Relaxations of the Maximum Cut Problem

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Abstract

Maximum cut is a classic NP-Hard problem in combinatorial optimization. In this work we present several relaxations for the maximum cut problem.

1 Introduction

2 Mathematical Formulations

3 Relaxations for Max Cut

3.1 Eigenvalue Relaxation

Proposition 1. *The optimal value of z^* in the maxcut problem defined on graph $G = (V, E)$ satisifes*

$$z^* \leq z^{ev} := \frac{n}{4}\lambda_{max}(L(G))$$

Proof. Consider the maximization problem

$$\max\{\mathbf{z}^\top L(G)\mathbf{z} \,|\, \mathbf{z} \in \{-1, 1\}^n\}$$

Since $L(G) = D(G) - A(G)$, then $L(G)_{ii} = \sum_{j \in V} w_{ij}$ for $i \in V$ and $L(G)_{ij} = -w_{ij}$ for $i, j \in V$ such that $i \neq j$. We expand the objective function and obtain

$$\max_{\mathbf{z} \in \{-1,1\}} \sum_{i \in V} \sum_{j \in V} w_{ij} z_i z_i - \sum_{i \in V} \sum_{j \in V} w_{ij} z_i z_j$$

Furthermore,

$$\max_{\mathbf{z} \in \{-1,1\}} \sum_{i \in V} \sum_{j \in V} w_{ij}(z_i z_i - z_i z_j)$$

Since $z_i \in \{-1, 1\}$ then $z_i z_i = 1$, we can simplify the objective function to

$$\max_{\mathbf{z} \in \{-1,1\}} \sum_{i \in V} \sum_{j \in V} w_{ij}(1 - z_i z_j)$$

From the previous section we saw an almost identical formulation for max cut. The only difference being a multiplier of $\frac{1}{4}$. Let z^* represent the optimial solution for max cut

$$z^* = \frac{1}{4} \max\{\mathbf{z}^\top L(G)\mathbf{z} \,|\, \mathbf{z} \in \{-1, 1\}^n\} \tag{1}$$

Taking the continuous relaxation of (1), namely $z_i \in [-1, 1]$ for $i \in V$, is equivalent to maximizing over the norm infinity, $\|\mathbf{z}\|_\infty \leq 1$. We can relax this further by maximizing over a ball of radius $\sqrt{n}$. In other words, our region is now defined where $\|\mathbf{z}\| \leq \sqrt{n}$.

$$z^* \leq \frac{1}{4} \max\{\mathbf{z}^\top L(G)\mathbf{z} \,|\, \mathbf{z} \leq \sqrt{n}\} \tag{2}$$

We can define our problem over the unit ball with a simple transformation. Let $\|\mathbf{z}\| = \sqrt{n}\mathbf{x}$ and we now have

$$z^* \leq \frac{n}{4} \max\{\mathbf{x}^\top L(G)\mathbf{x} \,|\, \|\mathbf{x}\| \leq 1\} \tag{3}$$

Since for any symmetric matrix $\mathbf{A}$, $\max\{\mathbf{x}^\top \mathbf{A}\mathbf{x} \,|\, \|\mathbf{x}\| \leq 1\} = \lambda_{\max(\mathbf{A})}$ and $L(G)$ is symmetric, then we have

$$z^* \leq z^{ev} := \frac{n}{4}\lambda_{max}(L(G)) \tag{4}$$

$\square$

Proposition 2. *The optimal value of z^* in the maxcut problem defined on graph $G = (V, E)$ satisifes*

$$z^* \leq z^{ev} := -\frac{1}{4}\sum_{i=1}^{n} u_i + \frac{n}{4}\lambda_{max}(L(G) + diag(\mathbf{u}))$$

for all $\mathbf{u} \in \mathbb{R}^n$

Proof. Consider the maximization problem

$$\max\{\mathbf{z}^\top(L(G) + diag(\mathbf{u}))\mathbf{z}|\mathbf{z} \in \{-1, 1\}\} \tag{5}$$

From the previous proof, we perform a similar technique to (5) to obtain the following optimization problem

$$\max_{\mathbf{z}\in\{-1,1\}} \sum_{i\in V}\left(\sum_{j\in V} w_{ij} + u_i\right) z_i z_i - \sum_{i\in V}\sum_{j\in V} w_{ij} z_i z_j$$

Rearranging terms and using the fact that $z_i z_i = 1$,

$$\max_{\mathbf{z}\in\{-1,1\}} \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(1 - z_i z_j) + \sum_{i=1} u_i \tag{6}$$

Which is closely related to the objective function in our prevoius formulation. We obtain the following relationship between (6) and z^*:

$$z^* = \frac{1}{4}\max\{\mathbf{z}^\top(L(G) + diag(\mathbf{u}))\mathbf{z} \,|\, \mathbf{z} \in \{-1, 1\}\} - \frac{1}{4}\sum_{i=1}^{n} u_i$$

