Machine Learning: Homework #1

Due on February 21, 2021

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Problem 1

[20 points] Suppose that $Y = \{0, 1\}, P(Y = 0) = P(Y = 1) = \frac{1}{2}$. X is a continuous random variable with probability density function given by

$$p_{X|Y}(x \mid y = 0) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

$$p_{X|Y}(x \mid y = 1) = \begin{cases} \frac{1}{2}, & x \in [a, a+2] \\ 0, & otherwise \end{cases}$$

where $a \geq 0$

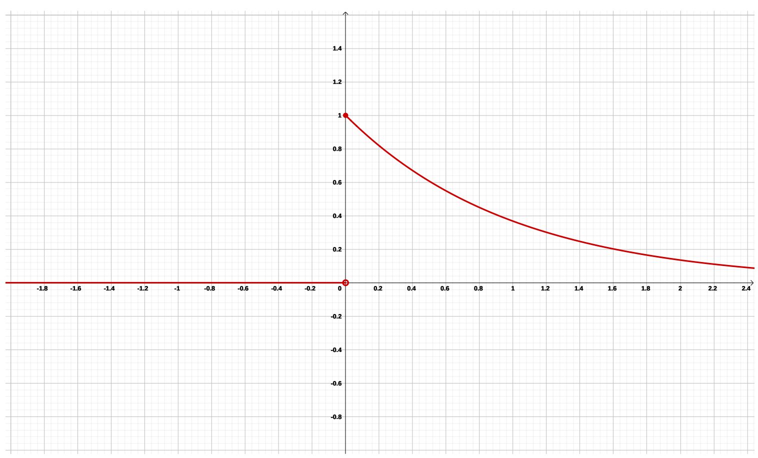
(a) Skecth $p_{X|Y}(x \mid y = 0)$ and $p_{X|Y}(x \mid y = 1)$

(b) If $X = \frac{1}{2}$, find the most likely value of Y by using Bayes' Theorem. What about $X = 1$?

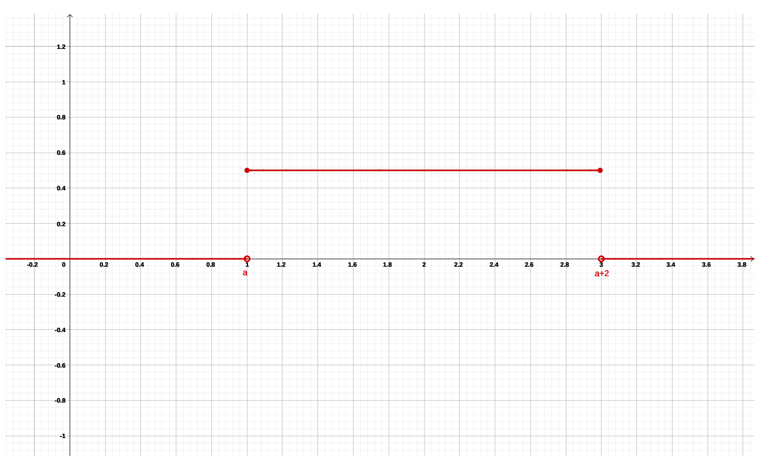
Solution

Subproblem (a)

The plots of $p_{X|Y}(x \mid y = 0)$ and $p_{X|Y}(x \mid y = 1)$ are showed in Figure 1.



(a) Probability density function 1



(b) Probability density function 2

Figure 1 The plot of probability density functions

Subproblem (b)

◦ When $X = \frac{1}{2}$, we can have

$$\begin{aligned} p\left(Y=0 \mid X=\frac{1}{2}\right) &= \frac{p\left(Y=0\right)\cdot p\left(X=\frac{1}{2}\mid Y=0\right)}{p\left(X=\frac{1}{2}\right)} = \frac{\frac{1}{2}\cdot e^{-\frac{1}{2}}}{p\left(X=\frac{1}{2}\right)} \\ p\left(Y=1 \mid X=\frac{1}{2}\right) &= \frac{p\left(Y=1\right)\cdot p\left(X=\frac{1}{2}\mid Y=1\right)}{p\left(X=\frac{1}{2}\right)} = \frac{\frac{1}{2}\cdot p\left(X=\frac{1}{2}\mid Y=1\right)}{p\left(X=\frac{1}{2}\right)} \\ &\le \frac{\frac{1}{2}\cdot\frac{1}{2}}{p\left(X=\frac{1}{2}\right)} < \frac{\frac{1}{2}\cdot e^{-\frac{1}{2}}}{p\left(X=\frac{1}{2}\right)} = p\left(Y=0 \mid X=\frac{1}{2}\right) \end{aligned} \tag{1}$$

