Time Series Analysis: Homework #1

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Problem 1

Let $\{e_t\}$ be an independent white noise process. Suppose that the observed process is $Y_t = e_t + \theta e_{t-1}$ where θ is either 5 or 1/5. Find the autocorrelation function for $\{Y_t\}$ both when $\theta = 5$ and when $\theta = 1/5$

Solution

Because $\{e_t\}$ is an independent white noise process, we suppose $e_t \sim \text{IWN}\big(0, \sigma^2\big)$. Then we can derive the autocovariance function and autocorrelation function for $\{Y_t\}$ as follow.

$$\begin{aligned}
\gamma_k &= \text{Cov}\left(Y_t, Y_{t+k}\right) \\
&= \text{Cov}\left(e_t + \theta e_{t-1}, e_{t+k} + \theta e_{t+k-1}\right) \\
\rho_k &= \frac{\gamma_k}{\sqrt{\text{Var}\left(Y_t\right)}\sqrt{\text{Var}\left(Y_{t+k}\right)}} \\
&= \frac{\gamma_k}{\sqrt{\text{Var}\left(e_t + \theta e_{t-1}\right)}\sqrt{\text{Var}\left(e_{t+k} + \theta e_{t+k-1}\right)}}
\end{aligned} \tag{1}$$

Then, we can have

$$\begin{aligned}
\gamma_k &= \begin{cases} \left(1+\theta^2\right)\sigma^2 & (k=0) \\ \theta\sigma^2 & (k=1) \\ 0 & (k>1) \end{cases} \\
\rho_k &= \begin{cases} 1 & (k=0) \\ \frac{\theta}{1+\theta^2} & (k=1) \\ 0 & (k>1) \end{cases}
\end{aligned} \tag{2}$$

Thus, when $\theta = 5$ or $\theta = 1/5$, their acf are the same. We can have

$$\rho_k = \begin{cases} 1 & (k=0) \\ \frac{5}{26} & (k=1) \\ 0 & (k>1) \end{cases} \tag{3}$$

Problem 2

Suppose $Y_t = 4 + 3t + X_t$ where $\{X_t\}$ is a zero mean stationary series with autocovariance function γ_k.

(a) Find the mean function of $\{Y_t\}$.

(b) Find the autocovariance function for $\{Y_t\}$.

(c) Is $\{Y_t\}$ stationary?(Why or why not?)

Solution

Subproblem(a)

We can find the mean of (Y_t) as follow

$$\begin{aligned}
\text{E}\left(Y_t\right) &= 4 + 3t + \text{E}\left(X_t\right) \\
&= 4 + 3t
\end{aligned} \tag{4}$$

Subproblem(b)

We can find the autocovariance function of (Y_t) as follow. We denote the autocovariance function of (Y_t) as Γ_k.

$$\begin{aligned}\Gamma_k &= \text{Cov}\left(Y_t, Y_{t+k}\right)\\ &= \text{Cov}\left(4+3t+X_t, 4+3\left(t+k\right)+X_{t+k}\right)\\ &= \text{Cov}\left(X_t, X_{t+k}\right)\\ &= \gamma_k\end{aligned} \tag{5}$$

Subproblem(c)
$\{Y_t\}$ is not stationary, because the mean of $\{Y_t\}$ is $4+3t$, which depends on t.

Problem 3

Suppose that $\{Y_t\}$ is stationary with autocovariance function γ_k.

(a) Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary by finding the mean and autocovariance function for $\{W_t\}$.

(b) Show that $U_t = \nabla\nabla Y_t = \nabla\left[Y_t - Y_{t-1}\right] = Y_t - 2Y_{t-1} + Y_{t-2}$ is stationary.

Solution

Subproblem(a)
The mean of $\{W_t\}$ is as follow

$$\begin{aligned}\text{E}\left(W_t\right) &= \text{E}\left(Y_t - Y_{t-1}\right)\\ &= \text{E}\left(Y_t\right) - \text{E}\left(Y_{t-1}\right)\\ &= 0\end{aligned} \tag{6}$$

The autocovariance of $\{W_t\}$ is as follow, we deonte the autocovariance of $\{W_t\}$ as Γ_k

$$\begin{aligned}\Gamma_k &= \text{Cov}\left(W_t, W_{t+k}\right)\\ &= \text{Cov}\left(Y_t - Y_{t-1}, Y_{t+k} - Y_{t+k-1}\right)\\ &= \begin{cases} 2\gamma_0 - 2\gamma_1 & k = 0\\ -\gamma_{k-1} + 2\gamma_k - \gamma_{k+1} & k \geq 1\end{cases}\end{aligned} \tag{7}$$

According to equation 6 and equation 7, we can find that the mean of $\{W_t\}$ is 0 which does not depend on t. Besides, the covariance of $\{W_t\}$, say Γ_k exists, is finite and depends only on k but not on t. Thus, we can find $\{W_t\}$ is stationary.

