

Gurobi Seminar 5

Traveling Salesman Problem

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Welcome!

- Welcome to the fifth installment of our Gurobi series!
- In this workshop we will cover the traveling salesman problem
- At the end of this workshop you will know how to implement lazy cuts in Gurobi
- Materials for this workshop can be found here:
`github.com/Dpapazaharias1/UB-INFORMS-Gurobi-Seminar`

Contents

1. Lazy Cuts

2. Traveling Salesman Problem

3. Lazy Cuts In Gurobi

User Cuts vs. Lazy Cuts

- In the last session we implemented user cuts for vertex packing
- Recall that user cuts are ordinary constraints which are added to improve the LP relaxation, but do not eliminate feasible solution. A formulation is valid with or without these cuts.
- **Lazy cuts**, in contrast, represent one portion of the constraint set that is essential in defining the formulation. Lazy cuts are only added once an integer feasible solution is identified.
- Lazy cuts are part of a very large constraint set where most constraints are unlikely to be violated. The term “lazy” is a reference to only adding the constraint once it is needed.

Contents

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2. Traveling Salesman Problem

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Problem Description

- **Input:** A list of cities and the distances between each pair of cities
- **Output:** Minimum cost route which visits each city and return to the origin
- We define this problem over a digraph $G = (N, A)$ where the nodes represent the cities and the distances are represented by arc weights c_{ij}
- There are many versions of the TSP but for this workshop we will focus on the symmetric TSP ($c_{ij} = c_{ji}$)

Building the model

- Define a binary variable for each arc

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is in the tour} \\ 0 & \text{otherwise} \end{cases}$$

- Minimize the cost of the tour

$$\min \sum_{(i,j) \in A} c_{ij} x_{ij}$$

- Each city must be visited once

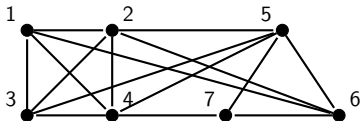
$$\sum_{j \in N} x_{ij} = 1 \quad i \in N$$

- Each city must be departed from once

$$\sum_{i \in N} x_{ij} = 1 \quad j \in N$$

Issues

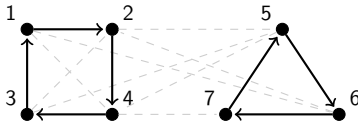
- Consider this instance of the TSP
- If we solve the model as is we may receive the solution:



- We visit and leave each city, but we have **subtours**
- We introduce a set of constraints to break subtours

Issues

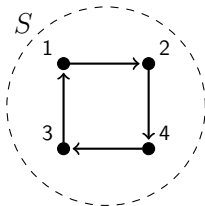
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- If we solve the model as is we may receive the solution:



- We visit and leave each city, but we have **subtours**
- We introduce a set of constraints to break subtours

Subtour Elimination

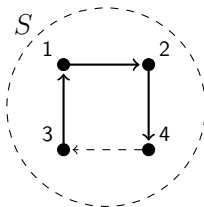
- Consider a subset of vertices S which are part of a subtour



- For any subtour the number of arcs (i, j) with $i, j \in S$ is equal to $|S|$
- We can break these subtours by restricting the number of arcs in S to be $|S| - 1$
- By breaking these subtours we will force one arc to enter S from $V \setminus S$ and one arc from S to $V \setminus S$.

Subtour Elimination

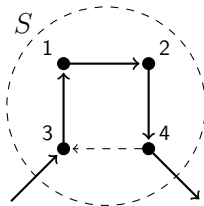
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Subtour Elimination

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- By breaking these subtours we will force one arc to enter S from $V \setminus S$ and one arc from S to $V \setminus S$.

DFJ Formulation

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is in the tour} \\ 0 & \text{otherwise} \end{cases}$$

$$\min \sum_{(i,j) \in A} c_{ij} x_{ij} \quad (1)$$

$$\text{s.t.} \quad \sum_{j \in N} x_{ij} = 1 \quad i \in N \quad (2)$$

$$\sum_{i \in N} x_{ij} = 1 \quad j \in N \quad (3)$$

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1 \quad S \subset N \quad (4)$$

$$x_{ij} \in \{0, 1\} \quad (i, j) \in A \quad (5)$$

Subtour Elimination

The complicating constraints for the TSP are the subtour elimination constraints:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1$$

- There are $O(2^n)$ subtour elimination constraints
- In practice, only a fraction of the subtour elimination constraints are necessary
- In order to solve the TSP, we will begin with none of the constraints and add them as needed

Separation Algorithm

For TSP, we will check if a current integer solution violates any of the subtour elimination constraints:

- Given a solution \bar{x}_{ij} , we construct a digraph $G' = (N, A')$ where $A' = \{(i, j) \in A | \bar{x}_{ij} = 1\}$
- Find the connected components of G'
- If there is more than one connected component, then G' contains subtours, add the corresponding constraint for each subtour S :

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1$$

and solve the LP relaxation.