As we can see from equation 1, the conditional probability $p\left(Y=0 \mid X=\frac{1}{2}\right)$ is larger than $p\left(Y=1 \mid X=\frac{1}{2}\right)$. Thus, the most likely value of Y is 0.

◦ When $X = 1$, we can have

$$\begin{aligned} p\left(Y=0 \mid X=1\right) &= \frac{p\left(Y=0\right)\cdot p\left(X=1\mid Y=0\right)}{p\left(X=1\right)} = \frac{\frac{1}{2}\cdot e^{-1}}{p\left(X=1\right)} \\ p\left(Y=1 \mid X=1\right) &= \frac{p\left(Y=1\right)\cdot p\left(X=1\mid Y=1\right)}{p\left(X=1\right)} = \frac{\frac{1}{2}\cdot p\left(X=1\mid Y=1\right)}{p\left(X=1\right)} \end{aligned} \tag{2}$$

According to equation 2, we can see that different value of a will result in different conclusion.

- $1 \in [a, a+2]$

  In this situation, we have

$$\begin{cases} a \ge 0 \\ a \le 1 \\ a+2 \ge 1 \end{cases} \Longrightarrow a \in [0, 1] \tag{3}$$

  Then, we can have

$$\begin{aligned} p\left(Y=1 \mid X=1\right) &= \frac{\frac{1}{2}\cdot p\left(X=1\mid Y=1\right)}{p\left(X=1\right)} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{p\left(X=1\right)} \\ &> \frac{\frac{1}{2}\cdot e^{-1}}{p\left(X=1\right)} = p\left(Y=0 \mid X=1\right) \end{aligned} \tag{4}$$

  As we can see from equation 3 and 4, when $a \in [0, 1]$, the conditional probability $p\left(Y=0 \mid X=1\right)$ is less than $p\left(Y=1 \mid X=1\right)$. Thus, the most likely value of Y is 1.

- $1 \notin [a, a+2]$

  In this situation, we have

$$\begin{cases} a \ge 0 \\ a > 1 \text{ or } a+2 < 1 \end{cases} \Longrightarrow a \in (1, +\infty) \tag{5}$$

  Then, we can have

$$\begin{aligned} p\left(Y=1 \mid X=1\right) &= \frac{\frac{1}{2}\cdot p\left(X=1\mid Y=1\right)}{p\left(X=1\right)} = \frac{\frac{1}{2}\cdot 0}{p\left(X=1\right)} = 0 \\ &< \frac{\frac{1}{2}\cdot e^{-1}}{p\left(X=1\right)} = p\left(Y=0 \mid X=1\right) \end{aligned} \tag{6}$$

  As we can see from equation 5 and 6, when $a \in (1, +\infty)$, the conditional probability $p\left(Y=0 \mid X=1\right)$ is larger than $p\left(Y=1 \mid X=1\right)$. Thus, the most likely value of Y is 0.

Problem 2

[10 points] Markov's inequality is the most elementary tail bound which means that if a non-negative random variable X has finite mean, then we have

$$\Pr[X \geq t] \leq \frac{E[X]}{t} \quad \forall t > 0$$

For a random variable X with finite variance, then show that it satifies the Chebyshev's inequality