Subproblem(b)
The mean of $\{U_t\}$ is as follow

$$\begin{aligned}\text{E}\left(U_t\right) &= \text{E}\left(Y_t - 2Y_{t-1} + Y_{t-2}\right)\\ &= \text{E}\left(Y_t\right) - 2\cdot\text{E}\left(Y_{t-1}\right) + \text{E}\left(Y_{t-2}\right)\\ &= 0\end{aligned} \tag{8}$$

The autocovariance of $\{U_t\}$ is as follow, we deonte the autocovariance of $\{U_t\}$ as Γ_k

$$\begin{aligned}\Gamma_k &= \text{Cov}\left(U_t, U_{t+k}\right)\\ &= \text{Cov}\left(Y_t - 2Y_{t-1} + Y_{t-2}, Y_{t+k} - 2Y_{t+k-1} + Y_{t+k-2}\right)\\ &= \begin{cases} 6\gamma_0 - 8\gamma_1 + 2\gamma_2 & k = 0\\ -4\gamma_0 + 7\gamma_1 - 4\gamma_2 + \gamma_3 & k = 1\\ \gamma_{k-2} - 4\gamma_{k-1} + 6\gamma_k - 4\gamma_{k+1} + \gamma_{k+2} & k \geq 2\end{cases}\end{aligned} \tag{9}$$

Problem 3 continued on next page. . .

According to equation 8 and equation 9, we can find that the mean of $\{U_t\}$ is 0 which does not depend on t. Besides, the covariance of $\{U_t\}$, say Γ_k exists, is finite and depends only on k but not on t. Thus, we can find $\{U_t\}$ is stationary.

Problem 4

Let $\{Y_t\}$ be an AR(2) process of the special form $Y_t = \varphi_2 Y_{t-2} + e_t$. Find the range of values of φ_2 for which the process is stationary.

Solution

We can rewrite the form of Y_t as follow

$$\begin{aligned} & Y_t = 0 \cdot Y_{t-1} + \varphi_2 Y_{t-2} + e_t \\ \Longrightarrow \quad & \theta_2(\mathbf{B}) Y_t = \left(1 - \varphi_2 \mathbf{B}^2\right) Y_t = e_t \end{aligned} \tag{10}$$

Then we can derive the characteristic equation as follow

$$\theta_2(\mathbf{B}) = 1 - \varphi_2 \mathbf{B}^2 = 0 \tag{11}$$

- $\varphi_2 = 0$

  In this situation, $Y_t = e_t$, which is not an AR(2) process, so $\varphi_2 = 0$ is not satisfiable.

- $\varphi_2 > 0$

  In this situation, by solving the characteristic equation, we have

$$\begin{aligned} & \mathbf{B} = \sqrt{\frac{1}{\varphi_2}} \\ \Longrightarrow \quad & |\mathbf{B}| = \sqrt{\frac{1}{\varphi_2}} > 1 \\ \Longrightarrow \quad & \varphi_2 < 1 \end{aligned} \tag{12}$$

  Thus, in this situation, we have $\varphi_2 \in (0, 1)$

- $\varphi_2 < 0$

  In this situation, by solving the characteristic equation, we have

$$\begin{aligned} & \mathbf{B} = \sqrt{\frac{1}{-\varphi_2}} i \\ \Longrightarrow \quad & |\mathbf{B}| = \sqrt{\frac{1}{-\varphi_2}} > 1 \\ \Longrightarrow \quad & \varphi_2 > -1 \end{aligned} \tag{13}$$

  Thus, in this situation, we have $\varphi_2 \in (-1, 0)$

Above all, we find the range of φ_2 is $(-1, 0) \cup (0, 1)$.

Problem 5

Suppose that $Y_t = A + Bt + X_t$ where $\{X_t\}$ is a random walk. Suppose that A and B are constants.