Connected Components

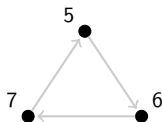
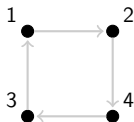
```

1: procedure CONNECTEDCOMPONENTS
2:    $C \leftarrow \emptyset$ 
3:   initialize each  $i \in N$  as undiscovered
4:   for  $i \in N$  do
5:      $K, Q \leftarrow \emptyset$ 
6:     if  $i$  is not discovered then
7:       label  $i$  as discovered
8:        $K \leftarrow K \cup \{i\}$ ,  $Q \leftarrow Q \cup \{i\}$ 
9:       while  $Q$  is not empty do
10:         $v \leftarrow Q.\text{dequeue}()$ 
11:        for  $u \in \text{Adj}(v)$  do
12:          if  $u$  is not discovered then
13:            label  $u$  as discovered
14:             $K \leftarrow K \cup \{u\}$ ,  $Q \leftarrow Q \cup \{u\}$ 
15:        $C \leftarrow C \cup \{K\}$ 

```


Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\}$$

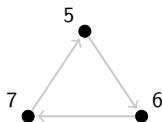
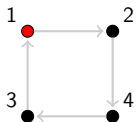
$$K = \{\}$$

$$Q = \{\}$$

$$D = \{0, 0, 0, 0, 0, 0, 0\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\}$$

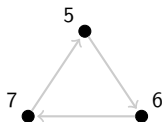
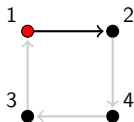
$$K = \{1\}$$

$$Q = \{1\}$$

$$D = \{1, 0, 0, 0, 0, 0, 0\}$$

Detecting Subtours

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$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\}$$

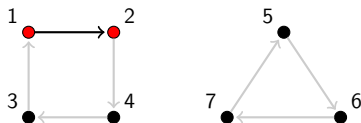
$$K = \{1\}$$

$$Q = \{\}$$

$$D = \{1, 0, 0, 0, 0, 0, 0\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

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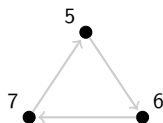
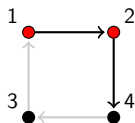
$$K = \{1, 2\}$$

$$Q = \{2\}$$

$$D = \{1, 1, 0, 0, 0, 0, 0\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\}$$

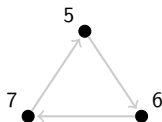
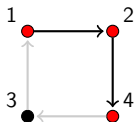
$$K = \{1, 2\}$$

$$Q = \{\}$$

$$D = \{1, 1, 0, 0, 0, 0, 0\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

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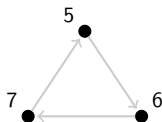
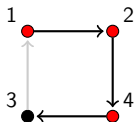
$$K = \{1, 2, 4\}$$

$$Q = \{4\}$$

$$D = \{1, 1, 0, 1, 0, 0, 0\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\}$$

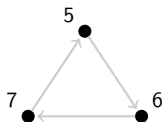
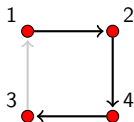
$$K = \{1, 2, 4\}$$

$$Q = \{\}$$

$$D = \{1, 1, 0, 1, 0, 0, 0\}$$

Detecting Subtours

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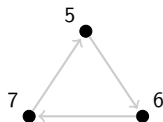
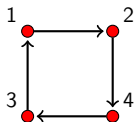
$$K = \{1, 2, 4, 3\}$$

$$Q = \{3\}$$

$$D = \{1, 1, 0, 1, 0, 0, 0\}$$

Detecting Subtours

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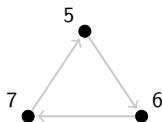
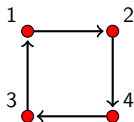
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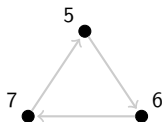
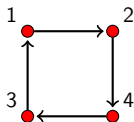
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$$N = \{1, \mathbf{2}, 3, 4, 5, 6, 7\}$$

$$C = \{\{1, 2, 4, 3\}\}$$

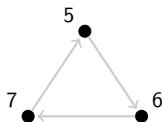
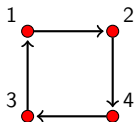
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$$N = \{1, 2, \mathbf{3}, 4, 5, 6, 7\}$$