$$\Pr[|X - \mu|] \geq t] \leq \frac{\mathrm{Var}(X)}{t^2} \quad \forall t > 0$$

Solution

We set $D = \{x : |x - \mu| \geq t\}$, thus we have

$$\frac{|x - \mu|}{t} \geq 1, \quad x \in D \tag{7}$$

Then, we set $f(x)$ as the probability density function of X, so we have

$$\begin{aligned}
\Pr\left[|X - \mu| \geq t\right] &= \int_D f(x)dx = \int_D 1 \cdot f(x)dx \\
&\leq \int_D \frac{|x - \mu|}{t} \cdot f(x)dx \quad \text{(euqation 7)} \\
&\leq \int_D \left(\frac{|x - \mu|}{t}\right)^2 \cdot f(x)dx \quad \text{(euqation 7)} \\
&= \frac{1}{t^2} \int_D (x - \mu)^2 \cdot f(x)dx \\
&\leq \frac{1}{t^2} \int_{-\infty}^{+\infty} (x - \mu)^2 \cdot f(x)dx \\
&= \frac{E\left[(X - \mu)^2\right]}{t^2} = \frac{\mathrm{Var}\,(X)}{t^2}
\end{aligned} \tag{8}$$

Problem 3

[20 points] Let $X \in R^{m \times n}$ be a matrix of full column rank. Show that

$$\min_{\theta \in R^n} \|y - X\theta\|_2^2 = \left\|P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} y\right\|_2^2,$$

where $b_1 = x_1, b_2 = P_{b_1}^{\perp} x_2, b_3 = P_{b_2}^{\perp} P_{b_1}^{\perp} x_3, \cdots, b_n = P_{b_n-1}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} x_n$.(Hint: $P_{b_i}^{\perp}$ is the projection of orthogonal complementary space of b_i.)

Solution

Because X is a matrix of full column rank, we have that $\{x_i\}, i = 1, 2, \ldots, n$ are linearly independent and the solution of $\mathrm{argmin}_{\theta \in R^n} \|y - X\theta\|_2^2$ is $\hat{\theta} = \left(X^T X\right)^{-1} X^T y$. Then we can have:

$$\min_{\theta \in R^n} \|y - X\theta\|_2^2 = \|y - X\left(X^T X\right)^{-1} X^T y\|_2^2 = \|\left(I - P\right) y\|_2^2 \tag{9}$$

where $P = X\left(X^T X\right)^{-1} X^T$ is the projection matrix onto the range space $\mathcal{R}(X)$, thus, $I - P$ is the projection matrix onto the orthogonal complement space of $\mathcal{R}(X)$.

Problem 3 continued on next page...

• Then, we would like to prove that $P_{b_k}^{\perp} b_k = 0$. We know that the vector b_k can be decomposed uniquely onto the space ${b_k}^{\perp}$ ans space $R^m - {b_k}^{\perp} + \{0\}$. So we have

$$\begin{aligned} b_k &= P_{b_k}^{\perp} b_k + P_{R^m - {b_k}^{\perp} + \{0\}} b_k \\ &= P_{b_k}^{\perp} b_k + b_k \end{aligned} \Longrightarrow P_{b_k}^{\perp} b_k = 0 \tag{10}$$

Thus, we can have (suppose $i > j$)

$$\begin{aligned} b_i \cdot b_j &= \left\langle P_{b_{i-1}}^{\perp} \dots P_{b_2}^{\perp} P_{b_1}^{\perp} x_i, b_j \right\rangle \\ &= \left\langle P_{b_{i-1}}^{\perp} \dots P_{b_{j-1}}^{\perp} P_{b_{j+1}}^{\perp} \dots P_{b_2}^{\perp} P_{b_1}^{\perp} x_i, P_{b_j}^{\perp} b_j \right\rangle \\ &= \left\langle P_{b_{i-1}}^{\perp} \dots P_{b_{j-1}}^{\perp} P_{b_{j+1}}^{\perp} \dots P_{b_2}^{\perp} P_{b_1}^{\perp} x_i, 0 \right\rangle \\ &= 0 \quad (\forall i \neq j) \end{aligned} \tag{11}$$

since $P_{b_k}^{\perp}$ is self-adjoint and $P_{b_k}^{\perp} x_k = 0$. Thus, $\{b_i\}, i = 1, 2, \dots, n$ are orthogonal to each other, so they are independent to each other. Then, we can have $W = \operatorname{span}\{b_1, b_2, \dots, b_n\} = \operatorname{span}\{x_1, x_2, \dots x_n\} = \mathcal{R}(X)$.