(a) Is $\{Y_t\}$ stationary?

(b) Is $\{\nabla Y_t\}$ stationary?

Problem 5 continued on next page. . .

Solution

Subproblem(a)

Because $\{X_t\}$ is a random walk, so $X_t = X_{t-1} + w_t = w_1 + w_2 + \cdots + w_t$, where w_t is a white noise series and we assume $w_t \sim \text{WN}\left(0, \sigma^2\right)$. Then we can derive the mean of $\{Y_t\}$ as follow

$$\begin{aligned}\mathrm{E}\left(Y_t\right) &= \mathrm{E}\left(A + Bt + X_t\right) \\ &= \mathrm{E}\left(A + Bt + w_1 + w_2 + \cdots + w_t\right) \\ &= \mathrm{E}\left(A + Bt\right) + \mathrm{E}\left(w_1 + w_2 + \cdots + w_t\right) \\ &= A + Bt\end{aligned} \tag{14}$$

Thus, we can derive that $\{Y_t\}$ is not stationary, because the mean of $\{Y_t\}$ is $A + Bt$, which depends on t.

Subproblem(b)

We can rewrite the form of $\{\nabla Y_t\}$ as follow

$$\begin{aligned}\nabla Y_t &= Y_t - Y_{t-1} \\ &= A + Bt + X_t - \left(A + B\left(t-1\right) + X_{t-1}\right) \\ &= B + X_t - X_{t-1} \\ &= B + w_t\end{aligned} \tag{15}$$

Then, we can derive the mean of $\{\nabla Y_t\}$ as follow

$$\begin{aligned}\mathrm{E}\left(\nabla Y_t\right) &= \mathrm{E}\left(B + w_t\right) \\ &= B\end{aligned} \tag{16}$$

The variance of $\{\nabla Y_t\}$ is as follow

$$\begin{aligned}\mathrm{Var}\left(\nabla Y_t\right) &= \mathrm{Var}\left(B + w_t\right) \\ &= \mathrm{Var}\left(w_t\right) \\ &= \sigma^2\end{aligned} \tag{17}$$

The autocovariance of $\{\nabla Y_t\}$ is as follow (for $k > 0$).

$$\begin{aligned}\mathrm{Cov}\left(\nabla Y_t, \nabla Y_{t-k}\right) &= \mathrm{Cov}\left(B + w_t, B + w_{t-k}\right) \\ &= \mathrm{Cov}\left(w_t, w_{t-k}\right) \\ &= 0\end{aligned} \tag{18}$$

Above all, we can derive that $\{\nabla Y_t\}$ is stationary.

Problem 6

For a random walk with random starting value, let $Y_t = Y_0 + e_t + e_{t-1} + \cdots + e_1$ for $t > 0$, where Y_0 has a distribution with mean μ_0 and variance σ_0^2. Suppose futher that $Y_0, e_1, \cdots, e_t$ are independent, $e_t \sim IWN\left(0, \sigma_e^2\right)$

(a) Show that $E\left(Y_t\right) = \mu_0$ for all t.

(b) Show that $\mathrm{Var}\left(Y_t\right) = t\sigma_e^2 + \sigma_0^2$.

(c) Show that $\mathrm{Cov}\left(Y_t, Y_s\right) = \min(t, s)\sigma_e^2 + \sigma_0^2$

(d) Show that $\mathrm{Corr}\left(Y_t, Y_s\right) = \sqrt{\frac{t\sigma_e^2 + \sigma_0^2}{s\sigma_e^2 + \sigma_0^2}}$, for $0 \leq t \leq s$.

Problem 6 continued on next page...

Solution

Subproblem(a)

We can derive $\mathrm{E}\left(Y_t\right)$ as follow

- For $t=0$

$$\mathrm{E}\left(Y_t\right)=\mathrm{E}\left(Y_0\right)=\mu_0 \tag{19}$$

- For $t>0$

$$\begin{aligned}\mathrm{E}\left(Y_t\right)&=\mathrm{E}\left(Y_0+e_t+e_{t-1}+\cdots+e_1\right)\\&=\mathrm{E}\left(Y_0\right)+\mathrm{E}\left(e_t\right)+\mathrm{E}\left(e_{t-1}\right)+\cdots+\mathrm{E}\left(e_1\right)\\&=\mu_0\end{aligned} \tag{20}$$

Thus, we have showed that $\mathrm{E}\left(Y_t\right)=\mu_0$ for all t.