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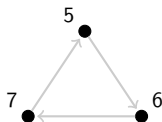
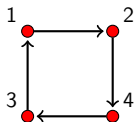
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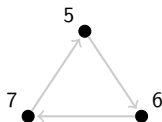
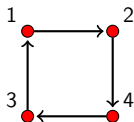
$$K = \{\}$$

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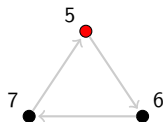
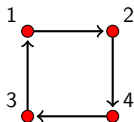
$$K = \{\}$$

$$Q = \{\}$$

$$D = \{1, 1, 1, 1, 0, 0, 0\}$$

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- Given a solution current feasible integer solution \bar{x}_{ij} .



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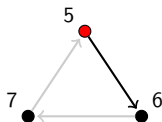
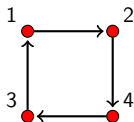
$$K = \{5\}$$

$$Q = \{5\}$$

$$D = \{1, 1, 1, 1, 1, 0, 0\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\{1, 2, 4, 3\}\}$$

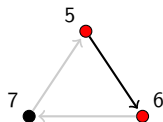
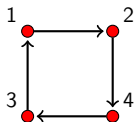
$$K = \{5\}$$

$$Q = \{\}$$

$$D = \{1, 1, 1, 1, 1, 0, 0\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\{1, 2, 4, 3\}\}$$

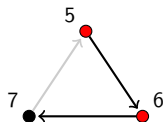
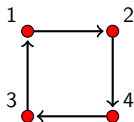
$$K = \{5, 6\}$$

$$Q = \{6\}$$

$$D = \{1, 1, 1, 1, 1, 1, 0\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\{1, 2, 4, 3\}\}$$

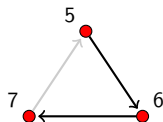
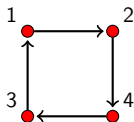
$$K = \{5, 6\}$$

$$Q = \{\}$$

$$D = \{1, 1, 1, 1, 1, 1, 0\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\{1, 2, 4, 3\}\}$$

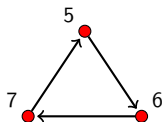
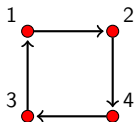
$$K = \{5, 6, 7\}$$

$$Q = \{7\}$$

$$D = \{1, 1, 1, 1, 1, 1, 1\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\{1, 2, 4, 3\}\}$$

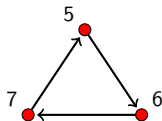
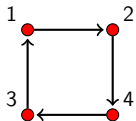
$$K = \{5, 6, 7\}$$

$$Q = \{\}$$

$$D = \{1, 1, 1, 1, 1, 1, 1\}$$

Detecting Subtours

- Given a solution current feasible integer solution \bar{x}_{ij} .



$$N = \{1, 2, 3, 4, 5, 6, 7\}$$

$$C = \{\{1, 2, 4, 3\}, \{5, 6, 7\}\}$$

$$K = \{\}$$

$$Q = \{\}$$

$$D = \{1, 1, 1, 1, 1, 1, 1\}$$

- This procedure allows us to identify all of the subtours at a current solution and add the cuts for each of them

Example

For the current solution



The corresponding subtour elimination cuts are

$$x_{12} + x_{21} + x_{13} + x_{31} + x_{14} + x_{41} + x_{23} + x_{32} + x_{24} + x_{42} + x_{34} + x_{43} \leq 3$$

$$x_{56} + x_{65} + x_{57} + x_{75} + x_{67} + x_{76} \leq 2$$

Visualization

Contents

1. Lazy Cuts
2. Traveling Salesman Problem
3. Lazy Cuts In Gurobi

Lazy Cuts

- We will implement our lazy cuts in a **callback function**
- Lazy cuts use different attributes than user cuts and occur at different points of the branch and bound tree
- The essential Gurobi attributes for user and lazy cuts:

Attribute	User Cuts	Lazy Cuts
parameters where current solution add constraint	<code>Model.Params.PreCrush</code> <code>GRB.Callback.MIPNODE</code> <code>Model.cbGetNodeRel()</code> <code>Model.cbCut()</code>	<code>Model.Params.lazyConstraints</code> <code>GRB.Callback.MIPSOL</code> <code>Model.cbGetSolution()</code> <code>Model.cbLazy()</code>

Future Work

- We will now go to the Jupyter notebook to solve the TSP
- In a supplemental workshop we will cover some other techniques used solve or approximate solutions of the TSP
- These techniques include:
 1. Lagrangian Relaxation
 2. Approximation Algorithms
 3. Heuristics