• Then, we would like to prove that $P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp}$ is the projection matrix onto the orthogonal complement space of W. Choose a vector $\boldsymbol{u} \in \mathbb{R}^m$, it is equal to prove that $\boldsymbol{v} = P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} \boldsymbol{u} \in W^{\perp}$. Then it is euqal to prove that $\boldsymbol{v}$ is orthogonal to any vector in W. We denote the vector $\boldsymbol{w_i} \in W$ as follow

$$\boldsymbol{w_i} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \tag{12}$$

Thus, we have

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{w_i} &= \left\langle P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} \boldsymbol{u}, a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \right\rangle \\ &= \left\langle P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} \boldsymbol{u}, a_1 b_1 \right\rangle + \left\langle P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} \boldsymbol{u}, a_2 b_2 \right\rangle + \cdots + \left\langle P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} \boldsymbol{u}, a_n b_n \right\rangle \\ &= \left\langle P_{b_n}^{\perp} \cdots P_{b_3}^{\perp} P_{b_2}^{\perp} \boldsymbol{u}, P_{b_1}^{\perp} a_1 b_1 \right\rangle + \left\langle P_{b_n}^{\perp} \cdots P_{b_3}^{\perp} P_{b_1}^{\perp} \boldsymbol{u}, P_{b_2}^{\perp} a_2 b_2 \right\rangle + \cdots + \left\langle P_{b_{n-1}}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} \boldsymbol{u}, P_{b_n}^{\perp} a_n b_n \right\rangle \\ &= 0 \end{aligned} \tag{13}$$

Thus, $P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp}$ is the projection matrix onto the orthogonal complement space of W, as well as the orthogonal complement space of $\mathcal{R}(X)$. Due to the unique of projection matrix, we have $P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} = I - P$, which shows that

$$\min_{\theta \in R^n} \|y - X\theta\|_2^2 = \|(I - P) y\|_2^2 = \left\| P_{b_n}^{\perp} \cdots P_{b_2}^{\perp} P_{b_1}^{\perp} y \right\|_2^2 \tag{14}$$

Problem 4

[50points] MLE for robust regression. Suppose we have the generative linear regression model

$$Y = X\theta^{\star} + \varepsilon$$

where ε is the error term and $\varepsilon \sim N(0, \Sigma)$. The maximum likelihood estimator for θ is:

$$\begin{aligned} \hat{\theta}_{LS} &= \operatorname{argmin}_{\theta \in R^d} \|X\theta - y\|_2^2 \\ &= \left(X^T X\right)^{-1} X^T y \end{aligned}$$

(a) Suppose the error term, $\varepsilon = [\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]$ follows the Laplace distribution, i.e. $\varepsilon_i \overset{\text{i.i.d}}{\sim} L(0, b), i = 1, 2, \cdots, n$ and the probability density function is $P(\varepsilon_i) = \frac{1}{2b} e^{-\frac{|\varepsilon_i - 0|}{b}}$ for some $b > 0$ Under the MLE principle, what is the learning problem? Please write out the derivation process. (15 points)

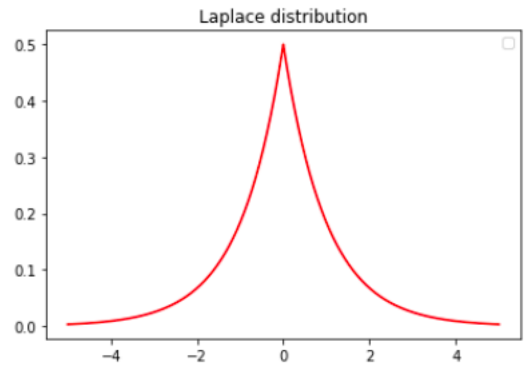


Figure 2 PDF of Laplace distribution

(b) **Huber-smoothing**. $L1-$ norm minimization

$$\hat{\theta}_{L1} = \underset{\theta}{\operatorname{argmin}} \|X\theta - y\|_1$$

is one possible solution for robust regression. However, it is nondifferentiable. We utilize smoothing technique for approximately solving the $L1-$ norm minimization. Huber function is one possibility. The definition and sketch map are shown as below.

$$h_\mu(z) \begin{cases} |z|, & |z| \geq \mu \\ \frac{z^2}{2\mu} + \frac{\mu}{2}, & |z| \leq \mu \end{cases}$$