We repeat the process from the previous proof of relaxing the variables so that they are continuous and then further relaxing this problem onto a ball of radius $\sqrt{n}$.

$$z^* \leq \frac{1}{4}\max\{\mathbf{z}^\top(L(G) + diag(\mathbf{u}))\mathbf{z} \,|\, \|\mathbf{z}\| \leq \sqrt{n}\} - \frac{1}{4}\sum_{i=1}^{n} u_i$$

Let $\mathbf{z} = \sqrt{n}\mathbf{x}$.

$$z^* \leq \frac{n}{4}\max\{\mathbf{x}^\top(L(G) + diag(\mathbf{u}))\mathbf{x} \,|\, \|\mathbf{x}\| \leq 1\} - \frac{1}{4}\sum_{i=1}^{n} u_i$$

Finally, we obtain the inequality

$$z^* \leq z^{ev}(\mathbf{u}) = -\frac{1}{4}\sum_{i=1}^{n} u_i + \frac{n}{4}\lambda_{max}(L(G) + diag(\mathbf{u})) \tag{7}$$

□

3.2 Lagrangian Dual

In the previous section we have proved several upper bounds on max cut based on the Laplace $L(G)$, including one which changes of the diagonal of $L(G)$ with some vector $\mathbf{u} \in \mathbb{R}^n$. In this section we will compute the Lagrangian dual bound of maximum cut, which corresponds to the values of $\mathbf{u}$ such that $z^{ev}(\mathbf{u})$ is minimized.

$$z^{LD} = \min \left\{ -\frac{1}{4}\sum_{i=1}^{n} u_i + \frac{n}{4}\lambda_{max}(L(G) + diag(\mathbf{u})) \,\middle|\, \mathbf{u} \in \mathbb{R}^n \right\}$$

Consider the gradient of our objective function, $\nabla z^{ev}(\mathbf{u})$. In order to compute $\nabla \lambda_{max}(L(G)+diag(\mathbf{u}))$ we first use the fact that $L(G)+diag(\mathbf{u})$ is a symmetric square matrix and therefore $\lambda_{max}(L(G) + diag(\mathbf{u})) = \max\{\mathbf{x}^\top(L(G) + diag(\mathbf{u}))\mathbf{x} \mid \|\mathbf{x}\| \leq 1\}$. This optimization problem can further be expanaded to

$$\max_{\|\mathbf{x}\|\leq 1} \sum_{i\in V}\left(\sum_{j\in V} w_{ij} + u_i\right) x_i x_i - \sum_{i\in V}\sum_{j\in V} w_{ij} x_i x_j$$

We can see that this objective value is maximized for $x_k = 1$ where $k = \arg\max_i\{\sum_{j\in V} w_{ij} + u_i \mid \sum_{j\in V} w_{ij} + u_i > 0\}$, and $x_i = 0$ for $i \in V \setminus \{k\}$. If there exists no positive value of $\sum_{j\in V} w_{ij} + u_i$ then the objective function is maximized for $\mathbf{x} = \mathbf{0}$. Therefore, $\nabla\lambda_{max}(L(G)+diag(\mathbf{u})) = e_k$ if $\lambda_{max}(L(G) + diag(\mathbf{u})) \geq 0$ and $\mathbf{0}$ otherwise and $\nabla z^{ev}(\mathbf{u})$ can be computed as

$$\nabla z^{ev}(\mathbf{u})_k = \begin{cases} \frac{n-1}{4} & \text{if } k = \arg\max_i \left\{ \sum_{j\in V} w_{ij} + u_i \,\middle|\, \sum_{j\in V} w_{ij} + u_i > 0 \right\} \\ -\frac{1}{4} & \text{otherwise} \end{cases}$$

3.2.1 Subgradient Algorithm

We implement the following subgradient algorithm in order to compute z^{LD}. We postpone details on our implementation decisions such as stopping criteria, initial solutions and choice of sequence h_k for a later section.

Algorithm 1: Subgradient Algorithm for MaxCut

Result: Computes the Lagrangian dual bound z^{LD}
Select an initial solution $\mathbf{u}_0$ and appropriate sequence $\{h_k\}_{k=0}^{\infty}$;
for $k \geq 0$ **do**
  Compute $z^{ev}(\mathbf{u})$ and $\nabla z^{ev}(\mathbf{u})$;
  $\mathbf{u}_{k+1} \leftarrow \mathbf{u}_k - h_k \frac{\nabla z^{ev}(\mathbf{u})}{\|\nabla z^{ev}(\mathbf{u})\|}$;
end