Subproblem(b)

- For $t=0$

$$\begin{aligned}\mathrm{Var}\left(Y_t\right)&=\mathrm{Var}\left(Y_0\right)=\sigma_0^2\\&=0\cdot\sigma_e^2+\sigma_0^2=t\sigma_e^2+\sigma_0^2\end{aligned} \tag{21}$$

- For $t>0$

$$\begin{aligned}\mathrm{Var}\left(Y_t\right)&=\mathrm{Var}\left(Y_0+e_t+e_{t-1}+\cdots+e_1\right)\\&=\mathrm{Var}\left(Y_0\right)+\mathrm{Var}\left(e_t\right)+\mathrm{Var}\left(e_{t-1}\right)+\cdots+\mathrm{Var}\left(e_1\right)\\&=\sigma_0^2+t\sigma_e^2\end{aligned} \tag{22}$$

Thus, we have showed that $\mathrm{Var}\left(Y_t\right)=t\sigma_e^2+\sigma_0^2$ for all t.

Subproblem(c)

Suppose $t\le s$

$$\begin{aligned}\mathrm{Cov}\left(Y_t,Y_s\right)&=\mathrm{Cov}\left(Y_0+e_t+e_{t-1}+\cdots+e_1,Y_0+e_s+e_{s-1}+\cdots+e_1\right)\\&=\mathrm{Cov}\left(Y_0+e_t+e_{t-1}+\cdots+e_1,Y_0+e_s+e_{s-1}+\cdots+e_t+\cdots+e_1\right)\\&=\mathrm{Cov}\left(Y_0\right)+\sum_{i=1}^{t}\sigma_e^2\\&=\sigma_0^2+t\sigma_e^2=\min\left(s,t\right)\sigma_e^2+\sigma_0^2\end{aligned} \tag{23}$$

Subproblem(d)

$$\begin{aligned}\mathrm{Corr}\left(Y_t,Y_s\right)&=\frac{\mathrm{Cov}\left(Y_t,Y_s\right)}{\sqrt{\mathrm{Var}\left(Y_t\right)}\cdot\sqrt{\mathrm{Var}\left(Y_s\right)}}\\&=\frac{t\sigma_e^2+\sigma_0^2}{\sqrt{t\sigma_e^2+\sigma_0^2}\cdot\sqrt{s\sigma_e^2+\sigma_0^2}}\\&=\sqrt{\frac{t\sigma_e^2+\sigma_0^2}{s\sigma_e^2+\sigma_0^2}}\end{aligned} \tag{24}$$

Problem 7

Suppose that $\{Y_t\}$ is an $AR(1)$ process with $-1 < \phi < +1$. $Y_t = \phi Y_{t-1} + e_t$, $e_t \sim IWN\left(0, \sigma_e^2\right)$

(a) Show that $\text{Var}\left(W_t\right) = 2\sigma_e^2/(1+\phi)$.

(b) Find the autocovariance function for $W_t = \nabla Y_t = Y_t - Y_{t-1}$ in terms of ϕ and σ_e^2.

Solution

Subproblem(a)

Firstly, we rewrite the form of $\{Y_t\}$ as follow

$$\begin{aligned}
& Y_t = \phi Y_{t-1} + e_t \\
\Longrightarrow\ & \left(1 - \phi\mathbf{B}\right) Y_t = e_t \\
\Longrightarrow\ & Y_t = \left(1 - \phi\mathbf{B}\right)^{-1} e_t \\
& \quad = \left(1 + \phi\mathbf{B} + \phi^2\mathbf{B}^2 + \phi^3\mathbf{B}^3 + \cdots\right) e_t \\
& \quad = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots \\
& \quad = \sum_{i=0}^{\infty} \phi^i e_{t-i}
\end{aligned} \tag{25}$$

Then, we can derive the varaince of $\{Y_t\}$ as follow

$$\begin{aligned}
\text{Var}\left(Y_t\right) &= \text{Var}\left(\sum_{i=0}^{\infty} \phi^i e_{t-i}\right) \\
&= \sum_{i=0}^{\infty} \text{Var}\left(\phi^i e_{t-i}\right) \\
&= \sum_{i=0}^{\infty} \left(\phi^2\right)^i \sigma_e^2 \\
&= \sigma_e^2 \sum_{i=0}^{\infty} \left(\phi^2\right)^i \\
&= \frac{\sigma_e^2}{1-\phi^2}
\end{aligned} \tag{26}$$