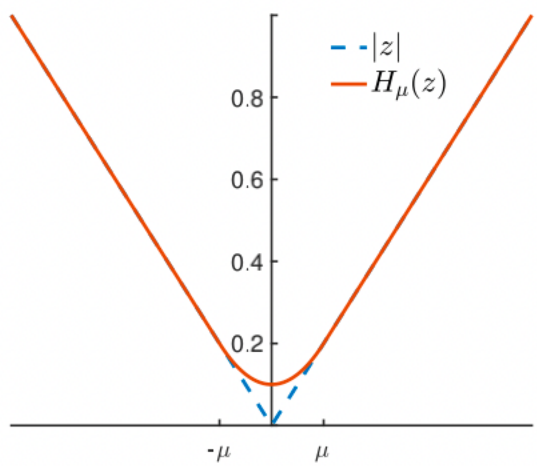


Figure 3 Huber smoothing

Then,

$$H_\mu(Z) = \sum_{j=1}^{n} h_\mu(z_j)$$

By using Huber smoothing, the approximation of the optimization of $L1-$ norm can be changed to

$$\min_\theta H_\mu(X\theta - y)$$

Let

$$f(\theta) = H_\mu(X\theta - y)$$

find the gradient $\nabla f(\theta)$. (10 points)

(c) Gradient descent for minimizing $f(\theta)$. The process of gradient descent algorithm is shown in the following table.

Algorithm 1: The Process of Gradient Descent Algorithm

```
Input: observed data X, y and initialization parameter θ_0,
Huber smoothing parameter μ,
total iteration number T,
learning rate α.
for k = 0, 1, 2, ···, T do
    θ_{k+1} = θ_k − α∇f(θ_k)
end for
return θ_T
```

The data set is generated by the linear model

$$Y = X\theta^\star + \varepsilon_1 + \varepsilon_2$$

where $\varepsilon_1 \in R^n$ follows Gaussian distribution, ε_2 are outliers. Given the observed data $(x, y) = \{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$ and true value $\theta^\star$,

(1) calculate the estimation $\hat{\theta}_{LS}$ by using linear least squares and compute $\left\|\hat{\theta}_{LS} - \theta^\star\right\|_2$. (5 points)

(2) suppose $n = 1000, d = 50$, use python to implement the gradient descent algorithm to minimize $f(\theta)$, the parameters are set as $\mu = 10^{-5}, \alpha = 0.001, T = 1000$, plot the error $\|\theta_k - \theta^\star\|_2$ as a function of iteraction number. You can download the data $\{Y, X, \theta^\star\}$ from Blackboard. (20 points)

Solution

Subproblem (a)

Our model is

$$y_i = \boldsymbol{x}_i^\top \boldsymbol{\theta} \tag{15}$$

To be more explicit, consider

$$y_i = \boldsymbol{x}_i^\top \boldsymbol{\theta} + \epsilon_i \quad \text{with} \quad \epsilon_i \sim L(0, b) \tag{16}$$

where ϵ_i are i.i.d. for $i = 1, \cdots, n$.
Equivalently,

$$\epsilon_i = y_i - \boldsymbol{x}_i^\top \boldsymbol{\theta} \sim L(0, b) \tag{17}$$

Thus, we can get the likelihood function as follow

$$\begin{aligned} L(\boldsymbol{\theta}) &= \prod_{i=1}^{n} P(\epsilon_i) = \prod_{i=1}^{n} \frac{1}{2b} e^{-\frac{|\epsilon_i|}{b}} \\ &= (2b)^{-n} e^{-\frac{\sum_{i=1}^{n} |\epsilon_i|}{b}} \end{aligned} \tag{18}$$

Then, we can get the log-likelihood function as follow

$$\begin{aligned}\log L(\boldsymbol{\theta}) &= -n\log(2b) - \frac{1}{b}\sum_{i=1}^{n}|\epsilon_i| \\ &= \text{Constant} - \frac{1}{b}\sum_{i=1}^{n}|\epsilon_i|\end{aligned} \tag{19}$$

In order to make log-likelihood function to reach maximum value, we can derive the learning problem as follow

$$\begin{aligned}\widehat{\boldsymbol{\theta}}_{MLE} &= \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\sum_{i=1}^{n}|\epsilon_i| \\ &= \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\sum_{i=1}^{n}\left|y_i - x_i^\top\boldsymbol{\theta}\right| \\ &= \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\theta}\|_1\end{aligned} \tag{20}$$

