + \sum_{i=1} u_i \tag{10}$$

Which is closely related to the objective function in our prevoius formulation. We can obtain a relationship between (10) and z^*:

$$z^* = \frac{1}{4}\max\{\mathbf{z}^\top (L(G) + diag(\mathbf{u}))\mathbf{z} \mid \mathbf{z} \in \{-1, 1\}\} - \frac{1}{4}\sum_{i=1}^{n} u_i$$

We repeat the process from the previous proof of relaxing the variables so that they are continuous and then further relaxing this problem onto a ball of radius $\sqrt{n}$.

$$z^* \leq \frac{1}{4}\max\{\mathbf{z}^\top (L(G) + diag(\mathbf{u}))\mathbf{z} \mid \|\mathbf{z}\| \leq \sqrt{n}\} - \frac{1}{4}\sum_{i=1}^{n} u_i$$

Let $\mathbf{z} = \sqrt{n}\mathbf{x}$.

$$z^* \leq \frac{n}{4} \max\{\mathbf{x}^\top (L(G) + diag(\mathbf{u}))\mathbf{x} \mid \|\mathbf{x}\| \leq 1\} - \frac{1}{4} \sum_{i=1}^{n} u_i$$

Finally, we obtain the inequality

$$z^* \leq z^{ev}(\mathbf{u}) = -\sum_{i=1}^{n} u_i + \frac{n}{4} \lambda_{max}(L(G) + diag(\mathbf{u})) \tag{11}$$

$\square$