Then, we can derive the autocovariance of Y_t and Y_{t-k} as follow

$$\begin{aligned}
\text{Cov}\left(Y_t, Y_{t-k}\right) &= \text{Cov}\left(\sum_{i=0}^{\infty} \phi^i e_{t-i}, \sum_{j=0}^{\infty} \phi^j e_{t-k-j}\right) \\
&= \sum_{i=j+k} \phi^i \phi^j \,\text{Cov}\left(e_{t-i}, e_{t-k-j}\right) \\
&= \phi \sum_{j=0}^{\infty} \left(\phi^2\right)^j \sigma_e^2 \\
&= \phi^k \sigma_e^2 \sum_{j=0}^{\infty} \left(\phi^2\right)^j \\
&= \frac{\phi^k \sigma_e^2}{1-\phi^2}
\end{aligned} \tag{27}$$

Thus, we can derive $\text{Var}(W_t)$ as follow

$$\begin{aligned}
\text{Var}(W_t) &= \text{Var}(Y_t - Y_{t-1}) \\
&= \text{Var}(Y_t) + \text{Var}(Y_{t-1}) - 2\,\text{Cov}(Y_t, Y_{t-1}) \\
&= 2\sigma_e^2 \cdot \frac{1}{1-\phi^2} - 2\phi\sigma_e^2 \cdot \frac{1}{1-\phi^2} \\
&= \frac{2\sigma_e^2(1-\phi)}{1-\phi^2} \\
&= \frac{2\sigma_e^2(1-\phi)}{(1+\phi)(1-\phi)} \\
&= \frac{2\sigma_e^2}{1+\phi}
\end{aligned} \tag{28}$$

Subproblem(b)

- For $k = 0$

$$\begin{aligned}
\gamma_k = \gamma_0 &= \text{Cov}(W_t, W_t) \\
&= \text{Cov}(Y_t - Y_{t-1}, Y_t - Y_{t-1}) \\
&= \text{Cov}(Y_t, Y_t) - \text{Cov}(Y_t, Y_{t-1}) - \text{Cov}(Y_{t-1}, Y_t) + \text{Cov}(Y_{t-1}, Y_{t-1}) \\
&= \frac{\sigma_e^2}{1-\phi^2} - \frac{\phi\sigma_e^2}{1-\phi^2} - \frac{\phi\sigma_e^2}{1-\phi^2} + \frac{\sigma_e^2}{1-\phi^2} \\
&= \frac{\sigma_e^2}{1-\phi^2} \cdot (2 - 2\phi)
\end{aligned} \tag{29}$$

- For $k \geq 1$

$$\begin{aligned}
\gamma_k &= \text{Cov}(W_t, W_{t+k}) \\
&= \text{Cov}(Y_t - Y_{t-1}, Y_{t+k} - Y_{t+k-1}) \\
&= \text{Cov}(Y_t, Y_{t+k}) - \text{Cov}(Y_t, Y_{t+k-1}) - \text{Cov}(Y_{t-1}, Y_{t+k}) + \text{Cov}(Y_{t-1}, Y_{t+k-1}) \\
&= \phi^k\sigma_e^2 \cdot \frac{1}{1-\phi^2} - \phi^{k-1}\sigma_e^2 \cdot \frac{1}{1-\phi^2} - \phi^{k+1}\sigma_e^2 \cdot \frac{1}{1-\phi^2} + \phi^k\sigma_e^2 \cdot \frac{1}{1-\phi^2} \\
&= \frac{\sigma_e^2}{1-\phi^2} \cdot \left(-\phi^{k-1} + 2\phi^k - \phi^{k+1}\right)
\end{aligned} \tag{30}$$

Above all, we can find the autocovariance function for W_t as follow

$$\gamma_k = \begin{cases} \frac{\sigma_e^2}{1-\phi^2} \cdot (2 - 2\phi) & (k = 0) \\ \frac{\sigma_e^2}{1-\phi^2} \cdot \left(-\phi^{k-1} + 2\phi^k - \phi^{k+1}\right) & (k \geq 1) \end{cases} \tag{31}$$