Subproblem (b)

We can derive the gradient of $f(\boldsymbol{\theta})$ as follow

$$\begin{aligned}\nabla f(\boldsymbol{\theta}) &= \begin{pmatrix}\frac{\partial f(\boldsymbol{\theta})}{\partial\theta_1} \\ \frac{\partial f(\boldsymbol{\theta})}{\partial\theta_2} \\ \vdots \\ \frac{\partial f(\boldsymbol{\theta})}{\partial\theta_d}\end{pmatrix} = \begin{pmatrix}\frac{\partial H_\mu(\boldsymbol{X}\boldsymbol{\theta}-\boldsymbol{y})}{\partial\theta_1} \\ \frac{\partial H_\mu(\boldsymbol{X}\boldsymbol{\theta}-\boldsymbol{y})}{\partial\theta_2} \\ \vdots \\ \frac{\partial H_\mu(\boldsymbol{X}\boldsymbol{\theta}-\boldsymbol{y})}{\partial\theta_d}\end{pmatrix} = \begin{pmatrix}\frac{\partial\sum_{i=1}^n h_u\left(\boldsymbol{x}_i^\top\boldsymbol{\theta}-y_i\right)}{\partial\theta_1} \\ \frac{\partial\sum_{i=1}^n h_u\left(\boldsymbol{x}_i^\top\boldsymbol{\theta}-y_i\right)}{\partial\theta_2} \\ \vdots \\ \frac{\partial\sum_{i=1}^n h_u\left(\boldsymbol{x}_i^\top\boldsymbol{\theta}-y_i\right)}{\partial\theta_d}\end{pmatrix} \\ &= \begin{pmatrix}\sum_{i=1}^n \frac{\boldsymbol{x}_i^\top\boldsymbol{\theta}-y_i}{\max\left\{\left|\boldsymbol{x}_i^\top\boldsymbol{\theta}-y_i\right|,\mu\right\}}x_{i1} \\ \sum_{i=1}^n \frac{\boldsymbol{x}_i^\top\boldsymbol{\theta}-y_i}{\max\left\{\left|\boldsymbol{x}_i^\top\boldsymbol{\theta}-y_i\right|,\mu\right\}}x_{i2} \\ \vdots \\ \sum_{i=1}^n \frac{\boldsymbol{x}_i^\top\boldsymbol{\theta}-y_i}{\max\left\{\left|\boldsymbol{x}_i^\top\boldsymbol{\theta}-y_i\right|,\mu\right\}}x_{id}\end{pmatrix} \\ &= \boldsymbol{X}^\top\cdot\begin{pmatrix}\frac{\boldsymbol{x}_1^\top\boldsymbol{\theta}-y_1}{\max\left\{\left|\boldsymbol{x}_1^\top\boldsymbol{\theta}-y_1\right|,\mu\right\}} \\ \frac{\boldsymbol{x}_2^\top\boldsymbol{\theta}-y_2}{\max\left\{\left|\boldsymbol{x}_2^\top\boldsymbol{\theta}-y_2\right|,\mu\right\}} \\ \vdots \\ \frac{\boldsymbol{x}_n^\top\boldsymbol{\theta}-y_n}{\max\{|\boldsymbol{x}_n^\top\boldsymbol{\theta}-y_n|,\mu\}}\end{pmatrix}\end{aligned} \tag{21}$$

Subproblem (c)

(1) Since $n > d$, we suppose $\boldsymbol{X} \in \mathbb{R}^{n\times d}$ has full column rank. Thus, we can derive that $\boldsymbol{X}^\top\boldsymbol{X} \in \mathbb{R}^{d\times d}$ is invertible. Then, we can calculate $\hat{\boldsymbol{\theta}}_{LS}$ as follow

$$\hat{\boldsymbol{\theta}}_{LS} = \left(\boldsymbol{X}^\top\boldsymbol{X}\right)^{-1}\boldsymbol{X}^\top\boldsymbol{y} \tag{22}$$

By using Python, we can compute $\left\|\hat{\boldsymbol{\theta}}_{LS} - \boldsymbol{\theta}^\star\right\|_2$, the result is as follow

$$\left\|\hat{\boldsymbol{\theta}}_{LS} - \boldsymbol{\theta}^\star\right\|_2 = 144.695 \tag{23}$$

The Python code to solve this problem is showed as follow

```
import pandas as pd
import  numpy as np
X = pd.read_csv("Sample data of X.csv", header=0, index_col=0)
y = pd.read_csv("Sample data of y.csv", header=None)
theta_star = pd.read_csv("data of theta_star.csv", header=None)


X = np.array(X)
y = np.array(y)
theta_star = np.array(theta_star)

theta_LS = np.dot(np.dot(np.linalg.inv(np.dot(X.T, X)), X.T), y)
error = np.linalg.norm(theta_star - theta_LS, ord = 2)
print(error)
```

(2) By using Python, we can plot the error $\|\theta_k - \theta^\star\|_2$ as a function of iteraction number as Figure 4

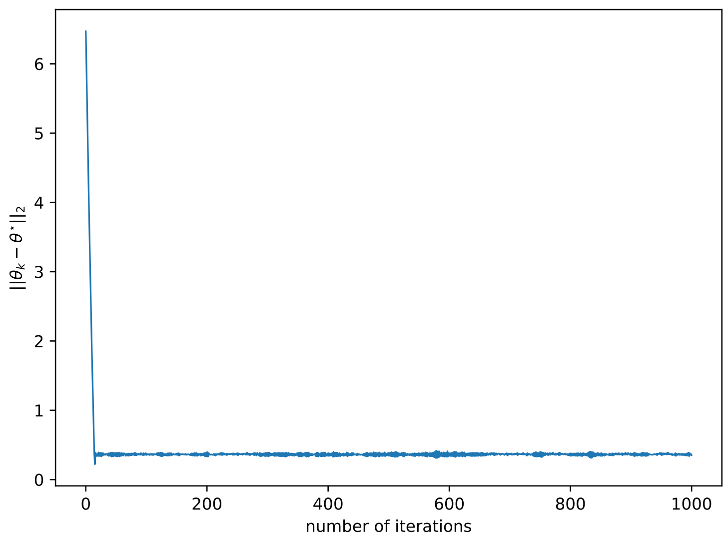


Figure 4 Error $\|\boldsymbol{\theta}_k - \boldsymbol{\theta}^\star\|_2$ vs. iteraction number

The Python code to solve this problem is showed as follow

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

def hu(z):
    if abs(z) >= mu:
        return abs(z)
    else:
        return z*z/(2*mu) + mu/2
def f(theta):
    return sum(map(hu, np.dot(X, theta) - y))[0]

def df(theta):
    t = np.dot(X, theta) - y
    t1 = np.dot(X, theta) - y
    t[abs(t)>=mu] = abs(t)[abs(t)>=mu]
    t[abs(t)<=mu] = mu
    return np.dot(X.T, t1/t)
```

```
def GM(theta):
    thetak = theta
    norm_list = []
    norm_list.append(np.linalg.norm(thetak-theta_star))
    for k in np.arange(T):
        thetak = thetak - alpha * df(thetak)
        norm_list.append(np.linalg.norm(thetak-theta_star))
    norm_list = np.array(norm_list)
    plot_convergence(norm_list)
    return thetak

def plot_convergence(norm_list):
    number = norm_list.size
    x = np.arange(number)
    #y = np.log(norm_list)
    y = norm_list
    plt.plot(x, y, linewidth=1)
    plt.xlabel('number of iterations')
    plt.ylabel(r'$||\theta_k-\theta^{\star}||_2$')
    plt.tight_layout()
    plt.savefig('./plt.pdf', dpi=1000)

X = pd.read_csv("Sample data of X.csv", header=0, index_col=0)
y = pd.read_csv("Sample data of y.csv", header=None)
theta_star = pd.read_csv("data of theta_star.csv", header=None)

X = np.array(X)
y = np.array(y)
theta_star = np.array(theta_star)

n = X.shape[0]
d = X.shape[1]
mu = 1e-5
alpha = 1e-3
T = 1000
theta0 = np.zeros(d).reshape(d,1)
thetaT = GM(theta0)
```